Dunkl harmonic oscillator and Witten's perturbation on strata

(Oscilador armónico de Dunkl y perturbación de Witten en estratos)

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Tesis realizada en el Departamento de Geometría y Topología de la Facultad de Matemáticas, bajo la dirección del profesor Jesús Antonio Álvarez López, para obtener el grado de Doctor en Ciencias Matemticas por la Universidad de Santiago de Compostela.

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Contents

Abstract

In the first part, eigenfunction estimates and embedding results are proved for the Dunkl harmonic oscillator on the line. These kind of results are generalized to operators on $\mathbb{R}_+$ of the form $P = -\frac{d^2}{dx^2} + sx^2 - 2f_1\frac{d}{dx} + f_2$, where $s > 0$, and f_1 and f_2 are functions satisfying $f_2 = \sigma(\sigma - 1)x^{-2} - f_1^2 - f_1'$ for some $\sigma > -1/2$.

The second part contains the main result, which is a version of Morse inequalities for the minimum and maximum ideal boundary conditions of the de Rham complex on strata endowed with adapted metrics, where compact Thom-Mather stratifications are considered. An adaptation of the analytic method of Witten is used in the proof. The local analysis is reduced to the study of the operator P of the first part.

Introduction

The main goal of this work is to use Witten's perturbation method to prove a version of Morse inequalities for the minimum and maximum ideal boundary conditions of the de Rham complex on strata, endowed with adapted metrics, where compact Thom-Mather stratifications are considered. For that purpose, we study first eigenfunction estimates and embedding results for the Dunkl harmonic oscillator on the line, which are generalized to other related operators on $\mathbb{R}_+$. The study of these operators is the key ingredient in our local analysis of the Witten's perturbation.

Thus this thesis has two main parts, Parts 1 and 2. Part 1 is devoted to the study of eigenfunction estimates and embedding results for the Dunkl harmonic oscillator and related operators. Part 2 deals with the Witten's perturbation on strata, where the first part is used.

This work is published in the preprints [**1, 2**].

Let us introduce those chapters separately and state their main results.

Eigenfunction estimates and embedding theorems

The Dunkl operator T_σ on $C^\infty(\mathbb{R})$, depending on some $\sigma > -1/2$, is the perturbation of the usual derivative that can be defined by setting $T_\sigma = \frac{d}{dx}$ on even functions and $T_\sigma = \frac{d}{dx} + 2\sigma\frac{1}{x}$ on odd functions. This kind of operator, more generally on $\mathbb{R}^n$, was introduced by C.F. Dunkl [**21, 22, 23, 24, 25**]. It gave rise to what is now called Dunkl theory (see the survey article [**57**]). This area had a big development in the last years, mainly due to its applications in Quantum Calogero-Moser-Sutherland models (see e.g. [**10, 52, 37, 38, 61, 3, 4**]). In particular, the Dunkl harmonic oscillator [**55, 26, 50, 49**] is $L_\sigma = -T_\sigma^2 + sx^2$, depending on $s > 0$; i.e., it is given by using T_σ instead of d/dx in the expression of the usual harmonic oscillator $H = -\frac{d^2}{dx^2} + sx^2$.

On the other hand, let p_k ($k \in \mathbb{N}$, including zero[1]) is the sequence of orthogonal polynomials for the measure $e^{-sx^2}|x|^{2\sigma}\,dx$, taken with norm one and positive leading coefficient. Up to normalization, these are the generalized Hermite polynomials [**59**, p. 380, Problem 25]; see also [**16, 20, 27, 17, 55, 56**]. Let $x_{k,k} < x_{k,k-1} < \cdots < x_{k,1}$ denote the roots of each p_k; in particular, $x_{k,k/2}$ is the smallest positive root if k is even. The corresponding generalized Hermite functions are $\phi_k = p_k e^{-sx^2/2}$.

It is known that L_σ, with domain the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$, is essentially self-adjoint in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Moreover the spectrum of its self-adjoint extension, denoted by $\mathcal{L}_\sigma$, consists of the eigenvalues $(2k + 1 + 2\sigma)s$, with corresponding eigenfunctions ϕ_k.

[1] We adopt the convention $0 \in \mathbb{N}$.

We show asymptotic estimates of the functions ϕ_k as $k\to\infty$, which are used to prove embedding theorems, and these results are extended to other related perturbations of H. Even though we consider only the Dunkl harmonic oscillator on the line to begin with, this work is more difficult than in the case of H, and has some new features. It may also give a hint of how to proceed for higher dimension.

To get uniform estimates, we consider the functions $\xi_k=|x|^\sigma\phi_k$ instead of ϕ_k. They satisfy the equation $\xi_k''+q_k\xi_k=0$, where $q_k=(2k+1+2\sigma)s-s^2x^2-\bar\sigma_kx^{-2}$ with $\bar\sigma_k=\sigma(\sigma-(-1)^k)$. Let $\widehat I_k=q^{-1}(\mathbb{R}_+)$ (the oscillation region), which is of the form: $(-b_k,-a_k)\cup(a_k,b_k)$ if $\bar\sigma_k>0$ (for $k>0$), $(-b_k,b_k)$ if $\bar\sigma_k=0$, or $(-b_k,0)\cup(0,b_k)$ if $\bar\sigma_k<0$, where $b_k>a_k>0$ with $b_k\in O(k^{1/2})$ and $a_k\in O(k^{-1/2})$ as $k\to\infty$. If $\bar\sigma_k\ge 0$, then set $\widehat J_k=\widehat I_k$. When $\bar\sigma_k<0$ and k is large enough, the equation $q_k(b)=4\pi/b^2$ has two positive solutions, $b_{k,+}<b_{k,-}$, with $b_{k,+}\in O(k^{-1/2})$. Then set $\widehat J_k=(-b_k,-b_{k,+}]\cup[b_{k,+},b_k)$. The first important estimate proved in Part 1 is the following.

Theorem A. *There exist $C,C',C''>0$, depending on σ and s, such that, for $k\ge 1$:*

(i) $\xi_k^2(x)\le C/\sqrt{q_k(x)}$ *for all $x\in\widehat J_k$;*
(ii) *if k is odd or $\sigma\ge 0$, then $\xi_k^2(x)\le C'k^{-1/6}$ for all $x\in\mathbb{R}$; and*
(iii) *if k is even and $\sigma<0$, then $\xi_k^2(x)\le C''k^{-1/6}$ if $|x|\ge x_{k,k/2}$.*

In the case of Theorem A-(iii), the estimate of ξ_k cannot be extended to $\mathbb{R}\setminus\{0\}$ because these functions are unbounded near zero. Therefore some condition of the type $|x|\ge x_{k,k/2}$ must be assumed; the meaning of this condition is clarified by pointing out that $x_{k,k/2}\in O(k^{-1/2})$ as $k\to\infty$. This weakness is complemented by the following result.

Theorem B. *Suppose that $\sigma<0$. There exist $C'''>0$, depending on σ and s, such that $\phi_k^2(x)\le C'''$ for all k even and all $x\in\mathbb{R}$.*

The following theorem asserts that the type of asymptotic estimates of Theorem A-(ii),(iii) are optimal.

Theorem C. *There exist $C^{(IV)},C^{(V)}>0$, depending on σ and s, such that, for $k\ge 1$:*

(i) $\max_{x\in\mathbb{R}}\xi_k^2(x)\ge C^{(IV)}k^{-1/6}$*; and,*
(ii) *if k is even and $\sigma<0$, then* $\max_{|x|\ge x_{k,k/2}}\xi_k^2(x)\ge C^{(V)}k^{-1/6}$.

To prove Theorems A–C, we apply the method that Bonan-Clark have used with H [**6**]. The estimates are satisfied by the functions ξ_k instead of ϕ_k because the method can be applied to the conjugation $K_\sigma=|x|^\sigma L_\sigma|x|^{-\sigma}$. This method has two steps: first, it estimates the distance from any point x in an oscillation region to some root $x_{k,i}$, and, second, the value of $\xi_k^2(x)$ is estimated by using $|x-x_{k,i}|$. These computations for K_σ become much more involved than in [**6**]; indeed, several cases are considered separately, some of them with significant differences; for instance, some roots $x_{k,i}$ may not be in the oscillation region $\widehat I_k$, and the functions ξ_k may not be bounded, as we said.

The asymptotic distribution of the roots $x_{k,i}$ as $k\to\infty$ also has a well known measure theoretic interpretation [**28, 62, 63**]; specially, the generalized Hermite polynomials are considered in [**62**, Section 4]. However the weak convergence of measures considered in those publications does not seem to give the asymptotic approximation of the roots needed in the first step.

For each $m \in \mathbb{N}$, let $\mathcal{S}^m$ be the Banach space of functions $\phi \in C^m(\mathbb{R})$ with $\sup_x |x^i\phi^{(j)}(x)| < \infty$ for $i + j \leq m$; thus $\mathcal{S} = \bigcap_m \mathcal{S}^m$ with the corresponding Fréchet topology. On the other hand, for each real $m \geq 0$, let W^m_σ be the version of the Sobolev space obtained as Hilbert space completion of $\mathcal{S}$ with respect to the scalar product defined by $\langle\phi,\psi\rangle_{W^m_\sigma} = \langle(1+\mathcal{L}_\sigma)^m\phi,\psi\rangle_\sigma$, where $\langle\ ,\ \rangle_\sigma$ denotes the scalar product of $L^2(\mathbb{R},|x|^{2\sigma}\,dx)$. Let also $W^\infty_\sigma = \bigcap_m W^m_\sigma$ with the corresponding Fréchet topology. The subindex ev/odd is added to any space of functions on $\mathbb{R}$ to indicate its subspace of even/odd functions. The following embedding theorems are proved in Part 1; the second one is a version of the Sobolev embedding theorem.

THEOREM D. *For each* $m \geq 0$, $\mathcal{S}^{M_{m',\mathrm{ev/odd}}}_{\mathrm{ev/odd}} \subset W^m_{\sigma,\mathrm{ev/odd}}$ *continuously*[2] *if* $m' \in \mathbb{N}$, $m' - m > 1/2$, *and*

$$M_{m',\mathrm{ev/odd}} = \begin{cases} \frac{3m'}{2} + \frac{m'}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } m' \text{ is even} \\ \frac{5m'}{2} & \text{if } \sigma < 0 \text{ and } m' \text{ is even}\,, \end{cases}$$

$$M_{m',\mathrm{ev}} = \begin{cases} \frac{3m'-1}{2} + \frac{m'-1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } m' \text{ is odd} \\ \frac{5m'+1}{2} & \text{if } \sigma < 0 \text{ and } m' \text{ is odd}\,, \end{cases}$$

$$M_{m',\mathrm{odd}} = \begin{cases} \frac{3m'+1}{2} + \frac{m'+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } m' \text{ is odd} \\ \frac{5m'+7}{2} & \text{if } \sigma < 0 \text{ and } m' \text{ is odd}\,. \end{cases}$$

THEOREM E. *For all* $m \in \mathbb{N}$, $W^{m'}_\sigma \subset \mathcal{S}^m$ *continuously if*

$$m' - m > \begin{cases} 4 + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+1) & \text{if } \sigma \geq 0 \\ 4 & \text{if } \sigma < 0\,. \end{cases}$$

Moreover $W^{m'}_{\sigma,\mathrm{ev}} \subset \mathcal{S}^0_{\mathrm{ev}}$ *continuously if* $\sigma < 0$ *and* $m' > 2$.

COROLLARY F. $\mathcal{S} = W^\infty_\sigma$ *as Fréchet spaces.*

In other words, Corollary F states that an element $\phi \in L^2(\mathbb{R},|x|^{2\sigma}\,dx)$ is in $\mathcal{S}$ if and only if the "Fourier coefficients" $\langle\phi,\phi_k\rangle_\sigma$ are rapidly decreasing on k. This also means that $\mathcal{S} = \bigcap_m \mathcal{D}(\mathcal{L}^m_\sigma)$ (the smooth core[3] $\mathcal{D}^\infty(\mathcal{L}^m_\sigma)$) because the sequence of eigenvalues of $\mathcal{L}_\sigma$ is in $O(k)$ as $k \to \infty$.

We introduce a perturbed version $\mathcal{S}^m_\sigma$ of every $\mathcal{S}^m$ (Chapter 3), whose definition involves T_σ instead of $\frac{d}{dx}$ and is inspired by the estimates of Theorems A and B. They satisfy much simpler embedding results (Chapter 4): $\mathcal{S}^{m'}_\sigma \subset W^m_\sigma$ if $m' - m > 1/2$, and $W^{m'}_\sigma \subset \mathcal{S}^m_\sigma$ if $m' - m > 1$. The proof of the second embedding uses the estimates of Theorems A and B. Even though $\mathcal{S} = \bigcap_m \mathcal{S}^m_\sigma$, the inclusion relations between the spaces $\mathcal{S}^m_\sigma$ and $\mathcal{S}^{m'}$ are complicated, which motivates the complexity of Theorems D and E.

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[2]Let X and Y be topological vector spaces. It is said that $X \subset Y$ continuously if X is a linear subspace of Y and the inclusion map $X \hookrightarrow Y$ is continuous.

[3]Recall that a *core* of a closed densely defined operator T between Hilbert spaces is any subspace of its domain $\mathcal{D}(T)$ which is dense with the graph norm. If T is self-adjoint, then $\mathcal{D}^\infty(T) = \bigcap_{k\geq 1}\mathcal{D}(T^k)$ is a core for T, which is called its *smooth core* [**11**].

Next, we consider other perturbations of H on $\mathbb{R}_+$ (Chapter 5). Let $\mathcal{S}_{\mathrm{ev},U}$ denote the space of restrictions of even Schwartz functions to some open set U, and set $\phi_{k,U} = \phi_k|_U$. The notation $\mathcal{S}_{\mathrm{ev},+}$ and $\phi_{k,+}$ is used if $U = \mathbb{R}_+$.

Theorem G. *Let*

$$P = H - 2f_1 \frac{d}{dx} + f_2 \tag{1}$$

where $f_1 \in C^1(U)$ *and* $f_2 \in C(U)$ *for some open subset* $U \subset \mathbb{R}_+$ *of full Lebesgue measure. Assume that*

$$f_2 = \sigma(\sigma - 1)x^{-2} - f_1^2 - f_1' \tag{2}$$

for some $\sigma > -1/2$. *Let*

$$h = x^\sigma e^{-F_1} \ , \tag{3}$$

where $F_1 \in C^2(U)$ *is a primitive of* f_1. *Then the following properties hold:*

(i) P*, with domain* $h\,\mathcal{S}_{\mathrm{ev},U}$*, is essentially self-adjoint in* $L^2(\mathbb{R}_+, e^{2F_1}\,dx)$*;*
(ii) *the spectrum of its self-adjoint extension, denoted by* $\mathcal{P}$*, consists of the eigenvalues* $(4k + 1 + 2\sigma)s$ $(k \in \mathbb{N})$ *with multiplicity one and normalized eigenfunctions* $\sqrt{2}\,h\phi_{2k,U}$*; and*
(iii) *the smooth core of* $\mathcal{P}$ *is* $h\,\mathcal{S}_{\mathrm{ev},U}$.

This theorem follows by showing that the stated condition on f_1 and f_2 characterizes the cases where P can be obtained by the following process: first, restricting L_σ to even functions, then restricting to U, and finally conjugating by h. The term of P with $\frac{d}{dx}$ can be removed by conjugation with the product of a positive function, obtaining the operator $H + \sigma(\sigma - 1)x^{-2}$.

Several examples of such type of operator P are given. For instance, we get the following.

Corollary H. *Let* $P = H - 2c_1x^{-1}\frac{d}{dx} + c_2x^{-2}$ *for some* $c_1, c_2 \in \mathbb{R}$. *If there is some* $a \in \mathbb{R}$ *such that*

$$a^2 + (2c_1 - 1)a - c_2 = 0 \ , \tag{4}$$

$$\sigma := a + c_1 > -1/2 \ , \tag{5}$$

then:

(i) P*, with domain* $x^a\,\mathcal{S}_{\mathrm{ev},+}$*, is essentially self-adjoint in* $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$*;*
(ii) *the spectrum of its self-adjoint extension, denoted by* $\mathcal{P}$*, consists of the eigenvalues* $(4k + 1 + 2\sigma)s$ $(k \in \mathbb{N})$ *with multiplicity one and normalized eigenfunctions* $\sqrt{2}\,x^a\phi_{2k,+}$*; and*
(iii) $\mathcal{D}^\infty(\mathcal{P}) = x^a\,\mathcal{S}_{\mathrm{ev},+}$.

In Corollary H, for some $c_1, c_2 \in \mathbb{R}$, there are two values of a satisfying the stated condition, obtaining two different self-adjoint operators defined by P in different Hilbert spaces. For instance, the Dunkl harmonic oscillator L_σ may define self-adjoint operators even when $\sigma \leq -1/2$.

Corollary H will be applied in Part 2 to prove our Morse inequalities on strata of compact Thom-Mather stratifications with adapted metrics.

Witten's perturbation on strata

A Hilbert complex [**11**] is a differential complex given by a densely defined closed operator $\mathbf{d}$ in a graded separable Hilbert space $\mathfrak{H}$. The corresponding Laplacian $\mathbf{\Delta} = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ is a self-adjoint operator in $\mathfrak{H}$. It is said that $\mathbf{d}$ is discrete when $\mathbf{\Delta}$ has a discrete spectrum[4]; in particular, its homology is of finite dimension by a version of the Hodge decomposition.

Let $(\Omega_0(M), d)$ be the compactly supported de Rham complex of a Riemannian manifold M. Its Hilbert complex extensions in $L^2\Omega(M)$ (the graded Hilbert space of square integrable differential forms) are called its ideal boundary conditions (i.b.c.). There is a minimum i.b.c., $d_{\mathrm{min}} = \overline{d}$, and a maximum i.b.c., $d_{\mathrm{max}} = \delta^*$, where δ is de Rham coderivative acting on $\Omega_0(M)$. The Laplacian defined by $d_{\mathrm{min/max}}$ is denoted by $\Delta_{\mathrm{min/max}}$. It is well known that $d_{\mathrm{min}} = d_{\mathrm{max}}$ if M is complete, but suppose that M may not be complete. The i.b.c. $d_{\mathrm{min/max}}$ defines the min/max-cohomology $H^\bullet_{\mathrm{min/max}}(M)$, min/max-Betti numbers $\beta^r_{\mathrm{min/max}} = \beta^r_{\mathrm{min/max}}(M)$, and min/max-Euler characteristic $\chi_{\mathrm{min/max}} = \chi_{\mathrm{min/max}}(M)$ (if the min/max-Betti numbers are finite); these are quasi-isometric invariants of M. These concepts can indeed be defined for arbitrary elliptic complexes [**11**].

From now on, assume that M is a stratum of a compact Thom-Mather stratification A [**60, 44, 45, 64**]. Roughly speaking, around each $x \in \overline{M}$, there is a chart of A with values in a product $\mathbb{R}^m \times c(L)$, where:

- L is a compact Thom-Mather stratification of lower depth, and $c(L) = L \times [0,\infty)/L \times \{0\}$ (the cone with link L);
- x corresponds to $(0, *)$, where $*$ is the vertex of $c(L)$; and,
- near x, M corresponds to $\mathbb{R}^m \times M'$ for some stratum M' of $c(L)$.

We have, either $M' = N \times \mathbb{R}_+$ for some stratum N of L, or $M' = \{*\}$. Note that $x \in M$ just when $M' = \{*\}$. Let $\rho : c(L) \to [0,\infty)$ be the canonical function induced by the second factor projection $L \times [0,\infty) \to [0,\infty)$. The sum of ρ and the norm of $\mathbb{R}^m$ will be also called the canonical function of $\mathbb{R}^m \times c(L)$.

Endow M with a Riemannian metric g, which is adapted in the following sense defined by induction on the depth of M [**13, 14**]: there is a chart around each $x \in \overline{M} \setminus M$ as above such that g is quasi-isometric to a model metric of the form $g_0 + \rho^2\tilde{g} + (d\rho)^2$ on $\mathbb{R}^m \times N \times \mathbb{R}_+$, where g_0 is the Euclidean metric on $\mathbb{R}^m$ and $\tilde{g}$ an adapted metric on N; this $\tilde{g}$ is well defined since $\operatorname{depth} N < \operatorname{depth} M$. Note that g may not be complete. More general adapted metrics are considered in [**47, 48, 8**]. The first main result of Part 2 is the following.

THEOREM I. *With the above notation, the following properties hold:*

(i) $d_{\mathrm{min/max}}$ *is discrete.*
(ii) *Let* $0 \le \lambda_{\mathrm{min/max},0} \le \lambda_{\mathrm{min/max},1} \le \cdots$ *be the eigenvalues of* $\Delta_{\mathrm{min/max}}$*, repeated according to their multiplicities. Then there is some* $\theta > 0$ *such that* $\liminf_k \lambda_{\mathrm{min/max},k}\, k^{-\theta} > 0$.

The discreteness of d_{min} is essentially due to J. Cheeger [**13, 14**]. Theorem I-(ii) is a weak version of the Weyl's assymptotic formula (see e.g. [**54**, Theorem 8.16]). Elliptic theory for the case of conformally conic manifolds was studied in [**12, 39**],

[4]Recall that a self-adjoint operator has a discrete spectrum when there is no essential spectrum; i.e., the spectrum consists of eigenvalues with finite multiplicity without accumulation points.

and a non-commutative index theorem for the case of conical pseudo-manifolds is given in [**19**].

A smooth function f on M is called relatively admissible (or rel-admissible) when the functions $|df|$ and $|\operatorname{Hess} f|$ are bounded. In this case, f may not have any continuous extension to $\overline{M}$, but it has a continuous extension to the (componentwise) metric completion $\widehat{M}$ of M. Then it makes sense to say that $x \in \widehat{M}$ is a rel-critical point of f when there is a sequence (y_k) in M such that $\lim_k y_k = x$ in $\widehat{M}$ and $\lim_k |df(y_k)| = 0$. To say that f is a rel-Morse function on M, it should be also required that $\operatorname{Hess} f$ is "rel-non-degenerate" at each rel-critical point x, but a "rel-Morse lemma" is missing. Thus, instead, we require the existence of a local model of $\widehat{M}$ centered at x of the form $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times c(L_+) \times c(L_-)$ so that:

- M corresponds to the stratum $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, where $M_\pm$ is a stratum of $c(L_\pm)$; and
- f corresponds to a constant plus the model function $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, where $\rho_\pm$ is the canonical function on $\mathbb{R}^{m_\pm} \times c(L_\pm)$.

Either $M_\pm$ is the vertex stratum $\{*_\pm\}$ of $c(L_\pm)$, or $M_\pm = N_\pm \times \mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$; in the second case, let $n_\pm = \dim N_\pm$. This local model makes sense because the product of two Thom-Mather stratifications can be endowed with a Thom-Mather structure; in particular, the product of two cones becomes a cone. There is no canonical choice of a product Thom-Mather structure, but all of them have the same adapted metrics.

For each rel-critical point x of f as above and every $r \in \mathbb{Z}$, define $\nu^r_{x,\mathrm{min/max}}$ in the following way. If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = N_- \times \mathbb{R}_+$, then let

$$\nu^r_{x,\mathrm{min/max}} = \sum_{r_+,r_-} \beta^{r_+}_{\mathrm{min/max}}(N_+)\,\beta^{r_-}_{\mathrm{min/max}}(N_-)\;,$$

where (r_+, r_-) runs in the subset of $\mathbb{Z}^2$ determined by the conditions:

$$r = m_- + r_+ + r_- + 1\;, \tag{6}$$

$$r_+ \le \begin{cases} \frac{n_+}{2} - 1 & \text{if } n_+ \text{ is even} \\ \frac{n_+-3}{2} & \text{if } n_+ \text{ is odd, in the minimum i.b.c. case} \\ \frac{n_+-1}{2} & \text{if } n_+ \text{ is odd, in the maximum i.b.c. case}\;, \end{cases} \tag{7}$$

$$r_- \ge \begin{cases} \frac{n_-}{2} & \text{if } n_- \text{ is even} \\ \frac{n_+-1}{2} & \text{if } n_- \text{ is odd, in the minimum i.b.c. case} \\ \frac{n_-+1}{2} & \text{if } n_- \text{ is odd, in the maximum i.b.c. case}\;, \end{cases} \tag{8}$$

If $M_+ = \{*_+\}$ and $M_- = N_- \times \mathbb{R}_+$, let $\nu^r_{x,\mathrm{min/max}} = \sum_{r_+} \beta^{r_+}_{\mathrm{min/max}}(N_+)$, where r_+ runs in the the set of integers satisfying $r = m_- + r_+$ and (7). If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = \{*_-\}$, let $\nu^r_{x,\mathrm{min/max}} = \sum_{r_-} \beta^{r_-}_{\mathrm{min/max}}(N_-)$, where r_- runs in the the set of integers satisfying $r = m_- + r_- + 1$ and (8). If $M_+ = \{*_+\}$ and $M_- = \{*_-\}$, let[5] $\nu^r_{x,\mathrm{min/max}} = \delta_{r,m_-}$. Finally, let $\nu^r_{\mathrm{min/max}} = \sum_x \nu^r_{x,\mathrm{min/max}}$, where x runs in the rel-critical point set of f. The second main result of Part 2 is the following.

[5]Kronecker's delta is used.

THEOREM J. *With the above notation, we have the inequalities*

$$\beta^0_{\min/\max} \le \nu^0_{\min/\max} ,$$
$$\beta^1_{\min/\max} - \beta^0_{\min/\max} \le \nu^1_{\min/\max} - \nu^0_{\min/\max} ,$$
$$\beta^2_{\min/\max} - \beta^1_{\min/\max} + \beta^0_{\min/\max} \le \nu^2_{\min/\max} - \nu^1_{\min/\max} + \nu^0_{\min/\max} ,$$

etc., and the equality

$$\chi_{\min/\max} = \sum_r (-1)^r \, \nu^r_{\min/\max} .$$

We also show that the existence of rel-Morse functions. For instance, for any smooth action of a compact Lie group G on a closed manifold M, any invariant Morse-Bott function on M whose critical manifolds are orbits induces a rel-Morse function on $G\backslash M$; this provides a rich family of examples where Theorem J can be applied.

To prove Theorem I, it is first shown that the stated properties are "rel-local" (Chapter 10), and it is well known that they are invariant by quasi-isometries. Then the spectrum is studied for the local models $\mathbb{R}^m \times N \times \mathbb{R}_+$ with the model metrics $g_0 + \rho^2\tilde{g} + (d\rho)^2$, assuming that the result holds for N with $\tilde{g}$ by induction. In fact, by the min-max principle, it is enough to make this argument for the minimum/maximum i.b.c. $d_{s,\min/\max}$ of the Witten's perturbation d_s ($s > 0$) of d defined by any rel-Morse function [**68**]; the Laplacian defined by $d_{s,\min/\max}$ is denoted by $\Delta_{s,\min/\max}$. In this way, the proof of Theorem I becomes a step in the proof of Theorem J by using the analytic method of E. Witten; specially, as it is described in [**54**, Chapters 9 and 14].

A part of that method is a local analysis around the rel-critical points; more explicitly, the spectral analysis of the perturbed Laplacian $\Delta_{s,\min/\max}$ defined with the model functions $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$. By the version of the Künneth formula for Hilbert complexes [**11**], this study can be reduced to the case of the functions $\pm\frac{1}{2}\rho^2$ on $N \times \mathbb{R}_+$, where ρ is the canonical function of $c(L)$. Then the discrete spectral decomposition for N with $\tilde{g}$ is used to split the Witten's perturbation of the de Rham complex of $N \times \mathbb{R}_+$ into direct sum of simple elliptic complexes of two types (Chapters 11, 14 and 15), whose Laplacians are given by the perturbation of the harmonic oscillator on $\mathbb{R}_+$ studied in Part 1, which is related to the Dunkl harmonic oscillator. We end up with the spectral properties of $\Delta_{s,\min/\max}$ needed to describe the "cohomological contribution" from the rel-critical points (Chapter 19, Section 3).

Another part of the adaptation of Witten's method is the proof of the "null cohomological contribution" away from the rel-critical points. In this part, some arguments of [**54**, Chapter 14] cannot be used because there is no version of the Sobolev embedding theorem with the Sobolev spaces $W^m(\Delta_{\min/\max})$ defined with $\Delta_{\min/\max}$; such a result may be true, but the usual way to prove it does not work since $W^m(\Delta_{\min/\max})$ may depend on the choice of the adapted metric (Chapter 21). Therefore a new method is applied in that part of the proof (Chapter 19, Section 2), which uses strongly Theorem I-(ii).

By extending f to $\widehat{M}$, Theorem J can be considered as Morse inequalities on the Thom-Mather stratification $\widehat{M}$. In this sense, it would be interesting to compare it with the Morse inequalities of Goresky-MacPherson [**30**, Chapter 6, Section 6.12], where they consider intersection homology with lower middle perversity of complex analytic varieties with Whitney stratifications. Another analytic proof of Morse inequalities was made by U. Luwig in [**41, 42, 43**] for the special case of conformally conic manifolds, but her admissible and Morse functions are different from ours: the norm of their differential is bounded away from zero around the frontier of the stratum, and the norm of their Hessian may be unbounded.

In the future, we hope to extend this work to the case of other types of adapted metrics (those considered in [**47, 48, 8**], or even more general ones); in the case of $d_{\min}$ with the adapted metrics of [**47, 48, 8**], it would give Morse inequalities for the intersection homology with arbitrary perversity. This will require the study of a perturbation of the harmonic oscillator on $\mathbb{R}_+$ more general than in Part 1.

It is also natural to try to extend this work to the case of "rel-Morse-Bott functions", where the rel-critical point set consists of "rel-non-degenerate rel-critical Thom-Mather substratifications".

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Part 1

Eigenfunction estimates and embedding theorems

CHAPTER 1

Preliminaries on the Dunkl harmonic oscillator

Most of the contents of this section are taken or adapted from [**55**].

1. Dunkl operator

Recall that, for any $\phi \in C^\infty = C^\infty(\mathbb{R})$, there is some $\psi \in C^\infty$ such that $\phi(x) - \phi(0) = x\psi(x)$, which also satisfies

$$\psi^{(m)}(x) = \int_0^1 t^m \phi^{(m+1)}(tx)\, dt \tag{9}$$

for all $m \in \mathbb{N}$ (see e.g. [**36**, Theorem 1.1.9]). The notation $\psi = x^{-1}\phi$ is used.

The Dunkl operator, in the case of dimension one, is the differential-difference operator T_σ on C^∞, depending on a parameter $\sigma \in \mathbb{R}$, defined by

$$(T_\sigma\phi)(x) = \phi'(x) + 2\sigma\, \frac{\phi(x) - \phi(-x)}{x}\ .$$

It can be considered as a perturbation of the derivative operator $\frac{d}{dx}$.

Consider the decomposition $C^\infty = C^\infty_{\text{ev}} \oplus C^\infty_{\text{odd}}$, as direct sum of subspaces of even and odd functions. The matrix expressions of operators on C^∞ will be considered with respect to this decomposition. The operator of multiplication by a function h will be denoted also by h. We can write

$$\frac{d}{dx} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}\ , \quad x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}\ ,$$

$$T_\sigma = \begin{pmatrix} 0 & \frac{d}{dx} + 2\sigma x^{-1} \\ \frac{d}{dx} & 0 \end{pmatrix} = \frac{d}{dx} + 2\sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}$$

on C^∞. With

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}\ ,$$

we have

$$[T_\sigma, x] = 1 + 2\Sigma\ , \tag{10}$$

$$T_\sigma\Sigma + \Sigma T_\sigma = x\,\Sigma + \Sigma\,x = 0\ . \tag{11}$$

Consider the perturbed factorial $m!_\sigma$ of each $m \in \mathbb{N}$, which is inductively defined by setting $0!_\sigma = 1$, and

$$m!_\sigma = \begin{cases} (m-1)!_\sigma m & \text{if } m \text{ is even} \\ (m-1)!_\sigma(m + 2\sigma) & \text{if } m \text{ is odd} \end{cases}$$

for $m > 0$. Observe that $m!_\sigma > 0$ if $\sigma > -1/2$, which will be the case of our interest; otherwise, $m!_\sigma$ may be ≤ 0. For $k \leq m$, even when $k!_\sigma = 0$, the quotient $m!_\sigma/k!_\sigma$

can be understood as the product of the factors from the definition of $m!_\sigma$ which are not included in the definition of $k!_\sigma$. For any $\phi \in C^\infty$ and $m \in \mathbb{N}$, we have

$$(T_\sigma^m \phi)(0) = \frac{m!_\sigma}{m!} \phi^{(m)}(0) . \tag{12}$$

This equality follows by (9) and induction on m.

2. Dunkl harmonic oscillator

Recall that, for dimension one, the harmonic oscillator, and the annihilation and creation operators are

$$H = -\frac{d^2}{dx^2} + s^2 x^2 , \quad A = sx + \frac{d}{dx} , \quad A' = sx - \frac{d}{dx}$$

on C^∞. By using T_σ instead of d/dx, we get a perturbations of H, A and A' called Dunkl harmonic oscillator, and Dunkl annihilation and creation operators:

$$L = -T_\sigma^2 + s^2 x^2 = H - 2\sigma \begin{pmatrix} x^{-1} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} x^{-1} \end{pmatrix} ,$$

$$B = sx + T_\sigma = A + 2\sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix} ,$$

$$B' = sx - T_\sigma = A' - 2\sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix} .$$

By (10) and (11),

$$L = BB' - (1 + 2\Sigma)s = B'B + (1 + 2\Sigma)s = \frac{1}{2}(BB' + B'B) , \tag{13}$$

$$[L, B] = -2sB , \quad [L, B'] = 2sB' , \tag{14}$$

$$[B, B'] = 2s(1 + 2\Sigma) , \tag{15}$$

$$[L, \Sigma] = B\Sigma + \Sigma B = B'\Sigma + \Sigma B' = 0 . \tag{16}$$

Recall also that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the space of functions $\phi \in C^\infty$ such that

$$\|\phi\|_{\mathcal{S}^m} = \sum_{i+j \le m} \sup_x |x^i \phi^{(j)}(x)|$$

is finite for all $m \in \mathbb{N}$. This defines a sequence of norms $\| \ \|_{\mathcal{S}^m}$ on $\mathcal{S}$, which is endowed with the corresponding Fréchet topology. The Banach space completion of $\mathcal{S}$ with respect to each norm $\| \ \|_{\mathcal{S}^m}$ will be denoted by $\mathcal{S}^m$. We have $\mathcal{S}^{m+1} \subset \mathcal{S}^m$ continuously, and $\mathcal{S} = \bigcap_m \mathcal{S}^m$. Let us remark that $\|\phi'\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}}$ for all m.

The above decomposition of C^∞ can be restricted to each $\mathcal{S}^m$ and $\mathcal{S}$, giving $\mathcal{S}^m = \mathcal{S}^m_{\text{ev}} \oplus \mathcal{S}^m_{\text{odd}}$ and $\mathcal{S} = \mathcal{S}_{\text{ev}} \oplus \mathcal{S}_{\text{odd}}$. The matrix expressions of operators on $\mathcal{S}$ will be considered with respect to this decomposition. For $\phi \in C^\infty_{\text{ev}}$, $\psi = x^{-1}\psi$ and $i, j \in \mathbb{N}$, it follows from (9) that

$$|x^i \psi^{(j)}(x)| \le \int_0^1 t^{j-i} |(tx)^i \phi^{(j+1)}(tx)| \, dt \le \sup_{y \in \mathbb{R}} |y^i \phi^{(j+1)}(y)|$$

for all $x \in \mathbb{R}$. Thus $\|\psi\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}}$ for all $m \in \mathbb{N}$, obtaining that $\mathcal{S}_{\text{odd}} = x\,\mathcal{S}_{\text{ev}}$ and $x^{-1} : C^\infty_{\text{odd}} \to C^\infty_{\text{ev}}$ restricts to a continuous operator $x^{-1} : \mathcal{S}_{\text{odd}} \to \mathcal{S}_{\text{ev}}$. Therefore $x : \mathcal{S}_{\text{ev}} \to \mathcal{S}_{\text{odd}}$ is an isomorphism of Fréchet spaces, and T_σ, B, B' and L define continuous operators on $\mathcal{S}$.

Let $\langle\ ,\ \rangle_\sigma$ and $\|\ \|_\sigma$ denote the scalar product and norm of $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Assume from now on that $\sigma > -1/2$, and therefore $\mathcal{S}$ is a dense subset of $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. In $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, with domain $\mathcal{S}$, $-T_\sigma$ is adjoint of T_σ, B' is adjoint of B, and L is essentially self-adjoint. The self-adjoint extension of L, with domain $\mathcal{S}$, will be denoted by $\mathcal{L}$, or $\mathcal{L}_\sigma$. Its spectrum consists of the eigenvalues $(2k+1+2\sigma)s$ ($k \in \mathbb{N}$). The corresponding normalized eigenfunctions ϕ_k are inductively defined by

$$\phi_0 = s^{(2\sigma+1)/4}\Gamma(\sigma+1/2)^{-1/2}e^{-sx^2/2}\ , \tag{17}$$

$$\phi_k = \begin{cases} (2ks)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \tag{18}$$

for $k \geq 1$. We also have

$$B\phi_0 = 0\ , \tag{19}$$

$$B\phi_k = \begin{cases} (2ks)^{1/2}\phi_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{1/2}\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \tag{20}$$

for $k \geq 1$. These assertions follow from (13)–(16) like in the case of H.

3. Generalized Hermite polynomials

From (17), (18) and the definition of B', it follows that the functions ϕ_k are the generalized Hermite functions $\phi_k = p_k e^{-sx^2/2}$, where p_k is the sequence of polynomials inductively defined by

$$p_0 = s^{(2\sigma+1)/4}\Gamma(\sigma+1/2)^{-1/2}\ , \tag{21}$$

$$p_k = \begin{cases} (2ks)^{-1/2}(2sxp_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{-1/2}(2sxp_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is odd}\ , \end{cases} \tag{22}$$

for $k \geq 1$. Up to normalization, these are the generalized Hermite polynomials; i.e., the orthogonal polynomials associated with the measure $|x|^{2\sigma}e^{-sx^2}\,dx$ [**59**, p. 380, Problem 25]. Each p_k is of precise degree k, even/odd if k is even/odd, and with positive leading coefficient, denoted by γ_k. By (22),

$$\gamma_k = \begin{cases} k^{-1/2}(2s)^{1/2}\gamma_{k-1} & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2}(2s)^{1/2}\gamma_{k-1} & \text{if } k \text{ is odd}\ . \end{cases} \tag{23}$$

We also have

$$T_\sigma p_0 = 0\ , \tag{24}$$

$$T_\sigma p_k = \begin{cases} (2ks)^{1/2}p_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{1/2}p_{k-1} & \text{if } k \text{ is odd}\ . \end{cases} \tag{25}$$

The following recursion formula follows directly from (22) and (25):

$$p_k = \begin{cases} k^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1+2\sigma)^{1/2}p_{k-2}\big) & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1)^{1/2}p_{k-2}\big) & \text{if } k \text{ is odd}\ . \end{cases} \tag{26}$$

We have $p_k(0)=0$ if and only if k is odd, and $p_k'(0)=0$ if and only if k is even. By (26) and induction on k,

$$p_k(0)=(-1)^{k/2}\sqrt{\frac{(k-1+2\sigma)(k-3+2\sigma)\cdots(1+2\sigma)}{k(k-2)\cdots 2}}\,p_0 \tag{27}$$

if k is even. When k is odd, by (25) and (27),

$$(T_\sigma p_k)(0)=(-1)^{(k-1)/2}\sqrt{\frac{(k+2\sigma)(k-2+2\sigma)\cdots(1+2\sigma)2s}{(k-1)(k-3)\cdots 2}}\,p_0\ ,$$

obtaining

$$p_k'(0)=\frac{(-1)^{(k-1)/2}}{1+2\sigma}\sqrt{\frac{(k+2\sigma)(k-2+2\sigma)\cdots(1+2\sigma)2s}{(k-1)(k-3)\cdots 2}}\,p_0 \tag{28}$$

by (12). From (26) and by induction on k, we also get

$$x^{-1}p_k=\sum_{\ell\in\{0,2,\dots,k-1\}}(-1)^{\frac{k-\ell-1}{2}}\sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}}\,p_\ell \tag{29}$$

if k is odd[1].

The following assertions come from the general theory of orthogonal polynomials [**59**, Chapter III]. All zeros of each polynomial p_k are real and of multiplicity one. Each open interval between consecutive zeros of p_k contains exactly one zero of p_{k+1}, and at least one zero of every p_ℓ with $\ell>k$. Moreover p_k has exactly $\lfloor k/2\rfloor$ positive zeros and $\lfloor k/2\rfloor$ negative zeros. The zeros of each p_k will be denoted $x_{k,1}>x_{k,2}>\cdots>x_{k,k}$. On each interval $(x_{k,i+1},x_{k,i})$, the function p_{k+1}/p_k is strictly increasing, and satisfies

$$\lim_{x\to x_{k,i}^{\pm}}\frac{p_{k+1}(x)}{p_k(x)}=\mp\infty\ .$$

For every polynomial p of degree $\le k-1$, we have

$$p^2(x)\le\int_{-\infty}^{\infty}p^2(t)\,|t|^{2\sigma}e^{-st^2}\,dt\cdot\sum_{\ell=0}^{k}p_\ell^2(x) \tag{30}$$

for all $x\in\mathbb{R}$. The Gauss-Jacobi formula states that there are $\lambda_{k,1},\lambda_{k,2},\dots,\lambda_{k,k}\in\mathbb{R}$ such that, for any polynomial p of degree $\le 2k-1$,

$$\int_{-\infty}^{\infty}p(x)\,|x|^{2\sigma}e^{-sx^2}\,dx=\sum_{i=1}^{k}p(x_{k,i})\lambda_{k,i}\ . \tag{31}$$

LEMMA 1.1. *We have*

$$p_k'^{\,2}(x_{k,i})\lambda_{k,i}=\begin{cases}2s & \textit{if } k \textit{ is even}\\ 2s/(1+2\sigma) & \textit{if } k \textit{ is odd}\ .\end{cases}$$

[1] As a convention, the product of an empty set of factors is 1. Thus $(k-1)(k-3)\cdots(\ell+2)=1$ for $\ell=k-1$ in (29). Similarly, (27) and (28) also hold for $k=0$ and $k=1$, respectively.

PROOF. This is a direct adaptation of the proof of [**6**, Corollary 3]. With

$$p = \frac{p_k p_{k-1}}{x - x_{k,i}} \;,$$

the formula (31) becomes

$$\frac{\gamma_k}{\gamma_{k-1}} = p'_k(x_{k,i}) p_{k-1}(x_{k,i}) \lambda_{k,i} \;,$$

and the result follows from (23)–(25). □

4. Proofs of the properties of the Dunkl harmonic oscillator

For the reader's convenience, we include in this part the formal statements and proofs of the spectral properties of L indicated in Section 2. The polynomials p_k of Section 3 are also used. The reader familiar with this type of arguments can skip this part; there will not be any further reference to it.

LEMMA 1.2. *With $\mathcal{S}$ as domain, $-T_\sigma$ is adjoint of T_σ in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$.*

PROOF. For $\phi \in \mathcal{S}_{\text{even}}$ and $\psi \in \mathcal{S}_{\text{odd}}$,

$$\begin{aligned}
\left\langle \frac{d}{dx}\phi, \psi \right\rangle_\sigma &= \int_{-\infty}^{\infty} \phi' \psi \, |x|^{2\sigma} \, dx \\
&= 2 \int_0^\infty \phi' \psi x^{2\sigma} \, dx \\
&= -2 \int_0^\infty \phi(\psi' x^{2\sigma} + \psi 2\sigma x^{2\sigma-1}) \, dx \\
&= -2 \int_0^\infty \phi(\psi' + 2\sigma x^{-1}\psi) x^{2\sigma} \, dx \\
&= - \int_{-\infty}^{\infty} \phi(\psi' + 2\sigma x^{-1}\psi) \, |x|^{2\sigma} \, dx \\
&= - \left\langle \phi, \left(\frac{d}{dx} + 2\sigma x^{-1} \right) \psi \right\rangle_\sigma \;. \quad \square
\end{aligned}$$

COROLLARY 1.3. *With $\mathcal{S}$ as domain, B' is adjoint of B in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$, and L is symmetric in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$.*

PROOF OF (19) AND (20). By (17),

$$B\phi_0 = s^{(2\sigma+1)/4}\, \Gamma((2\sigma+1)/2)^{-1/2} \left(sx + \frac{d}{dx} \right) e^{-sx^2/2} = 0 \;.$$

Next, we proceed by induction on $k \geq 1$. By (13) and (18),

$$\begin{aligned}
B\phi_1 &= (2(1+2\sigma)s)^{-1/2}\, BB'\phi_0 \\
&= (2(1+2\sigma)s)^{-1/2}\, (B'B + 2(1+2\Sigma)s)\phi_0 \\
&= (2(1+2\sigma)s)^{-1/2}\, 2(1+2\sigma)s\phi_0) \\
&= (2(1+2\sigma)s)^{1/2}\, \phi_0 \;.
\end{aligned}$$

Now, let $k \geq 2$ and suppose that the statement holds for ϕ_{k-1}. To simplify the notation, let $\nu_k = 1 - (-1)^k$. Observe that $\nu_k = \nu_{k-1} + 2\,(-1)^{k-1}$. Then, by (13) and (18) again,

$$\begin{aligned}
B\phi_k &= (2(k+\nu_k\sigma)s)^{-1/2}\,BB'\phi_{k-1}\\
&= (2(k+\nu_k\sigma)s)^{-1/2}\,(B'B+2(1+2\Sigma)s)\phi_{k-1}\\
&= (2(k+\nu_k\sigma)s)^{-1/2}\,((2(k-1+\nu_{k-1}2\sigma)s)^{1/2}B'\phi_{k-2}\\
&\quad + 2(1+(-1)^{k-1}2\sigma)s\phi_{k-1})\\
&= (2(k+\nu_k\sigma)s)^{-1/2}\,(2(k-1+\nu_{k-1}2\sigma)s+2(1+(-1)^{k-1}2\sigma)s)\phi_{k-1}\\
&= (2(k+\nu_k\sigma)s)^{1/2}\,\phi_{k-1}\;. \quad \square
\end{aligned}$$

PROPOSITION 1.4. *For each $k \in \mathbb{N}$, ϕ_k is an eigenfunction of L, normalized in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, with corresponding eigenvalue $(2k+1+2\sigma)s$.*

PROOF. This follows by induction on k. For $k=0$,

$$\begin{aligned}
L\phi_0 = \|\psi_0\|_\sigma^{-1}\,L\psi_0 &= \|\psi_0\|_\sigma^{-1}\,(H\psi_0 - 2\sigma x^{-1}\,\psi_0')\\
&= \|\psi_0\|_\sigma^{-1}\,(1+2\sigma)s\,\psi_0 = (1+2\sigma)s\,\phi_0\;,
\end{aligned}$$

and $\|\phi_0\|_\sigma = 1$ because

$$\int_{-\infty}^{\infty} e^{-sx^2}|x|^{2\sigma}\,dx = 2\int_0^\infty e^{-sx^2}x^{2\sigma}\,dx = s^{-(2\sigma+1)/2}\,\Gamma((2\sigma+1)/2)\;.$$

Now suppose that $k \ge 1$ and the result holds for ϕ_{k-1}. Let $\nu_k = 1-(-1)^k$, like in the proof of (19) and (20). By (13), (14), (18) and Corollary 1.3,

$$\begin{aligned}
L\phi_k &= (2(k+\nu_k\sigma)s)^{-1/2}\,LB'\phi_{k-1}\\
&= (2(k+\nu_k\sigma)s)^{-1/2}\,(B'L+2sB')\phi_{k-1}\\
&= (2(k+\nu_k\sigma)s)^{-1/2}\,((2(k-1)+1+2\sigma)s+2s)B'\phi_{k-1}\\
&= (2k+1+2\sigma)s\,\phi_k\;,\\
\|\phi_k\|_\sigma^2 &= (2(k+\nu_k\sigma)s)^{-1}\,\langle BB'\phi_{k-1},\phi_{k-1}\rangle_\sigma\\
&= (2(k+\nu_k\sigma)s)^{-1}\,\langle (L+(1+2\Sigma)s)\phi_{k-1},\phi_{k-1}\rangle_\sigma\\
&= (2(k+\nu_k\sigma)s)^{-1}\,2(k+\sigma+(-1)^{k-1}\sigma)s\,\|\phi_{k-1}\|_\sigma^2\\
&= 1\;. \quad \square
\end{aligned}$$

The functions ϕ_k form a base of the linear subspace

$$\mathcal{P} = \left\{\,p\,e^{-sx^2/2} \mid p \text{ is a polynomial}\,\right\} \subset \mathcal{S}\;.$$

The density of $\mathcal{P}$ in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$ does not follow from the general theory of orthogonal polynomials [**59**, Section 3.1], and therefore a particular proof must be given like in the case of the Hermite polynomials [**59**, Theorem 5.7.1].

PROPOSITION 1.5. *$\mathcal{P}$ is dense in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$.*

PROOF. For each integer $j \geq 0$, let $f_j(x) = x^j\, e^{-sx^2/2}$. We have

$$\begin{aligned}\|f_j\|_\sigma^2 &= \int_{\infty}^{\infty} x^{2j}\, e^{-sx^2}\, |x|^{2\sigma}\, dx \\ &= 2\int_0^\infty x^{2(j+\sigma)}\, e^{-sx^2}\, dx \\ &= s^{-1/2}\int_0^\infty y^{j+\frac{2\sigma-1}{2}}\, e^{-y}\, dy \\ &= s^{-1/2}\,\Gamma\left(j+\frac{2\sigma+1}{2}\right) \\ &\leq s^{-1/2}\,(j+\lfloor\sigma\rfloor)!\,,\end{aligned}$$

where we have used the substitution $y = sx^2$. Hence

$$\|(i\lambda)^j (j!)^{-1/2}\, f_j\|_\sigma \leq s^{-1/4}(\lfloor\sigma\rfloor!\, 2^{\lfloor\sigma\rfloor})^{1/2}(2^{1/2}|\lambda|)^j (j!)^{-1/2}$$

for each $\lambda \in \mathbb{R}$ because

$$\frac{(j+\lfloor\sigma\rfloor)!}{j!} = \lfloor\sigma\rfloor!\binom{j+\lfloor\sigma\rfloor}{j} \leq \lfloor\sigma\rfloor!\, 2^{j+\lfloor\sigma\rfloor}\,.$$

It follows that the series

$$e^{i\lambda x - sx^2/2} = \sum_{j=0}^\infty \frac{(i\lambda)^j}{j!}\, f_j$$

is convergent in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$; indeed, it belongs to $\overline{\mathcal{P}}$ because $f_j \in \mathcal{P}$. Therefore any f orthogonal to $\overline{\mathcal{P}}$ in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$ satisfies

$$\int_{-\infty}^\infty f(x)\, e^{i\lambda x - sx^2/2}\, |x|^{2\sigma}\, dx = 0$$

for all $\lambda \in \mathbb{R}$, obtaining $f(x)\, e^{-sx^2/2}\, |x|^{2\sigma} = 0$ almost everywhere with respect to dx by Plancherel's theorem. So $f = 0$ almost everywhere with respect to $|x|^{2\sigma}\, dx$. □

The following result is a direct consequence of Propositions 1.4 and 1.5, and Corollary 1.3.

COROLLARY 1.6. *With domain $\mathcal{S}$, the operator L is essentially self-adjoint in $L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$, and its spectrum consists of the eigenvalues and eigenfunctions stated in Proposition 1.4.*

CHAPTER 2

Estimates of the generalized Hermite functions

To get uniform estimates of the functions ϕ_k, they are multiplied by $|x|^\sigma$, obtaining eigenfunctions of another perturbation of H.

1. Second perturbation of H

Now, consider the perturbed derivative,

$$E_\sigma = |x|^\sigma T_\sigma |x|^{-\sigma} = \begin{pmatrix} 0 & \frac{d}{dx} + \sigma x^{-1} \\ \frac{d}{dx} - \sigma x^{-1} & 0 \end{pmatrix} ,$$

and the perturbed harmonic oscillator,

$$K = |x|^\sigma L |x|^{-\sigma} = -E_\sigma^2 + s^2 x^2 = \begin{pmatrix} H + \sigma(\sigma-1)x^{-2} & 0 \\ 0 & H + \sigma(\sigma+1)x^{-2} \end{pmatrix} ,$$

defined on

$$|x|^\sigma \mathcal{S} = |x|^\sigma \mathcal{S}_{\mathrm{ev}} \oplus |x|^\sigma \mathcal{S}_{\mathrm{odd}} .$$

According to Sections 2 and 3 of Chapter 1, and since $|x|^\sigma : L^2(\mathbb{R}, |x|^{2\sigma}\,dx) \to L^2(\mathbb{R}, dx)$ is a unitary isomorphism, K is essentially self-adjoint in $L^2(\mathbb{R}, dx)$, and the spectrum of its self-adjoint extension, denoted by $\mathcal{K}$, or $\mathcal{K}_\sigma$, consists of the eigenvalues $(2k+1+2\sigma)s$ ($k \in \mathbb{N}$) of multiplicity one, and corresponding normalized eigenfunctions

$$\xi_k = |x|^\sigma \phi_k = p_k |x|^\sigma e^{-sx^2/2} .$$

Each ξ_k is C^∞ on $\mathbb{R} \setminus \{0\}$, and it is C^∞ on $\mathbb{R}$ if and only if $\sigma \in \mathbb{N}$. If $\sigma > 0$ or k is odd, then ξ_k is defined and continuous on $\mathbb{R}$, and $\xi_k(0) = 0$. If $\sigma < 0$ and k is even, then ξ_k is only defined on $\mathbb{R} \setminus \{0\}$; in fact, by (27),

$$\lim_{x\to 0} \xi_k(x) = (-1)^{k/2}\infty .$$

By (25) and (26),

$$\begin{aligned} \xi_k' &= \left(p_k' + \left(\frac{\sigma}{x} - sx\right) p_k\right) |x|^\sigma e^{-sx^2/2} && (32) \\ &= \begin{cases} (\sqrt{2ks}\, p_{k-1} + (\frac{\sigma}{x} - sx)p_k)\, |x|^\sigma e^{-sx^2/2} & \text{if } k \text{ is even} \\ (\sqrt{2(k+2\sigma)s}\, p_{k-1} - (\frac{\sigma}{x} + sx)p_k)\, |x|^\sigma e^{-sx^2/2} & \text{if } k \text{ is odd} \end{cases} \\ &= \begin{cases} ((sx + \frac{\sigma}{x})p_k - \sqrt{2(k+1+2\sigma)s}\, p_{k+1})\, |x|^\sigma e^{-sx^2/2} & \text{if } k \text{ is even} \\ ((sx - \frac{\sigma}{x})p_k - \sqrt{2(k+1)s}\, p_{k+1})\, |x|^\sigma e^{-sx^2/2} & \text{if } k \text{ is odd} . \end{cases} && (33) \end{aligned}$$

By (32), (27) and (28),

$$\lim_{x\to 0^\pm} \xi_k'(x) = \begin{cases} 0 & \text{if } \sigma > 1 \text{ or } \sigma = 0 \\ \pm p_k(0) & \text{if } \sigma = 1 \\ \pm(-1)^{k/2}\infty & \text{if } 0 < \sigma < 1 \\ \mp(-1)^{k/2}\infty & \text{if } -1/2 < \sigma < 0 \end{cases}$$

if k is even,

$$\lim_{x\to 0} \xi_k'(x) = \begin{cases} 0 & \text{if } \sigma > 0 \\ p_k'(0) & \text{if } \sigma = 0 \\ (-1)^{(k-1)/2}\infty & \text{if } -1/2 < \sigma < 0 \end{cases}$$

if k is odd, and

$$\lim_{x\to 0^\pm} (\xi_k \xi_k')(x) = \begin{cases} 0 & \text{if } k \text{ is odd or } \sigma = 0 \text{ or } \sigma > 1/2 \\ \pm p_k^2(0)/2 & \text{if } k \text{ is even and } \sigma = 1/2 \\ \pm\infty & \text{if } k \text{ is even and } 0 < \sigma < 1/2 \\ \mp\infty & \text{if } k \text{ is even and } -1/2 < \sigma < 0\ . \end{cases} \tag{34}$$

By (33),

$$\frac{\xi_k'}{\xi_k} = \begin{cases} sx + \frac{\sigma}{x} - \sqrt{2(k+1+2\sigma)s}\,\frac{p_{k+1}}{p_k} & \text{if } k \text{ is even} \\ sx - \frac{\sigma}{x} - \sqrt{2(k+1)s}\,\frac{p_{k+1}}{p_k} & \text{if } k \text{ is odd}\ , \end{cases} \tag{35}$$

which generalizes a formula of [**31**] for the Hermite functions.

For the sake of simplicity, let

$$\bar\sigma_k = \sigma(\sigma - (-1)^k)\ .$$

Each ξ_k satisfies

$$\xi_k'' + q_k\xi_k = 0\ , \tag{36}$$

where

$$q_k = (2k+1+2\sigma)s - s^2x^2 - \bar\sigma_k x^{-2}\ .$$

2. Description of q_k

The following elementary analysis of the functions q_k will be used in Sections 3 and 4. If k is even, then $\bar\sigma_k = 0$ if $\sigma \in \{0,1\}$, and $\bar\sigma_k < 0$ if $0 < \sigma < 1$, and $\bar\sigma_k > 0$ otherwise. When k is odd, we have $\bar\sigma_k = 0$ if $\sigma = 0$, and $\sigma\bar\sigma_k > 0$ if $\sigma \neq 0$. Each q_k is defined and smooth on $\mathbb{R}$ just when $\bar\sigma_k = 0$, otherwise it is defined and smooth only on $\mathbb{R}\setminus\{0\}$. Moreover q_k is even and

$$q_k' = -2s^2x + 2\bar\sigma_k x^{-3}\ .$$

Observe that

$$\lim_{x\to\pm\infty} q_k(x) = -\infty\ , \qquad \lim_{x\to 0} q_k(x) = \begin{cases} -\infty & \text{if } \bar\sigma_k > 0 \\ \infty & \text{if } \bar\sigma_k < 0\ , \end{cases}$$

$$\lim_{x\to\pm\infty} q_k'(x) = \mp\infty\ , \quad \lim_{x\to 0^\pm} q_k'(x) = \begin{cases} \pm\infty & \text{if } \bar\sigma_k > 0 \\ \mp\infty & \text{if } \bar\sigma_k < 0\ . \end{cases}$$

We have the following cases for the zeros of q_k':

- If $\bar\sigma_k > 0$, then q_k' has two zeros, which are
$$\pm x_{\max} = \pm\sqrt{\sqrt{\bar\sigma_k}/s}\ ,$$
At these points, q_k reaches its maximum, which equals $c_{\max}s$ for
$$c_{\max} = 2k + 1 + 2\sigma - 2\sqrt{\bar\sigma_k}\ .$$
Notice that, in this case, $c_{\max} = 0$ if $k = 0$ and $\sigma = -1/8$, $c_{\max} < 0$ if $k = 0$ and $-1/2 < \sigma < -1/8$, and $c_{\max} > 0$ otherwise.
- If $\bar\sigma_k = 0$, then q_k' has one zero, which is 0, where q_k reaches its maximum $c_{\max}s$ as above with $c_{\max} = 2k + 1 + 2\sigma > 0$.
- If $\bar\sigma_k < 0$, then $q_k' > 0$ on $\mathbb{R}_-$ and $q_k' < 0$ on $\mathbb{R}_+$.

We have the following possibilities for the zeros of q_k:

- If $\bar\sigma_k > 0$ and $c_{\max} > 0$, then q_k has four zeros, which are
$$\pm a_k = \pm\sqrt{\frac{2k+1+2\sigma-\sqrt{(2k+1+2\sigma)^2-4\bar\sigma_k}}{2s}}\ ,$$
$$\pm b_k = \pm\sqrt{\frac{2k+1+2\sigma+\sqrt{(2k+1+2\sigma)^2-4\bar\sigma_k}}{2s}}\ . \tag{37}$$
- If $\bar\sigma_k > 0$ and $c_{\max} = 0$, or $\bar\sigma_k \le 0$, then q_k has two zeros, $\pm b_k$, defined by (37).
- If $\bar\sigma_k > 0$ and $c_{\max} < 0$, then $q_k < 0$.

If q_k has four zeros, $\pm a_k$ and $\pm b_k$, then
$$s(b_k - a_k)^2 = c_{\max}\ , \tag{38}$$
and
$$2sa_k^2 = \frac{4\bar\sigma_k}{2k+1+2\sigma+\sqrt{(2k+1+2\sigma)^2-4\bar\sigma_k}}\ ,$$
obtaining
$$a_k \in O(k^{-1/2}) \tag{39}$$
as $k \to \infty$.

If q_k has at least two zeros, $\pm b_k$, then
$$2s(b_k^2 - b_\ell^2) = 2 + 4\,\frac{k^2-\ell^2+(1+2\sigma)(k-\ell)+\bar\sigma_\ell-\bar\sigma_k}{\sqrt{(2k+1+2\sigma)^2-4\bar\sigma_k}+\sqrt{(2\ell+1+2\sigma)^2-4\bar\sigma_\ell}}$$
for $\ell \le k$, obtaining
$$b_{k+1} - b_k \in O(k^{-1/2}) \tag{40}$$
as $k \to \infty$, and
$$b_k - b_\ell \ge C(k-\ell)k^{-1/2} \tag{41}$$
for some $C > 0$ if k and ℓ are large enough. If $\bar\sigma_k = 0$, then $sb_k^2 = c_{\max}$.

Like in [**6**], the maximal open intervals where q_k is defined and > 0 (respectively, < 0) will be called *oscillation* (respectively, *non-oscillation*) intervals of ξ_k; this terminology is justified by Lemma 2.1 bellow. We have the following possibilities for the oscillation intervals:

- If $\bar\sigma_k > 0$ and $c_{\max} > 0$, then ξ_k has two oscillation intervals, (a_k, b_k) and $(-b_k, -a_k)$, containing $x_{\max}$ and $-x_{\max}$, respectively.
- If $\bar\sigma_k > 0$ and $c_{\max} \le 0$, then ξ_k has no oscillation intervals.
- If $\bar\sigma_k < 0$, then ξ_k has two oscillation intervals, $(-b_k, 0)$ and $(0, b_k)$.

- If $\bar{\sigma}_k = 0$, then ξ_k has one oscillation interval, $(-b_k, b_k)$.

3. Location of the zeros of ξ_k and ξ_k'

In $\mathbb{R} \setminus \{0\}$, the functions ξ_k and p_k have the same zeros. Then ξ_k and ξ_k' have no common zeros by (32). The functions ξ_0 and ξ_1 have no zeros in $\mathbb{R} \setminus \{0\}$, and the two zeros $\pm x_{2,1}$ of ξ_2 are in $\mathbb{R} \setminus \{0\}$.

LEMMA 2.1. *On $\mathbb{R} \setminus \{0\}$:*

(i) *the zeros of ξ_k' belong to the oscillation intervals of ξ_k;*
(ii) *if k is odd or $\sigma \geq 0$, the zeros of ξ_k belong to the oscillation intervals of ξ_k; and*
(iii) *if k is even and $\sigma < 0$, the zeros of ξ_k, possibly except $\pm x_{k,k/2}$, belong to the oscillation intervals of ξ_k.*

PROOF. It is enough to consider the zeros in $\mathbb{R}_+$ because ξ_k is either even or odd. We can also assume that $\xi_k \xi_k'$ has zeros on $\mathbb{R}_+$, otherwise there is nothing to prove.

Let x_* and x^* denote the minimum and maximum of the zeros of $\xi_k \xi_k'$ in $\mathbb{R}_+$. By (36),

$$(\xi_k \xi_k')' = {\xi_k'}^2 - q_k \xi_k^2 > 0$$

on the non-oscillation intervals, and therefore $\xi_k \xi_k'$ is strictly increasing on those intervals. In particular, since $\xi_k \xi_k'$ is strictly increasing on (b_k, ∞) and $(\xi_k \xi_k')(x) \to 0$ as $x \to \infty$, it follows that $x^* < b_k$. This shows the statement when there is one oscillation interval of the form $(-b_k, b_k)$. So it remains to consider the case where there is an oscillation interval of ξ_k in $\mathbb{R}_+$ of the form (a_k, b_k). This holds when k is odd and $\sigma > 0$, $k = 0$ and $\sigma \in (-1/8, 0) \cup (1, \infty)$, or $k \in 2\mathbb{Z}_+$ and $\sigma \in (-1/2, 0) \cup (1, \infty)$.

If k is odd and $\sigma > 0$, or k is even and $\sigma \in (1, \infty)$, then $x_* > a_k$ because $\xi_k \xi_k'$ is strictly increasing on $(0, a_k)$ and $(\xi_k \xi_k')(x) \to 0$ as $x \to 0^+$ by (34).

Finally, assume that $k \in 2\mathbb{Z}_+$ and $\sigma \in (-1/2, 0)$. Then the above arguments do not work because $(\xi_k \xi_k')(x) \to -\infty$ as $x \to 0^+$ by (34). Let f be the function on $\mathbb{R}_+$ defined by $f(x) = sx + \frac{\sigma}{x}$. We have $f(x) \to -\infty$ as $x \to 0^+$, and $f' = s - \frac{\sigma}{x^2} > 0$ on $\mathbb{R}_+$. Moreover $\sqrt{-\sigma/s}$ is the unique zero of f in $\mathbb{R}_+$.

