Positive solutions of a discontinuous one dimensional beam equation

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Abstract

We provide sufficient conditions for the existence of at least a positive solution for a fourth–order beam equation with a discontinuous nonlinear term. Also a multiplicity result is established. They are based on a recent generalization of the Krasnosel'skiĭ fixed point theorem in cones.

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1 Introduction

We study the existence of positive solutions for the following fourth-order equation

$$\begin{aligned} &u^{(4)}(t) = g(t)f(u(t)), \quad t \in (0,1), \\ &u(0) = u(1) = 0 = u''(0) = u''(1), \end{aligned} \tag{1.1}$$

where $g \geq 0$ a.e. on $I = [0,1]$ and $g \in L^1(0,1)$ and the function $f : [0,\infty) \to [0,\infty)$ is such that $u \mapsto f(u)$ is measurable for every $u \in \mathcal{C}^2([0,1])$ and $f \in L^\infty_{loc}([0,\infty))$.

Problem (1.1) was intensively studied in the literature (see, for example [4, 6, 13]) and it arises in many applications. For instance, fourth-order problems appear in nonlinear suspension bridge models (see [5, 13] and the references therein).

However continuity assumptions are usually imposed about f. Our goal is to weaken this hypothesis without using monotonicity conditions and, even in that case, to obtain positive Carathéodory solutions for (1.1). We achieve an existence result when f has a superlinear or sublinear behavior. Also a result concerning the existence of two positive solutions for problem (1.1) is obtained in Section 4.

Our approach is based on a multivalued version of the well–known Krasnosel'skiĭ compression–expansion fixed point theorem [10] which we apply in order to get fixed points of a regularization of the integral operator associated to (1.1). The mentioned multivalued mapping is constructed by 'convexifying' the fixed point operator related to the fourth-order problem (1.1). These ideas can be found in [8] and we will detail them in Section 2. They recall the classical ideas of Filippov and Krasovskij envelopes which regularize the differential equation and transform it into a differential inclusion, see [9, 11].

2 Fixed point theorem

In the sequel we need the following definitions. A subset K of a Banach space $(X, \|\cdot\|)$ is a cone if it is closed, $K+K \subset K$, $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone K defines the partial order in X given by $x \preceq y$ if and only if $y - x \in K$. We will denote $K_c = \{x \in K : \|x\| < c\}$ and $\overline{K}_c$ its closure, with $0 < c < \infty$.

Let U be a relatively open subset of K and let $T : \overline{U} \subset K \to K$ be an operator, not necessarily continuous.

Definition 2.1 *The closed–convex envelope of an operator $T : \overline{U} \longrightarrow K$ is the multivalued mapping $\mathbb{T} : \overline{U} \longrightarrow 2^K$ given by*

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap \overline{U}\right) \quad \text{for every } x \in \overline{U}, \tag{2.2}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\text{co}}$ means closed convex hull.

In other words, we say that $y \in \mathbb{T}x$ if for every $\varepsilon > 0$ and every $\rho > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $x_i \in \overline{B}_\varepsilon(x) \cap \overline{U}$ and coefficients $\lambda_i \in [0,1]$ $(i = 1, 2, \ldots, m)$ such that $\sum \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^{m} \lambda_i\, T x_i \right\| < \rho.$$

Now we present a discontinuous version of Krasnosel'skiĭ theorem which is a straightforward consequence of the multivalued version given by Fitzpatrick and Petryshyn [10].

Proposition 2.2 *Let $r_i \leq R$ $(i = 1, 2)$ with $r_1 \neq r_2$ positive numbers and $T : \overline{K}_R \to K$ a mapping such that $T\,\overline{K}_R$ is relatively compact and fulfills condition*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in \overline{K}_R, \tag{2.3}$$

where $\mathbb{T}$ is the closed–convex envelope of T as defined in (2.2).

Suppose that

(a) *$\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \geq 1$,*

(b) *there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for all $\lambda \geq 0$ and all $x \in K$ with $\|x\| = r_2$.*

Then T has at least a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

Observe that condition (2.3) is equivalent to $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$, where $\mathrm{Fix}(S)$ denotes the set of fixed points of the operator S. For more details about the previous fixed point theorem, see [8].

