GEOMETRY OF FOUR-DIMENSIONAL KÄHLER AND PARA-KÄHLER LIE GROUPS

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Abstract. We classify four-dimensional para-Kähler Lie algebras and study their geometry showing that they are symmetric or simply harmonic special recurrent, in the semi-symmetric case. The non semi-symmetric case allows essentially two distinct geometries in addition to the 3-symmetric spaces, without a Kählerian counterpart.

1. Introduction

Let $(M, g, J_\pm)$ be a (para-)Kähler manifold, where $J_\pm$ denotes the parallel (para) complex structure ($J_\pm^2 = \mp\,\mathrm{Id}$) and $g(J_\pm X, J_\pm Y) = \pm g(X, Y)$ for all vector fields X, Y on M. While Kähler geometry has been intensively studied for a long time, the interest in para-Kähler geometry raised considerably during the last decades (see, for example, [1, 2, 14, 18]). Besides the formal analogy in their structural definition, there are substantial differences between Kähler and para-Kähler geometries. Let $\Omega_\pm(X, Y) = g(J_\pm X, Y)$ denote the associated (para-)Kähler 2-form. In both situations it is a closed 2-form whose orientation agrees with that of the paracomplex structure in the para-Kähler case and is the opposite of the orientation induced by the complex structure in the indefinite Kähler case. The Kähler and para-Kähler conditions require the integrability of the associated almost (para)complex structure $J_\pm$. While integrability of an almost complex structure ensures the existence of an underlying holomorphic structure, the integrability of the almost paracomplex structure shows that the symplectic manifold is locally diffeomorphic to a product of two complementary Lagrangian submanifolds. Considering the induced metric on the space of 2-forms $(\Lambda^2(TM), \langle\!\langle , \rangle\!\rangle)$, the fundamental form $\Omega_\pm$ is a spacelike section in the Kähler case but a timelike one in the para-Kähler situation. Since in the four-dimensional case the induced metric on $\Lambda^2(TM)$ has signature $(++----)$ one may expect para-Kähler structures to be more common than Kähler structures.

On the other hand, certain links between Kähler and para-Kähler structures have been broadly investigated in the literature. The case of anti-commuting Kähler and para-Kähler structures, i.e., hypersymplectic and paraquaternionic structures, has special interest (see [3, 19] and references therein). Also the case of commuting Kähler and para-Kähler structures is relevant in different geometric situations and it is related to the existence of anti-Kähler structures (see, for example, [5, 15]).

Lie groups not only constitute a basic tool for producing examples, but also are essential for the classification of homogeneous structures. Since any symplectic Lie group is necessarily solvable [13], Ovando classified left-invariant Kähler and

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symplectic structures on four-dimensional Lie groups (see [21, 22]). The situation is very rigid in the positive definite case and a four-dimensional homogeneous Kähler structure is either symmetric or isometric to the only 3-symmetric space [4, 6, 12, 20]. The neutral signature case allows other possibilities with a para-Kähler counterpart. The description of four-dimensional para-Kähler Lie algebras has been recently approached in different works (see [7, 8, 24]) but it still seams not to be clear. Working at the purely algebraic level, one equivalently describes the decomposition of all four-dimensional symplectic Lie algebras as a direct sum of two-dimensional Lagrangian subalgebras. Although two-dimensional Lie algebras reduce to the Abelian algebra and the Affine one, they may behave quite differently giving rise to a number of different geometric structures.

The curvature tensor of a (para-)Kähler surface is codified by the Ricci operator and the Weyl curvature operator. If the Ricci operator of a (para-)Kähler surface is not diagonalizable then either it is complex diagonalizable with a single complex eigenvalue or the Ricci operator has a single eigenvalue which is a double root of the corresponding minimal polynomial (see [15]). Moreover, the self-dual (resp., anti-self-dual) Weyl curvature operator of a para-Kähler (resp., indefinite Kähler) surface is diagonalizable with eigenvalues $\{\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}\}$, where τ denotes the scalar curvature, and thus the possible geometries are determined by the anti-self-dual (resp., self-dual) Weyl curvature operator. All the algebraic possibilities but W^- having complex eigenvalues are realizable as para-Kähler Lie groups. In contrast, not all of them are realized in the Kähler situation (see Section 18).

The purpose of this work is firstly to describe all left-invariant para-Kähler structures on four-dimensional Lie groups and secondly to analyze their geometry, thus complementing the analysis in [9] by considering homogeneous spaces with no isotropy. The geometry of left-invariant Kähler structures obtained by Ovando in [22] is also clarified. For a fixed symplectic structure Ω on a Lie algebra $\mathfrak{g}$ we describe all the para-Kähler structures $(J, \langle,\rangle, \Omega)$ up to isometric automorphisms preserving the symplectic structure and modulo reversing the metric, since in any of these two cases the corresponding Lagrangian decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L}'$ is preserved. The different Lagrangian decompositions, being essential in the para-Kähler setting, are explicitly described in each case.

The geometry of four-dimensional para-Kähler Lie groups naturally splits into the symmetric and the non-symmetric cases. The latter further splits into the semi-symmetric and the non semi-symmetric situations. The symmetric case is summarized as follows.

Theorem 1.1. *Let $(G, \langle,\rangle, J)$ be a non-flat locally symmetric four-dimensional para-Kähler Lie group. Then, one has two distinct situations:*

(1) *If the Ricci operator is diagonalizable, then one of the following holds:*
  - (1.a) *The Ricci operator vanishes and the anti-self-dual Weyl curvature operator is two-step nilpotent.*
  - (1.b) *The paraholomorphic sectional curvature is a non-zero constant.*
  - (1.c) *The metric is Einstein with non-zero scalar curvature, and the self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.*
  - (1.d) *The manifold is locally a product of two surfaces of constant Gaussian curvature. The self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.*

(2) *If the Ricci operator is non-diagonalizable, then one of the following holds:*
  (2.a) *The Ricci operator has complex eigenvalues. The self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.*
  (2.b) *The Ricci operator is two-step nilpotent and the anti-self-dual Weyl curvature operator vanishes or it is two-step nilpotent.*

Structures in Assertion (1.a), which do not have a Kählerian counterpart, are realized on $\mathfrak{r}_{4,-1}$, $\mathfrak{r}_{4,-1,-1}$, and $\mathfrak{d}_{4,1}$, and they correspond to symmetric Osserman manifolds with non-diagonalizable Jacobi operators [17].

A para-Kähler manifold (M, g, J) is said to be *opposite para-Kähler* if there is a para-Kähler structure (J', g) so that $JJ' = J'J$. In the four-dimensional case the corresponding Kähler forms Ω and Ω' induce opposite orientations and, moreover, $Q = JJ'$ is a parallel product structure. Hence, (M, g) splits locally as a product of two oriented surfaces $M = N_1 \times N_2$ so that $J = J_1 \oplus J_2$ and $J' = J_1 \oplus -J_2$, where J_i is the paracomplex structure on N_i determined by the volume form. Conversely, a four-dimensional product of two oriented Lorentzian surfaces naturally inherits a para-Kähler and opposite para-Kähler structure. This is the case of the structures corresponding to Assertion (1.d) in Theorem 1.1.

An immediate consequence of Theorem 1.1 and the results in [15] shows that all locally conformally flat para-Kähler structures have a left-invariant model as follows.

Corollary 1.2. *Let (M, g, J) be a locally conformally flat four-dimensional para-Kähler manifold. Then, it is flat or locally isometric to the para-Kähler Lie group determined by one of the following:*

(1) *The symplectic Lie algebra $(\mathfrak{r}'_2, \Omega)$ determined by $\Omega = e^{14} + e^{23}$, with metric $\langle,\rangle = \frac{2}{\kappa}\, e^1 \circ e^2 - 2\kappa\, e^3 \circ e^4$.*
(2) *The symplectic Lie algebra $(\mathfrak{d}_{4,1}, \Omega)$ determined by $\Omega = e^{12} - e^{34}$, with metric $\langle,\rangle = 2(\, e^1 \circ e^3 + e^2 \circ e^4)$.*
(3) *The symplectic Lie algebra $(\mathfrak{r}_2\mathfrak{r}_2, \Omega)$ determined by $\Omega = e^{12} + e^{34}$, with metric $\langle,\rangle = \frac{1}{\kappa}(e^1 \circ e^1 - e^3 \circ e^3) - \kappa(e^2 \circ e^2 - e^4 \circ e^4)$.*

A pseudo-Riemannian manifold (M, g) is *semi-symmetric* if its curvature tensor satisfies $R(X, Y) \cdot R = 0$ for all vector fields X, Y on M, where $R(X, Y)$ acts on the curvature tensor R as a derivation. The special significance of the semi-symmetry condition is that a curvature tensor is semi-symmetric if it is pointwise the curvature tensor of a symmetric space. However, the model symmetric space may change from point to point and there are examples of semi-symmetric manifolds which are not even locally homogeneous. Another generalization of symmetric spaces is given by manifolds of *recurrent curvature*, in which case there is a 1-form ξ so that the covariant derivative of the curvature tensor satisfies $\nabla R = \xi \otimes R$. Furthermore, a recurrent manifold is said to be *special* if there are 2-forms $\alpha \wedge \beta$ and $\gamma \wedge \delta$ so that the curvature tensor acting on the space of 2-forms is completely described by $\mathcal{R}(\alpha \wedge \beta) = \pm\gamma \wedge \delta$. Special recurrent manifolds are simply harmonic (see [23] for more information). The local structure of special recurrent manifolds was given in [23] showing that there exist local coordinates $(x^1, \dots, x^4)$ so that the metric expresses as

$$g = \Psi(x^1, x^2) dx^1 \circ dx^1 + 2dx^1 \circ dx^4 + 2dx^2 \circ dx^3, \tag{1}$$

where $\Psi(x^1,x^2)$ is an arbitrary function with non-constant $\partial_2\partial_2\Psi$, in which case is locally symmetric. An easy calculation shows that special recurrent manifolds are semi-symmetric since their curvature tensor reflects that of the locally symmetric metrics corresponding to $\Psi(x^1,x^2)=\pm(x^2)^2$. Moreover, any metric (1) is (locally) hypersymplectic since

$$J_K=\partial_{x^2}\otimes dx^1+\partial_{x^3}\otimes dx^4-\partial_{x^1}\otimes dx^2-\partial_{x^4}\otimes dx^3+\tfrac{1}{2}\Psi(\partial_{x^3}\otimes dx^1+\partial_{x^4}\otimes dx^2),$$
$$J_{pK}=\partial_{x^2}\otimes dx^2+\partial_{x^4}\otimes dx^4-\partial_{x^1}\otimes dx^1-\partial_{x^3}\otimes dx^3+\Psi\partial_{x^4}\otimes dx^1$$

are anti-commuting Kähler and para-Kähler structures with corresponding symplectic structures $\omega_{J_K}=\frac{1}{2}\Psi\,dx^1\wedge dx^2+dx^1\wedge dx^3-dx^2\wedge dx^4$ and $\omega_{J_{pK}}=dx^1\wedge dx^4-dx^2\wedge dx^3$. Hence, any special recurrent manifold admits a locally defined Kähler structure (g,J_K) and two locally defined para-Kähler structures (g,J_{pK}) and (g,J_KJ_{pK}). Hypersymplectic structures on four-dimensional Lie algebras were classified by Andrada [3] who showed that besides the Abelian algebra $\mathfrak{r}^4$, these structures may only occur on $\mathfrak{rh}_3$, $\mathfrak{r}_{4,-1,-1}$ or $\mathfrak{d}_{4,2}$. The non-flat cases are also described in Section 11.2 and Section 17.1.2.

Note that, in addition to the hypersymplectic structure, any special recurrent manifold is locally para-Kähler and opposite almost para-Kähler, i.e., there exists an almost paracomplex structure J'_{pK}, given by

$$J'_{pK}=\partial_{x^1}\otimes dx^1+\partial_{x^2}\otimes dx^2-\partial_{x^3}\otimes dx^3-\partial_{x^4}\otimes dx^4-\Psi\partial_{x^4}\otimes dx^1\,,$$

which commutes with J_{pK} so that (g,J'_{pK}) is a locally defined almost para-Hermitian structure with closed fundamental form $\Omega_{J'_{pK}}=-dx^1\wedge dx^4-dx^2\wedge dx^3$. Para-Kähler and opposite almost para-Kähler structures on non-flat Lie groups were considered in [11] where it is shown that the corresponding Ricci operator either vanishes or it is diagonalizable with two-dimensional kernel. In sharp contrast with the Kähler situation [4, 10] there are many para-Kähler Lie groups which admit an opposite almost para-Kähler structure, even in the symmetric case.

The geometry of para-Kähler Lie groups is quite rigid in the non-symmetric case. One has two essentially different possibilities depending on whether the curvature tensor is semi-symmetric or not. The semi-symmetric case is as follows.

Theorem 1.3. *Let $(G,\langle,\rangle,J)$ be a four-dimensional para-Kähler Lie group which is not locally symmetric. Then, the curvature tensor is semi-symmetric if and only if the Ricci operator vanishes. Moreover, all these structures are recurrent and harmonic.*

The result of Theorem 1.3 also holds true in the Kähler situation (cf. Theorem 18.3). Finally, the class of non-symmetric para-Kähler Lie groups whose curvature tensor is not semi-symmetric is given as follows.

Theorem 1.4. *Let $(G,\langle,\rangle,J)$ be a four-dimensional para-Kähler Lie group whose curvature tensor is not semi-symmetric. Then, one of the following holds:*

- (1.a) *The Ricci operator has a single eigenvalue which is a double root of the minimal polynomial. The anti-self-dual Weyl curvature operator is three-step nilpotent.*
- (1.b) *The Ricci operator is diagonalizable with two-dimensional kernel. Moreover,*
  - (1.b.i) *the self-dual and anti-self-dual Weyl curvature operators have the same eigenvalues and W^- has a double root of the minimal polynomial, or*

(1.b.ii) *the self-dual and anti-self-dual Weyl curvature operators have opposite eigenvalues and both operators $W^\pm$ are diagonalizable.*

The only geometry in Theorem 1.4 with a Kähler counterpart corresponds to Assertion (1.b.ii) as shown in Theorem 18.3.