If x_* is a zero of ξ_k', then ξ_k (and p_k as well) has no zeros in $[-x_*, x_*]$. Therefore 0 is the unique zero of p_{k+1} in this interval. So $p_{k+1}/p_k > 0$ on $(0, x_*]$. Since

$$0 = f(x_*) - \sqrt{2(k + 1 + 2\sigma)s}\, \frac{p_{k+1}(x_*)}{p_k(x_*)}$$

by (35), it follows that $f(x_*) > 0$, obtaining $x_* > \sqrt{-\sigma/s}$. But

$$a_k^2 = \frac{2k + 1 + 2\sigma - \sqrt{(2k+1)^2 + 8(k+1)\sigma}}{2s} < -\frac{\sigma}{s}$$

because $k > 1$, obtaining $x_* > a_k$.

If x_* is a zero of ξ_k (i.e., $x_* = x_{k,k/2}$), then the other positive zeros of $\xi_k \xi_k'$ are $> a_k$ because this function is strictly increasing on $(0, a_k)$. □

In the case of Lemma 2.1-(iii), the zero $\pm x_{k,k/2}$ of ξ_k may be in an oscillation interval, in a non-oscilation intervals or in their common boundary point. For instance, for $k = 2$,

$$p_2 = \left(\sqrt{\frac{2}{1+2\sigma}} s x^2 - \sqrt{\frac{1+2\sigma}{2}} \right) p_0$$

by (22), obtaining

$$x_{2,1}^2 = \frac{1+2\sigma}{2s} \,.$$

Moreover

$$a_2^2 = \frac{5+2\sigma-\sqrt{25+24\sigma}}{2s} \,.$$

So

$$x_{2,1} - a_2 = \frac{-4+\sqrt{25+24\sigma}}{2s} \,,$$

and therefore $\sigma > -3/8$ if and only if $x_{2,1} > a_2$. So (a_2, b_2) contains no zero of ξ_2 when $\sigma \in (-1/2, -3/8]$. For $k > 2$, every oscillation interval of ξ_k contains some zero of ξ_k by Lemma 2.1.

Lemma 2.2. *There exist $C_0, C_1, C_2 > 0$, depending on σ, such that, if $k \geq C_0$ and I is any oscillation interval of ξ_k, then there is some subinterval $J \subset I$ so that:*

(i) *for every $x \in J$, there exists some zero $x_{k,i}$ of ξ_k in I such that*

$$|x - x_{k,i}| \leq \frac{C_1}{\sqrt{q_k(x)}} \,;$$

(ii) *each connected component of $I \setminus J$ is of length $\leq C_2 k^{-1/2}$.*

Proof. According to Section 2, for any $c > 0$ with $cs \in q_k(I)$, the set $I_c = I \cap q_k^{-1}([cs, \infty))$ is a subinterval of I, whose boundary in I is $I \cap q_k^{-1}(cs)$.

Claim 1. If $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$, then each boundary point of I_c in I satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$.

Let f_c be the function on $\mathbb{R}$ defined by $f_c(x) = \sin(\sqrt{cs}\, x)$, whose zeros are $\ell\pi/\sqrt{cs}$ for $\ell \in \mathbb{Z}$. Since $f_c'' + cs f_c = 0$ and $cs \leq q_k$ on I_c, the zeros of ξ_k in I_c separate the zeros of f_c in I_c by Sturm's comparison theorem. If $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$, then each boundary point x of I_c is at a distance $\leq 2\pi/\sqrt{cs}$ of two consecutive zeros of f_c in I_c, and there is some zero of ξ_k between them, which shows Claim 1 because $q_k(x) = cs$.

Now we have to analyze each type of oscillation interval separately, corresponding to the possibilities for $\bar\sigma_k$ and $c_{\max}$. When there are two oscillation intervals of ξ_k, it is enough to consider only the oscillation interval contained in $\mathbb{R}_+$ because the function ξ_k is either even or odd.

The first type of oscillation interval is of the form $I = (a_k, b_k)$, which corresponds to the conditions $\bar\sigma_k > 0$ and $c_{\max} > 0$. We have $cs \in q_k(I)$ when $0 < c \leq c_{\max}$. Then $q_k^{-1}(cs)$ consists of the points

$$\pm a_{k,c} = \pm\sqrt{\frac{2k+1+2\sigma-c-\sqrt{(2k+1+2\sigma-c)^2-4\bar\sigma_k}}{2s}} \,,$$

$$\pm b_{k,c} = \pm\sqrt{\frac{2k+1+2\sigma-c+\sqrt{(2k+1+2\sigma-c)^2-4\bar\sigma_k}}{2s}} \,, \tag{42}$$

and we get $I_c = [a_{k,c}, b_{k,c}]$. Since

$$s(b_{k,c} - a_{k,c})^2 = c_{\max} - c\ , \tag{43}$$

we have $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c(c_{\max} - c) \geq 4\pi^2$, which means that $c_{\max} \geq 4\pi$ and $c_- \leq c \leq c_+$ for

$$c_\pm = \frac{c_{\max} \pm \sqrt{c_{\max}^2 - 16\pi^2}}{2}\ .$$

Since $c_{\max} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on σ, such that $c_{\max} \geq 4\pi$ for all $k \geq C_0$. Assuming $k \geq C_0$, let $a_{k,\pm} = a_{k,c_\pm}$ and $b_{k,\pm} = b_{k,c_\pm}$, which satisfy

$$a_k < a_{k,-} < a_{k,+} < b_{k,+} < b_{k,-} < b_k\ .$$

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [a_{k,-}, a_{k,+}] \cup [b_{k,+}, b_{k,-}]$ if and only if $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$, and in this case x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $x \in (a_k, a_{k,-}) \cup (b_{k,-}, b_k)$, then $\operatorname{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $x \in (a_{k,+}, b_{k,+})$, then

$$s(b_{k,+} - a_{k,+})^2 = c_{\max} - c_+ = c_- = \frac{16\pi^2}{c_+} \leq \frac{32\pi^2}{c_{\max}} \leq \frac{32\pi^2}{c}$$

by (43), obtaining

$$\operatorname{length}(I_{c_+}) \leq \frac{4\sqrt{2}\pi}{\sqrt{cs}}\ .$$

Since $I_c \subset I_{c_+}$ and it is already proved that I_{c_+} contains some zero of ξ_k, it follows that x also satisfies the condition of (i) with $C_1 = 4\sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 4\sqrt{2}\pi$ if $c_{\max} \geq 4\pi$. In this case, (ii) is obvious because $J = I$.

The second type of oscillation interval is of the form $I = (0, b_k)$, which corresponds to the condition $\bar\sigma_k < 0$. Now, $cs \in q_k(I)$ for any $c > 0$, the set $q_k^{-1}(cs)$ consists of the points $\pm b_{k,c}$, defined like in (42), and we have $I_c = (0, b_{k,c}]$. The equality $cs = q_k(2\pi/\sqrt{cs})$ holds when

$$(2k + 1 + 2\sigma)^2 - 4\bar\sigma_k - 16\pi^2 > 0 \tag{44}$$

and c is

$$c_\pm = 2\pi^2\,\frac{2k + 1 + 2\sigma \pm \sqrt{(2k + 1 + 2\sigma)^2 - 4\bar\sigma_k - 16\pi^2}}{\bar\sigma_k - 4\pi^2}\ .$$

Assuming (44), we have $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c_- \leq c \leq c_+$. Let $b_{k,\pm} = b_{k,c_\pm}$, satisfying $0 < b_{k,+} < b_{k,-} < b_k$.

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [b_{k,+}, b_{k,-}]$ if and only if $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$; in this case, x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. And second, if $x \in (b_{k,-}, b_k)$, then $\operatorname{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. So, when (44) is true, (i) holds with $J = [b_{k,+}, b_k)$ and $C_1 = 2\pi$.

Notice that $c_+ \in O(k)$ as $k \to \infty$. Then there are some $C_0, C_2 > 0$, depending on σ, such that, if $k \geq C_0$, then (44) holds and $sb_{k,+}^2 = 4\pi^2/c_+ \leq C_2 k^{-1}$, showing (ii) in this case.

The third and final type of oscillation interval is $I = (-b_k, b_k)$, which corresponds to the condition $\bar\sigma_k = 0$. We have $cs \in q_k(I)$ when $0 < c \leq c_{\max}$. Then $q_k^{-1}(cs)$ consists of the points $\pm b_{k,c}$, defined like in (42), and we get $I_c = [-b_{k,c}, b_{k,c}]$. Since

$$sb_{k,c}^2 = c_{\max} - c\ , \tag{45}$$

we have $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c(c_{\max} - c) \geq \pi^2$, which means that $c_{\max} \geq \pi$ and $c_- \leq c \leq c_+$ for

$$c_\pm = \frac{c_{\max} \pm \sqrt{c_{\max}^2 - 4\pi^2}}{2}\ .$$

Since $c_{\max} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on σ, such that $c_{\max} \geq 4\pi$ for all $k \geq C_0$. Assuming $k \geq C_0$, let $b_{k,\pm} = b_{k,c_\pm}$, which satisfy $0 < b_{k,+} < b_{k,-} < b_k$.

Fix any $x \in I$ and let $q_k(x) = cs$. First, $b_{k,+} \leq |x| \leq b_{k,-}$ if and only if $\operatorname{length}(I_c) \geq 2\pi/\sqrt{cs}$; in this case, x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $|x| > b_{k,-}$, then $\operatorname{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $|x| < b_{k,+}$, then

$$sb_{k,+}^2 = c_{\max} - c_+ = c_- = \frac{4\pi^2}{c_+} \leq \frac{8\pi^2}{c_{\max}} \leq \frac{8\pi^2}{c}$$

by (45), obtaining

$$\operatorname{length}(I_{c_+}) \leq \frac{\sqrt{2}\,\pi}{\sqrt{cs}}\ .$$

Since $I_c \subset I_{c_+}$ and it is already proved that I_{c_+} contains some zero of ξ_k, it follows that x also satisfies the condition of (i) with $C_1 = \sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 2\pi$. In this case, (ii) is also obvious because $J = I$. □

LEMMA 2.3. *There exist $C_0', C_1', C_2' > 0$, depending on σ and s, such that, if $k \geq C_0'$ and I is any oscillation interval of ξ_k, then there is some subinterval $J' \subset I$ so that:*

(i) *$q_k \geq C_1' k^{1/3}$ on J'; and*
(ii) *each connected component of $I \setminus J'$ is of length $\leq C_2' k^{-1/6}$.*

PROOF. We use the notation of the proof of Lemma 2.2. The same type of argument can be used for all types of oscillation intervals. Thus, e.g., suppose that I is of the type $(0, b_k)$. Since $b_k \in O(k^{1/2})$ as $k \to \infty$, we have $b_k' = b_k - k^{-1/6} \in I$ for k large enough, and

$$q_k(b_k') = -s^2(k^{-1/3} - 2b_k k^{-1/6}) - 4\bar\sigma_k((b_k - k^{-1/6})^{-2} - b_k^{-2}) \in O(k^{1/3})$$

as $k \to \infty$. So there are $C_0', C_1' > 0$, depending on σ and s, such that $b_k' \in I$ and $c' = q_k(b_k') \geq C_1' k^{1/3}$ for $k \geq C_0'$. Then (i) and (ii) hold with $J' = I_{c'} = (0, b_k']$. □

COROLLARY 2.4. *There exist $C_0'', C_1'' > 0$, depending on σ and s, such that, if $k \geq C_0''$ and I is any oscillation interval of ξ_k, then, for each $x \in I$, there exists some zero $x_{k,i}$ of ξ_k in I so that*

$$|x - x_{k,i}| \leq C_1'' k^{-1/6}\ .$$

PROOF. With the notation of Lemmas 2.2 and 2.3, let $C_0'' = \max\{C_0, C_0'\}$ and $C_2'' = \max\{C_2, C_2'\}$. Assume $k \geq C_0''$ and consider the subinterval $J'' = J \cap J' \subset I$. By Lemmas 2.2-(ii) and 2.3-(ii), each connected component of $I \setminus J''$ is of length $\leq C_2'' k^{-1/6}$. Then, for each $x \in I$, there is some $x'' \in J''$ such that $|x - x''| \leq C_2'' k^{-1/6}$. By Lemmas 2.2-(i) and 2.3-(i), there is some zero $x_{k,i}$ of ξ_k in I such that

$$|x'' - x_{k,i}| = \frac{C_1}{\sqrt{q_k(x'')}} \leq \frac{C_1}{\sqrt{C_1'}}\, k^{-1/6} \; .$$

Hence

$$|x - x_{k,i}| \leq \left(C_2'' + C_1/\sqrt{C_1'}\right) k^{-1/6} \; . \quad \square$$

4. Estimates of ξ_k

LEMMA 2.5. *Let I be an oscillation interval of ξ_k, let $x \in I$ and let $x_{k,i}$ be a zero of ξ_k in I. Then*

$$\xi_k^2(x) \leq \begin{cases} \frac{8s}{3}\, |x - x_{k,i}| & \text{if } k \text{ is even} \\ \frac{8s}{3(1+2\sigma)}\, |x - x_{k,i}| & \text{if } k \text{ is odd} \; . \end{cases}$$

PROOF. We can assume that there are no zeros of ξ_k between x and $x_{k,i}$. For the sake of simplicity, suppose also that $x_{k,i} < x$ and $\xi_k > 0$ on $(x_{k,i}, x)$; the other cases are analogous. The key observation of [**6**] is that then the graph of ξ_k on $[x_{k,i}, x]$ is concave down, and therefore

$$\frac{1}{2}\xi_k(x)(x - x_{k,i}) \leq \int_{x_{k,i}}^{x} \xi_k(t)\, dt \; .$$

By Schwartz's inequality and (31), it follows that

$$\left(\frac{1}{2}\xi_k(x)(x - x_{k,i})\right)^2 \leq \left(\int_{-\infty}^{\infty} \frac{p_k^2(t)\, |t|^{2\sigma} e^{-st^2}}{(t - x_{k,i})^2}\, dt\right) \left(\int_{x_{k,i}}^{x} (t - x_{k,i})^2\, dt\right)$$

$$= {p_k'}^2(x_{k,i})\, \lambda_{k,i}\, \frac{(x - x_{k,i})^3}{3} \; ,$$

and the result follows by Lemma 1.1. $\square$

With the notation of Lemma 2.2, for each $k \geq C_0$, let $\widehat{I}_k$ denote the union of the oscillation intervals of ξ_k, and let $\widehat{J}_k \subset \widehat{I}_k$ denote the union of the corresponding subintervals J defined in the proof of Lemma 2.2. More precisely:

- if $\bar{\sigma}_k > 0$ and $c_{\max} > 0$, then $\widehat{J}_k = \widehat{I}_k = (-a_k, -b_k) \cup (a_k, b_k)$;
- if $\bar{\sigma}_k < 0$, then $\widehat{I}_k = (-b_k, 0) \cup (0, b_k)$ and $\widehat{J}_k = (-b_k, -b_{k,+}] \cup [b_{k,+}, b_k)$; and
- if $\bar{\sigma}_k = 0$, then $\widehat{J}_k = \widehat{I}_k = (-b_k, b_k)$.

If $k < C_0$, we also use the notation $\widehat{J}_k = \widehat{I}_k$ for the union of the oscillation intervals, which may be empty if there are no oscillation intervals.

PROOF OF THEOREM A. Part (i) follows from Lemmas 2.2 and 2.5.

In any case, $\xi_k(x) \to 0$ as $x \to \infty$. If moreover k is odd or $\sigma \geq 0$, then ξ_k is continuous on $\mathbb{R}$. Thus ξ_k^2 is bounded and reaches its maximum at some point $\bar{x} \in \mathbb{R}$. Since $\xi_k(0) = 0$ (if $\bar{\sigma}_k \neq 0$) or $0 \in \widehat{I}_k$ (if $\bar{\sigma}_k = 0$), it follows from Lemma 2.1 that $\bar{x} \in \widehat{I}_k$. Then (ii) follows by Corollary 2.4 and Lemma 2.5.

If k is even and $\sigma < 0$, then ξ_k is not defined at 0 and $\xi_k^2(x) \to \infty$ as $x \to 0$. So we can only conclude as above that the restriction of ξ_k^2 to the set defined by $|x| \geq x_{k,k/2}$ is bounded, and reaches its maximum at some point $\bar{x}$ of this set. Then $\bar{x} \in \widehat{I}_k$ by Lemma 2.1, and therefore (iii) holds by Corollary 2.4 and Lemma 2.5. □

Consider the case $\sigma < 0$ and k even, when Theorem A does not provide any estimate of ξ_k^2 around zero. According to Section 3 of Chapter 1, the function $p_k^2(x)$ on the region $|x| \leq x_{k,k/2}$ reaches its maximum at $x = 0$, and moreover $p_k^2(0) < p_0^2$ by (27). Hence $\phi_k^2(x) < p_0^2$ for $|x| \leq x_{k,k/2}$, which complements Theorem A-(iii). On the other hand, $\phi_k^2(x) \leq \xi_k^2(x)$ for $|x| \leq 1$. Moreover $x_{k,k/2} \leq 1$ for k large enough by Corollary 2.4 since $a_k \to 0$ as $k \to \infty$. So Theorem B follows from Theorem A-(iii).

The following lemmas will be used in the proof of Theorem C.

Lemma 2.6. *There is some $F > 0$ such that, for $k \geq 1$ and $x \geq b_{k+1}$,*

$$\xi_k(x) \leq \frac{F k^{-5/12}}{(x - b_k)^2} .$$

Proof. Let $x_0 \in (x_{k,1}, b_k)$ such that $\xi_k'(x_0) = 0$. Since

$$\xi_k'(x) = \int_{x_0}^{x} \xi_k''(t)\, dt$$

and $\xi_k'(x) < 0$ for $x > b_k$, we get

$$\int_{x_0}^{x} q_k(t)\xi_k(t)\, dt > 0$$

for $x > b_k$. Because $\xi_k(x) > 0$ for $x > x_0$, $q_k(x) > 0$ for $x_0 < x < b_k$ and $q_k(x) < 0$ for $x > b_k$, it follows that

$$\int_{x_0}^{b_k} q_k(t)\xi_k(t)\, dt > -\int_{b_k}^{x} q_k(t)\xi_k(t)\, dt . \tag{46}$$

According to Corollary 2.4 and Theorem A-(ii),(iii), for $k \geq C_0''$ and with $\bar{C} = \max\{C', C''\}$, we get

$$\begin{aligned}
\int_{x_0}^{b_k} q_k(t)\xi_k(t)\,dt &\leq \bar{C}^{1/2}k^{-1/12}\int_{x_0}^{b_k} q_k(t)\,dt \\
&= \bar{C}^{1/2}k^{-1/12}\Big((2k+1+2\sigma)s(b_k-x_0) \\
&\quad - \frac{s^2}{3}(b_k^3-x_0^3)+\bar{\sigma}_k(b_k^{-1}-x_0^{-1})\Big) \\
&\leq \bar{C}^{1/2}k^{-1/12}\Big((2k+1+2\sigma)sC_1''k^{-1/6} \\
&\quad - \frac{s^2}{3}(b_k^3-(b_k-C_1''k^{-1/6})^3)+\frac{|\bar{\sigma}_k|\,C_1''k^{-1/6}}{b_k(b_k-C_1''k^{-1/6})}\Big) \\
&\leq \bar{C}^{1/2}k^{-1/12}\Big((2k+1+2\sigma)sC_1''k^{-1/6} \\
&\quad - s^2\Big(C_1''b_k^2k^{-1/6}-C_1''^2b_kk^{-1/3}-\frac{C_1''^3k^{-1/2}}{3}\Big) \\
&\quad + \frac{|\bar{\sigma}_k|\,C_1''k^{-1/6}}{b_k(b_k-C_1''k^{-1/6})}\Big)\,.
\end{aligned}$$

Since

$$2k+1+2\sigma - sb_k^2 = \frac{\bar{\sigma}_k}{sb_k^2}\ ,$$

there is some $F_0 > 0$ such that

$$\int_{x_0}^{b_k} q_k(t)\xi_k(t)\,dt \leq F_0 k^{1/12} \tag{47}$$

for all $k \in \mathbb{N}$.

On the other hand,

$$-\int_{b_k}^{x} q_k(t)\xi_k(t)\,dt \geq -\xi_k(x)\int_{b_k}^{x} q_k(t)\,dt\,.$$

With the substitution $u = t - b_k$, we get

$$q_k(t) = -s^2u(u+2b_k)+\frac{\bar{\sigma}_k}{b_k^2}-\bar{\sigma}_k(u+b_k)^{-2}\ ,$$

giving

$$\begin{aligned}
-\xi_k(x)\int_{b_k}^{x} q_k(t)\,dt &= \xi_k(x)\Big(s^2\Big(\frac{1}{3}(x-b_k)^3+b_k(x-b_k)^2\Big) \\
&\quad - \frac{\bar{\sigma}_k}{b_k^2}(x-b_k)-\bar{\sigma}_k(x^{-1}-b_k^{-1})\Big) \\
&\geq \xi_k(x)\Big(s^2b_k(x-b_k)^2-\frac{|\bar{\sigma}_k|}{b_k^2}(x-b_k)-|\bar{\sigma}_k|\,b_k^{-1}\Big) \\
&\geq \xi_k(x)\Big(\Big(s^2b_k-\frac{|\bar{\sigma}_k|}{b_k^2(b_{k+1}-b_k)}\Big)(x-b_k)^2-|\bar{\sigma}_k|\,b_k^{-1}\Big)
\end{aligned}$$

for $x \geq b_{k+1}$. By (40), it follows that there is some $F_1 > 0$ such that

$$-\int_{b_k}^{x} q_k(t)\xi_k(t)\,dt \geq F_1\xi_k(x)k^{1/2}(x-b_k)^2 \tag{48}$$

for all k and $x \geq b_{k+1}$. Now the result follows from (46)–(48). $\square$

LEMMA 2.7. *For each $\epsilon > 0$, there is some $G > 0$ such that, for all $k \in \mathbb{N}$,*

$$\max_{|x-x_{k,1}|\leq \epsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_\ell^2(x) \leq Gk^{1/6}\,.$$

PROOF. Take any $x \in \mathbb{R}$ such that $|x-x_{k,1}| \leq \epsilon k^{-1/6}$. By Corollary 2.4,

$$|x-b_k| \leq |x-x_{k,1}| + |x_{k,1}-b_k| \leq (\epsilon + C_1'')k^{-1/6} \tag{49}$$

for $k \geq C_0''$. In particular, $b_k < x$ if k is large enough. With this assumption, let $\ell_0, \ell_1, \ell_2 \in \mathbb{N}$ satisfying $0 < \ell_0 < \ell_1 < \ell_2 - 1$, where ℓ_0 and ℓ_1 will be determined later, and ℓ_2 is the maximum of the naturals $\ell < k$ with $b_{\ell'} \leq x$ for all $\ell' \leq \ell$. Let

$$f_\pm(t) = \sqrt{2t+1+2\sigma \pm 1}$$

for $t \geq 1$. We have

$$f_\pm(\ell) - \sqrt{s}b_\ell = 2\,\frac{\pm(2\ell+1+2\sigma)+1+\bar\sigma_\ell}{\left(2\ell+1+2\sigma \pm 2 - \sqrt{(2\ell+1+2\sigma)^2 - 4\bar\sigma_\ell}\right)\left(f_\pm(\ell)+\sqrt{s}b_\ell\right)}$$

for $\ell \in \mathbb{Z}_+$. So, assuming that k is large enough, we can fix ℓ_0, independently of k and x, so that

$$f_-(\ell) < \sqrt{s}b_\ell < f_+(\ell)$$

for all $\ell \geq \ell_0$. We have $f_+(\ell_1) < f_-(\ell_2)$ because $\ell_1 < \ell_2 - 1$. Moreover observe that

$$f_+'(t) = (2(t+1+\sigma))^{-1/2} > 0\,,$$
$$f_+''(t) = -(2(t+1+\sigma))^{-3/2} < 0$$

for all $t \geq 1$. Then, by Lemma 2.6,

$$\sum_{\ell=\ell_0}^{\ell_1-1} \xi_\ell^2(x) \leq \sum_{\ell=\ell_0}^{\ell_1-1} \frac{F^2\ell^{-5/6}}{(x-b_\ell)^4} \leq F^2 \sum_{\ell=\ell_0}^{\ell_1-1} \frac{\ell^{-5/6}}{(b_{\ell_2}-b_\ell)^4}$$
$$\leq F^2\sqrt{s}\sum_{\ell=\ell_0}^{\ell_1-1} \frac{\ell^{-5/6}}{(f_-(\ell_2)-f_+(\ell))^4} \leq F^2\sqrt{s}\int_{\ell_0}^{\ell_1} \frac{t^{-5/6}\,dt}{(f_-(\ell_2)-f_+(t))^4}\,.$$

After integrating by parts four times, we get

$$\int_{\ell_0}^{\ell_1} \frac{t^{-5/6}\,dt}{(f_-(\ell_2)-f_+(t))^4} \leq \frac{\ell_1^{-5/6} {f_+'}^{-1}(\ell_1)}{3(f_-(\ell_2)-f_+(\ell_1))^3} + \frac{5\ell_1^{-11/6} {f_+'}^{-2}(\ell_1)}{36\,(f_-(\ell_2)-f_+(\ell_1))^2}$$
$$+ \frac{55\,\ell_1^{-17/6} {f_+'}^{-3}(\ell_1)}{216\,(f_-(\ell_2)-f_+(\ell_1))} + \frac{935}{1296}\ell_1^{-23/6} {f_+'}^{-4}(\ell_1)\ln(f_-(\ell_2))$$
$$+ \frac{21505}{7776}\ln(f_-(\ell_2))\int_{\ell_0}^{\ell_1} t^{-29/6} {f_+'}^{-4}(t)\,dt\,.$$

Therefore, since $f'_+(t) \in O(t^{-1/2})$ as $t \to \infty$, there exists some $G_1 > 0$, independent of k and x, such that

$$\sum_{\ell=\ell_0}^{\ell_1-1} \xi_\ell^2(x) \le G_1\biggl(\frac{\ell_1^{-1/3}}{(f_-(\ell_2)-f_+(\ell_1))^3} + \frac{\ell_1^{-5/6}}{(f_-(\ell_2)-f_+(\ell_1))^2} + \frac{\ell_1^{-4/3}}{f_-(\ell_2)-f_+(\ell_1)} + \ell_1^{-11/6}\ln(f_-(\ell_2)) + \ln(f_-(\ell_2))\biggr) .$$

We have

$$\ell_1^{-11/6}\ln(f_-(\ell_2)) + \ln(f_-(\ell_2)) \le \ell_2^{1/6}$$

for k large enough. Then $\sum_{\ell=1}^{\ell_0-1}\xi_\ell^2(x)$ has an upper bound of the type of the statement if ℓ_1 satisfies

$$\max\left\{ \frac{\ell_1^{-1/3}}{(f_-(\ell_2)-f_+(\ell_1))^3}, \frac{\ell_1^{-5/6}}{(f_-(\ell_2)-f_+(\ell_1))^2}, \frac{\ell_1^{-4/3}}{f_-(\ell_2)-f_+(\ell_1)} \right\} \le \ell_2^{1/6} . \tag{50}$$

On the other hand, according to Theorem A-(ii),(iii),

$$\sum_{\ell_1}^{\ell_2}\xi_\ell^2(x) \le \bar{C}\sum_{\ell_1}^{\ell_2}\ell^{-1/6} \le \bar{C}\int_{\ell_1}^{\ell_2} y^{-1/6}\,dy = \frac{6\bar{C}}{5}\,(\ell_2^{5/6}-\ell_1^{5/6}) ,$$

where $\bar{C} = \max\{C', C''\}$. Then $\sum_{\ell=\ell_1}^{\ell_2}\xi_\ell^2(x)$ has an upper bound of the type of the statement if

$$\ell_2^{5/6} - \ell_1^{5/6} \le G_2\ell_2^{1/6}$$

for some $G_2 > 0$, independent of k and x, which is equivalent to

$$\ell_1 \ge \ell_2\bigl(1 - G_2\ell_2^{-2/3}\bigr)^{6/5} . \tag{51}$$

Thus we must check the compatibility of (50) with (51) for some ℓ_1 and G_2. By (51) and since, for each $G_2, \delta > 0$, we have $G_2\ell_2^{-2/3} \le \ell_2^{-\frac{2}{3}+\delta}$ for k large enough, we can replace (50) with

$$\max\left\{ \frac{\ell_2^{-1/3}\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^{-2/5}}{(f_-(\ell_2)-f_+(\ell_1))^3}, \frac{\ell_2^{-5/6}\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^{-1}}{(f_-(\ell_2)-f_+(\ell_1))^2}, \frac{\ell_2^{-4/3}\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^{-8/5}}{f_-(\ell_2)-f_+(\ell_1)} \right\} \le \ell_2^{1/6}$$

for some $\delta > 0$, which is equivalent to

$$\ell_1 \le \frac{1}{2}\left(\sqrt{2(\ell_2+\sigma)} - \ell_2^a\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^b\right)^2 - 1 - \sigma$$

for

$$(a,b) \in \{(-1/6,-2/15),(-1/2,-1/2),(-3/2,-8/5)\} .$$

Thus the compatibility of (50) with (51) holds if there is some $G_2, \delta > 0$ such that

$$\ell_2\bigl(1-G_2\ell_2^{-2/3}\bigr)^{6/5} \le \frac{1}{2}\left(\sqrt{2(\ell_2+\sigma)} - \ell_2^a\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^b\right)^2 - 2 - \sigma ,$$

which is equivalent to

$$G_2 \ge \ell_2^{2/3}\left(1 - \left(\frac{1}{2}\left(\sqrt{2(1+\sigma\ell_2^{-1})} - \ell_2^{a-\frac{1}{2}}\bigl(1-\ell_2^{-\frac{2}{3}+\delta}\bigr)^b\right)^2 - (2+\sigma)\ell_2^{-1}\right)^{5/6}\right) .$$

There is some $G_2 > 0$ satisfying this condition because the l'Hôspital rule shows that, for δ small enough, each function

$$t^{2/3}\left(1 - \left(\frac{1}{2}\left(\sqrt{2(1+\sigma t^{-1})} - t^{a-\frac{1}{2}}\left(1-t^{-\frac{2}{3}+\delta}\right)^b\right)^2 - (2+\sigma)t^{-1}\right)^{5/6}\right)$$

is convergent in $\mathbb{R}$ as $t \to \infty$.

Now, if $\ell_2 < k-1$, let ℓ_3 denote the minimum integer $\ell < k$ such that $b_{\ell'} > x$ for all $\ell' \geq \ell$. Also, let $\bar{\sigma}_{\mathrm{min/max}}$ denote the minimum/maximum values of $\bar{\sigma}_\ell$ for $\ell \in \mathbb{N}$. Then

$$\sqrt{\frac{2(\ell_3-1)+1+2\sigma+\sqrt{(2(\ell_3-1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{min}}}}{2s}} \leq x$$
$$< \sqrt{\frac{2(\ell_2+1)+1+2\sigma+\sqrt{(2(\ell_2+1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{max}}}}{2s}}\,,$$

obtaining

$$2(\ell_3-\ell_2)-4$$
$$< \sqrt{(2(\ell_2+1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{max}}} - \sqrt{(2(\ell_3-1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{min}}}\,.$$

If $\ell_3 > \ell_2 + 1$, it follows that

$$(2(\ell_2+1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{max}} > (2(\ell_3-1)+1+2\sigma)^2+4\bar{\sigma}_{\mathrm{min}}\,,$$

giving

$$2\sqrt{\bar{\sigma}_{\mathrm{max}}-\bar{\sigma}_{\mathrm{min}}} > \sqrt{(2(\ell_2+1)+1+2\sigma)^2-(2(\ell_3-1)+1+2\sigma)^2}$$
$$\geq 2(\ell_3-\ell_2)-4\,.$$

Therefore $\sum_{\ell=\ell_2+1}^{\ell_3}\xi^2(x)$ has an upper bound of the type of the statement by Theorem A-(ii),(iii).

Let

$$h(t) = (2t+1+2\sigma)s - s^2x^2 - \bar{\sigma}_{\mathrm{max}}x^{-2}$$

for $t \geq 0$. According to Theorem A-(i), if $\ell_3 < k-1$, then

$$\sum_{\ell=\ell_3+1}^{k-1}\xi_\ell^2(x) \leq C\sum_{\ell=\ell_3+1}^{k-1}\frac{1}{\sqrt{q_\ell(x)}} \leq C\sum_{\ell=\ell_3+1}^{k-1}\frac{1}{\sqrt{h(\ell)}} \leq C\int_{\ell_3}^{k-1}\frac{dt}{\sqrt{h(t)}}$$
$$= \frac{C}{2s}\left(\sqrt{h(k-1)}-\sqrt{h(\ell_3)}\right) \leq \frac{C}{2s}\sqrt{2(k-1-\ell_3)}\,.$$

Hence $\sum_{\ell=\ell_3+1}^{k-1}\xi_\ell^2(x)$ also has an upper bound like in the statement because, by (41), (40) and (49), there is some $G_3, G_4 > 0$ such that

$$G_3(k-1-\ell_3)k^{-1/2} \leq b_{k-1}-b_{\ell_3} \leq b_{k-1}-x \leq G_4k^{-1/6}\,. \quad \square$$

Proof of Theorem C. By (31),

$$1 = \int_{-\infty}^{\infty}\left(\frac{p_k(x)}{x-x_{k,1}}\right)^2\frac{|x|^{2\sigma}e^{-sx^2}}{{p'_k}^2(x_{k,1})\,\lambda_{k,1}}\,dx\,.$$

Thus, by (30) and Lemma 2.7,

$$\int_{|x-x_{k,1}|\le\epsilon k^{-1/6}}\left(\frac{p_k(x)}{x-x_{k,1}}\right)^2\frac{|x|^{2\sigma}e^{-sx^2}}{{p'_k}^2(x_{k,1})\,\lambda_{k,1}}\,dx$$

$$\le\int_{|x-x_{k,1}|\le\epsilon k^{-1/6}}\sum_{\ell=0}^{k-1}\xi_\ell^2(x)\,dx\le 2\epsilon k^{-1/6}\max_{|x-x_{k,1}|\le\epsilon k^{-1/6}}\sum_{\ell=0}^{k-1}\xi_\ell^2(x)\le 2\epsilon G$$

for any $\epsilon>0$. It follows that

$$\int_{|x-x_{k,1}|\ge\epsilon k^{-1/6}}\left(\frac{p_k(x)}{x-x_{k,1}}\right)^2\frac{|x|^{2\sigma}e^{-sx^2}}{{p'_k}^2(x_{k,1})\,\lambda_{k,1}}\,dx\ge\frac{1}{2}\tag{52}$$

when $\epsilon\le\frac{1}{4G}$, which implies part (i) by Lemma 1.1.

When k is even and $\sigma<0$, either $0<x_{k,k/2}<a_k$ or $|x_{k,k/2}-a_k|\le C_1''k^{-1/6}$ for k large enough according to Corollary 2.4. Moreover $|x_{k,1}-b_k|\le C_1''k^{-1/6}$ for k large enough by Corollary 2.4 as well. So, by (39) and (38), there are some $C_0,C_1>0$, independent of k, such that

$$x_{k,k/2}\le a_k+C_1''k^{-1/6}\le C_0k^{-1/2}\ ,$$

$$x_{k,1}-x_{k,k/2}\ge b_k-a_k-2C_1''k^{-1/6}=\sqrt{\frac{c_{\max}}{s}}-2C_1''k^{-1/6}\ge C_1k^{1/2}$$

On the other hand, by (27), there is some $C_2>0$, independent of k, such that $\xi_k^2(x)\le C_2\,|x|^{2\sigma}$ for $|x|\le x_{k,k/2}$. Therefore

$$\int_{|x|\le x_{k,k/2}}\frac{\xi_k^2(x)\,dx}{(x-x_{k,1})^2}\le\frac{C_2}{(x_{k,1}-x_{k,k/2})^2}\int_{|x|\le x_{k,k/2}}|x|^{2\sigma}\,dx$$

$$=\frac{2C_2x_{k,k/2}^{2\sigma+1}}{(2\sigma+1)(x_{k,1}-x_{k,k/2})^2}\le\frac{2C_2C_0^{2\sigma+1}}{(2\sigma+1)C_1^2}\,k^{-\frac{2\sigma+3}{2}}<\frac{2C_2C_0^{2\sigma+1}}{(2\sigma+1)C_1^2}\,k^{-1}\ .$$

This inequality and (52) imply part (ii). □

CHAPTER 3

Perturbed Schwartz space

We introduce a perturbed version $\mathcal{S}_\sigma$ of $\mathcal{S}$. It will be shown that $\mathcal{S}_\sigma = \mathcal{S}$ after all, but the relevance of this new definition to study L will become clear in the next section; in particular, the norms used to define $\mathcal{S}_\sigma$ will be appropriate to show embedding results, like a version of the Sobolev embedding theorem. Since $\mathcal{S}_\sigma$ must contain the functions ϕ_k, Theorems A and B indicate that different definitions must be given for $\sigma \geq 0$ and $\sigma < 0$.

When $\sigma \geq 0$, for any $\phi \in C^\infty$ and $m \in \mathbb{N}$, let

$$\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j\leq m} \sup_x |x|^\sigma \left|x^i T_\sigma^j \phi(x)\right| . \tag{53}$$

This defines a norm $\| \ \|_{\mathcal{S}_\sigma^m}$ on the linear space of functions $\phi \in C^\infty$ with $\|\phi\|_{\mathcal{S}_\sigma^m} < \infty$, and let $\mathcal{S}_\sigma^m$ denote the corresponding Banach space completion. There is a canonical inclusion $\mathcal{S}_\sigma^{m+1} \subset \mathcal{S}_\sigma^m$, and the perturbed Schwartz space is defined as $\mathcal{S}_\sigma = \bigcap_m \mathcal{S}_\sigma^m$, endowed with the corresponding Fréchet topology. In particular, $\mathcal{S}_0$ is the usual Schwartz space $\mathcal{S}$. Like in the case of $\mathcal{S}$, there are direct sum decompositions into subspaces of even and odd functions, $\mathcal{S}_\sigma^m = \mathcal{S}_{\sigma,\mathrm{ev}}^m \oplus \mathcal{S}_{\sigma,\mathrm{odd}}^m$ for each $m \in \mathbb{N}$, and $\mathcal{S}_\sigma = \mathcal{S}_{\sigma,\mathrm{ev}} \oplus \mathcal{S}_{\sigma,\mathrm{odd}}$.

When $\sigma < 0$, the spaces of even and odd functions are considered separately. Let

$$\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j\leq m,\ i+j \text{ even}} \sup_x \left|x^i (T_\sigma^j \phi)(x)\right| + \sum_{i+j\leq m,\ i+j \text{ odd}} \sup_{x\neq 0} |x|^\sigma \left|x^i (T_\sigma^j \phi)(x)\right| \tag{54}$$

for $\phi \in C_{\mathrm{ev}}^\infty$, and let

$$\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j\leq m,\ i+j \text{ even}} \sup_{x\neq 0} |x|^\sigma \left|x^i (T_\sigma^j \phi)(x)\right| + \sum_{i+j\leq m,\ i+j \text{ odd}} \sup_x \left|x^i \,(T_\sigma^j \phi)(x)\right| \tag{55}$$

for $\phi \in C_{\mathrm{odd}}^\infty$. These expressions define a norm $\| \ \|_{\mathcal{S}_\sigma^m}$ on the linear spaces of functions ϕ in C_{odd}^∞ and C_{ev}^∞ with $\|\phi\|_{\mathcal{S}_\sigma^m} < \infty$. The corresponding Banach space completions will be denoted by $\mathcal{S}_{\sigma,\mathrm{odd}}^m$ and $\mathcal{S}_{\sigma,\mathrm{ev}}^m$. Let $\mathcal{S}_\sigma^m = \mathcal{S}_{\sigma,\mathrm{ev}}^m \oplus \mathcal{S}_{\sigma,\mathrm{odd}}^m$, which is also a Banach space by considering e.g. the norm, also denoted by $\| \ \|_{\mathcal{S}_\sigma^m}$, defined by the maximum of the norms on both components. There are canonical inclusions $\mathcal{S}_\sigma^{m+1} \subset \mathcal{S}_\sigma^m$, and let $\mathcal{S}_\sigma = \bigcap_m \mathcal{S}_\sigma^m$, endowed with the corresponding Fréchet topology. We have $\mathcal{S}_\sigma = \mathcal{S}_{\sigma,\mathrm{ev}} \oplus \mathcal{S}_{\sigma,\mathrm{odd}}$ for $\mathcal{S}_{\sigma,\mathrm{ev}} = \bigcap_m \mathcal{S}_{\sigma,\mathrm{ev}}^m$ and $\mathcal{S}_{\sigma,\mathrm{odd}} = \bigcap_m \mathcal{S}_{\sigma,\mathrm{odd}}^m$.

From these definitions, it easily follows that $\mathcal{S}_\sigma$ consists of functions which are C^∞ on $\mathbb{R}\setminus\{0\}$ but *a priori* possibly not even defined at zero, and $\mathcal{S}_\sigma^m \cap C^\infty$ is dense in $\mathcal{S}_\sigma^m$ for all m; thus $\mathcal{S}_\sigma \cap C^\infty$ is dense in $\mathcal{S}_\sigma$.

Obviously, Σ defines a bounded operator on each $\mathcal{S}_\sigma^m$. It is also easy to see that T_σ defines a bounded operator $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ for any m; notice that, when $\sigma < 0$, the role played by the parity of $i+j$ fits well to prove this property. Similarly, x defines a bounded operator $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ for any m because

$$[T_\sigma^j, x] = \begin{cases} jT_\sigma^{j-1} & \text{if } j \text{ is even} \\ (j+2\Sigma)T_\sigma^{j-1} & \text{if } j \text{ is odd} \end{cases}$$

by (10) and (11). So B and B' define bounded operators $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ too, and L defines a bounded operator $\mathcal{S}_\sigma^{m+2} \to \mathcal{S}_\sigma^m$. Therefore T_σ, x, Σ, B, B' and L define continuous operators on $\mathcal{S}_\sigma$.

In order to prove Theorems D and E, we introduce an intermediate weakly perturbed Schwartz space $\mathcal{S}_{w,\sigma}$. Like $\mathcal{S}_\sigma$, it is defined as a Fréchet space of the form $\mathcal{S}_{w,\sigma} = \bigcap_m \mathcal{S}_{w,\sigma}^m$, where each $\mathcal{S}_{w,\sigma}^m$ is the Banach space defined like $\mathcal{S}_\sigma^m$ by using $\frac{d}{dx}$ instead of T_σ in the right hand sides of (53)–(55); in particular, $\mathcal{S}_{w,\sigma}^0 = \mathcal{S}_\sigma^0$ as Banach spaces. The notation $\|\ \|_{\mathcal{S}_{w,\sigma}^m}$ will be used for the norm of $\mathcal{S}_{w,\sigma}^m$. As before, $\mathcal{S}_{w,\sigma}$ consists of functions which are C^∞ on $\mathbb{R}\setminus\{0\}$ but *a priori* possibly not even defined at zero, $\mathcal{S}_{w,\sigma} \cap C^\infty$ is dense in $\mathcal{S}_{w,\sigma}$, there is a canonical decomposition $\mathcal{S}_{w,\sigma} = \mathcal{S}_{w,\sigma,\mathrm{ev}} \oplus \mathcal{S}_{w,\sigma,\mathrm{odd}}$ given by the subspaces of even and odd functions, and $\frac{d}{dx}$ and x define bounded operators on $\mathcal{S}_{w,\sigma}^{m+1} \to \mathcal{S}_{w,\sigma}^m$. Thus $\frac{d}{dx}$ and x define continuous operators on $\mathcal{S}_{w,\sigma}$.

Lemma 3.1. *If $\sigma \geq 0$, then $\mathcal{S}^{m+\lceil\sigma\rceil} \subset \mathcal{S}_{w,\sigma}^m$ continuously for all m.*

Proof. Let $\phi \in \mathcal{S}$. For all i and j, we have

$$|x|^\sigma \left|x^i \phi^{(j)}(x)\right| \leq \left|x^{i+\lceil\sigma\rceil} \phi^{(j)}(x)\right|$$

for $|x| \geq 1$, and

$$|x|^\sigma \left|x^i \phi^{(j)}(x)\right| \leq \left|x^i \phi^{(j)}(x)\right|$$

for $|x| \leq 1$. So

$$\|\phi\|_{\mathcal{S}_{w,\sigma}^m} \leq \|\phi\|_{\mathcal{S}^{m+\lceil\sigma\rceil}}$$

for all m. □

Lemma 3.2. *If $\sigma \geq 0$, then $\mathcal{S}_{w,\sigma}^{m'} \subset \mathcal{S}^m$ continuously for all m, where*

$$m' = m + 1 + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil + 1)$$

Proof. Let $\phi \in \mathcal{S}_{w,\sigma}$. For all i and j,

$$\left|x^i \phi^{(j)}(x)\right| \leq |x|^\sigma \left|x^i \phi^{(j)}(x)\right| \tag{56}$$

for $|x| \geq 1$. It remains to prove an inequality of this type for $|x| \leq 1$, which is the only difficult part of the proof. It will be a consequence of the following assertion.

CLAIM 2. For each $n \in \mathbb{N}$, there are finite families of real numbers, $c^n_{a,b}$, $d^n_{k,\ell}$ and $e^n_{u,v}$, where the indices a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$ with $b, \ell, v \leq M_n = 1 + \frac{n(n+1)}{2}$ and $k \geq n$, such that

$$\phi(x) = \sum_{a,b} c^n_{a,b} x^a \phi^{(b)}(1) + \sum_{k,\ell} d^n_{k,\ell} x^k \phi^{(\ell)}(x) + \sum_{u,v} e^n_{u,v} x^u \int_x^1 t^n \phi^{(v)}(t)\, dt$$

for all $\phi \in C^\infty$.

Assuming that Claim 2 is true, the proof can be completed as follows. Let $\phi \in \mathcal{S}_{w,\sigma}$ and set $n = \lceil \sigma \rceil$. For $|x| \leq 1$, according to Claim 2,

$$\begin{aligned}
|\phi(x)| &\leq \sum_{a,b} |c^n_{a,b}| \left|\phi^{(b)}(1)\right| + \sum_{k,\ell} |d^n_{k,\ell}| \left|x^k \phi^{(\ell)}(x)\right| \\
&\quad + \sum_{u,v} |e^n_{u,v}|\, 2 \max_{|t|\leq 1} \left|t^n \phi^{(v)}(t)\right| \\
&\leq \sum_{i,j} |c^n_{a,b}| \left|\phi^{(b)}(1)\right| + \sum_{k,\ell} |d^n_{k,\ell}|\, |x|^\sigma \left|\phi^{(\ell)}(x)\right| \\
&\quad + \sum_{u,v} |e^n_{u,v}|\, 2 \max_{|t|\leq 1} |t|^\sigma \left|\phi^{(v)}(t)\right| .
\end{aligned}$$

Let $m, i, j \in \mathbb{N}$ with $i + j \leq m$. By applying the above inequality to the function $x^i \phi^{(j)}$, and expressing each derivative $(x^i \phi^{(j)})^{(r)}$ as a linear combination of functions of the form $x^p \phi^{(q)}$ with $p+q \leq i+j+r$, it follows that there is some $C \geq 1$, depending only on σ and m, such that

$$\left|x^i \phi^{(j)}(x)\right| \leq C \, \|\phi\|_{\mathcal{S}^{i+j+M_n}_{w,\sigma}} \tag{57}$$

for $|x| \leq 1$. By (56) and (57),

$$\|\phi\|_{\mathcal{S}^m} \leq C \, \|\phi\|_{\mathcal{S}^{m'}_{w,\sigma}}$$

with $m' = m + M_n$.

Now, let us prove Claim 2. By induction on n and using integration by parts, it is easy to prove that

$$\int_x^1 t^n \phi^{(n+1)}(t)\, dt = \sum_{r=0}^{n} (-1)^{n-r} \frac{n!}{r!} \left(\phi^{(r)}(1) - x^r \phi^{(r)}(x)\right) . \tag{58}$$

This shows directly Claim 2 for $n \in \{0, 1\}$. Proceeding by induction, let $n > 1$ and assume that Claim 2 holds for $n-1$. By (58), it is enough to find appropriate expressions of $x^r \phi^{(r)}(x)$ for $0 < r < n$. For that purpose, apply Claim 2 for $n-1$ to each function $\phi^{(r)}$, and multiply the resulting equality by x^r to get

$$\begin{aligned}
x^r \phi^{(r)}(x) = \sum_{a,b} c^{n-1}_{a,b} x^{r+a} \phi^{(r+b)}(1) &+ \sum_{k,\ell} d^{n-1}_{k,\ell} x^{r+k} \phi^{(r+\ell)}(x) \\
&+ \sum_{u,v} e^{n-1}_{u,v} x^{r+u} \int_x^1 t^{n-1} \phi^{(r+v)}(t)\, dt ,
\end{aligned}$$

where a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$ with $b, \ell, v \leq M_{n-1}$ and $k \geq n-1$; thus $r + k \geq n$ and

$$r + b, r + \ell, r + v \leq n - 1 + M_{n-1} = M_n - 1 .$$

Therefore it only remains to rise the exponent of t by a unit in the integrals of the last sum. Once more, integration by parts makes the job:

$$\int_x^1 t^n \phi^{(r+v+1)}(t)\,dt = \phi^{(r+v)}(1) - x^n \phi^{(r+v)}(x) - n \int_x^1 t^{n-1} \phi^{(r+v)}\,dt\ . \quad \square$$

LEMMA 3.3. *If $\sigma < 0$, then $\mathcal{S}^{m+1}_{w,\sigma} \subset \mathcal{S}^m$ continuously for all m.*

PROOF. Let $i, j \in \mathbb{N}$ such that $i + j \le m$. Since

$$\left|x^i \phi^{(j)}(x)\right| \le \begin{cases} |x|^\sigma \left|x^i \phi^{(j)}(x)\right| & \text{if } 0 < |x| \le 1 \\ |x|^\sigma \left|x^{i+1} \phi^{(j)}(x)\right| & \text{if } |x| \ge 1\ . \end{cases}$$

for any $\phi \in C^\infty$, we get $\|\phi\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}_{w,\sigma}}$. $\square$

LEMMA 3.4. *If $\sigma < 0$, then $\mathcal{S}^{m+2} \subset \mathcal{S}^m_{w,\sigma}$ continuously for all m.*

PROOF. This is proved by induction on m. We have $\|\ \|_{\mathcal{S}^0_{w,\sigma}} = \|\ \|_{\mathcal{S}^0}$ on C^∞_{ev}. On the other hand, for $\phi \in C^\infty_{\text{odd}}$ and $\psi = x^{-1}\phi \in C^\infty_{\text{ev}}$, we get

$$|x|^\sigma |\phi(x)| \le \begin{cases} |\psi(x)| & \text{if } 0 < |x| \le 1 \\ |\phi(x)| & \text{if } |x| \ge 1\ . \end{cases}$$

So, by (9),

$$\|\phi\|_{\mathcal{S}^0_{w,\sigma}} \le \max\{\|\phi\|_{\mathcal{S}^0}, \|\psi\|_{\mathcal{S}^0}\} \le \|\phi\|_{\mathcal{S}^1}\ .$$

Now, assume that $m > 0$ and the result holds for $m-1$. Let $i, j \in \mathbb{N}$ such that $i + j \le m$, and let $\phi \in \mathcal{S}_{\text{ev}}$. If $i = 0$ and j is odd, then $\phi^{(j)} \in \mathcal{S}_{\text{odd}}$. Thus there is some $\psi \in \mathcal{S}_{\text{ev}}$ such that $\phi^{(j)} = x\psi$, obtaining

$$|x|^\sigma \left|\phi^{(j)}(x)\right| \le \begin{cases} |\psi(x)| & \text{if } 0 < |x| \le 1 \\ \left|\phi^{(j)}(x)\right| & \text{if } |x| \ge 1\ . \end{cases}$$

If $i + j$ is odd and $i > 0$, then

$$|x|^\sigma \left|x^i \phi^{(j)}(x)\right| \le \begin{cases} \left|x^{i-1} \phi^{(j)}(x)\right| & \text{if } 0 < |x| \le 1 \\ \left|x^i \phi^{(j)}(x)\right| & \text{if } |x| \ge 1\ . \end{cases}$$

Hence, by (9), there is some $C > 0$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le C \max\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^0}\} \le C \max\left\{\|\phi\|_{\mathcal{S}^m}, \|\phi^{(j)}\|_{\mathcal{S}^1}\right\} \le C\,\|\phi\|_{\mathcal{S}^{m+1}}\ .$$

Finally, let $\phi \in \mathcal{S}_{\text{odd}}$. There is some $\psi \in \mathcal{S}_{\text{ev}}$ such that $\phi = x\psi$. If i is even and $j = 0$, then

$$|x|^\sigma \left|x^i \phi(x)\right| \le \begin{cases} |x^i \psi(x)| & \text{if } 0 < |x| \le 1 \\ |x^i \phi(x)| & \text{if } |x| \ge 1\ . \end{cases}$$

If $i + j$ is even and $j > 0$, then

$$|x|^\sigma \left|x^i \phi^{(j)}(x)\right| \le \begin{cases} \left|x^i \psi^{(j)}(x)\right| + j\,|x|^\sigma \left|x^i \psi^{(j-1)}(x)\right| & \text{if } 0 < |x| \le 1 \\ \left|x^{i+1} \psi^{(j)}(x)\right| + j\,|x|^\sigma \left|x^i \psi^{(j-1)}(x)\right| & \text{if } |x| \ge 1 \end{cases}$$

because

$$\left[\frac{d^j}{dx^j}, x\right] = j\,\frac{d^{j-1}}{dx^{j-1}}\ .$$

Therefore, by (9) and the induction hypothesis, there are some $C', C'' > 0$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le C' \max\left\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^{m+1}} + \|\psi\|_{\mathcal{S}^{m-1}_{w,\sigma}}\right\} \le C'' \|\phi\|_{\mathcal{S}^{m+2}} . \quad \square$$

Corollary 3.5. *$\mathcal{S} = \mathcal{S}_{w,\sigma}$ as Fréchet spaces.*

Corollary 3.6. *x^{-1} defines a bounded operator $\mathcal{S}^{m'}_{w,\sigma,\mathrm{odd}} \to \mathcal{S}^m_{w,\sigma,\mathrm{ev}}$, where*

$$m' = \begin{cases} m + 2 + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil + 3) & \text{if } \sigma \ge 0 \\ m + 4 & \text{if } \sigma < 0 . \end{cases}$$

Proof. If $\sigma \ge 0$, the composite

$$\mathcal{S}^{m+2+\frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+3)}_{w,\sigma,\mathrm{odd}} \hookrightarrow \mathcal{S}^{m+\lceil\sigma\rceil+1}_{\mathrm{odd}} \xrightarrow{x^{-1}} \mathcal{S}^{m+\lceil\sigma\rceil}_{\mathrm{ev}} \hookrightarrow \mathcal{S}^m_{w,\sigma,\mathrm{ev}}$$

is bounded by Lemmas 3.3 and 3.4. If $\sigma < 0$, the composite

$$\mathcal{S}^{m+4}_{w,\sigma,\mathrm{odd}} \hookrightarrow \mathcal{S}^{m+3}_{\mathrm{odd}} \xrightarrow{x^{-1}} \mathcal{S}^{m+2}_{\mathrm{ev}} \hookrightarrow \mathcal{S}^m_{w,\sigma,\mathrm{ev}} ,$$

is bounded by Lemmas 3.1 and 3.2. $\square$

Corollary 3.7. *x^{-1} defines a continuous operator $\mathcal{S}_{w,\sigma,\mathrm{odd}} \to \mathcal{S}_{w,\sigma,\mathrm{ev}}$.*

Lemma 3.8. *$\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{w,\sigma,\mathrm{ev/odd}} \subset \mathcal{S}^m_{\sigma,\mathrm{ev/odd}}$ continuously for all m, where*

$$M_{m,\mathrm{ev/odd}} = \begin{cases} \frac{3m}{2} + \frac{m}{4}\lceil\sigma\rceil(\lceil\sigma\rceil + 3) & \text{if } \sigma \ge 0 \text{ and } m \text{ is even} \\ \frac{5m}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is even} , \end{cases}$$

$$M_{m,\mathrm{ev}} = \begin{cases} \frac{3m-1}{2} + \frac{m-1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil + 3) & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd} \\ \frac{5m-3}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} , \end{cases}$$

$$M_{m,\mathrm{odd}} = \begin{cases} \frac{3m+1}{2} + \frac{m+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil + 3) & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd} \\ \frac{5m+3}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} . \end{cases}$$

Proof. The result follows by induction on m. The statement is true for $m = 0$ because $\mathcal{S}^0_{w,\sigma} = \mathcal{S}^0_\sigma$ as Banach spaces. Now, take any $m > 0$, and assume that the result holds for $m - 1$.

For $\phi \in C^\infty_{\mathrm{ev}}$, $i + j \le m$ with $j > 0$ and $x \in \mathbb{R}$, we have

$$\left|x^i T^j_\sigma \phi(x)\right| = \left|x^i T^{j-1}_\sigma \phi'(x)\right| ,$$

obtaining

$$\|\phi\|_{\mathcal{S}^m_\sigma} \le \|\phi'\|_{\mathcal{S}^{m-1}_\sigma} + \|\phi\|_{\mathcal{S}^m_{w,\sigma}} .$$

But, by the induction hypothesis and since $M_{m,\mathrm{ev}} = M_{m-1,\mathrm{odd}} + 1$, there are some $C, C' > 0$, independent of ϕ, such that

$$\|\phi'\|_{\mathcal{S}^{m-1}_\sigma} \le C \|\phi'\|_{\mathcal{S}^{M_{m-1,\mathrm{odd}}}_{w,\sigma}} \le C' \|\phi\|_{\mathcal{S}^{M_{m,\mathrm{ev}}}_{w,\sigma}} .$$

For $\phi \in C^\infty_{\mathrm{odd}}$, and i, j and x as above, we have

$$\left|x^i T^j_\sigma \phi(x)\right| \le \left|x^i T^{j-1}_\sigma \phi'(x)\right| + 2|\sigma| \left|x^i T^{j-1}_\sigma x^{-1}\phi(x)\right| ,$$

obtaining

$$\|\phi\|_{\mathcal{S}^m_\sigma} \le \|\phi'\|_{\mathcal{S}^{m-1}_\sigma} + 2|\sigma| \|x^{-1}\phi\|_{\mathcal{S}^{m-1}_\sigma} + \|\phi\|_{\mathcal{S}^m_{w,\sigma}} .$$

But, by the induction hypothesis, Corollary 3.6, and since

$$M_{m,\mathrm{odd}} = \begin{cases} M_{m-1,\mathrm{ev}} + 2 + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil + 3) & \text{if } \sigma \geq 0 \\ M_{m-1,\mathrm{ev}} + 4 & \text{if } \sigma < 0 \ , \end{cases}$$

there are some $C, C > 0$, independent of ϕ, such that

$$\begin{aligned} \|\phi'\|_{\mathcal{S}_\sigma^{m-1}} + 2\,|\sigma|\,\|x^{-1}\phi\|_{\mathcal{S}_\sigma^{m-1}} &\leq C\left(\|\phi'\|_{\mathcal{S}_{w,\sigma}^{M_{m-1,\mathrm{ev}}}} + \|x^{-1}\phi\|_{\mathcal{S}_{w,\sigma}^{M_{m-1,\mathrm{ev}}}}\right) \\ &\leq C'\,\|\phi\|_{\mathcal{S}_{w,\sigma}^{M_{m,\mathrm{odd}}}} \ . \quad \square \end{aligned}$$

COROLLARY 3.9. *$\mathcal{S}_{w,\sigma} \subset \mathcal{S}_\sigma$ continuously.*

CHAPTER 4

Perturbed Sobolev spaces

Observe that $\mathcal{S}_\sigma \subset L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Like in the case where $\mathcal{S}$ is considered as domain, it is easy to check that, in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, with domain $\mathcal{S}_\sigma$, B is adjoint of B' and L is symmetric.

LEMMA 4.1. *$\mathcal{S}_\sigma$ is a core[1] of $\mathcal{L}$.*

PROOF. Let R denote the restriction of L to $\mathcal{S}_\sigma$. Then $\mathcal{L} \subset \overline{R} \subset R^* \subset \mathcal{L}^* = \mathcal{L}$ in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$ because $\mathcal{S} \subset \mathcal{S}_\sigma$ by Corollaries 3.5 and 3.9. □

For each $m \in \mathbb{N}$, let W^m_σ be the Hilbert space completion of $\mathcal{S}$ with respect to the scalar product $\langle\ ,\ \rangle_{W^m_\sigma}$ defined by

$$\langle \phi, \psi \rangle_{W^m_\sigma} = \langle (1 + L)^m \phi, \psi \rangle_\sigma\ .$$

The corresponding norm will be denoted by $\|\ \|_{W^m_\sigma}$, whose equivalence class is independent of the parameter s used to define L. In particular, $W^0_\sigma = L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. As usual, $W^{m'}_\sigma \subset W^m_\sigma$ when $m' > m$, and let $W^\infty_\sigma = \bigcap_m W^m_\sigma$, which is endowed with the induced Fréchet topology. Once more, there are direct sum decompositions into subspaces of even and odd (generalized) functions, $W^m_\sigma = W^m_{\sigma,\mathrm{ev}} \oplus W^m_{\sigma,\mathrm{odd}}$ and $W^\infty_\sigma = W^\infty_{\sigma,\mathrm{ev}} \oplus W^\infty_{\sigma,\mathrm{odd}}$.

According to Lemma 4.1, the space W^m_σ can be defined for any real number m by using $(1 + \mathcal{L})^m$, and moreover $\mathcal{S}_\sigma$ can be used instead of $\mathcal{S}$ in its definition.

Obviously, L defines a bounded operator $W^{m+2}_\sigma \to W^m_\sigma$ for each $m \geq 0$, and therefore a continuous operator on W^∞_σ. Moreover, by (16), Σ defines a bounded operator on each W^m_σ, and therefore a continuous operators on W^∞_σ.

LEMMA 4.2. *B and B' define bounded operators $W^{m+1}_\sigma \to W^m_\sigma$ for each m.*

PROOF. This follows by induction on m. For $m = 0$, by (13), for each $\phi \in \mathcal{S}$,

$$\|B\phi\|^2_\sigma = \|B'\phi\|^2_\sigma = \langle B'B\phi, \phi\rangle_\sigma = \langle (L - (1 + 2\Sigma)s)\phi, \phi\rangle_\sigma \leq C_0\, \|\phi\|^2_{W^1_\sigma}$$

for some $C_0 > 0$ independent of ϕ. It follows that B and B' define bounded operators $W^1_\sigma \to L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$.

Now take $m > 0$ and assume that there are some $C_{m-1}, C'_{m-1} > 0$ so that

$$\|B\phi\|^2_{W^{m-1}_\sigma} \leq C_{m-1}\, \|\phi\|^2_{W^m_\sigma}\ , \quad \|B'\phi\|^2_{W^{m-1}_\sigma} \leq C'_{m-1}\, \|\phi\|^2_{W^m_\sigma}$$

[1]Recall that a *core* of a closed densely defined operator T between Hilbert spaces is any subspace of its domain $\mathcal{D}(T)$ which is dense with the graph norm.

for all $\phi \in \mathcal{S}$. Then, by (14),

$$\begin{aligned}\|B\phi\|^2_{W^m_\sigma} &= \langle (1+L)B\phi, B\phi\rangle_{W^{m-1}_\sigma}\\ &= \|B\phi\|^2_{W^{m-1}_\sigma} + \langle LB\phi, B\phi\rangle_{W^{m-1}_\sigma}\\ &= (1-2s)\,\|B\phi\|^2_{W^{m-1}_\sigma} + \langle BL\phi, B\phi\rangle_{W^{m-1}_\sigma}\\ &\le (1-2s)\,\|B\phi\|^2_{W^{m-1}_\sigma} + \|BL\phi\|_{W^{m-1}_\sigma}\,\|B\phi\|_{W^{m-1}_\sigma}\\ &\le C_{m-1}((1-2s)\,\|\phi\|^2_{W^{m-1}_\sigma} + \|L\phi\|_{W^m_\sigma}\,\|\phi\|_{W^m_\sigma})\\ &\le C_m\,\|\phi\|^2_{W^{m+1}_\sigma}\end{aligned}$$

for some $C_m > 0$ independent of ϕ. Similarly,

$$\|B'\phi\|^2_{W^m_\sigma} \le C'_m\,\|\phi\|^2_{W^{m+1}_\sigma}$$

for some $C'_m > 0$ independent of ϕ. □

Remark 1. B' is not adjoint of B in W^m_σ for $m > 0$.

L and Σ preserve $W^m_{\sigma,\mathrm{ev}}$ and $W^m_{\sigma,\mathrm{odd}}$ for each m, whilst B and B' interchange these subspaces.

The motivation of our tour through perturbed Schwartz spaces is the following embedding results; the second one is a version of the Sobolev embedding theorem.

Proposition 4.3. *$\mathcal{S}^{m'}_\sigma \subset W^m_\sigma$ continuously if $m' - m > 1/2$.*

Proposition 4.4. *$W^{m'}_\sigma \subset \mathcal{S}^m_\sigma$ continuously if $m' - m > 1$.*

Corollary 4.5. *$\mathcal{S}_\sigma = W^\infty_\sigma$ as Fréchet spaces.*

For each non-commutative polynomial p (of two variables, X and Y), let p' denote the non-commutative polynomial obtained by reversing the order of the variables in p; e.g., if $p(X,Y) = X^2Y^3X$, then $p'(X,Y) = XY^3X^2$. It will be said that p is *symmetric* if $p(X,Y) = p'(Y,X)$. Notice that any non-commutative polynomial of the form $p'(Y,X)p(X,Y)$ is symmetric. Given any non-commutative polynomial p, the continuous operators $p(B,B')$ and $p'(B',B)$ on $\mathcal{S}_\sigma$ are adjoint from each other in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$; thus $p(B,B')$ is a symmetric operator if p is symmetric. The following lemma will be used in the proof of Proposition 4.3

Lemma 4.6. *For each non-negative integer m, we have*

$$(1+L)^m = \sum_a q'_a(B',B)q_a(B,B')$$

for some finite family of homogeneous non-commutative polynomials q_a of degree $\le m$.