Remark 2.3 *Condition* (a) *in Proposition 2.2 is satisfied if one of the following two conditions holds:*

(i) $y \not\geq x$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_1$,

(ii) $\|y\| < \|x\|$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_1$.

Analogously, assumption (b) *in Proposition 2.2 holds if*

(I) $y \not\leq x$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_2$, *or*

(II) $\|y\| > \|x\|$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_2$.

3 Positive solutions

In this section we establish sufficient conditions for the existence of positive solutions for the simply supported beam equation (1.1).

Technical reasons make that we need to work in the Banach space $(\mathcal{C}^2([0,1]), \|\cdot\|)$, where $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ and $\|\cdot\|_\infty$ is the usual supremum norm.

We shall look for fixed points of the operator $T : \mathcal{C}^2([0,1]) \to \mathcal{C}^2([0,1])$ given by

$$Tu(t) := \int_0^1 G(t,s)g(s)f(u(s))\,ds,$$

where G is the Green's function. It is given by

$$G(t,s) = \begin{cases} \dfrac{1}{6}s(1-t)(2t - s^2 - t^2), & s \leq t, \\ \dfrac{1}{6}t(1-s)(2s - t^2 - s^2), & s > t, \end{cases}$$

which is non negative and satisfies (see [4, 13])

$$\begin{aligned} G(t,s) &\leq \Phi(s), \quad \text{for } t, s \in [0,1], \\ c\,\Phi(s) &\leq G(t,s), \quad \text{for } t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right], s \in [0,1], \end{aligned}$$

where

$$\Phi(s) = \begin{cases} \dfrac{\sqrt{3}}{27} s(1-s^2)^{3/2}, & \text{for } 0 \leq s \leq 1/2, \\ \dfrac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2}, & \text{for } 1/2 \leq s \leq 1, \end{cases}$$

and $c = 45\sqrt{3}/128 \approx 0.608924$.

We shall look for fixed points of T in the cone

$$K = \left\{ u \in \mathcal{C}^2([0,1]) : u \geq 0, \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u(t) \geq c\,\|u\|_\infty \right\}.$$

PROPOSITION 3.1 *The operator $T : K \to K$ is well-defined and maps bounded sets into relatively compact sets.*

Proof. The fact that $TK \subset K$ can be verified by using the properties of the Green's function G together with the mapping Φ. In addition, from the hypothesis about f and g and the regularity of the Green's function it is routine to conclude that T maps bounded sets into relatively compact ones by means of the Áscoli–Arzela's theorem. □

Now we define the points where we allow the function f to be discontinuous. The following definition is an adjustment to the admissible discontinuity curves of [7, 12] in the case of a fourth order problem and a function f only dependent on the space variable u.

DEFINITION 3.2 *An admissible discontinuity point is a nonnegative real number x satisfying one of the following conditions:*

(a) $f(x) = 0$ *(x is said a viable point),*

(b) *There exist $\varepsilon > 0$ and $\psi \in L^1(0,1)$, $\psi(t) > 0$ for a.a. $t \in [0,1]$ such that*

$$\psi(t) < g(t)f(y) \quad \text{for a.a. } t \in [0,1] \text{ and all } y \in [x-\varepsilon, x+\varepsilon] \quad (x \text{ is inviable}). \tag{3.4}$$

REMARK 3.3 *Notice that the hypothesis about the admissible discontinuity points defined here are similar to the condition*

$$0 \in F(x) = \bigcap_{\varepsilon>0} \overline{\text{co}} f(\overline{B}_\varepsilon(x)) \quad \text{implies} \quad f(x) = 0;$$

given in [11] for first-order discontinuous autonomous systems. Indeed, they are equivalent whenever $g \equiv 1$.

Moreover, observe that if $g(t) > 0$ for a.a. $t \in [0,1]$, then the fact that there exists $\varepsilon > 0$ such that

$$\inf_{y \in [x-\varepsilon, x+\varepsilon]} f(y) > 0$$

implies condition (3.4) in Definition 3.2. Similar assumptions were required in [1, 2] in the study of second order problems, but the approach there relies on critical point theory for non smooth operators.