2. Para-Kähler Lie algebras

A para-Kähler Lie algebra is a triple $(\mathfrak{g}, J, \langle,\rangle)$ so that

$$J^2 = \mathrm{Id}, \quad \langle Jx, Jy\rangle = -\langle x, y\rangle, \quad \nabla J = 0,$$

for all vectors $x, y \in \mathfrak{g}$. The para-Kähler 2-form $\Omega(x, y) = \langle Jx, y\rangle$ is non-degenerate and parallel. Thus, $d\Omega = 0$ and hence $(\mathfrak{g}, \Omega)$ is a symplectic Lie algebra which satisfies $\Omega(Jx, Jy) = -\Omega(x, y)$. Moreover, the eigenspaces $\ker(J \pm \mathrm{Id})$ are Lagrangian subalgebras and $\mathfrak{g} = \mathfrak{L}\oplus\mathfrak{L}' = \ker(J-\mathrm{Id})\oplus\ker(J+\mathrm{Id})$ is a Lagrangian decomposition of $(\mathfrak{g}, \Omega)$.

Four-dimensional symplectic Lie algebras were classified by Ovando [21] and we follow the notation therein to denote the different four-dimensional solvable Lie algebras. For each symplectic Lie algebra $(\mathfrak{g}, \Omega)$ in [21] we proceed conversely by determining all almost paracomplex structures $J = (a_{ij})$ satisfying $\Omega(Jx, Jy) = -\Omega(x, y)$, so that $\langle x, y\rangle = \Omega(Jx, y)$ determines a para-Kähler metric. Note that all the required conditions $J^2 = \mathrm{Id}$, $\Omega(Jx, Jy) = -\Omega(x, y)$ and $N_J(x, y) = 0$, where N_J is the Nijenhuis tensor $N_J(x, y) = [Jx, Jy] - J[Jx, y] - J[x, Jy] + [x, y]$, reduce to systems of polynomial equations on the unknowns $\{a_{ij}\}$ which we explicitly solve in each case. We omit the details of the calculations and emphasize the different para-Kähler structures and their curvature.

First of all, we point out that not all symplectic Lie algebras $(\mathfrak{g}, \Omega)$ admit a para-Kähler structure. Indeed, although the following Lie algebras admit symplectic structures, a straightforward calculation shows that none of them admits any para-Kähler structure:

$\mathfrak{r}\mathfrak{r}'_{3,0}$: $[e_1, e_2] = -e_3$, $[e_1, e_3] = e_2$,

$\mathfrak{n}_4$: $[e_1, e_4] = -e_2$, $[e_2, e_4] = -e_3$,

$\mathfrak{r}'_{4,0,\delta}$: $[e_1, e_4] = -e_1$, $[e_2, e_4] = \delta e_3$, $[e_3, e_4] = -\delta e_2$,

$\mathfrak{d}'_{4,\delta}$: $[e_1, e_2] = e_3$, $[e_1, e_4] = e_2 - \frac{\delta}{2}e_1$, $[e_2, e_4] = -e_1 - \frac{\delta}{2}e_2$, $[e_3, e_4] = -\delta e_3$,

where $\{e_1, \ldots, e_4\}$ is a basis of each Lie algebra with $\delta > 0$. All the other symplectic Lie algebras in Ovando's classification [21] admit para-Kähler structures. We study all of them separately in the subsequent sections.

3. Para-Kähler structures on $\mathfrak{r}_2\mathfrak{r}_2$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ determined by $[e_1, e_2] = e_2$ and $[e_3, e_4] = e_4$. The automorphisms of the Lie algebra are given by

$$\Phi_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & z_{23} & z_{24} \\ 1 & 0 & 0 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{pmatrix} \quad \text{or} \quad \Phi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & z_{43} & z_{44} \end{pmatrix},$$

with $z_{24}z_{42} \neq 0$ in the first case and $z_{22}z_{44} \neq 0$ in the second one. The symplectic structures on $\mathfrak{r}_2\mathfrak{r}_2$ are given by (see [21]) $\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{34}e^{34}$ with

$\alpha_{12}\alpha_{34} \neq 0$. It was shown by Ovando [21] that any symplectic structure $(\mathfrak{r}_2\mathfrak{r}_2, \omega)$ is symplectomorphically equivalent to $\Omega_\lambda = e^{12}+e^{34}+\lambda e^{13}$, with $\lambda \geq 0$. Furthermore, the symplectic form Ω_λ is preserved by any automorphism Φ_2 with $z_{22} = z_{44} = 1$ (in the special case $\lambda = 0$, the automorphisms Φ_1 with $z_{24} = z_{42} = 1$ also preserve the symplectic form Ω_0). Considering the action of the symplectomorphisms Φ_2 preserving Ω_λ one may restrict to paracomplex structures $J = (a_{ij})$ satisfying one of the following:

Case 1. $a_{12} = 0$ and $a_{14} = 0$.

Case 2. $a_{12} = 0$ and $a_{14} \neq 0$, in which case one may also assume $a_{13} = 0$.

Case 3. $a_{12} \neq 0$ and $a_{14} = 0$, in which case one may also assume $a_{11} = 0$.

Case 4. $a_{12} \neq 0$ and $a_{14} \neq 0$, in which case one may also assume $a_{11} = a_{13} = 0$.

We distinguish the two possibilities corresponding to the symplectic structures Ω_λ (with $\lambda \neq 0$) and Ω_0 since they give rise to different geometries. In each case we examine the compatibility of the paracomplex structure with the symplectic structure (i.e., $\Omega_\lambda(J\cdot, J\cdot) = -\Omega_\lambda(\cdot,\cdot)$) and the integrability of the paracomplex structure (i.e., the integrability of the eigenspaces $\ker(J \mp \mathrm{Id})$ corresponding to the eigenvalues ± 1 of J).

3.1. **Para-Kähler structures on $(\mathfrak{r}_2\mathfrak{r}_2, \Omega_\lambda)$ with $\Omega_\lambda = e^{12} + e^{34} + \lambda e^{13}$ $(\lambda \neq 0)$.** A straightforward calculation shows that no para-Kähler structures may correspond to cases 2, 3, and 4 above. Hence, assume $a_{12} = a_{14} = 0$. Any para-Kähler structure $(J, \langle,\rangle)$ compatible with Ω_λ is equivalent to one of the following:

$$(J_{11}, \langle,\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1, \quad J_{11}e_2 = e_2, \quad J_{11}e_3 = e_3, \quad J_{11}e_4 = -e_4, \\ \langle,\rangle_{11} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 - e^3 \circ e^4), \end{cases}$$

$$(J_{12}, \langle,\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1 + 2\lambda e_2 - 2e_3, \ \ J_{12}e_2 = e_2, \\ J_{12}e_3 = e_3, \ \ J_{12}e_4 = 2e_2 - e_4, \\ \langle,\rangle_{12} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 + 2e^1 \circ e^4 - e^3 \circ e^4), \end{cases}$$

$$(J_{13}, \langle,\rangle_{13}) : \begin{cases} J_{13}e_1 = -e_1, \ \ J_{13}e_2 = e_2 - 2e_4, \\ J_{13}e_3 = 2e_1 + e_3 + 2\lambda e_4, \ \ J_{13}e_4 = -e_4, \\ \langle,\rangle_{13} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 - 2e^2 \circ e^3 - e^3 \circ e^4). \end{cases}$$

The structures above correspond to the decomposition $\mathfrak{r}_2\mathfrak{r}_2 = \mathfrak{L}_{1i} \oplus \mathfrak{L}'_{1i}$ as a direct sum of Lagrangian subalgebras as follows:

$$\begin{aligned} \mathfrak{r}_2\mathfrak{r}_2 &= \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \mathrm{span}\{e_2, e_3\} \oplus \mathrm{span}\{e_1, e_4\} \\ &= \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} = \mathrm{span}\{e_2, e_3\} \oplus \mathrm{span}\{e_4 - e_2, e_1 + e_3 - \lambda e_2\} \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \mathrm{span}\{e_4 - e_2, e_1 + e_3 + \lambda e_2\} \oplus \mathrm{span}\{e_1, e_4\}. \end{aligned}$$

Moreover, a straightforward calculation shows that all structures above are non-flat and Ricci-flat with recurrent curvature (i.e., $\nabla_{1i}R_{1i} = \xi_{1i} \otimes R_{1i}$), with recurrence 1-forms given by

$$\xi_{11} = 2(e^1 + e^3), \quad \xi_{12} = 2e^3, \quad \xi_{13} = 2e^1.$$

Furthermore, the corresponding curvature operators $\mathcal{R}_{1i} : \Lambda^2 \to \Lambda^2$ acting on the space of 2-forms are given by

$$\mathcal{R}_{1i}(e^1 \wedge e^2) = -\lambda e^2 \wedge e^4 = R_{1i}(e_1, e_3, e_3, e_1) e^2 \wedge e^4, \quad \text{for } i = 1, 2, 3,$$

from where it follows that all para-Kähler structures above are special recurrent and thus locally modelled on (1). Hence, their curvature tensor is semi-symmetric.

Finally, the structures $(J_{11}, \langle,\rangle_{11})$ admit opposite almost para-Kähler structures compatible with the opposite symplectic form $\Omega'_{11} = e^{12} - e^{34} + (\mu - \lambda)e^{13}$, with $\mu \in \mathbb{R}$, while the structures $(J_{12}, \langle,\rangle_{12})$ and $(J_{13}, \langle,\rangle_{13})$ do not admit opposite almost para-Kähler structures.

3.2. **Para-Kähler structures on $(\mathfrak{r}_2\mathfrak{r}_2, \Omega_0)$ with $\Omega_0 = e^{12} + e^{34}$.** Proceeding as in the previous case, one gets that no para-Kähler structures may exist in Case 2. The other three cases give rise to three essentially different geometries as follows.

3.2.1. *Para-Kähler structures of constant paraholomorphic sectional curvature.* Assuming $a_{12} \neq 0$ and $a_{14} \neq 0$ as in Case 4, any para-Kähler structure is equivalent to $(J_{21}, \langle,\rangle_{21})$ given by

$$\begin{aligned}
&J_{21}e_1 = e_3 - \tfrac{1}{\kappa}e_2, \;\; J_{21}e_2 = -\kappa(e_1 + e_3), \;\; J_{21}e_3 = -e_3,\\
&J_{21}e_4 = e_4 - e_2 - \kappa(e_1 + e_3),\\
&\langle,\rangle_{21} = \kappa(e^2 \circ e^2 + e^4 \circ e^4 + 2e^2 \circ e^4) - 2e^1 \circ e^4 + 2e^3 \circ e^4 - \tfrac{1}{\kappa}e^1 \circ e^1, \quad \kappa \neq 0,
\end{aligned}$$

which corresponds to the Lagrangian decomposition

$$\mathfrak{r}_2\mathfrak{r}_2 = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \operatorname{span}\{e_4 - e_2, e_1 + e_3 - \tfrac{1}{\kappa}e_2\} \oplus \operatorname{span}\{e_3, e_2 + \kappa e_1\}.$$

Furthermore, the paraholomorphic sectional curvature is constant $H = \kappa$.

3.2.2. *Flat para-Kähler structures.* Assuming $a_{12} = 0$ and $a_{14} = 0$ as in Case 1, if $a_{34} = 0$, then one has three inequivalent flat para-Kähler structures as follows:

$$(J_{22}, \langle,\rangle_{22}) : \begin{cases} J_{22}e_1 = -e_1, \;\; J_{22}e_2 = e_2 - 2e_4, \;\; J_{22}e_3 = 2e_1 + e_3, \;\; J_{22}e_4 = -e_4,\\ \langle,\rangle_{22} = 2(e^1 \circ e^2 - 2e^2 \circ e^3 - e^3 \circ e^4), \end{cases}$$

$$(J_{23}, \langle,\rangle_{23}) : \begin{cases} J_{23}e_1 = -e_1, \;\; J_{23}e_2 = e_2, \;\; J_{23}e_3 = \epsilon e_3, \;\; J_{23}e_4 = -\epsilon e_4,\\ \langle,\rangle_{23} = 2(e^1 \circ e^2 - \epsilon e^3 \circ e^4), \quad \epsilon = \pm 1, \end{cases}$$

$$(J_{24}, \langle,\rangle_{24}) : \begin{cases} J_{24}e_1 = -e_1 - 2e_3, \;\; J_{24}e_2 = e_2, \;\; J_{24}e_3 = e_3, \;\; J_{24}e_4 = 2e_2 - e_4,\\ \langle,\rangle_{24} = 2(e^1 \circ e^2 + 2e^1 \circ e^4 - e^3 \circ e^4), \end{cases}$$

which correspond to the Lagrangian decompositions $\mathfrak{r}_2\mathfrak{r}_2 = \mathfrak{L}_{2i} \oplus \mathfrak{L}'_{2i}$ as follows

$$\begin{aligned}
\mathfrak{r}_2\mathfrak{r}_2 &= \mathfrak{L}_{22} \oplus \mathfrak{L}'_{22} = \operatorname{span}\{e_1 + e_3, e_4 - e_2\} \oplus \operatorname{span}\{e_1, e_4\}\\
&= \mathfrak{L}_{23} \oplus \mathfrak{L}'_{23} = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1, e_4\} \qquad (\epsilon = 1)\\
&= \mathfrak{L}_{23} \oplus \mathfrak{L}'_{23} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\} \qquad (\epsilon = -1)\\
&= \mathfrak{L}_{24} \oplus \mathfrak{L}'_{24} = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1 + e_3, e_4 - e_2\}.
\end{aligned}$$