Proof. The result follows easily from the following assertions.

Claim 3. If m is even, then $L^m = g_m(B,B')^2$ for some symmetric homogeneous non-commutative polynomial g_m of degree m.

Claim 4. If m is odd, then

$$L^m = g'_{m,1}(B',B)g_{m,1}(B,B') + g'_{m,2}(B',B)g_{m,2}(B,B')$$

for some homogeneous non-commutative polynomials $g_{m,1}$ and $g_{m,2}$ of degree m.

If m is even, then $L^{m/2} = g_m(B,B')$ for some symmetric homogeneous non-commutative polynomial g_m of degree $\leq m$ by (13). So $L^m = g_m(B,B')^2$, showing Claim 3.

If m is odd, then write $L^{\lfloor m/2 \rfloor} = f_m(B,B')$ as above for some symmetric homogeneous non-commutative polynomial f_m of degree $\leq m-1$. Then, by (13),

$$L^m = \frac{1}{2}\, f_m(B,B')(BB' + B'B) f_m(B,B') \ .$$

Thus Claim 4 follows with

$$g_{m,1}(B,B') = \frac{1}{\sqrt{2}}\, B' f_m(B,B') \ , \quad g_{m,2}(B,B') = \frac{1}{\sqrt{2}}\, B f_m(B,B') \ . \quad \square$$

Proof of Proposition 4.3 when $\sigma \geq 0$. By the definitions of B and B', for each non-commutative polynomial p of degree $\leq m'$ (of three variables), there exists some $C_p > 0$ such that $|x|^\sigma\, |p(x,B,B')\phi|$ is uniformly bounded by $C_p\, \|\phi\|_{\mathcal{S}_\sigma^{m'}}$ for all $\phi \in \mathcal{S}_\sigma$. Write

$$(1+L)^m = \sum_a q'_a(B',B) q_a(B,B')$$

according to Lemma 4.6, and let

$$\bar{q}_a(x,B,B') = x^{m'-m} q_a(B,B') \ .$$

Then, for each $\phi \in \mathcal{S}_\sigma$,

$$\begin{aligned} \|\phi\|^2_{W_\sigma^m} &= \sum_a \|q_a(B,B')\phi\|^2_\sigma \\ &= \sum_a \int_{-\infty}^{\infty} |(q_a(B,B')\phi)(x)|^2\, |x|^{2\sigma}\, dx \\ &\leq 2 \sum_a \left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)}\, dx \right) \|\phi\|^2_{\mathcal{S}_\sigma^{m'}} \ , \end{aligned}$$

where the integral is finite because $-2(m'-m) < -1$. $\square$

Proof of Proposition 4.3 when $\sigma < 0$. Now, for each homogeneous non-commutative polynomial p of degree $d \leq m'$, there is some $C_p > 0$ such that:

- $|p(x,B,B')\phi|$ is uniformly bounded by $C_p\, \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{ev}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$ if d is even, and by $C_p\, \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ if d is odd; and
- $|x|^\sigma\, |p(x,B,B')\phi|$ is uniformly bounded by $C_p\, \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ if d is even, and by $C_p\, \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{ev}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$ if d is odd.

With the notation of Lemma 4.6, let d_a denote the degree of each homogenous non-commutative polynomial q_a, and let $\bar{q}_a(x,B,B')$ be defined like in the previous case. Then, as above,

$$\begin{aligned} \|\phi\|^2_{W_\sigma^m} \leq 2 \sum_{a \text{ with } d_a \text{ even}} &\left(C^2_{q_a} \int_0^1 x^{2\sigma}\, dx + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)+2\sigma}\, dx \right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{ev}}} \\ + 2 \sum_{a \text{ with } d_a \text{ odd}} &\left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)}\, dx \right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{ev}}} \end{aligned}$$

for $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$, and

$$\|\phi\|^2_{W^m_\sigma} \leq 2 \sum_{a \text{ with } d_a \text{ even}} \left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)}\, dx \right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$$
$$+ 2 \sum_{a \text{ with } d_a \text{ odd}} \left(C^2_{q_a} \int_0^1 x^{2\sigma}\, dx + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)+2\sigma}\, dx \right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$$

for $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$, where the integrals are finite because $-1/2 < \sigma < 0$ and $-2(m'-m) < -1$. □

Let $\mathcal{C}$ denote the space of rapidly decreasing sequences of real numbers. Recall that a sequence $c = (c_k) \in \mathbb{R}^{\mathbb{N}}$ is rapidly decreasing if

$$\|c\|_{\mathcal{C}_m} = \sup_k |c_k|(1+k)^m$$

is finite for all $m \geq 0$. These expressions define norms $\| \ \|_{\mathcal{C}_m}$ on $\mathcal{C}$. Let $\mathcal{C}_m$ denote the completion of $\mathcal{C}$ with respect to $\| \ \|_{\mathcal{C}_m}$, which consists of the sequences $c \in \mathbb{R}^{\mathbb{N}}$ with $\|c\|_{\mathcal{C}_m} < \infty$. So $\mathcal{C} = \bigcap_m \mathcal{C}_m$ with the induced Fréchet topology. Also, for each $m \geq 0$, let ℓ^2_m denote the Hilbert space completion of $\mathcal{C}$ with respect to the scalar product $\langle \ , \ \rangle_{\ell^2_m}$ defined by

$$\langle c, c' \rangle_{\ell^2_m} = \sum_k c_k c'_k (1+k)^m$$

for $c = (c_k)$ and $c' = (c'_k)$. The corresponding norm will be denoted by $\| \ \|_{\ell^2_m}$. Thus ℓ^2_m is a weighted version of ℓ^2; in particular, $\ell^2_0 = \ell^2$. Let $\ell^2_\infty = \bigcap_m \ell^2_m$ with the corresponding Fréchet topology.

A sequence $c = (c_k)$ will be called even/odd if $c_k = 0$ for all odd/even k. We get the following direct sum decompositions into subspaces of even and odd sequences:

$$\mathcal{C}_m = \mathcal{C}_{m,\mathrm{ev}} \oplus \mathcal{C}_{m,\mathrm{odd}}\,, \qquad \mathcal{C} = \mathcal{C}_{\mathrm{ev}} \oplus \mathcal{C}_{\mathrm{odd}}\,,$$
$$\ell^2_m = \ell^2_{m,\mathrm{ev}} \oplus \ell^2_{m,\mathrm{odd}}\,, \qquad \ell^2_\infty = \ell^2_{\infty,\mathrm{ev}} \oplus \ell^2_{\infty,\mathrm{odd}}\,.$$

Lemma 4.7. *$\ell^2_{2m} \subset \mathcal{C}_m$ and $\mathcal{C}_{m'} \subset \ell^2_m$ continuously for all m if $2m' - m > 1$.*

Proof. It is easy to see that

$$\|c\|_{\mathcal{C}_m} \leq \|c\|_{\ell^2_{2m}}\,, \quad \|c\|_{\ell^2_m} \leq \|c\|_{\mathcal{C}_{m'}} \left(\sum_k (1+k)^{m-2m'} \right)^{1/2}$$

for any $c \in \mathcal{C}$, where the last series is convergent because $m - 2m' < -1$. □

Corollary 4.8. *$\ell^2_\infty = \mathcal{C}$ as Fréchet spaces.*

According to Section 2 of Chapter 1, the "Fourier coefficients" mapping $\phi \mapsto (\langle \phi_k, \phi \rangle_\sigma)$ defines a quasi-isometry $W^m_\sigma \to \ell^2_m$ for all m, and therefore an isomorphism $W^\infty_\sigma \to \mathcal{C}$ of Fréchet espaces. Notice that the "Fourier coefficients" mapping can be restricted to the even and odd subspaces.

Corollary 4.9. *Any $\phi \in L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$ is in $\mathcal{S}_\sigma$ if and only if its "Fourier coefficients" $\langle \phi_k, \phi \rangle_\sigma$ are rapidly degreasing on k.*

Proof. By Corollary 4.5, the "Fourier coefficients" mapping defines an isomorphism $\mathcal{S}_\sigma \to \mathcal{C}$ of Fréchet spaces. □

There is also a version of the Rellich theorem stated as follows.

PROPOSITION 4.10. *The operator* $W_\sigma^{m'} \hookrightarrow W_\sigma^m$ *is compact for* $m' > m$.

By using the "Fourier coefficients" mapping, Proposition 4.10 follows from the following lemma (see e.g. [**54**, Theorem 5.8]).

LEMMA 4.11. *The operator* $\ell^2_{m'} \hookrightarrow \ell^2_m$ *is compact for* $m' > m$.

PROOF OF PROPOSITION 4.4. For $\phi \in \mathcal{S}_\sigma$, its "Fourier coefficients" $c_k = \langle \phi_k, \phi \rangle_\sigma$ form a sequence $c = (c_k)$ in $\mathcal{C}$, and

$$\sum_k |c_k| \, (1+k)^{m/2} \le \|c\|_{\ell^2_{m'}} \left(\sum_k (1+k)^{m-m'} \right)^{1/2}$$

by Cauchy-Schwartz inequality, where the last series is convergent since $m - m' < -1$. Therefore

$$\sum_k |c_k| \, (1+k)^{m/2} \le C \, \|\phi\|_{W_\sigma^{m'}} \tag{59}$$

for some $C > 0$ independent of ϕ.

On the other hand, for all $i, j \in \mathbb{N}$ with $i + j \le m$, there is some homogeneus non-commutative polynomial $p_{i,j}$ of degree $i+j$ such that $x^i T_\sigma^j = p_{i,j}(B, B')$. Then, by (18)–(20),

$$|\langle \phi_k, x^i T_\sigma^j \phi \rangle_\sigma| \le C_{i,j} (1+k)^{m/2} \sum_{|\ell - k| \le m} |c_\ell| \tag{60}$$

for some $C_{i,j} > 0$ independent of ϕ.

Now suppose that $\sigma \ge 0$. By (59), (60) and Theorem A-(ii), there is some $C'_{i,j} > 0$ independent of ϕ and x so that

$$\begin{aligned} |x|^\sigma \, |x^i T_\sigma^j \phi(x)| &\le |x|^\sigma \sum_k |\langle \phi_k, x^i T_\sigma^j \phi \rangle_\sigma| \, |\phi_k(x)| \\ &= \sum_k |\langle \phi_k, x^i T_\sigma^j \phi \rangle_\sigma| \, |\xi_k(x)| \le C'_{i,j} \, \|\phi\|_{W_\sigma^{m'}} \end{aligned} \tag{61}$$

for all x. Hence $\|\phi\|_{\mathcal{S}_\sigma^m} \le C' \|\phi\|_{W_\sigma^{m'}}$ for some $C' > 0$ independent of ϕ.

Finally assume that $\sigma < 0$. By (59), (60) and Theorem B, there is some $C'_{i,j} > 0$, independent of ϕ and x, so that

$$|x^i T_\sigma^j \phi(x)| \le \sum_k |\langle \phi_k, x^i T_\sigma^j \phi \rangle_\sigma| \, |\phi_k(x)| \le C'_{i,j} \, \|\phi\|_{W_\sigma^{m'}}$$

for all x if $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$ and $i+j$ is even, or $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ and $i+j$ is odd. On the other hand, by (59), (60) and Theorem A-(ii), there is some $C''_{i,j} > 0$, independent of ϕ and x, such that, like in (61),

$$|x|^\sigma \, |x^i T_\sigma^j \phi(x)| \le C''_{i,j} \, \|\phi\|_{W_\sigma^{m'}}$$

for all $x \ne 0$ if $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ and $i+j$ is even, or $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$ and $i+j$ is odd. Therefore there is some $C' > 0$ such that $\|\phi\|_{\mathcal{S}^m_{\sigma,\mathrm{ev}}} \le C' \|\phi\|_{W_\sigma^{m'}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{ev}}$, and $\|\phi\|_{\mathcal{S}^m_{\sigma,\mathrm{odd}}} \le C' \|\phi\|_{W_\sigma^{m'}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$. □

As suggested by (29), consider the mapping $c = (c_k) \mapsto \Xi(c) = (d_\ell)$, where c is odd and $\Xi(c)$ is even with

$$d_\ell = \sum_{k \in \{\ell+1, \ell+3, \dots\}} (-1)^{\frac{k-\ell-1}{2}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}} \, c_k$$

for ℓ even, assuming that this series is convergent.

LEMMA 4.12. *Ξ defines a bounded map $\ell^2_{m',\mathrm{odd}} \to \mathcal{C}_{m,\mathrm{ev}}$ if $m'-m>1$.*

PROOF. By the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|d\|_{\mathcal{C}_m} &= \sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}}\,|c_k|\,(1+\ell)^m \\
&\le \sqrt{2s}\sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} |c_k|\,(1+\ell)^m \\
&\le \sqrt{2s}\,\|c\|_{\ell^2_{m'}} \sup_\ell \Bigg(\sum_{k\in\{\ell+1,\ell+3,\dots\}} (1+k)^{-m'}(1+\ell)^m\Bigg)^{1/2} \\
&\le \sqrt{2s}\,\|c\|_{\ell^2_{m'}} \Bigg(\sum_k (1+k)^{m-m'}\Bigg)^{1/2},
\end{aligned}$$

where the last series is convergent since $m-m'<-1$. □

COROLLARY 4.13. *x^{-1} defines a bounded operator $\mathcal{S}^{m'}_{\sigma,\mathrm{odd}} \to \mathcal{S}^m_{\sigma,\mathrm{ev}}$ if $2m' > m+5$.*

PROOF. Since $2m'>m+5$, there are $m_1,m_2,m_3\ge 0$ such that

$$m'-m_3>1/2\;,\quad m_3-m_2>1\;,\quad 2m_2-m_1>1\;,\quad m_1-m>1\;.$$

Then, by Propositions 4.3 and 4.4, Lemmas 4.7 and 4.12, and using the "Fourier coefficients" mapping, we get the following composition of bounded maps:

$$\mathcal{S}^{m'}_{\sigma,\mathrm{odd}} \hookrightarrow W^{m_3}_{\sigma,\mathrm{odd}} \to \ell^2_{m_3,\mathrm{odd}} \xrightarrow{\Xi} \mathcal{C}_{m_2,\mathrm{ev}} \hookrightarrow \ell^2_{m_1,\mathrm{ev}} \to W^{m_1}_{\sigma,\mathrm{ev}} \hookrightarrow \mathcal{S}^m_{\sigma,\mathrm{ev}}\;.$$

By (29), this composite is an extension of the map $x^{-1}:\mathcal{S}_{\mathrm{odd}}\to\mathcal{S}_{\mathrm{ev}}$. □

QUESTION 4.14. The proof of Corollary 4.13 is very indirect. Is it possible to prove it without using (29) and the perturbed Sobolev spaces?

COROLLARY 4.15. *x^{-1} defines a continuous operator $\mathcal{S}_{\sigma,\mathrm{odd}} \to \mathcal{S}_{\sigma,\mathrm{ev}}$.*

LEMMA 4.16. *$\mathcal{S}^1_{\sigma,\mathrm{ev}} \subset \mathcal{S}^1_{w,\sigma,\mathrm{ev}}$ and $\mathcal{S}^{m+2}_\sigma \subset \mathcal{S}^m_{w,\sigma}$ continuously for $m\ge 1$.*

PROOF. Let us construct a sequence of naturals $M_{m,\mathrm{ev/odd}}$ such that $\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{\sigma,\mathrm{ev/odd}} \subset \mathcal{S}^m_{w,\sigma,\mathrm{ev/odd}}$ continuously for all m. Like in the proof of Lemma 3.8, we proceed by induction on m, with $M_{0,\mathrm{ev/odd}}=0$. For $m>0$, assume that the terms $M_{m-1,\mathrm{ev/odd}}$ are constructed.

For $\phi\in C^\infty_{\mathrm{ev}}$, $i+j\le m$ with $j>0$ and $x\in\mathbb{R}$, we have

$$\left|x^i\phi^{(j)}(x)\right| = \left|x^i(T_\sigma\phi)^{(j-1)}(x)\right|\;,$$

obtaining

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le \|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + \|\phi\|_{\mathcal{S}^m_\sigma}\;.$$

But there are some $C,C'>0$, independent of ϕ, such that

$$\|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} \le C\,\|T_\sigma\phi\|_{\mathcal{S}^{M_{m-1,\mathrm{odd}}}_\sigma} \le C'\,\|\phi\|_{\mathcal{S}^{M_{m,\mathrm{ev}}}_\sigma}$$

with

$$M_{m,\mathrm{ev}} = M_{m-1,\mathrm{odd}}+1\;. \tag{62}$$

For $\phi \in C^\infty_{\mathrm{odd}}$, and i, j and x as above, we have

$$\left|x^i\phi^{(j)}(x)\right| \le \left|x^i(T_\sigma\phi)^{(j-1)}(x)\right| + 2\sigma\left|x^i(x^{-1}\phi)^{(j-1)}(x)\right| ,$$

obtaining

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le \|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + 2\,|\sigma|\,\|x^{-1}\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + \|\phi\|_{\mathcal{S}^m_\sigma} ,$$

But, by Corollary 4.13, there are some $C, C' > 0$, independent of ϕ, such that

$$\begin{aligned}\|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + 2\,|\sigma|\,\|x^{-1}\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} &\le C\left(\|\phi'\|_{\mathcal{S}_\sigma^{M_{m-1,\mathrm{ev}}}} + \|x^{-1}\phi\|_{\mathcal{S}_\sigma^{M_{m-1,\mathrm{ev}}}}\right)\\ &\le C'\,\|\phi\|_{\mathcal{S}_\sigma^{M_{m,\mathrm{odd}}}}\end{aligned}$$

if

$$M_{m,\mathrm{odd}} \ge M_{m-1,\mathrm{ev}} + 1 , \quad 2M_{m,\mathrm{odd}} > M_{m-1,\mathrm{ev}} + 5 . \tag{63}$$

The conditions (62) and (63) are satisfied with $M_{1,\mathrm{ev}} = 1$, $M_{1,\mathrm{odd}} = 3$ and $M_{m,\mathrm{ev/odd}} = m + 2$ for $m \ge 2$. □

Corollary 4.17. *$\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{\mathrm{ev/odd}} \subset \mathcal{S}^m_{\sigma,\mathrm{ev/odd}}$ continuously for all m, where*

$$M_{m,\mathrm{ev/odd}} = \begin{cases} \frac{3m}{2} + \frac{m}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is even}\\ \frac{5m}{2} + 2 & \text{if } \sigma < 0 \text{ and } m \text{ is even} ,\end{cases}$$

$$M_{m,\mathrm{ev}} = \begin{cases} \frac{3m-1}{2} + \frac{m-1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd}\\ \frac{5m+1}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} ,\end{cases}$$

$$M_{m,\mathrm{odd}} = \begin{cases} \frac{3m+1}{2} + \frac{m+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd}\\ \frac{5m+7}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} .\end{cases}$$

Proof. This follows from Lemmas 3.1, 3.3 and 3.8. □

Corollary 4.18. *$\mathcal{S}^{m'}_\sigma \subset \mathcal{S}^m$ continuously for all m, where*

$$m' = m + 3 + \frac{\lceil\sigma\rceil(\lceil\sigma\rceil+1)}{2} .$$

Moreover $\mathcal{S}^1_{\sigma,\mathrm{ev}} \subset \mathcal{S}^0_{\mathrm{ev}}$ continuously.

Proof. This follows from Lemmas 3.2, 3.4 and 4.16. □

Corollary 4.19. *$\mathcal{S}_\sigma = \mathcal{S}$ as Fréchet spaces.*

Proof. This is a consequence of Corollaries 4.17 and 4.18 □

Now, Theorems D and E follow from Corollaries 4.17 and 4.18 and Propositions 4.3 and 4.4.

CHAPTER 5

Perturbation of H on $\mathbb{R}_+$

More general perturbations of H can be obtained with conjugation of L by the operator of multiplication by functions which are defined and positive almost everywhere (with respect to the Lebesgue measure), like we did in Section 1 of Chapter 2 with the function $|x|^\sigma$. We will only consider conjugations of the even and odd components of L separately, and acting on spaces of functions on $\mathbb{R}_+$. This will be also enough for the application in Part 2.

Let $L_{\mathrm{ev/odd}}$, or $L_{\sigma,\mathrm{ev/odd}}$, denote the restriction of L to $\mathcal{S}_{\mathrm{ev/odd}}$. Since the function $|x|^{2\sigma}$ is even, there is an orthogonal decomposition $L^2(\mathbb{R},|x|^{2\sigma}\,dx) = L^2_{\mathrm{ev}}(\mathbb{R},|x|^{2\sigma}\,dx) \oplus L^2_{\mathrm{odd}}(\mathbb{R},|x|^{2\sigma}\,dx)$ as direct sum of subspaces of even and odd functions. Then $L_{\mathrm{ev/odd}}$ is essentially self-adjoint in $L^2_{\mathrm{ev/odd}}(\mathbb{R},|x|^{2\sigma}\,dx)$, and its self-adjoint extension $\mathcal{L}_{\mathrm{ev/odd}}$, or $\mathcal{L}_{\sigma,\mathrm{ev/odd}}$, is obtained by restriction of $\mathcal{L}$. We also get an obvious version of Corollary F for $\mathcal{L}_{\mathrm{ev/odd}}$.

Fix open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure. Let $\mathcal{S}_{\mathrm{ev/odd},U}$ denote the linear subspace of $C^\infty(\mathbb{R}_+)$ consisting of the restrictions to U of the functions in $\mathcal{S}_{\mathrm{ev/odd}}$. The restriction to U defines a linear isomorphism

$$\mathcal{S}_{\mathrm{ev/odd}} \cong \mathcal{S}_{\mathrm{ev/odd},U}\ , \tag{64}$$

and a unitary isomorphism

$$L^2_{\mathrm{ev/odd}}(\mathbb{R},|x|^{2\sigma}\,dx) \cong L^2(\mathbb{R}_+,2x^{2\sigma}\,dx)\ . \tag{65}$$

Let $L_{\mathrm{ev/odd},U}$, or $L_{\sigma,\mathrm{ev/odd},U}$, denote the operator defined by $L_{\mathrm{ev/odd}}$ on $\mathcal{S}_{\mathrm{ev/odd},U}$ via (64). Let also $\phi_{k,U} = \phi_k|_U$, whose norm in $L^2(\mathbb{R}_+,x^{2\sigma}\,dx)$ is $1/\sqrt{2}$ since (65) is unitary. When $U = \mathbb{R}_+$, the notation $\mathcal{S}_{\mathrm{ev/odd},+}$, $L_{\mathrm{ev/odd},+}$, or $L_{\sigma,\mathrm{ev/odd},+}$, and $\phi_{k,+}$ will be used. Moreover let $\mathcal{L}_{\mathrm{ev/odd},+}$, or $\mathcal{L}_{\sigma,\mathrm{ev/odd},+}$, be the self-adjoint operator in $L^2(\mathbb{R}_+,x^{2\sigma}\,dx)$ that corresponds to $L_{\mathrm{ev/odd}}$ via (65).

Going one step further, for any positive function $h \in C^2(U)$, the operator (of multiplication by) h defines a unitary isomorphism

$$h : L^2(\mathbb{R}_+,x^{2\sigma}\,dx) \xrightarrow{\cong} L^2(\mathbb{R}_+,x^{2\sigma}h^{-2}\,dx)\ . \tag{66}$$

We get that $hL_{\mathrm{ev/odd},U}h^{-1}$, with domain $h\,\mathcal{S}_{\mathrm{ev/odd},U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,x^{2\sigma}h^{-2}\,dx)$, and its self-adjoint extension is $h\mathcal{L}_{\mathrm{ev/odd},+}h^{-1}$. Via (65) and (66), we obtain an obvious version of Corollary F for $h\mathcal{L}_{\mathrm{ev/odd},+}h^{-1}$. By using

$$\left[\frac{d}{dx},h\right] = h'\ ,\quad \left[\frac{d^2}{dx^2},h\right] = 2h'\frac{d}{dx} + h''\ , \tag{67}$$

it easily follows that $hL_{\mathrm{ev/odd},U}h^{-1}$ is of the form (1) with $f_1 \in C^1(U)$ and $f_2 \in C(U)$. Then Theorem G is a consequence of the following.

Lemma 5.1. *For $\sigma > -1/2$, a positive function $h \in C^2(U)$, and an operator P of the form* (1) *with $f_1 \in C^1(U)$ and $f_2 \in C(U)$, we have $P = hL_{\sigma,\mathrm{ev},U}h^{-1}$ on*

$h\,\mathcal{S}_{\mathrm{ev},U}$ *if and only if* (2) *and* (3) *are satisfied with some primitive* $F_1 \in C^2(U)$ *of* f_1.

PROOF. By (67),

$$\begin{aligned} h^{-1}Ph &= -h^{-1}\frac{d^2}{dx^2}\,h + sx^2 - 2h^{-1}f_1\frac{d}{dx}\,h + f_2 \\ &= -\frac{d^2}{dx^2} - h^{-1}\left(2h'\frac{d}{dx} + h''\right) + sx^2 \\ &\quad - 2f_1\frac{d}{dx} - 2h^{-1}f_1h' + f_2 \\ &= H - 2(h^{-1}h' + f_1)\frac{d}{dx} - h^{-1}h'' - 2h^{-1}f_1h' + f_2\ . \end{aligned}$$

So $P = hL_{\sigma,\mathrm{ev},U}h^{-1}$ if and only if

$$h^{-1}h' = \sigma x^{-1} - f_1\ , \tag{68}$$

$$f_2 = h^{-1}h'' + 2h^{-1}h'f_1\ . \tag{69}$$

The equality (68) is equivalent to (3), and gives

$$h^{-1}h'' = (\sigma x^{-1} - f_1)^2 - \sigma x^{-2} + f_1'\ .$$

So, by (69),

$$\begin{aligned} f_2 &= (\sigma x^{-1} - f_1)^2 - \sigma x^{-2} + f_1' + 2(\sigma x^{-1} - f_1)f_1 \\ &= \sigma(\sigma - 1)x^{-2} - f_1^2 - f_1'\ . \end{aligned}$$

It follows that (68) and (69) are equivalent to (3) and (2). □

REMARK 2. By (67), we get an operator of the same type if h and $\frac{d}{dx}$ is interchanged in (1).

REMARK 3. By using (67) with $h = x^{-1}$ on $\mathbb{R}_+$, it is easy to check that $L_{\sigma,\mathrm{odd},+} = xL_{1+\sigma,\mathrm{ev},+}x^{-1}$ on $\mathcal{S}_{\mathrm{odd},+} = x\,\mathcal{S}_{\mathrm{ev},+}$ for all $\sigma > -1/2$. So no new operators are obtained with the conjugation $L_{\sigma,\mathrm{odd},U}$ by h.

REMARK 4. If f_1 is a rational function, then the function f_2, given by (2), is also rational.

REMARK 5. The term of P with $\frac{d}{dx}$ can be removed by conjugation, obtaining the operator $H + \sigma(\sigma-1)x^{-2}$, given by restricting K_σ, first to even functions and second to $\mathbb{R}_+$. In this way, we get all operators of the form $H + cx^{-2}$ with $c > -1/4$.

CHAPTER 6

Examples

1. Case where f_1 is a multiple of x^{-1}

A particular class of (1) is given by the operators of the form

$$P = H - 2c_1 x^{-1} \frac{d}{dx} + c_2 x^{-2} \tag{70}$$

for $c_1, c_2 \in \mathbb{R}$. In this case, we can take $F_1 = c_1 \log x$. Then $e^{F_1} = x^{c_1}$, (3) gives $h = x^a$ with $a = \sigma - c_1$, and (2) becomes $c_2 x^{-2} = (a^2 + a(2c_1 - 1))x^{-2}$. Therefore Corollary H follows from Theorem G.

Remark 6. According to Remark 2, we get an operator of the same type if x^{-1} and $\frac{d}{dx}$ is interchanged in (70). We may also use that, with the function x^a ($a \in \mathbb{R}$), (67) becomes

$$\left[\frac{d}{dx}, x^a\right] = a x^{a-1}\ , \quad \left[\frac{d^2}{dx^2}, x^a\right] = 2a x^{a-1} \frac{d}{dx} + a(a-1)x^{a-2}\ . \tag{71}$$

Remark 7. By Corollary H-(iii), we have $h\,\mathcal{D}^\infty(\mathcal{P}) \subset \mathcal{D}^\infty(\mathcal{P})$ for all $h \in C^\infty(\mathbb{R}_+)$ such that $h' \in C_0^\infty(\mathbb{R}_+)$.

The existence of $a \in \mathbb{R}$ satisfying (4) is characterized by the condition

$$(2c_1 - 1)^2 + 4c_2 \geq 0\ . \tag{72}$$

Observe that (72) is satisfied if $c_2 \geq \min\{0, 2c_1\}$. In particular, we have the following special cases.

Example 6.1. Suppose that $c_2 = 0$; i.e., $P = H - 2c_1 x^{-1} \frac{d}{dx}$. Thus $P = L_{c_1,\mathrm{ev},+}$ if $c_1 > -1/2$; however, this inequality is not required *a priori*. Then (4) means that $a \in \{0, 1 - 2c_1\}$, and (5) gives

$$\sigma = \begin{cases} c_1 & \text{if } a = 0 \\ 1 - c_1 & \text{if } a = 1 - 2c_1\ . \end{cases}$$

In the case $a = 0$ and $\sigma = c_1$, the condition $c_1 > -1/2$ is needed to apply Corollary H. In this case, Corollary H holds for $P = L_{c_1,\mathrm{ev},+}$ on $\mathcal{S}_{\mathrm{ev},+}$, which is a direct consequence of the known properties of L_{c_1} (Section 2 of Chapter 1 and Corollary F).

Nevertheless, Corollary H gives new information in the case $a = 1 - 2c_1$ and $\sigma = 1 - c_1$: we have $\sigma > -1/2$ just when $c_1 < 3/2$ ($c_1 \leq -1/2$ is allowed!). When this inequality is satisfied, Corollary H states that P, with domain $x^{1-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$, is also essentially self-adjoint in $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+3-2c_1)s$ ($k \in \mathbb{N}$) with multiplicity one; the corresponding normalized eigenfunctions are $\sqrt{2}\,x^{1+2c_1}\phi_{2k,+}$; and $\mathcal{D}^\infty(\mathcal{P}) = x^{1-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$.

Thus, when $-1/2 < c_1 < 3/2$, we have got two essentially self-adjoint operators in $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$ defined by P, with domains $\mathcal{S}_{\mathrm{ev},+}$ and $x^{1-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$, which are equal just when $c_1 = 1/2$. In particular, if $c_1 = 0$, these operators are defined by H with domains $\mathcal{S}_{\mathrm{ev},+}$ and $x\,\mathcal{S}_{\mathrm{ev},+} = \mathcal{S}_{\mathrm{odd},+}$.

EXAMPLE 6.2. Suppose that $c_2 = 2c_1$; i.e., $P = H - 2c_1x^{-1}\frac{d}{dx} + 2c_1x^{-2}$. Then (4) means that $a \in \{1, -2c_1\}$, and (5) gives

$$\sigma = \begin{cases} 1 + c_1 & \text{if } a = 1 \\ -c_1 & \text{if } a = -2c_1 \, . \end{cases}$$

In the case $a = 1$ and $\sigma = 1 + c_1$, we have $\sigma > -1/2$ if and only if $c_1 > -3/2$. When this inequality is satisfied, Corollary H states that P, with domain $x\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k + 3 + 2c_1)s$ ($k \in \mathbb{N}$) with multiplicity one; the corresponding normalized eigenfunctions are $\sqrt{2}\,x\phi_{2k,+}$; and $\mathcal{D}^\infty(\mathcal{P}) = x\,\mathcal{S}_{\mathrm{ev},+}$.

In the case $a = -2c_1$ and $\sigma = -c_1$, we have $\sigma > -1/2$ just when $c_1 < 1/2$. When this inequality is satisfied, Corollary H states that P, with domain $x^{-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k + 1 - 2c_1)s$ ($k \in \mathbb{N}$) with multiplicity one; the corresponding normalized eigenfunctions are $\sqrt{2}\,x^{-2c_1}\phi_{2k,+}$; and $\mathcal{D}^\infty(\mathcal{P}) = x^{-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$.

Thus, when $-3/2 < c_1 < 1/2$, we have got two essentially self-adjoint operators in $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$ defined by P, with domains $x\,\mathcal{S}_{\mathrm{ev},+}$ and $x^{-2c_1}\,\mathcal{S}_{\mathrm{ev},+}$, which are equal just when $c_1 = -1/2$. In particular, if $c_1 = 0$, we get again that these operators are defined by H with domains $x\,\mathcal{S}_{\mathrm{ev},+} = \mathcal{S}_{\mathrm{odd},+}$ and $\mathcal{S}_{\mathrm{ev},+}$.

In this case, we will use the notation $\chi_{s,a,\sigma,k} = \sqrt{2}\,x^a\phi_{2k,+}$ (or simply χ_k) for the eigenfunctions. The following property of $\chi_{s,a,\sigma,0}$ will be also used.

LEMMA 6.3. *If h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle h\chi_{s,a,\sigma,0}, \chi_{s,a,\sigma,0}\rangle_{c_1} \to 1$ as $s \to \infty$.*

PROOF. Given any $\epsilon > 0$, take some $x_0 > 0$ such that $|h(x) - 1| \leq \epsilon/2$ for $x \leq x_0$. For s large enough, we have

$$\int_{x_0}^\infty e^{-sx^2}\,x^{2\sigma}\,dx \leq \frac{\epsilon}{4p_0^2\,\max|h-1|}$$

Hence, for s large enough,

$$\begin{aligned} |\langle(1-h)\chi_{s,a,\sigma,0}, \chi_{s,a,\sigma,0}\rangle_{c_1}| &\leq 2p_0^2\int_0^\infty |1-h(x)|\,e^{-sx^2}\,x^{2\sigma}\,dx \\ &= p_0^2\epsilon\int_0^{x_0} e^{-sx^2}\,x^{2\sigma}\,dx + 2p_0^2\,(\max|1-h|)\int_{x_0}^\infty e^{-sx^2}\,x^{2\sigma}\,dx \\ &< p_0^2\epsilon\int_0^\infty e^{-sx^2}\,x^{2\sigma}\,dx + \frac{\epsilon}{2} = \frac{\epsilon}{2}\,\|\chi_{s,a,\sigma,0}\|_{c_1}^2 + \frac{\epsilon}{2} = \epsilon\,. \quad \square \end{aligned}$$

2. Case where f_1 is a multiple of other potential functions

Suppose that $f_1 = cx^r$ for $c, r \in \mathbb{R}$ with $r \neq -1$. Given any $\sigma > -1/2$, now (2) becomes

$$f_2 = \sigma(\sigma-1)x^{-2} + c^2x^{2r} - crx^{r-1}\,.$$

Moreover we can take $F_1 = \frac{cx^{r+1}}{r+1}$, obtaining

$$h = x^\sigma \exp\left(-\frac{cx^{r+1}}{r+1}\right)$$

according to (3). Then Theorem G asserts that the operator

$$P = H - 2cx^r \frac{d}{dx} + \sigma(\sigma-1)x^{-2} - c^2x^{2r} - crx^{r-1} ,$$

with domain $h\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2F_1}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ with multiplicity one and normalized eigenfunctions $\sqrt{2}\,h\phi_{2k,+}$; and the smooth core of $\mathcal{P}$ is $h\,\mathcal{S}_{\mathrm{ev},+}$.

3. Case where f_1 is a multiple of g'/g for some function g

The operators of Section 1 can be generalized as follows. For an open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure, take $f_1 = cg'/g$ for $c \in \mathbb{R}$ and some non-vanishing function $g \in C^2(U)$. Given any $\sigma > -1/2$, the equality (2) gives

$$f_2 = \sigma(\sigma-1)x^{-2} - c(c-1)\,\frac{g'^2}{g^2} - c\,\frac{g''}{g} .$$

In this case, we can take $F_1 = c\log|g|$, obtaining $h = x^\sigma\,|g|^{-c}$ by (3). Then Theorem G states that the operator

$$P = H - 2c\,\frac{g'}{g}\,\frac{d}{dx} + \sigma(\sigma-1)x^{-2} + c(c-1)\,\frac{g'^2}{g^2} - c\,\frac{g''}{g} ,$$

with domain $x^\sigma\,|g|^{-c}\,\mathcal{S}_{\mathrm{ev},U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, |g|^{2c}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ with multiplicity one and normalized eigenfunctions $\sqrt{2}\,x^\sigma\,|g|^{-c}\phi_{2k,U}$; and the smooth core of $\mathcal{P}$ is $x^\sigma\,|g|^{-c}\,\mathcal{S}_{\mathrm{ev},U}$. This agrees with Corollary H when $g = x$.

EXAMPLE 6.4. If we take $g = \cos x$, which does not vanish on $U = \mathbb{R}_+ \setminus (2\mathbb{N}+1)\frac{\pi}{2}$, we get that, for any $\sigma > -1/2$, the operator

$$P = H - 2c\,\tan x\,\frac{d}{dx} + \sigma(\sigma-1)x^{-2} + c(c-1)\tan^2 x - c ,$$

with domain $x^\sigma\,|\cos x|^{-c/2}\,\mathcal{S}_{\mathrm{ev},U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, |\cos x|^{2c}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ with multiplicity one and normalized eigenfunctions $\sqrt{2}\,x^\sigma|\cos x|^{-c}\phi_{2k,U}$; and the smooth core of $\mathcal{P}$ is $x^\sigma\,|\cos x|^{-c/2}\,\mathcal{S}_{\mathrm{ev},U}$.

Similar examples can be given with other trigonometric and hyperbolic functions.

EXAMPLE 6.5. For $g = e^x$, it follows that, for any $\sigma > -1/2$, the operator

$$P = H - 2c\,\frac{d}{dx} + \sigma(\sigma-1)x^{-2} - c^2 ,$$

with domain $x^\sigma e^{-cx}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2cx}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ with multiplicity one and normalized eigenfunctions $\sqrt{2}\,x^\sigma e^{-cx/2}\phi_{2k,+}$; and the smooth core of $\mathcal{P}$ is $x^\sigma e^{-cx}\,\mathcal{S}_{\mathrm{ev},+}$.

EXAMPLE 6.6. With more generality, for $g = e^{x^n}$ ($0 \neq n \in \mathbb{Z}$) and any $\sigma > -1/2$, the operator

$$P = H - 2cnx^{n-1}\frac{d}{dx} + \sigma(\sigma-1)x^{-2} - c(c-1)n^2x^{2(n-1)} - c\Big(n(n-1)x^{n-2} + n^2x^{2(n-1)}\Big) ,$$

with domain $x^\sigma e^{-cx^n}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2cx^n}\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k+1+2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2}\,x^\sigma e^{-cx^n}\phi_{2k,+}$; and the smooth core of $\mathcal{P}$ is $x^\sigma e^{-cx^n}\,\mathcal{S}_{\mathrm{ev},+}$.

4. Transformation of P by changes of variables

We can use arbitrary changes of variables to provide a larger family of essentially self-adjoint operators whose spectrum can be described. For instance, the operator P on $\mathbb{R}_+$, given in Chapter 5, can be transformed into a differential operator on $\mathbb{R}$ with the change of variable $x = \log y$, where now y denotes the standard coordinate of $\mathbb{R}_+$. Since $dx/dy = 1/y = e^{-x}$, we get

$$\frac{d}{dy} = e^{-y}\frac{d}{dx} , \quad \frac{d^2}{dy^2} = e^{-2y}\left(\frac{d^2}{dx^2} - \frac{d}{dx}\right) .$$

So this change of variables transforms the operator P of (1) (on functions of y) into the operator

$$P_1 = -e^{-2x}\frac{d^2}{dx^2} + s^2e^{2x} - 2\left(f_1(e^x)e^{-x} - e^{-2x}\right)\frac{d}{dx} + f_2(e^x)$$

(on functions of x), and transforms $L^2(\mathbb{R}_+, e^{2F_1(y)}\,dy)$ into $L^2(\mathbb{R}_+, e^{2F_1(e^x)}e^x\,dx)$. Suppose that f_1 and f_2 satisfy (2) for some $\sigma > -1/2$, and let $h \in C^2(U)$ be defined by (3) for some primitive $F_1 \in C^2(U)$ of f_1. Let also $V = \{\, x \in \mathbb{R} \mid e^x \in U \,\}$. Then P_1, with domain

$$\{\, h(e^x)\phi(e^x) \mid \phi \in \mathcal{S}_{\mathrm{ev},U} \,\} \subset C^2(V) , \tag{73}$$

is essentially self-adjoint in $L^2(\mathbb{R}, e^{2F_1(e^x)}e^x\,dx)$; the spectrum of its self-adjoint extension $\mathcal{P}_1$ consists of the eigenvalues $(4k+1+2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2}\,h(e^x)\phi_{2k,U}(e^x)$; and the smooth core of $\mathcal{P}_1$ is (73).

Part 2

Witten's perturbation on strata

CHAPTER 7

Preliminaries on Thom-Mather stratifications

1. Thom-Mather stratifications

Here, we recall the needed concepts introduced by R. Thom [**60**] and J. Mather [**44**]. We mainly follow [**64**], and some new remarks are also made, specially concerning products.

1.1. Thom-Mather stratifications and their morphisms. Let A be Hausdorff, locally compact and second countable topological space. Let $X \subset A$ be a locally closed subset. Two subsets $Y, Z \subset A$ are said to be *equal near* X (or $Y = Z$ *near* X) if $Y \cap U = Z \cap U$ for some neighborhood U of X in A. It is also said that two maps, $f : Y \to B$ and $g : Z \to B$, are *equal near* X (or $f = g$ *near* X) when there is some neighborhood U of X in A such that $Y \cap U = Z \cap U$, and the restrictions of f and g to $Y \cap U$ are equal.

Consider triples (T, π, ρ), where T is an open neighborhood of X in A, $\pi : T \to X$ is a continuous retraction, and $\rho : X \to [0, \infty)$ is a continuous function such that $\rho^{-1}(0) = X$. Two such triples, (T, π, ρ) and (T', π', ρ'), are said to be *equal near* X when $T = T'$, $\pi = \pi'$ and $\rho = \rho'$ near X. This defines an equivalence relation whose equivalence classes are called *tubes* of X in A. The notation $[T, \pi, \rho]$ is used for the tube represented by (T, π, ρ). If X is open in A, then $[X, \mathrm{id}_X, 0]$ is its unique tube (the *trivial tube*).

Definition 7.1. A *Thom-Mather stratification*[1] (or *Thom-Mather stratified space*) is a triple $(A, \mathcal{S}, \tau)$, where:

(i) A is a Hausdorff, locally compact and second countable space,
(ii) $\mathcal{S}$ is a partition of A into locally closed subspaces with the additional structure of smooth (C^∞) manifolds, called *strata*, and
(iii) τ is the assignment of a tube τ_X of each $X \in \mathcal{S}$ in A,

such that the following conditions are satisfied with some choice of $(T_X, \pi_X, \rho_X) \in \tau_X$ for each $X \in \mathcal{S}$:

(iv) For all $X, Y \in \mathcal{S}$, if $X \cap \overline{Y} \neq \emptyset$, then $X \subset \overline{Y}$. The notation $X \leq Y$ is used in this case, and this defines a partial order relation on $\mathcal{S}$. As usual, $X < Y$ means that $X \leq Y$ but $X \neq Y$.
(v) If $Y \neq X$ in $\mathcal{S}$ and $T_X \cap Y \neq \emptyset$, then $X < Y$ and $(\pi_X, \rho_X) : T_X \cap Y \to X \times \mathbb{R}_+$ is a smooth submersion; in particular, $\dim X < \dim Y$.
(vi) If $X < Y$ in $\mathcal{S}$, then $\pi_Y(T_X \cap T_Y) \subset T_X$, and $\pi_X\, \pi_Y = \pi_X$ and $\rho_X\, \pi_Y = \rho_X$ on $T_X \cap T_Y$.

It may be also said that $(\mathcal{S}, \tau)$ is a *Thom-Mather stratification* of A.

Remark 8. (i) A is paracompact and normal.

[1]This is called *abstract prestratification* in [**44**] and *abstract stratification* in [**64**].

(ii) By the normality of A, we can also assume that, if $X, Y \in \mathcal{S}$ and $T_X \cap T_Y \neq \emptyset$, then $X \leq Y$ or $Y \leq X$.
(iii) The frontier of a stratum X equals the union of the strata $Y < X$.
(iv) The connected components of each stratum may have different dimensions.
(v) The connected components of the strata, with the corresponding restrictions of the tubes, define an induced Thom-Mather stratification $A_{\mathrm{con}} \equiv (A, \mathcal{S}_{\mathrm{con}}, \tau_{\mathrm{con}})$; in this way, we can assume that the strata are connected if desirable.

Remark 9. The following are some variants of the concept "stratification" and related notions:
(i) A *weak Thom-Mather stratification* is defined by removing the condition $\rho_X \, \pi_Y = \rho_X$ from Definition 7.1-(vi).
(ii) A *stratification* is a pair $(A, \mathcal{S})$ satisfying Definition 7.1-(i),(ii),(iv); it is also said that $\mathcal{S}$ is a *stratification* of A. Definition 7.1-(iv) is called the *frontier condition*. If moreover τ satisfies the other conditions of Definition 7.1, then it is called *Thom-Mather structure* on $(A, \mathcal{S})$.
(iii) If A is a subspace of a smooth manifold M, then a stratification $\mathcal{S}$ of A is usually required to consist of regular submanifolds of M; the term *stratified subspace* of M is used in this case. In [**29**], a weaker version of this notion is defined by requiring local finiteness of $\mathcal{S}$ instead of the frontier condition.
(iv) For a stratified subspace $(A, \mathcal{S})$ of a smooth manifold M, the *condition* (B), introduced by H. Whitney [**66, 67**], is defined as follows[2]. In the case $M = \mathbb{R}^m$, it requires that, for all $X \neq Y$ in $\mathcal{S}$, if (x_i) and (y_i) are sequences in X and Y, respectively, both of them converging in A to some $x \in X$, if the sequence of tangent spaces $T_{y_i}Y$ converges[3] to a linear subspace $T \subset \mathbb{R}^n$, and if the sequence of lines $\mathbb{R}(x_i - y_i)$ converges to a line $L \subset \mathbb{R}^m$, then $L \subset T$. This property is preserved by local diffeomorphisms of $\mathbb{R}^m$, and therefore generalizes to arbitrary smooth manifolds. This condition gives rise to the concept of *Whitney stratification* of a subspace (or *Whitney stratified subspace*) of M.

Example 7.2. (i) Any smooth manifold is a Thom-Mather stratification with one stratum and the trivial tube.
(ii) Any smooth manifold with boundary is a stratification with two strata, the interior and the boundary. It can be endowed with a Thom-Mather structure by using a collar of the boundary.
(iii) Any subanalytic subset of $\mathbb{R}^m$ has a primary and secondary stratifications; the secondary one satisfies condition (B) [**40, 45, 33, 32, 34**].
(iv) J. Mather [**44**] has proved that any Whitney stratified subspace of a smooth manifold admits a Thom-Mather structure (see also [**29**, Proposition 2.6 and Corollary 2.7]).

For a stratification $A \equiv (A, \mathcal{S})$, the *depth* of any $X \in \mathcal{S}$, denoted by $\operatorname{depth} X$, is the supremum of the naturals n such that there exist strata $X_0, \dots, X_n$ with $X_0 < X_1 < \dots < X_n = X$. Notice that $\operatorname{depth} X \leq \dim X$. Moreover $\operatorname{depth} X = 0$

[2] Certain condition (A) was also introduced by H. Whitney in [**66, 67**], but J. Mather [**44**] has observed that it follows from condition (B).

[3] The convergence of linear subspaces of $\mathbb{R}^m$ is considered in the appropriate Grassmannians.

(X is minimal in $\mathcal{S}$) if and only if X is closed in A. The *depth* and *dimension* of A are the supremum of the depths and dimensions of its strata, respectively. The dimension of A equals its topological dimension, which may be infinite. The depth of A is zero if and only if all strata are open and closed.

Let $A \equiv (A, \mathcal{S}, \tau)$ be a Thom-Mather stratification. Let $B \subset A$ be a locally closed subset. Suppose that, for all $X \in \mathcal{S}$, $X \cap B$ is a smooth submanifold of X, and $B \cap \pi_X^{-1}(X \cap B)$, endowed with the restrictions of π_X and ρ_X, defines a tube $\tau_{X\cap B}$ of $X \cap B$ in B. Then let $\mathcal{S}|_B = \{\, X \cap B \mid X \in \mathcal{S}\,\}$, and let $\tau|_B$ be defined by the assignment of $\tau_{X\cap B}$ to each $X \cap B \in \mathcal{S}|_B$. If $(B, \mathcal{S}|_B, \tau|_B)$ satisfies the conditions of a stratification, it is said that the stratification A (or $(\mathcal{S}, \tau)$) can be *restricted* to B, and $B \equiv (B, \mathcal{S}|_B, \tau|_B)$ is called a *restriction* of A (or $(\mathcal{S}|_B, \tau|_B)$ is called the *restriction* of $(\mathcal{S}, \tau)$); it may be also said that B is a *Thom-Mather substratification* of A. For instance, A can be restricted to any open subset and to any locally closed union of strata. A restriction of a restriction of A is a restriction of A.

For a stratum X of A, we can consider the restriction of A to $\overline{X}$. In this way, to study X, we can assume that X is dense in A and $\dim X = \dim A$ if desirable.

A locally closed subset $B \subset A$ is said to be *saturated* if the stratification A can be restricted to B and, for every $X \in \mathcal{S}$, there is a representative (T_X, π_X, ρ_X) of τ_X such that $\pi_X^{-1}(X \cap B) = T_X \cap B$.

Let $A' \equiv (A', \mathcal{S}', \tau')$ be another Thom-Mather stratification. A continuous map $f : A \to A'$ is called a *morphism* if, for any $X \in \mathcal{S}$, there is some $X' \in \mathcal{S}'$ such that $f(X) \subset X'$, the restriction $f : X \to X'$ is smooth, and there are $(T_X, \pi_X, \rho_X) \in \tau_X$ and $(T'_{X'}, \pi'_{X'}, \rho'_{X'}) \in \tau'_{X'}$ such that $f(T_X) \subset T'_{X'}$, $f\pi_X = \pi'_{X'}f$ and $f\rho_X = \rho'_{X'}$. Notice that the continuity of a morphism follows from the other conditions. Morphisms between stratifications form a category with the operation of composition; in particular, we have the corresponding concepts of *isomorphism* and *automorphism*. The set of morphisms $A \to A'$ is denoted by $\operatorname{Mor}(A, A')$, and the group of automorphisms of A is denoted by $\operatorname{Aut}(A)$. The other variants of the concept "stratification" given in Remark 9 also have obvious corresponding versions of morphisms, isomorphisms and automorphisms; in particular, we get the concept of *weak morphism* between weak Thom-Mather stratifications. A (weak) morphism is called *submersive* when it restricts to smooth submersions between the strata.

EXAMPLE 7.3. Let G be a compact Lie group G acting smoothly on a closed manifold M. Consider the orbit type stratifications of M and $G\backslash M$ [**9**]. It is well known that $G\backslash M$ admits a Thom-Mather structure [**64**, Introduction], which can be seen as follows. $G\backslash M$ is locally isomorphic to a semi-algebraic subset of an Euclidean space whose primary and secondary stratifications are equal [**5**]. By using an invariant smooth partition of unity of M, like in the Whitney's embedding theorem, it follows that $G\backslash M$ is isomorphic to a Whitney stratified subspace of some Euclidean space, and therefore it admits a Thom-Mather structure. This can also be seen by observing that the stratification of M satisfies condition (B), and the proof of [**29**, Proposition 2.6] can be adapted to produce an invariant[4] Thom-Mather structure on M, which induces a Thom-Mather structure on $G\backslash M$.

The following two lemmas are easy to prove.

[4] G acts by automorphisms.

LEMMA 7.4. *Let A be a Hausdorff, locally compact and second countable space, $\{U_i\}$ an open covering of A, and $(\mathcal{S}_i, \tau_i)$ a Thom-Mather stratification of each U_i.*

(i) *If $(\mathcal{S}_i, \tau_i)$ and $(\mathcal{S}_j, \tau_j)$ have the same restrictions to $U_{ij} := U_i \cap U_j$ for all i and j, then there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on A whose restriction to each U_i is $(\mathcal{S}_i, \tau_i)$.*
(ii) *If $((\mathcal{S}_i|_{U_{ij}})_{\mathrm{con}}, (\tau_i|_{U_{ij}})_{\mathrm{con}}) = ((\mathcal{S}_j|_{U_{ij}})_{\mathrm{con}}, (\tau_j|_{U_{ij}})_{\mathrm{con}})$ for all i and j, then there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on A with connected strata such that $((\mathcal{S}|_{U_i})_{\mathrm{con}}, (\tau|_{U_i})_{\mathrm{con}}) = (\mathcal{S}_{i,\mathrm{con}}, \tau_{i,\mathrm{con}})$.*

LEMMA 7.5. *Let $(A', \mathcal{S}', \tau')$ be another Thom-Mather stratification.*

(i) *With the notation of Lemma 7.4-(i), let $f_i : (U_i, \mathcal{S}_i, \tau_i) \to (A', \mathcal{S}', \tau')$ be a morphism for each i. If $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all i and j, then the combination of the maps f_i is a morphism $f : (A, \mathcal{S}, \tau) \to (A', \mathcal{S}', \tau')$.*
(ii) *With the notation of Lemma 7.4-(ii), let $f_i : (U_i, \mathcal{S}_{i,\mathrm{con}}, \tau_{i,\mathrm{con}}) \to (A', \mathcal{S}', \tau')$ be a morphism for each i. If $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all i and j, then the combination of the maps f_i is a morphism $f : (A, \mathcal{S}, \tau) \to (A', \mathcal{S}', \tau')$.*

REMARK 10. As a particular case of Lemma 7.4, given a countable family of Thom-Mather stratifications, $\{A_i \equiv (A_i, \mathcal{S}_i, \tau_i)\}$, there is a unique Thom-Mather stratification $(\mathcal{S}, \tau)$ on the topological sum $\bigsqcup_i A_i$ whose restriction to each A_i is $(\mathcal{S}_i, \tau_i)$; this $(\mathcal{S}, \tau)$ will be called the *sum* of the Thom-Mather stratifications $(\mathcal{S}_i, \tau_i)$.

1.2. Products. The product of two weak Thom-Mather stratifications, A and A', has a weak Thom-Mather stratification $A \times A' \equiv (A \times A', \mathcal{S}'', \tau'')$ with $\mathcal{S}'' = \{ X \times X' \mid X \in \mathcal{S},\ X' \in \mathcal{S}' \}$ and $\tau''_{X\times X'} = [T''_{X\times X'}, \pi''_{X\times X'}, \rho''_{X\times X'}]$, where $T''_{X\times X'} = T_X \times T'_{X'}$, $\pi''_{X\times X'} = \pi_X \times \pi'_{X'}$ and $\rho''_{X\times X'}(x, x') = \rho_X(x) + \rho'_{X'}(x')$.

If A and A' are Thom-Mather stratifications and the depth of at least one of them is zero, then $A \times A'$ is a Thom-Mather stratification, but this is not true when the depths of A and A' are positive [**64**, Section 1.2.9, pp. 5–6]. Another choice of $\rho_{X\times X'}$ is needed to get the second equality of Definition 7.1-(vi). For instance, $\rho''_{X\times X'} = \max\{\rho_X, \rho'_{X'}\}$ satisfies that condition, but it is not smooth on the intersection of the strata with $T''_{X\times X'}$. To solve this problem, pick up a function $h : [0,\infty)^2 \to [0,\infty)$ that is continuous, homogeneous of degree one, smooth on $\mathbb{R}^2_+$, with $h^{-1}(0) = \{(0,0)\}$, and such that, for some $C > 1$, we have $h(r,s) = \max\{r,s\}$ if $C \min\{r,s\} < \max\{r,s\}$. Then $A \times A'$ becomes a Thom-Mather stratification by setting $\rho''_{X\times X'}(x,x') = h(\rho_X(x), \rho'_{X'}(x'))$; it will be called *a product* of A and A'.

1.3. Cones. Recall that the *cone* with *link* a non-empty topological space L is the quotient space $c(L) = L \times [0,\infty)/L \times \{0\}$. The class $* = L \times \{0\}$ is called the *vertex* or *summit* of $c(L)$. The element of $c(L)$ represented by each $(x,\rho) \in L \times [0,\infty)$ will be denoted by $[x,\rho]$. The function on $c(L)$ induced by the second factor projection $L \times [0,\infty) \to [0,\infty)$ will be called its *canonical function*, and will be usually denoted by ρ. Notice that $c(L)$ is locally compact if and only if L is compact. It is also declared that $c(\emptyset)$ is the singleton space $\{*\}$, and the above terminology can be obviously adapted to this case.

Now, suppose that L is a compact Thom-Mather stratification. Then $c(L)$ has a canonical Thom-Mather stratification so that $\{*\}$ is a stratum, its restriction to $c(L) \setminus \{*\} = L \times \mathbb{R}_+$ is the product Thom-Mather stratification, and the tube of $\{*\}$ is $[c(L), \pi, \rho]$, where ρ is the canonical function and π is the unique map $c(L) \to \{*\}$.

If $L \neq \emptyset$, then $\operatorname{depth} c(L) = \operatorname{depth} L + 1$ and $\dim c(L) = \dim L + 1$. For any $\epsilon > 0$, let $c_\epsilon(L) = \rho^{-1}([0,\epsilon))$.

Let L' be another compact Thom-Mather stratification, and let $*'$ denote the vertex of $c(L')$. If $L \neq \emptyset$, the *cone* of any morphism $f : L \to L'$ is the morphism $c(f) : c(L) \to c(L')$ induced by $f \times \mathrm{id} : L \times [0,\infty) \to L' \times [0,\infty)$. If $L = \emptyset$, $c(f)$ is defined by mapping $*$ to $*'$. Reciprocally, it is easy to check that, for any morphism $h : c(L) \to c(L')$, there is some morphism $f : L \to L'$ such that $h = c(f)$ near $*$; in particular, $h(*) = *'$. Let $c(\operatorname{Aut}(L)) = \{\, c(f) \mid f \in \operatorname{Aut}(L) \,\} \subset \operatorname{Aut}(c(L))$.

Example 7.6. For each integer $m \geq 1$, there is a canonical homeomorphism $\mathrm{can} : c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$ defined by $\mathrm{can}([x,\rho]) = \rho x$. Of course, this is not an isomorphism of Thom-Mather stratifications, but it restricts to a diffeomorphism of the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$ to $\mathbb{R}^m \setminus \{0\}$. Via $\mathrm{can} : c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$, the canonical function of $c(\mathbb{S}^{m-1})$ corresponds to the function $\rho_0(x) = |x|$ on $\mathbb{R}^m$, which will be also called the *canonical function* on $\mathbb{R}^m$ for the scope of this work. If ρ_1 is the canonical function on $c(L)$ for some compact Thom-Mather stratification L, then the function $\rho = \sqrt{\rho_0^2 + \rho_1^2}$ will be called the *canonical function* on $\mathbb{R}^m \times c(L)$.

The following argument shows that a product of two cones is isomorphic to a cone. With the above notation, let $\rho : c(L) \to [0,\infty)$ and $\rho' : c(L') \to [0,\infty)$ be the canonical functions, and let $\rho'' = h(\rho \times \rho') : c(L) \times c(L') \to [0,\infty)$ for a function h like in Section 1.2. Since the restrictions $\rho : L \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\rho' : L' \times \mathbb{R}_+ \to \mathbb{R}_+$ are submersive weak morphisms, and $h : \mathbb{R}^2_+ \to \mathbb{R}_+$ is non-singular, it follows that $\rho'' : c(L) \times c(L') \setminus \{(*,*')\} \to \mathbb{R}_+$ is a submersive weak morphism. Hence $L'' = \rho''^{-1}(1)$ is saturated in $c(L) \times c(L')$ [**64**, Lemma 2.9, p. 17]. Let $*''$ denote the vertex of $c(L'')$. Since h is homogeneous of degree one, the mapping

$$[([x,r],[x',r']),s] \mapsto ([x,rs],[x',r's])$$

defines an isomorphism $c(L'') \to c(L) \times c(L')$, whose inverse is given by $(*,*') \mapsto *''$, and

$$([x,r],[x',r']) \mapsto \left[\left(\left[x, \frac{r}{h(r,r')}\right], \left[x', \frac{r'}{h(r,r')}\right]\right), h(r,r')\right]$$

if $(r,r') \neq (0,0)$.

1.4. Conic bundles. Let X be a smooth manifold, L a compact Thom-Mather stratification, and $\pi : T \to X$ a fiber bundle whose typical fiber is $c(L)$ and whose structural group can be reduced to $c(\operatorname{Aut}(L))$. Thus there is a family of local trivializations of π, $\{(U_i,\phi_i)\}$, such that the corresponding transition functions define a cocycle with values in $c(\operatorname{Aut}(L))$; i.e., for all i and j, there is a map $h_{ij} : U_{ij} := U_i \cap U_j \to c(\operatorname{Aut}(L))$ such that $\phi_j\phi_i^{-1}(x,y) = (x, h_{ij}(x)(y))$ for every $x \in U_{ij}$ and $y \in c(L)$. Thus we get another cocycle consisting of maps $g_{ij} : U_{ij} \to \operatorname{Aut}(L)$ so that $h_{ij}(x) = c(g_{ij}(x))$ for all $x \in U_{ij}$. Consider the Thom-Mather stratification on each open subset $\pi^{-1}(U_i) \subset T$ that corresponds by ϕ_i to the product Thom-Mather stratification on $U_i \times c(L)$. For each connected open $V \subset U_{ij}$ and every stratum N_0 of L, there is an stratum N_1 of L such that $g_{ij}(x)(N_0) = N_1$ for all $x \in V$, and suppose also that, in this case, the map $V \times N_0 \to N_1$, $(x,y) \mapsto g_{ij}(x)(y)$, is smooth. Then each mapping $(x,y) \mapsto (x, g_{ij}(x)(y))$ defines an automorphism of $U_{ij} \times L$. This means that the induced Thom-Mather stratifications on $\pi^{-1}(U_i)$ and $\pi^{-1}(U_j)$ have the same restriction to $\pi^{-1}(U_{ij})$. By Lemma 7.4-(i), it follows that there is a unique Thom-Mather stratification on T whose restriction to each $\pi^{-1}(U_i)$ is the above Thom-Mather stratification. Furthermore there is a canonical section of π, called the *vertex* (or *summit*) *section*, which is well defined by $x \mapsto *_x = \phi_i^{-1}(x,*)$ if $x \in U_i$, where $*$ denotes the vertex of $c(L)$; each $*_x$ can be called the *vertex* of the fiber over x. The image of the vertex section is a stratum of T, called the *vertex* (or *summit*) *stratum*, which is diffeomorphic to X.

If $\pi : T \to X$ is endowed with a maximal family Φ of trivializations satisfying the above conditions, it will be called a *conic bundle*, and the corresponding Thom-Mather stratification on T is called its *conic bundle Thom-Mather stratification*. It will be also said that Φ is the *conic bundle structure* of π.

Let $\rho : c(L) \to [0,\infty)$ be the canonical function. Its lift to each $U_i \times c(L)$ is also denoted by ρ. The functions $\phi^*\rho$ on the sets $\pi_X^{-1}(U_i)$ can be combined to define a function $\rho : T \to [0,\infty)$. The tubular neighborhood of X in T is $[T,\pi,\rho]$, and (T,π,ρ) is called its *canonical representative*.

Let $\pi' : T' \to X'$ be another conic bundle, whose structure is given by a family Φ' of trivializations as above. Let $F : T \to T'$ be a fiber bundle morphism over a map $f : X \to X'$. Then we can choose $\{(U_i,\phi_i)\}$ as above and a family $\{(U_i',\phi_i')\} \subset \Phi'$ such that $f(U_i) \subset U_i'$ for all i, and therefore $F(\pi^{-1}(U_i)) \subset {\pi'}^{-1}(U_i')$. Let $h_{ij}' = c(g_{ij}') : U_{ij}' := U_i' \cap U_j' \to c(\mathrm{Aut}(L'))$ be the maps defined by the transition maps $\phi_j'\, {\phi_i'}^{-1}$ as above. Suppose that there are maps $\kappa_i : U_i \to \mathrm{Mor}(L,L')$ such that $\kappa_j(x)\, g_{ij}(x) = g_{ij}'(f(x))\, \kappa_j(x)$ for all $x \in U_{ij}$. For each connected open $V \subset U_i$ and every stratum N of L, there is an stratum N' of L' such that $\kappa_i(x)(N) \subset N'$ for all $x \in V$, and assume also that, in this case, the map $V \times N \to N'$, $(x,y) \mapsto \kappa_i(x)(y)$, is smooth. Then F is called a *morphism of conic bundles*. In this case, each mapping $(x,y) \mapsto (f(x),\kappa_i(x)(y))$ defines a morphism $U_i \times c(L) \to U_i' \times c(L')$. So each restriction $F : \pi^{-1}(U_i) \to {\pi'}^{-1}(U_i')$ is a morphism of Thom-Mather stratifications, and therefore $F : T \to T'$ is a morphism of Thom-Mather stratifications by Lemma 7.5-(i). According to Section 1.3, any morphism of Thom-Mather stratifications between conic bundles, preserving the vertex stratum, equals a conic bundle morphism near the vertex stratum.