Now we enunciate three technical results whose proofs can be looked up in [12].

LEMMA 3.4 ([12, LEMMA 4.1]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a,b)$, $g \geq 0$ a.e., and $h > 0$ a.e. in (a,b). For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have*

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

COROLLARY 3.5 ([12, COROLLARY 4.2]) *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ a.e. in (a,b). For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have*

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J} h(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

COROLLARY 3.6 ([12, COROLLARY 4.3]) *Let $a, b \in \mathbb{R}$, $a < b$, and let f, $f_n : [a, b] \longrightarrow \mathbb{R}$ be absolutely continuous functions on $[a, b]$ ($n \in \mathbb{N}$), such that $f_n \to f$ uniformly on $[a, b]$ and for a measurable set $A \subset [a, b]$ with $m(A) > 0$ we have*

$$\lim_{n\to\infty} f_n'(t) = g(t) \quad \text{for a.a. } t \in A.$$

If there exists $M \in L^1(a, b)$ such that $|f'(t)| \le M(t)$ a.e. in $[a, b]$ and also $|f_n'(t)| \le M(t)$ a.e. in $[a, b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

We shall also need the following result whose proof is similar to that of Lemma 3.11 in [8].

LEMMA 3.7 *If $M \in L^1(0, 1)$, $M \ge 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^3([0,1]) : \left|u'''(t) - u'''(s)\right| \le \int_s^t M(r)\,dr \quad \text{whenever } 0 \le s \le t \le 1 \right\},$$

is closed in $\mathcal{C}^2([0, 1])$.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ in the $\mathcal{C}^2$ norm, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the $\mathcal{C}^3$ norm.

Following the notation of [3], we define

$$\gamma_* = \inf_{t\in[1/4,3/4]} \int_{1/4}^{3/4} G(t,s)g(s)\,ds, \qquad \gamma^* = \sup_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds,$$

$$\gamma_1^* = \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| g(s)\,ds, \qquad \gamma_2^* = \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| g(s)\,ds$$

and we suppose $\gamma_* > 0$.

Now we prove the main result of this section.

THEOREM 3.8 *Assume that the functions f and g satisfy the following hypotheses:*

(H_1) *$g \ge 0$ a.e. on $I = [0, 1]$ and $g \in L^1(0, 1)$;*

(H_2) *$f : [0, \infty) \to [0, \infty)$ is such that*

- *$u \mapsto f(u)$ is measurable for every $u \in \mathcal{C}^2([0, 1])$, and*
- *f is locally bounded;*

(H_3) *There exist admissible discontinuity points $x_n \ge 0$ such that the function $u \mapsto f(u)$ is continuous in $[0, \infty) \setminus \bigcup_{n\in\mathbb{N}} \{x_n\}$.*

Moreover, assume that either

(i) *$f_0 := \lim_{u\to 0^+} \dfrac{f(u)}{u} = +\infty$ and $f_\infty := \lim_{u\to\infty} \dfrac{f(u)}{u} = 0$ (sublinear case); or*

(ii) *$f_0 = 0$ and $f_\infty = \infty$ (superlinear case).*

Then BVP (1.1) has at least a positive solution.

Proof. We are going to prove that the conditions of Proposition 2.2 are satisfied. We suppose that f satisfies (i) (it is similar if the nonlinearity f is in the superlinear case). Claims 1 and 2 are standard (see, e.g. [4, Theorem 3.1]), but here some changes are necessary due to the use of the set–valued operator $\mathbb{T}$, and the last one is a technical result which follows the ideas of [12, Theorem 4.4].

Claim 1: There exists $r_1 > 0$ such that $\|y\| < \|u\|$ for all $y \in \mathbb{T}\, u$ and all $u \in K$ with $\|u\| = r_1$.