3.2.3. *Para-Kähler and opposite para-Kähler structures.* The remaining possibilities corresponding to Case 1 with $a_{34} \neq 0$ and Case 3, give rise to two families of para-Kähler structures determined by

$$(J_{25}, \langle,\rangle_{25}) : \begin{cases} J_{25}e_1 = -\frac{1}{\kappa_1}e_2, \quad J_{25}e_2 = -\kappa_1 e_1, \quad J_{25}e_3 = -e_3, \quad J_{25}e_4 = e_4, \\ \langle,\rangle_{25} = -\frac{1}{\kappa_1}e^1 \circ e^1 + \kappa_1 e^2 \circ e^2 + 2e^3 \circ e^4, \end{cases}$$

$$(J_{26}, \langle,\rangle_{26}) : \begin{cases} J_{26}e_1 = -\frac{1}{\kappa_1}e_2, \; J_{26}e_2 = -\kappa_1 e_1, \; J_{26}e_3 = -\frac{1}{\kappa_2}e_4, \; J_{26}e_4 = -\kappa_2 e_3, \\ \langle,\rangle_{26} = -\frac{1}{\kappa_1}e^1 \circ e^1 + \kappa_1 e^2 \circ e^2 - \frac{1}{\kappa_2}e^3 \circ e^3 + \kappa_2 e^4 \circ e^4, \end{cases}$$

where $\kappa_1\kappa_2 \neq 0$. The corresponding Lagrangian decompositions are given by

$$\begin{aligned} \mathfrak{r}_2\mathfrak{r}_2 &= \mathfrak{L}_{25} \oplus \mathfrak{L}'_{25} = \operatorname{span}\{e_4, e_2 - \kappa_1 e_1\} \oplus \operatorname{span}\{e_3, e_2 + \kappa_1 e_1\} \\ &= \mathfrak{L}_{26} \oplus \mathfrak{L}'_{26} = \operatorname{span}\{e_4 - \kappa_2 e_3, e_2 - \kappa_1 e_1\} \oplus \operatorname{span}\{e_4 + \kappa_2 e_3, e_2 + \kappa_1 e_1\}. \end{aligned}$$

Moreover, both metrics are locally symmetric and the Ricci operators are diagonalizable on the basis $\{e_1, \dots, e_4\}$ with Ricci curvatures $\mathrm{Ric}_{25} = \mathrm{diag}[\kappa_1, \kappa_1, 0, 0]$ and $\mathrm{Ric}_{26} = \mathrm{diag}[\kappa_1, \kappa_1, \kappa_2, \kappa_2]$. Hence, the underlying pseudo-Riemannian manifolds split locally as a product of two Lorentzian surfaces of constant Gaussian curvature $N_1(\kappa_1) \times \mathbb{L}^2$ or $N_1(\kappa_1) \times N_2(\kappa_2)$, where $\mathbb{L}^2$ denotes the Minkowskian plane, and thus they also admit opposite para-Kähler structures. Furthermore, the metric $\langle,\rangle_{26}$ is Einstein if $\kappa_1 = \kappa_2$ and locally conformally flat if $\kappa_1 = -\kappa_2$.

4. Para-Kähler structures on $\mathfrak{rh}_3$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{rh}_3$ determined by $[e_1, e_2] = e_3$. Symplectic forms on $\mathfrak{rh}_3$ are of the form $\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{23}e^{23} + \alpha_{24}e^{24}$ with $\alpha_{14}\alpha_{23} - \alpha_{13}\alpha_{24} \neq 0$, as shown in [21]. All these symplectic structures $(\mathfrak{rh}_3, \omega)$ are symplectomorphically equivalent to $\Omega = e^{14} + e^{23}$ through an automorphism of the form

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ z_{31} & z_{32} & z_{11}z_{22} - z_{12}z_{21} & z_{34} \\ z_{41} & z_{42} & 0 & z_{44} \end{pmatrix}, \quad \text{with} \quad (z_{11}z_{22} - z_{12}z_{21})z_{44} \neq 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{rh}_3, \Omega)$ are given by Φ with $z_{11}z_{22}^2 = 1$, $z_{12}z_{22} = -z_{34}$, $z_{21} = 0$, $z_{42} = z_{22}^3 z_{31} - z_{22}z_{34}z_{41}$ and $z_{44} = z_{22}^2$. A long but straightforward calculation shows that para-Kähler structures on $\mathfrak{rh}_3$ are flat and equivalent to one of the following:

$$(J_1, \langle,\rangle_1) : \begin{cases} J_1e_1 = e_1, \quad J_1e_2 = -e_2, \quad J_1e_3 = e_3, \quad J_1e_4 = -e_4, \\ \langle,\rangle_1 = 2(-e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

$$(J_2, \langle,\rangle_2) : \begin{cases} J_2e_1 = -e_2, \quad J_2e_2 = -e_1, \quad J_2e_3 = e_4, \quad J_2e_4 = e_3, \\ \langle,\rangle_2 = 2(e^1 \circ e^3 + e^2 \circ e^4). \end{cases}$$

In each case, $\mathfrak{rh}_3$ decomposes as a direct sum of Lagrangian subalgebras as follows:

$$\begin{aligned} \mathfrak{rh}_3 &= \mathfrak{L}_1 \oplus \mathfrak{L}'_1 = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\} \\ &= \mathfrak{L}_2 \oplus \mathfrak{L}'_2 = \operatorname{span}\{e_3 + e_4, e_2 - e_1\} \oplus \operatorname{span}\{e_1 + e_2, e_4 - e_3\}. \end{aligned}$$

5. Para-Kähler structures on $\mathfrak{rr}_{3,0}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{rr}_{3,0}$ determined by $[e_1, e_2] = e_2$. Symplectic structures on $\mathfrak{rr}_{3,0}$ are given by $\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{34}e^{34}$ with $\alpha_{12}\alpha_{34} \neq 0$, as shown in [21]. All these symplectic structures $(\mathfrak{rr}_{3,0}, \omega)$ are symplectomorphically equivalent to $\Omega = e^{12} + e^{34}$ through an automorphism of the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ z_{31} & 0 & z_{33} & z_{34} \\ z_{41} & 0 & z_{43} & z_{44} \end{pmatrix}, \quad \text{with} \quad (z_{33}z_{44} - z_{34}z_{43})z_{22} \neq 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{rr}_{3,0}, \Omega)$ are the ones above satisfying

$$\Phi_1 : z_{22} = 1, \quad z_{31} = 0, \quad z_{41} = 0, \quad z_{44}z_{33} = 1 + z_{34}z_{43} \quad \text{with } z_{33} \neq 0,$$
$$\Phi_2 : z_{22} = 1, \quad z_{31} = 0, \quad z_{41} = 0, \quad z_{33} = 0, \quad z_{43}z_{34} = -1.$$

A straightforward calculation, proceeding as in the previous cases, shows that any para-Kähler structure on $(\mathfrak{rr}_{3,0}, \Omega)$ is equivalent to one of the following:

$$(J_1, \langle,\rangle_1) : \begin{cases} J_1e_1 = \epsilon\, e_1, \quad J_1e_2 = -\epsilon\, e_2, \quad J_1e_3 = e_3, \quad J_1e_4 = -e_4, \\ \langle,\rangle_1 = -2(\epsilon\, e^1 \circ e^2 + e^3 \circ e^4), \quad \epsilon = \pm 1, \end{cases}$$

$$(J_2, \langle,\rangle_2) : \begin{cases} J_2e_1 = \frac{1}{\kappa}e_2, \quad J_2e_2 = \kappa e_1, \quad J_2e_3 = e_3, \quad J_2e_4 = -e_4, \\ \langle,\rangle_2 = \frac{1}{\kappa}e^1 \circ e^1 - \kappa e^2 \circ e^2 - 2e^3 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

and they correspond to the Lagrangian decompositions

$$\begin{aligned} \mathfrak{rr}_{3,0} &= \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\} && (\epsilon = 1) \\ &= \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1, e_4\} && (\epsilon = -1) \\ &= \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_2 + \kappa e_1, e_3\} \oplus \operatorname{span}\{e_2 - \kappa e_1, e_4\}. \end{aligned}$$

The structures $(J_1, \langle,\rangle_1)$ are flat, while the structures $(J_2, \langle,\rangle_2)$ are locally symmetric with diagonalizable Ricci operator $\operatorname{Ric}_2 = -\operatorname{diag}[\kappa, \kappa, 0, 0]$. Thus, the underlying pseudo-Riemannian manifold splits locally as a product $N \times \mathbb{L}^2$, where N is a Lorentzian surface of constant Gaussian curvature $K_N = -\kappa$ and $\mathbb{L}^2$ denotes the Minkowskian plane. Hence, they are para-Kähler and opposite para-Kähler structures.

6. Para-Kähler structures on $\mathfrak{rr}_{3,-1}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{rr}_{3,-1}$ determined by $[e_1, e_2] = e_2$ and $[e_1, e_3] = -e_3$. Symplectic structures on $\mathfrak{rr}_{3,-1}$ are given by $\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{23}e^{23}$ with $\alpha_{14}\alpha_{23} \neq 0$, as shown in [21]. All these symplectic structures $(\mathfrak{rr}_{3,-1}, \omega)$ are symplectomorphically equivalent to $\Omega = e^{14} + e^{23}$ through an automorphism of the form

$$\Phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ z_{31} & 0 & z_{33} & 0 \\ z_{41} & 0 & 0 & z_{44} \end{pmatrix} \quad \text{or} \quad \Phi_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ z_{21} & 0 & z_{23} & 0 \\ z_{31} & z_{32} & 0 & 0 \\ z_{41} & 0 & 0 & z_{44} \end{pmatrix},$$

with $z_{22}z_{33}z_{44} \neq 0$ in the first case and $z_{23}z_{32}z_{44} \neq 0$ in the second one. Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{rr}_{3,-1}, \Omega)$ are given by

$$\Phi_1 : z_{21} = 0, \quad z_{31} = 0, \quad z_{33}z_{22} = 1, \quad z_{44} = 1,$$
$$\Phi_2 : z_{21} = 0, \quad z_{31} = 0, \quad z_{32}z_{23} = -1, \quad z_{44} = -1.$$

A straightforward calculation shows that a para-Kähler structure on $(\mathfrak{rr}_{3,-1}, \Omega)$ corresponds to one of the following.

6.1. *Flat para-Kähler structures on* $\mathfrak{rr}_{3,-1}$. Assuming the component of the paracomplex structure $a_{14} \neq 0$, all para-Kähler structures are flat and equivalent to

$$(J_1, \langle,\rangle_1) : \begin{cases} J_1e_1 = \frac{1}{\kappa}e_4, \quad J_1e_2 = -e_2, \quad J_1e_3 = e_3, \quad J_1e_4 = \kappa e_1, \\ \langle,\rangle_1 = \frac{1}{\kappa}e^1 \circ e^1 + 2e^2 \circ e^3 - \kappa e^4 \circ e^4, \qquad \kappa \neq 0. \end{cases}$$

They correspond to the decomposition $\mathfrak{rr}_{3,-1} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_4 + \kappa e_1, e_3\} \oplus \operatorname{span}\{e_4 - \kappa e_1, e_2\}$ of $(\mathfrak{rr}_{3,-1}, \Omega)$ as a direct sum of Lagrangian subalgebras.

6.2. *Ricci-flat para-Kähler structures on* $\mathfrak{rr}_{3,-1}$. On the other hand, if the component of the paracomplex structure $a_{14} = 0$, then any para-Kähler structure is equivalent to

$$(J_2, \langle,\rangle_2) : \begin{cases} J_2e_1 = -e_1, \quad J_2e_2 = -e_2, \quad J_2e_3 = e_3 - \kappa e_2, \quad J_2e_4 = e_4, \\ \langle,\rangle_2 = 2(e^1 \circ e^4 + e^2 \circ e^3) + \kappa e^3 \circ e^3, \quad \kappa = 0, \pm 1. \end{cases}$$

The symplectic Lie algebra $(\mathfrak{rr}_{3,-1}, \Omega)$ decomposes as a direct sum of Lagrangian subalgebras as $\mathfrak{rr}_{3,-1} = \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_4, e_3 - \frac{\kappa}{2}e_2\} \oplus \operatorname{span}\{e_1, e_2\}$.

The para-Kähler structures $(J_2, \langle,\rangle_2)$ are flat if $\kappa = 0$ and Ricci-flat otherwise. Moreover, if $\kappa \neq 0$, then the metric has recurrent curvature with recurrence 1-form $\xi = 2e^1$ (i.e., $\nabla_2 R_2 = 2e^1 \otimes R_2$), and the corresponding curvature operator acting on the space of 2-forms is given by $\mathcal{R}_2(e^1 \wedge e^3) = 2\,\kappa\, e^2 \wedge e^4 = R_2(e_1, e_3, e_3, e_1)e^2 \wedge e^4$. Therefore, they are special recurrent and simply harmonic manifolds locally isometric to a metric in (1). Hence, their curvature tensor is semi-symmetric. Furthermore, the structures $(J_2, \langle,\rangle_2)$ with $\kappa \neq 0$ admit opposite almost para-Kähler structures compatible with the opposite symplectic form $\Omega_2' = e^{14} - e^{23} + \mu e^{13}$, with $\mu \in \mathbb{R}$.