The case of conic bundles is specially important because, as pointed out in [**7**, Chapitre A, Remarque 3], the proof of [**64**, Theorem 2.6, pp. 16–17] can be easily adapted to get the following.

PROPOSITION 7.7. *Let $A \equiv (A,\mathcal{S},\tau)$ be a Thom-Mather stratification with connected strata. Then, for any $X \in \mathcal{S}$, there is some $(T,\pi,\rho) \in \tau_X$ such that $\pi : T \to X$ admits a structure Φ of conic bundle such that the corresponding conic bundle Thom-Mather stratification is $(\mathcal{S}|_T,\tau|_T)$.*

REMARK 11.
(i) The notation T_X, π_X, ρ_X, L_X and Φ_X will be used when a reference to the stratum X is desired.
(ii) The connectedness of the strata is assumed for the sake of simplicity. In the general case, the description of Proposition 7.7 holds around the connected components of the strata.
(iii) We can choose ρ so that (T,π,ρ) is the canonical representative of the tube around X in T with its conic bundle Thom-Mather stratification.

DEFINITION 7.8. A *chart* or *distinguished neighborhood* of A is a pair (O,ξ), where O is open in A and, for some $X \in \mathcal{S}$ and $\epsilon > 0$, with the notation and conditions of Proposition 7.7, ξ is an isomorphism $O \to B \times c_\epsilon(L)$ defined by some $(U,\phi) \in \Phi$ and some chart (U,ζ) of X with $\zeta(U) = B$, where B is an open subset of $\mathbb{R}^m$ for $m = \dim X$. It is said that (O, ξ) is said to be *centered* at $x \in X$ if B is an open ball centered at 0 and $\xi(x) = (0, *)$, where $*$ is the vertex of $c(L)$. A collection of charts that cover A is called an *atlas* of A.

REMARK 12. Definition 7.8 also includes the case where any factor of the product $\mathbb{R}^m \times c(L)$ is missing by taking $m = 0$ or $L = \emptyset$.

REMARK 13. The following two assertions follow by using charts and induction on the depth of the strata:

(i) In any Thom-Mather stratification, there is at most one dense stratum, which is open.
(ii) Any stratum with compact closure has a finite number of connected components.

1.5. Uniqueness of Thom-Mather stratifications.

LEMMA 7.9. *Let A be a Hausdorff, locally compact and second countable space, let $(A', \mathcal{S}', \tau')$ be a Thom-Mather stratification with connected strata, and let $f : A \to A'$ be a continuous map. Then there is at most one Thom-Mather stratification $(\mathcal{S}, \tau)$ on A with connected strata so that $f : (A, \mathcal{S}, \tau) \to (A', \mathcal{S}', \tau')$ is a morphism that restricts to local diffeomorphism between corresponding strata.*

PROOF. Let $(\mathcal{S}, \tau)$ be a Thom-Mather stratification on A satisfying the conditions of the statement. Then the elements of $\mathcal{S}$ are the connected components X of the sets $f^{-1}(X')$ for $X' \in \mathcal{S}$, endowed with the differential structure so that $f : X \to X'$ is a local diffeomorphism. Thus the elements of $\mathcal{S}$ are determined by f and the elements of $\mathcal{S}'$.

Let $X \in \mathcal{S}$ and $X' \in \mathcal{S}'$ with $f(X) \subset X'$, and let $(T, \pi, \rho) \in \tau_X$ and $(T', \pi', \rho') \in \tau'_{X'}$ with $f(T) \subset T'$, $\pi' f = f \pi$ and $\rho' f = \rho$; in particular, ρ is determined by f and ρ'. Let $x \in T$ and $x' = f(x) \in T'$, and let $Y \in \mathcal{S}$ such that $x \in Y$. Then $f \pi(x) = \pi'(x')$, obtaining that $\pi(x)$ is the unique point of $X \cap f^{-1}(\pi'(x'))$ that is contained in the connected component of x in $f^{-1}\pi'^{-1}(\pi'(x'))$. It follows that π is also determined by f and π', and therefore τ_X is determined by f and $\tau'_{X'}$. □

1.6. Relatively local properties on strata. The following kind of terminology will be used for a subspace X of an arbitrary topological space A. Let $\mathcal{P}$ be a property that may hold on open subsets $U \subset X$; for the sake of simplicity, let us say that "U is $\mathcal{P}$" when $\mathcal{P}$ holds on U. It is is said that X is *relatively locally* (or simply, *rel-locally*) $\mathcal{P}$ *at* some $x \in \overline{X}$ if there is a base $\mathcal{U}$ of open neighborhoods of x in A such that $U \cap X$ is $\mathcal{P}$ for all $U \in \mathcal{U}$; if X is rel-locally $\mathcal{P}$ at all points of $\overline{X}$, then X is said to be *relatively locally* (or simply, *rel-locally*) $\mathcal{P}$. Similarly, $\mathcal{P}$ is said to be a *relatively local* (or simply, *rel-local*) property when X is $\mathcal{P}$ if and only if it is rel-locally $\mathcal{P}$.

We will apply this terminology to the case where A is a Thom-Mather stratification and X is a stratum of A. For instance, on X, we will consider functions that are rel-locally bounded or rel-locally bounded away from zero, rel-locally finite open coverings, and rel-local connectedness at points of $\overline{X}$. Any locally finite covering of $\overline{X}$ by open subsets of A restricts to a rel-locally finite open covering of X; thus there exist rel-locally finite open coverings of X by the paracompactness of A. Observe that $\overline{X}$ is compact if and only if any rel-locally finite open covering of X is finite.

2. Adapted metrics on strata

The definition of adapted metrics was given for the regular stratum of any Thom-Mather stratification that is a pseudomanifold [**13, 14, 47, 48**]. But its definition has an obvious version for any stratum of a Thom-Mather stratification. In this work, we will consider only the simplest type of adapted metrics, whose definition is recalled. The corresponding (componentwise) metric completion of strata will be specially studied.

2.1. Adapted metrics on strata and local quasi-isometries between Thom-Mather stratifications. Let A be a Thom-Mather stratification. The adapted metrics on its strata are combinations of the adapted metrics on their connected components with respect to the Thom-Mather stratification defined by those connected components. Thus we can assume that the strata of A are connected to define adapted metrics. This definition is given by induction on the depth of the strata.

Definition 7.10. Let M be a stratum of A. If $\operatorname{depth} M = 0$, then M is a closed manifold, and any Riemannian metric on M is called *adapted*. If $\operatorname{depth} M > 0$ and adapted metrics are defined for strata of lower depth, then an *adapted metric* on M is a Riemannian metric g such that, for any point $x \in \overline{M} \setminus M$, there is some chart (O, ξ) of A centered at x, with $\xi(O) = B \times c_\epsilon(L)$ and $\xi(O \cap M) = B \times N \times (0, \epsilon)$ for some stratum N of L, so that g is quasi-isometric to $\xi^*(g_0 + \rho^2\tilde{g} + (d\rho)^2)$ on O, where g_0 is the standard Riemannian metric on $\mathbb{R}^m$, ρ is the standard coordinate of $\mathbb{R}_+$, and $\tilde{g}$ is some adapted metric on N, which is defined because the depth of N in L is smaller than the depth of M in A.

Remark 14. Since all Riemannian metrics on a smooth manifold are locally quasi-isometric, any metric on $\mathbb{R}^m$ could be used in Definition 7.10 instead of g_0.

Remark 15. The following properties follow by taking charts and using induction on the depth of the strata:

(i) Any pair of adapted metrics on M, g and g', are rel-locally quasi-isometric; in particular, if $\overline{M}$ is compact, then any pair of adapted metrics on M are quasi-isometric.
(ii) Any point in $\overline{M}$ has a countable base $\{\, O_m \mid m \in \mathbb{N} \,\}$ of open neighborhoods such that, with respect to any adapted metric, $\operatorname{vol}(M \cap O_m) \to 0$ and $\max\{\, \operatorname{diam} P \mid P \in \pi_0(M \cap O_m) \,\} \to 0$ as $m \to \infty$; in particular, if $\overline{M}$ is compact, then, with respect to any adapted metric, we have $\operatorname{vol} M < \infty$ and $\operatorname{diam} P < \infty$ for all $P \in \pi_0(M)$.
(iii) Any morphism of Thom-Mather stratifications restricts to rel-locally uniformly continuous maps between corresponding strata with respect to arbitrary adapted metrics.
(iv) If g and g' are adapted metrics on strata M and M' of Thom-Mather stratifications A and A', respectively, then $g \oplus g'$ is an adapted metric on the stratum $M \times M'$ of any product Thom-Mather stratification on $A \times A'$ (Section 1.2).

In [**8**, Appendix], it was proved that there exist adapted metrics on the regular stratum of any Thom-Mather stratification that is a pseudomanifold. It can be easily checked that the same argument proves the existence of adapted metrics on any stratum M of every Thom-Mather stratification A.

EXAMPLE 7.11. The proof in [**8**, Appendix] also shows the following:

(i) With the notation of Definition 7.10, the metric $g = g_0 + \rho^2\tilde{g} + (d\rho)^2$ is adapted on the stratum $M = \mathbb{R}^m \times N \times \mathbb{R}_+$ of $c(L)$; it will be called a *model adapted metric*.

(ii) Given a rel-locally finite atlas $\{(O_a, \xi_a)\}$ of $\overline{M}$, a smooth partition of unity $\{\lambda_a\}$ subordinated to the open covering $\{M \cap O_a\}$ of M, and an adapted metric g_a on each $M \cap O_a$, then the metric $\sum_a \lambda_a g_a$ on M is adapted.

EXAMPLE 7.12. For an integer $m \geq 1$, let $\tilde{g}_0$ be the restriction to $\mathbb{S}^{m-1}$ of the standard metric g_0 of $\mathbb{R}^m$. Then, via can : $c(\mathbb{S}^{m-1}) \to \mathbb{R}^m$ (Example 7.6), the model adapted metric $g_1 = \rho^2\tilde{g}_0 + (d\rho)^2$ on the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$ corresponds to g_0 on $\mathbb{R}^m \setminus \{0\}$.

EXAMPLE 7.13. With the notation of Example 7.3, for any invariant Riemannian metric g on M, consider the Riemannian metric $\bar{g}$ on the strata of $G\backslash M$ so that the canonical projection of the strata of M to the strata of $G\backslash M$ is a Riemannian submersion. The proof of [**29**, Proposition 2.6] can be easily adapted to produce an invariant Thom-Mather structure on M so that the restriction of g to any stratum is adapted. Hence $\bar{g}$ is adapted for the induced Thom-Mather structure of $G\backslash M$.

A weak isomorphism between Thom-Mather stratifications is called a *local quasi-isometry* if it restricts to rel-local quasi-isometries between their strata with respect to adapted metrics; this is independent of the choice of adapted metrics by Remark 15-(i). In particular, a local quasi-isometry between compact Thom-Mather stratifications restricts to quasi-isometries between their strata; thus a local quasi-isometry between compact Thom-Mather stratifications will be called a *quasi-isometry*. The condition of being locally quasi-isometric defines an equivalence relation on the family of Thom-Mather stratifications on any Hausdorff, locally compact and second countable space; each equivalence class will be called a *quasi-isometry type* of Thom-Mather stratifications. By Remark 15-(iv), the product of Thom-Mather stratifications is unique up to local quasi-isometries.

DEFINITION 7.14. Consider an adapted metric on a connected stratum M of a Thom-Mather stratification A, and let d denote the corresponding distance function on M. For each $x \in \overline{M}$ and $\rho > 0$, the *relative ball* (or *rel-ball*) of *radius* ρ and *center* x is the set consisting of the points $y \in M$ such that there is a sequence (z_k) in M with $\lim_k z_k = x$ in $\overline{M}$ and $\limsup_k d(y, z_k) < \rho$. The term *$\rho$-relative neighborhood* (or *ρ-rel-neighborhood*) of x will be also used for this concept.

EXAMPLE 7.15. (i) The rel-balls centered at points of M are the usual balls.

(ii) In the case of a model adapted metric on the stratum $M = N \times \mathbb{R}_+$ of $c(L)$, the ρ-rel-neighborhood of the vertex $*$ is $N \times (0, \rho)$.

2.2. Relatively local completion. Let M be a stratum of a Thom-Mather stratification A, and fix an adapted metric g on M.

DEFINITION 7.16. Assume first that M is connected, and consider the distance function d on M induced by g. The *relatively local completion* (or simply, *rel-local completion*) is the subspace $\widehat{M}$ of the metric completion of M whose points can be represented by Cauchy sequences in M that converge in A; the limits in $\overline{M}$ of those sequences define a canonical continuous map $\lim : \widehat{M} \to \overline{M}$. The canonical dense injection of M into its metric completion restricts to a canonical dense injection $\iota : M \to \widehat{M}$ satisfying $\lim \iota = \mathrm{id}_M$. The more specific notation $\lim_M$ and ι_M may be also used.

If M is not connected, then $\widehat{M}$ is defined as the disjoint union of the rel-local completions of its connected components.

Remark 16. (i) If $\overline{M}$ is compact, then $\widehat{M}$ is independent of the choice of the adapted metric by Remark 15-(i).

(ii) For any open $O \subset A$, $\widehat{M \cap O}$ can be canonically identified to the open subspace $\lim^{-1}(\overline{M} \cap O) \subset \widehat{M}$.

Example 7.17 (Relatively local completion of the strata of cones). Let L be a compact Thom-Mather stratification and M a stratum of $c(L)$. With the notation of Section 1.3, if $M = \{*\}$, then $\widehat{M} = M$, obviously. Now, suppose that $M = N \times \mathbb{R}_+$ for some stratum N of L. Consider the model adapted metric $g = \rho^2 \tilde{g} + (d\rho)^2$ for some adapted metric $\tilde{g}$ on N, and the corresponding rel-local completion $\widehat{M}$. $\pi_0(N)$ is finite by Remark 13-(ii). For each $P \in \pi_0(N)$, let $\widehat{P}$ denote the rel-local completion of P with respect to L_{con}, which is independent of the choice of $\tilde{g}$. Then it is easy to check that

$$M \equiv \bigsqcup_P P \times \mathbb{R}_+ \xrightarrow{\bigsqcup_P \iota_P \times \mathrm{id}} \bigsqcup_P \widehat{P} \times \mathbb{R}_+ \hookrightarrow \bigsqcup_P c(\widehat{P})$$

extends to a homeomorphism $\widehat{M} \to \bigsqcup_{P \in \pi_0(N)} c(\widehat{P})$.

Remark 17. The following properties follow easily by using charts, induction on the depth of the strata, Example 7.17 and Remark 15-(ii):

(i) $\lim : \widehat{M} \to \overline{M}$ is surjective with finite fibers.

(ii) M is rel-locally connected with respect to $\widehat{M}$.

(iii) If $\overline{M}$ is compact, then $\widehat{M}$ is compact, and therefore its connected components are the metric completions of the connected components of M.

Proposition 7.18. (i) *$\widehat{M}$ has a unique Thom-Mather stratification with connected strata such that $\lim : \widehat{M} \to \overline{M}$ is a morphism that restricts to local diffeomorphisms between corresponding strata. In particular, the connected components of M can be considered as strata of $\widehat{M}$ via ι_M.*

(ii) *The restriction of g to the connected components of M are adapted metrics with respect to $\widehat{M}$.*

(iii) *Let M' be a connected stratum of another Thom-Mather stratification A' endowed with an adapted metric. Then, for any morphism $f : A \to A'$ with $f(M) \subset M'$, the restriction $f : M \to M'$ extends to a morphism $\hat{f} : \widehat{M} \to \widehat{M}'$. Moreover $\hat{f}$ is an isomorphism if f is an isomorphism.*

Proof. This is proved by induction on $\operatorname{depth} M$. If $\operatorname{depth} M = 0$, then $\widehat{M} \equiv \overline{M} = M$, and there is nothing to prove.

Suppose that $\operatorname{depth} M > 0$ and the statement holds for strata of lower depth. We can assume that the strata of $\overline{M}$ is connected. For each stratum X of $\overline{M}$, let (T_X, π_X, ρ_X) be a representative of the tube around X in $\overline{M}$ satisfying the conditions of Section 1.4 with a compact Thom-Mather stratification L_X and a family $\{(U_i, \phi_i)\}$ of local trivializations of π_X. The corresponding cocycle with values in $c(\mathrm{Aut}(L_X))$ consists of the maps $h_{ij} : U_i \cap U_j \to c(\mathrm{Aut}(L_X))$ defined by

$h_{ij}(x) = (\phi_j\,\phi_i^{-1})(x,\cdot)$. We have $h_{ij}(x) = c(g_{ij}(x))$ for a cocycle consisting of maps $g_{ij} : U_i \cap U_j \to \operatorname{Aut}(L_x)$.

By the density of M in $\overline{M}$ and Remark 13-(i), there is a dense stratum N of L_X so that $\phi_i(M \cap \pi_X^{-1}(U_i)) = U_i \times N \times \mathbb{R}_+$ for all i. Consider triples (x, i, P) such that $x \in U_i$ and $P \in \pi_0(N)$. Two triples of this type, (x, i, P) and (y, j, Q), are declared to be equivalent if $x = y$ and $g_{ij}(x)(P) = Q$. The equivalence class of each triple (x, i, P) is denoted by $[x, i, P]$, and let X' denote the corresponding quotient set. There is a canonical map $f_X : X' \to X$, defined by $f_X([x, i, P]) = x$. Consider the topology on X' determined by requiring that the sets $U'_{i,P} = \{\, [x, i, P] \mid x \in U_i \,\}$ are open, and the restrictions $f_X : U'_{i,P} \to U_i$ are homeomorphisms. Notice that f_X is a finite fold covering map; in particular, in the case $X = M$, f_M is a homeomorphism. Consider the differential structure on each X' so that f_X is a local diffeomorphism.

By the induction hypothesis, for each $P \in \pi_0(N)$, $\widehat{P}$ satisfies the statement of the proposition with some Thom-Mather stratification. Consider quadruples (x, i, P, u) such that $x \in U_i$, $P \in \pi_0(N)$ and $u \in c(\widehat{P})$. Two such quadruples, (x, i, P, u) and (y, j, Q, v), are said to be equivalent if $x = y$, $g_{ij}(x)(P) = Q$ and $c(\widehat{g_{ij}(x)})(u) = v$. The equivalence class of each quadruple (x, i, P, u) is denoted by $[x, i, P, u]$, and let T'_X denote the corresponding quotient set. There are canonical maps, $\pi'_X : T'_X \to X'$, $\lim'_X : T'_X \to T_X$, $\rho'_X : T'_X \to [0, \infty)$ and $\iota'_X : M \cap T_X \to T'_X$ defined by $\pi'_X([x, i, P, u]) = [x, i, P]$, $\lim'_X([x, i, P, u]) = \phi_i^{-1}(x, c(\lim_P)(u))$, $\rho'_X([x, i, P, u]) = \rho(u)$, and $\iota'_X(z) = [x, i, P, (\iota_P(v), r)]$ if $z \in M \cap \pi_X^{-1}(U_i)$ and $\phi_i(z) = (x, v, r) \in U_i \times P \times \mathbb{R}_+$. Notice that $f_X\,\pi'_X = \pi_X \lim'_X$ and $\rho_X\,\pi'_X = \rho'_X$.

Let $G \subset \operatorname{Aut}(L_X)$ be the subgroup generated by the above elements $g_{ij}(x)$. Since the canonical action of G on L_X preserves N, we get an induced action of G on $\pi_0(N)$. Since X is connected, there is a bijection between $G\backslash\pi_0(N)$ and the set $\pi_0(X')$ of connected components of X', where any orbit $\mathcal{O} \in G\backslash\pi_0(N)$ corresponds to the connected component $X'_{\mathcal{O}} \in \pi_0(X')$ consisting of the points $[x, i, P] \in X'$ with $P \in \mathcal{O}$. Also, let $T'_{X,\mathcal{O}} = (\pi'_X)^{-1}(X'_{\mathcal{O}}) \subset T'_X$.

Given any $\mathcal{O} \in G\backslash\pi_0(N)$, fix some $P_0 \in \mathcal{O}$. For any other $P \in \mathcal{O}$, there is some $g_P \in G$ such that $g_P(P) = P_0$. Thus the restriction $g_P : P \to P_0$ induces a map $\widehat{g_P} : \widehat{P} \to \widehat{P_0}$, and let $\phi'_{i,P} : (\pi'_X)^{-1}(U'_{i,P}) \to U'_{i,P} \times c(\widehat{P_0})$ be the bijection defined by $\phi'_{i,P}([x, i, P, u]) = ([x, i, P], c(\widehat{g_P})(u))$. Consider the topology on $T'_{X,\mathcal{O}}$ determined by requiring that the sets $(\pi'_X)^{-1}(U'_{i,P})$ are open, and the maps $\phi'_{i,P}$ are homeomorphisms. Then the maps $\phi'_{i,P}$ are local trivializations of the restriction $\pi'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to X'_{\mathcal{O}}$ of π'_X, obtaining that $\pi'_{X,\mathcal{O}}$ is a fiber bundle with typical fiber $c(\widehat{P_0})$. The associated cocycle has values in $c(\operatorname{Aut}(\widehat{P_0}))$; in fact, it consists of the functions $h'_{i,P;j,Q} : U'_{i,P} \cap U'_{j,Q} \to c(\operatorname{Aut}(\widehat{P_0}))$ defined by

$$h'_{i,P;j,Q}([x, i, P])(u) = c(g'_{i,P;j,Q}([x, i, P]))(u)\ ,$$

where $g'_{i,P;j,Q} : U'_{i,P} \cap U'_{j,Q} \to \operatorname{Aut}(\widehat{P_0})$ is the cocycle given by

$$g'_{i,P;j,Q}([x, i, P]) = \widehat{g_Q}\,\widehat{g_{ij}(x)}\,\widehat{g_P}^{-1}\ .$$

The conditions of Section 1.4 are satisfied, obtaining that $\pi'_{X,\mathcal{O}}$ is a conic bundle, and therefore $T'_{X,\mathcal{O}}$ can be endowed with the corresponding conic bundle Thom-Mather stratification.

Since $N_{X,\mathcal{O}} := \bigcup_{P \in \mathcal{O}} P$ is G-invariant, the set $N_{X,\mathcal{O}} \times \mathbb{R}_+$ is invariant by all transformations $h_{ij}(x)$ for $x \in U_{ij}$, and therefore it defines an open subspace

$M_{X,\mathcal{O}} \subset M \cap T_X$. Let $\lim'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to T_X$, $\rho'_{X,\mathcal{O}} : T'_{X,\mathcal{O}} \to [0,\infty)$ and $\iota'_{X,\mathcal{O}} : M_{X,\mathcal{O}} \to T'_{X,\mathcal{O}}$ be defined by restricting $\lim'_X$, ρ'_X and ι'_X. Then $(T'_{X,\mathcal{O}}, \pi'_{X,\mathcal{O}}, \rho'_{X,\mathcal{O}})$ is the canonical representative of the tube of X' in $T'_{X,\mathcal{O}}$, $\iota'_{X,\mathcal{O}}$ is a dense open embedding, $\lim'_{X,\mathcal{O}} \iota'_{X,\mathcal{O}} = \mathrm{id}$, and $\lim'_{X,\mathcal{O}}$ is the conic bundle morphism over $f_X : X'_{\mathcal{O}} \to X$ induced by the maps $\kappa_{i,P} : U'_{i,P} \to \mathrm{Mor}(\widehat{P_0}, L_X)$ given by $\kappa_{i,P}([x,i,P]) = \lim_P \widehat{g_P}^{-1}$ (Section 1.4). By the induction hypothesis, $\kappa_{i,P}([x,i,P])$ restricts to local diffeomorphisms between corresponding strata, and therefore $\lim'_{X,\mathcal{O}}$ restricts to local diffeomorphisms between corresponding strata.

On $T'_X \equiv \bigsqcup_{\mathcal{O} \in G\backslash\pi_0(N)} T'_{X,\mathcal{O}}$, consider the sum of the topologies and Thom-Mather stratifications of the spaces $T'_{X,\mathcal{O}}$ (Remark 10). By Lemma 7.5-(i), $\lim'_X : T'_X \to T_X$ is a morphism that restricts to local diffeomorphisms between corresponding strata. Observe that the strata of T'_X are connected.

By using the local trivializations of π_X and each $\pi'_{X,\mathcal{O}}$, and Example 7.17, it follows that $\iota'_{X,\mathcal{O}} : M_{X,\mathcal{O}} \to T'_{X,\mathcal{O}}$ extends to an isomorphism $\widehat{M_{X,\mathcal{O}}} \to T'_{X,\mathcal{O}}$ such that $\lim'_{X,\mathcal{O}}$ corresponds to $\lim_{M_{X,\mathcal{O}}}$. Hence $\iota'_X : M \cap T_X \to T'_X$ extends to an isomorphism $\widehat{M \cap T_X} \to T'_X$ such that $\lim'_X$ corresponds to $\lim_{M\cap T_X}$. Then, according to Remark 16-(ii), we can consider the spaces T'_X as open subspaces of $\widehat{M}$, obtaining an open covering of $\widehat{M}$ as X runs in the family of strata of $\overline{M}$. Moreover each restriction $\lim_M : T'_X \to \overline{M} \cap T_X$ restricts to local diffeomorphisms between the corresponding strata. Hence, by Lemma 7.9, for strata X and Y of $\overline{M}$, the restrictions of the Thom-Mather stratifications of T'_X and T'_Y to $T'_X \cap T'_Y$ induce the same Thom-Mather stratification with connected strata. By Lemma 7.4-(ii), it follows that there is a unique Thom-Mather stratification with connected strata on $\widehat{M}$ whose restriction to each T'_X induces the above conic bundle Thom-Mather stratification. By Lemma 7.5-(ii), $\lim_M$ is a morphism because its restriction to each T'_X is a morphism. This completes the proof of (i).

In the above construction, consider every $U'_{i,P} \times P_0$ as a stratum of each $U'_{i,P} \times c(\widehat{P_0})$ via $\mathrm{id} \times \iota_{P_0}$. Let $g'_{i,P}$ be any Riemannian metric on $U'_{i,P}$, and let $\tilde{g}_0$ be an adapted metric on P_0 with respect to $\overline{P_0} \subset L_X$. Thus $g'_{i,P} + \tilde{g}_0$ is an adapted metric on $U'_{i,P} \times P_0$, and therefore, by the induction hypothesis, it is also adapted with respect to $U'_{i,P} \times c(\widehat{P_0})$. Hence, considering each $M_{X,\mathcal{O}}$ as a stratum of $T'_{X,\mathcal{O}}$ via $\iota'_{X,\mathcal{O}}$, the restriction of g to each $M_{X,\mathcal{O}}$ is adapted with respect to $T'_{X,\mathcal{O}}$, and (ii) follows.

Part (iii) follows from (i), (ii) and Remark 15-(iii). □

CHAPTER 8

Relatively Morse functions

Our version of Morse functions on strata is introduced and studied in this section.

Let M be a stratum of a Thom-Mather stratification A, and fix an adapted metric g on M. Identify M and its image by the canonical dense open embedding $\iota : M \to \widehat{M}$. Let $f \in C^\infty(M)$.

DEFINITION 8.1. (i) It is said that f is *relatively admissible* (or simply, *rel-admissible*) with respect to g if f, $|df|$ and $|\nabla df|$ are rel-locally bounded.

(ii) A point $x \in \widehat{M}$ is called *relatively critical* (or simply, *rel-critical*) if

$$\liminf_{y \in M,\ y \to x} |df(y)| = 0$$

for some adapted metric. The set of rel-critical points of f is denoted by $\mathrm{Crit}_{\mathrm{rel}}(f)$.

(iii) A point $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$ is said to be *relatively non-degenerate* (or simply, *rel-non-degenerate*) if there is some neighborhood O of x in $\widehat{M}$ and some $c > 0$ such that $|\nabla_v df| \geq c\,|v|$ for all $v \in T(M \cap O)$.

REMARK 18. (i) Let O be any open subset of A. If $f \in C^\infty(M)$ is rel-admissible with respect to g, then $f|_{M\cap O}$ is rel-admissible with respect to $g|_{M\cap O}$.

(ii) The rel-local boundedness of $|df|$ is invariant by rel-local quasi-isometries, and therefore it is independent of g, but the rel-local boundedness of $|\nabla df|$ depends on the choice of g. However it follows from Lemma 8.4 and Proposition 8.5 below that the existence of g so that f is rel-admissible with respect to g is a rel-local property.

(iii) If depth $M = 0$, then any smooth function is admissible, and its (rel-non-degenerate) rel-critical points are its (non-degenerate) critical points.

(iv) A rel-admissible function on M may not have any continuous extension to $\overline{M}$, but it has a continuous extension to $\widehat{M}$ by the rel-local boundedness of $|df|$. Thus it becomes natural to define its rel-critical points in $\widehat{M}$.

(v) The admissible functions on M form a unital subalgebra of $C^\infty(M)$ because d is a derivation and, for $f, h \in C^\infty(M)$,

$$\nabla d(fh) = df \otimes dh + f\nabla dh + dh \otimes df + h\nabla df\ .$$

EXAMPLE 8.2. With the notation of Example 7.11-(i), for any $h \in C_0^\infty(\mathbb{R}_+)$, the function $h(\rho)$ is rel-admissible on the stratum $\mathbb{R}^m \times N \times \mathbb{R}_+$ of $\mathbb{R}^m \times c(L)$ with respect to any model adapted metric.

Example 8.3. With the notation of Examples 7.3 and 7.13, for any G-invariant smooth function f on M, let $\bar{f}$ denote the induced function on $G\backslash M$, whose restriction to each stratum is smooth, and df is the pull-back of $d\bar{f}$ on corresponding strata of M and $G\backslash M$. Fix any invariant metric on M and consider the induced adapted metric on the strata of $G\backslash M$. The restriction of $\operatorname{Hess} f$ to horizontal tangent vectors on the strata of M corresponds via the canonical projection to $\operatorname{Hess}\bar{f}$ on the strata of $G\backslash M$ by [**51**, Lemma 1]. It easily follows that $\bar{f}$ is rel-admissible on the strata of $G\backslash M$.

Lemma 8.4. *For any rel-locally finite covering $\{\, O_a \mid a \in \mathcal{A}\,\}$ of $\overline{M}$ by open subsets of A, there is a smooth partition of unity $\{\lambda_a\}$ on M subordinated to the open covering $\{M \cap O_a\}$ such that, for any adapted metric on M, each function $|d\lambda_a|$ is rel-locally bounded.*

Proof. If $\operatorname{depth} M = 0$, then the rel-locally bounded smooth functions on M are the locally bounded ones, and therefore the statement holds in this case because any continuous function is locally bounded. Thus suppose that $\operatorname{depth} M > 0$. For $0 \le k \le \operatorname{depth} M$, let $\mathfrak{F}_k$ denote the union of all strata $X < M$ with $\operatorname{depth} X \le k$. The lemma is given by the case $k = \operatorname{depth} M$ in the following assertion.

Claim 5. For $0 \le k \le \operatorname{depth} M$, there is a family of smooth functions $\{\lambda_{a,k}\}$ on M such that:

- (i) $0 \le \sum_a \lambda_{a,k} \le 1$ for all k;
- (ii) $\lambda_{a,k}$ is supported in $M \cap O_a$ for all $a \in \mathcal{A}$;
- (iii) there is some open neighborhood U_k of $\mathfrak{F}_k$ in A so that $\sum_a \lambda_{a,k} = 1$ on $U_k \cap M$; and,
- (iv) for any adapted metric on M, each function $|d\lambda_{a,k}|$ is rel-locally bounded.

This claim is proved by induction on k. To simplify its proof, observe that it is also satisfied for $k = -1$ with $\mathfrak{F}_{-1} = U_{-1} = \emptyset$, and $\lambda_{a,-1} = 0$ for all $a \in \mathcal{A}$.

Now, assume that Claim 5 holds for some $k \in \{-1, 0, \dots, \operatorname{depth} M - 1\}$. Let V_k be another open neighborhood of $\mathfrak{F}_k$ in A such that $\overline{V_k} \subset U_k$. We can assume that the strata of A are connected by Remark 8-(v).

$\mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$ is the union of the strata X that satisfy $\overline{X} \setminus X \subset \mathfrak{F}_k$, and therefore the sets $X \setminus \overline{V_k}$ are closed in $A \setminus \overline{V_k}$ and disjoint from each other. For the strata $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, choose representatives $(T_X, \pi_X, \rho_X) \in \tau_X$ satisfying the properties of Definition 7.1-(iv)–(vi), Proposition 7.7 and Remark 11-(iii). Let Φ_X denote the conic bundle structure of π_X. Moreover, like in Remark 8-(ii), we can assume that the sets $T_X \setminus \overline{V_k}$ are disjoint one another.

By refining $\{O_a\}$ if necessary, we can suppose that, for each stratum $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, any point in $X \setminus \overline{V_k}$ is in some set O_a such that there is a chart of A of the form (O_a, ξ_a), obtained from a local trivialization in Φ_X according to Definition 7.8; in this case, let $\xi_a(O_a) = B_a \times c_{\epsilon_a}(L_X)$ for some open subset $B_a \subset \mathbb{R}^{m_X}$ and some $\epsilon_a > 0$, where $m_X = \dim X$; let $\mathcal{A}_X$ be the family the indices $a \in \mathcal{A}$ that satisfy this condition. For each $a \in \mathcal{A}_X$, take a smooth function $h_a : [0, \infty) \to [0, 1]$ supported in $[0, \epsilon_a)$ and such that $h_a = 1$ around 0. Let $\{\, \mu_a \mid a \in \mathcal{A}_X\,\}$ be a smooth partition of unity on $\mathfrak{F}_{k+1} \setminus \overline{V_k}$ subordinated to the open covering $\{\, O_a \setminus \overline{V_k} \mid a \in \mathcal{A}_X\,\}$. Set $\lambda_k = \sum_a \lambda_{a,k}$. Then define

$$\lambda_{a,k+1} = \lambda_{a,k} + (1 - \lambda_k) \cdot \rho_X^* h_a \cdot \pi_X^* \mu_a$$

if $a \in \mathcal{A}_X$ for some stratum $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, and $\lambda_{a,k+1} = \lambda_{a,k}$ otherwise. These functions are smooth on M because λ_k is smooth and equals 1 on U_k. It is easy to check that they also satisfy Claim 5-(i)–(iv). □

PROPOSITION 8.5. *Let $\{ O_a \mid a \in \mathcal{A} \}$ be a rel-locally finite covering of $\overline{M}$ by open subsets of A, let $\{\lambda_a\}$ be a partition of unity on M subordinated to the open covering $\{M \cap O_a\}$ satisfying the conditions of Lemma 8.4, and let $f \in C^\infty(M)$ such that each $f|_{M\cap O_a}$ is rel-admissible with respect to some metric g_a on $M \cap O_a$. Then f is rel-admissible with respect to the adapted metric $g = \sum_a \lambda_a g_a$ on M.*

To prove Proposition 8.5, we will use the following lemma.

LEMMA 8.6. *Let X be a Riemannian manifold of dimension n, and let $f \in C^\infty(X)$ and $p \in X$. If $(df)(p) \neq 0$, then there is a system of coordinates $(x^1, \dots, x^n)$ of X around p such that $(\partial_1(p), \dots, \partial_n(p))$ is an orthonormal reference and $\partial_i\partial_j f = 0$ for all $i, j \in \{1, \dots, n\}$, where $\partial_i = \partial/\partial x^i$.*

PROOF. Because $(df)(p) \neq 0$, the 1-form df defines a codimension one foliation around p (its tangent bundle is $\ker df$). By using a foliation chart around p, it follows that there is a system of coordinates $(x^1, \dots, x^n)$ around p such that the vectors $\partial_1(p), \dots, \partial_{n-1}(p)$ are orthonormal, and $x^n = f/|(df)(p)|$. It is easy to check that these coordinates satisfy the stated properties. □

PROOF OF PROPOSITION 8.5. Let $|\ |_a$ and ∇^a denote the norm and Levi-Civita connection of each g_a, and let $|\ |$ and ∇ denote the norm and Levi-Civita connection of g. On every $M \cap O_a$, the functions $|df|_a$ and $|\nabla^a df|_a$ are rel-locally bounded. Since g and g_a are rel-locally quasi-isometric on $M \cap O_a$, we get that $|df|$ and $|\nabla^a df|$ are rel-locally bounded on $M \cap O_a$. By shrinking $\{O_a\}$ if necessary, we can assume that there are constants $K_a \geq 0$ and $C_a \geq 1$ such that

$$|df|, |\nabla^a df|, |d\lambda_a| \leq K_a \quad \text{on} \quad M \cap O_a\ , \tag{74}$$

$$\frac{1}{C_a}\,|X|_a \leq |X| \leq C_a\,|X|_a \quad \forall X \in T(M \cap O_a)\ . \tag{75}$$

For any fixed $a_0 \in \mathcal{A}$, it is enough to prove that $|\nabla df|$ is bounded on $M \cap O_{a_0}$. For each $p \in M \cap O_{a_0}$, take any system of coordinates $(x^1, \dots, x^n)$ on some open neighborhood U of p in M such that $(\partial_1(p), \dots, \partial_n(p))$ is an orthonormal reference with respect to g. Let $g_{a,ij}$ and g_{ij} be the corresponding metric coefficients of g_a and g on $O_a \cap U$ and U, respectively; thus $g_{ij}(p) = \delta_{ij}$, and we can write $g_{ij} = \sum_a \lambda_a\, g_{a,ij}$ on U. As usual, the inverses of the matrices $(g_{a,ij})$ and (g_{ij}) are denoted by (g_a^{ij}) and (g^{ij}). By (75) and since $g_{ij}(p) = \delta_{ij}$, we have

$$\frac{1}{C_a^2}\, g_{a,ii}(p) \leq 1 \leq C_a^2\, g_{a,ii}(p)$$

for all $i \in \{1, \dots, n\}$ if $p \in O_a$, giving

$$\begin{aligned} |g_{a,ij}(p)| &= \frac{1}{2}\,|\,|\partial_i(p) + \partial_j(p)|_a^2 - g_{a,ii}(p) - g_{a,jj}(p)\,| \\ &\leq \frac{1}{2}\,(|\partial_i(p) + \partial_j(p)|_a^2 + g_{a,ii}(p) + g_{a,jj}(p)) \\ &\leq \frac{C_a^2}{2}\,(|\partial_i(p) + \partial_j(p)|^2 + 2) = 2C_a^2 \end{aligned}$$

for all $i,j \in \{1,\dots,n\}$. Since O_{a_0} meets a finite number of sets O_a, it follows that $|g_{a,ij}(p)|$ and $|g_a^{ij}(p)|$ are bounded by some $C \geq 1$, independent of the point $p \in O_{a_0}$. Similarly, by (74), we get that $|(df)(p)|$, $|(\nabla^a df)(p)|$ and $|(d\lambda_a)(p)|$ are bounded by some $K \geq 0$ independent of the point $p \in O_{a_0}$.

Let $\Gamma^k_{a,ij}$ and Γ^k_{ij} be the Christoffel symbols of g_a and g on $O_a \cap U$ and U, respectively, corresponding to $(x^1,\dots,x^n)$. Since $g_{ij}(p) = \delta_{ij}(p)$, we have[1]

$$\begin{aligned}\Gamma^k_{ij}(p) &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(p)\\ &= \frac{1}{2}\sum_a (g_{a,jk}\,\partial_i\lambda_a + \lambda_a\,\partial_i g_{a,jk} + g_{a,ik}\,\partial_j\lambda_a + \lambda_a\,\partial_j g_{a,ik}\\ &\qquad - g_{a,ij}\,\partial_k\lambda_a - \lambda_a\,\partial_k g_{a,ij})(p)\\ &\left.\begin{aligned}&= \frac{1}{2}\sum_a (g_{a,jk}\,\partial_i\lambda_a + g_{a,ik}\,\partial_j\lambda_a - g_{a,ij}\,\partial_k\lambda_a)(p)\\ &\quad + \sum_a \lambda_a(p)\,\Gamma^\ell_{a,ij}(p)\,g_{a,\ell k}(p)\ .\end{aligned}\right\}\end{aligned} \tag{76}$$

On the other hand,

$$\begin{aligned}\nabla df &= dx^i \otimes \nabla_i(\partial_k f\, dx^k)\\ &= \partial_i\partial_k f\, dx^i \otimes dx^k - \partial_k f\,\Gamma^k_{ij}\, dx^i \otimes dx^j\\ &= (\partial_i\partial_j f - \partial_k f\,\Gamma^k_{ij})\, dx^i \otimes dx^j\ .\end{aligned} \tag{77}$$

Similarly,

$$\nabla^a df = (\partial_i\partial_j f - \partial_k f\,\Gamma^k_{a,ij})\, dx^i \otimes dx^j\ . \tag{78}$$

If $(df)(p) = 0$, then

$$(\nabla df)(p) = (\partial_i\partial_j f\, dx^i \otimes dx^j)(p) = (\nabla^a df)(p)$$

by (77) and (78), and therefore $|(\nabla df)(p)| \leq K$.

If $(df)(p) \neq 0$, by Lemma 8.6, we can assume that the coordinates $(x^1,\dots,x^n)$ also satisfy $(\partial_i\partial_j f)(p) = 0$ for all $i,j \in \{1,\dots,n\}$. So, by (77) and (78),

$$\begin{aligned}(\nabla df)(p) &= -(\partial_k f\,\Gamma^k_{ij}\, dx^i \otimes dx^j)(p)\ ,\\ (\nabla^a df)(p) &= -(\partial_k f\,\Gamma^k_{a,ij}\, dx^i \otimes dx^j)(p)\ .\end{aligned}$$

Since $g^{ij}(p) = \delta_{ij}$, it follows that $|(\partial_k f\,\Gamma^k_{a,ij})(p)| \leq K$ for all $i,j \in \{1,\dots,n\}$, and it is enough to find a similar bound for each $|(\partial_k f\,\Gamma^k_{ij})(p)|$. But, by (76),

$$\begin{aligned}|(\partial_k f\,\Gamma^k_{ij})(p)| &\leq \frac{1}{2}|(df)(p)|\sum_a |(d\lambda_a)(p)|\,(|g_{a,jk}(p)| + |g_{a,ik}(p)| + |g_{a,ij}(p)|)\\ &\quad + \sum_a \lambda_a(p)\,|(\partial_k f\,\Gamma^\ell_{a,ij})(p)|\,|g_{a,\ell k}(p)|\\ &\leq \left(\frac{3}{2}K^2 C + KC\right)\cdot \#\{\, a \in \mathcal{A} \mid O_a \cap O_{a_0} \neq \emptyset \,\}\ . \quad \square\end{aligned}$$

[1] Einstein convention is used for the sums involving local coefficients.

We would like to define relatively Morse functions on M as rel-admissible functions whose rel-critical points are rel-non-degenerate. However an appropriate version of the Morse lemma [**46**, Lemma 2.2] is missing (see Problem 8.9 below), and therefore they are defined by giving their "rel-local models" around their rel-critical points.

DEFINITION 8.7. It is said that $f \in C^\infty(M)$ is a *relatively Morse function* (or *rel-Morse function*) if it is rel-admissible with respect to some adapted metric and, for every $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, there exists a chart (O, ξ) of $\widehat{M}$ centered at x, with $\xi(O) = B \times c_\epsilon(L)$, such that, for some $m_\pm \in \mathbb{N}$ and compact Thom-Mather stratifications $L_\pm$, there exists a pointed diffeomorphism $\theta_0 : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m_+} \times \mathbb{R}^{m_-}, (0,0))$, and a local quasi-isometry $\theta_1 : c(L) \to c(L_+) \times c(L_-)$ so that $f_{M\cap O}$ corresponds to a constant plus $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ via $(\theta_0 \times \theta_1)\,\xi$, where $\rho_\pm$ is the canonical function on $\mathbb{R}^{m_\pm} \times c(L_\pm)$ (Example 7.6).

EXAMPLE 8.8. With the notation of Examples 7.2-(v), 7.13 and (8.3), the invariant Morse-Bott functions on M whose critical submanifolds are orbits form a dense subset of the space of invariant smooth functions [**65**, Lemma 4.8]. They induce rel-Morse functions on every orbit type stratum of $G\backslash M$.

Let f be a rel-Morse function on M. For each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, with the notation of Definition 8.7, let $M_\pm$ be the strata of $c(L_\pm)$ so that $(\theta_0 \times \theta_1)\,\xi$ defines an open embedding of $M \cap O$ into $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, where either $M_\pm$ is the vertex stratum of $c(L_\pm)$, or $M_\pm = N_\pm \times \mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$ with $n_\pm = \dim M_\pm$. Using this local data, for each $r \in \mathbb{Z}$, the number $\nu^r_{x,\min/\max} = \nu^r_{x,\min/\max}(f)$ was defined in the Introduction, before Theorem J, page **??**. Recall also that $\nu^r_{\min/\max} = \nu^r_{\min/\max}(f)$ was defined as the sum of the numbers $\nu^r_{x,\min/\max}$ for $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$.

REMARK 19. (i) Every rel-Morse function on M is a Morse function, and the rel-critical points in M are the usual critical points. For such a critical point $x \in M$ with index m_-, we have $\nu^r_{x,\min/\max} = \delta_{r,m_-}$; thus $\sum_{x\in\operatorname{Crit}(f)} \nu^r_{x,\min/\max}$ is the number of critical points with index r. If $\operatorname{depth} M = 0$, then any Morse function on M is a rel-Morse function by the Morse lemma.
(ii) The rel-critical points of rel-Morse functions are isolated.
(iii) The function $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$ is rel-Morse, and will be called a *model rel-Morse function*.

PROBLEM 8.9 ("Rel-Morse lemma"). Let x be a rel-non-degenerate rel-critical point of a rel-admissible function f on M. Does there exist a chart (O, ξ) of $\widehat{M}$ centered at x and maps θ_0 and θ_1 satisfying the conditions of Definition 8.7? An affirmative answer may require a stronger condition in Definition 8.1-(i); for instance, the rel-local boundedness of $|\nabla^k f|$ for all $k \in \mathbb{N}$.

The existence, and indeed certain abundance, of rel-Morse functions is guaranteed by the following result.

PROPOSITION 8.10. *Let $\mathcal{F} \subset C^\infty(M)$ denote the subset of functions with continuous extensions to $\overline{M}$ that restrict to rel-Morse functions on all strata $\leq M$. Then $\mathcal{F}$ is dense in $C^\infty(M)$ with the weak C^∞ topology.*

Proof. If $\operatorname{depth} M = 0$, then the statement holds by the density of the Morse functions in $C^\infty(M)$ with the strong C^∞ topology [**35**, Theorem 6.1.2]. Thus suppose that $\operatorname{depth} M > 0$. Let the sets $\mathfrak{F}_k$ be defined like in the proof of Lemma 8.4.

Claim 6. For $0 \le k \le \operatorname{depth} M$, there is an open neighborhood U_k of $\mathfrak{F}_k$ in A and some $f_k \in C(U_k \cap \overline{M})$ such that, for each stratum $X \le M$,

(i) f_k restricts to a rel-Morse function on $U_k \cap X$; and,
(ii) if $\operatorname{depth} X > k$, then:
  (a) the restriction of f_k to $U_k \cap X$ has no critical points, and
  (b) there is some $(T_X, \pi_X, \rho_X) \in \tau_X$ such that f_k is constant on the fibers of $\pi_X : U_k \cap \overline{M} \cap T_X \to X$.

This assertion is proved by induction on k. To simplify its proof, observe that it is also satisfied for $k = -1$ with $\mathfrak{F}_{-1} = U_{-1} = \emptyset$ and $f_{-1} = \emptyset$.

Now, assume that Claim 6 holds for some $k \in \{-1, 0, \dots, \operatorname{depth} M - 1\}$. Let V_k be another open neighborhood of $\mathfrak{F}_k$ in A such that $\overline{V_k} \subset U_k$. We can assume that the strata of A are connected by Remark 8-(v). For the strata $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$, choose representatives $(T_X, \pi_X, \rho_X) \in \tau_X$ satisfying the properties stated in the proof of Claim 5. We can also suppose that these (T_X, π_X, ρ_X) satisfy Claim 6-(ii)-(b) with f_k. A fixed adapted metric g on M will be used.

Let X be a stratum contained in $\mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$. By the density of the Morse functions in $C^\infty(X)$ with the strong C^∞ topology and since the restriction of f_k to $U_k \cap X$ has no critical points by Claim 6-(iii), it is easy to construct a Morse function h_X on X such that $h_X = f_k$ on $V_k \cap X$. Since (T_X, π_X, ρ_X) satisfies Claim 6-(ii)-(b) with f_k, we get $\pi_X^* h_X = f_k$ on $U_k \cap \overline{M} \cap T_X$.

Let U_{k+1} be the open neighborhood of $\mathfrak{F}_{k+1}$ given as the union of V_k and the sets T_X for strata $X \subset \mathfrak{F}_{k+1} \setminus \mathfrak{F}_k$. The function f_k on $V_k \cap \overline{M}$ and the functions $\pi_X^* h_X + \rho_X^2$ on the sets $T_X \cap \overline{M}$ can be combined to define a function $f_{k+1} \in C(U_{k+1} \cap \overline{M})$. The function f_{k+1} satisfies Claim 6-(i) and Claim 6-(ii)-(a). Moreover it satisfies Claim 6-(ii)-(b) by Definition 7.1-(vi).

Finally, let us complete the proof of Proposition 8.10. A basic neighborhood $\mathcal{N}$ of any $h \in C^\infty(M)$ with respect to the weak C^∞ topology can be determined by a finite family of charts (U_i, ϕ_i) of M, compact subsets $K_i \subset U_i$, some $k \in \mathbb{N}$ and some $\epsilon > 0$. Precisely, $\mathcal{N}$ consists of the functions $h' \in C^\infty(M)$ such that $|D^\ell((h' - h)\,\phi_i^{-1})| < \epsilon$ on $\phi_i(K_i)$ for all i and $0 \le \ell \le k$. By Claim 6, there is some open neighborhood U of $\overline{M} \setminus M$ in A and some $f \in C(U \cap \overline{M})$ that restricts to rel-Morse functions on $U \cap X$ for all strata $X \le M$, and whose restriction to $U \cap M$ has no critical points. By shrinking U if necessary, we can assume that $\overline{U} \cap K_i = \emptyset$ for all i. Let V be another open neighborhood of $\overline{M} \setminus M$ in A so that $\overline{V} \subset U$. By the density of the Morse functions in $C^\infty(M)$ with the strong C^∞ topology, it is easy to check that there is a Morse function $h' \in \mathcal{N}$ such that $h' = f$ on $V \cap M$. Therefore $h' \in \mathcal{F} \cap \mathcal{N}$. □

For rel-Morse functions, a much better density result should be true as suggested by the following.

Problem 8.11. By using the ideas of this section, define and study a "rel-strong C^∞ topology" on the set of rel-admissible functions on M, and show that the rel-Morse functions form a dense subset.

An approach to Problems 8.9 and 8.11 would take us too far from the main goals of the work.

CHAPTER 9

Preliminaries on Hilbert complexes

Here, we recall from [**11**] some basic definitions and needed results about Hilbert and elliptic complexes. Some elementary observations are also made.

1. Hilbert complexes

For each $r \in \mathbb{N}$, let $\mathfrak{H}_r$ be a separable (real or complex) Hilbert space such that, for some $N \in \mathbb{N}$, we have $\mathfrak{H}_r = 0$ for all $r > N$. They give rise to the graded Hilbert space $\mathfrak{H} = \bigoplus_r \mathfrak{H}_r$, where the terms $\mathfrak{H}_r$ are mutually orthogonal. For each degree r, let $\mathbf{d}_r$ be a densely defined closed operator of $\mathfrak{H}_r$ to $\mathfrak{H}_{r+1}$. Let $\mathcal{D}_r = \mathcal{D}(\mathbf{d}_r)$ and $\mathcal{R}_r = \mathbf{d}_r(\mathcal{D}_r)$ for each r, and let $\mathcal{D} = \bigoplus_r \mathcal{D}_r$ and $\mathbf{d} = \bigoplus_r \mathbf{d}_r$. Assume that $\mathcal{R}_r \subset \mathcal{D}_{r+1}$ and $\mathbf{d}_{r+1}\mathbf{d}_r = 0$ for all r. Then the complex

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{\mathbf{d}_0} \mathcal{D}_1 \xrightarrow{\mathbf{d}_1} \cdots \xrightarrow{\mathbf{d}_{N-1}} \mathcal{D}_N \longrightarrow 0$$

is called a *Hilbert complex*; its notation is abbreviated as $(\mathcal{D}, \mathbf{d})$, or simply as $\mathbf{d}$. Assuming that $\mathcal{D}_0 \neq 0$, the maximum $N \in \mathbb{N}$ such that $\mathcal{D}_N \neq 0$ will be called the *length* of $(\mathcal{D}, \mathbf{d})$. We may also consider Hilbert complexes with spaces of negative degree or with homogeneous operators of degree -1 without any essential change.

For the adjoint operator $\mathbf{d}_r^*$ of each $\mathbf{d}_r$, let $\mathcal{D}_r^* = \mathcal{D}(\mathbf{d}_r^*) \subset \mathfrak{H}_{r+1}$ and $\mathcal{R}_r^* = \mathbf{d}_r^*(\mathcal{D}_r^*) \subset \mathfrak{H}_r$, and set $\mathcal{D}^* = \bigoplus_r \mathcal{D}_r^*$ and $\mathbf{d}^* = \bigoplus_r \mathbf{d}_r^*$. Then we get a Hilbert complex

$$0 \longleftarrow \mathcal{D}_{-1}^* \xleftarrow{\mathbf{d}_0^*} \mathcal{D}_0^* \xleftarrow{\mathbf{d}_1^*} \cdots \xleftarrow{\mathbf{d}_{N-1}^*} \mathcal{D}_{N-1}^* \longleftarrow 0\,,$$

denoted by $(\mathcal{D}^*, \mathbf{d}^*)$ (or simply $\mathbf{d}^*$), which is called *dual* or *adjoint* of $(\mathcal{D}, \mathbf{d})$.

If $(\mathcal{D}', \mathbf{d}')$ is another Hilbert complex in the graded Hilbert space $\mathfrak{H}' = \bigoplus_r \mathfrak{H}'_r$, a homomorphism of complexes, $\zeta = \bigoplus_r \zeta_r : (\mathcal{D}, \mathbf{d}) \to (\mathcal{D}', \mathbf{d}')$, is called a *map of Hilbert complexes* if it is the restriction of a bounded map $\zeta : \mathfrak{H} \to \mathfrak{H}'$. If moreover ζ is an isomorphism of complexes and ζ^{-1} is a Hilbert complex map, then ζ is called an *isomorphism of Hilbert complexes*. If $\zeta : (\mathcal{D}, \mathbf{d}) \to (\widetilde{\mathcal{D}}', \mathbf{d}')$ is an isomorphism, where $\widetilde{\mathcal{D}}'_r = \mathcal{D}'_{r+r_0}$ for all r and some fixed $r_0 \neq 0$, then it will be said that $\zeta : (\mathcal{D}, \mathbf{d}) \to (\mathcal{D}', \mathbf{d}')$ is an *isomorphismm up to a shift of degree*.

Let

$$\mathfrak{H}_{\mathrm{ev}} = \bigoplus_r \mathfrak{H}_{2r}\,, \quad \mathfrak{H}_{\mathrm{odd}} = \bigoplus_r \mathfrak{H}_{2r+1}\,,$$
$$\mathcal{D}_{\mathrm{ev}} = \bigoplus_r \mathcal{D}_{2r}\,, \quad \mathcal{D}_{\mathrm{odd}}^* = \bigoplus_r \mathcal{D}_{2r-1}^*\,,$$
$$\mathbf{d}_{\mathrm{ev}} = \bigoplus_r \mathbf{d}_{2r}\,, \quad \mathbf{d}_{\mathrm{odd}}^* = \bigoplus_r \mathbf{d}_{2r-1}^*\,.$$

Note that $\mathcal{D}^*_{\rm odd}\subset\mathfrak{H}_{\rm ev}$. The operator $\mathbf{D}_{\rm ev}=\mathbf{d}_{\rm ev}+\mathbf{d}^*_{\rm odd}$, with domain $\mathcal{D}_{\rm ev}\cap\mathcal{D}^*_{\rm odd}$, is a densely defined closed operator of $\mathfrak{H}_{\rm ev}$ to $\mathfrak{H}_{\rm odd}$, whose adjoint is $\mathbf{D}_{\rm odd}=\mathbf{d}_{\rm odd}+\mathbf{d}^*_{\rm ev}$. Thus

$$\mathbf{D}=\begin{pmatrix}0 & \mathbf{D}_{\rm ev}\\ \mathbf{D}_{\rm odd} & 0\end{pmatrix}=\mathbf{d}+\mathbf{d}^*$$

is a self-adjoint operator in $\mathfrak{H}=\mathfrak{H}_{\rm ev}\oplus\mathfrak{H}_{\rm odd}$ with $\mathcal{D}(\mathbf{D})=\mathcal{D}\cap\mathcal{D}^*$, and

$$\boldsymbol{\Delta}=\mathbf{D}^2=\mathbf{D}_{\rm odd}\mathbf{D}_{\rm ev}\oplus\mathbf{D}_{\rm ev}\mathbf{D}_{\rm odd}=\mathbf{d}^*\mathbf{d}+\mathbf{d}\mathbf{d}^*$$

is a self-adjoint non-negative operator, which can be called the *Laplacian* of $(\mathcal{D},\mathbf{d})$. Observe that $(\mathcal{D},\mathbf{d})$ and $(\mathcal{D}^*,\mathbf{d}^*)$ define the same Laplacian. The Hilbert complex $(\mathcal{D},\mathbf{d})$ can be reconstructed from $\mathbf{D}_{\rm ev}$ [**11**, Lemma 2.3]. The restriction of $\boldsymbol{\Delta}$ to each space $\mathcal{D}_r$ will be denoted by $\boldsymbol{\Delta}_r$. Notice that $\ker\boldsymbol{\Delta}_r=\ker\mathbf{d}_r\cap\ker\mathbf{d}^*_{r-1}$ for all r. Moreover we have a weak Hodge decomposition [**11**, Lemma 2.1]

$$\mathfrak{H}_r=\ker\boldsymbol{\Delta}_r\oplus\overline{\mathcal{R}_{r-1}}\oplus\overline{\mathcal{R}^*_r}\;.$$

The smooth core $\mathcal{D}^\infty(\boldsymbol{\Delta})$, also denoted by $\mathcal{D}^\infty(\mathbf{d})$ or $\mathcal{D}^\infty$, is a subcomplex of $(\mathcal{D},\mathbf{d})$, and $(\mathcal{D}^\infty,\mathbf{d})\hookrightarrow(\mathcal{D},\mathbf{d})$ induces an isomorphism in homology [**11**, Theorem 2.12]. It will be also said that $\mathcal{D}^\infty$ (respectively, $\mathcal{D}^\infty_r$) is the *smooth core* of $\mathbf{d}$ (respectively, $\mathbf{d}_r$); notice that it is a core of $\mathbf{d}$ (respectively, $\mathbf{d}_r$). Let $\mathcal{R}^\infty_r=\mathbf{d}_r(\mathcal{D}^\infty_r)$ and $\mathcal{R}^{*\infty}_r=\mathbf{d}^*_r(\mathcal{D}^\infty_r)$, which are dense subspaces of $\mathcal{R}_r$ and $\mathcal{R}^*_r$.

The following properties are equivalent [**11**, Theorem 2.4]:

- The homology of $(\mathcal{D},\mathbf{d})$ is of finite dimension and $\mathcal{R}$ is closed in $\mathfrak{H}$.
- The homology of $(\mathcal{D},\mathbf{d})$ is of finite dimension.
- $\mathbf{D}_{\rm ev}$ is a Fredholm operator.
- $0\notin\operatorname{spec}_{\rm ess}(\boldsymbol{\Delta})$ (the essential spectrum of $\boldsymbol{\Delta}$).

In this case, $(\mathcal{D},\mathbf{d})$ is called a *Fredholm complex* and satisfies the following properties:

- $\mathcal{R}$ and $\mathcal{R}^*$ are closed in $\mathfrak{H}$ [**11**, Corollary 2.5], obtaining the stronger Hodge decompositions
$$\mathfrak{H}_r=\ker\boldsymbol{\Delta}_r\oplus\mathcal{R}_{r-1}\oplus\mathcal{R}^*_r\;,\quad \mathcal{D}^\infty=\ker\boldsymbol{\Delta}_r\oplus\mathcal{R}^\infty_{r-1}\oplus\mathcal{R}^{*\infty}_r\;.$$
- $\mathbf{d}_r:\mathcal{R}^{*\infty}_r\to\mathcal{R}^\infty_r$ and $\mathbf{d}^*_r:\mathcal{R}^\infty_r\to\mathcal{R}^{*\infty}_r$ are isomorphisms.
- $\ker\boldsymbol{\Delta}_r$ is isomorphic to the homology of degree r of $(\mathcal{D},\mathbf{d})$.

It is said that $(\mathcal{D},\mathbf{d})$ is *discrete* when $\boldsymbol{\Delta}$ has a discrete spectrum ($\operatorname{spec}_{\rm ess}(\boldsymbol{\Delta})=\emptyset$). The following properties hold when $(\mathcal{D},\mathbf{d})$ is discrete:

- For each $\lambda\in\operatorname{spec}(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})$, we get isomorphisms
$$\mathbf{d}_r:E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^{*\infty}_r})\to E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})\;,\quad \mathbf{d}^*_r:E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})\to E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^{*\infty}_r})$$
between the corresponding eigenspaces. Thus $\operatorname{spec}(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})=\operatorname{spec}(\boldsymbol{\Delta}|_{\mathcal{R}^{*\infty}_r})$.
- We have
$$\operatorname{spec}(\mathbf{d}_r|_{\mathcal{R}^{*\infty}_r}\oplus\mathbf{d}^*_r|_{\mathcal{R}^\infty_r})=\{\,\pm\sqrt{\lambda}\mid\lambda\in\operatorname{spec}(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})\,\}\;,$$
and, for each $\lambda\in\operatorname{spec}(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})$, $E_{\pm\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}^\infty_r}\oplus\mathbf{d}^*_r|_{\mathcal{R}^{*\infty}_r})$ consists of the elements of the form $u\pm v$ with $u\in E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^\infty_r})$ and $v\in E_\lambda(\boldsymbol{\Delta}|_{\mathcal{R}^{*\infty}_r})$ satisfying $\mathbf{d}^*u=\sqrt{\lambda}\,v$ and $\mathbf{d}v=\sqrt{\lambda}\,u$. Moreover the mapping $u+v\mapsto u-v$, for u and v as above, defines an isomorphism
$$E_{\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}^{*\infty}_r}\oplus\mathbf{d}^*_r|_{\mathcal{R}^\infty_r})\to E_{-\sqrt{\lambda}}(\mathbf{d}_r|_{\mathcal{R}^{*\infty}_r}\oplus\mathbf{d}^*_r|_{\mathcal{R}^\infty_r})\;.$$

- Any Hilbert complex $(\mathcal{D}',\mathbf{d}')$ isomorphic to $(\mathcal{D},\mathbf{d})$ is also discrete, and, if $\operatorname{spec}(\mathbf{\Delta}_r)$ and $\operatorname{spec}(\mathbf{\Delta}'_r)$ consist of the eigenvalues $0\le\lambda_0\le\lambda_1\le\cdots$ and $0\le\lambda'_0\le\lambda'_1\le\cdots$, respectively, then there is some $C\ge 1$ such that $C^{-1}\lambda_k\le\lambda'_k\le C\lambda_k$ for all $k\in\mathbb{N}$ [**11**, Lemma 2.17].

Consider Hilbert complexes, $(\mathcal{D}',\mathbf{d}')$ and $(\mathcal{D}'',\mathbf{d}'')$, in respective graded Hilbert spaces, $\mathfrak{H}'$ and $\mathfrak{H}''$. The Hilbert space tensor product[1], $\mathfrak{H}=\mathfrak{H}'\widehat{\otimes}\mathfrak{H}''$, has a canonical grading ($\mathfrak{H}_r=\bigoplus_{p+q=r}\mathfrak{H}'_p\widehat{\otimes}\mathfrak{H}''_q$), and

$$\widetilde{\mathcal{D}}=(\mathcal{D}'\otimes\mathfrak{H}'')\cap(\mathfrak{H}'\otimes\mathcal{D}'')\subset\mathfrak{H}$$

is a dense graded subspace. Let $\tilde{\mathbf{d}}=\mathbf{d}'\otimes 1+\mathbf{w}\otimes\mathbf{d}''$ with domain $\widetilde{\mathcal{D}}$, where $\mathbf{w}$ denotes the degree involution on $\mathfrak{H}'$, and let $\mathbf{d}=\overline{\tilde{\mathbf{d}}}$, whose domain is denoted by $\mathcal{D}$. Then $(\mathcal{D},\mathbf{d})$ is a Hilbert complex in $\mathfrak{H}$ called the *tensor product* of $(\mathcal{D}',\mathbf{d}')$ and $(\mathcal{D}'',\mathbf{d}'')$. If $\mathbf{\Delta}'$, $\mathbf{\Delta}''$ and $\mathbf{\Delta}$ denote the Laplacians of $(\mathcal{D}',\mathbf{d}')$, $(\mathcal{D}'',\mathbf{d}'')$ and $(\mathcal{D},\mathbf{d})$, respectively, then $\mathbf{\Delta}=\mathbf{\Delta}'\otimes 1+1\otimes\mathbf{\Delta}''$ on $\widetilde{\mathcal{D}}$. The following result is elementary.