Since $f_\infty = 0$, for each $L > 0$ there exists $M > 0$ such that

$$f(s) \leq M + Ls \quad \text{ for } s \geq 0.$$

We can choose $L > 0$ small enough such that $5 \max\{\gamma^*, \gamma_1^*, \gamma_2^*\} L < 2$ and $r_1 > 0$ large enough such that $2 \max\{\gamma^*, \gamma_1^*, \gamma_2^*\} M < r_1$. Suppose that $u \in K$ with $\|u\| = r_1$, then for every finite family $u_i \in \overline{B}_r(u) \cap K$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$ and $r = \|u\|_\infty /4$, we have

$$\begin{aligned} v(t) = \sum_{i=1}^m \lambda_i T u_i(t) &\leq \sum_{i=1}^m \lambda_i \int_0^1 G(t, s) g(s) \left[M + L u_i(s)\right] ds \\ &\leq \sum_{i=1}^m \lambda_i \gamma^* [M + L \|u_i\|_\infty] \leq \gamma^* [M + 5L \|u\|_\infty /4] < \|u\| . \end{aligned}$$

In addition,

$$\left|v'(t)\right| = \left|\sum_{i=1}^m \lambda_i (Tu_i)'(t)\right| \leq \sum_{i=1}^m \lambda_i \int_0^1 \left|\frac{\partial G}{\partial t}(t, s)\right| g(s) \left[M + L u_i(s)\right] ds \leq \gamma_1^* [M + 5L \|u\|_\infty /4] < \|u\| ,$$

and

$$\left|v''(t)\right| = \left|\sum_{i=1}^m \lambda_i (Tu_i)''(t)\right| \leq \sum_{i=1}^m \lambda_i \int_0^1 \left|\frac{\partial^2 G}{\partial t^2}(t, s)\right| g(s) \left[M + L u_i(s)\right] ds \leq \gamma_2^* [M + 5L \|u\|_\infty /4] < \|u\| .$$

Hence, if $y \in \mathbb{T}\, u$, then it is the limit of a sequence of functions v as above, so $\|y\| < \|u\|$ for all $y \in \mathbb{T}\, u$ and all $u \in K$ with $\|u\| = r_1$.

Claim 2: There exists $r_2 > 0$ such that $y \not\preceq u$ for all $y \in \mathbb{T}\, u$ and all $u \in K$ with $\|u\| = r_2$.

Hypothesis (i) $f_0 = \infty$ guarantees that we can choose $L > 0$ large enough such that $\gamma_* L c > 2$ and $C > 0$ satisfying $f(s) \geq Ls$ provided that $0 \leq s \leq C$. Suppose that $u \in K$ with $\|u\| = C/2 =: r_2$, then for every finite family $u_i \in \overline{B}_r(u) \cap K$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$ and $r = \|u\|_\infty /2$, we have $\|u_i\|_\infty \leq 3r_2/2 < C$, so $0 \leq u_i(t) \leq C$ for all $t \in [1/4, 3/4]$ and

$$\sum_{i=1}^m \lambda_i T u_i(t) \geq \sum_{i=1}^m \lambda_i \int_{1/4}^{3/4} G(t, s) g(s) f(u_i(s))\, ds \geq \gamma_* L c \sum_{i=1}^m \lambda_i \|u_i\|_\infty \geq \gamma_* L c \left(\|u\|_\infty - r\right) > \|u\|_\infty ,$$

which implies that $y \not\preceq u$ for all $y \in \mathbb{T}u$ with $u \in K$ and $\|u\| = r_2$.

Claim 3: The operator T satisfies the condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in \overline{K}_R$ with $R \geq r_1$.

First, notice that there exists $\tilde{R} > 0$ such that $f(u) \leq \tilde{R}$ for all $u \in \overline{K}_R$. Therefore, there exists $M \in L^1(0, 1)$ such that

$$g(t) f(u) \leq M(t) \quad \text{ for a.a. } t \in [0, 1] \text{ and all } u \in \overline{K}_R. \tag{3.5}$$

Now we consider the set

$$Q = \left\{ u \in \mathcal{C}^3([0,1]) : \left| u'''(t) - u'''(s) \right| \le \int_s^t M(r)\, dr \quad (s \le t) \right\}, \tag{3.6}$$

which is a closed and convex subset of $\mathcal{C}^2([0,1])$ by Lemma 3.7. It is immediate to see that $T\,K \subset Q$, by the definition of the operator T, and since Q is a closed and convex subset of X we have that $\mathbb{T}\,K \subset Q$. In particular, $\mathbb{T}\,\overline{K}_R \subset Q$. We note that condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ needs only to be verified for every $u \in \overline{K}_R \cap \mathbb{T}\,\overline{K}_R \subset \overline{K}_R \cap Q$.

