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Impulses in Differential Equations and Dynamical Systems

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TESE DE DOUTORAMENTO

IMPULSES IN DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

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ESCOLA DE DOUTORAMENTO INTERNACIONAL DA UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

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Juan José Nieto Roig

Contents

Summary

This work is devoted to the study of impulsive differential equations and impulsive dynamical systems. These differential equations and dynamical systems arise from the fact that a lot of modelled phenomena undergo a rapid change in their evolution, and these perturbations can be assumed to be instantaneous. For example, they can be used in population dynamics, in aircraft control, or in economics, where the sudden changes can represent natural disasters, external forces, etc. Moreover, this theory is, in general, more complicated, because there can be unexpected behaviors which do not appear on the classical theory.

Two main problems are explored: the existence of solutions for several boundary value problems for impulsive differential equations, and the asymptotic behavior of dynamical systems with impulses, studying the attractors, with an emphasis on the nonautonomous case.

In the first part of the manuscript, we provide several existence results for impulsive boundary value problems. We explore both problems with impulses at fixed times and problems with impulses at variable times. We obtain several results in both cases, using different techniques, such as variational methods or some fixed point theorems.

In the second part, we study impulsive nonautonomous dynamical systems and their asymptotic behavior. We mainly focus on impulsive evolution processes and the pullback attractor, obtaining conditions to guarantee its existence and some properties. Later, we extend our study to the multivalued situation, and to other type of nonautonomous systems.

Introduction

The aim of this thesis is to study both differential equations and dynamical systems which are under the influence of impulses. These impulses act as a short-term perturbation on the system whose duration is insignificant, in comparison with the rest of the process, so it is assumed that they are instantaneous. They normally occur when the state of the system reaches a certain set. Differential equations and dynamical systems with impulses provide a good framework to study a lot of real-world phenomena. Applications can be found, for example, in population dynamics, in epidemiology, or in aeronautics, to name a few of them.

We can say that both approaches consist of three basic elements: a differential equation or a dynamical system, which is responsible for the behavior of the system "most of the time", an impulsive set, which "decides" when those perturbations appear, and an application, which is responsible for making those instantaneous changes. Those rapid perturbations imply that solutions are going to be, in general, discontinuous. Therefore, this is one of the difficulties which are introduced in these studies. Moreover, some new and unexpected phenomena could appear.

The purpose of this manuscript is twofold: on the one hand, we obtain several results regarding the existence of solutions for some boundary value problems for impulsive differential equations, using different techniques. On the other hand, we study the asymptotic behavior of impulsive dynamical systems, mainly the nonautonomous case and, in particular, using evolution processes.

We present now a small summary of the three chapters of the thesis.

Chapter 1

This first chapter is a little collection of preliminary notions and results, with the intention of making a self-contained work.

We start with a small introduction to some topological results and some basic results on critical point theory and functional analysis. Next, we consider some classical fixed point theorems, in finite and infinite dimensions, and also a small introduction to some results on classical degree theory. We continue with a summary of results on continuous dynamical systems. We begin with the autonomous case and the notion of semigroup and some of its properties. In order to study the asymptotic behavior, we consider the global attractor, which helps understand a lot of qualitative properties of the solutions. For the nonautonomous case, we start with a small introduction to the skew product formulation and to evolution processes. We focus on evolution processes and in the study of the pullback attractor, which is one of the possible generalizations of the global attractor.

Chapter 2

The second chapter focuses on the study of impulsive differential equations. We start with an introduction to this type of differential equations, and seeing some of its properties. We treat two cases, with impulses at fixed times, and with state-dependent impulses (or impulses at variable times). Furthermore, we analyze some similarities and differences in these cases, as well as some unexpected behavior that can arise, particularly when we are dealing with impulses at variable times.

In the next section, we obtain some existence results for some problems with impulses at fixed times. We begin with a first order problem with impulses and singularities, and we prove the existence of periodic solutions. Later, we study a second order problem. We combine a variational approach with some fixed point results in order to guarantee the existence of solutions, even though the problem has dependence on the derivative. We finish the section with an analysis of the topological structure of the solution set. Then we use this result to apply a multivalued fixed point result in finite dimension, which will imply the existence of solutions for both Dirichlet and Neumann boundary conditions.

We consider the case of impulses at variable times in the final section. We start with a first order problem, and we explain the construction of an ad-hoc space of functions. Then, we obtain the structure of the solution set for this problem. Using the same approach as the previous section, we get the existence of periodic solutions. Later, we consider a second order differential equation of Duffing-type with impulses at variable times. We use a time-map approach in combination with a study of the Poincaré map, in order to obtain the existence of periodic solutions under different assumptions.

We provide some examples, distributed through the chapter, for the different existence results which have been proved.

Chapter 3

The third chapter consists of a collection of results about impulsive dynamical systems, in particular about the nonautonomous case. We start with an introduction to the autonomous case, with some definitions as well as some recent results about the study of attractors. Then, we define the impulsive trajectories, obtain some results about the impact time map, and we adapt some results from the continuous case to this framework. We continue with the definition of the pullback attractor, as well as some conditions to guarantee its existence. In particular, the invariance property is not easy to obtain, and we propose two different but related approaches, although we are not able to obtain a sufficient and necessary condition.

The next section is devoted to an initial study of perturbations for impulsive evolution processes. We obtain a result about the upper semicontinuity of the pullback attractor, as well as a result about the weak-lower semicontinuity. We continue by extending some previous result to the multivalued framework, using the notion of multivalued impulsive evolution process.