LEMMA 9.1. *If $(\mathcal{D}',\mathbf{d}')$ and $(\mathcal{D}'',\mathbf{d}'')$ are discrete, then $(\mathcal{D},\mathbf{d})$ is discrete. More precisely, given complete orthonormal systems of $\mathfrak{H}'$ and $\mathfrak{H}''$ consisting of eigenvectors e'_k and e''_k ($k\in\mathbb{N}$) of $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$, with corresponding eigenvalues λ'_k and λ''_k, respectively, we get a complete orthonormal system of $\mathfrak{H}$ consisting of the eigenvectors $e'_k\otimes e''_\ell\in\widetilde{\mathcal{D}}$ of $\mathbf{\Delta}$ with corresponding eigenvalues $\lambda'_k+\lambda''_\ell$.*

Let $(\mathcal{E},d)$ be a densely defined complex in a graded separable Hilbert space $\mathfrak{H}$ ($\mathcal{E}$ is a dense graded linear subspace of $\mathfrak{H}$). Consider the family of Hilbert complexes $(\mathcal{D},\mathbf{d})$ in $\mathfrak{H}$ extending $(\mathcal{E},d)$ ($(\mathcal{E},d)$ is a subcomplex of $(\mathcal{D},\mathbf{d})$) endowed with the order relation defined by “being a subcomplex”. We will be interested in its minimum/maximum elements. Notice that, if $(\mathcal{E},d)$ has some Hilbert complex extension, then $\overline{d}$ is a Hilbert complex; thus, in this case, $\overline{d}$ is the minimum Hilbert complex extension of $(\mathcal{E},d)$. Another complex of the form $(\mathcal{E},\delta)$, with $\delta_r:\mathcal{E}_{r+1}\to\mathcal{E}_r$ for each degree r, will be called a *formal adjoint* of $(\mathcal{E},d)$ if $\langle du,v\rangle=\langle u,\delta v\rangle$ for all $u,v\in\mathcal{E}$; there is at most one formal adjoint by the density of $\mathcal{E}$ in $\mathfrak{H}$. In this case, if $(\mathcal{E},\delta)$ has some Hilbert complex extension, then the adjoint of the minimum Hilbert complex extension of $(\mathcal{E},\delta)$ is the maximum Hilbert complex extension of $(\mathcal{E},d)$.

Now, consider a countable family of densely defined complexes $(\mathcal{E}^a,d^a)$ in separable graded Hilbert spaces $\mathfrak{H}^a$ ($a\in\mathbb{N}$), and let $(\mathcal{D}^a,\mathbf{d}^a)$ be a Hilbert complex extension of each $(\mathcal{E}^a,d^a)$ in $\mathfrak{H}^a$. Suppose that the Hilbert complexes $(\mathcal{D}^a,\mathbf{d}^a)$ are of uniformly finite length (there is some $N\in\mathbb{N}$ such that $\mathcal{D}^a_r=0$ for all $r\ge N$ and all a). Let $(\mathcal{E},d)$ be the complex defined by $\mathcal{E}=\bigoplus_a\mathcal{E}^a$ and $d=\bigoplus_a d^a$. The Hilbert space direct sum[2], $\mathfrak{H}=\widehat{\bigoplus}_a\mathfrak{H}^a$, has an induced grading ($\mathfrak{H}_r=\widehat{\bigoplus}_a\mathfrak{H}^a_r$). Let $\mathbf{d}=\widehat{\bigoplus}_a\mathbf{d}^a$ (the graph of $\mathbf{d}$ is the Hilbert space direct sum of the graphs of the maps $\mathbf{d}^a$). The domain $\mathcal{D}$ of $\mathbf{d}$ consists of the points $(u^a)\in\mathfrak{H}$ such that $u^a\in\mathcal{D}^a$ for all a and $(\mathbf{d}^a u^a)\in\mathfrak{H}$. Moreover $\mathbf{d}$ is defined by $(u^a)\mapsto(\mathbf{d}^a u^a)$. Clearly, $(\mathcal{D},\mathbf{d})$

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[1]Recall that this is the Hilbert space completion of the algebraic tensor product $\mathfrak{H}'\otimes\mathfrak{H}''$ with respect to the scalar product defined by $\langle u'\otimes u'',v'\otimes v''\rangle=\langle u',v'\rangle'\,\langle u'',v''\rangle''$, where $\langle\ ,\ \rangle'$ and $\langle\ ,\ \rangle''$ are the scalar products of $\mathfrak{H}'$ and $\mathfrak{H}''$, respectively.

[2]Recall that this is the Hilbert space completion of the algebraic direct sum, $\bigoplus_a\mathfrak{H}^a$, with respect to the scalar product $\langle(u^a),(v^a)\rangle=\sum_a\langle u^a,v^a\rangle_a$, where each $\langle\ ,\ \rangle_a$ is the scalar product of $\mathfrak{H}^a$. We have $\mathfrak{H}=\bigoplus_a\mathfrak{H}^a$ if the number of terms $\mathfrak{H}^a$ is finite.

is a Hilbert complex extension of $(\mathcal{E}, d)$ in $\mathfrak{H}$ with

$$\mathcal{D}^\infty(\mathbf{d}) = \widehat{\bigoplus_a} \mathcal{D}^\infty(\mathbf{d}^a) \;, \tag{79}$$

$$\mathbf{d}^* = \widehat{\bigoplus_a} \mathbf{d}^{a*} \;. \tag{80}$$

LEMMA 9.2. (i) *If each $(\mathcal{D}^a, \mathbf{d}^a)$ is a minimum Hilbert complex extension of $(\mathcal{E}^a, \mathbf{d}^a)$ in $\mathfrak{H}^a$, then $(\mathcal{D}, \mathbf{d})$ is a minimum Hilbert complex extension of $(\mathcal{E}, \mathbf{d})$ in $\mathfrak{H}$.*

(ii) *If each $(\mathcal{E}^a, d^a)$ has a formal adjoint $(\mathcal{E}^a, \delta^a)$ with some Hilbert complex extension, and each $(\mathcal{D}^a, \mathbf{d}^a)$ is a maximum Hilbert complex extension of $(\mathcal{E}^a, \mathbf{d}^a)$ in $\mathfrak{H}^a$, then $(\mathcal{D}, \mathbf{d})$ is a maximum Hilbert complex extension of $(\mathcal{E}, \mathbf{d})$ in $\mathfrak{H}$.*

PROOF. Property (i) follows because d is dense in $\mathbf{d}$ if each d^a is dense in $\mathbf{d}^a$.

Now, assume the conditions of (ii) and let $\delta = \bigoplus_a \delta^a$. Then each $\mathbf{d}^{a*}$ is a minimum Hilbert complex extension of $(\mathcal{E}^a, \delta^a)$. So, by (80) and (i), $(\mathcal{D}^*, \mathbf{d}^*)$ is a minimum Hilbert complex extension of $(\mathcal{E}, \delta)$, and therefore $(\mathcal{D}, \mathbf{d})$ is a maximum Hilbert complex extension of $(\mathcal{E}, d)$. □

2. Elliptic complexes

Let M be a possibly non-complete Riemannian manifold, and let $E = \bigoplus_r E_r$ be a graded Riemannian (or Hermitean) vector bundle over M, with $E_r = 0$ if $r < 0$ or $r > N$ for some $N \in \mathbb{N}$. The space of smooth sections of each E_r will be denoted by $C^\infty(E_r)$, its subspace of compactly supported smooth sections will be denoted by $C_0^\infty(E_r)$, and the Hilbert space of square integrable sections of E_r will be denoted by $L^2(E_r)$; then $C^\infty(E) = \bigoplus_r C^\infty(E_r)$, $C_0^\infty(E) = \bigoplus_r C_0^\infty(E_r)$ and $L^2(E) = \bigoplus_r L^2(E_r)$. For each r, let $d_r : C^\infty(E_r) \to C^\infty(E_{r+1})$ be a first order differential operator, and set $d = \bigoplus_r d_r$. Suppose that $(C^\infty(E), d)$ is an elliptic complex[3]; however, ellipticity is not needed for several elementary properties stated in this section. The simpler notation (E, d) (or even d) will be preferred. Elliptic complexes with non-zero terms of negative degrees or homogeneous differential operators of degree -1 may be also considered without any essential change.

Consider the formal adjoint $\delta_r = {}^t d_r : C^\infty(E_{r+1}) \to C^\infty(E_r)$ for each r, and set $\delta = \bigoplus_r \delta_r$. Then (E, δ) is another elliptic complex that will be called the *formal adjoint* of (E, d), and its subcomplex $(C_0^\infty(E), \delta)$ is formal adjoint of $(C_0^\infty(E), d)$ in $L^2(E)$ in the sense of Section 1. Let $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ on $C^\infty(E)$; Δ can be called the *Laplacian* defined by (E, d). The components of Δ are $\Delta_r = d_{r-1}\delta_{r-1} + \delta_r d_r$.

Any Hilbert complex extension of $(C_0^\infty(E), d)$ in $L^2(E)$ is called an *ideal boundary condition* (shortly, *i.b.c.*) of (E, d). There always exist a minimum and maximum i.b.c., $d_{\min} = \overline{d}$ and $d_{\max} = \delta_{\min}^*$ [**11**, Lemma 3.1]. The complex $d_{\min/\max}$ defines the operator $D_{\min/\max} = d_{\min/\max} + \delta_{\max/\min}$ and the Laplacian $\Delta_{\min/\max} = D_{\min/\max}^2$, which extend D and Δ on $C_0^\infty(E)$. The homogeneous components of $\Delta_{\min/\max}$ are

$$\Delta_{\min/\max,r} = \delta_{\max/\min,r}\, d_{\min/\max,r} + d_{\min/\max,r-1}\, \delta_{\max/\min,r-1} \;. \tag{81}$$

[3]Recall that this means that it is a complex and the sequence of principal symbols of the operators d_r is exact in the fiber over each non-zero cotangent vector

The notation $d_{r,\min/\max}$ and $\delta_{r,\max/\min}$ also makes sense for $d_{\min/\max,r}$ and $\delta_{\max/\min,r}$ by considering d_r and δ_r as differential complexes of length one (ellipticity is not needed here); similarly, any first order differential operator can be considered as a differential complex of length one and denote its minimum/maximum i.b.c. with the the min/max subindex, regardless of ellipticity.

For any i.b.c. $(\mathcal{D}, \mathbf{d})$ of (E, d), the map of complexes, $(\mathcal{D}\cap C^\infty(E), d) \hookrightarrow (\mathcal{D}, \mathbf{d})$, induces an isomorphism in homology [**11**, Theorem 3.5]. We have $\mathcal{D}^\infty \subset \mathcal{D}\cap C^\infty(E)$ by elliptic regularity.

Let (E', d') be another elliptic complex over another Riemannian manifold M'. Consider a vector bundle isomorphism $\zeta : E \to E'$ over a quasi-isometric diffeomorphism $\xi : M \to M'$ such that the restrictions of ζ to the fibers are quasi-isometries. It induces a map $\zeta : C^\infty(E) \to C^\infty(E')$ defined by $(\zeta u)(x') = \zeta(u(\xi^{-1}(x'))$ for $u \in C^\infty(E)$ and $x' \in M'$. If moreover $\zeta : (C^\infty(E'), d') \to (C^\infty(E), d)$ is a homomorphism of complexes, then it will be called a *quasi-isometric isomorphism* of elliptic complexes, and the simpler notation $\zeta : (E', d') \to (E, d)$ will be preferred. In this case, ζ induces a quasi-isometric isomorphism $\zeta : L^2(E') \to L^2(E)$, which restricts to an isomorphism of complexes, $\zeta : (C_0^\infty(E'), d') \to (C_0^\infty(E), d)$. Moreover, for any i.b.c. $(\mathcal{D}', \mathbf{d}')$ of (E', d'), there is a unique i.b.c. $(\mathcal{D}, \mathbf{d})$ of (E, d) so that $\zeta : L^2(E') \to L^2(E)$ restricts to a Hilbert complex isomorphism $\zeta : (\mathcal{D}', \mathbf{d}') \to (\mathcal{D}, \mathbf{d})$. In particular, ζ induces Hilbert complex isomorphisms between the corresponding minimum/maximum i.b.c. If ξ is isometric and the restrictions to the fibers of ζ are isometries, then $\zeta : (E', d') \to (E, d)$ is called an *isometric isomorphism* of elliptic complexes. For instance, for any quasi-isometric (respectively, isometric) diffeomorphism $\xi : M \to M'$, the induced isomorphism ξ^* between the corresponding de Rham complexes is quasi-isometric (respectively, isometric).

Now, let (E', d') and (E'', d'') be elliptic complexes on Riemannian manifolds M' and M'', respectively, and consider the exterior tensor product $E = E' \boxtimes E''$ on $M = M' \times M''$ with its canonical grading ($E_r = \bigoplus_{p+q=r} E'_p \boxtimes E''_q$). With the weak C^∞ topology, $C^\infty(E') \otimes C^\infty(E'')$ can be canonically realized as a dense subspace of $C^\infty(E)$. Then $d = d' \otimes 1 + \mathbf{w} \otimes d''$ on $C^\infty(E') \otimes C^\infty(E'')$ has a unique continuous extension to $C^\infty(E)$, also denoted by d. It turns out that (E, d) is an elliptic complex. Moreover the minimum/maximum i.b.c. of (E, d) is the tensor product, in the sense of Section 1, of the minimum/maximum i.b.c. of (E', d') and (E'', d'') [**11**, Lemma 3.6].

Example 9.3. A particular case of elliptic complex on M is its de Rham complex $(\Omega(M), d)$. In this case, δ is the de Rham coderivative, the subcomplex of compactly supported differential forms is denoted by $\Omega_0(M)$, and the Hilbert space of L^2 differential forms is denoted by $L^2\Omega(M)$. Let $H_{\min/\max}(M)$ denote the cohomology of the minimum/maximum i.b.c., $d_{\min/\max}$, of $(\Omega_0(M), d)$, which is a quasi-isometric invariant of M. $H_{\min}(M)$ is canonically isomorphic to the L^2-cohomology $H_{(2)}(M)$ [**13**]; (a generalization to arbitrary elliptic complexes is given in [**11**, Theorem 3.5]). The dimensions $\beta^r_{\min/\max}(M) = \dim H^r_{\min/\max}(M)$ can be called *min/max-Betti numbers*; if they are finite, then $\chi_{\min/\max}(M) = \sum_r (-1)^r \beta^r_{\min/\max}(M)$ is defined and can be called *min/max-Euler characteristic*; the simpler notation $\beta^r_{\min/\max}$ and $\chi_{\min/\max}$ may be used. Is is known that $d_{\min/\max}$ satisfies the following properties for special classes of Riemannian manifolds:

- If M is complete, then $d_{\min} = d_{\max}$ (a particular case of [**11**, Lemma 3.8]).

- If M is the interior of a compact manifold with boundary, then $d_{\mathrm{min/max}}$ is given by the relative/absolute boundary conditions [**11**, Theorem 4.1].
- Suppose that $M = \widetilde{M} \setminus \Sigma$, where $\widetilde{M}$ is a closed Riemannian manifold of dimension > 2 and Σ is a closed finite union of submanifolds with codimension ≥ 2. Then $d_{\mathrm{min}} = d_{\mathrm{max}}$ [**11**, Theorem 4.4].
- Let A be a compact Tham-Mather stratification that is a pseudomanifold. If M is the regular stratum of A endowed with an adapted metric, then $H_{(2)}(M)$ is isomorphic to the intersection homology of A with lower middle perversity [**15**]. There is a more general isomorphism of this type involving more general types of adapted metrics and intersection homologies with other perversities [**47, 48, 8**].

CHAPTER 10

Sobolev spaces defined by an i.b.c.

Let T be a self-adjoint operator in a Hilbert space $\mathfrak{H}$. For each $m \in \mathbb{N}$, the *Sobolev space* of *order* k associated to T is the Hilbert space completion $W^m = W^m(T)$ of $\mathcal{D}^\infty = \mathcal{D}^\infty(T)$ with respect to the scalar product $\langle\ ,\ \rangle_m$ on $\mathcal{D}^\infty$ defined by $\langle u, v\rangle_m = \langle u, (1+T)^m v\rangle$. The notation $\|\ \|_m$ and Cl_m (or $\|\ \|_{W^m}$ and Cl_{W^m}) will be used for the norm and closure in W^m. There are continuous inclusions $W^{m+1} \hookrightarrow W^m$, and we have $\mathcal{D}^\infty = \bigcap_m W^m$. Moreover T defines a bounded operator $W^{m+2} \to W^m$.

Now, let $(\mathcal{D}, \mathbf{d})$ be an i.b.c. of an elliptic complex (E, d) on a Riemannian manifold M. Its adjoint $(\mathcal{D}^*, \mathbf{d}^*)$ is an i.b.c. of the elliptic complex (E, δ), where $\delta = {}^t d$. We get the operators $D = d+\delta$ and $\mathbf{D} = \mathbf{d}+\mathbf{d}^*$, and the Laplacians $\Delta = D^2$ and $\mathbf{\Delta} = \mathbf{D}^2$. Then $W^m = W^m(\mathbf{\Delta})$ can be called the *Sobolev space* of *order* m associated to $(\mathcal{D}, \mathbf{d})$, and may be also denoted by $W^m(\mathbf{d})$; the notation $W^m(\mathbf{d}_r)$ will be also used when we consider its subspace of homogeneous elements of degree r. Since $(\mathcal{D}, \mathbf{d})$ and $(\mathcal{D}^*, \mathbf{d}^*)$ define the same Laplacian, we have $W^m(\mathbf{d}) = W^m(\mathbf{d}^*)$ for all m. For $u \in \mathcal{D}^\infty_r$, we have

$$\|u\|_1^2 = \|u\|^2 + \|Du\|^2 = \|u\|^2 + \|d_r u\|^2 + \|\delta_{r-1} u\|^2 \ .$$

So

$$W^1 = \mathcal{D}(\mathbf{D}) = \mathcal{D} \cap \mathcal{D}^* \ , \tag{82}$$

$$\|u\|_1^2 = \|u\|^2 + \|\mathbf{D}u\|^2 = \|u\|^2 + \|\mathbf{d}_r u\|^2 + \|\mathbf{d}^*_{r-1} u\|^2 \tag{83}$$

for $u \in W^1(\mathbf{d}_r)$.

LEMMA 10.1. *The following properties are equivalent:*

(i) $(\mathcal{D}, \mathbf{d})$ *is discrete.*
(ii) $W^1 \hookrightarrow W^0 = L^2(E)$ *is compact.*
(iii) $W^{m+1} \hookrightarrow W^m$ *is compact for all* m.

PROOF. The part "(i) ⇒ (iii)" follows with the arguments of the proof of the Rellich's theorem on a torus (see e.g. [**54**, Theorem 5.8]). The part "(ii) ⇒ (i)" follows with the arguments to prove that any Dirac operator on a closed manifold has a discrete spectrum (see e.g. [**54**, pp. 81–82]). □

The following refinement of Lemma 10.1 is obtained with a deeper analysis.

LEMMA 10.2. *Suppose that* $(\mathcal{D}, \mathbf{d})$ *is discrete, and let* $0 \le \lambda_1 \le \lambda_2 \le \cdots$ *be the eigenvalues of* $\mathbf{\Delta}$*, repeated according to their multiplicities. Let* B^1 *be the standard unit ball of* W^1*, and* B_r *the standard ball of radius* $r > 0$ *in* $L^2(E)$*. Then the following properties are equivalent for* $\theta > 0$*:*

(i) $\liminf_k \lambda_k k^{-\theta} > 0$.

(ii) *There are some* $C_0, C_1 > 0$ *such that, for all* $n \in \mathbb{Z}_+$, *there is a linear subspace* $Z_n \subset L^2(E)$ *so that:*
   (a) Z_n *is closed and of codimension* $\le C_0\, n^{1/\theta}$ *in* $L^2(E)$*;*
   (b) $\mathbf{D}(W^1 \cap Z_n) \subset Z_n$*; and*
   (c) $B^1 \cap Z_n \subset B_{C_1/n}$.

(iii) *There are some* $C_0, \dots, C_4 > 0$ *and* $A \in \mathbb{Z}_+$ *such that, for all* $n \in \mathbb{Z}_+$, *there is a linear map*[1] $R_n = (R_n^1, \dots, R_n^A) : L^2(E) \to \bigoplus_A L^2(E)$ *so that:*
   (a) $\dim\ker R_n \le C_0\, n^{1/\theta}$*;*
   (b) $\|R_n u\| \le C_1 \|u\|$ *for all* $u \in L^2(E)$*;*
   (c) $\|R_n u\| \ge C_2 \|u\|$ *for all* $u \in (\ker R_n)^\perp$*;*
   (d) $R_n^a(W^1) \subset W^1$ *and* $\|[\mathbf{D}, R_n^a]u\| \le C_3 \|u\|$ *for all* $u \in W^1$*; and*
   (e) $B^1 \cap R_n^a(L^2(E)) \subset B_{C_4/n}$.

Proof. Let (e_i) $(i \in \mathbb{Z})$ be a complete orthonormal system of $L^2(E)$ such that $e_{\pm k}$ is a $\pm\sqrt{\lambda_k}$-eigenvector of $\mathbf{D}$ for each $k \in \mathbb{N}$. The mapping $u = \sum_i u_i e_i \mapsto (u_i)$ defines a unitary isomorphism $L^2(E) \cong \ell^2(\mathbb{Z})$. Moreover W^1 consists of the elements $u \in L^2(E)$ with $\sum_k (1+\lambda_k) u_{\pm k}^2 < \infty$, and $\|u\|_1^2 = \sum_k (1+\lambda_k)(u_k^2 + u_{-k}^2)$ for $u \in W^1$.

Suppose that (i) holds. Then there is some $C > 0$ so that $1 + \lambda_k \ge C k^\theta$ for all k. For each $n \in \mathbb{Z}_+$, the linear subspace

$$Z_n = \left\{\, u \in L^2(E) \mid u_{\pm k} = 0 \text{ if } k \le (n/C)^{1/\theta} \,\right\}$$

of $L^2(E)$ satisfies (ii)-(a),(b) with $C_0 = 2/C^{1/\theta}$. Furthermore, for every $u \in B^1 \cap Z_n$,

$$\begin{aligned} \|u\|^2 = \sum_{k>(n/C)^{1/\theta}} (u_k^2 + u_{-k}^2) &< \frac{C}{n} \sum_{k>(n/C)^{1/\theta}} k^\theta (u_k^2 + u_{-k}^2) \\ &\le \frac{1}{n} \sum_{k>(n/C)^{1/\theta}} (1+\lambda_k)(u_k^2 + u_{-k}^2) = \frac{\|u\|_1^2}{n} < \frac{1}{n}\,, \end{aligned}$$

completing the proof of (ii)-(c) with $C_1 = 1$.

Now, assume that (ii) is satisfied. By (ii)-(a),

$$L^2(E) = Z_n^\perp \oplus Z_n \tag{84}$$

as topological vector space [**58**, Chapter I, 3.5]. Furthermore, by (ii)-(a) and the canonical linear isomorphism

$$\frac{W^1}{W^1 \cap Z_n} \cong \frac{W^1 + Z_n}{Z_n}\,,$$

we also get that $W^1 \cap Z_n$ is a closed linear subspace of finite codimension in W^1. Hence

$$W^1 = Y_n \oplus (W^1 \cap Z_n) \tag{85}$$

as topological vector spaces for any linear complement Y_n of $W^1 \cap Z_n$ in W^1 [**58**, Chapter I, 3.5].

[1] For $A \in \mathbb{Z}_+$ and any topological vector space L, the notation $\bigoplus_A L$ is used for the direct sum of A copies of L. Similarlarly, for any linear map between topological vector spaces, $T : L \to L'$, the notation $\bigoplus_A T : \bigoplus_A L \to \bigoplus_A L'$ is used for the direct sum of A copies of T.

On the other hand, for each $u \in Z_n^\perp$, the linear mapping $v \mapsto \langle u, \mathbf{D}v\rangle$ is bounded on Y_n because Y_n is of finite dimension, and $\langle u, \mathbf{D}w\rangle = 0$ for all $w \in W^1 \cap Z_n$ by (ii)-(b). So $v \mapsto \langle u, \mathbf{D}v\rangle$ is bounded on W^1 by (85), obtaining that $u \in W^1$ by (82) since $\mathbf{D}$ is self-adjoint. Hence $Z_n^\perp \subset W^1$, and therefore we can take $Y_n = Z_n^\perp$ in (85), obtaining

$$W^1 = Z_n^\perp \oplus (W^1 \cap Z_n) \tag{86}$$

as topological vector spaces. Note that $W^1 \cap Z_n$ is dense in Z_n by (84) and (86). So, since $\mathbf{D}$ is self-adjoint, it follows from (ii)-(b) and (86) that $\mathbf{D}$ preserves $Z_n^\perp$.

To get (iii), take $A = 1$ and R_n equal to the orthogonal projection of $L^2(E)$ to Z_n. Then (iii)-(a) follows from (ii)-(a), and properties (iii)-(b),(c) hold with $C_1 = C_2 = 1$ because R_n is an orthogonal projection. By (ii)-(b) and since $\mathbf{D}$ preserves $Z_n^\perp$, we get $R_n(W^1) \subset W^1$ and $DR_n = R_nD$ on W^1, showing (iii)-(d). Property (iii)-(e) is a consequence of (ii)-(c).

Finally, assume that (iii) is true. The following general assertion will be used.

Claim 7. Let $\mathfrak{H}$ be a (real or complex) Hilbert space, Π an orthogonal projection of $\mathfrak{H}$ with finite rank p, and $0 < C < 1$. Then the cardinality of any orthonormal set contained in

$$U_C = \{\, u \in \mathfrak{H} \mid \|\Pi u\| > C\,\|u\|\,\}$$

is $\le p/C^2$.

Suppose $v_1, \dots, v_p$ is an orthonormal basis of $\Pi(\mathfrak{H})$. Let $u_1, \dots, u_k$ be orthonormal vectors in U_C, and Π' the orthogonal projection of $\mathfrak{H}$ to the linear subspace generated by them. We have

$$kC^2 \le \sum_{j=1}^{k} \|\Pi u_j\|^2 = \sum_{j=1}^{k}\sum_{i=1}^{p} |\langle v_i, u_j\rangle|^2 = \sum_{i=1}^{p} \|\Pi' v_i\|^2 \le p\ ,$$

showing Claim 7.

Let $p_n = \lfloor C_0\, n^{1/\theta}\rfloor$ and $0 < C < 1$.

Claim 8. There is some $I \subset \mathbb{Z}$ with $\#I \le p_n/C^2$ and $\|R_n e_i\| \ge C_2 C$ for all $i \in \mathbb{Z} \setminus I$.

Let Π_n and $\widetilde{\Pi}_n$ be the orthogonal projections of $L^2(E)$ to $\ker R_n$ and $(\ker R_n)^\perp$, respectively. By Claim 7, the cardinality of the set

$$I = \{\, i \in \mathbb{Z} \mid \|\widetilde{\Pi}_n e_i\| > C\,\}$$

is $\le p_n/C^2$. For $i \in \mathbb{Z} \setminus I$, we have

$$\|R_n e_i\| = \|R_n \Pi_n e_i\| \ge C_2\, \|\Pi_n e_i\| \ge C_2 C$$

by (iii)-(b), showing Claim 8.

From Claim 8, it follows that there is some $i_n \in \mathbb{Z}$ such that

$$|i_n| \le \frac{p_n}{C^2} + 1\ , \tag{87}$$

$$\|R_n e_{i_n}\| \ge C_2 C\ . \tag{88}$$

We have

$$\begin{aligned}\|R_n^a e_{i_n}\|_1^2 &= \|R_n^a e_{i_n}\|^2 + \|\mathbf{D} R_n^a e_{i_n}\|^2 \\ &\le \|R_n^a e_{i_n}\|^2 + (\|R_n^a \mathbf{D} e_{i_n}\| + \|[\mathbf{D}, R_n^a] e_{i_n}\|)^2 \\ &\le C_1^2 + \left(C_1\sqrt{\lambda_{|i_n|}} + C_3\right)^2 .\end{aligned}$$

Hence

$$u_{n,r}^a = \frac{r}{\sqrt{C_1^2 + \left(C_1\sqrt{\lambda_{|i_n|}} + C_3\right)^2}} R_n^a e_{i_n} \in B^1 \cap Z_n$$

for all $r \in [0,1)$, giving

$$\begin{aligned}\frac{rC_2C}{\sqrt{C_1^2 + \left(C_1\sqrt{\lambda_{|i_n|}} + C_3\right)^2}} &\le \frac{r\,\|R_n e_{i_n}\|}{\sqrt{C_1^2 + \left(C_1\sqrt{\lambda_{|i_n|}} + C_3\right)^2}} \\ &\le \frac{r\sum_a \|R_n^a e_{i_n}\|}{\sqrt{C_1^2 + \left(C_1\sqrt{\lambda_{|i_n|}} + C_3\right)^2}} = \sum_a \|u_{n,r}^a\| < \frac{AC_4}{n}\end{aligned}$$

for all $r \in [0,1)$ by (88) and (iii)-(e). So there is some $C' > 0$, independent of n, such that

$$\lambda_{|i_n|} \ge \frac{1}{C_2^2}\left(\sqrt{\frac{C_2^2C^2}{AC_4^2}\,n^2 - C_1^2} - C_3^2\right)^2 \ge C'n^2 \tag{89}$$

for n large enough. If $|i_{n-1}| \le k < |i_n|$ for n large enough and $k \in \mathbb{N}$, then

$$\lambda_k \ge \lambda_{|i_{n-1}|} \ge C'(n-1)^2 \ge C'n \ge C'\left(\frac{C^2(|i_n|-1)}{C_0}\right)^\theta \ge C'\left(\frac{C^2k}{C_0}\right)^\theta$$

by (89) and (87). This shows (i) because, since $|i_n| \to \infty$ as $n \to \infty$ by (89), there is an increasing sequence (n_ℓ) in $\mathbb{Z}_+$ such that $[|i_{n_0-1}|, \infty) = \bigcup_\ell [|i_{n_\ell - 1}|, |i_{n_\ell}|)$. □

For any fixed $f \in C^\infty(M)$, let f also denote the operator of multiplication by f on $C^\infty(E)$ (or on $L^2(E)$ if f is bounded). Observe that $[d,f]$ is of order zero because d is of first order; moreover $[d,f]^* = -[\delta, f]$.

Lemma 10.3. *If f and $|[d,f]|$ are bounded, then:*

(i) *$f\,\mathcal{D}(d_{\min/\max}) \subset \mathcal{D}(d_{\min/\max})$ and $[d_{\min/\max}, f] = [d,f]$; and*
(ii) *$f\,W^1(d_{\min/\max}) \subset W^1(d_{\min/\max})$.*

Proof. For each $u \in \mathcal{D}(d_{\min})$, there is a sequence (u_n) in $C_0^\infty(E)$ such that $u_n \to u$ and (du_n) is convergent in $L^2(E)$; in fact, $d_{\min}u = \lim_n du_n$. Then $fu_n \to fu$ and

$$d(fu_n) = f\,du_n + [d,f]u_n \to f\,d_{\min}u + [d,f]u$$

in $L^2(E)$ because f and $|[d,f]|$ are bounded. So $fu \in \mathcal{D}(d_{\min})$ and $d_{\min}(fu) = f\,d_{\min}u + [d,f]u$.

Now, suppose that $u \in \mathcal{D}(d_{\max})$. Thus there is some $v \in L^2(E)$ such that $\langle u, \delta w\rangle = \langle v, w\rangle$ for all $w \in C_0^\infty(E)$; indeed, $v = d_{\max}u$. Then

$$\begin{aligned}\langle fu, \delta w\rangle = \langle u, f\delta w\rangle &= \langle u, \delta(fw) - [\delta, f]w\rangle \\ &= \langle v, fw\rangle - \langle u, [\delta,f]w\rangle = \langle fv + [d,f]u, w\rangle\end{aligned}$$

for all $w \in C_0^\infty(E)$. So $fu \in \mathcal{D}(d_{\max})$ and $d_{\max}(fu) = f\,d_{\max}u + [d,f]u$. This completes the proof of (i).

Property (ii) follows from (82) by applying (i) to d and δ. □

Let (E', d') be another elliptic complex on a Riemannian manifold M'. The scalar product of $L^2(E')$ will be denoted by $\langle\ ,\ \rangle'$, and let $\delta' = {}^t d'$. Let U and U' be open subsets of M and M', respectively, so that $U \supset \operatorname{supp} f$, and let $\zeta : (E|_U, d) \to (E'|_{U'}, d')$ be a quasi-isometric isomorphism of elliptic complexes whose underlying quasi-isometric diffeomorphism is $\xi : U \to U'$. For each $u \in L^2(E)$, identify fu to $fu|_U$, and identify $\zeta(fu) \in L^2(E'|_{U'})$ with its extension by zero to the whole of M'; in this way, we get a subspace $\zeta(f\,\mathcal{D}(d_{\min/\max})) \subset L^2(E')$.

LEMMA 10.4. *If f and $|[d, f]|$ are bounded, then the following properties hold:*

(i) *We have $\zeta(f\,\mathcal{D}(d_{\min/\max})) \subset \mathcal{D}(d'_{\min/\max})$ and $d'_{\min/\max}\zeta = \zeta d_{\min/\max}$ on $f\,\mathcal{D}(d_{\min/\max})$*

(ii) *If moreover ζ is isometric, then $\zeta(f\,W^1(d_{\min/\max})) \subset W^1(d'_{\min/\max})$.*

PROOF. Let $u \in f\,\mathcal{D}(d_{\min})$. Then $u \in \mathcal{D}(d_{\min})$ by Lemma 10.3-(i); in fact, according to its proof, there is a sequence (u_n) in $C_0^\infty(E)$ such that $u_n \to u$ and $du_n \to d_{\min}u$ in $L^2(E)$, and with $\operatorname{supp} u_n \subset \operatorname{supp} f$ for all n. Then $\zeta u_n \in C_0^\infty(E')$, $\zeta u_n \to \zeta u$ and $d'\zeta u_n = \zeta du_n \to \zeta d_{\min}u$ in $L^2(E')$. Hence $\zeta u \in \mathcal{D}(d'_{\min})$ and $d'_{\min}\zeta u = \zeta d_{\min}u$.

To prove the case of $d_{\max}$, since $\mathcal{D}(d'_{\max})$ is invariant by quasi-isometric changes of the metrics of M' and E', after shrinking U and U' if necessary, we can assume that $\zeta : (E|_U, d) \to (E'|_{U'}, d')$ is an isometric isomorphism of elliptic complexes. Such a change of metrics can be achieved by taking an open subset $V' \subset M'$ so that $\xi(\operatorname{supp} f) \subset V'$ and $\overline{V'} \subset U'$, and using a smooth partition of unity of M' subordinated to $\{V', M' \setminus \xi(\operatorname{supp} f)\}$ to combine metrics. Let $u \in f\,\mathcal{D}(d_{\max})$. Then $u \in \mathcal{D}(d_{\max})$ by Lemma 10.3-(i); indeed, according to its proof, the support of $v := d_{\max}u$ is contained in $\operatorname{supp} f$. Thus

$$\langle \zeta u, \delta'\zeta w\rangle' = \langle \zeta u, \zeta\delta w\rangle' = \langle u, \delta w\rangle = \langle v, w\rangle = \langle \zeta v, \zeta w\rangle'$$

for each $u \in f\,\mathcal{D}(d_{\max})$ and all $w \in C_0^\infty(E|_U)$. So $\langle \zeta u, \delta' w'\rangle' = \langle \zeta v, w'\rangle'$ for all $w' \in C_0^\infty(E')$, giving $\zeta u \in \mathcal{D}(d'_{\max})$ and $d_{\max}(\zeta u) = \zeta d_{\max}u$. This completes the proof of (i).

If ζ is isometric, then it is also an isometric isomorphism $(E|_U, \delta) \to (E'|_{U'}, \delta')$. So (ii) follows from (82) by applying (i) to d and δ. □

PROPOSITION 10.5. *Let (E, d) be an elliptic complex on a Riemannian manifold M. Let $\{U_a\}$ be a finite open covering of M, and let $\{f_a\}$ be a smooth partition of unity on M subordinated to $\{U_a\}$ such that each $|[d, f_a]|$ is bounded. Assume also that there is another family $\{\tilde f_a\} \subset C^\infty(M)$ such that $\tilde f_a$ and $|[d, \tilde f_a]|$ are bounded, $\tilde f_a = 1$ on $\operatorname{supp} f_a$, and $\operatorname{supp} \tilde f_a \subset U_a$. For each a, let (E^a, d^a) be an elliptic complex on a Riemannian manifold M_a, let $V_a \subset M_a$ be an open subset, and let $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ be a quasi-isometric isomorphism of elliptic complexes over $\xi_a : U_a \to V_a$. Then the following properties hold:*

(i) $\mathcal{D}(d_{\min/\max}) = \{\, u \in L^2(E) \mid \zeta_a(f_a u) \in \mathcal{D}(d^a_{\min/\max})\ \forall a \,\}$.

(ii) *If $d^a_{\min/\max}$ is discrete for all a, then $d_{\min/\max}$ is discrete.*

PROOF. The inclusion "$\subset$" of (i) follows from properties (i) of Lemmas 10.3 and 10.4.

Now, take any $u \in L^2(E)$ such that $\zeta_a(f_a u) \in \mathcal{D}(d^a_{\mathrm{min/max}})$ for all a. Let g_a and $\tilde g_a$ be the smooth functions on each M_a, supported in V_a, that correspond to f_a and $\tilde f_a$ via ξ_a. By Lemma 10.3-(i),

$$f_a u = \zeta_a^{-1}\zeta_a(f_a u) = \zeta_a^{-1}(\tilde g_a\, \zeta_a(f_a u)) \in \mathcal{D}(d_{\mathrm{min/max}})\ .$$

So $u = \sum_a f_a u \in \mathcal{D}(d_{\mathrm{min/max}})$, completing the proof of (i).

To prove (ii), we can make the following reduction. Since discreteness is invariant by quasi-isometric isomorphisms of elliptic complexes, like in the proof of Lemma 10.4-(i), after shrinking $\{U_a\}$ if necessary, we can assume that each $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric. If every $d^a_{\mathrm{min/max}}$ is discrete, then each $W^1(d^a_{\mathrm{min/max}}) \hookrightarrow L^2(F^a)$ is compact by Lemma 10.1. So

$$\mathrm{Cl}_1(g_a\, W^1(d^a_{\mathrm{min/max}})) \hookrightarrow \mathrm{Cl}_0(g_a\, L^2(E^a))$$

is compact for all a by Lemma 10.3-(ii). Therefore

$$\mathrm{Cl}_1(f_a\, W^1(d_{\mathrm{min/max}})) \hookrightarrow \mathrm{Cl}_0(f_a\, L^2(E))$$

is compact by Lemma 10.4-(ii). Since $W^1(d_{\mathrm{min/max}}) = \sum_a f_a W^1(d_{\mathrm{min/max}})$ by Lemma 10.3-(ii), it follows that $W^1(d_{\mathrm{min/max}}) \hookrightarrow L^2(E)$ is compact. Hence $d_{\mathrm{min/max}}$ is discrete by Lemma 10.1. □

Proposition 10.6. *With the notation of Proposition 10.5, suppose that every $d^a_{\mathrm{min/max}}$ is discrete, and therefore $d_{\mathrm{min/max}}$ is also discrete. Let*

$$0 \le \lambda^a_{\mathrm{min/max},0} \le \lambda^a_{\mathrm{min/max},1} \le \cdots\ , \quad 0 \le \lambda_{\mathrm{min/max},0} \le \lambda_{\mathrm{min/max},1} \le \cdots$$

denote the eigenvalues, repeated according to their multiplicities, of the Laplacians $\Delta^a_{\mathrm{min/max}}$ and $\Delta_{\mathrm{min/max}}$ defined by $d^a_{\mathrm{min/max}}$ and $d_{\mathrm{min/max}}$, respectively. If there exist some[2] $\theta_a > 0$ for all a such that $\liminf_k \lambda^a_{\mathrm{min/max},k} k^{-\theta_a} > 0$, then $\liminf_k \lambda_{\mathrm{min/max},k}\, k^{-\theta} > 0$ with $\theta = \min_a \theta_a$.

Proof. According to Sections 1 and 2 of Chapter 9, $\liminf_k \lambda^a_{\mathrm{min/max},k} k^{-\theta_a} > 0$ is a condition invariant by quasi-isometric isomorphisms of elliptic complexes. Therefore, like in the proof of Proposition 10.5-(ii), we can suppose that $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric. Set $D^a_{\mathrm{min/max}} = d^a_{\mathrm{min/max}} + \delta^a_{\mathrm{max/min}}$ and $W^{1,a} = W^1(d^a_{\mathrm{min/max}})$. Let $B^{1,a}$ denote the standard unit ball in $W^{1,a}$, and B^a_r the standard ball of radius $r > 0$ in $L^2(E^a)$. By Lemma 10.2, we get the following.

Claim 9. There are some $C_{a,0}, C_{a,1} > 0$ for every a such that, for all $n \in \mathbb{Z}_+$, there is a linear subspace $Z^a_n \subset L^2(E^a)$ so that:

(a) Z^a_n is closed and of codimension $\le C_{a,0}\, n^{1/\theta_a}$ in $L^2(E^a)$;
(b) $D^a_{\mathrm{min/max}}(W^{1,a} \cap Z^a_n) \subset Z^a_n$; and
(c) $B^{1,a} \cap Z^a_n \subset B^a_{C_{a,1}/n}$.

For each a, fix an open subset $O_a \subset M$ such that $\operatorname{supp} f_a \subset O_a$, $\overline{O_a} \subset U_a$ and the frontier of O_a has zero Riemannian measure. Let $P_a = \xi_a(O_a)$,

$$\mathcal{P}^a = \left\{\, v \in L^2(E^a) \mid v \text{ is essentially supported in } \overline{P_a}\,\right\}\ ,$$

[2]The notation $\theta_{a,\mathrm{min/max}}$ would be more correct, but, for the sake of simplicity, reference to the maximum/minimum i.b.c. is omitted here and in most of the notation of the proof.

and $Z_n^a{}' = Z_n^a \cap \mathcal{P}^a$. Each $\mathcal{P}^a$ is a closed linear subspace of $L^2(E^a)$ satisfying

$$D^a_{\text{min/max}}(W^{1,a} \cap \mathcal{P}^a) \subset \mathcal{P}^a \ . \tag{90}$$

CLAIM 10. (a) $Z_n^a{}'$ is closed and of codimension $\leq C_{a,0}\, n^{1/\theta_a}$ in $\mathcal{P}^a$;
(b) $D^a_{\text{min/max}}(W^{1,a} \cap Z_n^a{}') \subset Z_n^a{}'$; and
(c) $B^{1,a} \cap Z_n^a{}' \subset B^a_{C_{a,1}/n} \cap \mathcal{P}^a$.

Claim 10-(a) follows from Claim 9-(a) and the canonical linear isomorphism

$$\frac{\mathcal{P}^a}{Z_n^a{}'} \cong \frac{\mathcal{P}^a + Z_n^a}{Z_n^a} \ .$$

Claim 10-(b) is a consequence of Claim 9-(b) and (90), and Claim 10-(c) follows from Claim 9-(c).

Now, consider the linear spaces

$$\mathcal{O}^a = \left\{\, u \in L^2(E) \mid u \text{ is essentially supported in } \overline{O_a} \,\right\} \ ,$$
$$Z_n^a{}'' = \left\{\, u \in \mathcal{O}^a \mid \exists v \in Z_n^a{}' \text{ so that } \zeta_a(u|_{U_a}) = v|_{V_a} \,\right\} \ .$$

Each $\mathcal{O}^a$ is a closed linear subspace of $L^2(E)$, and we have $L^2(E) = \sum_a \mathcal{O}^a$. Set $D_{\text{min/max}} = d_{\text{min/max}} + \delta_{\text{max/min}}$ and $W^1 = W^1(d_{\text{min/max}})$. Let B^1 denote the standard unit ball in W^1, and B_r the standard ball of radius $r > 0$ in $L^2(E)$. Since $\zeta_a : (E|_{U_a}, d) \to (E^a|_{V_a}, d^a)$ is isometric for all a, Claim 10 gives the following.

CLAIM 11. (a) $Z_n^a{}''$ is closed and of codimension $\leq C_{a,0}\, n^{1/\theta_a}$ in $\mathcal{O}^a$;
(b) $D_{\text{min/max}}(W^1 \cap Z_n^a{}'') \subset Z_n^a{}''$; and
(c) $B^1 \cap Z_n^a{}'' \subset B_{C_{a,1}/n} \cap \mathcal{O}^a$.

Let Y_n^a be a linear complement of each $Z_n^a{}''$ in $\mathcal{O}^a$. By Claim 11-(a), we have

$$\mathcal{O}^a = Y_n^a \oplus Z_n^a{}'' \tag{91}$$

as topological vector spaces [**58**, Chapter I, 3.5]. On the other hand, for any $m \in \mathbb{Z}_+$, $W^m \cap \mathcal{O}^a$ is dense in $\mathcal{O}^a$ because it contains all sections $u \in C_0^\infty(E)$ with $\operatorname{supp} u \subset O_a$. So we can choose $Y_n^a \subset W^m$ by Claim 11-(a), obtaining

$$W^m \cap \mathcal{O}^a = Y_n^a \oplus (W^m \cap Z_n^a{}'') \tag{92}$$

as topological vector spaces with respect to the topology induced by $\|\ \|$. The following assertion follows from (91), (92) and the density of $W^m \cap \mathcal{O}^a$ in $\mathcal{O}^a$.

CLAIM 12. $W^m \cap Z_n^a{}''$ is $\|\ \|$-dense in $Z_n^a{}''$.

For the case $m = 1$, observe that (92) is satisfied with

$$Y_n^a = \mathcal{O}^a \cap (W^1 \cap Z_n^a{}'')^{\perp_1} \ , \tag{93}$$

where $\perp_1$ denotes $\langle\ ,\ \rangle_1$-orthogonality, and therefore (92) also holds with respect to the topology induced by $\|\ \|_1$. From now on, consider the choice (93) for Y_n^a.

CLAIM 13. $D_{\text{min/max}}(Y_n^a) \subset W^1$.

Since the Riemannian measure of the frontier of O_a is zero, $\mathcal{O}^{a\,\perp}$ consists of the sections $u \in L^2(E)$ whose essential support is contained in $M \setminus O_a$. Hence the set

$$(W^1 \cap \mathcal{O}^{a\,\perp}) + Y_n^a + (W^1 \cap Z_n^a{}'')$$

is dense in $L^2(E)$ by (92) for $m = 1$. It follows that, given any $u \in Y_n^a$, to check that $D_{\min/\max}u \in W^1$, its enough to check that the mapping

$$v \mapsto \langle D_{\min/\max}u, D_{\min/\max}v\rangle$$

is bounded on $W^1 \cap \mathcal{O}^{a\,\perp}$, Y_n^a and $W^1 \cap Z_n^{a\,\prime\prime}$. This mapping vanishes on $W^1 \cap \mathcal{O}^{a\,\perp}$ because

$$D_{\min/\max}(W^1 \cap \mathcal{O}^a) \subset \mathcal{O}^a\ , \quad D_{\min/\max}(W^1 \cap \mathcal{O}^{a\,\perp}) \subset \mathcal{O}^{a\,\perp}\ .$$

Moreover it is bounded on Y_n^a because this space is of finite dimension. Finally, for $v \in W^1 \cap Z_n^{a\,\prime\prime}$, we have

$$\langle D_{\min/\max}u, D_{\min/\max}v\rangle = -\langle u, v\rangle$$

because $u \perp_1 v$. Thus the above mapping is bounded on $W^1 \cap Z_n^{a\,\prime\prime}$, which completes the proof of Claim 13.

Claim 14. $D_{\min/\max}(Y_n^a) \subset Y_n^a$.

For $u \in Y_n^a$ and $v \in W^2 \cap Z_n^{a\,\prime\prime}$, we have

$$\begin{aligned}\langle D_{\min/\max}u, v\rangle_1 &= \langle D_{\min/\max}u, v\rangle + \langle \Delta_{\min/\max}u, D_{\min/\max}v\rangle\\ &= \langle u, D_{\min/\max}v\rangle + \langle D_{\min/\max}u, \Delta_{\min/\max}v\rangle = \langle u, D_{\min/\max}v\rangle_1 = 0\end{aligned}$$

by Claim 13 and because $D_{\min/\max}$ is self-adjoint. Then Claim 14 follows by Claim 12.

Claim 15. $Y_n^a = \mathcal{O}^a \cap (Z_n^{a\,\prime\prime})^\perp$.

Let $u \in Y_n^a$ and $v \in W^1 \cap Z_n^{a\,\prime\prime}$. By Claim 14, $\Delta_{\min/\max}$ is a self-adjoint operator on Y_n^a. Then $u = (1 + \Delta_{\min/\max})u_0$ for $u_0 = (1 + \Delta_{\min/\max})^{-1}u \in Y_n^a$, obtaining

$$\langle u, v\rangle = \langle (1 + \Delta_{\min/\max})u_0, v\rangle = \langle u_0, v\rangle_1 = 0\ .$$

This shows Claim 15 by Claim 12 and (91).

Let $\Pi_n^a : \mathcal{O}^a \to Z_n^{a\,\prime\prime}$ denote the orthogonal projection. The following claim follows from (92) for $m = 1$, and Claims 11-(b), 14 and 15.

Claim 16. $\Pi_n^a(W^1 \cap \mathcal{O}^a) \subset W^1 \cap \mathcal{O}^a$, and $[D_{\min/\max}, \Pi_n^a] = 0$ on $W^1 \cap \mathcal{O}^a$.

Consider each function f_a as the corresponding bounded multiplication operator on $L^2(E)$. Assuming that a runs in $\{1, \dots, A\}$ for some $A \in \mathbb{Z}_+$, we get the bounded operator $T = (f_1, \dots, f_A) : L^2(E) \to \bigoplus_A L^2(E)$. Also, let $\Sigma : \bigoplus_A L^2(E) \to L^2(E)$ be the bounded operator defined by $\Sigma(u_1, \dots, u_A) = \sum_a u_a$. We have $\Sigma T = 1$ because $\{f_a\}$ is a partition of unity.

Claim 17. The image of T is closed.

Let (u^i) be a sequence in $L^2(E)$ such that (Tu^i) converges to some v in $\bigoplus_A L^2(E)$. Then $u^i = \Sigma T u^i \to \Sigma v$ as $i \to \infty$, obtaining $Tu^i \to T\Sigma v$ as $i \to \infty$. Hence $v = T\Sigma v \in T(L^2(E))$, showing Claim 17.

By Claim 17 and the open mapping theorem (see e.g. [**18**, Chapter III, 12.1] or [**58**, Chapter III, 2.1]), we get that T is a topological homomorphism[3]. So $T : L^2(E) \to T(L^2(E))$ is a quasi-isometric isomorphism; its inverse is $\Sigma : T(L^2(E)) \to$

[3]Recall that a bounded operator between topological vector spaces, $T : \mathfrak{H} \to \mathfrak{G}$, is called a topological homomorphism if the map $T : \mathfrak{H} \to T(\mathfrak{H})$ is open, where $T(\mathfrak{H})$ is endowed with the restriction of the topology of $\mathfrak{G}$.

$L^2(E)$. Since $\Pi_n := \bigoplus_a \Pi_n^a$ is an orthogonal projection of $\bigoplus_A L^2(E)$, it follows that $R_n := \Pi_n T$ satisfies Lemma 10.2-(iii)-(b),(c). Moreover, by Claim 11-(a),

$$\dim\ker R_n \leq \dim\ker \Pi_n = \sum_a \dim\ker \Pi_n^a \leq \sum_a C_{0,a}\, n^{1/\theta_a} \leq C_0\, n^{1/\theta}$$

with $C_0 = \sum_a C_{0,a}$ and $\theta = \min_a \theta_a$, which shows that R_n satisfies Lemma 10.2-(iii)-(a).

We have $R_n = (R_n^1, \dots, R_n^A)$ with $R_n^a = \Pi_n^a\, f_a$. Since each function $|[d, f_a]|$ is uniformly bounded, it follows that $f_a\, W^1 \subset W^1$ and $[D_{\min/\max}, f_a] : W^1 \to L^2(E)$ extends to a bounded operator on $L^2(E)$. Therefore each R_n^a satisfies Lemma 10.2-(iii)-(d) by Claim 16.

Finally, R_n^a satisfies Lemma 10.2-(iii)-(e) by Claim 11-(c). Now, the result follows from Lemma 10.2. □

CHAPTER 11

Two simple types of elliptic complexes

Here, we study the two types of simple elliptic complexes. They will show up in the direct sum splitting of the local model of Witten's perturbation (Chapter 15). We could describe better the spectra of the Laplacians associated to the minimum/maximum i.b.c. of these simple elliptic complexes, but this will be done with the local model of the Witten's perturbation (Chapter 14).

1. Some more results on general elliptic complexes

Consider the notation of the beginning of Section 2 in Chapter 9.

LEMMA 11.1. *Let $\mathcal{G} \subset C^\infty(E) \cap L^2(E)$ be a graded linear subspace containing $C_0^\infty(E)$, preserved by d and δ, and such that $\langle du, v\rangle = \langle u, \delta v\rangle$ for all $u, v \in \mathcal{G}$. Let $d_{\mathcal{G}}$, $\delta_{\mathcal{G}}$ and $\Delta_{\mathcal{G}}$ denote the restrictions of d, δ and Δ to $\mathcal{G}$. Assume that $\Delta_{\mathcal{G}}$ is essentially self-adjoint in $L^2(E)$, and $\mathcal{G}$ is the smooth core of $\overline{\Delta_{\mathcal{G}}}$. Then the following properties hold:*

- (i) *If $\mathcal{G}_r \subset \mathcal{D}(d_{\min,r})$ and $\mathcal{G}_{r-1} \subset \mathcal{D}(d_{\min,r-1})$ for some degree r, then $\mathcal{G}_r$ is the smooth core of $d_{\min,r}$.*
- (ii) *If $\mathcal{G}_r \subset \mathcal{D}(\delta_{\min,r-1})$ and $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\min,r})$ for some degree r, then $\mathcal{G}_r$ is the smooth core of $d_{\max,r}$.*

PROOF. For each degree r, the restrictions $d_r : \mathcal{G}_r \to \mathcal{G}_{r+1}$, $\delta_r : \mathcal{G}_{r+1} \to \mathcal{G}_r$ and $\Delta_r : \mathcal{G}_r \to \mathcal{G}_r$ will be denoted by $d_{\mathcal{G},r}$, $\delta_{\mathcal{G}}, r$ and $\Delta_{\mathcal{G},r}$, respectively. Suppose that $\mathcal{G}_r \subset \mathcal{D}(d_{\min,r})$ and $\mathcal{G}_{r-1} \subset \mathcal{D}(d_{\min,r-1})$, and therefore $d_{\mathcal{G},r} \subset d_{\min,r}$ and $d_{\mathcal{G},r-1} \subset d_{\min,r-1}$. Since $C_0^\infty(E) \subset \mathcal{G}$ and $\langle du, v\rangle = \langle u, \delta v\rangle$ for all $u, v \in \mathcal{G}$, it follows that $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\max,r})$ and $\mathcal{G}_r \subset \mathcal{D}(\delta_{\max,r-1})$, and therefore $\delta_{\mathcal{G},r} \subset \delta_{\max,r}$ and $\delta_{\mathcal{G},r-1} \subset \delta_{\max,r-1}$. By (81), we get $\Delta_{\mathcal{G},r} \subset \Delta_{\min,r}$. So $\overline{\Delta_{\mathcal{G},r}} \subset \Delta_{\min,r}$, and therefore $\overline{\Delta_{\mathcal{G},r}} = \Delta_{\min,r}$ because these operators are self-adjoint in $L^2(E_r)$. Then $\mathcal{G}_r$ is the smooth core of $d_{\min,r}$, completing the proof of (i).

Now, assume that $\mathcal{G}_r \subset \mathcal{D}(\delta_{\min,r-1})$ and $\mathcal{G}_{r+1} \subset \mathcal{D}(\delta_{\min,r})$, and therefore $\delta_{\mathcal{G},r-1} \subset \delta_{\min,r-1}$ and $\delta_{\mathcal{G},r} \subset \delta_{\min,r}$. As above, it follows that $d_{\mathcal{G},r-1} \subset d_{\max,r-1}$ and $d_{\mathcal{G},r} \subset d_{\max,r}$. By (81), we get $\Delta_{\mathcal{G},r} \subset \Delta_{\max,r}$. So $\overline{\Delta_{\mathcal{G},r}} \subset \Delta_{\max,r}$, obtaining $\overline{\Delta_{\mathcal{G},r}} = \Delta_{\max,r}$ as before. Thus $\mathcal{G}_r$ is the smooth core of $d_{\max,r}$, completing the proof of (ii). □

Now, suppose that there is an orthogonal decomposition $E_{r+1} = E_{r+1,1} \oplus E_{r+1,2}$ for some degree $r + 1$. Thus

$$C^\infty(E_{r+1}) \equiv C^\infty(E_{r+1,1}) \oplus C^\infty(E_{r+1,2}) ,$$
$$C_0^\infty(E_{r+1}) \equiv C_0^\infty(E_{r+1,1}) \oplus C_0^\infty(E_{r+1,2}) ,$$
$$L^2(E_{r+1}) \equiv L^2(E_{r+1,1}) \oplus L^2(E_{r+1,2}) ,$$

giving

$$d_r = \begin{pmatrix} d_{r,1} \\ d_{r,2} \end{pmatrix} \;, \quad \delta_r = \begin{pmatrix} \delta_{r,1} & \delta_{r,2} \end{pmatrix} \;.$$

LEMMA 11.2. *We have:*

$$\mathcal{D}(d_{\max,r}) = \mathcal{D}(d_{r,1,\max}) \cap \mathcal{D}(d_{r,2,\max}) \;, \quad d_{\max,r} = \begin{pmatrix} d_{r,1,\max}|_{\mathcal{D}(d_{\max,r})} \\ d_{r,2,\max}|_{\mathcal{D}(d_{\max,r})} \end{pmatrix} \;.$$

PROOF. Let $u \in L^2(E_r)$. We have $u \in \mathcal{D}(d_{\max,r})$ if and only if there is some $w \in L^2(E_{r+1})$ such that $\langle u, \delta v\rangle = \langle w, v\rangle$ for all $v \in C_0^\infty(E_{r+1})$, and moreover $d_{\max,r}u = w$ in this case. Writing $w = w_1 \oplus w_2$ and $v = v_1 \oplus v_2$, this condition on u means that $\langle u, \delta_{0,i}v_i\rangle = \langle w_i, v_i\rangle$ for all $v_i \in C_0^\infty(E^i_{r+1})$ and $i \in \{1, 2\}$. In turn, this is equivalent to $u \in \mathcal{D}(d_{r,1,\max}) \cap \mathcal{D}(d_{r,2,\max})$ with $d_{r,i,\max}u = w_i$. □

For $i \in \{1, 2\}$, let $\Delta_{r,i} = \delta_{r,i}d_{r,i} + d_{r-1}\delta_{r-1}$ on $C^\infty(E_r)$.

COROLLARY 11.3. *If $a\,\Delta_r = b\,\Delta_{r,i} + c$ for some $a, b, c \in \mathbb{R}$ with $a, b \neq 0$, $d_{\min,r}$ and $d_{r,i,\min}$ have the same smooth core, and $d_{r,i,\min} = d_{r,i,\max}$ for some $i \in \{0, 1\}$, then $d_{\min,r} = d_{\max,r}$.*

PROOF. By Lemma 11.2 and since $d_{r,i,\min} = d_{r,i,\max}$, we get $\mathcal{D}(d_{\max,r}) \subset \mathcal{D}(d_{r,i,\min})$. Because $a\,\Delta_r = b\,\Delta_{r,i} + c$ for some $a, b, c \in \mathbb{R}$ with $a, b \neq 0$, it follows that

$$\{\, u \in \mathcal{D}(d_{\max,r}) \cap C^\infty(E_r) \mid \Delta_r^k u \in L^2(E_r)\ \forall k \in \mathbb{N} \,\} \\ \subset \{\, u \in \mathcal{D}(d_{r,i,\min}) \cap C^\infty(E_r) \mid \Delta_{r,i}^k u \in L^2(E_r)\ \forall k \in \mathbb{N} \,\} \;.$$

This means that the smooth core of $d_{\max,r}$ is contained in the smooth core of $d_{r,i,\min}$, which equals the smooth core of $d_{\min,r}$. Then $d_{\max,r} = d_{\min,r}$. □

2. An elliptic complex of length two

Consider the standard metric on $\mathbb{R}_+$. Let E be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose non-zero terms are E_0 and E_1, which are real/complex trivial line bundles endowed with the standard Riemannian/Hemitian metrics. Thus

$$C^\infty(E_0) \equiv C^\infty(\mathbb{R}_+) \equiv C^\infty(E_1) \;, \quad L^2(E_0) \equiv L^2(\mathbb{R}_+, d\rho) \equiv L^2(E_1) \;,$$

where real/complex valued functions are considered in $C^\infty(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+, d\rho)$. For any fixed $s > 0$ and $\kappa \in \mathbb{R}$, let

$$C^\infty(E_0) \underset{\delta}{\overset{d}{\rightleftarrows}} C^\infty(E_1)$$

be the differential operators defined by

$$d = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho \;, \quad \delta = -\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho \;.$$

It is easy to check that (E,d) is an elliptic complex, whose formal adjoint is (E,δ). By (71), the homogeneous components of the corresponding Laplacian Δ are:

$$\begin{aligned}\Delta_0 = \delta d &\equiv H + \kappa\left[\frac{d}{d\rho},\rho^{-1}\right] \mp s\left[\frac{d}{d\rho},\rho\right] + \kappa^2\rho^{-2} \mp 2s\kappa\\ &= H + \kappa(\kappa-1)\rho^{-2} \mp s(1+2\kappa)\ , \qquad (94)\\ \Delta_1 = d\delta &= H - \kappa\left[\frac{d}{d\rho},\rho^{-1}\right] \pm s\left[\frac{d}{d\rho},\rho\right] + \kappa^2\rho^{-2} \mp 2s\kappa\\ &= H + \kappa(\kappa+1)\rho^{-2} \pm s(1-2\kappa)\ , \qquad (95)\end{aligned}$$

where H is the harmonic oscillator on $C^\infty(\mathbb{R}_+)$ defined with the constant s. Then Δ_0 and Δ_1 are of the form of P in (1) (with $c_1 = 0$) plus a constant; in particular, for $\kappa = 0$, they are equal to H plus a constant.

For Δ_0, the condition (4) means that $a \in \{\kappa, 1-\kappa\}$, and (5) gives $\sigma = \kappa$ if $a = \kappa$, and $\sigma = 1-\kappa$ if $a = 1-\kappa$. By Corollary H, the following holds:

- If $\kappa > -1/2$, then Δ_0, with domain $\rho^\kappa\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^\kappa\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 3/2$, then Δ_0, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

For Δ_1, the condition (4) means that $a \in \{1+\kappa, -\kappa\}$, and (5) becomes $\sigma = 1+\kappa$ if $a = 1+\kappa$, and $\sigma = -\kappa$ if $a = -\kappa$. Now Corollary H states the following:

- If $\kappa > -3/2$, then Δ_1, with domain $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 1/2$, then Δ_1, with domain $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

When $\kappa > -1/2$, let $\mathcal{E}_1 \subset C^\infty(E) \cap L^2(E)$ be the dense graded linear subspace with

$$\mathcal{E}_1^0 \equiv \rho^\kappa\,\mathcal{S}_{\mathrm{ev},+}\ ,\quad \mathcal{E}_1^1 \equiv \rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\ .$$

When $\kappa < 1/2$, let $\mathcal{E}_2 \subset C^\infty(E) \cap L^2(E)$ be the dense graded linear subspace with

$$\mathcal{E}_2^0 \equiv \rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\ ,\quad \mathcal{E}_2^1 \equiv \rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\ .$$

Observe that, by restricting d and δ, we get complexes $(\mathcal{E}_1, d)$ and $(\mathcal{E}_1, \delta)$ when $\kappa > -1/2$, and complexes $(\mathcal{E}_2, d)$ and $(\mathcal{E}_2, \delta)$ when $\kappa < 1/2$. Thus Δ preserves $\mathcal{E}_1$ when $\kappa > -1/2$, and preserves $\mathcal{E}_2$ when $\kappa < 1/2$.

Proposition 11.4. (i) *If $|\kappa| < 1/2$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are the smooth cores of $d_{\max}$ and $d_{\min}$, respectively.*

(ii) *If $|\kappa| \geq 1/2$, then (E,d) has a unique i.b.c., whose smooth core is $\mathcal{E}_1$ when $\kappa \geq 1/2$, and $\mathcal{E}_2$ when $\kappa \leq -1/2$.*

The following lemma will be used in the proof of Proposition 11.4.

Lemma 11.5. *Suppose that $\theta \geq 1/2$. Then, for each $\xi \in \rho^\theta\,\mathcal{S}_{\mathrm{ev},+}$, considered as subspace of $C^\infty(E_0)$ (respectively, $C^\infty(E_1)$), there is a sequence (ξ_n) in $C_0^\infty(E_0)$ (respectively, $C_0^\infty(E_1)$), independent of κ, such that $\lim_n \xi_n = \xi$ and $\lim_n d\xi_n =$*

$d\xi$ in $L^2(E_0)$ (respectively, $\lim_n \delta\xi_n = \delta\xi$ in $L^2(E_1)$). In particular, $\rho^\theta\,\mathcal{S}_{\mathrm{ev},+}$ is contained in $\mathcal{D}(d_{\min})$ (respectively, $\mathcal{D}(\delta_{\min})$).

Remark 20. In Lemma 11.5, the independence of κ means that (ξ_n) depends only on θ and ξ, whilst the convergences $\lim_n d\xi_n = d\xi$ and $\lim_n \delta\xi_n = \delta\xi$ hold with d and δ defined by any κ.

Proof of Lemma 11.5. The proof is made for $\mathcal{D}(d_{\min})$; the case of $\mathcal{D}(\delta_{\min})$ is analogous.