Therefore we fix $u \in \overline{K}_R \cap Q$ and we consider the following three cases.

Case 1: $m(\{t \in [0,1] : u(t) = x_n\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that then T is continuous at u.

The assumption implies that for a.a. $t \in [0,1]$ the function $f(\cdot)$ is continuous at $u(t)$. Hence, if $u_k \to u$ in Q, then

$$f(u_k(t)) \to f(u(t)) \quad \text{ for a.a. } t \in [0,1],$$

which, along with (3.5), yield $Tu_k \to Tu$ in $\mathcal{C}^2([0,1])$.

Case 2: $m(\{t \in [0,1] : u(t) = x_n\}) > 0$ for some $n \in \mathbb{N}$ such that x_n is inviable. In this case, we can prove that $u \notin \mathbb{T}u$.

Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in [0,1] : u(t) = x_n\}) > 0$ and we will simply denote x instead of x_n. There exist $\varepsilon > 0$ and $\psi \in L^1(0,1)$, $\psi(t) > 0$ for a.a. $t \in [0,1]$ such that (3.4) holds.

We denote $J = \{t \in [0,1] : u(t) = x\}$ and we deduce from Lemma 3.4 that there exists a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} M(s)\, ds}{(1/4)\int_{\tau_0}^t \psi(s)\, ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} M(s)\, ds}{(1/4)\int_t^{\tau_0} \psi(s)\, ds}. \tag{3.7}$$

By Corollary 3.5 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\, ds}{\int_{\tau_0}^t \psi(s)\, ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\, ds}{\int_t^{\tau_0} \psi(s)\, ds}. \tag{3.8}$$

Let us now fix a point $\tau_0 \in J_1$. From (3.7) and (3.8) we deduce that there exist $t_- < \tilde{t}_- < \tau_0$ and $t_+ > \tilde{t}_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied for all $t \in [\tilde{t}_+, t_+]$:

$$\int_{[\tau_0,t]\setminus J} M(s)\, ds < \frac{1}{4}\int_{\tau_0}^t \psi(s)\, ds, \tag{3.9}$$

$$\int_{[\tau_0,t]\cap J} \psi(s)\, ds \ge \int_{[\tau_0,t]\cap J_0} \psi(s)\, ds > \frac{1}{2}\int_{\tau_0}^t \psi(s)\, ds, \tag{3.10}$$

and for all $t \in [t_-, \tilde{t}_-]$:

$$\int_{[t,\tau_0]\setminus J} M(s)\, ds < \frac{1}{4}\int_t^{\tau_0} \psi(s)\, ds, \tag{3.11}$$

$$\int_{[t,\tau_0]\cap J} \psi(s)\, ds > \frac{1}{2}\int_t^{\tau_0} \psi(s)\, ds. \tag{3.12}$$

Finally, we define a positive number

$$\tilde{\rho} = \min\left\{\frac{1}{4}\int_{\tilde{t}_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{\tilde{t}_+} \psi(s)\,ds\right\}, \tag{3.13}$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. It suffices to prove the following claim:

Claim – Let $\varepsilon > 0$ be given by our assumptions over x and let $\rho = \frac{\tilde{\rho}}{2}\min\{\tilde{t}_- - t_-, t_+ - \tilde{t}_+\}$ be where $\tilde{\rho}$ is as in (3.13). For every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \ldots, m)$, with $\sum \lambda_i = 1$, we have $\|u - \sum \lambda_i T u_i\| \geq \rho$.