7. Para-Kähler structures on $\mathfrak{r}_2'$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the symplectic Lie algebra $\mathfrak{r}_2'$ determined by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3.$$

Symplectic forms on $\mathfrak{r}_2'$ are of the form $\omega = \alpha_{12}e^{12} + \alpha_{13}(e^{13} - e^{24}) + \alpha_{14}(e^{14} + e^{23})$ with $\alpha_{13}^2 + \alpha_{14}^2 \neq 0$, as shown in [21]. The automorphisms of $\mathfrak{r}_2'$ are given by

$$\Phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & z_{34} \\ -z_{32} & z_{31} & -z_{34} & z_{33} \end{pmatrix} \quad \text{or} \quad \Phi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{32} & -z_{31} & z_{34} & -z_{33} \end{pmatrix},$$

with $z_{33}^2 + z_{34}^2 \neq 0$, from where it follows that any symplectic form is symplectomorphically equivalent to $\Omega_\lambda = \lambda e^{12} + e^{14} + e^{23}$, for some $\lambda \in \mathbb{R}$. We emphasize that although there exist Kähler structures with associated symplectic form Ω_λ for

$\lambda \neq 0$ [22], a straightforward calculation shows that Ω_λ does not support any para-Kähler structure for $\lambda \neq 0$. On the contrary, there exist para-Kähler structures whose associated symplectic structure is $\Omega_0 = e^{14} + e^{23}$. In order to describe these structures, we recall that the automorphisms preserving the symplectic structure $(\mathfrak{r}_2', \Omega_0)$ are given by Φ_1 (resp., Φ_2) above with $z_{33} = 1$ and $z_{34} = 0$ (resp., $z_{33} = -1$ and $z_{34} = 0$). A long but straightforward calculation shows that any para-Kähler structure on $(\mathfrak{r}_2', \Omega_0)$ corresponds to one of the different situations given by $a_{13} \neq 0$ or $a_{13} = 0$. While the Ricci operator is diagonalizable in the case $a_{13} = 0$, this does not happen when $a_{13} \neq 0$. If $a_{13} = 0$, then one has three essentially different cases depending on whether the component of the paracomplex structure $a_{14} = 0$ or $a_{14} \neq 0$.

7.1. *Para-Kähler structures on $\mathfrak{r}_2'$ with complex Ricci operator.* Assuming the component of the paracomplex structure $a_{13} \neq 0$, then para-Kähler structures are equivalent to $(J_1, \langle,\rangle_1)$ given by

$$J_1 e_1 = \widehat{\beta} e_3 + \widehat{\alpha} e_4, \quad J_1 e_2 = -\widehat{\alpha} e_3 + \widehat{\beta} e_4, \quad J_1 e_3 = \beta e_1 - \alpha e_2, \quad J_1 e_4 = \alpha e_1 + \beta e_2,$$
$$\langle,\rangle_1 = \widehat{\alpha}(e^1 \circ e^1 - e^2 \circ e^2) + \alpha(e^3 \circ e^3 - e^4 \circ e^4) + 2(\widehat{\beta} e^1 \circ e^2 - \beta e^3 \circ e^4),$$

where $\widehat{\beta} = \frac{\beta}{\beta^2+\alpha^2}$ and $\widehat{\alpha} = \frac{\alpha}{\beta^2+\alpha^2}$, with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. In this situation $\mathfrak{r}_2'$ decomposes into a direct sum of Lagrangian subalgebras $\mathfrak{r}_2' = \mathfrak{L}_1 \oplus \mathfrak{L}_1'$ where

$$\mathfrak{L}_1 = \operatorname{span}\{e_4 + \alpha e_1 + \beta e_2, e_3 + \beta e_1 - \alpha e_2\},$$
$$\mathfrak{L}_1' = \operatorname{span}\{e_4 - \alpha e_1 - \beta e_2, e_3 - \beta e_1 + \alpha e_2\}.$$

The Ricci operator is complex diagonalizable with eigenvalues $-2(\alpha \pm \beta\sqrt{-1})$. The underlying pseudo-Riemannian structure is locally symmetric. Furthermore, the self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues. Therefore, the metric is locally conformally flat if and only if $\alpha = 0$.

We note that the Ricci tensor ρ_1 defines another para-Kähler structure $(\mathfrak{r}_2', J_1, \rho_1)$ with the same Levi-Civita connection as $(\mathfrak{r}_2', J_1, \langle,\rangle_1)$. A straightforward calculation shows that $(\mathfrak{r}_2', J_1, \rho_1)$ is Einstein of non-constant paraholomorphic sectional curvature, and thus it corresponds (after normalization of the associated symplectic structure) to one of the para-Kähler structures $(J_4, \langle,\rangle_4)$ described below.

7.2. *Flat para-Kähler structures on $\mathfrak{r}_2'$.* If $a_{13} = 0$ and $a_{14} = 0$, then one has that para-Kähler structures are equivalent to the flat structure

$$(J_2, \langle,\rangle_2) : \begin{cases} J_2 e_1 = e_1, \quad J_2 e_2 = e_2, \quad J_2 e_3 = -e_3, \quad J_2 e_4 = -e_4, \\ \langle,\rangle_2 = -2(e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

so that $\mathfrak{r}_2' = \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_1, e_2\} \oplus \operatorname{span}\{e_3, e_4\}$.

7.3. *Para-Kähler Einstein structures on $\mathfrak{r}_2'$ of non-zero curvature.* Assume that $a_{13} = 0$ and $a_{14} \neq 0$. Then any para-Kähler structure is equivalent to one of the following:

$$(J_3,\langle,\rangle_3):\begin{cases} J_3e_1=-\frac{1}{\kappa}e_4, \quad J_3e_2=\frac{2}{\kappa}e_3-e_2, \quad J_3e_3=e_3, \quad J_3e_4=-\kappa e_1,\\ \langle,\rangle_3=-\frac{1}{\kappa}(e^1\circ e^1-2e^2\circ e^2)+2e^2\circ e^3+\kappa e^4\circ e^4, \quad \kappa\neq 0,\end{cases}$$

$$(J_4,\langle,\rangle_4):\begin{cases} J_4e_1=-\frac{1}{\kappa}e_4, \quad J_4e_2=\frac{1}{\kappa}e_3, \quad J_4e_3=\kappa e_2, \quad J_4e_4=-\kappa e_1,\\ \langle,\rangle_4=-\frac{1}{\kappa}\,(e^1\circ e^1-e^2\circ e^2)-\kappa(e^3\circ e^3-e^4\circ e^4), \quad \kappa\neq 0,\end{cases}$$

where the symplectic Lie algebra $(\mathfrak{r}_2',\Omega_0)$ decomposes as a direct sum of Lagrangian subalgebras

$$\begin{aligned}\mathfrak{r}_2'=\mathfrak{L}_3\oplus\mathfrak{L}_3' &= \operatorname{span}\{e_3,e_4-\kappa e_1\}\oplus\operatorname{span}\{e_3-\kappa e_2,e_4+\kappa e_1\}\\ =\mathfrak{L}_4\oplus\mathfrak{L}_4' &= \operatorname{span}\{e_3+\kappa e_2,e_4-\kappa e_1\}\oplus\operatorname{span}\{e_4+\kappa e_1,e_3-\kappa e_2\}.\end{aligned}$$

The structures $(J_3,\langle,\rangle_3)$ have non-zero constant paraholomorphic sectional curvature $H_3=\kappa$.

The structures $(J_4,\langle,\rangle_4)$ are Einstein with $\operatorname{Ric}_4=2\kappa\operatorname{Id}(\neq 0)$. Moreover, the self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues, which shows that the paraholomorphic sectional curvature is not constant. A straightforward calculation shows that structures $(J_4,\langle,\rangle_4)$ have a compatible anti-Kähler structure (see [5]) defined by a complex structure $\mathfrak{J}$ given by $\mathfrak{J}e_1=e_2$, $\mathfrak{J}e_3=e_4$, which commutes with the paracomplex structure J_4. Hence, the structure $(\langle,\rangle^*,J_4)$ is also a locally symmetric para-Kähler structure, where $\langle X,Y\rangle^*=\langle\mathfrak{J}X,Y\rangle_4$ is the twin metric of $\langle,\rangle_4$. A straightforward calculation shows that the Ricci operator Ric^* has complex eigenvalues $\pm 2\kappa\sqrt{-1}$ and thus it corresponds to a metric in Section 7.1 after renormalization of the corresponding symplectic structure.

8. Para-Kähler structures on $\mathfrak{r}_{4,0}$

Let $\{e_1,e_2,e_3,e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,0}$ determined by $[e_1,e_4]=-e_1$ and $[e_3,e_4]=-e_2$. Symplectic structures on $\mathfrak{r}_{4,0}$ are given by (see [21]) $\omega=\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ with $\alpha_{14}\alpha_{23}\neq 0$. Using the automorphisms of $\mathfrak{r}_{4,0}$ given by

$$\Phi=\begin{pmatrix} z_{11} & 0 & 0 & z_{14}\\ 0 & z_{22} & z_{23} & z_{24}\\ 0 & 0 & z_{22} & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{with } z_{11}z_{22}\neq 0,$$

it follows that any symplectic form on $\mathfrak{r}_{4,0}$ is symplectomorphically equivalent to $\Omega_\epsilon=e^{14}+\epsilon e^{23}$, where $\epsilon=\pm 1$. In order to describe the para-Kähler structures observe that the automorphisms preserving the symplectic structures $(\mathfrak{r}_{4,0},\Omega_\epsilon)$ are given by Φ above with $z_{11}=1$, $z_{24}=0$, $z_{34}=0$ and $z_{22}=\pm 1$. A straightforward calculation shows that any para-Kähler structure is equivalent to

$$(J,\langle,\rangle):\begin{cases} Je_1=-e_1, \quad Je_2=e_2, \quad Je_3=-e_3, \quad Je_4=e_4,\\ \langle,\rangle=2(e^1\circ e^4-\epsilon\, e^2\circ e^3),\end{cases}$$

which correspond to the Lagrangian decomposition of $(\mathfrak{r}_{4,0},\Omega_\epsilon)$

$$\mathfrak{r}_{4,0}=\mathfrak{L}\oplus\mathfrak{L}'=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_1,e_3\}.$$

The structures $(\mathfrak{r}_{4,0},J,\langle,\rangle)$ are Ricci-flat and recurrent with recurrence 1-form $\xi=2e^4$. Moreover, the curvature tensor acting on the space of 2-forms is given by $\mathcal{R}(e^3 \wedge e^4) = -e^1 \wedge e^2 = \epsilon R(e_3, e_4, e_3, e_4) e^1 \wedge e^2$. Hence, the para-Kähler structures $(\mathfrak{r}_{4,0}, J, \langle , \rangle)$ are special recurrent and simply harmonic, and they are locally modelled on (1). Thus, their curvature tensor is semi-symmetric. Moreover, the structures $(J, \langle , \rangle)$ admit opposite almost para-Kähler structures compatible with the opposite symplectic forms $\Omega' = e^{14} - \epsilon e^{23} - \mu e^{34}$, with $\mu \in \mathbb{R}$.

9. Para-Kähler structures on $\mathfrak{r}_{4,-1}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3 - e_2.$$

Any symplectic form on $\mathfrak{r}_{4,-1}$ is of the form $\omega = \alpha_{13} e^{13} + \alpha_{14} e^{14} + \alpha_{24} e^{24} + \alpha_{34} e^{34}$ with $\alpha_{13}\alpha_{24} \neq 0$, as shown in [21]. Moreover, the automorphisms of $\mathfrak{r}_{4,-1}$ are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & z_{23} & z_{24} \\ 0 & 0 & z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad z_{11} z_{22} \neq 0,$$

from where it follows that any symplectic structure is symplectomorphically equivalent to $\Omega = e^{13} + e^{24}$. Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-1}, \Omega)$ are given by Φ above with $z_{11} = 1$, $z_{22} = 1$, $z_{34} = 0$ and $z_{23} = z_{14}$. It now follows that any para-Kähler structure on $\mathfrak{r}_{4,-1}$ is equivalent to

$$(J, \langle , \rangle) : \begin{cases} Je_1 = -e_1, \quad Je_2 = e_2, \quad Je_3 = -\kappa e_1 + e_3, \quad Je_4 = -e_4, \\ \langle , \rangle = 2(e^1 \circ e^3 - e^2 \circ e^4) + \kappa\, e^3 \circ e^3, \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decomposition $\mathfrak{r}_{4,-1} = \mathfrak{L} \oplus \mathfrak{L}' = \operatorname{span}\{e_2, e_3 - \frac{\kappa}{2} e_1\} \oplus \operatorname{span}\{e_1, e_4\}$. The structures above are flat if $\kappa = 0$. Otherwise, they are Ricci-flat, locally symmetric and the underlying structures are locally modelled on (1) with $\Psi(x^1, x^2) = \pm (x^2)^2$. Furthermore, the structures above admit opposite almost para-Kähler structures compatible with the opposite symplectic form $\Omega' = e^{13} - e^{24} + \mu e^{34}$, with $\mu \in \mathbb{R}$.

10. Para-Kähler structures on $\mathfrak{r}_{4,-1,\beta}$ $(-1 < \beta < 0)$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1,\beta}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -\beta e_3.$$

Any symplectic form on this Lie algebra must be of the form $\omega = \alpha_{12} e^{12} + \alpha_{14} e^{14} + \alpha_{24} e^{24} + \alpha_{34} e^{34}$ with $\alpha_{12}\alpha_{34} \neq 0$, as shown in [21]. Moreover, the automorphisms of $\mathfrak{r}_{4,-1,\beta}$ are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11} z_{22} z_{33} \neq 0,$$

from where it follows that any symplectic form is symplectomorphically equivalent to $\Omega = e^{12} + e^{34}$. The automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-1,\beta}, \Omega)$ are the ones given by $z_{14} = z_{24} = 0$, $z_{33} = 1$ and $z_{22} z_{11} = 1$.

The para-Kähler structures on $(\mathfrak{r}_{4,-1,\beta}, \Omega)$ split into two different classes depending on the value of the coefficient a_{43} of the paracomplex structure as follows.