Let $0 < a < b$ and $f \in C_0^\infty(\mathbb{R}_+)$ such that $0 \le f \le 1$, $f(\rho) = 1$ for $\rho \le a$, and $f(\rho) = 0$ for $\rho \ge b$. For each $n \in \mathbb{N}$, let $g_n, h_n \in C^\infty(\mathbb{R}_+)$ be defined by $g_n(\rho) = f(n\rho)$ and $h_n(\rho) = f(\rho/n)$. It is clear that

$$\chi_{[\frac{b}{n},na]} \le (1-g_n)h_n \le \chi_{[\frac{a}{n},nb]}\ , \tag{96}$$

where χ_S denotes the characteristic function of each subset $S \subset \mathbb{R}_+$.

Let $\phi \in \mathcal{S}_{\mathrm{ev},+}$. From (96), we get $(1-g_n)h_n\rho^\theta\phi \in C_0^\infty(E_0)$ and $(1-g_n)h_n\rho^\theta\phi \to \rho^\theta\phi$ in $L^2(E_0)$ as $n \to \infty$. Observe that

$$d((1-g_n)h_n\rho^\theta\phi) = -g_n' h_n\rho^\theta\phi + (1-g_n)h_n'\rho^\theta\phi + (1-g_n)h_n\, d(\rho^\theta\phi)\ .$$

In right hand side of this equality, the last term converges to $d(\rho^\theta\phi)$ in $L^2(E_1)$ as $n \to \infty$ by (96). Moreover

$$\begin{aligned}\|(1-g_n)h_n'\rho^\theta\phi\|^2 &= \int_0^\infty (1-g_n)^2 {h_n'}^2(\rho)\rho^{2\theta}\phi^2(\rho)\,d\rho\\ &\le (\max\rho^{2\theta}\phi^2)\,n^{-2}\int_0^\infty {f'}^2(\rho/n)\,d\rho = (\max\rho^{2\theta}\phi^2)\,n^{-1}\int_0^\infty {f'}^2(x)\,dx\\ &= (\max\rho^{2\theta}\phi^2)\,n^{-1}\,\|f'\|^2\ ,\end{aligned}$$

which converges to zero as $n \to \infty$, and

$$\begin{aligned}\|g_n' h_n\rho^\theta\phi\|^2 &= \int_0^\infty {g_n'}^2(\rho)h_n^2(\rho)\rho^{2\theta}\phi^2(\rho)\,d\rho \le (\max\phi^2)\,n^2\int_0^\infty {f'}^2(n\rho)\rho^{2\theta}\,d\rho\\ &= (\max\phi^2)\,n^{1-2\theta}\int_0^\infty {f'}^2(x)x^{2\theta}\,dx = (\max\phi^2)\,n^{1-2\theta}\,\|f'\rho^\theta\|^2\ ,\end{aligned}$$

which converges to zero as $n \to \infty$ if $\theta > 1/2$.

In the case $\theta = 1/2$, it is enough to prove that f can be chosen so that $\|f'\rho^{1/2}\|$ is as small as desired. For $m > 1$ and $0 < \epsilon < 1$, observe that there is some f as above such that:

- the support of f' is contained in $[e^{-\epsilon}, e^m]$,
- $-\frac{1}{m\rho} \le f' \le 0$, and
- $f'(\rho) = -\frac{1}{m\rho}$ if $1 \le \rho \le e^{m-\epsilon}$.

Then

$$\|f'\rho^{1/2}\|^2 = \int_{e^{-\epsilon}}^{e^m} {f'}^2(\rho)\rho\,d\rho \le \frac{1}{m^2}\int_{e^{-\epsilon}}^{e^m}\frac{d\rho}{\rho} = \frac{m+\epsilon}{m^2}\ ,$$

which converges to zero as $m \to \infty$. □

Proof of Proposition 11.4. Suppose that $|\kappa| < 1/2$. Since $1 \pm \kappa > 1/2$, by Lemma 11.5, $\mathcal{E}_2^0 \subset \mathcal{D}(d_{\min})$ and $\mathcal{E}_1^1 \subset \mathcal{D}(\delta_{\min})$. The other conditions of Lemma 11.1 are satisfied by d with $\mathcal{G} = \mathcal{E}_2$, and by δ with $\mathcal{G} = \mathcal{E}_1$ by the discussion previous to Proposition 11.4. So $\mathcal{E}_2$ is the smooth core of $d_{\min}$ and $\mathcal{E}_1$ is the smooth core of $d_{\max}$ by Lemma 11.1.

Now, assume that $\kappa \geq 1/2$ (respectively, $\kappa \leq -1/2$), giving also $1+\kappa > 1/2$ (respectively, $1-\kappa > 1/2$). Then, by Lemma 11.5, $\mathcal{E}_1^0 \subset \mathcal{D}(d_{\min})$ and $\mathcal{E}_1^1 \subset \mathcal{D}(\delta_{\min})$ (respectively, $\mathcal{E}_2^0 \subset \mathcal{D}(d_{\min})$ and $\mathcal{E}_2^1 \subset \mathcal{D}(\delta_{\min})$). By the discussion previous to Proposition 11.4, the other conditions of Lemma 11.1 are satisfied by d and δ with $\mathcal{G} = \mathcal{E}_1$ (respectively, $\mathcal{G} = \mathcal{E}_2$). So, by Lemma 11.1, $\mathcal{E}_1$ (respectively, $\mathcal{E}_2$) is the smooth core of $d_{\min}$ and $d_{\max}$. □

Remark 21. In the proof of Lemma 11.5 and Proposition 11.4, we have borrowed ideas from the proof of [**11**, Theorem 4.1]; in fact, in the case with $\kappa = 0$, Proposition 11.4 could be proved exactly like [**11**, Theorem 4.1].

3. An elliptic complex of length three

Consider again the standard metric on $\mathbb{R}_+$. Let F be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose non-zero terms are F_0, F_1 and F_2, which are trivial real/complex vector bundles of ranks 1, 2 and 1, respectively, endowed with the standard Riemannian/Hermitian metrics. Thus

$$
\begin{gathered}
C^\infty(F_0) \equiv C^\infty(\mathbb{R}_+) \equiv C^\infty(F_2) , \quad C^\infty(F_1) \equiv C^\infty(\mathbb{R}_+) \oplus C^\infty(\mathbb{R}_+) , \\
L^2(F_0) \equiv L^2(\mathbb{R}_+, d\rho) \equiv L^2(F_2) , \quad L^2(F_1) \equiv L^2(\mathbb{R}_+, d\rho) \oplus L^2(\mathbb{R}_+, d\rho) ,
\end{gathered}
$$

where real/complex valued functions are considered in $C^\infty(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+, d\rho)$. Fix $s, c > 0$ and $\kappa \in \mathbb{R}$, and let

$$
C^\infty(F_0) \underset{\delta_0}{\overset{d_0}{\rightleftarrows}} C^\infty(F_1) \underset{\delta_1}{\overset{d_1}{\rightleftarrows}} C^\infty(F_2)
$$

be the differential operators defined by

$$
d_0 = \begin{pmatrix} d_{0,1} \\ d_{0,2} \end{pmatrix} , \quad \delta_0 = \begin{pmatrix} \delta_{0,1} & \delta_{0,2} \end{pmatrix} , \quad d_1 = \begin{pmatrix} d_{1,1} & d_{1,2} \end{pmatrix} , \quad \delta_1 = \begin{pmatrix} \delta_{1,1} \\ \delta_{1,2} \end{pmatrix} ,
$$

where

$$d_{0,1} = \frac{c}{\sqrt{1+c^2}}\left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right) ,$$
$$d_{0,2} = \frac{1}{\sqrt{1+c^2}}\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right) ,$$
$$\delta_{0,1} = \frac{c}{\sqrt{1+c^2}}\left(-\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right) ,$$
$$\delta_{0,2} = \frac{1}{\sqrt{1+c^2}}\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right) ,$$
$$d_{1,1} = \frac{1}{\sqrt{1+c^2}}\left(\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right) ,$$
$$d_{1,2} = \frac{c}{\sqrt{1+c^2}}\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right) ,$$
$$\delta_{1,1} = \frac{1}{\sqrt{1+c^2}}\left(-\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right) ,$$
$$\delta_{1,2} = \frac{c}{\sqrt{1+c^2}}\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right) .$$

A direct computation shows that d_0 and d_1 define an elliptic complex (F, d) of length three. Its formal adjoint is the complex (F, δ) given by δ_0 and δ_1. The homogeneous components Δ_0 and Δ_2 of the corresponding Laplacian Δ can be computed as follows, where the notation of Section 1 is used. By (94) and (95),

$$\begin{aligned}
\Delta_{0,1} = \delta_{0,1}d_{0,1} &= \frac{c^2}{1+c^2}\left(-\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right)\left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right) \\
&= \frac{c^2}{1+c^2}\left(H + \kappa(\kappa+1)\rho^{-2} \mp s(1-2\kappa)\right) , \\
\Delta_{0,2} = \delta_{0,2}d_{0,2} &= \frac{1}{1+c^2}\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right)\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right) \\
&= \frac{1}{1+c^2}\left(H + (\kappa+1)\kappa\rho^{-2} \mp s(1+2(\kappa+1))\right) , \\
\Delta_{2,1} = d_{1,1}\delta_{1,1} &= \frac{1}{1+c^2}\left(\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right) \\
&= \frac{1}{1+c^2}\left(H + \kappa(\kappa+1)\rho^{-2} \pm s(1-2\kappa)\right) , \\
\Delta_{2,2} = d_{1,2}\delta_{1,2} &= \frac{c^2}{1+c^2}\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right)\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right) \\
&= \frac{c^2}{1+c^2}\left(H + (\kappa+1)\kappa\rho^{-2} \pm s(1+2(\kappa+1))\right) , \\
\Delta_0 = \delta_0 d_0 = \Delta_{0,1} + \Delta_{0,2} &= H + \kappa(\kappa+1)\rho^{-2} \mp s\left(2 + \frac{1-c^2}{1+c^2}(1+2\kappa)\right) , \\
\Delta_2 = d_1\delta_1 = \Delta_{2,1} + \Delta_{2,2} &= H + \kappa(\kappa+1)\rho^{-2} \pm s\left(2 + \frac{1-c^2}{1+c^2}(1+2\kappa)\right) .
\end{aligned}$$

Thus Δ_0 can be identified to Δ_2, and they are of the form of P in (1) (with $c_1 = 0$) plus a constant.

For Δ_0 and Δ_2, the condition (4) means that $a \in \{1+\kappa, -\kappa\}$, and (5) gives $\sigma = 1+\kappa$ if $a = 1+\kappa$, and $\sigma = -\kappa$ if $a = -\kappa$. By Corollary H, the following holds:

- If $\kappa > -3/2$, then Δ_0 and Δ_2, with domain $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectra of their closures are discrete, and the smooth core of their closures is $\rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 3/2$, then Δ_0 and Δ_2, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectra of their closures are discrete, and the smooth core of their closures is $\rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

Write

$$\begin{aligned}\Delta_1 &= d_0\delta_0 + \delta_1 d_1\\ &\qquad\qquad = \begin{pmatrix} d_{0,1}\delta_{0,1} + \delta_{1,1}d_{1,1} & d_{0,1}\delta_{0,2} + \delta_{1,1}d_{1,2} \\ d_{0,2}\delta_{0,1} + \delta_{1,2}d_{1,1} & d_{0,2}\delta_{0,2} + \delta_{1,2}d_{1,2}\end{pmatrix} = \begin{pmatrix}\Delta_{1,1} & A \\ B & \Delta_{1,2}\end{pmatrix}\ .\end{aligned}$$

By (94) and (95),

$$\begin{aligned}\Delta_{1,1} &= \frac{1}{1+c^2}\left(c^2\left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right)\right.\\
&\qquad\qquad \left. + \left(-\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right)\left(\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right)\right)\\
&= \frac{1}{1+c^2}\big(c^2\left(H + \kappa(\kappa-1)\rho^{-2} \mp s(1+2\kappa)\right)\\
&\qquad\qquad + H + \kappa(\kappa-1)\rho^{-2} \mp s(1+2\kappa)\big)\\
&= H + \kappa(\kappa-1)\rho^{-2} \mp s\frac{1-c^2}{1+c^2}(1+2\kappa)\ ,\\
\Delta_{1,2} &= \frac{1}{1+c^2}\left(\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right)\right.\\
&\qquad\qquad \left. + c^2\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right)\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right)\right)\\
&= \frac{1}{1+c^2}\big(H + (\kappa+1)(\kappa+2)\rho^{-2} \pm s(1-2(\kappa+1))\\
&\qquad\qquad + c^2\left(H + (\kappa+1)(\kappa+2)\rho^{-2} \mp s(1-2(\kappa+1))\right)\big)\\
&= H + (\kappa+1)(\kappa+2)\rho^{-2} \mp s\frac{1-c^2}{1+c^2}(1+2\kappa)\ .\end{aligned}$$

So $\Delta_{1,1}$ and $\Delta_{1,2}$ also are of the form of P in (1) (with $c_1 = 0$) plus a constant.

For $\Delta_{1,1}$, the condition (4) means that $a \in \{\kappa, 1-\kappa\}$, and (5) gives $\sigma = \kappa$ if $a = \kappa$, and $\sigma = 1-\kappa$ if $a = 1-\kappa$. By Corollary H, the following holds:

- If $\kappa > -1/2$, then $\Delta_{1,1}$, with domain $\rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < 3/2$, then $\Delta_{1,1}$, with domain $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

For $\Delta_{1,2}$, the condition (4) means that $a \in \{2+\kappa, -1-\kappa\}$, and (5) becomes $\sigma = 2+\kappa$ if $a = 2+\kappa$, and $\sigma = -1-\kappa$ if $a = -1-\kappa$. Then Corollary H states the following:

- If $\kappa > -5/2$, then $\Delta_{1,2}$, with domain $\rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.
- If $\kappa < -1/2$, then $\Delta_{1,2}$, with domain $\rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, d\rho)$, the spectrum of its closure is discrete, and the smooth core of its closure is $\rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}$.

Finally, by (71),

$$\begin{aligned}
A &= \frac{c}{1+c^2}\left(\left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right)\right.\\
&\qquad\left. + \left(-\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right)\right)\\
&= \pm\frac{2cs}{1+c^2}\left(\left[\frac{d}{d\rho}, \rho\right] - 1\right) = 0\,,\\
B &= \frac{c}{1+c^2}\left(\left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho\right)\left(-\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right)\right.\\
&\qquad\left. + \left(\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right)\left(\frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\right)\right)\\
&= \pm\frac{2cs}{1+c^2}\left(\left[\frac{d}{d\rho}, \rho\right] - 1\right) = 0\,.
\end{aligned}$$

When $\kappa > -1/2$, let $\mathcal{F}_1 \subset C^\infty(F) \cap L^2(F)$ be the dense graded linear subspace with

$$\mathcal{F}_1^0 = \rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,, \quad \mathcal{F}_1^1 = \rho^{\kappa}\,\mathcal{S}_{\mathrm{ev},+} \oplus \rho^{2+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,, \quad \mathcal{F}_1^2 = \rho^{1+\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,.$$

When $\kappa < -1/2$, let $\mathcal{F}_2 \subset C^\infty(F) \cap L^2(F)$ be the dense graded linear subspace with

$$\mathcal{F}_2^0 = \rho^{-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,, \quad \mathcal{F}_2^1 = \rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+} \oplus \rho^{-1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,, \quad \mathcal{F}_2^2 = \rho^{1-\kappa}\,\mathcal{S}_{\mathrm{ev},+}\,.$$

By restricting d and δ, we get complexes $(\mathcal{F}_1, d)$ and $(\mathcal{F}_1, \delta)$ when $\kappa > -1/2$, and complexes $(\mathcal{F}_2, d)$ and $(\mathcal{F}_2, \delta)$ when $\kappa < 1/2$. Thus Δ preserves $\mathcal{F}_1$ when $\kappa > -1/2$, and preserves $\mathcal{F}_2$ when $\kappa < -1/2$.

Proposition 11.6. *Suppose that $\kappa \neq -1/2$. Then (F, d) has a unique i.b.c., whose smooth core is $\mathcal{F}_1$ if $\kappa > -1/2$, and $\mathcal{F}_2$ if $\kappa < -1/2$.*

Proof. We prove only the case with $\kappa > -1/2$; the other case is analogous.

By Lemma 11.5 (using the independence of (ξ_n) on κ in its statement), we get $\mathcal{F}_1^0 \subset \mathcal{D}(d_{0,\min})$ and $\mathcal{F}_1^2 \subset \mathcal{D}(\delta_{1,\min})$. Then, by the discussion previous to this proposition, the other conditions of Lemma 11.1 are satisfied by the complexes defined by d and δ with $\mathcal{G} = \mathcal{F}_1$, obtaining that $\mathcal{F}_1^0$ and $\mathcal{F}_1^2$ are the smooth cores of $d_{0,\min}$ and $\delta_{1,\min}$, respectively. By Proposition 11.4 and since $1+\kappa, 2+\kappa > 1/2$, we get $d_{0,2,\min} = d_{0,2,\max}$ with smooth core $\mathcal{F}_1^0$, and $\delta_{2,2,\min} = \delta_{2,2,\max}$ with smooth core $\mathcal{F}_1^2$. So, according to the discussion previous to this proposition, the conditions of Corollary 11.3 are satisfied with d and δ, obtaining $d_{0,\min} = d_{0,\max}$ and $\delta_{1,\min} = \delta_{1,\max}$, which also gives $d_{1,\min} = d_{1,\max}$. □

4. Finite propagation speed of the wave equation

For the Hermitian bundle versions of E and F, consider the wave equation

$$\frac{du_t}{dt} - iDu_t = 0 \tag{97}$$

on any open subset of $\mathbb{R}_+$, where $D = d + \delta$ and u_t is in $C^\infty(E)$ or $C^\infty(F)$, depending smoothly on $t \in \mathbb{R}$.

PROPOSITION 11.7. *For $0 < a < b$, suppose that $u_t \in \mathcal{D}^\infty(d_{\min/\max})$, depending smoothly on $t \in \mathbb{R}$, satisfies* (97) *on* $(0, b)$. *The following properties hold:*

(i) *If* $\operatorname{supp} u_0 \subset [a, \infty)$, *then* $\operatorname{supp} u_t \subset [a - |t|, \infty)$ *for* $0 < |t| \le a$.
(ii) *If* $\operatorname{supp} u_0 \subset (0, a]$, *then* $\operatorname{supp} u_t \subset (0, a + |t|]$ *for* $0 < |t| \le b - a$.

PROOF. We prove Proposition 11.7 only for E; the proof is clearly analogous for F, but with more cases because F is of length three. Let $u_{t,0} \in C^\infty(E^0) \equiv C^\infty(\mathbb{R}_+)$ and $u_{t,1} \in C^\infty(E^1) \equiv C^\infty(\mathbb{R}_+)$ be the homogeneous components of u_t. From the description of the smooth core of $d_{\min/\max}$ in Proposition 11.4, it follows that

$$\lim_{\rho \downarrow 0} (u_{t,0}\, u_{t,1})(\rho) = 0 \ . \tag{98}$$

We have

$$\begin{aligned} \frac{d}{dt} \int_0^{a-t} |u_t(\rho)|^2 \, d\rho &= \int_0^{a-t} ((iDu_t, u_t) + (u_t, iDu_t))(\rho) \, d\rho - |u_t(a-t)|^2 \\ &= i \int_0^{a-t} ((Du_t, u_t) - (u_t, Du_t))(\rho) \, d\rho - |u_t(a-t)|^2 \ . \end{aligned}$$

But, since d and δ are respectively equal to $d/d\rho$ and $-d/d\rho$ up to the sum of multiplication operators by the same real valued functions,

$$\begin{aligned} (Du_t, u_t) - (u_t, Du_t) &= \frac{du_{t,0}}{dt} \cdot \overline{u_{t,1}} - \frac{du_{t,1}}{dt} \cdot \overline{u_{t,0}} - u_{t,1} \cdot \frac{d\overline{u_{t,0}}}{dt} + u_{t,0} \cdot \frac{d\overline{u_{t,1}}}{dt} \\ &= 2\, \Im \left(\frac{du_{t,0}}{d\rho} \cdot \overline{u_{t,1}} + u_{t,0} \cdot \frac{d\overline{u_{t,1}}}{d\rho} \right) = 2\, \Im \frac{d}{d\rho} (u_{t,0}\, \overline{u_{t,1}}) \ , \end{aligned}$$

giving

$$\begin{aligned} \left| \int_0^{a-t} ((Du_t, u_t) - (u_t, Du_t))(\rho) \, d\rho \right| &\le 2 \left| (u_{t,0}\, \overline{u_{t,1}})(a-t) - \lim_{\rho \downarrow 0} (u_{t,0}\, \overline{u_{t,1}})(\rho) \right| \\ &= 2 \left| (u_{t,0}\, \overline{u_{t,1}})(a-t) \right| \le |u_{t,0}(a-t)|^2 + |u_{t,1}(a-t)|^2 = |u_t(a-t)|^2 \end{aligned}$$

by (98). So

$$\frac{d}{dt} \int_0^{a-t} |u_t(\rho)|^2 \, d\rho \le 0 \ ,$$

giving

$$\int_0^{a-t} |u_t(\rho)|^2 \, d\rho \le \int_0^a |u_0(\rho)|^2 \, d\rho = 0 \ ,$$

and (i) follows.

Property (ii) can be proved with the same kind of arguments, but using that

$$\lim_{\rho \to \infty} u(\rho) = 0 \tag{99}$$

for all $u \in \mathcal{D}^\infty(d_{\min/\max})$ instead of (98). □

REMARK 22. The proof of Proposition 11.7 is an adaptation of [**54**, Proposition 7.20], where (98) and (99) are used to settle the lack of compact support.

CHAPTER 12

Preliminaries on Witten's perturbation of the de Rham complex

Let $M \equiv (M,g)$ be a Riemannian manifold of dimension n. For any $x \in M$ and any $\alpha \in T_xM^*$, let

$$\alpha\lrcorner = (-1)^{nr+n+1} \star \alpha\wedge \star \quad \text{on} \quad \bigwedge^r T_xM^* ,$$

involving the Hodge star operator $\star$ on $\bigwedge T_xM^*$ defined by any choice of orientation of T_xM. Writing $\alpha = g(X,\cdot)$ for $X \in T_xM$, we have $\alpha\lrcorner = -\iota_X$, where ι_X denotes the inner product by X. Moreover let

$$R_\alpha = \alpha\wedge\, - \alpha\lrcorner\,, \quad L_\alpha = \alpha\wedge\, + \alpha\lrcorner$$

on $\bigwedge T_xM^*$. Recall that there is an isomorphism between the underlying linear spaces of the exterior and Clifford algebras of T_xM^*,

$$\bigwedge T_xM^* \to \mathrm{Cl}(T_xM^*)\,, \quad e_{i_1}\wedge\cdots\wedge e_{i_r} \mapsto e_{i_1}\bullet\cdots\bullet e_{i_r}\,,$$

where $(e_1,\dots,e_n)$ is an orthonormal frame of T_xM^* and "$\bullet$" denotes Clifford multiplication. By this linear isomorphism, L_α and $R_\alpha\mathbf{w}$ correspond to left and right Clifford multiplication by α. So L_α and R_β anticommute for any $\alpha,\beta \in T_xM^*$. Any symmetric bilinear form $H \in T_xM^*\otimes T_xM^*$ induces an endomorphism $\mathbf{H}$ of $\bigwedge T_xM^*$ defined by

$$\mathbf{H} = \sum_{i,j=1}^{n} H(e_i,e_j)\, L_{e_i}\, R_{e_j}\,, \tag{100}$$

by using an orthonormal frame $(e_1,\dots,e_n)$ of T_xM^*. Observe that $|\mathbf{H}| = |H|$.

On the graded algebra of differential forms, $\Omega(M)$, let d and δ be the derivative and coderivative, let $D = d+\delta$ (the de Rham operator), and let $\Delta = D^2 = d\delta + \delta d$ (the Laplacian on differential forms). For any $f \in C^\infty(M)$, E. Witten [**68**] has introduced the following perturbations of the above operators, depending on a parameter $s \geq 0$:

$$d_s = e^{-sf}\, d\, e^{sf} = d + s\, df\wedge\,, \tag{101}$$

$$\delta_s = e^{sf}\, \delta\, e^{-sf} = \delta - s\, df\lrcorner\,, \tag{102}$$

$$D_s = d_s + \delta_s = D + sR\,,$$

$$\Delta_s = D_s^2 = d_s\delta_s + \delta_s d_s = \Delta + s(RD+DR) + s^2R^2\,, \tag{103}$$

where $R = R_{df}$. Notice that δ_s is the formal adjoint of d_s, and therefore D_s and Δ_s are formally self-adjoint.

The Hessian of f, with respect to g, is the smooth section of $TM^*\otimes TM^*$ defined by $\operatorname{Hess} f = \nabla df$, which is symmetric and induces an endomorphism $\mathbf{Hess}f$

of $\bigwedge TM^*$ according to (100). Then [**54**, Lemma 9.17]

$$RD + DR = \mathbf{Hess} f \ , \quad R^2 = |df|^2 \ ,$$

obtaining that (103) becomes

$$\Delta_s = \Delta + s\,\mathbf{Hess} f + s^2\,|df|^2 \ . \tag{104}$$

The Witten's perturbed operators also make sense with complex valued differential forms, and the above equalities hold as well.

EXAMPLE 12.1. Let $d^{\pm}_{0,s}$, $\delta^{\pm}_{0,s}$, $D^{\pm}_{0,s}$, $\Delta^{\pm}_{0,s}$ denote the Witten's perturbed operators on $\Omega(\mathbb{R}^m)$ defined by the model Morse function $\pm\frac{1}{2}\,\rho_0^2$ and the standard metric g_0. According to [**54**, Proposition 9.18 and the proof of Lemma 14.11], $\Delta^{\pm}_{0,s}$, with domain $\Omega_0(\mathbb{R}^m)$, is essentially self-adjoint in $L^2\Omega(\mathbb{R}^m, g_0)$, and its self-adjoint extension has a discrete spectrum of the following form:

- 0 is an eigenvalue of multiplicity one, and the corresponding eigenforms are of degree zero in the case of $\Delta^{+}_{0,s}$, and of degree m in the case of $\Delta^{-}_{0,s}$.
- Let $e^{\pm}_s$ be a 0-eigenform of $\Delta^{\pm}_{0,s}$ with norm one, and let h be a bounded measurable function on $\mathbb{R}^m$ such that $h(x) \to 1$ as $x \to 0$. Then $\langle h e^{\pm}_s, e^{\pm}_s \rangle \to 1$ as $s \to \infty$.
- All non-zero eigenvalues, as functions of s, are in $O(s)$ as $s \to \infty$.

Therefore $(\bigwedge T\mathbb{R}^{m*}, d^{\pm}_{0,s})$ has a unique i.b.c., which is discrete.

CHAPTER 13

Witten's perturbation on a cone

For our version of Morse functions, the local analysis of the Witten's perturbed Laplacian will be reduced to the case of the functions $\pm\frac{1}{2}\rho^2$ on a stratum of a cone with a model adapted metric, where ρ denotes the canonical function. That kind of local analysis begins in this section.

1. Laplacian on a cone

Let L be a non-empty compact Thom-Mather stratification, let ρ be the canonical function on $c(L)$, let N be a stratum of L of dimension $\tilde{n}$, let $M = N \times \mathbb{R}_+$ be the corresponding stratum of $c(L)$ with dimension $n = \tilde{n} + 1$, and let $\pi : M \to N$ denote the second factor projection. From $\bigwedge TM^* = \bigwedge TN^* \boxtimes \bigwedge T\mathbb{R}_+^*$, we get a canonical identity

$$\bigwedge^r TM^* \equiv \pi^* \bigwedge^r TN^* \oplus d\rho \wedge \pi^* \bigwedge^{r-1} TN^* \equiv \pi^* \bigwedge^r TN^* \oplus \pi^* \bigwedge^{r-1} TN^* \tag{105}$$

for each degree r, obtaining

$$\Omega^r(M) \equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus d\rho \wedge C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) \tag{106}$$

$$\equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) \; . \tag{107}$$

Here, smooth functions $\mathbb{R}_+ \to \Omega(N)$ are defined by considering $\Omega(N)$ as Fréchet space with the weak C^∞ topology. Let d and $\tilde{d}$ denote the exterior derivatives on $\Omega(M)$ and $\Omega(N)$, respectively. The following lemma is elementary.

Lemma 13.1. *According to* (107),

$$d \equiv \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} & -\tilde{d} \end{pmatrix} \; .$$

Fix an adapted metric $\tilde{g}$ on N, and let $g = \rho^2\tilde{g} + (d\rho)^2$ be the corresponding adapted metric on M. The induced metrics on $\bigwedge TM^*$ and $\bigwedge TN^*$ are also denoted by g and $\tilde{g}$, respectively. According to (105),

$$g = \rho^{-2r}\, \tilde{g} \oplus \rho^{-2(r-1)}\, \tilde{g} \tag{108}$$

on $\bigwedge^r TM^*$.

Given an orientation on an open subset $W \subset N$, and denoting by $\tilde{\omega}$ the corresponding $\tilde{g}$-volume form on W, consider the orientation on $W \times \mathbb{R}_+ \subset M$ so that the corresponding g-volume form is

$$\omega = \rho^{n-1}\, d\rho \wedge \tilde{\omega} \; . \tag{109}$$

The corresponding star operators on $\bigwedge T(W \times \mathbb{R}_+)^*$ and $\bigwedge TW^*$ will be denoted by $\star$ and $\tilde{\star}$, respectively.

LEMMA 13.2. *According to* (105),

$$\star \equiv \begin{pmatrix} 0 & \rho^{n-2r+1}\tilde{\star} \\ (-1)^r \rho^{n-2r-1}\tilde{\star} & 0 \end{pmatrix}$$

on $\bigwedge^r T(W \times \mathbb{R}_+)^*$.

PROOF. Let $\alpha, \alpha' \in \pi^* \bigwedge TN^*$, at the same point $(\rho, x) \in \mathbb{R}_+ \times W$. If α and α' are of degree r, then

$$\begin{aligned} \alpha' \wedge \rho^{n-2r-1}\, d\rho \wedge \tilde{\star}\alpha &= (-1)^r \rho^{n-2r-1}\, d\rho \wedge \alpha' \wedge \tilde{\star}\alpha \\ &= (-1)^r \rho^{n-2r-1} \tilde{g}(\alpha', \alpha)\, d\rho \wedge \tilde{\omega} = (-1)^r g(\alpha', \alpha)\, \omega \end{aligned}$$

by (108) and (109), giving $\star\alpha = (-1)^r \rho^{n-2r-1} d\rho \wedge \tilde{\star}\alpha$. Similarly, if α and α' are of degree $r-1$, then

$$d\rho \wedge \alpha' \wedge \rho^{n-2r+1}\tilde{\star}\alpha = \rho^{n-2r+1}\tilde{g}(\alpha', \alpha)\, d\rho \wedge \tilde{\omega} = g(d\rho \wedge \alpha', d\rho \wedge \alpha)\, \omega\ ,$$

obtaining $\star(d\rho \wedge \alpha) = \rho^{n-2r+1}\tilde{\star}\alpha$. □

Let $L^2\Omega^r(M, g)$ and $L^2\Omega^r(N, \tilde{g})$ be simply denoted by $L^2\Omega^r(M)$ and $L^2\Omega^r(N)$. From (108) and (109), it follows that (107) induces a unitary isomorphism

$$\begin{aligned} L^2\Omega^r(M) \cong (L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho) \,\widehat{\otimes}\, L^2\Omega^r(N)) \\ \oplus (L^2(\mathbb{R}_+, \rho^{n-2r+1}\, d\rho) \,\widehat{\otimes}\, L^2\Omega^{r-1}(N))\ , \quad (110) \end{aligned}$$

which will be considered as an identity.

Let δ and $\tilde{\delta}$ denote the exterior coderivatives on $\Omega(M)$ and $\Omega(N)$, respectively.

LEMMA 13.3. *According to* (107),

$$\delta \equiv \begin{pmatrix} \rho^{-2}\,\tilde{\delta} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \\ 0 & -\rho^{-2}\,\tilde{\delta} \end{pmatrix}$$

on $\Omega^r(M)$.

PROOF. For an oriented open subset $W \subset N$, consider the orientation on $W \times \mathbb{R}_+$ defined as above, and let $\star$ and $\tilde{\star}$ denote the corresponding star operators on $\bigwedge T(W \times \mathbb{R}_+)^*$ and $\bigwedge TW^*$. By Lemmas 13.1 and 13.2, on $\Omega^r(W \times \mathbb{R}_+)$,

$$\begin{aligned} \delta &= (-1)^{nr+n+1} \star d \star \\ &\equiv (-1)^{nr+n+1} \begin{pmatrix} 0 & \rho^{-n+2r-1}\tilde{\star} \\ (-1)^{n-r+1}\rho^{-n+2r-3}\tilde{\star} & 0 \end{pmatrix} \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} & -\tilde{d} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & \rho^{n-2r+1}\tilde{\star} \\ (-1)^r \rho^{n-2r-1}\tilde{\star} & 0 \end{pmatrix} \\ &= (-1)^{nr+n+1} \begin{pmatrix} -(-1)^r \rho^{-2}\tilde{\star}\tilde{d}\tilde{\star} & \rho^{-n+2r-1}\, \frac{d}{d\rho}\, \rho^{n-2r+1}\tilde{\star}^2 \\ 0 & (-1)^{n-r+1}\rho^{-2}\tilde{\star}\tilde{d}\tilde{\star} \end{pmatrix} \\ &= \begin{pmatrix} \rho^{-2}\tilde{\delta} & -\rho^{-n+2r-1}\, \frac{d}{d\rho}\, \rho^{n-2r+1} \\ 0 & -\rho^{-2}\tilde{\delta} \end{pmatrix}\ , \end{aligned}$$

which equals the matrix of the statement by (71). □

Let Δ and $\widetilde{\Delta}$ denote the Laplacians on $\Omega(M)$ and $\Omega(N)$, respectively.

COROLLARY 13.4. *According to* (107),

$$\Delta \equiv \begin{pmatrix} P & -2\rho^{-1}\,\tilde{d} \\ -2\rho^{-3}\,\tilde{\delta} & Q \end{pmatrix}$$

on $\Omega^r(M)$, *where*

$$\begin{aligned} P &= \rho^{-2}\,\widetilde{\Delta} - \frac{d^2}{d\rho^2} - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho}\ , \\ Q &= \rho^{-2}\,\widetilde{\Delta} - \frac{d^2}{d\rho^2} - (n-2r+1)\,\frac{d}{d\rho}\,\rho^{-1}\ . \end{aligned}$$

PROOF. By Lemmas 13.1 and 13.3,

$$\begin{aligned} \delta d &\equiv \begin{pmatrix} \rho^{-2}\,\tilde{\delta} & -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \\ 0 & -\rho^{-2}\,\tilde{\delta} \end{pmatrix} \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} & -\tilde{d} \end{pmatrix} \\ &= \begin{pmatrix} \rho^{-2}\,\tilde{\delta}\tilde{d} - \frac{d^2}{d\rho^2} - (n-2r-1)\rho^{-1}\frac{d}{d\rho} & (\frac{d}{d\rho} + (n-2r-1)\rho^{-1})\tilde{d} \\ -\rho^{-2}\tilde{\delta}\frac{d}{d\rho} & \rho^{-2}\,\tilde{\delta}\tilde{d} \end{pmatrix}\ , \\ d\delta &\equiv \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} & -\tilde{d} \end{pmatrix} \begin{pmatrix} \rho^{-2}\,\tilde{\delta} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \\ 0 & -\rho^{-2}\,\tilde{\delta} \end{pmatrix} \\ &= \begin{pmatrix} \rho^{-2}\,\tilde{d}\tilde{\delta} & -\tilde{d}(\frac{d}{d\rho} + (n-2r+1)\rho^{-1}) \\ \frac{d}{d\rho}\rho^{-2}\tilde{\delta} & -\frac{d^2}{d\rho^2} - (n-2r+1)\frac{d}{d\rho}\rho^{-1} + \rho^{-2}\,\tilde{d}\tilde{\delta} \end{pmatrix} \\ &= \begin{pmatrix} \rho^{-2}\,\tilde{d}\tilde{\delta} & -\tilde{d}(\frac{d}{d\rho} + (n-2r+1)\rho^{-1}) \\ \rho^{-2}\frac{d}{d\rho}\tilde{\delta} - 2\rho^{-3}\tilde{\delta} & -\frac{d^2}{d\rho^2} - (n-2r+1)\frac{d}{d\rho}\rho^{-1} + \rho^{-2}\,\tilde{d}\tilde{\delta} \end{pmatrix}\ . \end{aligned}$$

The sum of these matrices is the matrix of the statement. □

2. Witten's perturbation on a cone

Let $d_s^{\pm}$, $\delta_s^{\pm}$, $D_s^{\pm}$ and $\Delta_s^{\pm}$ ($s \geq 0$) denote the Witten's perturbations of d, δ, D and Δ induced by the function $f = \pm\frac{1}{2}\rho^2$ on M. In this case, $df = \pm\rho\,d\rho$. According to (107),

$$\rho\,d\rho\wedge \equiv \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix}\ , \quad -\rho\,d\rho\lrcorner \equiv \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix}\ .$$

So the following is a consequence of Lemmas 13.1 and 13.3, (101) and (102).

COROLLARY 13.5. *According to* (107),

$$\begin{aligned} d_s^{\pm} &\equiv \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} \pm s\rho & -\tilde{d} \end{pmatrix}\ , \\ \delta_s^{\pm} &\equiv \begin{pmatrix} \rho^{-2}\,\tilde{\delta} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \pm s\rho \\ 0 & -\rho^{-2}\,\tilde{\delta} \end{pmatrix} \end{aligned}$$

on $\Omega^r(M)$.

With the notation of Chapter 12,

$$R = \pm\rho(d\rho\wedge - d\rho\lrcorner) \equiv \pm\begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}\ ,$$

and therefore

$$R^2 \equiv \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \equiv \rho^2 \; . \tag{111}$$

LEMMA 13.6. *$RD + DR = \pm(2r-n)$ on $\Omega^r(M)$.*

PROOF. By Lemmas 13.1 and 13.3, and according to (107),

$$\begin{aligned}
RD &\equiv \pm \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} \begin{pmatrix} \tilde{d} + \rho^{-2}\tilde{\delta} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \\ \frac{d}{d\rho} & -\tilde{d} - \rho^{-2}\tilde{\delta} \end{pmatrix} \\
&= \pm \begin{pmatrix} \rho \, \frac{d}{d\rho} & -\rho\tilde{d} - \rho^{-1}\tilde{\delta} \\ \rho \, \tilde{d} + \rho^{-1}\tilde{\delta} & -\rho \, \frac{d}{d\rho} - n + 2r - 1 \end{pmatrix} \; , \\
DR &\equiv \pm \begin{pmatrix} \tilde{d} + \rho^{-2} \, \tilde{\delta} & -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \\ \frac{d}{d\rho} & -\tilde{d} - \rho^{-2} \, \tilde{\delta} \end{pmatrix} \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} \\
&= \pm \begin{pmatrix} -\frac{d}{d\rho} \, \rho - n + 2r + 1 & \rho \, \tilde{d} + \rho^{-1} \, \tilde{\delta} \\ -\rho \, \tilde{d} - \rho^{-1}\tilde{\delta} & \frac{d}{d\rho} \, \rho \end{pmatrix} \; .
\end{aligned}$$

So

$$RD + DR \equiv \pm \begin{pmatrix} 2r - n & 0 \\ 0 & 2r - n \end{pmatrix} \equiv \pm(2r-n)$$

by (71). □

REMARK 23. The expression of $RD + DR$ can be also obtained by computing $\operatorname{Hess} f$ (Chapter 12).

The following is a consequence of (104), Corollary 13.4 and Lemma 13.6.

COROLLARY 13.7. *According to* (107),

$$\Delta_s^{\pm} \equiv \begin{pmatrix} P_s^{\pm} & -2u\rho^{-1}\tilde{d} \\ -2\rho^{-3}\tilde{\delta} & Q_s^{\pm} \end{pmatrix}$$

on $\Omega^r(M)$, where

$$\begin{aligned}
P_s^{\pm} &= \rho^{-2}\widetilde{\Delta} + H - (n-2r-1)\rho^{-1} \, \frac{d}{d\rho} \mp s(n-2r) \; , \\
Q_s^{\pm} &= \rho^{-2}\widetilde{\Delta} + H - (n-2r+1)\rho^{-1} \, \frac{d}{d\rho} + (n-2r+1)\rho^{-2} \mp s(n-2r) \; .
\end{aligned}$$

CHAPTER 14

Domains of the Witten's Laplacian on a cone

Theorem I is proved by induction on the dimension. Thus, with the notation of Chapter 13, suppose that $\tilde{d}_{\min/\max}$ satisfies the statement of Theorem I. Let

$$\widetilde{\mathcal{H}}_{\min/\max} = \ker \widetilde{D}_{\min/\max} = \ker \widetilde{\Delta}_{\min/\max} \;,$$

which is a graded subspace of $\Omega(N)$. For each degree r, let

$$\widetilde{\mathcal{R}}_{\min/\max,r-1}, \widetilde{\mathcal{R}}^*_{\min/\max,r} \subset L^2\Omega^r(N)$$

be the images of $\tilde{d}_{\min/\max,r-1}$ and $\tilde{\delta}_{\min/\max,r}$, respectively, whose intersections with $\mathcal{D}^\infty(\widetilde{\Delta})$ are denoted by $\widetilde{\mathcal{R}}^\infty_{\min/\max,r-1}$ and $\widetilde{\mathcal{R}}^{*\infty}_{\min/\max,r}$. According to Section 1 of Chapter 9, $\widetilde{\Delta}$ preserves $\widetilde{\mathcal{R}}^\infty_{\min/\max,r-1}$ and $\widetilde{\mathcal{R}}^{*\infty}_{\min/\max,r}$, and its restrictions to these spaces have the same eigenvalues. For any eigenvalue $\tilde{\lambda}$ of the restriction of $\widetilde{\Delta}$ to $\widetilde{\mathcal{R}}^\infty_{\min/\max,r-1}$, let

$$\widetilde{\mathcal{R}}_{\min/\max,r-1,\tilde{\lambda}} = E_{\tilde{\lambda}}(\widetilde{\Delta}_{\min/\max}) \cap \widetilde{\mathcal{R}}^\infty_{\min/\max,r-1} \;,$$
$$\widetilde{\mathcal{R}}^*_{\min/\max,r,\tilde{\lambda}} = E_{\tilde{\lambda}}(\widetilde{\Delta}_{\min/\max}) \cap \widetilde{\mathcal{R}}^{*\infty}_{\min/\max,r} \;.$$

Moreover

$$L^2\Omega^r(N) = \widetilde{\mathcal{H}}^r_{\min/\max} \oplus \widehat{\bigoplus_{\tilde{\lambda}}} \left(\widetilde{\mathcal{R}}_{\min/\max,r-1,\tilde{\lambda}} \oplus \widetilde{\mathcal{R}}^*_{\min/\max,r,\tilde{\lambda}}\right) \;, \tag{112}$$

where $\tilde{\lambda}$ runs in the spectrum of $\widetilde{\Delta}_{\min/\max}$ on $\widetilde{\mathcal{R}}^\infty_{\min/\max,r-1}$; i.e., the positive spectrum of $\widetilde{\Delta}_{\min/\max,r}$.

Now, consider the Witten's perturbed Laplacian $\Delta^\pm_s$. In the following, suppose that $s > 0$.

1. Domains of first type

For some degree r, let $0 \neq \gamma \in \widetilde{\mathcal{H}}^r_{\min/\max}$. By Corollary 13.7,

$$\Delta^\pm_s \equiv H - (n-2r-1)\rho^{-1}\frac{d}{d\rho} \mp s(n-2r)$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\gamma \subset \Omega^r(M)$. This operator is of the type of P in (1) with $c_2 = 0$. Thus (72) is satisfied, and (4) means that $a \in \{0, -n+2r+2\}$.

For $a = 0$, we have $2\sigma = n - 2r - 1$. When $\sigma > -1/2$, which means $r \leq \frac{n-1}{2}$, Corollary H asserts that $\Delta^\pm_s$, with domain $\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + (1 \mp 1)(n-2r))s \tag{113}$$

of multiplicity one, with corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (113) becomes $4ks$, which is ≥ 0 for all k and $= 0$ just for $k = 0$. For Δ_s^-, (113) becomes $(4k + 2(n - 2r))s$, which is > 0 for all k.

For $a = -n + 2r + 2$, we have $2\sigma = -n + 2r + 3$. When $\sigma > -1/2$, which means $r \geq \frac{n-3}{2}$, Corollary H asserts that $\Delta_s^\pm$, with domain $\rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + 4 - (1 \pm 1)(n - 2r))s \tag{114}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (114) becomes $(4k + 4 - 2(n - 2r))s$, which is:

- > 0 for all k if $r \geq \frac{n-1}{2}$,
- ≥ 0 for all k and $= 0$ just for $k = 0$ if $r = \frac{n}{2} - 1$, and
- < 0 for $k = 0$ if $r = \frac{n-3}{2}$.

For Δ_s^-, (114) becomes $(4k + 4)s$, which are > 0 for all k.

When $\frac{n-3}{2} \leq r \leq \frac{n-1}{2}$, we have got two essentially self-adjoint operators, with $a = 0$ and $a = -n + 2r + 2$. These two operators are equal just when $r = \frac{n}{2} - 1$.

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *first type*.

2. Domains of second type

With the notation of Section 1,

$$\Delta_s^\pm \equiv H - (n - 2r - 1)\rho^{-1}\,\frac{d}{d\rho} + (n - 2r - 1)\rho^{-2} \mp s(n - 2r - 2)$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,d\rho \wedge \gamma \subset \Omega^{r+1}(M)$ by Corollary 13.7. This is an operator of the type of P in (1) with $c_2 = c_1$. Thus (72) is also satisfied, and (4) becomes $a \in \{1, -n + 2r + 1\}$.

For $a = 1$, we have $2\sigma = n - 2r + 1$ according to (5). When $\sigma > -1/2$, which means $r \leq \frac{n+1}{2}$, Corollary H asserts that $\Delta_s^\pm$, with domain $\rho\,\mathcal{S}_{\mathrm{ev},+} = \mathcal{S}_{\mathrm{odd},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k + 4 + (1 \mp 1)(n - 2r - 2))s \tag{115}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (115) is > 0 for all k. For Δ_s^-, (115) is:

- > 0 for all k if $r \leq \frac{n-1}{2}$,
- ≥ 0 for all k and $= 0$ just for $k = 0$ if $r = \frac{n}{2}$, and
- < 0 for $k = 0$ if $r = \frac{n+1}{2}$.

For $a = -n+2r+1$, we have $2\sigma = -n+2r+1$ according to (5). When $\sigma > -1/2$, which means $r \geq \frac{n-1}{2}$, Corollary H asserts that $\Delta_s^\pm$, with domain $\rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$(4k - (1 \pm 1)(n - 2r - 2))s \tag{116}$$

of multiplicity one, with normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (116) is > 0 for all k. For Δ_s^-, (116) is ≥ 0 for all k and $= 0$ just for $k = 0$.

For $\frac{n-1}{2} \leq r \leq \frac{n+1}{2}$, we have obtained two essentially self-adjoint operators, with $a = 1$ and $a = -n + 2r + 1$. These operators are equal just when $r = \frac{n}{2}$.

All of the above operators defined by $\Delta_s^{\pm}$, as well as their domains, will be said to be of *second type*.

3. Domains of third type

Let $\mu = \sqrt{\tilde{\lambda}}$ for an eigenvalue $\tilde{\lambda}$ of the restriction of $\widetilde{\Delta}_{\text{min/max}}$ to $\widetilde{\mathcal{R}}^{\infty}_{\text{min/max},r-1}$. According to Section 1 of Chapter 9, there are non-zero differential forms,

$$\alpha \in \widetilde{\mathcal{R}}_{\text{min/max},r-1,\lambda} \subset \Omega^r(N) , \quad \beta \in \widetilde{\mathcal{R}}^*_{\text{min/max},r-1,\lambda} \subset \Omega^{r-1}(N) ,$$

such that $\tilde{d}\beta = \mu\alpha$ and $\tilde{\delta}\alpha = \mu\beta$. By Corollary 13.7,

$$\Delta_s^{\pm} \equiv -\frac{d^2}{d\rho^2} - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + \mu^2\rho^{-2} \mp (n-2r+2)s$$

such that $\tilde{d}\beta = \mu\alpha$ and $\tilde{\delta}\alpha = \mu\beta$. By Corollary 13.7,

$$\Delta_s^{\pm} \equiv -\frac{d^2}{d\rho^2} - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + \mu^2\rho^{-2} \mp (n-2r+2)s$$

on $C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\beta \subset \Omega^{r-1}(M)$. This operator is of the type of P in (1) with $c_2 = \mu^2 > 0$. Thus (72) is satisfied, and (4) becomes

$$a = \frac{-n + 2r \pm \sqrt{(n-2r)^2 + 4\mu^2}}{2} . \tag{117}$$

These two possibilities for a have different sign because $\mu > 0$.

For the choice of positive square root in (117), we get

$$\sigma = \frac{1 + \sqrt{(n-2r)^2 + 4\mu^2}}{2} > \frac{1}{2} \tag{118}$$

according to (5). Then Corollary H asserts that $\Delta_s^{\pm}$, with domain $\rho^a\,\mathcal{S}_{\text{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k + 2 + \sqrt{(n-2r)^2 + 4\mu^2} \mp (n-2r+2)\right) s , \tag{119}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\text{ev},+}$. Notice that (119) is > 0 for all k.

For the choice of negative square root in (117), we get

$$\sigma = \frac{1 - \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{120}$$

according to (5). Then $\sigma > -1/2$ if and only if

$$\mu < 1 \quad \text{and} \quad |n - 2r| < 2\sqrt{1-\mu^2} , \tag{121}$$

which is equivalent to $\frac{\sqrt{3}}{2} \leq \mu < 1$ and $r = \frac{n}{2}$, or $\mu < \frac{\sqrt{3}}{2}$ and $\frac{n-1}{2} \leq r \leq \frac{n+1}{2}$. In this case, Corollary H asserts that $\Delta_s^{\pm}$, with domain $\rho^a\,\mathcal{S}_{\text{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, \rho^{n-2r+1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k + 2 - \sqrt{(n-2r)^2 + 4\mu^2} \mp (n-2r+2)\right) s , \tag{122}$$

with multiplicity one and corresponding normalized eigenfunctions $\rho^a\,\phi_{2k,+}$; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (122) is <0 for $k=0$. For Δ_s^-, (122) is >0 for all k.

When (121) is satisfied, we have got two different essentially self-adjoint operators defined by the two different choices of a in (117).

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *third type*.

4. Domains of fourth type

Let μ, α and β be like in Section 3. By Corollary 13.7,

$$\begin{aligned}\Delta_s^\pm \equiv &-\frac{d^2}{d\rho^2}+s^2\rho^2-(n-2r-1)\rho^{-1}\,\frac{d}{d\rho}+(\mu^2+n-2r-1)\rho^{-2}\\ &\mp(n-2r-2)s\end{aligned}$$

on $C^\infty(\mathbb{R}_+)\equiv C^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha\subset\Omega^{r+1}(M)$. This is another operator of the type of P in (1), which satisfies (72) because

$$(1-(n-2r-1))^2+4(\mu^2+n-2r-1)=(n-2r)^2+4\mu^2>0\;.$$

Moreover (4) becomes

$$a=\frac{-n+2r+2\pm\sqrt{(n-2r)^2+4\mu^2}}{2}\;. \tag{123}$$

These two possibilities for a are different because $\mu>0$.

With the choice of positive square root in (123) and according to (5), σ is also given by (118), which is $>1/2$. Then Corollary H asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k+2+\sqrt{(n-2r)^2+4\mu^2}\mp(n-2r-2)\right)s\;, \tag{124}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. Observe that (124) is >0 for all k.

With the choice of negative square root in (123) and according to (5), σ is also given by (120), which is $>-1/2$ if and only if (121) is satisfied. In this case, Corollary H asserts that $\Delta_s^\pm$, with domain $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)$; the spectrum of its closure consists of the eigenvalues

$$\left(4k+2-\sqrt{(n-2r)^2+4\mu^2}\mp(n-2r-2)\right)s\;, \tag{125}$$

with multiplicity one and corresponding normalized eigenfunctions χ_k; and the smooth core of its closure is $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$. For Δ_s^+, (125) is >0 for all k. For Δ_s^-, (125) is <0 for $k=0$.

When (121) is satisfied, we have got two different essentially self-adjoint operators defined by the two different choices of a in (123).

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *fourth type*.

5. Domains of fifth type

Let μ, α and β be like in Sections 3 and 4. By Corollary 13.7,

$$\Delta_s^\pm \equiv \begin{pmatrix} P_{\mu,s}^\pm & -2\rho^{-1}\mu \\ -2\rho^{-3}\mu & Q_{\mu,s}^\pm \end{pmatrix}$$

on

$$C^\infty(\mathbb{R}_+) \oplus C^\infty(\mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+)\,\alpha + C^\infty(\mathbb{R}_+)\,d\rho\wedge\beta \subset \Omega^r(M)\ ,$$

where

$$\begin{aligned} P_{\mu,s}^\pm &= H - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + \mu^2\rho^{-2} \mp (n-2r)s\ ,\\ Q_{\mu,s}^\pm &= H - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + (\mu^2+n-2r+1)\rho^{-2} \mp (n-2r)s\ . \end{aligned}$$

We will conjugate this matrix expression of $\Delta_s^\pm$ by some non-singular matrix Θ, whose entries are functions of ρ, to get a diagonal matrix whose diagonal entries are operators of the type of P in (1). This matrix will be of the form $\Theta = BC$ with

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}\ ,\quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\ ,$$

where c_{ij} are constants to be determined. Let $P_{\mu,s}^\pm$ and $Q_{\mu,s}^\pm$ be simply denoted by P and Q. A key observation here is that, by (71),

$$\begin{aligned} Q - \rho^{-1}\,P\,\rho &= -\frac{d^2}{d\rho^2} - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + (n-2r+1)\rho^{-2}\\ &\quad + \rho^{-1}\,\frac{d^2}{d\rho^2}\,\rho + (n-2r-1)\rho^{-2}\,\frac{d}{d\rho}\,\rho\\ &= -\frac{d^2}{d\rho^2} - (n-2r+1)\rho^{-1}\,\frac{d}{d\rho} + (n-2r+1)\rho^{-2}\\ &\quad + \frac{d^2}{d\rho^2} + 2\frac{d}{d\rho} + (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + (n-2r-1)\rho^{-2}\\ &= 2(n-2r)\rho^{-2}\ , \end{aligned}$$

obtaining

$$\begin{aligned} B^{-1}\,\Delta_s^\pm\,B &= \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}\begin{pmatrix} P & -2\mu\,\rho^{-1} \\ -2\mu\,\rho^{-3} & Q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}\\ &= \begin{pmatrix} P & -2\mu\rho^{-2} \\ -2\mu\rho^{-2} & \rho\,Q\,\rho^{-1} \end{pmatrix}\\ &= \begin{pmatrix} P & -2\mu\rho^{-2} \\ -2\mu\rho^{-2} & P + 2(n-2r)\rho^{-2} \end{pmatrix}\ . \end{aligned}$$

On the other hand, C must be non-singular and

$$C^{-1} = \frac{1}{\det C}\begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix}\ .$$

Therefore $\Theta^{-1}\Delta_s^{\pm}\Theta = (X_{ij})$ with

$$\begin{aligned} X_{11} &= P + \frac{2}{\det C}\,\left(\mu\,(-c_{22}c_{21} + c_{12}c_{11}) - (n-2r)c_{12}c_{21}\right)\rho^{-2}\,, \\ X_{12} &= \frac{2}{\det C}\,\left(\mu\,(-c_{22}^2 + c_{12}^2) - (n-2r)c_{12}c_{22}\right)\rho^{-2}\,, \\ X_{21} &= \frac{2}{\det C}\,\left(\mu\,(c_{21}^2 - c_{11}^2) + (n-2r)c_{11}c_{21}\right)\rho^{-2}\,, \\ X_{22} &= P + \frac{2}{\det C}\,\left(\mu(c_{21}c_{22} - c_{11}c_{12}) + (n-2r)c_{11}c_{22}\right)\rho^{-2}\,. \end{aligned}$$

We want (X_{ij}) to be diagonal, so we require

$$\mu(c_{12}^2 - c_{22}^2) - (n-2r)c_{12}c_{22} = \mu(c_{11}^2 - c_{21}^2) - (n-2r)c_{11}c_{21} = 0\,.$$

Both of these equations are of the form

$$\mu(x^2 - y^2) - (n-2r)xy = 0\,, \tag{126}$$

with $x = c_{12}$ and $y = c_{22}$ in the first equation, and $x = c_{11}$ and $y = c_{21}$ in the second one. There is some $c \in \mathbb{R} \smallsetminus \{0\}$ such that

$$x^2 - y^2 - \frac{n-2r}{\mu}\,xy = (x+cy)\left(x - \frac{y}{c}\right)\,. \tag{127}$$

In fact, since

$$(x+cy)\left(x - \frac{y}{c}\right) = x^2 - y^2 + \left(c - \frac{1}{c}\right)xy\,,$$

we need

$$c - \frac{1}{c} = -\frac{n-2r}{\mu}\,,$$

giving

$$\mu c^2 + (n-2r)c - \mu = 0\,, \tag{128}$$

whose solutions are

$$c_{\pm} = \frac{-n + 2r \pm \sqrt{(n-2r)^2 + 4\mu^2}}{2\mu}\,. \tag{129}$$

Observe that $c_+c_- = -1$. Let $c = c_+ > 0$, and therefore $-1/c = c_-$. By (127), the solutions of (126) are given by $x + cy = 0$ and $cx - y = 0$. Then we can take

$$C = \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix}\,,$$

with $\det C = 1 + c^2 > 0$. So, for

$$\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}\begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & -c \\ c\rho^{-1} & \rho^{-1} \end{pmatrix}\,,$$

we get $X_{12} = X_{21} = 0$, and

$$\begin{aligned} X_{11} &= P + \frac{2(-2\mu c + (n-2r)c^2)}{1+c^2}\,\rho^{-2}\,, \\ X_{22} &= P + \frac{2(2\mu c + n - 2r)}{1+c^2}\,\rho^{-2}\,. \end{aligned}$$

The notation $X = X_{11}$ and $Y = X_{22}$ will be used; thus $\Theta^{-1}\Delta_s^{\pm}\Theta = X \oplus Y$. The above expressions of X and Y can be simplified as follows. We have

$$1 + c^2 = 2 - \frac{n-2r}{\mu}\,c = \frac{2\mu - (n-2r)c}{\mu}$$

by (128), obtaining

$$\frac{2(-2\mu c + (n-2r)c^2)}{1+c^2} = \frac{2\mu c(-2\mu + (n-2r)c)}{2\mu - (n-2r)c} = -2\mu c\ ,$$
$$\frac{2(2\mu c + n - 2r)}{1+c^2} = \frac{2\mu(2\mu c + n - 2r)}{2\mu - (n-2r)c}\ .$$

Moreover

$$(2\mu c + n - 2r)^2 = (n-2r)^2 + 4\mu^2 > 0$$

by (129), and

$$\begin{aligned}(2\mu - (n-2r)c)&(2\mu c + n - 2r)\\ &= 4\mu^2 c + 2\mu(n-2r) - (n-2r)2\mu c^2 - (n-2r)^2 c\\ &= 4\mu^2 c + 2\mu(n-2r) - (n-2r)2\mu(1 - \frac{n-2r}{\mu}\,c) - (n-2r)^2 c\\ &= c(4\mu^2 c + (n-2r)2)\end{aligned}$$

by (128). Therefore

$$\begin{aligned}\frac{2(2\mu c + n - 2r)}{1+c^2} &= \frac{2\mu(2\mu c + n - 2r)^2}{(2\mu - (n-2r)c)(2\mu c + n - 2r)}\\ &= \frac{2\mu((n-2r)^2 + 4\mu^2)}{c(4\mu^2 c + (n-2r)2)} = \frac{2\mu}{c}\ .\end{aligned}$$

It follows that

$$\begin{aligned}X &= P - 2\mu c\rho^{-2}\\ &= H + s^2\rho^2 - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + (\mu^2 - 2\mu c)\rho^{-2} \mp (n-2r)s\ ,\\ Y &= P + \frac{2\mu}{c}\rho^{-2}\\ &= H + s^2\rho^2 - (n-2r-1)\rho^{-1}\,\frac{d}{d\rho} + (\mu^2 + \frac{2\mu}{c})\rho^{-2} \mp (n-2r)s\ .\end{aligned}$$

These operators are of the type of P in (1), and satisfy (72) because

$$\begin{aligned}(1 - (n-2r-1))^2 &+ 4(\mu^2 - 2\mu c)\\ &= 4 + (n-2r)^2 + 4\mu^2 - 4\sqrt{(n-2r)^2 + 4\mu^2}\\ &= (2 - \sqrt{(n-2r)^2 + 4\mu^2})^2 \geq 0\end{aligned}$$

and

$$\begin{aligned}(1 - (n-2r-1))^2 &+ 4(\mu^2 + \frac{2\mu}{c})\\ &= 4 + (n-2r)^2 + 4\mu^2 + 4\sqrt{(n-2r)^2 + 4\mu^2})\\ &= (2 + \sqrt{(n-2r)^2 + 4\mu^2})^2 > 0\ .\end{aligned}$$

So, for X and Y, the constants (4) and (5) become

$$a = \frac{2-n+2r \pm (2-\sqrt{(n-2r)^2+4\mu^2})}{2}\ , \tag{130}$$

$$b = \frac{2-n+2r \pm (2+\sqrt{(n-2r)^2+4\mu^2})}{2}\ , \tag{131}$$

$$\sigma = \frac{1 \pm (2-\sqrt{(n-2r)^2+4\mu^2})}{2}\ , \tag{132}$$

$$\tau = \frac{1 \pm (2+\sqrt{(n-2r)^2+4\mu^2})}{2}\ . \tag{133}$$

Suppose that $\sigma,\tau > -1/2$. By Corollary H, X and Y, with respective domains $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$ and $\rho^b\,\mathcal{S}_{\mathrm{ev},+}$, are essentially self-adjoint in $L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)$; the spectra of their closures consist of the eigenvalues

$$(4k+2a+(1\mp 1)(n-2r))s\ , \tag{134}$$

$$(4k+2+2b+(1\mp 1)(n-2r))s\ , \tag{135}$$

with multiplicity one and corresponding normalized eigenfunctions $\chi_{s,a,\sigma,k}$ and $\chi_{s,b,\tau,k}$, respectively, and the smooth cores of their closures are $\rho^a\,\mathcal{S}_{\mathrm{ev},+}$ and $\rho^b\,\mathcal{S}_{\mathrm{ev},+}$.

Since $\frac{1}{\sqrt{1+c^2}}C$ is an orthogonal matrix, it defines a unitary isomorphism

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\ ,$$

and we already know that

$$B = 1\oplus\rho^{-1} : L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)$$

is a unitary isomorphism too. So $\frac{1}{\sqrt{1+c^2}}\Theta$ is a unitary isomorphism

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\ .$$

Therefore, when $\sigma,\tau > -1/2$, the operator $\Delta_s^{\pm}$, with domain

$$\Theta(\rho^a\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^b\,\mathcal{S}_{\mathrm{ev},+}) = \{(\rho^a\phi - c\rho^b\psi, c\rho^{a-1}\phi+\rho^{b-1}\psi) \mid \phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\}\ , \tag{136}$$

is essentially self-adjoint in

$$L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho) \equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,\alpha + L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,d\rho\wedge\beta\ , \tag{137}$$

which is a Hilbert subspace of $L^2\Omega^r(M,g)$; the spectrum of its closure consists of the eigenvalues (134) and (135), with multiplicity one and corresponding normalized eigenvectors

$$\frac{1}{\sqrt{1+c^2}}\,\Theta(\chi_{s,a,\sigma,k},0)\ ,\qquad \frac{1}{\sqrt{1+c^2}}\,\Theta(0,\chi_{s,b,\tau,k})\ ,$$

respectively; and the smooth cores of its closure is (136).

The condition $\tau > -1/2$ only holds with the choice

$$\tau = \frac{3+\sqrt{(n-2r)^2+4\mu^2}}{2}$$

in (133), which corresponds to the choice

$$b = \frac{4 - n + 2r + \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{138}$$

in (131). With this choice, the eigenvalues (135) become

$$\left(4k + 6 \mp (n-2r) + \sqrt{(n-2r)^2 + 4\mu^2}\right) s \ , \tag{139}$$

which are > 0 for all k.

Consider the choice

$$a = \frac{-n + 2r + \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{140}$$

in (130), and, correspondingly,

$$\sigma = \frac{-1 + \sqrt{(n-2r)^2 + 4\mu^2}}{2} > -\frac{1}{2}$$

in (132). Then the eigenvalues (134) become

$$\left(4k \mp (n-2r) + \sqrt{(n-2r)^2 + 4\mu^2}\right) s \ , \tag{141}$$

which are > 0 for all k.