Let u_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T u_i$. For a.a. $t \in J = \{t \in [0,1] \,:\, u(t) = x\}$ we have

$$y^{(4)}(t) = \sum_{i=1}^{m} \lambda_i (Tu_i)^{(4)}(t) = \sum_{i=1}^{m} \lambda_i\, g(t) f(u_i(t)). \tag{3.14}$$

On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and every $t \in J$ we have

$$|u_i(t) - x| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on x ensure that for a.a. $t \in J$ we have

$$y^{(4)}(t) = \sum_{i=1}^{m} \lambda_i\, g(t) f(u_i(t)) > \sum_{i=1}^{m} \lambda_i\, \psi(t) = \psi(t) = \psi(t) + u^{(4)}(t). \tag{3.15}$$

Now for $t \in [t_-, \tilde{t}_-]$ we compute

$$\begin{aligned} y'''(\tau_0) - y'''(t) &= \int_t^{\tau_0} y^{(4)}(s)\,ds = \int_{[t,\tau_0]\cap J} y^{(4)}(s)\,ds + \int_{[t,\tau_0]\setminus J} y^{(4)}(s)\,ds \\ &> \int_{[t,\tau_0]\cap J} \psi(s)\,ds + \int_{[t,\tau_0]\cap J} u^{(4)}(s)\,ds \qquad \text{(by (3.15) and (3.14))} \\ &= \int_{[t,\tau_0]\cap J} \psi(s)\,ds + u'''(\tau_0) - u'''(t) - \int_{[t,\tau_0]\setminus J} u^{(4)}(s)\,ds \\ &\geq \int_{[t,\tau_0]\cap J} \psi(s)\,ds + u'''(\tau_0) - u'''(t) - \int_{[t,\tau_0]\setminus J} M(s)\,ds \\ &> u'''(\tau_0) - u'''(t) + \frac{1}{4}\int_t^{\tau_0} \psi(s)\,ds \quad \text{(by (3.11) and (3.12))}, \end{aligned}$$

hence $u'''(t) - y'''(t) \geq \tilde{\rho}$ provided that $u'''(\tau_0) \geq y'''(\tau_0)$. Therefore, by integration we obtain

$$u''(\tilde{t}_-) - y''(\tilde{t}_-) = u''(t_-) - y''(t_-) + \int_{t_-}^{\tilde{t}_-} (u'''(t) - y'''(t))\,dt \geq u''(t_-) - y''(t_-) + \tilde{\rho}(\tilde{t}_- - t_-).$$

If $u''(t_-) - y''(t_-) \leq -\rho$, then $\|y'' - u''\|_\infty \geq \rho$ and thus $\|y - u\| \geq \rho$ too. Otherwise, that is, if $u''(t_-) - y''(t_-) > -\rho$, then we have $u''(\tilde{t}_-) - y''(\tilde{t}_-) > \rho$ and hence $\|y - u\| \geq \rho$ too.

Similar computations in the interval $[\tilde{t}_+, t_+]$ instead of $[t_-, \tilde{t}_-]$ show that if $u'''(\tau_0) \leq y'''(\tau_0)$ then we have $y'''(t) - u'''(t) \geq \tilde{\rho}$ for all $t \in [\tilde{t}_+, t_+]$ and this also implies $\|y - u\| \geq \rho$. The claim is proven.

Case 3: $m(\{t \in [0,1] \,:\, u(t) = x_n\}) > 0$ only for some of those $n \in \mathbb{N}$ such that x_n is viable. Let us prove that in this case the relation $u \in \mathbb{T}u$ implies $u = Tu$.

Let us consider the subsequence of all viable admissible discontinuity points in the conditions of Case 3, which we denote again by $\{x_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in [0,1] \,:\, u(t) = x_n\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have

$$u^{(4)}(t) = 0 = g(t)f(x_n) = g(t)f(u(t)),$$

and therefore $u^{(4)}(t) = g(t)f(u(t))$ a.e. in $J = \cup_{n\in\mathbb{N}} J_n$.

Now we assume that $u \in \mathbb{T}u$ and we prove that it implies that $u^{(4)}(t) = g(t)f(u(t))$ a.e. in $[0,1] \setminus J$, thus showing that $u = Tu$.