10.1. *Ricci-flat para-Kähler structures on* $\mathfrak{r}_{4,-1,\beta}$ $(-1<\beta<0)$. Assuming $a_{43}=0$, the corresponding para-Kähler structures are equivalent to one of the following:

$$(J_1,\langle,\rangle_1):\begin{cases} J_1e_1=-e_1+\kappa\, e_2, \quad J_1e_2=e_2, \quad J_1e_3=-e_3, \quad J_1e_4=e_4, \\ \langle,\rangle_1=\kappa\, e^1\circ e^1+2(e^1\circ e^2+e^3\circ e^4), \quad \kappa=0,\pm1, \end{cases}$$

$$(J_2,\langle,\rangle_2):\begin{cases} J_2e_1=-e_1, \quad J_2e_2=-\kappa\, e_1+e_2, \quad J_2e_3=e_3, \quad J_2e_4=-e_4, \\ \langle,\rangle_2=\kappa\, e^2\circ e^2+2(e^1\circ e^2-e^3\circ e^4), \quad \kappa=0,\pm1. \end{cases}$$

They correspond to the Lagrangian decompositions

$$\begin{aligned} \mathfrak{r}_{4,-1,\beta} &= \mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_3,e_2-\tfrac{2}{\kappa}e_1\} \quad (\kappa\neq0)\\ &= \mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_1,e_3\} \quad (\kappa=0)\\ &= \mathfrak{L}_2\oplus\mathfrak{L}_2'=\operatorname{span}\{e_3,e_2-\tfrac{\kappa}{2}e_1\}\oplus\operatorname{span}\{e_1,e_4\}. \end{aligned}$$

The structures above are flat if $\kappa=0$. Otherwise, they are Ricci-flat and recurrent with recurrence 1-forms $\xi_1=2(\beta-1)e^4$ and $\xi_2=2(\beta+1)e^4$, respectively. The corresponding curvature operators acting on the space of 2-forms are given by

$$\begin{aligned} \mathcal{R}_1(e^1\wedge e^4) &= \kappa(\beta-2)e^2\wedge e^3=R_1(e_1,e_4,e_1,e_4)e^2\wedge e^3,\\ \mathcal{R}_2(e^2\wedge e^4) &= \kappa(\beta+2)e^1\wedge e^3=R_2(e_2,e_4,e_4,e_2)e^1\wedge e^3. \end{aligned}$$

Therefore, the metrics above are simply harmonic and special recurrent locally isometric to a metric given in (1). Hence, their curvature tensor is semi-symmetric. Furthermore, the structures $(J_1,\langle,\rangle_1)$ and $(J_2,\langle,\rangle_2)$ admit opposite almost para-Kähler structures compatible with the opposite symplectic forms $\Omega_1'=e^{12}-e^{34}+\mu e^{14}$ and $\Omega_2'=e^{12}-e^{34}+\mu e^{24}$ respectively, where $\mu\in\mathbb{R}$.

10.2. *Para-Kähler and opposite para-Kähler structures on* $\mathfrak{r}_{4,-1,\beta}$ $(-1<\beta<0)$. If $a_{43}\neq0$, then para-Kähler structures are equivalent to

$$(J_3,\langle,\rangle_3):\begin{cases} J_3e_1=-e_1, \quad J_3e_2=e_2, \quad J_3e_3=\kappa e_4, \quad J_3e_4=\tfrac{1}{\kappa}e_3, \\ \langle,\rangle_3=2e^1\circ e^2+\kappa e^3\circ e^3-\tfrac{1}{\kappa}e^4\circ e^4, \quad \kappa\neq0, \end{cases}$$

so that $\mathfrak{r}_{4,-1,\beta}=\mathfrak{L}_3\oplus\mathfrak{L}_3'=\operatorname{span}\{e_2,e_4+\tfrac{1}{\kappa}e_3\}\oplus\operatorname{span}\{e_1,e_4-\tfrac{1}{\kappa}e_3\}$.

The metrics above are locally symmetric with diagonalizable Ricci operator $\operatorname{Ric}_3=\operatorname{diag}[0,0,\kappa\beta^2,\kappa\beta^2]$. Hence, the underlying manifold is locally isometric to a product $\mathbb{L}^2\times N$ of the Minkowskian plane and a Lorentzian surface of constant sectional curvature $K_N=\kappa\beta^2$, thus being para-Kähler and opposite para-Kähler.

11. Para-Kähler structures on $\mathfrak{r}_{4,-1,-1}$

Let $\{e_1,e_2,e_3,e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1,-1}$ determined by

$$[e_1,e_4]=-e_1, \quad [e_2,e_4]=e_2, \quad [e_3,e_4]=e_3.$$

Any symplectic structure on $\mathfrak{r}_{4,-1,-1}$ is of the form $\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ with $\alpha_{13}\alpha_{24}-\alpha_{12}\alpha_{34}\neq0$, and all of them are symplectomorphically equivalent to $\Omega=e^{12}+e^{34}$ through an automorphism of $\mathfrak{r}_{4,-1,-1}$ given by

$$\Phi=\begin{pmatrix} z_{11} & 0 & 0 & z_{14}\\ 0 & z_{22} & z_{23} & z_{24}\\ 0 & z_{32} & z_{33} & z_{34}\\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}(z_{22}z_{33}-z_{23}z_{32})\neq0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-1,-1},\Omega)$ are the ones above with $z_{23}=z_{24}=0$, $z_{33}=1$, $z_{32}=z_{14}z_{22}$ and $z_{22}z_{11}=1$. The associated para-Kähler structures split into two cases depending on whether $a_{43}\neq 0$ or $a_{43}=0$ as follows.

11.1. *Para-Kähler and opposite para-Kähler structures on* $\mathfrak{r}_{4,-1,-1}$. Any para-Kähler structure with $a_{43}\neq 0$ is equivalent to

$$(J_1,\langle,\rangle_1):\begin{cases} J_1e_1=-e_1, \quad J_1e_2=e_2, \quad J_1e_3=\kappa e_4\ , \quad J_1e_4=\frac{1}{\kappa}e_3,\\ \langle,\rangle_1=2e^1\circ e^2+\kappa e^3\circ e^3-\frac{1}{\kappa}e^4\circ e^4, \quad \kappa\neq 0,\end{cases}$$

so that $\mathfrak{r}_{4,-1,-1}=\mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{\frac{1}{\kappa}e_3+e_4,e_2\}\oplus\operatorname{span}\{e_1,e_4-\frac{1}{\kappa}e_3\}$.

The structures above are locally symmetric and the Ricci operator is diagonalizable with eigenvalues $\{0,0,\kappa,\kappa\}$. Hence, the underlying pseudo-Riemannian structures split locally as a product $\mathbb{L}^2\times N$ of a Lorentzian surface N of constant Gaussian curvature $K_N=\kappa$ and the Minkowskian plane. Thus, they also admit an opposite para-Kähler structure.

11.2. *Ricci-flat para-Kähler structures on* $\mathfrak{r}_{4,-1,-1}$. Assuming $a_{43}=0$ para-Kähler structures on $(\mathfrak{r}_{4,-1,-1},\Omega)$ are equivalent to one of the following, which correspond to different geometric situations:

$$(J_2,\langle,\rangle_2):\begin{cases} J_2e_1=-e_1, \quad J_2e_2=-\kappa e_1+e_2, \quad J_2e_3=e_3, \quad J_2e_4=-e_4,\\ \langle,\rangle_2=\kappa e^2\circ e^2+2(e^1\circ e^2-e^3\circ e^4), \quad \kappa=0,\pm1,\end{cases}$$

inducing the decomposition $\mathfrak{r}_{4,-1,-1}=\mathfrak{L}_2\oplus\mathfrak{L}_2'=\operatorname{span}\{e_3,e_2-\frac{\kappa}{2}e_1\}\oplus\operatorname{span}\{e_4,e_1\}$,

$$(J_3,\langle,\rangle_3):\begin{cases} J_3e_1=-e_1+\kappa e_2, \quad J_3e_2=e_2, \quad J_3e_3=-e_3, \quad J_3e_4=e_4,\\ \langle,\rangle_3=\kappa e^1\circ e^1+2(e^1\circ e^2+e^3\circ e^4), \quad \kappa=0,\pm1,\end{cases}$$

so that the Lie algebra decomposes as

$$\begin{aligned}\mathfrak{r}_{4,-1,-1}=\mathfrak{L}_3\oplus\mathfrak{L}_3'&=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_3,e_2-\tfrac{2}{\kappa}e_1\} \quad (\kappa\neq 0)\\ &=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_3,e_1\} \qquad\qquad (\kappa=0),\end{aligned}$$

or

$$(J_4,\langle,\rangle_4):\begin{cases} J_4e_1=\kappa e_2+e_4, \quad J_4e_2=e_3, \quad J_4e_3=e_2, \quad J_4e_4=e_1-\kappa e_3,\\ \langle,\rangle_4=\kappa(e^1\circ e^1+e^4\circ e^4)+2(e^1\circ e^3-e^2\circ e^4), \quad \kappa\in\mathbb{R},\end{cases}$$

so that $(\mathfrak{r}_{4,-1,-1},\Omega)$ decomposes as a direct sum of Lagrangian subalgebras

$$\mathfrak{r}_{4,-1,-1}=\mathfrak{L}_4\oplus\mathfrak{L}_4'=\operatorname{span}\{e_2+e_3,e_1+e_4+\kappa e_2\}\oplus\operatorname{span}\{e_3-e_2,e_4-e_1+\kappa e_2\}.$$

The structures $(J_2,\langle,\rangle_2)$ are flat if $\kappa=0$. Otherwise, they are Ricci-flat and locally symmetric, thus locally modelled on (1) with $\Psi(x^1,x^2)=\pm(x^2)^2$. Moreover, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic 2-form $\Omega_2'=e^{12}-e^{34}+\mu e^{24}$ with $\mu\in\mathbb{R}$.

The structures $(J_3,\langle,\rangle_3)$ are flat if $\kappa=0$ and Ricci-flat otherwise. Moreover, they are special recurrent with recurrence 1-form $\xi_3=-4e^4$ and curvature operator determined by $\mathcal{R}_3(e^1\wedge e^4)=-3\kappa e^2\wedge e^3=R_3(e_1,e_4,e_1,e_4)e^2\wedge e^3$. Hence, these structures are simply harmonic locally modelled on (1) and thus their curvature tensor is semi-symmetric. Furthermore, they admit opposite almost para-Kähler structures compatible with the opposite symplectic 2-form $\Omega_3'=e^{12}-e^{34}+\mu e^{14}$,

$\mu \in \mathbb{R}$, and have an associated one-parameter family of hypersymplectic structures $(J_3, J_\delta, \langle,\rangle_3)$ given by the Kähler structures

$$J_\delta e_1 = -\tfrac{\kappa\delta}{2} e_3 - \tfrac{1}{\delta} e_4, \quad J_\delta e_2 = -\delta e_3, \quad J_\delta e_3 = \tfrac{1}{\delta} e_2, \quad J_\delta e_4 = \delta e_1 - \tfrac{\kappa\delta}{2} e_2,$$

so that $J_\delta J_3 = -J_3 J_\delta$ for any $\delta \neq 0$ (see also [3]).

Finally, the structures $(J_4, \langle,\rangle_4)$ are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and special recurrent with recurrence 1-form $\xi_4 = -4e^4$ and curvature operator given by $\mathcal{R}_4(e^1 \wedge e^4) = -3\kappa e^2 \wedge e^3 = R_4(e_1, e_4, e_1, e_4) e^2 \wedge e^3$. Therefore, they are simply harmonic and locally modelled on (1) and hence their curvature tensor is semi-symmetric. Moreover, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic 2-form $\Omega_4' = e^{13} + e^{24} + \mu e^{14}$ with $\mu \in \mathbb{R}$.

12. Para-Kähler structures on $\mathfrak{r}_{4,-\alpha,\alpha}$ with $0 < \alpha < 1$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-\alpha,\alpha}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = \alpha e_2, \quad [e_3, e_4] = -\alpha e_3.$$

Symplectic forms on $\mathfrak{r}_{4,-\alpha,\alpha}$ are given by $\omega = \alpha_{14} e^{14} + \alpha_{23} e^{23} + \alpha_{24} e^{24} + \alpha_{34} e^{34}$ with $\alpha_{14}\alpha_{23} \neq 0$ (see [21]) and all of them are symplectomorphically equivalent to $\Omega = e^{14} + e^{23}$ through a Lie algebra automorphism of the form

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad z_{11} z_{22} z_{33} = 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-\alpha,\alpha}, \Omega)$ are given by Φ above with $z_{11} = 1$, $z_{24} = z_{34} = 0$ and $z_{33} z_{22} = 1$. The different classes of para-Kähler structures on this Lie algebra arise from the cases $a_{41} \neq 0$ and $a_{41} = 0$, which we study separately in what follows.

12.1. *Para-Kähler and opposite para-Kähler structures on* $\mathfrak{r}_{4,-\alpha,\alpha}$. Any para-Kähler structure with $a_{41} \neq 0$ is equivalent to

$$(J_1, \langle,\rangle_1) : \begin{cases} J_1 e_1 = \kappa e_4, \quad J_1 e_2 = -e_2, \quad J_1 e_3 = e_3, \quad J_1 e_4 = \frac{1}{\kappa} e_1, \\ \langle,\rangle_1 = \kappa e^1 \circ e^1 - \frac{1}{\kappa} e^4 \circ e^4 + 2 e^2 \circ e^3, \quad \kappa \neq 0, \end{cases}$$

with Lagrangian decomposition $\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_3, e_4 + \frac{1}{\kappa} e_1\} \oplus \operatorname{span}\{e_2, e_4 - \frac{1}{\kappa} e_1\}$. The structures above are locally symmetric and the Ricci operator is diagonalizable with eigenvalues $\{0, 0, \kappa, \kappa\}$. Hence, the underlying manifold splits locally as a product $\mathbb{L}^2 \times N$ of a Lorentzian surface N of constant Gaussian curvature $K_N = \kappa$ and the Minkowskian plane, and thus it also admits an opposite para-Kähler structure.

12.2. *Ricci-flat para-Kähler structures.* Assuming $a_{41} = 0$, para-Kähler structures are equivalent to one of the following:

$$(J_2, \langle,\rangle_2) : \begin{cases} J_2 e_1 = -e_1, \quad J_2 e_2 = -e_2 + \kappa e_3, \quad J_2 e_3 = e_3, \quad J_2 e_4 = e_4, \\ \langle,\rangle_2 = \kappa e^2 \circ e^2 + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa = 0, \pm 1, \end{cases}$$

which induces the Lagrangian decompositions

$$\begin{aligned}\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_2 \oplus \mathfrak{L}_2' &= \operatorname{span}\{e_3, e_4\} \oplus \operatorname{span}\{e_1, e_3 - \tfrac{2}{\kappa} e_2\} \quad (\kappa \neq 0)\\ &= \operatorname{span}\{e_3, e_4\} \oplus \operatorname{span}\{e_1, e_2\} \qquad\qquad (\kappa = 0),\end{aligned}$$

or

$$(J_3, \langle,\rangle_3) : \begin{cases} J_3 e_1 = e_1, \quad J_3 e_2 = -e_2, \quad J_3 e_3 = -\kappa e_2 + e_3, \quad J_3 e_4 = -e_4, \\ \langle,\rangle_3 = \kappa e^3 \circ e^3 + 2(e^2 \circ e^3 - e^1 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

so that $\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_1, e_3 - \frac{\kappa}{2} e_2\} \oplus \operatorname{span}\{e_2, e_4\}$.