Now, consider the choice

$$a = \frac{4 - n + 2r - \sqrt{(n-2r)^2 + 4\mu^2}}{2} \tag{142}$$

in (130), and therefore

$$\sigma = \frac{3 - \sqrt{(n-2r)^2 + 4\mu^2}}{2}$$

in (132). In this case, the condition $\sigma > -1/2$ means that

$$\mu < 2 \quad \text{and} \quad |n - 2r| < 2\sqrt{4 - \mu^2} \ . \tag{143}$$

The eigenvalues (134) become

$$\left(4k + 4 \mp (n-2r) - \sqrt{(n-2r)^2 + 4\mu^2}\right) s \ . \tag{144}$$

For Δ_s^+, (144) is:

- ≥ 0 for all k if and only if $n - 2r \leq 2 - \mu^2/2$, and
- $= 0$ just when $k = 0$ and $n - 2r = 2 - \mu^2/2$.

For Δ_s^-, (144) is:

- ≥ 0 for all k if and only if $n - 2r \geq \mu^2/2 - 2$, and
- $= 0$ just when $k = 0$ and $n - 2r = \mu^2/2 - 2$.

All of the above operators defined by $\Delta_s^\pm$, as well as their domains, will be said to be of *fifth type*.

CHAPTER 15

Splitting of the Witten complex on a cone

1. Subcomplexes defined by domains of first and second types

Consider the notation of Sections 1 and 2 of Chapter 14. The following result follows from Corollary 13.5.

LEMMA 15.1. *For $s \geq 0$, $d_s^{\pm}$ and $\delta_s^{\pm}$ define maps*

$$0 \underset{\delta^{\pm}_{s,r-1}}{\overset{d^{\pm}_{s,r-1}}{\rightleftarrows}} C^{\infty}(\mathbb{R}_+)\,\gamma \underset{\delta^{\pm}_{s,r}}{\overset{d^{\pm}_{s,r}}{\rightleftarrows}} C^{\infty}(\mathbb{R}_+)\,d\rho\wedge\gamma \underset{\delta^{\pm}_{s,r+1}}{\overset{d^{\pm}_{s,r+1}}{\rightleftarrows}} 0\ ,$$

which are given by

$$d^{\pm}_{s,r} = \frac{d}{d\rho} \pm s\rho\ , \quad \delta^{\pm}_{s,r} = -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \pm s\rho\ ,$$

using to the canonical identities

$$C^{\infty}(\mathbb{R}_+)\,\gamma \equiv C^{\infty}(\mathbb{R}_+)\,d\rho\wedge\gamma \equiv C^{\infty}(\mathbb{R}_+)\ .$$

According to Sections 1 and 2 of Chapter 14, γ can be used to define the following domains of first and second type:

$$\begin{aligned}
\mathcal{E}^{r}_{\gamma,1} &= \mathcal{S}_{\mathrm{ev},+}\,\gamma && \text{for} \quad r \leq \frac{n-1}{2}\ ,\\
\mathcal{E}^{r}_{\gamma,2} &= \rho^{-n+2r+2}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma && \text{for} \quad r \geq \frac{n-3}{2}\ ,\\
\mathcal{E}^{r+1}_{\gamma,1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma && \text{for} \quad r \leq \frac{n+1}{2}\ ,\\
\mathcal{E}^{r+1}_{\gamma,2} &= \rho^{-n+2r+1}\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma && \text{for} \quad r \geq \frac{n-1}{2}\ .
\end{aligned}$$

The following is a direct consequence of Lemma 15.1.

LEMMA 15.2. *For any $s \geq 0$, $d_s^{\pm}$ and $\delta_s^{\pm}$ define maps*

$$0 \underset{\delta^{\pm}_{s,r-1}}{\overset{d^{\pm}_{s,r-1}}{\rightleftarrows}} \mathcal{E}^{r}_{\gamma,i} \underset{\delta^{\pm}_{s,r}}{\overset{d^{\pm}_{s,r}}{\rightleftarrows}} \mathcal{E}^{r+1}_{\gamma,i} \underset{\delta^{\pm}_{s,r+1}}{\overset{d^{\pm}_{s,r+1}}{\rightleftarrows}} 0\ ,$$

where $i = 1$ if $r \leq \frac{n-1}{2}$, and $i = 2$ if $r \geq \frac{n-1}{2}$.

REMARK 24. If n is odd, by Lemma 15.1 and (110), and since $\mathcal{S}_{\mathrm{ev},+}$ is contained in $L^2(\mathbb{R}_+, \rho^{2\sigma}\,d\rho)$ if and only if $\sigma > -1/2$, we get

$$\begin{aligned}
d_s^{\pm}(\mathcal{E}^{r}_{\gamma,2}) &\not\subset L^2\Omega^r(M) && \text{for} \quad r = \frac{n-3}{2}\ ,\\
\delta_s^{\pm}(\mathcal{E}^{r+1}_{\gamma,1}) &\not\subset L^2\Omega^{r+1}(M) && \text{for} \quad r = \frac{n+1}{2}\ .
\end{aligned}$$

This is compatible with $\Delta_s^+ \ngeq 0$ on $\mathcal{E}^r_{\gamma,2}$ when $r = \frac{n-3}{2}$ (Section 1 of Chapter 14), and $\Delta_s^- \ngeq 0$ on $\mathcal{E}^{r+1}_{\gamma,1}$ when $r = \frac{n+1}{2}$ (Section 2 of Chapter 14).

REMARK 25. If n is even, notice that

$$\begin{aligned} \mathcal{E}^r_{\gamma,1} &= \mathcal{E}^r_{\gamma,2} = \mathcal{S}_{\mathrm{ev},+}\,\gamma && \text{for} \quad r = \frac{n}{2} - 1 \,, \\ \mathcal{E}^{r+1}_{\gamma,1} &= \mathcal{E}^{r+1}_{\gamma,2} = \rho\, \mathcal{S}_{\mathrm{ev},+}\, d\rho \wedge \gamma && \text{for} \quad r = \frac{n}{2} \,. \end{aligned}$$

By Lemma 15.2, $\mathcal{E}_{\gamma,i} = \mathcal{E}^r_{\gamma,i} \oplus \mathcal{E}^{r+1}_{\gamma,i}$ is a subcomplex of length two of $\Omega(M)$ with $d_s^\pm$ and $\delta_s^\pm$, even for $s = 0$, where $i = 1$ for $r \le \frac{n-1}{2}$, and $i = 2$ for $r \ge \frac{n-1}{2}$. Moreover let $\mathcal{E}_{\gamma,0}$ denote the dense subcomplex of $\mathcal{E}_{\gamma,i}$ defined by

$$\begin{aligned} \mathcal{E}^r_{\gamma,0} &= C_0^\infty(\mathbb{R}_+)\,\gamma \equiv C_0^\infty(\mathbb{R}_+) \,, \\ \mathcal{E}^{r+1}_{\gamma,0} &= C_0^\infty(\mathbb{R}_+)\, d\rho \wedge \gamma \equiv C_0^\infty(\mathbb{R}_+) \,. \end{aligned}$$

The closure of $\mathcal{E}_{\gamma,i}$ (and $\mathcal{E}_{\gamma,0}$) in $L^2\Omega(M)$ is denoted by $L^2\mathcal{E}_\gamma$. We have

$$\begin{aligned} L^2\mathcal{E}^r_\gamma &= L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho)\,\gamma \equiv L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho) \,, \\ L^2\mathcal{E}^{r+1}_\gamma &= L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho)\, d\rho \wedge \gamma \equiv L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho) \,. \end{aligned}$$

Assume now that $s > 0$. With the notation of Section 2 of Chapter 11, consider the real version of the elliptic complex (E, d) determined by the constants s and

$$\kappa = \frac{n - 2r - 1}{2} \,, \tag{145}$$

and also its subcomplexes $\mathcal{E}_i$, where $i = 1$ if $\kappa > -1/2$ ($r \le \frac{n-1}{2}$), and $i = 2$ if $\kappa < 1/2$ ($r \ge \frac{n-1}{2}$).

PROPOSITION 15.3. *There is a unitary isomorphism $L^2\mathcal{E}_\gamma \to L^2(E)$, which restricts to isomorphisms of complexes up to a shift of degree, $(\mathcal{E}_{\gamma,0}, d_s^\pm) \to (C_0^\infty(E), d)$ and $(\mathcal{E}_{\gamma,i}, d_s^\pm) \to (\mathcal{E}_i, d)$, where $i = 1$ if $r \le \frac{n-1}{2}$, and $i = 2$ if $r \ge \frac{n-1}{2}$.*

PROOF. The unitary isomorphism

$$\rho^\kappa : L^2(\mathbb{R}_+, \rho^{n-2r-1}\, d\rho) \to L^2(\mathbb{R}_+, d\rho)$$

induces a unitary isomorphism $L^2\mathcal{E}_\gamma \to L^2(E)$, which restricts to an isomorphism $\mathcal{E}_{\gamma,0} \to C_0^\infty(E)$. Furthermore

$$\begin{aligned} \rho^\kappa \mathcal{E}^r_{\gamma,1} &= \rho^\kappa \mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^\kappa \mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^0_1 \,, \\ \rho^\kappa \mathcal{E}^{r+1}_{\gamma,1} &= \rho^{1+\kappa}\, \mathcal{S}_{\mathrm{ev},+}\, d\rho \wedge \gamma \equiv \rho^{1+\kappa} \mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^1_1 \end{aligned}$$

if $r \le \frac{n-1}{2}$, and

$$\begin{aligned} \rho^\kappa \mathcal{E}^r_{\gamma,2} &= \rho^{\kappa-n+2r+2}\, \mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^{1-\kappa}\, \mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^0_2 \,, \\ \rho^\kappa \mathcal{E}^{r+1}_{\gamma,2} &= \rho^{\kappa-n+2r+1}\, \mathcal{S}_{\mathrm{ev},+}\,\gamma \equiv \rho^{-\kappa}\, \mathcal{S}_{\mathrm{ev},+} \equiv \mathcal{E}^1_2 \end{aligned}$$

if $r \ge \frac{n-1}{2}$. By Lemma 15.1 and (71), we also have

$$\rho^\kappa\, d^\pm_{s,r}\, \rho^{-\kappa} = \rho^\kappa \left(\frac{d}{d\rho} \pm s\rho \right) \rho^{-\kappa} = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho \,,$$

which is the operator d of Section 2 of Chapter 11. $\square$

COROLLARY 15.4. (i) *If* $r \neq \frac{n-1}{2}$, *then* $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$ *has a unique Hilbert complex extension in* $L^2\mathcal{E}_\gamma$, *whose smooth core is* $\mathcal{E}_{\gamma,i}$, *where* $i = 1$ *if* $r < \frac{n-1}{2}$, *and* $i = 2$ *if* $r > \frac{n-1}{2}$.

(ii) *If* $r = \frac{n-1}{2}$, *then* $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$ *has minimum and maximum Hilbert complex extensions in* $L^2\mathcal{E}_\gamma$, *whose smooth cores are* $\mathcal{E}_{\gamma,2}$ *and* $\mathcal{E}_{\gamma,1}$, *respectively.*

PROOF. This follows from Propositions 11.4 and 15.3. □

For each degree r, we will choose one of the possible domains of first and second type defined by γ, denoted by $\mathcal{E}_\gamma^r$ and $\mathcal{E}_\gamma^{r+1}$, so that $\mathcal{E}_\gamma = \mathcal{E}_\gamma^r \oplus \mathcal{E}_\gamma^{r+1}$ is a subcomplex of $(\Omega(U), d_s^{\pm})$ according to Lemma 15.2.

If n is even, there is only one choice of domains of first and second types by Remark 25. Thus $\mathcal{E}_\gamma^r$ and $\mathcal{E}_\gamma^{r+1}$ have only one possible definition in this case.

If n is odd, there are two possible choices of domains of first and second types just for the following values of r:

$$\left.\begin{aligned} \mathcal{E}_{\gamma,1}^r &= \mathcal{S}_{\mathrm{ev},+}\,\gamma \\ \mathcal{E}_{\gamma,2}^r &= \rho^{-1}\,\mathcal{S}_{\mathrm{ev},+}\,\gamma \end{aligned}\right\} \quad \text{for} \quad r = \frac{n-3}{2}\,,$$

$$\left.\begin{aligned} \mathcal{E}_{\gamma,1}^r &= \mathcal{S}_{\mathrm{ev},+}\,\gamma \\ \mathcal{E}_{\gamma,2}^r &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,\gamma \\ \mathcal{E}_{\gamma,1}^{r+1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \\ \mathcal{E}_{\gamma,2}^{r+1} &= \mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \end{aligned}\right\} \quad \text{for} \quad r = \frac{n-1}{2}\,,$$

$$\left.\begin{aligned} \mathcal{E}_{\gamma,1}^{r+1} &= \rho\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \\ \mathcal{E}_{\gamma,2}^{r+1} &= \rho^2\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\gamma \end{aligned}\right\} \quad \text{for} \quad r = \frac{n+1}{2}\,.$$

By Remark 24 and Corollary 15.4, we choose

$$\begin{aligned} \mathcal{E}_\gamma^r &= \mathcal{E}_{\gamma,1}^r \quad \text{for} \quad r = \frac{n-3}{2}\,, \\ \mathcal{E}_\gamma^{r+1} &= \mathcal{E}_{\gamma,2}^{r+1} \quad \text{for} \quad r = \frac{n+1}{2}\,. \end{aligned}$$

In order to get the minimum and maximum i.b.c. of $(\bigwedge TM^*, d)$, according to Corrollary 15.3, we choose

$$\left.\begin{aligned} &\left.\begin{aligned} \mathcal{E}_\gamma^r &= \mathcal{E}_{\gamma,2}^r \\ \mathcal{E}_\gamma^{r+1} &= \mathcal{E}_{\gamma,2}^{r+1} \end{aligned}\right\} \quad \text{if} \quad \gamma \in \widetilde{\mathcal{H}}_{\min}^r \\ &\left.\begin{aligned} \mathcal{E}_\gamma^r &= \mathcal{E}_{\gamma,1}^r \\ \mathcal{E}_\gamma^{r+1} &= \mathcal{E}_{\gamma,1}^{r+1} \end{aligned}\right\} \quad \text{if} \quad \gamma \in \widetilde{\mathcal{H}}_{\max}^r \end{aligned}\right\} \quad \text{for} \quad r = \frac{n-1}{2}\,.$$

According to Corollary 15.4, the above choices of $\mathcal{E}_\gamma$ satisfy the following.

COROLLARY 15.5. (i) *If* $r \neq \frac{n-1}{2}$, *then* $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$ *has a unique Hilbert complex extension in* $L^2\mathcal{E}_\gamma$, *whose smooth core is* $\mathcal{E}_\gamma$.

(ii) *If* $r = \frac{n-1}{2}$, *then* $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$ *has different minimum and maximum Hilbert complex extensions in* $L^2\mathcal{E}_\gamma$. *If* $\gamma \in \widetilde{\mathcal{H}}_{\min/\max}$, *then* $\mathcal{E}_\gamma$ *is the smooth core of the minimum/maximum Hilbert complex extension of* $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$.

Let $(\mathcal{D}_\gamma, \mathbf{d}_{s,\gamma}^{\pm})$ denote the Hilbert complex extension of $(\mathcal{E}_{\gamma,0}, d_s^{\pm})$ with core $\mathcal{E}_\gamma$, let $\mathbf{\Delta}_{s,\gamma}^{\pm}$ be the corresponding Laplacian, and let $\mathcal{H}_{s,\gamma}^{\pm} = \mathcal{H}_{s,\gamma}^{\pm,r} \oplus \mathcal{H}_{s,\gamma}^{\pm,r+1} = \ker \mathbf{\Delta}_{s,\gamma}^{\pm}$.

The following result follows from Sections 1 and 2 of Chapter 14, Lemma 6.3 and the choices made to define $\mathcal{E}_\gamma$.

PROPOSITION 15.6. (i) *$(\mathcal{D}_\gamma, \mathbf{d}^\pm_{s,\gamma})$ is discrete.*

(ii) *$\mathcal{H}^{+,r+1}_{s,\gamma} = 0$, $\dim \mathcal{H}^{+,r}_{s,\gamma} = 1$ if*

$$r \leq \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-3}{2} & \text{if } n \text{ is odd and } \gamma \in \widetilde{\mathcal{H}}^r_{\min} \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \gamma \in \widetilde{\mathcal{H}}^r_{\max}\ , \end{cases}$$

and $\mathcal{H}^{+,r}_{s,\gamma} = 0$ otherwise.

(iii) *$\mathcal{H}^{-,r}_{s,\gamma} = 0$, $\dim \mathcal{H}^{-,r+1}_{s,\gamma} = 1$ if*

$$r \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \gamma \in \widetilde{\mathcal{H}}^r_{\min} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma \in \widetilde{\mathcal{H}}^r_{\max}\ , \end{cases}$$

and $\mathcal{H}^{-,r+1}_{s,\gamma} = 0$ otherwise.

(iv) *If $e^\pm_s \in \mathcal{H}^\pm_{s,\gamma}$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle h e^\pm_s, e^\pm_s \rangle \to 1$ as $s \to \infty$.*

(v) *All non-zero eigenvalues of $\mathbf{\Delta}^\pm_{s,\gamma}$ are in $O(s)$ as $s \to \infty$.*

2. Subomplexes defined by domains of third, fourth and fifth types

Consider the notation of Sections 3–5 of Chapter 14. The following result follows from Corollary 13.5.

LEMMA 15.7. *For $s \geq 0$, $d^\pm_s$ and $\delta^\pm_s$ define maps*

$$0 \underset{\delta^\pm_{s,r-2}}{\overset{d^\pm_{s,r-2}}{\rightleftarrows}} C^\infty(\mathbb{R}_+)\,\beta \underset{\delta^\pm_{s,r-1}}{\overset{d^\pm_{s,r-1}}{\rightleftarrows}} C^\infty(\mathbb{R}_+)\,\alpha + C^\infty(\mathbb{R}_+)\,d\rho\wedge\beta$$
$$\underset{\delta^\pm_{s,r}}{\overset{d^\pm_{s,r}}{\rightleftarrows}} C^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha \underset{\delta^\pm_{s,r+1}}{\overset{d^\pm_{s,r+1}}{\rightleftarrows}} 0\ ,$$

which are given by

$$\begin{aligned} d^\pm_{s,r-1} &= \begin{pmatrix} \mu \\ \frac{d}{d\rho} \pm s\rho \end{pmatrix}\ , \\ \delta^\pm_{s,r-1} &= \begin{pmatrix} \mu\rho^{-2} & -\frac{d}{d\rho} - (n-2r+1)\rho^{-1} \pm s\rho \end{pmatrix}\ , \\ d^\pm_{s,r} &= \begin{pmatrix} \frac{d}{d\rho} \pm s\rho & -\mu \end{pmatrix}\ , \\ \delta^\pm_{s,r} &= \begin{pmatrix} -\frac{d}{d\rho} - (n-2r-1)\rho^{-1} \pm s\rho \\ -\mu\rho^{-2} \end{pmatrix}\ , \end{aligned}$$

according to the canonical identities

$$C^\infty(\mathbb{R}_+)\,\beta \equiv C^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha \equiv C^\infty(\mathbb{R}_+)\ ,$$
$$C^\infty(\mathbb{R}_+)\,\alpha + C^\infty(\mathbb{R}_+)\,d\rho\wedge\beta \equiv C^\infty(\mathbb{R}_+) \oplus C^\infty(\mathbb{R}_+)\ .$$

Consider only the choices of a given by the positive square roots in (117) and (123) for domains of third and fourth types, and (140) for domains of fifth type; the other choices of a are rejected because they are very restrictive on μ and r, and give rise to some negative eigenvalues. If these values of a are denoted by a_3, a_4 and a_5 according to the types of domains, then $a_5 = a_3 = a_4 - 1$, and therefore the notation $a_5 = a_3 = a$ and $a_4 = a + 1$ will be used. Recall also that we only have the choice (138) for b, which equals $a + 2$. So we only consider the following domains of third, fourth and fifth types defined by α and β:

$$
\begin{aligned}
\mathcal{F}^{r-1}_{\alpha,\beta} &= \rho^a\,\mathcal{S}_{\mathrm{ev},+}\,\beta \equiv \rho^a\,\mathcal{S}_{\mathrm{ev},+}\ ,\\
\mathcal{F}^{r+1}_{\alpha,\beta} &= \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\,d\rho\wedge\alpha \equiv \rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\ ,\\
\mathcal{F}^{r}_{\alpha,\beta} &= \rho^a\,\left\{\,(\phi - c\rho^2\psi)\,\alpha + (c\rho^{-1}\phi + \rho\psi)\,d\rho\wedge\beta \mid \phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\,\right\}\\
&\equiv \rho^a\,\left\{\,(\phi - c\rho^2\psi, c\rho^{-1}\phi + \rho\psi) \mid \phi,\psi\in\mathcal{S}_{\mathrm{ev},+}\,\right\}\ .
\end{aligned}
$$

LEMMA 15.8. *For any $s \geq 0$, $d^\pm_s$ and $\delta^\pm_s$ define maps*

$$
0 \underset{\delta^\pm_{s,r-2}}{\overset{d^\pm_{s,r-2}}{\rightleftarrows}} \mathcal{F}^{r-1}_{\alpha,\beta} \underset{\delta^\pm_{s,r-1}}{\overset{d^\pm_{s,r-1}}{\rightleftarrows}} \mathcal{F}^{r}_{\alpha,\beta} \underset{\delta^\pm_{s,r}}{\overset{d^\pm_{s,r}}{\rightleftarrows}} \mathcal{F}^{r+1}_{\alpha,\beta} \underset{\delta^\pm_{s,r+1}}{\overset{d^\pm_{s,r+1}}{\rightleftarrows}} 0
$$

PROOF. Lemma 15.7 gives $\delta^\pm_s(\mathcal{F}^{r-1}_{\alpha,\beta}) = d^\pm_s(\mathcal{F}^{r+1}_{\alpha,\beta}) = 0$.
Observe that

$$
a = c\mu\ , \tag{146}
$$

obtaining

$$
c(a + n - 2r) = \mu \tag{147}
$$

by (128). By Lemma 15.7, (146) and (147), for $h \in \mathcal{S}_{\mathrm{ev},+}$,

$$
d^\pm_s(\rho^a h\,\beta) = \rho^a\left(\mu h\,\alpha + \left(\frac{d}{d\rho} + c\mu\rho^{-1} \pm s\rho\right)(h)\,d\rho\wedge\beta\right)\ , \tag{148}
$$

$$
\delta^\pm_s(\rho^{a+1}h\,d\rho\wedge\alpha) = \rho^a\left(\left(-\rho\frac{d}{d\rho} - \frac{\mu}{c} \pm s\rho^2\right)(h)\,\alpha - \mu\rho^{-1}h\,d\rho\wedge\beta\right)\ . \tag{149}
$$

The inclusion $d^\pm_s(\mathcal{F}^{r-1}_{\alpha,\beta}) \subset \mathcal{F}^{r}_{\alpha,\beta}$ follows from (148) if we can find $\phi,\psi\in\mathcal{S}_{\mathrm{ev},+}$ such that

$$
\phi - c\rho^2\psi = \mu h\ , \tag{150}
$$

$$
c\rho^{-1}\phi + \rho\psi = \left(\frac{d}{d\rho} + c\mu\rho^{-1} \pm s\rho\right)(h)\ . \tag{151}
$$

Subtract $c\rho^{-2}$ times (150) from ρ^{-1} times (151) to get

$$
\psi = \frac{1}{1+c^2}\left(\rho^{-1}\frac{d}{d\rho} \pm s\right)(h)\ ,
$$

which is well defined in $\mathcal{S}_{\mathrm{ev},+}$. Then

$$
\phi = \mu h + c\rho^2\psi
$$

by (150). These functions ϕ and ψ satisfy (150) and (151).

The inclusion $\delta_s^\pm(\mathcal{F}^{r+1}_{\alpha,\beta}) \subset \mathcal{F}^r_{\alpha,\beta}$ follows from (149) if we can find $\phi, \psi \in \mathcal{S}_{\mathrm{ev},+}$ such that

$$\phi - c\rho^2\psi = \left(-\rho\,\frac{d}{d\rho} - \frac{\mu}{c} \pm s\rho^2\right)(h)\ , \tag{152}$$

$$c\rho^{-1}\phi + \rho\psi = -\mu\rho^{-1}h\ . \tag{153}$$

The sum (152) and $c\rho$ times (153) gives

$$\phi = \frac{1}{1+c^2}\left(-\rho\,\frac{d}{d\rho} - \frac{1+c^2}{c}\,\mu \pm s\rho^2\right)(h)\ ,$$

which belongs to $\mathcal{S}_{\mathrm{ev},+}$. The even extensions of h and ϕ to $\mathbb{R}$, also denoted by h and ϕ, satisfy $c\phi(0) = -\mu h(0)$, and therefore $\mu h + c\phi \in \rho^2\,\mathcal{S}_{\mathrm{ev}}$. It follows that

$$\psi = \rho^{-2}(\mu h + c\phi)\ ,$$

obtained from (153), is well defined in $\mathcal{S}_{\mathrm{ev},+}$. These functions ϕ and ψ satisfy (152) and (153).

For arbitrary $\phi, \psi \in \mathcal{S}_{\mathrm{ev},+}$, let

$$\zeta = \rho^a\left(\left(\phi - c\rho^2\psi\right)\,\alpha + \left(c\rho^{-1}\phi + \rho\psi\right)\,d\rho\wedge\beta\right)\ . \tag{154}$$

By Corollary 13.5, (146) and (147),

$$\begin{aligned} d_s^\pm(\zeta) &= \rho^{a+1}\Bigg(\left(\rho^{-1}\,\frac{d}{d\rho} \pm s\right)(\phi)\\ &\qquad + c\left(-\rho^{-1}\,\frac{d}{d\rho} - \left(\frac{c^2+1}{c}\,\mu + 2\right) \pm s\rho^2\right)(\psi)\Bigg)\,d\rho\wedge\alpha\ ,\\ \delta_s^\pm(\zeta) &= \rho^a\Bigg(c\left(-\rho^{-1}\,\frac{d}{d\rho} \pm s\right)(\phi)\\ &\qquad + \left(-\rho\,\frac{d}{d\rho} - \left(\frac{c^2+1}{c}\,\mu + 2\right) \pm s\rho^2\right)(\psi)\Bigg)\,\beta\ , \end{aligned}$$

showing $d_s^\pm(\mathcal{F}^r_{\alpha,\beta}) \subset \mathcal{F}^{r+1}_{\alpha,\beta}$ and $\delta_s^\pm(\mathcal{F}^r_{\alpha,\beta}) \subset \mathcal{F}^{r-1}_{\alpha,\beta}$. □

By Lemma 15.8,

$$\mathcal{F}_{\alpha,\beta} = \mathcal{F}^{r-1}_{\alpha,\beta} \oplus \mathcal{F}^r_{\alpha,\beta} \oplus \mathcal{F}^{r+1}_{\alpha,\beta}$$

is a subcomplex of length three of $\Omega(M)$ with $d_s^\pm$ and $\delta_s^\pm$. Moreover let $\mathcal{F}_{\alpha,\beta,0}$ denote the dense subcomplex of $\mathcal{F}_{\alpha,\beta}$ defined by

$$\begin{aligned} \mathcal{F}^{r-1}_{\alpha,\beta,0} &= C_0^\infty(\mathbb{R}_+)\,\beta \equiv C_0^\infty(\mathbb{R}_+)\ ,\\ \mathcal{F}^{r+1}_{\alpha,\beta,0} &= C_0^\infty(\mathbb{R}_+)\,d\rho\wedge\alpha \equiv C_0^\infty(\mathbb{R}_+)\ ,\\ \mathcal{F}^{r}_{\alpha,\beta,0} &= C_0^\infty(\mathbb{R}_+)\,\alpha + C_0^\infty(\mathbb{R}_+)\,d\rho\wedge\beta \equiv C_0^\infty(\mathbb{R}_+)\oplus C_0^\infty(\mathbb{R}_+)\ . \end{aligned}$$

The closure of $\mathcal{F}_{\alpha,\beta}$ (and $\mathcal{F}_{\alpha,\beta,0}$) in $L^2\Omega(M)$ is denoted by $L^2\mathcal{F}_{\alpha,\beta}$. We have

$$\begin{aligned} L^2\mathcal{F}^{r-1}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,\beta \equiv L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\ ,\\ L^2\mathcal{F}^{r+1}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,d\rho\wedge\alpha \equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\ ,\\ L^2\mathcal{F}^{r}_{\alpha,\beta} &= L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\,\alpha + L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\,d\rho\wedge\beta\\ &\equiv L^2(\mathbb{R}_+,\rho^{n-2r+1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\ . \end{aligned}$$

Assume now that $s>0$. With the notation of Section 3 in Chapter 11, consider the real version of the elliptic complex (F,d), as well as its subcomplex $\mathcal{F}_1$, determined by the constants s, c and

$$\kappa=\frac{-1+\sqrt{(n-2r)^2+4\mu^2}}{2}>-\frac{1}{2}\;. \tag{155}$$

By (129),

$$\kappa=c\mu+\frac{n-2r-1}{2}=\frac{\mu}{c}-\frac{n-2r+1}{2}\;. \tag{156}$$

PROPOSITION 15.9. *There is a unitary isomorphism* $L^2\mathcal{F}_{\alpha,\beta}\to L^2(F)$, *which restricts to isomorphisms of complexes up to a shift of degree,* $(\mathcal{F}_{\alpha,\beta},d_s^\pm)\to(\mathcal{F}_1,d)$ *and* $(\mathcal{F}_{\alpha,\beta,0},d_s^\pm)\to(C_0^\infty(F),d)$.

PROOF. As an intermediate step, let

$$\begin{gathered}\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}=\rho\,\mathcal{F}^{r-1}_{\alpha,\beta}=\rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\;,\quad \widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}=\mathcal{F}^{r+1}_{\alpha,\beta}=\rho^{a+1}\,\mathcal{S}_{\mathrm{ev},+}\;,\\ \widehat{\mathcal{F}}^{r}_{\alpha,\beta}=\Theta^{-1}(\mathcal{F}^{r}_{\alpha,\beta})=\rho^{a}\,\mathcal{S}_{\mathrm{ev},+}\oplus\rho^{a+2}\,\mathcal{S}_{\mathrm{ev},+}\;,\\ \widehat{\mathcal{F}}_{\alpha,\beta}=\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\oplus\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\oplus\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}\;,\quad \widehat{\mathcal{F}}_{\alpha,\beta,0}=\mathcal{F}_{\alpha,\beta,0}\;,\\ L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}=L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}=L^2\mathcal{F}^{r+1}_{\alpha,\beta}=L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,\\ L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\equiv L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\oplus L^2(\mathbb{R}_+,\rho^{n-2r-1}\,d\rho)\;,\\ L^2\widehat{\mathcal{F}}_{\alpha,\beta}=L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\oplus L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\oplus L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}\;.\end{gathered}$$

Moreover let $\Xi:L^2\mathcal{F}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}_{\alpha,\beta}$ be the unitary isomorphism defined by

$$\rho:L^2\mathcal{F}^{r-1}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\;,\qquad \frac{1}{\sqrt{1+c^2}}\,\Theta^{-1}:L^2\mathcal{F}^{r}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r}_{\alpha,\beta}$$

and the identity map $L^2\mathcal{F}^{r+1}_{\alpha,\beta}\to L^2\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}$. It restricts to isomorphisms $\mathcal{F}_{\alpha,\beta}\to\widehat{\mathcal{F}}_{\alpha,\beta}$ and $\mathcal{F}_{\alpha,\beta,0}\to\widehat{\mathcal{F}}_{\alpha,\beta,0}$. Thus, by Lemma 15.8, $(\mathcal{F}_{\alpha,\beta},d_s^\pm)$ induces via Ξ a complex

$$0\xrightarrow{\hat d^\pm_{s,r-2}}\widehat{\mathcal{F}}^{r-1}_{\alpha,\beta}\xrightarrow{\hat d^\pm_{s,r-1}}\widehat{\mathcal{F}}^{r}_{\alpha,\beta}\xrightarrow{\hat d^\pm_{s,r}}\widehat{\mathcal{F}}^{r+1}_{\alpha,\beta}\xrightarrow{\hat d^\pm_{s,r+2}}0\;.$$

By Lemma 15.7 and (71),

$$\begin{aligned}\hat d^\pm_{s,r-1}&=\frac{1}{\sqrt{1+c^2}}\,\Theta^{-1}\,d^\pm_{s,r-1}\,\rho^{-1}\\ &=\frac{1}{\sqrt{1+c^2}}\begin{pmatrix}1&c\rho\\-c&\rho\end{pmatrix}\begin{pmatrix}\mu\\ \frac{d}{d\rho}\pm s\rho\end{pmatrix}\rho^{-1}\\ &=\frac{1}{\sqrt{1+c^2}}\begin{pmatrix}c\,\frac{d}{d\rho}+(\mu-c)\rho^{-1}\pm cs\rho\\ \frac{d}{d\rho}+(c\mu+1)\rho^{-1}\pm s\rho\end{pmatrix}\;,\end{aligned}\tag{157}$$

$$\begin{aligned}\hat d^\pm_{s,r}&=\frac{1}{\sqrt{1+c^2}}\,\Theta\,d^\pm_{s,r}\\ &=\frac{1}{\sqrt{1+c^2}}\begin{pmatrix}\frac{d}{d\rho}\pm s\rho&-\mu\end{pmatrix}\begin{pmatrix}1&-c\\c\rho^{-1}&\rho^{-1}\end{pmatrix}\\ &=\frac{1}{\sqrt{1+c^2}}\begin{pmatrix}\frac{d}{d\rho}-c\mu\rho^{-1}\pm s\rho&-c\,\frac{d}{d\rho}-\mu\rho^{-1}\mp cs\rho\end{pmatrix}\;.\end{aligned}\tag{158}$$

Now, the unitary isomorphism

$$\rho^{\frac{n-2r-1}{2}}: L^2(\mathbb{R}_+, \rho^{n-2r-1}\,d\rho) \to L^2(\mathbb{R}_+, d\rho)$$

induces a unitary isomorphism $L^2\widehat{\mathcal{F}}_{\alpha,\beta} \to L^2(F)$, which restricts to isomorphisms $\widehat{\mathcal{F}}_{\alpha,\beta} \to \mathcal{F}_1$ and $\widehat{\mathcal{F}}_{\alpha,\beta,0} \to C_0^\infty(F)$. Moreover, by (157), (158), (71) and (156),

$$\begin{aligned}
&\rho^{\frac{n-2r-1}{2}}\,\hat{d}^{\pm}_{s,r-1}\,\rho^{-\frac{n-2r-1}{2}} \\
&\qquad = \frac{1}{\sqrt{1+c^2}}\,\rho^{\frac{n-2r-1}{2}} \begin{pmatrix} c\,\frac{d}{d\rho} + (\mu-c)\rho^{-1} \pm cs\rho \\ \frac{d}{d\rho} + (c\mu+1)\rho^{-1} \pm s\rho \end{pmatrix} \rho^{-\frac{n-2r-1}{2}} \\
&\qquad = \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} c\left(\frac{d}{d\rho} + \kappa\rho^{-1} \pm s\rho\right) \\ \frac{d}{d\rho} - (\kappa+1)\rho^{-1} \pm s\rho \end{pmatrix}\,,
\end{aligned}$$

$$\begin{aligned}
&\rho^{\frac{n-2r-1}{2}}\,\hat{d}^{\pm}_{s,r}\,\rho^{-\frac{n-2r-1}{2}} \\
&\quad = \frac{1}{\sqrt{1+c^2}}\,\rho^{\frac{n-2r-1}{2}} \begin{pmatrix} \frac{d}{d\rho} - c\mu\rho^{-1} \pm s\rho & -c\,\frac{d}{d\rho} - \mu\rho^{-1} \mp cs\rho \end{pmatrix} \rho^{-\frac{n-2r-1}{2}} \\
&\quad = \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho & c\left(-\frac{d}{d\rho} - (\kappa+1)\rho^{-1} \mp s\rho\right) \end{pmatrix}\,,
\end{aligned}$$

which are the operators d_0 and d_1 of Section 3 in Chapter 11. □

COROLLARY 15.10. *$(\mathcal{F}_{\alpha,\beta,0}, d^{\pm}_s)$ has a unique Hilbert complex extension in $L^2\mathcal{F}_{\alpha,\beta}$, whose smooth core is $\mathcal{F}_{\alpha,\beta}$.*

PROOF. This follows from Propositions 11.6 and 15.9. □

Let $(\mathcal{D}_{\alpha,\beta}, \mathbf{d}^{\pm}_{s,\alpha,\beta})$ denote the unique Hilbert complex extension of $(\mathcal{F}_{\alpha,\beta,0}, d^{\pm}_s)$, according to Corollary 15.10, and let $\mathbf{\Delta}^{\pm}_{s,\alpha,\beta}$ denote the corresponding Laplacian. The following result follows from Sections 3–5 of Chapter 14.

PROPOSITION 15.11. (i) *$(\mathcal{D}_{\alpha,\beta}, \mathbf{d}^{\pm}_{s,\alpha,\beta})$ is discrete.*
(ii) *The eigenvalues of $\mathbf{\Delta}^{\pm}_{s,\alpha,\beta}$ are positive and in $O(s)$ as $s \to \infty$.*

3. Splitting into subcomplexes

Let $\mathcal{B}_{\min/\max,0}$ denote an orthonormal frame of $\widetilde{\mathcal{H}}_{\min/\max}$ consisting of homogeneous differential forms. For each positive eigenvalue μ of $\widetilde{D}_{\min/\max}$, let $\mathcal{B}_{\min/\max,\mu}$ be an orthonormal frame of $E_\mu(\widetilde{D}_{\min/\max})$ consisting of differential forms $\alpha+\beta$ like in Section 2. Then let

$$\mathbf{d}^{\pm}_{s,\min/\max} = \bigoplus_{\gamma} \mathbf{d}^{\pm}_{s,\gamma} \oplus \widehat{\bigoplus_{\mu}} \bigoplus_{\alpha+\beta} \mathbf{d}^{\pm}_{s,\alpha,\beta}\,,$$

where γ runs in $\mathcal{B}_{\min/\max,0}$, μ runs in the positive spectrum of $\widetilde{D}_{\min/\max}$, and $\alpha+\beta$ runs in $\mathcal{B}_{\min/\max,\mu}$. Observe that the domain of $\mathbf{d}^{\pm}_{s,\min/\max}$ is independent of s, and therefore it is denoted by $\mathcal{D}_{\min/\max}$. Let also

$$\mathcal{G}_{\min/\max} = \bigoplus_{\gamma} \mathcal{E}_{\gamma,0} \oplus \bigoplus_{\mu} \bigoplus_{\alpha+\beta} \mathcal{F}_{\alpha,\beta,0}\,.$$

PROPOSITION 15.12. *$\mathcal{D}(d^{\pm}_{s,\min/\max}) = \mathcal{D}_{\min/\max}$ and $d^{\pm}_{s,\min/\max} = \mathbf{d}^{\pm}_{s,\min/\max}$.*

PROOF. By Corollaries 15.5 and 15.10, Lemma 9.2 and (112), $(\mathcal{D}_{\min/\max}, \mathbf{d}^{\pm}_{s,\min/\max})$ is the minimum/maximum Hilbert complex extension of $(\mathcal{G}_{\min/\max}, d^{\pm}_s)$. Then the result easily follows from the following assertions.

CLAIM 18. $\mathcal{G}_{\min/\max} \subset \mathcal{D}(d^{\pm}_{s,\min/\max})$.

CLAIM 19. $\Omega_0(M) \subset \mathcal{D}_{\min/\max}$.

Let $\hat{d}^{\pm}_{s,\min/\max}$ denote the minimum/maximum Hilbert complex extension of $(\Omega_0(M), d^{\pm}_s)$ with respect to the product metric $\hat{g} = \tilde{g} + (d\rho)^2$ on $M = N \times \mathbb{R}_+$. With the terminology of [**11**, p. 110], observe that $(\Omega(M), d^{\pm}_s)$ is the product complex of the de Rham complex of N, $(\Omega(N), \tilde{d})$, and the Witten deformation of the de Rham complex of $\mathbb{R}_+$, defined by the function $\frac{1}{2}\rho^2$. Then, by [**11**, Lemma 3.6 and (2.38b)],

$$\begin{aligned}\mathcal{D}(\hat{d}^{\pm}_{s,\min/\max}) &\supset C^\infty_0(\mathbb{R}_+)\,\mathcal{D}(\tilde{d}_{\min/\max}) + C^\infty_0(\mathbb{R}_+)\,d\rho\wedge\mathcal{D}(\tilde{d}_{\min/\max})\\ &\supset \mathcal{G}_{\min/\max}\ . \end{aligned} \tag{159}$$

On the other hand, for $0 < a < b < \infty$, let $L^2_{a,b}\Omega(M,g)$ and $L^2_{a,b}\Omega(M,\hat{g})$ denote the Hilbert subspaces of $L^2\Omega(M,g)$ and $L^2\Omega(M,\hat{g})$, respectively, consisting of L^2 differential forms supported in $N \times [a,b]$. Since g and $\hat{g}$ are quasi-isometric on $N \times (a', b')$ for $0 < a' < a$ and $b < b' < \infty$, it follows that

$$\mathcal{D}(d^{\pm}_{s,\min/\max}) \cap L^2_{a,b}\Omega(M,g) = \mathcal{D}(\hat{d}^{\pm}_{s,\min/\max}) \cap L^2_{a,b}\Omega(M,\hat{g})\ . \tag{160}$$

Moreover

$$\mathcal{G}_{\min/\max} \subset \bigcup_{0<a<b<\infty} L^2_{a,b}\Omega(M,g)\ . \tag{161}$$

Now Claim 18 follows from (159)–(161).

Finally, Claim 19 follows from

$$\Omega_0(M) \subset \bigoplus_{\gamma} \mathcal{E}_{\gamma,0} \oplus \widehat{\bigoplus_{\mu}} \bigoplus_{\alpha+\beta} \mathcal{F}_{\alpha,\beta,0}\ , \tag{162}$$

where γ, μ and $\alpha+\beta$ vary as above. The inclusion (162) can be proved as follows. According to (106), any $\xi \in \Omega_0(M)$ can be written as $\xi = \xi_0 + d\rho \wedge \xi_1$, where $\xi_0, \xi_1 \in C^\infty_0(\mathbb{R}_+, \Omega_0(N))$. Then, by (112), we get functions $f_{k,\gamma}, f_{k,\ell,\alpha,\beta} \in C^\infty_0(\mathbb{R}_+)$, for $k, \ell \in \{0,1\}$, defined by

$$\begin{gathered} f_{k,\gamma}(\rho) = \langle \xi_k(\rho), \gamma\rangle_{\tilde{g}}\ ,\\ f_{k,0,\alpha,\beta}(\rho) = \langle \xi_k(\rho), \beta\rangle_{\tilde{g}}\ ,\quad f_{k,1,\alpha,\beta}(\rho) = \langle \xi_k(\rho), \alpha\rangle_{\tilde{g}}\ , \end{gathered}$$

and moreover

$$\begin{aligned}\alpha = {} & \sum_{\gamma} (f_{0,\gamma}\,\gamma + f_{1,\gamma}\,d\rho\wedge\gamma)\\ & + \sum_{\mu}\sum_{\alpha+\beta} (f_{0,0,\alpha,\beta}\,\beta + f_{1,0,\alpha,\beta}\,\alpha + f_{1,0,\alpha,\beta}\,d\rho\wedge\beta + f_{1,1,\alpha,\beta}\,d\rho\wedge\alpha)\end{aligned}$$

in $L^2\Omega(M,g)$, where γ, μ and $\alpha+\beta$ vary as above. Thus ξ belongs to the space in the right hand side of (162). □

REMARK 26. From (79), Remark 7, and Propositions 11.4, 11.6 and 15.12, it follows that, with the notation of Example 8.2, $h(\rho)\,\mathcal{D}^\infty(d^{\pm}_{s,\min/\max}) \subset \mathcal{D}^\infty(d^{\pm}_{s,\min/\max})$ for all $h \in C^\infty(\mathbb{R}_+)$ such that $h' \in C^\infty_0(\mathbb{R}_+)$.

Let $\mathcal{H}^{\pm}_{s,\min/\max} = \bigoplus_r \mathcal{H}^{\pm,r}_{s,\min/\max} = \ker \Delta^{\pm}_{s,\min/\max}$.

COROLLARY 15.13. (i) *$d^{\pm}_{s,\min/\max}$ is discrete.*

(ii) *$\mathcal{H}^{+,r}_{\min} \cong H^r_{\min}(N)$ if*

$$r \le \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-3}{2} & \text{if } n \text{ is odd}, \end{cases}$$

and $\mathcal{H}^{+,r}_{\min} = 0$ otherwise.

(iii) *$\mathcal{H}^{+,r}_{\max} \cong H^r_{\max}(N)$ if*

$$r \le \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases}$$

and $\mathcal{H}^{+,r}_{\max} = 0$ otherwise.

(iv) *$\mathcal{H}^{-,r+1}_{\min} \cong H^r_{\min}(N)$ if*

$$r \ge \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases}$$

and $\mathcal{H}^{+,r+1}_{\min} = 0$ otherwise.

(v) *$\mathcal{H}^{-,r+1}_{\max} \cong H^r_{\max}(N)$ if*

$$r \ge \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases}$$

and $\mathcal{H}^{+,r+1}_{\max} = 0$ otherwise.

(vi) *If $e^{\pm}_s \in \mathcal{H}^{\pm}_{s,\min/\max}$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle h e^{\pm}_s, e^{\pm}_s \rangle \to 1$ as $s \to \infty$.*

(vii) *Let $0 \le \lambda^{\pm}_{s,\min/\max,0} \le \lambda^{\pm}_{s,\min/\max,1} \le \cdots$ be the eigenvalues of $\Delta_{s,\min/\max}$, repeated according to their multiplicities. Given $k \in \mathbb{N}$, if $\lambda^{\pm}_{s,\min/\max,k} > 0$ for some s, then $\lambda^{\pm}_{s,\min/\max,k} \in O(s)$ as $s \to \infty$.*

(viii) *There is some $\theta > 0$ such that $\liminf_k \lambda^{\pm}_{s,\min/\max,k} k^{-\theta} > 0$.*

PROOF. For γ, μ and $\alpha + \beta$ as above, the spectra of $\Delta^{\pm}_s$ on $\mathcal{E}_\gamma$ and $\mathcal{F}_{\alpha,\beta}$ is discrete by Propositions 15.6-(i) and 15.11-(i). Moreover the union of all of these spectra has no accumulation points according to Sections 1–5 of Chapter 14 and since $\widetilde{\Delta}_{\min/\max}$ has a discrete spectrum. Then (i) follows by Proposition 15.12.

Now, properties (ii)–(vii) follow directly from Propositions 15.6, 15.11 and 15.12.

To prove (viii), let $0 \le \tilde{\lambda}_{\min/\max,0} \le \tilde{\lambda}_{\min/\max,1} \le \cdots$ denote the eigenvalues of $\widetilde{\Delta}_{s,\min/\max}$, repeated according to their multiplicities, and let $\mu_{\min/\max,\ell} = \sqrt{\tilde{\lambda}_{\min/\max,\ell}}$ for each $\ell \in \mathbb{N}$. Since N satisfies Theorem I-(ii) with $\tilde{g}$, there is some $C_0, \tilde{\theta} > 0$ such that

$$\tilde{\lambda}_{\min/\max,\ell} \ge C_0^2 \ell^{\tilde{\theta}} \tag{163}$$

for all ℓ. Consider the counting function

$$\mathfrak{N}^{\pm}_{s,\min/\max}(\lambda) = \#\left\{ k \in \mathbb{N} \mid \lambda^{\pm}_{s,\min/\max,k} < \lambda \right\}$$

for $\lambda > 0$. From (113)–(116), (119), (124), (139), (141) and (163), and the choices made in Chapter 15, it follows that there are some $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}\mathfrak{N}^{\pm}_{s,\min/\max}(\lambda) &\le \#\left\{\, (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2\, \mu_{\min/\max,\ell} + C_3 \le \lambda \,\right\}\\ &\le \#\{\, (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2 C_0 \ell^{\tilde\theta/2} + C_3 \le \lambda \,\}\\ &\le \#\left\{ (k,\ell) \in \mathbb{N}^2 \,\middle|\, \ell \le \left(\frac{\lambda - C_3}{C_2 C_0} - \frac{C_1 k}{C_2 C_0}\right)^{2/\tilde\theta} \right\}\\ &\le \int_0^{\frac{\lambda - C_3}{C_1}} \left(\frac{\lambda - C_3}{C_2 C_0} - \frac{C_1 x}{C_2 C_0}\right)^{2/\tilde\theta} dx\\ &= \frac{\tilde\theta(\lambda - C_3)^{(2+\tilde\theta)/\tilde\theta}}{(2+\tilde\theta)(C_2C_0)^{2/\tilde\theta} C_1} \;.\end{aligned}$$

So $\mathfrak{N}^{\pm}_{s,\min/\max}(\lambda) \le C\lambda^{(2+\tilde\theta)/\tilde\theta}$ for some $C > 0$ and all large enough λ, giving (viii) with $\theta = \frac{\tilde\theta}{2+\tilde\theta}$. □

EXAMPLE 15.14. Consider the notation of Examples 7.6, 7.12 and 12.1. On the stratum $\mathbb{S}^{m-1} \times \mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$, the model rel-Morse function $\pm\frac{1}{2}\rho^2$ and the metric g_1 define the Witten's perturbed operators $d_s^{\pm}$, $\delta_s^{\pm}$, $D_s^{\pm}$ and $\Delta_s^{\pm}$. Since ρ_0 and g_0 respectively correspond to ρ and g_1 by $\mathrm{can} : \mathbb{S}^{m-1} \times \mathbb{R}_+ \to \mathbb{R}^m \setminus \{0\}$, it follows that $d_s^{\pm}$, $\delta_s^{\pm}$, $D_s^{\pm}$ and $\Delta_s^{\pm}$ respectively correspond to $d_{0,s}^{\pm}$, $\delta_{0,s}^{\pm}$, $D_{0,s}^{\pm}$, $\Delta_{0,s}^{\pm}$ by $\mathrm{can}^* : \Omega(\mathbb{R}^m \setminus \{0\}) \to \Omega(\mathbb{S}^{m-1} \times \mathbb{R}_+)$, and moreover

$$L^2\Omega(\mathbb{R}^m, g_0) \equiv L^2\Omega(\mathbb{R}^m \setminus \{0\}, g_0) \xrightarrow{\ \mathrm{can}^*\ } L^2\Omega(\mathbb{S}^{m-1} \times \mathbb{R}_+, g_1) \tag{164}$$

is a unitary isomorphism. The extension by zero defines a canonical injection $\Omega_0(\mathbb{R}^m \setminus \{0\}) \to \Omega_0(\mathbb{R}^m)$, whose composite with $(\mathrm{can}^*)^{-1}$ is an injective homomorphism of complexes, $(\Omega_0(\mathbb{S}^{m-1} \times \mathbb{R}_+), d_s^{\pm}) \to (\Omega_0(\mathbb{R}^m), d_{0,s}^{\pm})$. Thus the unique i.b.c. of $(\bigwedge T\mathbb{R}^{m*}, d_{0,s}^{\pm})$ in $L^2\Omega(\mathbb{R}^m, g_0)$ corresponds to $d_{s,\max}^{\pm}$ via (164).

If $m \ge 2$, then $H^{\frac{m-1}{2}}(\mathbb{S}^{m-1}) = 0$ for odd m. So $(\bigwedge T(\mathbb{S}^{m-1} \times \mathbb{R}_+)^*, d_s^{\pm})$ has a unique i.b.c. by Corollaries 15.5 and 15.10, and Proposition 15.12.

If $m = 1$, then $\Omega(\mathbb{S}^0) = \Omega^0(\mathbb{S}^0) \equiv \mathbb{R}^2$, and therefore, according to (106), (107) and Corollary 13.5,

$$\begin{gathered}\Omega^0(\mathbb{S}^0 \times \mathbb{R}_+) \equiv C^\infty(\mathbb{R}_+, \mathbb{R}^2)\;,\\ \Omega^1(\mathbb{S}^0 \times \mathbb{R}_+) \equiv d\rho \wedge C^\infty(\mathbb{R}_+, \mathbb{R}^2) \equiv C^\infty(\mathbb{R}_+, \mathbb{R}^2)\;,\\ d_s^{\pm} \equiv \frac{d}{d\rho} \pm s\rho\;, \quad \delta_s^{\pm} \equiv -\frac{d}{d\rho} \pm s\rho\;,\end{gathered}$$

giving $d_{s,\min}^{\pm} \ne d_{s,\max}^{\pm}$ by Proposition 11.4-(i).

CHAPTER 16

Local model of the Witten's perturbation

The local model of our version of Morse functions around their critical points will be as follows. Let $m_\pm \in \mathbb{N}$, let $L_\pm$ be a compact Thom-Mather stratification, and let $M_\pm$ be a stratum in $c(L_\pm)$. Thus, either $M_\pm = N_\pm \times \mathbb{R}_+$ for some stratum $N_\pm$ of $L_\pm$, or $M_\pm$ is the vertex stratum of $c(L_\pm)$. On the stratum $M = \mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$ of the Thom-Mather stratification $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times c(L_+) \times c(L_-)$ (for any choice of product Thom-Mather structure on $c(L_+) \times c(L_-)$), consider an adapted metric given as product of standard metrics on the Euclidean spaces $\mathbb{R}^{m_\pm}$ and model adapted metrics on the strata $M_\pm$. Let d_s denote the Witten's perturbed differential map on $\Omega(M)$ induced by the model rel-Morse function $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ (Remark 19-(iii)). Let $\Delta_{s,\min/\max}$ be the Laplacian defined by $d_{s,\min/\max}$, and $\mathcal{H}_{s,\min/\max} = \bigoplus_r \mathcal{H}^r_{s,\min/\max} = \ker \Delta_{s,\min/\max}$. The following result is a direct consequence of Example 12.1, Corollary 15.13 and Lemma 9.1.

Corollary 16.1. (i) *$d_{s,\min/\max}$ is discrete.*

(ii) *If $M_+ = N_+ \times \mathbb{R}_+$ and $M_- = N_- \times \mathbb{R}_+$, then*

$$\mathcal{H}^r_{s,\min/\max} \cong \bigoplus_{r_+,r_-} H^{r_+}_{\min/\max}(N_+) \otimes H^{r_-}_{\min/\max}(N_-) ,$$

where (r_+, r_-) runs in the subset of $\mathbb{Z}^2$ defined by (6)–(8).

(iii) *If M_+ is the vertex stratum of $c(L_+)$ and $M_- = N_- \times \mathbb{R}_+$, then*

$$\mathcal{H}^r_{s,\min/\max} \cong \bigoplus_{r_-} H^{r_-}_{\min/\max}(N_-) ,$$

where r_- runs in the subset of $\mathbb{Z}$ defined by $r = m_- + r_- + 1$ and (8).

(iv) *If $M_+ = N_+ \times \mathbb{R}_+$ and M_- is the vertex stratum of $c(L_+)$, then*

$$\mathcal{H}^r_{s,\min/\max} \cong \bigoplus_{r_+} H^{r_+}_{\min/\max}(N_+) ,$$

where r_+ runs in the subset of $\mathbb{Z}$ defined by $r = m_- + r_+$ and (7).

(v) *If M_+ and M_- are the vertex strata of $c(L_+)$ and $c(L_-)$, then we have* $\dim \mathcal{H}^r_{s,\min/\max} = \delta_{r,m_-}$.

(vi) *If $e_s \in \mathcal{H}_{s,\min/\max}$ with norm one for each s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle he_s^\pm, e_s^\pm \rangle \to 1$ as $s \to \infty$.*

(vii) *Let $0 \le \lambda_{s,\min/\max,0} \le \lambda_{s,\min/\max,1} \le \cdots$ be the eigenvalues of $\Delta_{s,\min/\max}$, repeated according to their multiplicities. Given $k \in \mathbb{N}$, if $\lambda_{s,\min/\max,k} > 0$ for some s, then $\lambda_{s,\min/\max,k} \in O(s)$ as $s \to \infty$.*

(viii) *There is some $\theta > 0$ such that $\liminf_k \lambda_{s,\min/\max,k}\, k^{-\theta} > 0$.*

REMARK 27. According to Example 15.14, except for the case $m = 1$ and $d_{s,\min}$, the above local study of $d_{s,\min/\max}$ could be simplified by using the homeomorphism $\mathrm{can} \times \mathrm{id} : \mathbb{R}^m \times c(L) \to c(\mathbb{S}^{m-1}) \times c(L)$ and an isomorphism $c(\mathbb{S}^{m-1}) \times c(L) \to c(L')$ for some compact Thom-Mather stratification L' (Section 1.3 of Chapter 7). This would allow to consider only a quasi-isometry $c(L') \to c(L'_+) \times c(L'_-)$ and the model rel-Morse function on $M'_+ \times M'_-$ for strata $M'_\pm$ of $c(L'_\pm)$. The factors $\mathbb{R}^{m_\pm}$ could be forgotten in this way.

CHAPTER 17

Spectral properties of $\Delta_{\min/\max}$

Here, we prove Theorem I. Consider the notation of that theorem: M be a stratum with compact closure of a Thom-Mather stratification A, and g is an adapted metric on M. Let $\{(O_a, \xi_a)\}$ be a finite covering of $\overline{M}$ by charts of A. For each a, we have $\xi_a(O_a) = B_a \times c_{\epsilon_a}(L_a)$, where B_a is an open subset of $\mathbb{R}^{m_a}$ for some $m_a \in \mathbb{N}$, L_a is a compact Thom-Mather stratification, and $\epsilon_a > 0$. Then each ξ_a defines an open embedding $M \cap O_a$ into $\mathbb{R}^{m_a} \times M_a$ for some stratum M_a of L_a. We have, either $M_a = N_a \times \mathbb{R}_+$ for some stratum N_a of L_a, or $M_a = \{*_a\}$, where $*_a$ is the vertex of $c(L_a)$. If $M_a = N_a \times \mathbb{R}_+$, then $\xi_a(M \cap O_a) = B_a \times N_a \times (0, \epsilon_a)$. If $M_a = \{*_a\}$, then $\xi_a(M \cap O_a) = B_a \times \{*_a\} \equiv B_a$. Thus every $\xi_a(M \cap O_a)$ is, either open in $\mathbb{R}^{m_a}$, or open in $\mathbb{R}^{m_a} \times N_a \times \mathbb{R}_+$. By shrinking $\{(O_a, \xi_a)\}$ if necessary, we can assume that each diffeomorphism $\xi_a : M \cap O_a \to \xi_a(M \cap O_a)$ is quasi-isometric with respect to a model adapted metric on $\mathbb{R}^{m_a} \times M_a$.

By Lemma 8.4, there is a smooth partition of unity $\{\lambda_a\}$ on M subordinated to the open covering $\{M \cap O_a\}$ such that each function $|d\lambda_a|$ is rel-locally bounded; indeed, by shrinking $\{(O_a, \xi_a)\}$ again if necessary, we can assume that each $|d_a\lambda_a|$ is bounded. Also, by using Example 8.2, it is easy to construct another family $\{\tilde\lambda_a\} \subset C^\infty(M)$ such that $\tilde\lambda_a$ and $|d\tilde\lambda_a|$ are bounded, $\tilde\lambda_a = 1$ on $\operatorname{supp}\lambda_a$, and $\operatorname{supp}\tilde\lambda_a \subset M \cap O_a$. The existence of such families $\{\lambda_a\}$ and $\{\tilde\lambda_a\}$ is required to apply Propositions 10.5 and 10.6.

Let $d_{a,s}$ be the Witten's perturbation of d_a induced by the function $f_a = \frac{1}{2}\rho_a^2$ on $\mathbb{R}^{m_a} \times M_a$, where ρ_a is the canonical function of $\mathbb{R}^{m_a} \times c(L_a)$. According to Corollary 16.1-(i),(viii), each $d_{a,s,\min/\max}$ satisfies the properties stated in Theorem I, and let $\Delta_{a,s,\min/\max}$ denote the corresponding Laplacian.

By using Example 8.2 again, it is easy to see that there is some rel-admissible function h_a on $\mathbb{R}^{m_a} \times M_a$ such that $h_a = 0$ on $\xi(M \cap O_a)$ and $h_a = 1$ on the complement of some rel-compact neighborhood of $\xi(M \cap O_a)$ in $\mathbb{R}^m \times M_a$. Let $\hat d_{a,s}$ and $\widehat\Delta_{a,s}$ be the Witten's perturbation of d_a and Δ_a induced by the function $\hat f_a = h_a f_a$. The functions $|d_a\hat f_a - d_a f_a|$ and $|\operatorname{Hess}\hat f_a - \operatorname{Hess} f_a|$ are uniformly bounded, and therefore $\widehat\Delta_{a,s} - \Delta_{a,s}$ is a homomorphism with uniformly bounded norm by (104). By the min-max principle (see e.g. [**53**, Theorem XIII.1]), we get that $\hat d_{a,s,\min/\max}$ satisfies the properties stated in Theorem I. Then Theorem I follows by Propositions 10.5 and 10.6.

CHAPTER 18

Functions of the perturbed Laplacian on strata

The first ingredient of Theorem J is the following properties of the functional calculus of the perturbed Laplacian on strata.

Let M be a stratum of a compact Thom-Mather stratification endowed with an adapted metric, and let d and Δ be the de Rham derivative and Laplacian on M. Let f be any rel-admissible function on M, and let d_s and Δ_s be the corresponding Witten's perturbations of d and Δ. Since f is rel-admissible, for each s, $\Delta_s - \Delta$ is a homomorphism with uniformly bounded norm by (104). Hence $d_{s,\mathrm{min/max}}$ defines the same Sobolev spaces as $d_{\mathrm{min/max}}$. Moreover the properties stated in Theorem I can be extended to the perturbation $d_{s,\mathrm{min/max}}$ by (104) and the min-max principle.

For any rapidly decreasing function ϕ on $\mathbb{R}$, we easily get that $\phi(\Delta_{s,\mathrm{min/max}})$ is a Hilbert-Schmidt operator on $L^2\Omega(M)$ by the version of Theorem I-(ii) for $d_{s,\mathrm{min/max}}$. In fact, $\phi(\Delta_{s,\mathrm{min/max}})$ is a trace class operator because ϕ can be given as the product of two rapidly decreasing functions, $|\phi|^{1/2}$ and $\mathrm{sign}(\phi)\,|\phi|^{1/2}$, where $\mathrm{sign}(\phi)(x) = \mathrm{sign}(\phi(x)) \in \{\pm 1\}$ if $\phi(x) \neq 0$.

The extension of Theorem I-(ii) to $d_{s,\mathrm{min/max}}$ also shows that $\phi(\Delta_{s,\mathrm{min/max}})$ is valued in $W^\infty(d_{\mathrm{min/max}})$. However we do not have a "rel-Sobolev embedding theorem" describing $W^\infty(d_{\mathrm{min/max}})$; for instance, we do not know whether the elements of $W^m(d_{\mathrm{min/max}})$ are uniformly bounded for m large enough (see Chapter 21). We can only assert that $W^\infty(d_{\mathrm{min/max}}) \subset \Omega(M)$ by the usual elliptic regularity.

Like in the case of closed manifolds (see e.g. [**54**, Chapters 5 and 8]), it can be easily proved that $\phi(\Delta_{s,\mathrm{min/max}})$ can be given by a Schwartz kernel K, and $\mathrm{Tr}\,\phi(\Delta_{s,\mathrm{min/max}})$ equals the integral of the pointwise trace of K on the diagonal. But we do not know whether K is uniformly bounded by the indicated lack of a "rel-Sobolev embedding theorem".

CHAPTER 19

Finite propagation speed of the wave equation on strata

Let M be a stratum of a compact Thom-Mather stratification, g an adapted metric on M, and f a rel-Morse function on M. Let d_s, δ_s, D_s and Δ_s ($s \geq 0$) be the corresponding Witten's perturbed operators on $\Omega(M)$, defined by f and g. These operators make sense on complex valued differential forms as well as real valued ones. Complex coefficients are needed to consider the induced wave equation

$$\frac{d\alpha_t}{dt} - iD_s\alpha_t = 0 \;, \tag{165}$$

where $i = \sqrt{-1}$ and $\alpha_t \in \Omega(M)$ depends smoothly on $t \in \mathbb{R}$. We may also consider that (165) is satisfied only on some open subset of M.

If (165) holds on the whole of M, then, given $\alpha \in \mathcal{D}^\infty(d_{s,\min/\max})$, a usual energy estimate shows the uniqueness of the solution of (165) with the initial conditions $\alpha_0 = \alpha$ (see e.g. [**54**, Proposition 7.4]). In this case the solution is given by

$$\alpha_t = \exp(itD_{s,\min/\max})\alpha \;,$$

which belongs to $\mathcal{D}^\infty(d_{s,\min/\max})$ for all t.

It is known that compactly supported smooth solutions of (165) propagate at finite speed (see e.g. [**54**, Proposition 7.20]). To prove Theorem J, we need a version of that result for strata, stating this finite propagation speed towards/from the rel-critical points of f with forms in $\mathcal{D}^\infty(d_{s,\min/\max})$. For that purpose, we show first the corresponding result for the simple elliptic complexes of Sections 2 and 3 in Chapter 11.