Since $u \in \mathbb{T}u$ then for each $k \in \mathbb{N}$ we can guarantee that we can find functions $u_{k,i} \in \overline{B}_{1/k}(u) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ $(i = 1, 2, \dots, m(k))$ such that $\sum \lambda_{k,i} = 1$ and

$$\left\| u - \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i} \right\| < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i}$, and notice that $y_k \to u$ in the $\mathcal{C}^2$ norm and $\|u_{k,i} - u\| \le 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$.

For every $k \in \mathbb{N}$ we have $y_k \in Q$ as defined in (3.6), and therefore Lemma 3.7 guarantees that $u \in Q$ and, up to a subsequence, $y_k \to u$ in the $\mathcal{C}^3$ topology.

For a.a. $t \in [0,1] \setminus J$ we have that $f(\cdot)$ is continuous at $u(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \ge k_0$, we have

$$g(t)|f(u_{k,i}(t)) - f(u(t))| < \varepsilon \quad \text{for all } i \in \{1, 2, \dots, m(k)\},$$

and therefore

$$|y_k^{(4)}(t) - g(t)f(u(t))| \le \sum_{i=1}^{m(k)} \lambda_{k,i} g(t) |f(u_{k,i}(t)) - f(u(t))| < \varepsilon.$$

Hence $y_k^{(4)}(t) \to g(t)f(u(t))$ for a.a. $t \in [0,1] \setminus J$, and then Corollary 3.6 guarantees that $u^{(4)}(t) = g(t)f(u(t))$ for a.a. $t \in [0,1] \setminus J$.

Therefore the conditions of Proposition 2.2 are satisfied and we can ensure that BVP (1.1) has at least a positive solution. □

Remark 3.9 *Observe that the boundary conditions (BCs) do not play an important role together to the discontinuities of the nonlinearity f in order to guarantee the existence of positive solutions, so our result may be generalized to other BCs whenever the Green's function satisfied suitable sign conditions.*

We illustrate our theory with an example inspired by [4, Example 2].

Example 3.10 *Consider the BVP*

$$\begin{cases} u^{(4)} = \lfloor 7u^3 - 18u^2 + 12u \rfloor e^{-u} + \sqrt{u}, \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

The mapping $f(u) = \lfloor 7u^3 - 18u^2 + 12u \rfloor e^{-u} + \sqrt{u}$ is discontinuous at infinitely many points and these points are admissible inviable discontinuity points (it suffices to take $\psi \equiv 0.1$ and $\varepsilon = 0.05$ in Definition 3.2). In addition, it is not monotone and, clearly, $f_0 = +\infty$ and $f_\infty = 0$.

Therefore, Theorem 3.8 guarantees the existence of a positive solution for this problem.

4 A multiplicity result

We establish the existence of two positive solutions for problem (1.1). Our multiplicity result is based on the following Lemma and a suitable asymptotic behavior of the function f at zero and at infinity.

Lemma 4.1 *Assume that the functions f and g satisfy conditions (H_1) and (H_2).*

If there exist $r_1 > 0$ and $\varepsilon > 0$ such that

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \sup_{x \in [0, r_1+\varepsilon]} f(x) < r_1, \tag{4.16}$$

then $\|y\| < \|u\|$ for all $y \in \mathbb{T}u$ and all $u \in K$ with $\|u\| = r_1$.

Proof. Suppose that $u \in K$ with $\|u\| = r_1$. Then for every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$, we have (by condition (4.16))

$$\begin{aligned} v(t) = \sum_{i=1}^m \lambda_i T u_i(t) &= \sum_{i=1}^m \lambda_i \int_0^1 G(t,s) g(s) f(u_i(s))\, ds \\ &\le \sum_{i=1}^m \lambda_i \gamma^* f(r_1 + \varepsilon) \le \gamma^* \sup_{x \in [0, r_1+\varepsilon]} f(x) < r_1 = \|u\|. \end{aligned}$$

In addition,

$$|v'(t)| = \left| \sum_{i=1}^m \lambda_i (Tu_i)'(t) \right| \le \sum_{i=1}^m \lambda_i \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| g(s) f(u_i(s))\, ds \le \gamma_1^* \sup_{x \in [0, r_1+\varepsilon]} f(x) < r_1 = \|u\|,$$

and

$$|v''(t)| = \left| \sum_{i=1}^m \lambda_i (Tu_i)''(t) \right| = \sum_{i=1}^m \lambda_i \int_0^1 -\frac{\partial^2 G}{\partial t^2}(t,s) g(s) f(u_i(s))\, ds \le \gamma_2^* \sup_{x \in [0, r_1+\varepsilon]} f(x) < r_1 = \|u\|.$$