The structures $(J_2, \langle,\rangle_2)$ are flat if $\kappa = 0$, while structures $(J_3, \langle,\rangle_3)$ are flat if either $\kappa = 0$ or $\alpha = \frac{1}{2}$. Otherwise, all the structures above are Ricci-flat and special recurrent with recurrence 1-form given by $\xi_2 = 2(1+\alpha)e^4$ and $\xi_3 = 2(1-\alpha)e^4$, respectively, and curvature operator determined by

$$\begin{aligned}\mathcal{R}_2(e^2 \wedge e^4) &= \alpha(1+2\alpha)\kappa e^1 \wedge e^3 = R_2(e_2, e_4, e_4, e_2) e^1 \wedge e^3,\\ \mathcal{R}_3(e^3 \wedge e^4) &= \alpha(1-2\alpha)\kappa e^1 \wedge e^2 = R_3(e_3, e_4, e_3, e_4) e^1 \wedge e^2.\end{aligned}$$

Therefore, they are simply harmonic and locally modelled on (1), so their curvature tensor is semi-symmetric. Furthermore, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic forms given by $\Omega_2' = e^{14} - e^{23} + \mu e^{24}$ and $\Omega_3' = e^{14} - e^{23} + \mu e^{34}$, where $\mu \in \mathbb{R}$.

13. Para-Kähler structures on $\mathfrak{h}_4$

Let $\mathfrak{h}_4$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\tfrac{1}{2} e_1, \quad [e_2, e_4] = -e_1 - \tfrac{1}{2} e_2, \quad [e_3, e_4] = -e_3.$$

Symplectic structures on $\mathfrak{h}_4$, which are given by $\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14} e^{14} + \alpha_{24} e^{24}$ with $\alpha_{12} \neq 0$, are equivalent to $\Omega_\epsilon = \epsilon(e^{12} - e^{34})$ through a Lie algebra automorphism

$$\Phi = \begin{pmatrix} z_{22} & z_{12} & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 2z_{22}z_{24} & 2(z_{12} + 2z_{22})z_{24} - 2z_{14}z_{22} & z_{22}^2 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{22} \neq 0,$$

where $\epsilon = \pm 1$ (see [21]). Moreover, the Lie algebra automorphisms preserving the symplectic structure Ω_ϵ are given by Φ above with $z_{24} = z_{14} = 0$ and $z_{22} = \pm 1$. Now, one has that any para-Kähler structure on $(\mathfrak{h}_4, \Omega_\epsilon)$ is equivalent to

$$(J, \langle,\rangle) : \begin{cases} J e_1 = -\epsilon\, e_1, \quad J e_2 = \epsilon\, e_2, \quad J e_3 = \epsilon\, e_3, \quad J e_4 = -\epsilon\, e_4, \\ \langle,\rangle = 2(e^1 \circ e^2 + e^3 \circ e^4), \end{cases}$$

so that they correspond to the Lagrangian decomposition $\mathfrak{h}_4 = \mathfrak{L} \oplus \mathfrak{L}' = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1, e_4\}$. A straightforward calculation shows that the structures $(J, \langle,\rangle)$ are Ricci-flat with recurrent curvature. Moreover, the recurrence 1-form is given by $\xi = e^4$ and the curvature operator is determined by $\mathcal{R}(e^2 \wedge e^4) = -e^1 \wedge e^3 = R(e_2, e_4, e_2, e_4) e^1 \wedge e^3$. Therefore, $(\mathfrak{h}_4, J, \langle,\rangle)$ determines a simply harmonic manifold with special recurrent curvature locally modelled on (1) and its curvature tensor is semi-symmetric. Furthermore, $(\mathfrak{h}_4, J, \langle,\rangle)$ does not admit any opposite almost para-Kähler structure.

14. Para-Kähler structures on $\mathfrak{d}_{4,1}$

Let $\mathfrak{d}_{4,1}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1, \quad [e_3, e_4] = -e_3.$$

Symplectic structures on $\mathfrak{d}_{4,1}$ are given by (see [21]) $\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{24}e^{24}$, with $\alpha_{12} \neq 0$. The automorphisms of $\mathfrak{d}_{4,1}$ are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & 0 \\ z_{31} & -z_{14}z_{22} & z_{11}z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad z_{11}z_{22} \neq 0,$$

and one has that any symplectic form ω is symplectomorphically equivalent to one of the following:

$$\Omega_1 = e^{12} - e^{34}, \qquad \Omega_2 = e^{12} - e^{34} + e^{24}.$$

Moreover, the automorphisms Φ_i preserving the symplectic structure $(\mathfrak{d}_{4,1}, \Omega_i)$, $i = 1, 2$, are determined by the conditions

$$\Phi_1 : z_{31} = 0, \quad z_{22}z_{11} = 1 \qquad \text{and} \qquad \Phi_2 : z_{31} = 0, \quad z_{22} = 1, \quad z_{11} = 1.$$

We study the two symplectic structures separately.

14.1. **Para-Kähler structures on $(\mathfrak{d}_{4,1}, \Omega_1)$.** There are three distinct situations corresponding to different geometries.

14.1.1. *Para-Kähler structures of non-zero constant paraholomorphic sectional curvature.* Assume $a_{43} \neq 0$. Then necessarily $a_{23} = 0$ and any para-Kähler structure is equivalent to

$$(J_{11}, \langle,\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1, \quad J_{11}e_2 = e_2, \quad J_{11}e_3 = -\kappa e_4, \quad J_{11}e_4 = -\frac{1}{\kappa}e_3, \\ \langle,\rangle_{11} = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,1} = \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \operatorname{span}\{e_2, e_4 - \frac{1}{\kappa}e_3\} \oplus \operatorname{span}\{e_1, e_4 + \frac{1}{\kappa}e_3\}$. A straightforward calculation now shows that the structures $(J_{11}, \langle,\rangle_{11})$ have constant paraholomorphic sectional curvature $H = \kappa$.

14.1.2. *Locally symmetric Ricci-flat para-Kähler structures.* Set $a_{43} = 0$, $a_{23} = 0$. Then, any para-Kähler structure corresponds to one of the following:

$$(J_{12}, \langle,\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1, \quad J_{12}e_2 = -\kappa\, e_1 + e_2, \quad J_{12}e_3 = e_3, \quad J_{12}e_4 = -e_4, \\ \langle,\rangle_{12} = \kappa\, e^2 \circ e^2 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

$$(J_{13}, \langle,\rangle_{13}) : \begin{cases} J_{13}e_1 = e_1 + \kappa\, e_2, \quad J_{13}e_2 = -e_2, \quad J_{13}e_3 = e_3, \quad J_{13}e_4 = -e_4, \\ \langle,\rangle_{13} = \kappa\, e^1 \circ e^1 - 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1. \end{cases}$$

The structures $(J_{12}, \langle,\rangle_{12})$ are all flat, while the structures $(J_{13}, \langle,\rangle_{13})$ are flat if and only if $\kappa = 0$. Otherwise, they are locally symmetric and Ricci-flat. Therefore, their underlying pseudo-Riemannian structure corresponds to that in (1) with $\Psi(x^1, x^2) = \pm(x^2)^2$. Moreover, they correspond to the Lagrangian decompositions

$$\begin{aligned} \mathfrak{d}_{4,1} &= \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_1, e_4\} \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_3, e_2 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_2, e_4\} \quad (\kappa \neq 0) \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\} \qquad\qquad (\kappa = 0). \end{aligned}$$

Furthermore, $(\mathfrak{d}_{4,1}, J_{13}, \langle,\rangle_{13})$ does not admit any opposite almost para-Kähler structure.

14.1.3. *Locally symmetric para-Kähler structures with nilpotent Ricci operator.* Setting $a_{43}=0$ and $a_{23}\neq 0$, one has that any para-Kähler structure is equivalent to

$$(J_{14}, \langle,\rangle_{14}): \begin{cases} J_{14}e_1 = \kappa e_2 - e_4, \quad J_{14}e_2 = e_3, \quad J_{14}e_3 = e_2, \quad J_{14}e_4 = -e_1 + \kappa e_3, \\ \langle,\rangle_{14} = \kappa(e^1\circ e^1 + e^4\circ e^4) + 2(e^1\circ e^3 + e^2\circ e^4), \quad \kappa\in\mathbb{R}, \end{cases}$$

which corresponds to the Lagrangian decomposition

$$\mathfrak{d}_{4,1} = \mathfrak{L}_{14} \oplus \mathfrak{L}'_{14} = \operatorname{span}\{e_2+e_3, e_4-e_1-\kappa e_2\} \oplus \operatorname{span}\{e_3-e_2, e_4+e_1-\kappa e_2\}.$$

The structures $(J_{14}, \langle,\rangle_{14})$ are locally symmetric with two-step nilpotent Ricci operator. Furthermore, the metric is locally conformally flat if and only if $\kappa=0$; otherwise, the anti-self-dual Weyl curvature operator is two-step nilpotent.

14.2. **Para-Kähler structures on $(\mathfrak{d}_{4,1}, \Omega_2)$.** All para-Kähler structures are flat in this case and they are equivalent to the structures

$$(J_{21}, \langle,\rangle_{21}): \begin{cases} J_{21}e_1 = -e_1, \quad J_{21}e_2 = -\kappa e_1 + e_2, \quad J_{21}e_3 = e_3, \quad J_{21}e_4 = -e_4, \\ \langle,\rangle_{21} = \kappa e^2\circ e^2 + 2(e^1\circ e^2 - e^2\circ e^4 + e^3\circ e^4), \quad \kappa\in\mathbb{R}, \end{cases}$$

which corresponds to the Lagrangian decomposition

$$\mathfrak{d}_{4,1} = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_1, e_4\}.$$

15. Para-Kähler structures on $\mathfrak{d}_{4,\frac{1}{2}}$

Let $\mathfrak{d}_{4,\frac{1}{2}}$ be the Lie algebra generated by $\{e_1,e_2,e_3,e_4\}$ with

$$[e_1,e_2]=e_3, \quad [e_1,e_4] = -\tfrac{1}{2}e_1, \quad [e_2,e_4] = -\tfrac{1}{2}e_2, \quad [e_3,e_4]=-e_3.$$

The automorphisms of $\mathfrak{d}_{4,\frac{1}{2}}$ correspond to

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & 0 & z_{14} \\ z_{21} & z_{22} & 0 & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \begin{array}{l} z_{31} = 2(z_{11}z_{24} - z_{14}z_{21}), \\ z_{32} = 2(z_{12}z_{24} - z_{14}z_{22}), \\ z_{33} = z_{11}z_{22} - z_{12}z_{21} \neq 0. \end{array}$$

Symplectic forms on $\mathfrak{d}_{4,\frac{1}{2}}$ are given by $\omega = \alpha_{12}(e^{12}-e^{34}) + \alpha_{14}e^{14} + \alpha_{24}e^{24}$ with $\alpha_{12}\neq 0$, and thus symplectomorphically equivalent to $\Omega = e^{12}-e^{34}$ (see [21]). Moreover, the Lie algebra automorphisms preserving Ω are obtained by specializing

$$z_{11}z_{22} - z_{12}z_{21} = 1, \quad z_{14}z_{21} - z_{11}z_{24} = 0, \quad z_{14}z_{22} - z_{12}z_{24} = 0.$$

Now, the description of para-Kähler structures depends on whether the component $a_{43}\neq 0$ or $a_{43}=0$.

15.1. *Para-Kähler structures of non-zero constant paraholomorphic sectional curvature.* If $a_{43}\neq 0$, then para-Kähler structures are equivalent to

$$(J_1, \langle,\rangle_1): \begin{cases} J_1e_1 = -e_1, \quad J_1e_2 = e_2, \quad J_1e_3 = -\kappa e_4, \quad J_1e_4 = -\frac{1}{\kappa}e_3, \\ \langle,\rangle_1 = 2e^1\circ e^2 + \kappa e^3\circ e^3 - \frac{1}{\kappa}e^4\circ e^4, \quad \kappa\neq 0, \end{cases}$$

which corresponds to the Lagrangian decomposition

$$\mathfrak{d}_{4,\frac{1}{2}} = \mathfrak{L}_1 \oplus \mathfrak{L}'_1 = \operatorname{span}\{e_2, e_4 - \tfrac{1}{\kappa}e_3\} \oplus \operatorname{span}\{e_1, e_4 + \tfrac{1}{\kappa}e_3\}.$$

Furthermore, the paraholomorphic sectional curvature is constant $H=\kappa$.

15.2. *Flat para-Kähler structures.* If $a_{43} = 0$, then the corresponding para-Kähler structures are flat and they are equivalent to

$$(J_2, \langle , \rangle_2) : \begin{cases} J_2 e_1 = e_1, \quad J_2 e_2 = -e_2, \quad J_2 e_3 = e_3, \quad J_2 e_4 = -e_4, \\ \langle , \rangle_2 = 2(-e^1 \circ e^2 + e^3 \circ e^4), \end{cases}$$

so that $\mathfrak{d}_{4,\frac{1}{2}} = \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_3, e_1\} \oplus \operatorname{span}\{e_4, e_2\}$.