Take a rel-Morse chart around each $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, like in Definition 8.7, with values in a stratum $M'_x = \mathbb{R}^{m_{x,+}} \times \mathbb{R}^{m_{x,-}} \times M_{x,+} \times M_{x,-}$ of a product $\mathbb{R}^{m_{x,+}} \times \mathbb{R}^{m_{x,-}} \times c(L_{x,+}) \times c(L_{x,-})$, where either $M_{x,\pm} = N_{x,\pm} \times \mathbb{R}_+$, or $M_{x,\pm}$ is the vertex stratum of $c(L_{x,\pm})$. We can assume that the domains of these rel-Morse charts are disjoint one another. Consider a model metric g_x on each M'_x. For each $\rho > 0$, let $B_{x,\pm,\rho}$ be the standard ball of radius ρ in $\mathbb{R}^{m_{x,\pm}}$. If $M_{x,+} = N_{x,+} \times \mathbb{R}_+$ and $M_{x,\pm} = N_{x,-} \times \mathbb{R}_+$, let

$$U_{x,\rho} = B_{x,+,\rho} \times B_{x,-,\rho} \times N_{x,+} \times (0,\rho) \times N_{x,-} \times (0,\rho) \subset M'_x \;.$$

If $M_{x,\pm}$ is the vertex stratum, remove the factor $N_{x,\pm} \times (0,\rho)$ from the definition of $U_{x,\rho}$ (or change it by the corresponding vertex stratum). Let $d'_{x,s}$, $\delta'_{x,s}$, $D'_{x,s}$ and $\Delta'_{x,s}$ denote Witten's perturbed operators on $\Omega(M'_x)$ defined by g_x and the model rel-Morse function (Chapter 16). The corresponding wave equation is

$$\frac{d\alpha_t}{dt} - iD'_{x,s}\alpha_t = 0 \;, \tag{166}$$

with $\alpha_t \in \Omega(M'_x)$ depending smoothly on $t \in \mathbb{R}$. By Propositions 15.3, 15.9 and 15.12, the following result clearly boils down to the case of Proposition 11.7.

PROPOSITION 19.1. *For $0 < a < b$, suppose that $\alpha_t \in \mathcal{D}^\infty(d'_{x,s,\min/\max})$, depending smoothly on $t \in \mathbb{R}$, satisfies* (166) *on $U_{x,b}$. The following properties hold:*

(i) *If $\operatorname{supp}\alpha_0 \subset M'_x \setminus U_{x,a}$, then $\operatorname{supp}\alpha_t \subset M'_x \setminus U_{x,a-|t|}$ for $0 < |t| \le a$.*
(ii) *If $\operatorname{supp}\alpha_0 \subset \overline{U_{x,a}}$, then $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$.*

There is some $\rho_0 > 0$ such that each $\overline{U_{x,\rho_0}}$ is contained in the image of the rel-Morse chart centered at x, and moreover these charts are disjoint one another. We will identify each U_{x,ρ_0} with an open subset of M via the rel-Morse chart. According to Example 7.11, we can choose g so that its restriction to each U_{x,ρ_0} is identified to the restriction of g_x.

PROPOSITION 19.2. *Let $0 < a < b < \rho_0$ and $\alpha \in L^2\Omega(M)$. The following properties hold for $\alpha_t = \exp(itD_{s,\min/\max})\alpha$:*

(i) *If $\operatorname{supp}\alpha \subset M \setminus U_{x,a}$, then $\operatorname{supp}\alpha_t \subset M \setminus U_{x,a-|t|}$ for $0 < |t| \le a$.*
(ii) *If $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$, then $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$.*

PROOF. Since $\exp(itD_{s,\min/\max})$ is bounded, we can take $\alpha \in \mathcal{D}^\infty(d_{s,\min/\max})$, and therefore $\alpha_t \in \mathcal{D}^\infty(d_{s,\min/\max})$ for all t. According to Remark 26, there is some $h \in C^\infty(M)$ such that $\operatorname{supp} h \subset U_{x,\rho_0}$, $h = 1$ on $U_{x,b}$, and $h\,\mathcal{D}^\infty(d_{s,\min/\max}) \subset \mathcal{D}^\infty(d_{s,\min/\max})$. Then $h\alpha_t$ satisfies (166) on $U_{x,b}$ and belongs to $\mathcal{D}^\infty(d'_{s,\min/\max})$. So, by Proposition 19.1,

- $h\alpha_t = 0$ on $U_{x,a-|t|}$ for $0 < |t| \le a$ if $\operatorname{supp}\alpha \subset M \setminus U_{x,a}$, and
- $\operatorname{supp} h\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \le b - a$ if $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$.

Thus the result follows because $h = 1$ on $U_{x,b}$. ∎

CHAPTER 20

Morse inequalities on strata

Here, we prove Theorem J. Consider the notation of Chapter 19.

1. Analytic inequalities

By (101), we have the isomorphism of complexes $e^{sf} : (\Omega_0(M), d_s) \to (\Omega_0(M), d)$. Since f is bounded, we also have the quasi-isometric isomorphism $e^{sf} : L^2\Omega(M) \to L^2\Omega(M)$. So we obtain the isomorphism of Hilbert complexes

$$e^{sf} : (\mathcal{D}(d_{s,\min/\max}), d_{s,\min/\max}) \to (\mathcal{D}(d_{\min/\max}), d_{\min/\max}) ,$$

and therefore

$$\beta^r_{\min/\max} = \dim H^r(\mathcal{D}(d_{s,\min/\max}), d_{s,\min/\max}) \tag{167}$$

for all $s \geq 0$. In fact, since $|df|$ is bounded, it also follows from (101) that

$$\mathcal{D}(d_{s,\min/\max}) = \mathcal{D}(d_{\min/\max}) , \quad d_{s,\min/\max} = d_{\min/\max} + s\, df \wedge \ .$$

Thus

$$e^{sf}\, \mathcal{D}(d_{\min/\max}) = \mathcal{D}(d_{\min/\max}) .$$

Let ϕ be a smooth rapidly decreasing function on $\mathbb{R}$ with $\phi(0) = 1$. Then the operator $\phi(\Delta_{s,\min/\max})$ is of trace class (Chapter 18), and set

$$\mu^r_{s,\min/\max} = \operatorname{Tr}(\phi(\Delta_{s,\min/\max,r})) .$$

By (167), the following result follows with the obvious adaptation of the proof of [**54**, Proposition 14.3].

Proposition 20.1. *We have the inequalities*

$$\beta^0_{\min/\max} \leq \mu^0_{\min/\max} ,$$
$$\beta^1_{\min/\max} - \beta^0_{\min/\max} \leq \mu^1_{s,\min/\max} - \mu^0_{s,\min/\max} ,$$
$$\beta^2_{\min/\max} - \beta^1_{\min/\max} + \beta^0_{\min/\max} \leq \mu^2_{s,\min/\max} - \mu^1_{s,\min/\max} + \mu^0_{s,\min/\max} ,$$

etc., and the equality

$$\chi_{\min/\max} = \sum_r (-1)^r\, \mu^r_{s,\min/\max} .$$

Proof. The proof is reproduced for the reader's convenience. By Theorem I, $\Delta_{\min/\max,r}$ is discrete with kernel of dimension $\beta^r_{\min/\max}$. Then there is some non-negative rapidly decreasing $\tilde{\phi} \in C^\infty(\mathbb{R})$ such that $\tilde{\phi}(0) = 1$ and $\tilde{\phi}(\lambda) = 0$ for all non-zero eigenvalue of $\Delta_{\min/\max,r}$; there is no loss of generality in assuming also that $\tilde{\phi} \leq \phi$. Then $\beta^r_{\min/\max} = \operatorname{Tr}(\tilde{\phi}(\Delta_{\min/\max,r}))$, so that

$$\mu^r_{\min/\max} - \beta^r_{\min/\max} = \operatorname{Tr}((\phi - \tilde{\phi})(\Delta_{\min/\max,r})) . \tag{168}$$

We may write $(\phi - \tilde{\phi})(\lambda) = \lambda\psi^2(\lambda)$, where $\psi \in C^\infty(\mathbb{R})$ is non-negative and rapidly decreasing, and vanishes at zero, obtaining

$$(\phi - \tilde{\phi})(\Delta_{\min/\max,r}) = \Delta_{\min/\max,r}\, \psi^2(\Delta_{\min/\max,r})\ . \tag{169}$$

Then

$$\begin{aligned}\operatorname{Tr}&\big(d_{\min/\max,r-1}\, \delta_{\max/\min,r-1}\, \psi^2(\Delta_{\min/\max,r})\big)\\ &= \operatorname{Tr}\big(\psi(\Delta_{\min/\max,r})\, d_{\min/\max,r-1}\, \delta_{\max/\min,r-1}\, \psi(\Delta_{\min/\max,r})\big)\\ &= \operatorname{Tr}\big(\, \delta_{\max/\min,r-1}\, \psi^2(\Delta_{\min/\max,r})\, d_{\min/\max,r-1}\big)\\ &= \operatorname{Tr}\big(\, \delta_{\max/\min,r-1}\, d_{\min/\max,r-1}\, \psi^2(\Delta_{\min/\max,r-1})\big)\ .\end{aligned}$$

By (168), (169) and (81), it follows that

$$\begin{aligned}(\mu^r_{\min/\max} - \beta^r_{\min/\max}) &- (\mu^{r-1}_{\min/\max} - \beta^{r-1}_{\min/\max}) + (\mu^{r-2}_{\min/\max} - \beta^{r-2}_{\min/\max}) - \cdots\\ &= \operatorname{Tr}\big(d_{\min/\max,r-1}\, \delta_{\max/\min,r-1}\, \psi^2(\Delta_{\min/\max,r})\big)\\ &= \operatorname{Tr}\big((d_{\min/\max,r}\, \psi(\Delta_{\min/\max,r}))^*\, d_{\min/\max,r}\, \psi(\Delta_{\min/\max,r})\big)\\ &\geq 0\ .\end{aligned}$$

For $r = n$, this is an equality. □

2. Null contribution away from the critical points

By (104) and because $|df|$ and $|\operatorname{Hess} f|$ are bounded on M, we have

$$\mathcal{D}(\Delta_{s,\min/\max}) = \mathcal{D}(\Delta_{\min/\max})\ , \tag{170}$$

$$\Delta_{s,\min/\max} = \Delta_{\min/\max} + s\,\mathbf{Hess} f + s^2\,|df|^2 \tag{171}$$

for all $s \geq 0$.

For $\rho \leq \rho_0$, let $U_\rho = \bigcup_x U_{x,\rho}$, with x running in $\operatorname{Crit}_{\mathrm{rel}}(f)$. Fix some $\rho_1 > 0$ such that $3\rho_1 < \rho_0$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Hilbert subspaces of $L^2\Omega(M)$ consisting of forms essentially supported in $M \setminus U_{\rho_1}$ and $M \setminus U_{2\rho_1}$, respectively. It follows from (170) and (171) that there is some $C > 0$ such that[1]

$$\Delta_{s,\min/\max} \geq \Delta_{\min/\max} + Cs^2 \quad \text{on} \quad \mathfrak{G} \cap \mathcal{D}(\Delta_{\min/\max}) \tag{172}$$

if s is large enough.

Let h be a rel-admissible function on M such that $h \leq 0$, $h \equiv 1$ on U_{ρ_0} and $h \equiv 0$ on $M \setminus U_{2\rho_1}$ (see Example 8.2). Then $T_{s,\min/\max} = \Delta_{s,\min/\max} + hCs^2$, with domain $\mathcal{D}(\widetilde{\Delta}_{\min/\max})$, is essentially self-adjoint in $L^2\Omega(M)$ with a discrete spectrum, and moreover

$$T_{s,\min/\max} \geq \Delta_{\min/\max} + Cs^2 \tag{173}$$

for s is large enough by (172).

Fix some[2] $\phi \in \mathcal{S}_{\mathrm{ev}}$ such that $\phi \geq 0$, $\phi(0) = 1$ and $\operatorname{supp}\hat{\phi} \subset [-\rho_1, \rho_1]$, and let $\psi \in \mathcal{S}$ such that $\phi(x) = \psi(x^2)$. By using Proposition 19.2-(i), the argument of the

[1] Recall that, for symmetric operators S and T in a Hilbert space, with the same domain $\mathcal{D}$, it is said that $S \leq T$ if $\langle Su, u\rangle \leq \langle Tu, u\rangle$ for all $u \in \mathcal{D}$.

[2] The Schwartz functions with compactly supported Fourier transform are characterized by the Paley-Wiener-Schwartz theorem (see e.g. [**36**, Theorem 7.3.1]); they form a dense subalgebra of $\mathcal{S}$, which is invariant by linear changes of variables.

first part of the proof of [**54**, Lemma 14.6] can be obviously adapted to show the following.

LEMMA 20.2. $\psi(\Delta_{s,\min/\max}) = \psi(T_{s,\min/\max})$ *on* $\mathfrak{H}$.

PROOF. This proof is also reproduced for the reader's convenience. For $\alpha \in \mathcal{D}^\infty(\Delta_{\min/\max})$ supported in $M \setminus U_{2\rho_1}$, consider the time-dependent differential form

$$\alpha_t = \cos(tD_{s,\min/\max})\alpha = \frac{1}{2}\left(e^{itD_{s,\min/\max}} + e^{-itD_{s,\min/\max}}\right)\alpha$$

in $\mathcal{D}^\infty(\Delta_{\min/\max})$. It is a solution of the differential equation

$$\frac{\partial^2\alpha_t}{\partial t^2} + \Delta_s\alpha_t = 0$$

with initial conditions $\alpha_0 = \alpha$ and $\dot\alpha_0 = 0$; in fact, it is the unique solution, as one can easily check by verifying that the "energy"

$$\left\|\frac{\partial\alpha_t}{\partial t}\right\|^2 + \langle\Delta_s\alpha_t, \alpha_t\rangle$$

is preserved. By Proposition 19.2-(i), α_t is supported on $M \setminus U_{\rho_1}$ if $|t| < \rho_1$, and therefore $\Delta_s\alpha_t = T_{s,\min/\max}\alpha_t$. Thus α_t for $|t| < \rho_1$ is also the unique solution to the equation

$$\frac{\partial^2\alpha_t}{\partial t^2} + T_{s,\min/\max}\alpha_t = 0$$

with the same initial conditions. So $\alpha_t = \cos(t\sqrt{T_{s,\min/\max}})\alpha$.

Now, $\hat\phi$ has support in $[-\rho_1, \rho_1]$ and is even (since ϕ is). Therefore

$$\begin{aligned}
\psi(\Delta_{s,\min/\max})\alpha &= \phi(D_{s,\min/\max})\alpha \\
&= \frac{1}{2\pi}\int_{-\rho_1}^{\rho_1} e^{itD_{s,\min/\max}}\alpha\,\hat\phi(t)\,dt \\
&= \frac{1}{\pi}\int_0^{\rho_1} \hat\phi(t)\,\cos(tD_{s,\min/\max})\alpha\,dt \\
&= \frac{1}{\pi}\int_0^{\rho_1} \hat\phi(t)\,\alpha_t\,dt \\
&= \frac{1}{\pi}\int_0^{\rho_1} \hat\phi(t)\,\cos\left(t\sqrt{T_{s,\min/\max}}\right)\alpha\,dt \\
&= \cdots = \psi(T_{s,\min/\max})\alpha\,.
\end{aligned}$$

Then the result follows because $\mathcal{D}^\infty(\Delta_{\min/\max}) \cap \mathfrak{H}$ is dense in $\mathfrak{H}$. □

Let $\Pi : L^2\Omega(M) \to \mathfrak{H}$ denote the orthogonal projection. According to Chapter 18, $\psi(\Delta_{s,\min/\max})$ is of trace class for all $s \geq 0$. Then the self-adjoint operator $\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi$ is also of trace class (see e.g. [**54**, Proposition 8.8]).

LEMMA 20.3. $\operatorname{Tr}(\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi) \to 0$ *as* $s \to \infty$.

PROOF. Let

$$0 \leq \lambda_{\min/\max,0} \leq \lambda_{\min/\max,1} \leq \cdots\,, \quad 0 \leq \lambda_{s,\min/\max,0} \leq \lambda_{s,\min/\max,1} \leq \cdots$$

be the eigenvalues of $\Delta_{\min/\max}$ and $T_{s,\min/\max}$, respectively, repeated according to their multiplicities. By (173) and the min-max principle, we have

$$\lambda_{s,\min/\max,k} \geq \lambda_{\min/\max,k} + Cs^2$$

for s large enough. So

$$\operatorname{Tr}(\psi(T_{s,\min/\max})) = \sum_k \psi(\lambda_{s,\min/\max,k}) \le \sum_k \psi(\lambda_{\min/\max,k} + Cs^2)$$

for s large enough, giving $\operatorname{Tr}(\psi(T_{s,\min/\max})) \to 0$ as $s \to \infty$ since ψ is rapidly decreasing. Then the result follows because

$$\operatorname{Tr}(\Pi\,\psi(\Delta_{s,\min/\max})\,\Pi) = \operatorname{Tr}(\Pi\,\psi(T_{s,\min/\max})\,\Pi) \le \operatorname{Tr}(\psi(T_{s,\min/\max}))$$

by Lemma 20.2. □

3. Contribution from the rel-critical points

The following is a direct consequence of Corollary 16.1.

COROLLARY 20.4. *If h is a bounded measurable function on $\mathbb{R}_+$ such that $h(\rho) \to 1$ as $\rho \to 0$, then*

$$\lim_{s\to\infty} \operatorname{Tr}(h(\rho)\,\phi(\Delta'_{x,s,\min/\max,r})) = \lim_{s\to\infty} \operatorname{Tr}\phi(\Delta'_{x,s,\min/\max,r}) = \nu^r_{x,\min/\max}\,.$$

For each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\mathfrak{H}}_x \subset L^2\Omega(M)$ be the Hilbert subspace of differential forms supported in $\overline{U_{x,2\rho_1}}$; it can be also considered as a Hilbert subspace of $L^2\Omega(M'_x)$ since g and g_x have identical restrictions to U_{x,ρ_0}. Moreover Δ_s and $\Delta'_{x,s}$ can be identified on differential forms supported in U_{x,ρ_0}. By using Proposition 19.2-(ii), the argument of Lemma 20.2 can be obviously adapted to show the following.

LEMMA 20.5. $\phi(\Delta_{s,\min/\max}) \equiv \phi(\Delta'_{x,s,\min/\max})$ *on $\widetilde{\mathfrak{H}}_x$ for all $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$.*

For each $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}}_x$ and $\widetilde{\Pi}'_x : L^2\Omega(M'_x) \to \widetilde{\mathfrak{H}}_x$ denote the orthogonal projections. Since the subspaces $\widetilde{\mathfrak{H}}_x$ are orthogonal to each other, $\Pi := \sum_x \widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}} := \sum_x \widetilde{\mathfrak{H}}_x$ is the orthogonal projection.

LEMMA 20.6. $\operatorname{Tr}(\widetilde{\Pi}\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}) \to \nu^r_{\min/\max}$ *as $s \to \infty$.*

PROOF. By Corollary 20.4 and Lemma 20.5, and because Π'_x is the multiplication operator by the characteristic function of U_{ρ_1} in M'_x,

$$\begin{aligned}
\lim_{s\to\infty} \operatorname{Tr}(\widetilde{\Pi}\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}) &= \lim_{s\to\infty} \sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)} \operatorname{Tr}(\widetilde{\Pi}_x\,\phi(\Delta_{s,\min/\max,r})\,\widetilde{\Pi}_x) \\
&= \lim_{s\to\infty} \sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)} \operatorname{Tr}(\widetilde{\Pi}'_x\,\phi(\Delta'_{x,s,\min/\max,r})\,\widetilde{\Pi}'_x) \\
&= \sum_{x\in\operatorname{Crit}_{\mathrm{rel}}(f)} \nu^r_{x,\min/\max} = \nu^r_{\min/\max}\,. \quad \square
\end{aligned}$$

Now,

$$\lim_{s\to\infty} \operatorname{Tr}(\phi(\Delta_{s,\min/\max,r})) = \nu^r_{\min/\max}$$

by Lemmas 20.3 and 20.6, and because $\Pi + \widetilde{\Pi} = 1$, showing Theorem J by Proposition 20.1.

CHAPTER 21

Remark on the Sobolev spaces on strata

Our version of the Sobolev spaces on strata, $W^m(d_{\min/\max})$, may depend on the chosen adapted metric; thus there is no "rel-version" of the elliptic estimate. By taking local charts and arguing like in Chapter 17, it is enough to check this assertion for the perturbed local models $d^{\pm}_{s,\min/\max}$.

With the notation of Section 1 in Chapter 14, consider the case where n is odd, $r = \frac{n-1}{2}$ and $a = 0$; thus $\sigma = 0$. We have $\chi_0\,\gamma \in W^\infty(d^{\pm}_{s,\min/\max})$ with the metric g. Let $\tilde g'$ be another adapted metric on N such that $\widetilde{\Delta}'\gamma \neq 0$, and consider the corresponding adapted metric $g' = \rho^{-2}\tilde g' + d\rho^2$ on M. Let $\widetilde{\Delta}'$ be the laplacian on $\Omega(N)$ defined by $\tilde g'$, Δ' the Laplacian on $\Omega(M)$ defined by g', and $\Delta'^{\pm}_s$ the Witten's perturbation of Δ' induced by the function $\pm\frac{1}{2}\rho^2$. Let $\langle\ ,\ \rangle^{\sim\prime}$ and $\langle\ ,\ \rangle'$ denote the scalar products of $L^2\Omega(N,\tilde g')$ and $L^2\Omega(M,g')$, respectively, and let $\|\ \|^{\sim\prime}$ denote the norm defined by $\langle\ ,\ \rangle^{\sim\prime}$. By Corollary 13.7, we have $\Delta'^{\pm}_s = \rho^{-2}\widetilde{\Delta}' + H \mp s$ on $C^\infty(\mathbb{R}_+)\,\gamma$. Then

$$\langle \Delta'^{\pm}_s(\chi_0\,\gamma), \chi_0\gamma\rangle' = \langle\widetilde{\Delta}'\gamma,\gamma\rangle^{\sim\prime}\int_0^\infty \rho^{-2}\chi_0^2\,d\rho + \|\gamma\|^{\sim\prime 2}(1\mp 1)s = \infty$$

according to (110) and Section 1 of Chapter 14, and because $\chi_0(\rho) = \sqrt{2}p_0e^{-s\rho^2/2}$ is bounded away from zero for $0 < \rho \leq 1$. So $\chi_0\,\gamma \notin W^1(d^{\pm}_{s,\min/\max})$ with the metric g', obtaining different spaces $W^1(d^{\pm}_{s,\min/\max})$ by using g and g'.

The above observation is related with the following problem.

Problem 21.1. Let M be a stratum of an arbitrary compact stratification endowed with an adapted metric, and let $L^1\Omega(M)$ denote the Banach space of uniformly bounded measurable differential forms on M. Is there a continuous inclusion of $W^m(d_{\min/\max})$ into $L^1\Omega(M)$ for m large enough?

For the perturbation P of harmonic oscillator indicated in Chapter 5, the corresponding version of this problem has an affirmative answer when $a \geq 0$ (Corollary H-(iii)). If the spaces $W^m(d_{\min/\max})$ were independent of the adapted metric, we could give an affirmative answer to Problem 21.1 by using the local arguments of this chapter and induction. An affirmative solution of Problem 21.1 would allow to adapt the nice arguments of [**54**, Lemma 14.6] to show a stronger version of Lemma 20.3: the Schwartz kernel of $\psi(\Delta_{s,\min/\max})$ would converge uniformly to zero on $(M\setminus U_{2\rho_1})\times(M\setminus U_{2\rho_1})$.

Conclusion

The main goal of the thesis was to prove a version of Morse inequalities for the minimum and maximum ideal boundary conditions of the de Rham complex on strata endowed with adapted metrics, taken in compact Thom-Mather stratifications. The analytic method of Witten was used, involving his perturbation of the de Rham complex induced by our version of Morse functions on strata. The cohomology of this minimum ideal boundary condition is isomorphic to the intersection homology with lower middle perversity; there are analogous isomorphisms with other types of adapted metrics and other perversities.

Several new features have shown up in this work. First, the local analysis around our version of critical points was reduced to the study of an operator related to the so called Dunkl harmonic oscillator, which was recently very much used in Quantum Mechanics to describe the interaction among several particles. This led us to prove eigenfunction estimates and embedding results for the Dunkl harmonic oscillator on the line, which have their own interest.

Second, it turns out that, surprisingly, the Sobolev spaces defined by the perturbed Laplacian depend on the choice of the metric, and therefore the usual way to prove Sobolev inequalities does not work, even though they could be true. Because of this lack of Sobolev inequalities, some of the arguments of Witten's method cannot be made. Thus new types of arguments were produced to solve that problem, mainly using certain weak version of the Weyl's assymptotic formula. This formula is proved first by showing that it has a local character and holds for the local models.

Our Morse inequalities on strata seem to be new. Their expressions are a priori different from those of the Morse inequalities of Goresky-MacPherson, which involve intersection homology with lower middle perversity on complex analytic varieties with Whitney stratifications. Also, our version of Morse functions is different from those of U. Luwig, who studied Witten's perturbation for the special case of conformally conic manifolds. We hope there will be future applications, specially when we consider functions canonically associated to geometric or physical situations.

In particular, our version of Morse inequalities applies to the case of a smooth action of a compact Lie group G on a closed manifold M, and functions on the orbit type strata of $G\backslash M$ induced by invariant Morse-Bott functions on M whose critical manifolds are orbits. This provides a rich family of examples.

Several open problems emerge from our work: a "rel-Morse lemma", a "rel-strong C^∞ topology" where the rel-Morse functions should form a dense subset, a "rel-Sobolev lemma", a version with "rel-Morse-Bott functions" , etc. But the main one is the possible generalization to other types of adapted metrics. We hope that even completely new types of adapted metrics could be tackled, which could correspond to generalizations of intersection homology still to be defined, whose perversities would be a sequence of functions instead of naturals. This would require a generalization of our results on the Dunkl harmonic oscillator to other kind of perturbations of the harmonic oscillator, which seem to be perfectly possible.

Resumen

El principal objetivo de la tesis es usar el método de la perturbación de Witten para probar una versión de las desigualdades de Morse para la condición ideal de frontera mínima y máxima del complejo de de Rham en estratos, dotados con métricas adaptadas, donde se consideran estratificaciones de Thom-Mather compactas. Para lograrlo, se estudian primero estimaciones de autofunciones y resultados de embebimiento para el oscilador armónico de Dunkl en la recta, que se generalizan a otros operadores en $\mathbb{R}_+$. El estudio de estos operadores es el ingrediente clave en nuestro análisis local de la perturbación de Witten.

Así, esta tesis tiene dos partes principales, Partes 1 y 2. La primera está dedicada al estudio de estimaciones de autofunciones y resultados de embebimiento para el oscilador harmónico de Dunkl y operadores relacionados. La segunda aborda el estudio de la perturbación de Witten en estratos, en donde se usa la primera parte.

Este trabajo aparece en los preprints [**1, 2**].

Pasamos a comentar los capítulos por separado y enunciar sus resultados principales.

Estimaciones de autofunciones y teoremas de embebimiento

El operador de Dunkl T_σ en $C^\infty(\mathbb{R})$, dependiendo de un parámetro $\sigma > -1/2$, es la perturbación de la derivada usual que se puede definir como $T_\sigma = \frac{d}{dx}$ en funciones pares y $T_\sigma = \frac{d}{dx} + 2\sigma\frac{1}{x}$ en funciones impares. Este tipo de operador, más generalmente en $\mathbb{R}^n$, fue introducido por C.F. Dunkl [**21, 22, 23, 24, 25**]. Dio lugar a lo que ahora se denomina teoría de Dunkl (véase el panorama presentado en [**57**]). Este área tuvo un gran desarrollo en los últimos años, principalmente debido a sus aplicaciones en modelos cuánticos de Calogero-Moser-Sutherland (véase por ejemplo [**10, 52, 37, 38, 61, 3, 4**]). En particular, el oscilador armónico de Dunkl [**55, 26, 50, 49**] es $L_\sigma = -T_\sigma^2 + sx^2$, dependiendo de $s > 0$; es decir, se define usando T_σ en vez de d/dx en la expresión del oscilador armónico usual, $H = -\frac{d^2}{dx^2} + sx^2$.

Por otra parte, sea p_k la sucesión de polinomios ortogonales para la medida $e^{-sx^2}|x|^{2\sigma}\,dx$, considerados con norma uno y coeficiente principal positivo. Salvo normalización, éstos son los polinomios de Hermite generalizados [**59**, p. 380, Problema 25]; véase también [**16, 20, 27, 17, 55, 56**]. Denotemos por $x_{k,k} < x_{k,k-1} < \cdots < x_{k,1}$ las raíces de cada p_k; en particular, $x_{k,k/2}$ es la raíz positiva más pequeña si k es par. Las correspondientes funciones de Hermite generalizadas son $\phi_k = p_k e^{-sx^2/2}$.

Se sabe que L_σ, con dominio el espacio de Schwartz $\mathcal{S} = \mathcal{S}(\mathbb{R})$, es esencialmente auto-adjunto en $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Además el espectro de su extensión autoadjunta, denotada por $\mathcal{L}_\sigma$, está formada por los autovalores $(2k+1+2\sigma)s$ ($k \in \mathbb{N}$), cuyas autofunciones correspondientes son las funciones ϕ_k.

Demostramos estimaciones asintóticas de las funciones ϕ_k según $k \to \infty$, que son usadas para probar teoremas de embebimiento, y estos resultados se extienden a otras perturbaciones relacionadas de H. Aunque consideramos sólo el oscilador armónico de Dunkl en la recta para empezar, este trabajo es más difícil que en el caso de H, y tiene algunos aspectos nuevos. Podría dar también una indicación de cómo proceder en dimensiones mayores.

Para conseguir estimaciones uniformes, se consideran las funciones $\xi_k = |x|^\sigma \phi_k$ en vez de ϕ_k. Cumplen la ecuación $\xi_k'' + q_k \xi_k = 0$, donde $q_k = (2k+1+2\sigma)s - s^2x^2 - \bar\sigma_k x^{-2}$ con $\bar\sigma_k = \sigma(\sigma - (-1)^k)$. Sea $\widehat{I}_k = q^{-1}(\mathbb{R}_+)$ (la región de oscilación), que es de la forma: $(-b_k, -a_k) \cup (a_k, b_k)$ si $\bar\sigma_k > 0$ (para $k > 0$), $(-b_k, b_k)$ si $\bar\sigma_k = 0$, o $(-b_k, 0) \cup (0, b_k)$ si $\bar\sigma_k < 0$, donde $b_k > a_k > 0$ con $b_k \in O(k^{1/2})$ y $a_k \in O(k^{-1/2})$ cuando $k \to \infty$. Si $\bar\sigma_k \geq 0$, entonces sea $\widehat{J}_k = \widehat{I}_k$. Cuando $\bar\sigma_k < 0$ y k es suficientemente grande, la ecuación $q_k(b) = 4\pi/b^2$ tiene dos soluciones positivas, $b_{k,+} < b_{k,-}$, con $b_{k,+} \in O(k^{-1/2})$. Entonces sea $\widehat{J}_k = (-b_k, -b_{k,+}] \cup [b_{k,+}, b_k)$. La primera estimación importante que se demuestra en la Parte 1 es la siguiente.

Teorema A. *Existen $C, C', C'' > 0$, dependiendo de σ y s, tal que, para $k \geq 1$:*

(i) $\xi_k^2(x) \leq C/\sqrt{q_k(x)}$ *para todo* $x \in \widehat{J}_k$ *;*
(ii) *si k es impar o $\sigma \geq 0$, entonces* $\xi_k^2(x) \leq C'k^{-1/6}$ *para todo* $x \in \mathbb{R}$ *; y,*
(iii) *si k es par y $\sigma < 0$, entonces* $\xi_k^2(x) \leq C''k^{-1/6}$ *si* $|x| \geq x_{k,k/2}$.

En el caso del Teorema A-(iii), la estimación de ξ_k no se puede extender a $\mathbb{R} \setminus \{0\}$ porque estas funciones no están acotadas cerca de cero. Por tanto alguna condición del tipo $|x| \geq x_{k,k/2}$ debe ser asumida; el significado de esta condición se clarifica indicando que $x_{k,k/2} \in O(k^{-1/2})$ cuando $k \to \infty$. Este punto débil es complementado por el resultado siguiente.

Teorema B. *Supongamos que $\sigma < 0$. Existe algún $C''' > 0$, dependiendo de σ y s, tal que $\phi_k^2(x) \leq C'''$ para todo k par y $x \in \mathbb{R}$.*

El siguiente teorema afirma que el tipo de estimaciones del Teorema A-(ii),(iii) son óptimas.

Teorema C. *Existen $C^{(IV)}, C^{(V)} > 0$, dependiendo de σ y s, tal que, para $k \geq 1$:*

(i) $\max_{x\in\mathbb{R}} \xi_k^2(x) \geq C^{(IV)}k^{-1/6}$*; y,*
(ii) *si k es par y $\sigma < 0$, entonces* $\max_{|x|\geq x_{k,k/2}} \xi_k^2(x) \geq C^{(V)}k^{-1/6}$.

Para demostrar los Teoremas A–C, aplicamos el método que Bonan-Clark han usado con H [**6**]. Las estimaciones se cumplen con ξ_k en vez de ϕ_k porque el método se puede aplicar a la conjugación $K_\sigma = |x|^\sigma L_\sigma |x|^{-\sigma}$. Este método tiene dos pasos: primero, se estima la distancia de cualquier punto x en una región de oscilación a alguna raíz $x_{k,i}$, y, segundo, el valor de $\xi_k^2(x)$ se estima usando $|x - x_{k,i}|$. Estos cálculos para K_σ son mucho más complicados que en [**6**]; de hecho, se consideran varios casos distintos, algunos con diferencias significativas; por ejemplo, algunas raíces $x_{k,i}$ pueden estar fuera de la región de oscilación $\widehat{I}_k$, y las funciones ξ_k pueden no estar acotadas, según se ha dicho.

La distribución asintótica de las raíces $x_{k,i}$ según $k \to \infty$ también tiene una interpretación como medida muy conocida [**28, 62, 63**]; especialmente, los polinomios de Hermite generalizados se consideran en [**62**, Sección 4]. Sin embargo la convergencia débil de medidas considerada en esas publicaciones no parece aportar la aproximación asintótica de las raíces necesaria en el primer paso.

Para cada $m \in \mathbb{N}$, sea $\mathcal{S}^m$ el espacio de Banach de funciones $\phi \in C^m(\mathbb{R})$ con $\sup_x |x^i \phi^{(j)}(x)| < \infty$ para $i + j \leq m$; por tanto $\mathcal{S} = \bigcap_m \mathcal{S}^m$ con la correspondiente topología de Fréchet. Por otra parte, para cada número real $m \geq 0$, sea W_σ^m la versión del espacio de Sobolev obtenido como compleción de Hilbert de $\mathcal{S}$ respecto del producto escalar definido por $\langle \phi, \psi \rangle_{W_\sigma^m} = \langle (1 + \mathcal{L}_\sigma)^m \phi, \psi \rangle_\sigma$, donde $\langle\ ,\ \rangle_\sigma$ denota el producto escalar de $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Sea también $W_\sigma^\infty = \bigcap_m W_\sigma^m$ con la correspondiente topología de Fréchet. El subíndice ev/odd se añade a cualquier espacio de funciones en $\mathbb{R}$ para indicar su subespacio de funciones pares/impares. Se demuestran los teoremas de embebimiento siguientes en la Parte 1; el segundo es una versión del teorema de embebimiento de Sobolev.

Teorema D. *Para cada $m \geq 0$, $\mathcal{S}_{\text{ev/odd}}^{M_{m',\text{ev/odd}}} \subset W_{\sigma,\text{ev/odd}}^m$ continuamente si $m' \in \mathbb{N}$, $m' - m > 1/2$, y*

$$M_{m',\text{ev/odd}} = \begin{cases} \frac{3m'}{2} + \frac{m'}{4} \lceil \sigma \rceil (\lceil \sigma \rceil + 3) + \lceil \sigma \rceil & \text{si } \sigma \geq 0 \text{ y } m' \text{ es par} \\ \frac{5m'}{2} & \text{si } \sigma < 0 \text{ y } m' \text{ es par}\,, \end{cases}$$

$$M_{m',\text{ev}} = \begin{cases} \frac{3m'-1}{2} + \frac{m'-1}{4} \lceil \sigma \rceil (\lceil \sigma \rceil + 3) + \lceil \sigma \rceil & \text{si } \sigma \geq 0 \text{ y } m' \text{ es impar} \\ \frac{5m'+1}{2} & \text{si } \sigma < 0 \text{ y } m' \text{ es impar}\,, \end{cases}$$

$$M_{m',\text{odd}} = \begin{cases} \frac{3m'+1}{2} + \frac{m'+1}{4} \lceil \sigma \rceil (\lceil \sigma \rceil + 3) + \lceil \sigma \rceil & \text{si } \sigma \geq 0 \text{ y } m' \text{ es impar} \\ \frac{5m'+7}{2} & \text{si } \sigma < 0 \text{ y } m' \text{ es impar}\,. \end{cases}$$

Teorema E. *Para todo $m \in \mathbb{N}$, $W_\sigma^{m'} \subset \mathcal{S}^m$ continuamente si*

$$m' - m > \begin{cases} 4 + \frac{1}{2} \lceil \sigma \rceil (\lceil \sigma \rceil + 1) & \text{si } \sigma \geq 0 \\ 4 & \text{si } \sigma < 0\,. \end{cases}$$

Además $W_{\sigma,\text{ev}}^{m'} \subset \mathcal{S}_{\text{ev}}^0$ continuamente si $\sigma < 0$ y $m' > 2$.

Corolario F. *$\mathcal{S} = W_\sigma^\infty$ como espacios de Fréchet.*

En otras palabras, Corolario F afirma que un elemento $\phi \in L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$ está en $\mathcal{S}$ si y sólo si los "coeficientes de Fourier" $\langle \phi, \phi_k \rangle_\sigma$ son de decrecimiento rápido en k. Esto también significa que $\mathcal{S} = \bigcap_m \mathcal{D}(\mathcal{L}_\sigma^m)$ (el "core" diferenciable[1] $\mathcal{D}^\infty(\mathcal{L}_\sigma^m)$) porque la sucesión de autovalores de $\mathcal{L}_\sigma$ es de orden $O(k)$ cuando $k \to \infty$.

Introducimos una versión perturbada $\mathcal{S}_\sigma^m$ de cada $\mathcal{S}^m$ (Capítulo 3), cuya definición involucra a T_σ en vez de $\frac{d}{dx}$ y está inspirada por los Teoremas A y B. Cumplen resultados de embebimiento mucho más simples (Capítulo 4): $S_\sigma^{m'} \subset W_\sigma^m$ si $m' - m > 1/2$, y $W_\sigma^{m'} \subset \mathcal{S}_\sigma^m$ si $m' - m > 1$. La demostración del segundo embebimiento usa las estimaciones de los Teoremas A y B. Aunque $\mathcal{S} = \bigcap_m \mathcal{S}_\sigma^m$, las relaciones de inclusión entre los espacios $\mathcal{S}_\sigma^m$ y $\mathcal{S}^{m'}$ son complicadas, lo que motiva la complejidad de los Teoremas D y E.

[1] Recuérdese que un "*core*" de un operador cerrado densamente definido T entre espacios de Hilbert es cualquier subespacio de su dominio $\mathcal{D}(T)$ que es denso con la norma de la gráfica. Si T es auto-adjunto, entonces $\mathcal{D}^\infty(T) = \bigcap_{k \geq 1} \mathcal{D}(T^k)$ es un "core" de T, que se llama su *"core" diferenciable* [**11**].

A continuación, consideramos otras perturbaciones de H en $\mathbb{R}_+$ (Capítulo 5). Sea $\mathcal{S}_{\mathrm{ev},U}$ el espacio de restricciones de funciones de Schwartz pares a algún abierto U, y sea $\phi_{k,U} = \phi_k|_U$. La notación $\mathcal{S}_{\mathrm{ev},+}$ y $\phi_{k,+}$ se usa si $U = \mathbb{R}_+$.

Teorema G. *Sea $P = H - 2f_1\frac{d}{dx} + f_2$, donde $f_1 \in C^1(U)$ y $f_2 \in C(U)$ para algún subconjunto abierto $U \subset \mathbb{R}_+$ de medida de Lebesgue completa. Asumimos que $f_2 = \sigma(\sigma-1)x^{-2} - f_1^2 - f_1'$ para algún $\sigma > -1/2$. Sea $h = x^\sigma e^{-F_1}$, donde $F_1 \in C^2(U)$ es una primitiva de f_1. Entonces se cumplen las siguientes propiedades:*

(i) *P, con dominio $h\,\mathcal{S}_{\mathrm{ev},U}$, es esencialmente auto-adjunto en $L^2(\mathbb{R}_+, e^{2F_1}\,dx)$;*

(ii) *el espectro de su extensión auto-adjunta, denotada por $\mathcal{P}$, está formado por los autovalores $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ con multiplicidad uno y autofunciones normalizadas $\sqrt{2}\,h\phi_{2k,U}$; y*

(iii) *$\mathcal{D}^\infty(\mathcal{P}) = h\,\mathcal{S}_{\mathrm{ev},U}$.*

Este teorema se deduce mostrando que la condición enunciada sobre f_1 y f_2 caracteriza los casos en los que P se puede obtener con el siguiente proceso: primero restringiendo L_σ a funciones pares, después restringiendo a U, y finalmente conjugando por h. El término de P con $\frac{d}{dx}$ se puede quitar conjugando con el producto de una función positiva, obteniendo el operador $H + \sigma(\sigma-1)x^{-2}$.

Se dan varios ejemplos de ese tipo de operador P. Por ejemplo, se obtiene lo siguiente.

Corolario H. *Sea $P = H - 2c_1x^{-1}\frac{d}{dx} + c_2x^{-2}$ con $c_1, c_2 \in \mathbb{R}$. Si hay algún $a \in \mathbb{R}$ tal que $a^2 + (2c_1-1)a - c_2 = 0$ y $\sigma := a + c_1 > -1/2$, entonces:*

(i) *P, con dominio $x^a\,\mathcal{S}_{\mathrm{ev},+}$, es esencialmente auto-adjunto en $L^2(\mathbb{R}_+, x^{2c_1}\,dx)$;*

(ii) *el espectro de su extensión auto-adjunta, denotada por $\mathcal{P}$, está formada por los auto-valores $(4k+1+2\sigma)s$ $(k \in \mathbb{N})$ con multiplicidad uno y autofunciones normalizadas $\sqrt{2}\,x^a\phi_{2k,+}$; y*

(iii) *$\mathcal{D}^\infty(\mathcal{P}) = x^a\,\mathcal{S}_{\mathrm{ev},+}$.*

En el Corolario H, para algunos $c_1, c_2 \in \mathbb{R}$, hay dos valores de a cumpliendo las condiciones enunciadas, obteniendo dos operadores auto-adjuntos distintos definidos por P en espacios de Hilbert diferentes. Por ejemplo, el oscilador armónico de Dunkl L_σ puede definir operadores auto-adjuntos incluso cuando $\sigma \le -1/2$.

El Corolario H se aplica para probar nuestras desigualdades de Morse en estratos de estratificaciones de Thom-Mather compactas con métricas adaptadas.

Perturbación de Witten en estratos

Un complejo de Hilbert [**11**] es un complejo diferencial dado por un operador cerrado $\mathbf{d}$ definido densamente en un espacio de Hilbert separable graduado $\mathfrak{H}$. El correspondiente Laplaciano $\mathbf{\Delta} = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ es un operador auto-adjunto en $\mathfrak{H}$. Se dice que $\mathbf{d}$ es discreto cuando $\mathbf{\Delta}$ tiene un espectro discreto[2]; en particular, su homología es de dimensión finita por una versión de la descomposición de Hodge.

Sea $(\Omega_0(M), d)$ el complejo de de Rham con soporte compacto de una variedad riemanniana M. Sus extensiones como complejos de Hilbert en $L^2\Omega(M)$ (el espacio de Hilbert graduado de formas diferenciales con cuadrado integrable) se llaman condiciones ideales de frontera (c.i.f.). Hay una c.i.f. mínima, $d_{\min} = \overline{d}$, y una c.i.f.

[2] Recuérdese que un operador auto-adjunto tiene un espectro discreto cuando no hay espectro esencial; es decir, el espectro está formado por autovalores con multiplicidad finita sin puntos de acumulación.

máxima, $d_{\max} = \delta^*$, donde δ es la codiferencial de de Rham actuando sobre $\Omega_0(M)$. El laplaciano definido por $d_{\min/\max}$ se denota por $\Delta_{\min/\max}$. Es bien conocido que $d_{\min} = d_{\max}$ si M es completa, pero supongamos que M puede no ser completa. La c.i.f. $d_{\min/\max}$ define la cohomología min/max $H^\bullet_{\min/\max}(M)$, los números de Betti min/max $\beta^r_{\min/\max} = \beta^r_{\min/\max}(M)$, y la característica de Euler min/max $\chi_{\min/\max} = \chi_{\min/\max}(M)$ (si los números de Betti min/max son finitos); éstos son invariantes quasi-isométricos de M. Estos conceptos pueden definirse de hecho para complejos elípticos arbitrarios [**11**].

Desde ahora en adelante, asumamos que M es un estrato de una estratificación de Thom-Mather compacta A [**60, 44, 45, 64**]. A grosso modo, alrededor de cada $x \in \overline{M}$, hay una carta de A con valores en un producto $\mathbb{R}^m \times c(L)$, donde:

- L es una estratificación de Thom-Mather compacta de profundidad inferior, y $c(L) = L \times [0,\infty)/L \times \{0\}$ (el cono con enlace L);
- x corresponde a $(0,*)$, donde $*$ es el vértice de $c(L)$; y,
- cerca de x, M corresponde a $\mathbb{R}^m \times M'$ para algún estrato M' de $c(L)$.

Se tiene que, o bien $M' = N \times \mathbb{R}_+$ para algún estrato N de L, o bien $M' = \{*\}$. Obsérvese que $x \in M$ justo cuando $M' = \{*\}$. Sea $\rho : c(L) \to [0,\infty)$ la función canónica inducida por la proyección en el segundo factor $L \times [0,\infty) \to [0,\infty)$. La suma de ρ y la norma de $\mathbb{R}^m$ también la denominamos función canónica de $\mathbb{R}^m \times c(L)$.

Dotemos a M con una métrica riemanniana g, que es adaptada en el sentido siguiente definido por inducción en la profundidad de M [**13, 14**]: hay una carta alrededor de cada $x \in \overline{M} \setminus M$ como antes tal que g es quasi-isométrica a una métrica modelo de la forma $g_0 + \rho^2 \tilde{g} + (d\rho)^2$ en $\mathbb{R}^m \times N \times \mathbb{R}_+$, donde g_0 es la métrica euclídea en $\mathbb{R}^m$ y $\tilde{g}$ una métrica adaptada en N; esta $\tilde{g}$ está bien definida ya que $\operatorname{depth} N < \operatorname{depth} M$. Obsérvese que g puede no ser completa. En [**47, 48, 8**] se consideran métricas adaptadas más generales. El primer resultado importante de la Parte 2 es el siguiente.

TEOREMA I. *Con la notación anterior, se cumplen las siguientes propiedades:*

(i) $d_{\min/\max}$ *es discreto.*
(ii) *Sean* $0 \le \lambda_{\min/\max,0} \le \lambda_{\min/\max,1} \le \cdots$ *los autovalores de* $\Delta_{\min/\max}$*, repetidos de acuerdo a sus multiplicidades. Entonces hay algún* $\theta > 0$ *tal que* $\liminf_k \lambda_{\min/\max,k}\, k^{-\theta} > 0$.

La discreción de $d_{\min}$ es esencialmente debida a J. Cheeger [**13, 14**]. Teorema I-(ii) es una versión débil de la fórmula asintótica de Weyl (véase por ejemplo [**54**, Teorema 8.16]). La teoría elíptica para el caso de variedades conformalmente cónicas fue estudiada en [**12, 39**], y una versión no conmutativa del teorema del índice para pseudo-variedades cónicas se da en [**19**].

Una función diferenciable f en M se denomina relativamente admisible cuando las funciones $|df|$ y $|\operatorname{Hess} f|$ están acotadas. En este caso, f podría no tener extensiones continuas a $\overline{M}$, pero tiene una extensión continua a la compleción métrica (por componentes) $\widehat{M}$ de M. Entonces tiene sentido decir que $x \in \widehat{M}$ es un punto relativamente crítico de f cuando hay una sucesión (y_k) en M tal que $\lim_k y_k = x$ en $\widehat{M}$ y $\lim_k |df(y_k)| = 0$. Para decir que f es una función relativamente de Morse en M, debería requerirse también que $\operatorname{Hess} f$ sea "relativamente no degenerado" en cada punto relativamente crítico x, pero no existe un "lema relativamente Morse".

Así que, en vez de eso, requerimos la existencia de un modelo local de $\widehat{M}$ centrado en x de la forma $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times c(L_+) \times c(L_-)$ tal que:

- M corresponde al estrato $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, donde $M_\pm$ es un estrato de $c(L_\pm)$; y
- f corresponde a una constante más la función modelo $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ en $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$, donde $\rho_\pm$ es la función canónica en $\mathbb{R}^{m_\pm} \times c(L_\pm)$.

O bien $M_\pm$ es el estrato vértice $\{*_\pm\}$ de $c(L_\pm)$, o bien $M_\pm = N_\pm \times \mathbb{R}_+$ para algún estrato $N_\pm$ de $L_\pm$; en el segundo caso, sea $n_\pm = \dim N_\pm$. Este modelo local tiene sentido porque el producto de dos estratificaciones de Thom-Mather se puede dotar con una estructura de Thom-Mather; en particular, el producto de dos conos se convierte en un cono. No existe una elección canónica de una estructura producto de Thom-Mather, pero todas ellas tienen las mismas métricas adaptadas.

Para cada punto crítico relativo x de f como antes y cada $r \in \mathbb{Z}$, se define $\nu^r_{x,\mathrm{min/max}}$ de la forma siguiente. Si $M_+ = N_+ \times \mathbb{R}_+$ y $M_- = N_- \times \mathbb{R}_+$, entonces sea

$$\nu^r_{x,\mathrm{min/max}} = \sum_{r_+,r_-} \beta^{r_+}_{\mathrm{min/max}}(N_+)\, \beta^{r_-}_{\mathrm{min/max}}(N_-)\ ,$$

donde (r_+, r_-) recorre el subconjunto de $\mathbb{Z}^2$ determinado por las condiciones:

$$r = m_- + r_+ + r_- + 1\ ,$$

$$r_+ \leq \begin{cases} \frac{n_+}{2} - 1 & \text{si } n_+ \text{ es par} \\ \frac{n_+-3}{2} & \text{si } n_+ \text{ es impar, en el caso de c.i.f. mínima} \\ \frac{n_+-1}{2} & \text{si } n_+ \text{ es impar, en el caso de c.i.f. máxima}\ , \end{cases}$$

$$r_- \geq \begin{cases} \frac{n_-}{2} & \text{si } n_- \text{ es par} \\ \frac{n_+-1}{2} & \text{si } n_- \text{ es impar, en el caso de c.i.f. mínima} \\ \frac{n_-+1}{2} & \text{si } n_- \text{ es impar, en el caso de c.i.f. máxima}\ , \end{cases}$$

En los otros casos, se modifica la definición de $\nu^r_{x,\mathrm{min/max}}$ como sigue. Si $M_+ = \{*_+\}$ y $M_- = N_- \times \mathbb{R}_+$, entonces se suprime todo lo referente a r_+, N_+ y n_+, tomando $r = m_- + r_- + 1$. Si $M_+ = N_+ \times \mathbb{R}_+$ y $M_- = \{*_-\}$, entonces se suprime todo lo referente a r_-, N_- y n_-, tomando $r = m_- + r_+$. Si $M_+ = \{*_+\}$ y $M_- = \{*_-\}$, entonces se define[3] $\nu^r_{x,\mathrm{min/max}} = \delta_{r,m_-}$. Finalmente, sea $\nu^r_{\mathrm{min/max}} = \sum_x \nu^r_{x,\mathrm{min/max}}$, donde x el conjunto de puntos relativamente críticos de f. El segundo resultado principal de la Parte 2 es el siguiente.

Teorema J. *Con la notación anterior, se tienen las desigualdades*

$$\beta^0_{\mathrm{min/max}} \leq \nu^0_{\mathrm{min/max}}\ ,$$
$$\beta^1_{\mathrm{min/max}} - \beta^0_{\mathrm{min/max}} \leq \nu^1_{\mathrm{min/max}} - \nu^0_{\mathrm{min/max}}\ ,$$
$$\beta^2_{\mathrm{min/max}} - \beta^1_{\mathrm{min/max}} + \beta^0_{\mathrm{min/max}} \leq \nu^2_{\mathrm{min/max}} - \nu^1_{\mathrm{min/max}} + \nu^0_{\mathrm{min/max}}\ ,$$

etc., y la igualdad

$$\chi_{\mathrm{min/max}} = \sum_r (-1)^r\, \nu^r_{\mathrm{min/max}}\ .$$

[3] Se usa la delta de Kronecker.

También se demuestra la existencia de funciones relativamente de Morse. Por ejemplo, para cualquier acción diferenciable de un grupo de Lie compacto G en una variedad diferenciable cerrada M, cualquier función de Morse-Bott invariante en M cuyas variedades críticas son órbitas induce una función relativamente de Morse en $G\backslash M$; esto proporciona una rica familia de ejemplos en los que el Teorema J se puede aplicar.

Para demostrar el Teorema I, se muestra primero que las propiedades enunciadas son "relativamente locales" (Capítulo 10), y es bien conocido que son invariables por quasi-isometrías. Entonces el espectro se estudia para los modelos locales $\mathbb{R}^m \times N \times \mathbb{R}_+$ con las métricas modelo $g_0 + \rho^2 \tilde{g} + (d\rho)^2$, asumiendo que el resultado se cumple para N con $\tilde{g}$ por inducción. De hecho, por el principio mini-max, es suficiente hacer este argumento para la c.i.f. mínima/máxima $d_{s,\min/\max}$ de la perturbación de Witten d_s ($s > 0$) de d definida por cualquier función relativamente de Morse [**68**]; el laplaciano definido por $d_{s,\min/\max}$ se denota por $\Delta_{s,\min/\max}$. De esta forma, la demostración del Teorema I se convierte en un paso de la demostración del Teorema J usando el método analítico de E. Witten; especialmente, según se describe en [**54**, Capítulos 9 y 14].

Una parte de ese método es el análisis local alrededor de los puntos relativamente críticos; más explícitamente, el análisis espectral del laplaciano perturbado $\Delta_{s,\min/\max}$ definido con las funciones modelo $\frac{1}{2}(\rho_+^2 - \rho_-^2)$ en $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times M_+ \times M_-$. Por la versión de la fórmula de Künneth para complejos de Hilbert [**11**], este estudio se puede reducir al caso de las funciones $\pm\frac{1}{2}\rho^2$ en $N \times \mathbb{R}_+$, donde ρ es la función canónica de $c(L)$. Entonces la descomposición espectral discreta de N con $\tilde{g}$ es usada para descomponer la perturbación de Witten del complejo de de Rham de $N \times \mathbb{R}_+$ en suma directa de complejos elípticos simples de dos tipos (Capítulos 11, 14 y 15), cuyos laplacianos están dados por la perturbación del oscilador armónico en $\mathbb{R}_+$ estudiada en la Parte 1, relacionada con el oscilador armónico de Dunkl. Se finaliza obteniendo las propiedades espectrales de $\Delta_{s,\min/\max}$ que se necesitan para describir la "contribución cohomológica" de los puntos relativamente críticos (Capítulo 19, Sección 3).

Otra parte de la adaptación del método de Witten es la demostración de la "contribución cohomológica nula" lejos de los puntos relativamente críticos. En esta parte, algunos argumentos de [**54**, Capítulo 14] no se pueden usar porque no hay una versión del teorema de embebimiento de Sobolev con los espacios de Sobolev $W^m(\Delta_{\min/\max})$ definidos con $\Delta_{\min/\max}$; tal resultado podría ser cierto, pero la forma usual de probarlo no funciona ya que $W^m(\Delta_{\min/\max})$ puede depender de la elección de la métrica adaptada (Capítulo 21). Por tanto se aplica un nuevo método en esa parte de la demostración (Capítulo 19, Sección 2), que usa fuertemente el Teorema I-(ii).

Extendiendo f a $\widehat{M}$, se puede considerar el Teorema J como desigualdades de Morse en la estratificación de Thom-Mather $\widehat{M}$. En este sentido, sería interesante compararlo con las desigualdades de Morse probadas por Goresky-MacPherson [**30**, Capítulo 6, Sección 6.12], donde consideran homología intersección con perversidad media inferior de variedades analíticas complejas con estratificaciones de Whitney. Otra demostración analítica de desigualdades de Morse fue hecha por U. Luwig [**41, 42, 43**] para el caso especial de variedades conformemente cónicas, pero sus funciones admisibles y de Morse son diferentes de las nuestras: la norma de sus diferenciales no se aproxima a cero alrededor de la frontera del estrato, y la norma de su hessiano puede no estar acotada.

En el futuro, esperamos poder extender este trabajo al caso de otras métricas adaptadas (las consideradas en [**47, 48, 8**], o incluso más generales); en el caso de $d_{\min}$ con las métricas adaptadas de [**47, 48, 8**], se obtendrían desigualdades de Morse para la homología intersección con perversidad arbitraria. Esto requerirá el estudio de una perturbación del oscilador armónico en $\mathbb{R}_+$ más general que en la Parte 1.

También es natural intentar extender este trabajo a "funciones relativamente Morse-Bott", en las que el conjunto relativamente crítico esté formado por "subestratificaciones de Thom-Mather relativamente críticas y relativamente no degeneradas".

Bibliography

1. J.A. Álvarez López and M. Calaza, *Eigenfunction estimates and embedding theorems for the Dunkl harmonic oscillator*, Preprint arXiv:1101.5022v6 [math.SP], 2012.
2. ———, *Witten's perturbation on strata*, Preprint arXiv:1205.0348v1 [math.DG], 2012.
3. T.H. Baker and P.J. Forrester, *The Calogero-Sutherland model and generalized classical polynomials*, Comm. Math. Phys. **188** (1997), 175–216.
4. ———, *The Calogero-Sutherland model and polynomials with prescribed symmetry*, Nucl. Phys. B **492** (1997), 682–716.
5. Bierstone, *Lifting isotopies from orbit spaces*, Topology **14** (1975), 245–252.
6. S.S. Bonan and D.S. Clark, *Estimates of the Hermite and the Freud polynomials*, J. of Approx. Theory **63** (1990), 210–224.
7. J.P. Brasselet, G. Hector, and M. Saralegi, *Théorème de De Rham pour les variétés stratifiées*, Ann. Global Anal. Geom. **9** (1991), 211–243.
8. ———, *L^2-cohomologie des espaces stratifiés*, Manuscripta Math. **76** (1992), 21–32.
9. G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
10. L. Brink, T.H. Hansson, S. Konstein, and M.A. Vasiliev, *The Calogero model—anyonic representation, fermionic extension and supersymmetry*, Nuclear Phys. B **401** (1993), 591–612.
11. J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), 88–132.
12. ———, *Kähler-Hodge theory for conformal complex cones*, Geom. Funct. Anal. **3** (1993), 439–473.
13. J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace Operator (Univ. Hawaii, Honolulu, Hawaii, 1979) (Providence, R.I., 1980), Proc. Sympos. Pure Math., vol. XXXVI, Amer. Math. Soc., 1980, pp. 91–146.
14. ———, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), 575–657. MR 85d:58083
15. J. Cheeger, M. Goresky, and R. MacPherson, *L^2-cohomology and intersection homology of singular varieties*, Seminar on Differential Geometry (Princeton, New Yersey), Ann. Math. Stud., vol. 102, Princeton University Press, 1982, pp. 302–340.
16. T.S. Chihara, *Generalized Hermite polynomials*, Ph.D. thesis, Purdue University, 1955.
17. ———, *An introduction to orthogonal polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978.
18. J.B. Conway, *A course in functiopnal analysis*, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.
19. C. Debord, J.-M. Lescure, and V. Nistor, *Groupoids and an index theorem for conical pseudomanifolds*, J. Reine Angew. Math. **628** (2009), 1–35.
20. D. Dickinson and S.A. Warsi, *On a generalized Hermite polynomial and a problem of Carlitz*, Boll. Un. Mat. Ital. **18** (1963), 256–259.
21. C.F. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. **197** (1988), 33–60.
22. ———, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.
23. ———, *Operators commuting with coxeter group actions on polynomials*, Invariant Theory and Tableaux (Minneapolis, MN, 1988) (New York) (D. Stanton, ed.), IMA Vol. Math. Appl., vol. 19, Springer, 1990, pp. 107–117.
24. ———, *Integral kernels with reflection group invariants*, Canad. J. Math. **43** (1991), 1213–1227.

25. ———, *Hankel transforms associated to finite reflection groups*, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991) (Providence, RI), Contemp. Math., vol. 138, Amer. Math. Soc., 1992, pp. 123–138.
26. ———, *Symmetric functions and B_N-invariant spherical harmonics*, J. Phys. A **35** (2002), 10391–10408.
27. M. Dutta, S.K. Chatterjea, and K.L. More, *On a class of generalized Hermite polynomials*, Bull. Inst. Math. Acad. Sinica **3** (1975), 377–381.
28. P. Erdös and P. Turán, *On interpolation. III*, Ann. Math. **41** (1940), 510–555.
29. C.G. Gibson, K. Wirthmüler, A.A. du Plessis, and E.J.N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math., no. 552, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
30. M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiet, vol. 14, Springer-Verlag, Berlin, Heidelberg, New York, 1988.
31. E. Hille, *A class of reciprocal functions*, Annals of Math. **27** (1926), no. 4, 427–464.
32. H. Hironaka, *Introduction to real-analytic sets and real-analytic maps*, Quaderni dei Gruppi di Ricerca Matematica del Consiglio Nazionale delle Ricerche, Istituto Matematico "L. Tonelli" dell'Università di Pisa, Pisa, 1973.
33. ———, *Subanalytic sets*, Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki (Tokyo), Kinokuniya, 1973, pp. 453–493.
34. ———, *Stratification and flatness*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976) (Alphen aan den Rijn), Sijthoff and Noordhoff, 1977, pp. 199–265.
35. M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, Heidelberg, Berlin, 1976.
36. L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, second ed., Grundlehren der mathematischen Wissenschaften, no. 256, Springer-Verlag, Berlin, 1990.
37. S. Kakei, *Common algebraic structure for the Calogero-Sutherland models*, J. Phys. A **29** (1996), L619–L624.
38. L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, Comm. Math. Phys. **178** (1996), 425–452.
39. M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik, vol. 136, B.G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.
40. S. Łojasiewicz, *Sur les ensembles semi-analytiques*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2 (Paris), Gauthier-Villars, 1971, pp. 237–241.
41. U. Ludwig, *The geometric complex for algebraic curves with cone-like singularities and admissible Morse functions*, C.R. Acad. Sci. Paris, Ser. I **347** (2009), 801–804.
42. ———, *The Witten complex for algebraic curves with cone-like singularities*, C.R. Acad. Sci. Paris, Ser. I **347** (2009), 651–654.
43. ———, *The Witten deformation for even dimensional conformally conic manifolds*, arXiv:1011.5357v1 [math.DG], November 2010.
44. J.N. Mather, *Notes on topological stability*, Mimeographed Notes, Hardvard University, 1970.
45. ———, *Stratifications and mappings*, Dynamical Systems, Academic Press, 1973, pp. 195–232.
46. J. Milnor, *Morse theory*, Annals of Mathematics Studies, vol. 51, Princeton University Press, Princeton, New Jersey, 1963.
47. N. Nagase, *L^2-cohomology and intersection cohomology of stratified spaces*, Duke Math. J. **50** (1983), 329–368.
48. ———, *Sheaf theoretic L^2-cohomology*, Adv. Stud. Pure Math. **8** (1986), 273–279.
49. A. Nowak and K. Stempak, *Imaginary powers of the Dunkl harmonic oscillator*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 016, 12 pp.
50. ———, *Riesz transforms for the Dunkl harmonic oscillator*, Math. Z. **262** (2009), 539–556.
51. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
52. V. Pasquier, *A lecture on the Calogero-Sutherland models*, Integrable models and strings (Espoo, 1993) (Berlin), Lecture Notes in Phys., vol. 436, Springer, 1994, pp. 36–48.
53. M. Reed and B. Simon, *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, New York, 1978.

54. J. Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics, vol. 395, Addison Wesley Longman Limited, Edinburgh Gate, Harlow, Essex CM20 2JE, England, 1998.
55. M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, Nonselfadjoint operators and related topics (Beer Sheva, 1992) (Basel) (A. Feintuch and I. Gohberg, eds.), Oper. Theory Adv. Appl., vol. 73, Birkhäuser, 1994, pp. 369–396.
56. M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. **192** (1998), 519–542.
57. ______, *Dunkl operators: theory and applications*, Orthogonal polynomials and special functions (Leuven, 2002) (Berlin) (E. Koelink and W. Van Assche, eds.), Lecture Notes in Math., vol. 1817, Springer, 2003, pp. 93–135.
58. H.H. Schaefer, *Topological vector spaces*, Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
59. G. Szegö, *Orthogonal polynomials*, fourth ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975.
60. R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. **75** (1969), 240—284.
61. H. Ujino and M. Wadati, *Rodrigues formula for Hi-Jack symmetric polynomials associated with the quantum Colegero model*, J. Phys. Soc. Japan **65** (1996), 2423–2439.
62. W. Van Assche, *Some results on the distribution of the zeros of orthogonal polynomials*, Journal of Computational and Applied Mathematics **12-13** (1985), 615–623.
63. W. Van Assche and J.L. Teugels, *Second order asymptotic behaviour of the zeros of orthogonal polynomials*, Rev. Roumaine Math. Pures Appl. **32** (1987), 15–26.
64. A. Verona, *Stratified mappings—structure and triangulability*, Lecture Notes in Math., vol. 1102, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
65. A.G. Wasserman, *Equivariant differential topology*, Topology **8** (1969), 127–150.
66. H. Whitney, *Local properties of analytic varieties*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) (Princeton, N. J.), Princeton Univ. Press, 1965, pp. 205–244.
67. ______, *Tangents to an analytic variety*, Annals of Math. **81** (1965), 496–549.
68. E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), 661–692.