Hence, if $y \in \mathbb{T}u$, then it is the limit of a sequence of functions v as above, so $\|y\| < r_1$. □

Remark 4.2 *Notice that if f is a nondecreasing function, then condition (4.16) can be simply written as*

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} f(r_1 + \varepsilon) < r_1.$$

Now we present our multiplicity result concerning the existence of a "small" and a "big" positive solutions for problem (1.1).

Theorem 4.3 *Assume that the functions f and g satisfy conditions (H_1)–(H_3). Moreover,*

(1) $f_0 = \infty$ *and* $f_\infty = \infty$*;*

(2) *there exist* $r_1 > 0$ *and* $\varepsilon > 0$ *such that*

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \sup_{x\in[0,r_1+\varepsilon]} f(x) < r_1.$$

Then problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that* $\|u_1\| < r_1$ *and* $\|u_2\| > r_1$.

Proof. First, as in Claim 3, Theorem 3.8, condition (H_3) guarantees that $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$.

On the other hand, $f_0 = \infty$ implies that there exists $0 < r_2 < r_1$ such that $y \not\preceq u$ for all $y \in \mathbb{T}u$ and all $u \in K$ with $\|u\| = r_2$ (see Claim 2 in Theorem 3.8). Analogously, since $f_\infty = \infty$, there exists $R_2 > r_1$ such that $y \not\preceq u$ for all $y \in \mathbb{T}u$ and all $u \in K$ with $\|u\| = R_2$.

Therefore, by applying Proposition 2.2 twice, we obtain that the operator T has at least two fixed points u_1 and u_2 such that $r_2 < \|u_1\| < r_1$ and $r_1 < \|u_2\| < R_2$. □

To finish we present a simple example which, as far as we are aware, is not covered by the previous literature.

Example 4.4 *Consider the problem*

$$\begin{cases} u^{(4)} = u^p + \lfloor u \rfloor^q, \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{4.17}$$

where $0 < p < 1$ *and* $q > 1$. *Here* $g \equiv 1$ *and* $f(u) = u^p + \lfloor u \rfloor^q$.

Observe that f *is discontinuous at* $x_n = n$, $n \in \mathbb{N}$, *and for each* $n \in \mathbb{N}$,

$$0 < \inf\left\{f(x)\,:\,x \in \left[\frac{1}{2}, \infty\right)\right\} \le \inf\left\{f(x)\,:\,x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]\right\},$$

so the points x_n *are inviable, see Definition 3.2 and Remark 3.3.*

Since $0 < p < 1$ *and* $q > 1$, *we have that*

$$f_0 = \lim_{u\to 0^+} \frac{1}{u^{1-p}} + \frac{\lfloor u \rfloor^q}{u} = \infty, \qquad f_\infty = \lim_{u\to\infty} \frac{1}{u^{1-p}} + \frac{\lfloor u \rfloor^q}{u} \ge \lim_{u\to\infty} \frac{(u-1)^q}{u} = \infty.$$

Moreover, $\gamma^* = 1/384$, $\gamma_1^* \le 1/6$ *and* $\gamma_2^* = 1/8$, *so* $\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \le 1/6$. *By taking* $r_1 = 1/2$ *and* $\varepsilon = 1/2$, *condition* (2) *in Theorem 4.3 holds since* $\sup\{f(x)\,:\,x \in [0,1]\} = 2$ *and thus*

$$\frac{1}{6} \sup\{f(x)\,:\,x \in [0,1]\} < \frac{1}{2}.$$

Therefore, Theorem 4.3 ensures that problem (4.17) *has at least two positive solutions* u_1 *and* u_2 *such that* $\|u_1\| < 1/2$ *and* $\|u_2\| > 1/2$ *for any* $0 < p < 1$ *and* $q > 1$.

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