16. Para-Kähler structures on $\mathfrak{d}_{4,\lambda}$ with $\lambda > \frac{1}{2}$, $\lambda \neq 1, 2$

Let $\mathfrak{d}_{4,\lambda}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ with the products

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\lambda e_1, \quad [e_2, e_4] = (\lambda - 1)e_2, \quad [e_3, e_4] = -e_3.$$

The automorphisms of the Lie algebra are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ \frac{z_{11} z_{24}}{1-\lambda} & -\frac{z_{22} z_{14}}{\lambda} & z_{11} z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11} z_{22} \neq 0.$$

Thus, any symplectic form $\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14} e^{14} + \alpha_{24} e^{24}$ with $\alpha_{12} \neq 0$, is symplectomorphically equivalent to $\Omega = e^{12} - e^{34}$, where $\lambda > \frac{1}{2}$ and $\lambda \neq 1, 2$ (see [21]). Moreover, the Lie algebra automorphisms preserving Ω are given by Φ with $z_{24} = z_{14} = 0$ and $z_{22} z_{11} = 1$.

The description of the para-Kähler structures depends on whether the component a_{43} vanishes or not.

16.1. *Para-Kähler structures of non-zero constant paraholomorphic sectional curvature.* If $a_{43} \neq 0$, then para-Kähler structures are equivalent to

$$(J_1, \langle , \rangle_1) : \begin{cases} J_1 e_1 = -e_1, \quad J_1 e_2 = e_2, \quad J_1 e_3 = -\kappa e_4, \quad J_1 e_4 = -\frac{1}{\kappa} e_3, \\ \langle , \rangle_1 = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa} e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

which corresponds to the Lagrangian decomposition

$$\mathfrak{d}_{4,\lambda} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_2, e_4 - \tfrac{1}{\kappa} e_3\} \oplus \operatorname{span}\{e_1, e_4 + \tfrac{1}{\kappa} e_3\}.$$

Moreover, a straightforward calculation shows that the paraholomorphic sectional curvature is constant $H = \kappa$.

16.2. *Ricci-flat para-Kähler structures.* If $a_{43} = 0$, then the resulting para-Kähler structures are Ricci-flat and they are equivalent to

$$(J_2, \langle , \rangle_2) : \begin{cases} J_2 e_1 = -e_1, \quad J_2 e_2 = -\kappa\, e_1 + e_2, \quad J_2 e_3 = e_3, \quad J_2 e_4 = -e_4, \\ \langle , \rangle_2 = 2(e^1 \circ e^2 + e^3 \circ e^4) + \kappa\, e^2 \circ e^2, \quad \kappa = 0, \pm 1, \end{cases}$$

$$(J_3, \langle , \rangle_3) : \begin{cases} J_3 e_1 = e_1 + \kappa\, e_2, \quad J_3 e_2 = -e_2, \quad J_3 e_3 = e_3, \quad J_3 e_4 = -e_4, \\ \langle , \rangle_3 = \kappa\, e^1 \circ e^1 - 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

with corresponding Lagrangian decompositions

$$\begin{aligned} \mathfrak{d}_{4,\lambda} &= \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2} e_1\} \oplus \operatorname{span}\{e_1, e_4\} \\ &= \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_3, e_2 + \tfrac{2}{\kappa} e_1\} \oplus \operatorname{span}\{e_4, e_2\} \quad (\kappa \neq 0) \\ &= \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_3, e_1\} \oplus \operatorname{span}\{e_4, e_2\} \qquad (\kappa = 0). \end{aligned}$$

The two structures above are flat if and only if $\kappa = 0$. Otherwise, they have recurrent curvature with recurrence 1-forms $\xi_2 = 2\lambda e^4$ and $\xi_3 = 2(1-\lambda)e^4$, respectively. Moreover, the corresponding curvature operators acting on the space of 2-forms are given by

$$\mathcal{R}_2(e^2 \wedge e^4) = -\kappa(2\lambda^2 - 3\lambda + 1)e^1 \wedge e^3 = R_2(e_2, e_4, e_2, e_4)e^1 \wedge e^3,$$
$$\mathcal{R}_3(e^1 \wedge e^4) = \kappa\lambda(2\lambda - 1)e^2 \wedge e^3 = R_3(e_1, e_4, e_4, e_1)e^2 \wedge e^3,$$

respectively. Therefore, the corresponding manifolds are simply harmonic with special recurrent curvature tensor, thus locally isometric to a metric given by (1). Hence, their curvature tensor is semi-symmetric. Furthermore, the structures $(J_2, \langle,\rangle_2)$ and $(J_3, \langle,\rangle_3)$ do not admit any opposite almost para-Kähler structure.

17. Para-Kähler structures on $\mathfrak{d}_{4,2}$

Let $\mathfrak{d}_{4,2}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ with Lie brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3.$$

Symplectic forms on $\mathfrak{d}_{4,2}$ are of the form $\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{23}e^{23} + \alpha_{24}e^{24}$ with $\alpha_{12}^2 - \alpha_{14}\alpha_{23} \neq 0$. Any such form is symplectomorphically equivalent to

$$\Omega_1 = e^{12} - e^{34}, \quad \Omega_2 = e^{14} + e^{23}, \quad \text{or} \quad \Omega_3 = e^{14} - e^{23},$$

through a Lie algebra automorphism given by (see [21])

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ -z_{11}z_{24} & -\frac{z_{14}z_{22}}{2} & z_{11}z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}z_{22} \neq 0. \tag{2}$$

The situation is different in each case and therefore we consider the three possibilities separately.

17.1. Para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_1)$.

Lie algebra automorphisms preserving the symplectic structure $(\mathfrak{d}_{4,2}, \Omega_1)$ with $\Omega_1 = e^{12} - e^{34}$ are the ones given by Φ above with $z_{14} = z_{24} = 0$ and $z_{22}z_{11} = 1$. Now, we consider separately the cases $a_{43} \neq 0$ and $a_{43} = 0$, which give rise to different geometries as follows.

17.1.1. *Para-Kähler structures of non-zero constant paraholomorphic sectional curvature.* If $a_{43} \neq 0$, then any para-Kähler structure is equivalent to

$$(J_{11}, \langle,\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1, \quad J_{11}e_2 = e_2, \quad J_{11}e_3 = -\kappa e_4, \quad J_{11}e_4 = -\frac{1}{\kappa}e_3, \\ \langle,\rangle_{11} = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \text{span}\{e_2, e_4 - \frac{1}{\kappa}e_3\} \oplus \text{span}\{e_1, e_4 + \frac{1}{\kappa}e_3\}$. A straightforward calculation shows that all these structures have constant paraholomorphic sectional curvature $H = \kappa$.

17.1.2. *Ricci-flat para-Kähler structures.* Assuming $a_{43} = 0$, the possible para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_1)$ are equivalent either to

$$(J_{12}, \langle,\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1 + \kappa e_2, \quad J_{12}e_2 = e_2, \quad J_{12}e_3 = -e_3, \quad J_{12}e_4 = e_4, \\ \langle,\rangle_{12} = \kappa e^1 \circ e^1 + 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

which corresponds to the Lagrangian decomposition

$$\begin{aligned} \mathfrak{d}_{4,2} = \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_2 - \tfrac{2}{\kappa} e_1\} \qquad (\kappa \neq 0) \\ &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\} \qquad (\kappa = 0), \end{aligned}$$

or to

$$(J_{13}, \langle,\rangle_{13}) : \begin{cases} J_{13}e_1 = -e_1, \quad J_{13}e_2 = -\kappa e_1 + e_2, \quad J_{13}e_3 = e_3, \quad J_{13}e_4 = -e_4, \\ \langle,\rangle_{13} = \kappa e^2 \circ e^2 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

where $\mathfrak{d}_{4,2} = \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_2 - \frac{\kappa}{2} e_1, e_3\} \oplus \operatorname{span}\{e_1, e_4\}$.

The structures $(J_{12}, \langle,\rangle_{12})$ and $(J_{13}, \langle,\rangle_{13})$ are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and recurrent, with recurrence 1-forms $\xi_{12} = -2e^4$ and $\xi_{13} = 4e^4$, respectively. The corresponding curvature operators are given by

$$\begin{aligned} \mathcal{R}_{12}(e^1 \wedge e^4) &= 6\kappa e^2 \wedge e^3 = R_{12}(e_1, e_4, e_4, e_1) e^2 \wedge e^3, \\ \mathcal{R}_{13}(e^2 \wedge e^4) &= -3\kappa e^1 \wedge e^3 = R_{13}(e_2, e_4, e_2, e_4) e^1 \wedge e^3. \end{aligned}$$

Hence, these two families are simply harmonic special recurrent, locally modelled by (1). Thus the curvature tensor is semi-symmetric and moreover, none of them admits an opposite almost para-Kähler structure.

Furthermore, the structures $(J_{13}, \langle,\rangle_{13})$ have an associated one-parameter family of hypersymplectic structures $(J_{13}, J_\delta, \langle,\rangle_{13})$ which are given by the Kähler structures

$$J_\delta e_1 = \tfrac{1}{\delta} e_3, \quad J_\delta e_2 = \tfrac{\kappa}{2\delta} e_3 + \delta e_4, \quad J_\delta e_3 = -\delta e_1, \quad J_\delta e_4 = \tfrac{1}{\delta}(\tfrac{\kappa}{2} e_1 - e_2),$$

so that $J_\delta J_{13} = -J_{13} J_\delta$, for any $\delta \neq 0$ (see [3]).

17.2. **Para-Kähler structures on** $(\mathfrak{d}_{4,2}, \Omega_2)$**.** The Lie algebra automorphisms preserving the symplectic structure $(\mathfrak{d}_{4,2}, \Omega_2)$ corresponding to $\Omega_2 = e^{14} + e^{23}$ are the ones given by Φ in (2) with $z_{24} = z_{34} = 0$, $z_{11} = 1$ and $z_{22} = \pm 1$. Any para-Kähler structure on this Lie algebra satisfies $a_{21} = 0$ and different situations may occur depending on whether $a_{23} = 0$ or $a_{23} \neq 0$. We study each case separately.

17.2.1. *Flat para-Kähler structures.* Assuming $a_{23} = 0$ and $a_{41} = 0$, any para-Kähler structure is equivalent to the flat structure

$$(J_{21}, \langle,\rangle_{21}) : \begin{cases} J_{21}e_1 = e_1, \quad J_{21}e_2 = -e_2, \quad J_{21}e_3 = e_3, \quad J_{21}e_4 = -e_4, \\ \langle,\rangle_{21} = 2(e^2 \circ e^3 - e^1 \circ e^4), \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\}$.

17.2.2. *Para-Kähler structures which are not semi-symmetric.* Assuming $a_{23} = 0$ and $a_{41} \neq 0$, para-Kähler structures are equivalent to

$$(J_{22}, \langle,\rangle_{22}) : \begin{cases} J_{22}e_1 = e_1 + \kappa e_4, \quad J_{22}e_2 = -e_2, \quad J_{22}e_3 = e_3, \quad J_{22}e_4 = -e_4, \\ \langle,\rangle_{22} = \kappa e^1 \circ e^1 + 2(e^2 \circ e^3 - e^1 \circ e^4), \quad \kappa \neq 0, \end{cases}$$

which induce the Lagrangian decomposition

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{22} \oplus \mathfrak{L}'_{22} = \operatorname{span}\{e_3, e_4 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_2, e_4\}.$$

The Ricci operators are diagonalizable with eigenvalues $\{0, 0, 4\kappa, 4\kappa\}$ and the self-dual and anti-self-dual Weyl curvature operators have the same eigenvalues. Moreover, the anti-self-dual Weyl curvature operator W^- has a double root of its minimal polynomial. Furthermore, their curvature tensor is not semi-symmetric and hence it does not correspond to the curvature of any symmetric space.

Assuming $a_{23} \neq 0$, para-Kähler structures are equivalent to

$$(J_{23}, \langle,\rangle_{23}) : \begin{cases} J_{23}e_1 = \tfrac{\kappa}{2}e_4, \;\; J_{23}e_2 = -\tfrac{1}{\kappa}e_3, \;\; J_{23}e_3 = -\kappa e_2, \;\; J_{23}e_4 = \tfrac{2}{\kappa}e_1, \\ \langle,\rangle_{23} = \tfrac{\kappa}{2}e^1 \circ e^1 - \tfrac{1}{\kappa}e^2 \circ e^2 + \kappa e^3 \circ e^3 - \tfrac{2}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{23} \oplus \mathfrak{L}'_{23} = \operatorname{span}\{e_3 - \kappa e_2, e_4 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_3 + \kappa e_2, e_4 - \tfrac{2}{\kappa}e_1\}$.

The Ricci operators are diagonalizable with eigenvalues $\{0, 0, 3\kappa, 3\kappa\}$ and the self-dual and anti-self-dual Weyl curvature operators are diagonalizable with opposite eigenvalues. Furthermore, their curvature tensor is not semi-symmetric.

Let Q be the almost product structure associated to the Ricci operator so that $Q = -\operatorname{Id}$ on $\ker \operatorname{Ric}$ and $Q = \operatorname{Id}$ on the orthogonal distribution corresponding to eigenspace of the non-zero Ricci curvature. Define an opposite almost paracomplex structure $J'_{2i} = J_{2i}Q_{2i}$, for $i = 2, 3$. A straightforward calculation shows that J'_{2i} is an opposite almost para-Kähler structure commuting with J_{2i} with associated symplectic form $\Omega'_{2i}(x, y) = \langle J'_{2i}x, y\rangle_{2i} = \Omega_3$, for $i = 2, 3$.

17.3. **Para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_3)$.** The symplectic structure $(\mathfrak{d}_{4,2}, \Omega_3)$ given by $\Omega_3 = e^{14} - e^{23}$ is preserved by the automorphisms Φ in (2) with $z_{24} = z_{34} = 0$, $z_{11} = 1$ and $z_{22} = \pm 1$. Considering the action of the automorphisms which preserve the symplectic structure, the different possibilities arise from the following cases:

Case 1. $a_{21} = a_{23} = a_{41} = 0$, $a_{22} \neq 0$, in which case one may assume $a_{32} = 0$.
Case 2. $a_{21} = a_{23} = a_{41} = a_{22} = 0$.
Case 3. $a_{21} = a_{23} = 0$, $a_{41} \neq 0$, in which case one may assume $a_{11} = 0$.
Case 4. $a_{21} = 0$, $a_{23} \neq 0$, in which case one may assume $a_{22} = 0$.
Case 5. $a_{21} \neq 0$, in which case one may assume $a_{13} = 0$.

We study these cases separately.

17.3.1. *Flat para-Kähler structures.* Assuming $a_{21} = a_{23} = a_{41} = 0$ and $a_{22} \neq 0$ as in Case 1, if $a_{31} = 0$ then the corresponding para-Kähler structures are equivalent to the flat para-Kähler structure given by

$$(J_{31}, \langle,\rangle_{31}) : \begin{cases} J_{31}e_1 = -e_1, \quad J_{31}e_2 = e_2, \quad J_{31}e_3 = -e_3, \quad J_{31}e_4 = e_4, \\ \langle,\rangle_{31} = 2(e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

with Lagrangian decomposition $\mathfrak{d}_{4,2} = \mathfrak{L}_{31} \oplus \mathfrak{L}'_{31} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\}$. The case where $a_{31} \neq 0$ is considered in Section 17.3.4.

17.3.2. *Ricci-flat para-Kähler structures.* Assuming $a_{21} = a_{23} = a_{41} = a_{22} = 0$ as in Case 2, the corresponding para-Kähler structures are equivalent to

$$(J_{32}, \langle , \rangle_{32}) : \begin{cases} J_{32}e_1 = -e_3,\ J_{32}e_2 = -\kappa e_3 + e_4,\ J_{32}e_3 = -e_1,\ J_{32}e_4 = -\kappa e_1 + e_2, \\ \langle , \rangle_{32} = \kappa(e^2 \circ e^2 + e^4 \circ e^4) + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decomposition

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{32} \oplus \mathfrak{L}'_{32} = \operatorname{span}\{e_3 - e_1, e_4 + e_2 - \kappa e_1\} \oplus \operatorname{span}\{e_3 + e_1, e_4 - e_2 + \kappa e_1\}.$$

These structures are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and special recurrent with recurrence 1-form $\xi_{32} = 4e^4$ and curvature operator $\mathcal{R}_{32}(e^2 \wedge e^4) = -3\kappa e^1 \wedge e^3 = R_{32}(e_2, e_4, e_2, e_4)e^1 \wedge e^3$. Hence, the underlying structures are locally modelled in (1) and thus their curvature tensor is semi-symmetric. Moreover, these structures do not admit any opposite almost para-Kähler structures.

17.3.3. *Para-Kähler structures which are not semi-symmetric with diagonalizable Ricci operator.* Assuming $a_{21} = a_{23} = 0$ and $a_{41} \neq 0$ as in Case 3, para-Kähler structures are equivalent to

$$(J_{33}, \langle , \rangle_{33}) : \begin{cases} J_{33}e_1 = -e_1 + \kappa e_4, \quad J_{33}e_2 = e_2, \quad J_{33}e_3 = -e_3, \quad J_{33}e_4 = e_4, \\ \langle , \rangle_{33} = \kappa e^1 \circ e^1 + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{33} \oplus \mathfrak{L}'_{33} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_4 - \frac{2}{\kappa}e_1\}$.

The Ricci operator is diagonalizable with eigenvalues $\{4\kappa, 4\kappa, 0, 0\}$ and the self-dual and anti-self-dual Weyl curvature operators have exactly the same eigenvalues. Moreover, W^- has a double root of its minimal polynomial, and the curvature tensor is not semi-symmetric.

The assumption $a_{21} = 0$ and $a_{23} \neq 0$ as in Case 4 leads to para-Kähler structures equivalent to

$$(J_{34}, \langle , \rangle_{34}) : \begin{cases} J_{34}e_1 = \frac{\kappa}{2}e_4, \quad J_{34}e_2 = -\frac{1}{\kappa}e_3, \quad J_{34}e_3 = -\kappa e_2, \quad J_{34}e_4 = \frac{2}{\kappa}e_1, \\ \langle , \rangle_{34} = \frac{\kappa}{2}e^1 \circ e^1 + \frac{1}{\kappa}e^2 \circ e^2 - \kappa e^3 \circ e^3 - \frac{2}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

corresponding to the Lagrangian decomposition

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{34} \oplus \mathfrak{L}'_{34} = \operatorname{span}\{e_4 + \tfrac{2}{\kappa}e_1, e_3 - \kappa e_2\} \oplus \operatorname{span}\{e_4 - \tfrac{2}{\kappa}e_1, e_3 + \kappa e_2\}.$$

The Ricci operator is diagonalizable with eigenvalues $\{3\kappa, 3\kappa, 0, 0\}$ and the self-dual and the anti-self-dual Weyl curvature operators are diagonalizable with opposite eigenvalues. Moreover, the curvature tensor is not semi-symmetric.

Let Q be the almost product structure associated to the Ricci operator so that $Q = -\operatorname{Id}$ on $\ker \operatorname{Ric}$ and $Q = \operatorname{Id}$ on the orthogonal distribution corresponding to the eigenspace of the non-zero Ricci curvature. Define an opposite almost paracomplex structure $J'_{3i} = J_{3i}Q_{3i}$, for $i = 2, 3$. A straightforward calculation shows that J'_{3i} is an opposite almost para-Kähler structure commuting with J_{3i} with associated symplectic form $\Omega'_{3i}(x, y) = \langle J'_{3i}x, y\rangle_{3i} = \Omega_2$, for $i = 3, 4$.

17.3.4. *Para-Kähler structures which are not semi-symmetric with non-diagonalizable Ricci operator.* Assuming $a_{21} = a_{23} = a_{41} = 0$ and $a_{22} \neq 0$ as in Case 1, if $a_{31} \neq 0$, one has the para-Kähler structures

$$(J_{35}, \langle , \rangle_{35}) : \begin{cases} J_{35}e_1 = -e_1 - 2e_3,\ J_{35}e_2 = 2e_4 - e_2 - \kappa e_3,\ J_{35}e_3 = e_3,\ J_{35}e_4 = e_4, \\ \langle , \rangle_{35} = \kappa e^2 \circ e^2 + 2(e^1 \circ e^4 + 2e^1 \circ e^2 - e^2 \circ e^3), \quad \kappa \in \mathbb{R}, \end{cases}$$

corresponding to the Lagrangian decompositions $\mathfrak{d}_{4,2} = \mathfrak{L}_{35} \oplus \mathfrak{L}'_{35} = \operatorname{span}\{e_3, e_4\} \oplus \operatorname{span}\{e_4 - e_2 + \frac{\kappa}{2} e_1, e_3 + e_1\}$. The Ricci operator is two-step nilpotent and the curvature tensor is not semi-symmetric. Moreover, the anti-self-dual Weyl curvature operator is three-step nilpotent.

On the other hand, if $a_{21} \neq 0$ as in Case 5, then para-Kähler structures are equivalent to

$$(J_{36}, \langle,\rangle_{36}) : \begin{cases} J_{36}e_1 = -e_1 + \kappa(e_2 + e_4), \quad J_{36}e_2 = e_2, \\ J_{36}e_3 = \kappa(e_2 + e_4) - e_3, \qquad J_{36}e_4 = e_4, \\ \langle,\rangle_{36} = \kappa(e^1 \circ e^1 + e^3 \circ e^3 + 2e^1 \circ e^3) + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{36} \oplus \mathfrak{L}'_{36} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_2 - \frac{2}{\kappa} e_1 + e_4, e_3 - e_1\}$.

The Ricci operator has a single eigenvalue $\frac{3}{2}\kappa \neq 0$ which is a double root of the minimal polynomial, and the anti-self-dual Weyl curvature operator is three-step nilpotent. Moreover, the curvature tensor is not semi-symmetric. Since their anti-self-dual Weyl curvature operator is three-step nilpotent the structures above do not admit any opposite almost para-Kähler structure commuting with the Ricci operator [11].

18. Kähler Lie algebras

Kähler Lie algebras were classified by Ovando in [22] and the geometry of the corresponding structures, although similar to that of para-Kähler Lie algebras, is more rigid, allowing less possibilities. The symmetric case is essentially the same, but there are no left-invariant locally symmetric Ricci-flat Kählerian structures in contrast with Theorem 1.1-(1.a).

Theorem 18.1. *Let $(G, \langle,\rangle, \Omega)$ be a non-flat locally symmetric four-dimensional indefinite Kähler Lie group. Then, one of the following holds:*

(1) *The Ricci operator is diagonalizable and one of the following holds:*
 (1.a) *The holomorphic sectional curvature is a non-zero constant.*
 (1.b) *The metric is Einstein with non-zero scalar curvature.*
 (1.c) *The manifold is locally a product of two surfaces of constant Gaussian curvature.*

(2) *The Ricci operator is non-diagonalizable and one of the following holds:*
 (2.a) *The Ricci operator has complex eigenvalues.*
 (2.b) *The Ricci operator is two-step nilpotent.*

Following Ovando's classification, structures in Assertion (1.a) correspond to the Kähler structures on the Lie groups determined by $\mathfrak{d}_{4,\frac{1}{2}}$ and $\mathfrak{d}'_{4,\delta}$, where $\delta > 0$. Kähler structures in Assertion (1.b) correspond to the metrics $\langle,\rangle = a(e^1 \circ e^1 - e^2 \circ e^2 + e^3 \circ e^3 - e^4 \circ e^4)$ on $\mathfrak{r}'_2$ with symplectic form $\Omega = a(e^{13} - e^{24})$. Kähler structures which admit an opposite Kähler structure as in Assertion (1.c) correspond to the Kähler structures on $\mathfrak{r}\mathfrak{r}_{3,0}$, $\mathfrak{r}'_{4,0,\delta}$ for $\delta > 0$, and the structures on $\mathfrak{r}_2\mathfrak{r}_2$ given by the metrics $\langle,\rangle = a(e^1 \circ e^1 + e^2 \circ e^2) + b(e^3 \circ e^3 + e^4 \circ e^4)$ with symplectic form $\Omega = ae^{12} + be^{34}$, where $ab < 0$. Kähler structures in Assertion (2.a) correspond to the metrics $\langle,\rangle = a(e^1 \circ e^1 - e^2 \circ e^2 + e^3 \circ e^3 - e^4 \circ e^4) + 2b(e^1 \circ e^2 + e^3 \circ e^4)$ on $\mathfrak{r}'_2$ with symplectic form $\Omega = a(e^{13} - e^{24}) + b(e^{14} + e^{23})$, $b \neq 0$. Assertion (2.b) corresponds to the Kähler structures on $\mathfrak{d}_{4,1}$.

Metrics corresponding to Assertions (1.b) and (2.a) are linked by anti-Kähler structures, so that they have the same Levi-Civita connection as in the para-Kähler case. Moreover, the structures above contain locally conformally flat Kähler Lie groups so that one has:

Corollary 18.2. *Let (M, g, J) be a locally conformally flat four-dimensional indefinite Kähler manifold. Then, it is flat or it is locally isometric to the Kähler Lie group determined by one of the following:*

(1) *The Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ with the metrics $\langle , \rangle = a(e^1 \circ e^1 + e^2 \circ e^2 - e^3 \circ e^3 - e^4 \circ e^4)$ and the symplectic structure $\Omega = a(e^{12} - e^{34})$.*
(2) *The Lie algebra $\mathfrak{r}_2'$ with the metrics $\langle , \rangle = 2b(e^1 \circ e^2 + e^3 \circ e^4)$ and the symplectic structure $\Omega = b(e^{14} + e^{23})$.*
(3) *The Lie algebra $\mathfrak{d}_{4,1}$ with the metrics $\langle , \rangle = 2a(e^2 \circ e^4 - e^1 \circ e^3)$ and the symplectic structure $\Omega = a(e^{12} - e^{34})$.*

Finally, the non-symmetric case is essentially simpler than in the para-Kähler situation since the existence of Kähler and opposite almost Kähler structures is much rigid than the corresponding para-Kähler analogue (see [10, 16]).

Theorem 18.3. *Let $(G, \langle , \rangle, \Omega)$ be a non-symmetric four-dimensional indefinite Kähler Lie group. Then, one of the following holds:*

(1) *$(G, \langle , \rangle)$ is semi-symmetric if and only if the Ricci operator vanishes, in which case the curvature tensor is special recurrent and the metric is simply harmonic.*
(2) *$(G, \langle , \rangle)$ is not semi-symmetric if and only if it corresponds to the 3-symmetric space determined by the Kähler metrics $\langle , \rangle = a(\frac{1}{2}e^1 \circ e^1 + 2e^4 \circ e^4) + b(e^2 \circ e^2 + e^3 \circ e^3)$ on $\mathfrak{d}_{4,2}$ with symplectic form $\Omega = ae^{14} + be^{23}$, where $ab < 0$.*

Assertion (1) corresponds to the metrics $\langle , \rangle = -c(e^1 \circ e^1 + e^2 \circ e^2) - 2a(e^1 \circ e^4 + e^2 \circ e^3) + 2b(e^1 \circ e^3 - e^2 \circ e^4)$ on $\mathfrak{r}_2'$ with symplectic form $\Omega = ce^{12} + a(e^{13} - e^{24}) + b(e^{14} + e^{23})$, where $c(a^2 + b^2) \neq 0$, the metrics $\langle , \rangle = -c(e^1 \circ e^1 + e^4 \circ e^4) + 2b(e^1 \circ e^2 - e^3 \circ e^4) - 2a(e^1 \circ e^3 + e^2 \circ e^4)$ on $\mathfrak{r}_{4,-1,-1}$ with symplectic form $\Omega = a(e^{12} + e^{34}) + b(e^{13} - e^{24}) + ce^{14}$, where $c(a^2 + b^2) \neq 0$, and the metrics $\langle , \rangle = b(e^2 \circ e^2 + e^4 \circ e^4) + 2a(e^1 \circ e^2 + e^3 \circ e^4)$ on $\mathfrak{d}_{4,2}$ with symplectic form $\Omega = be^{24} + a(e^{14} + e^{23})$, where $a \neq 0$.

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