Then we consider a nonautonomous dynamical system given by a cocycle and a driving semigroup. We introduce impulses on the driving semigroup, and we obtain some results on the uniform attractor. We also consider an associated evolution process and its pullback attractor, and obtain a relation between both attractors.

Finally, in the last section, we start by proving three results which help in the applications. Then, we provide several applications of some theoretical results proved through the chapter. These applications include an "integrate-and-fire" model, an impulsive two-dimensional Navier–Stokes equations, an impulsive multivalued reaction-diffusion equation, and cascade systems.

Objectives and Methodology

The main aim of this thesis is to study the influence of impulses in the fields of differential equations and dynamical systems. In particular, some objectives are:

- To analyze impulsive differential equations, impulsive dynamical systems, and their variants.
- To study the different techniques used in the fields of differential equations and dynamical systems.
- To extend some results of the general theory of differential equations and obtain new results about the existence of solutions for impulsive differential equations.
- To study the asymptotic behavior of impulsive dynamical systems, in particular the nonautonomous case.
- To obtain some result about the existence of attractors for impulsive dynamical systems, as well as some properties of these attractors.

The methods used in this thesis will include several results of mathematical analysis, functional analysis, topology, and nonlinear analysis. Some techniques used include fixed point results, degree theory, critical point theory, as well as some results on classical asymptotic behavior for dynamical systems.

Chapter 1

Preliminaries

This chapter is intended to serve as an introduction of some notation and a review of concepts and results which will be used through the rest of the manuscript. In particular, we do not include the proofs of the results that we present here.

In Section 1.1 we include a background on some topological definitions and results, as well as some results on set-valued maps and critical point theory. We recall some classical fixed point theory results in Section 1.2, both in finite and infinite dimensional spaces. In Section 1.3 we introduce some notions of degree theory in finite dimensional spaces (the Brouwer degree) and in infinite dimensional spaces (the Leray–Schauder degree). Then, we use these degree results to study other type of problems, with the development of the coincidence degree theory. In Section 1.4 the goal is to make a brief introduction to the concept of continuous semigroup and some of its properties. We introduce the global attractor in order to study the asymptotic dynamics of these systems. Finally, in Section 1.5 we comment the two most typical formulations of nonautonomous dynamical systems. We will focus on an introduction to some results about evolution processes and the pullback attractor.

1.1 Topological and variational results

In this section, we recall some definitions and results on topological spaces, set-valued maps, functional analysis, and critical point theory. Most of the result are taken from [27, 55, 71, 92, 120].

Definition 1.1. Let X be a topological space and A a subspace. A continuous map $r\colon X \longrightarrow A$ is a retraction if $r(a) = a$ for all $a \in A$. In this case, it is said that A is a retract of X.

Note that A is a retract of X if and only if the identity Id_A has a continuous extension into X.

Definition 1.2. Let X be a topological space and A a subspace. The subspace A is called a neighborhood retract of X if it is closed and it is a retract of an open subset of X that contains A, that is, A is closed and there exists a subset $Y \subset X$ with $A \subset Y$ and A a retract of Y.

Definition 1.3. A topological space X is an absolute (neighborhood) retract if for any topological space Y and any homeomorphism $i\colon X \longrightarrow Y$ of X onto $i(X)$ with $i(X)$ closed in Y, the set $i(X)$ is a (neighborhood) retract of Y.

Definition 1.4. A topological space X is called contractible if there exist $a \in X$ and $h\colon X \times [0,1] \longrightarrow X$ a continuous map such that $h(x,0) = x$ and $h(x,1) = a$ for every $x \in X$.

Definition 1.5. A nonempty compact space X is an R_δ-set if there is a decreasing sequence $\{X_n\}_n$ of compact contractible spaces such that X is the intersection of all X_n.

It is easy to see that the intersection of any decreasing sequence of R_δ-sets is also R_δ. We also have the following result.

Theorem 1.6. *Let X be a complete metric space and A a nonempty subset of X. Then A is an R_δ-set if and only if A is compact and absolutely neighborhood contractible.*

Definition 1.7. Let X and Y be metric spaces and $\phi\colon X \longrightarrow Y$ a set-valued map. The map ϕ is called upper semicontinuous if for each open set $V \subset Y$, the set $\{x \in X : \phi(x) \subset V\}$ is open.

Definition 1.8. Let X and Y be metric spaces and $\phi\colon X \longrightarrow Y$ a set-valued map. The map ϕ is an R_δ-map if ϕ is upper semicontinuous and $\phi(x)$ is an R_δ-set for each $x \in X$.

Proposition 1.9. *Let X and Y be metric spaces and ϕ be a set-valued map from X to Y. Then ϕ is upper semicontinuous with compact values (meaning that $\phi(x)$ is a compact subset of Y for all $x \in X$) if and only if for every sequence $\{x_n\}_n$ convergent to x and every sequence $\{y_n\}_n$ with $y_n \in F(x_n)$, there exist a subsequence $\{y_{n_k}\}_k$ and $y \in F(x)$ such that $\{y_{n_k}\}_k$ converges to y.*

Theorem 1.10. *Let X be a metric space, E a Banach space with norm $\|\cdot\|$, and $f\colon X \longrightarrow E$ a proper map (f continuous and for every compact $K \subset E$, the set $f^{-1}(K)$ is compact). Assume that for each $\varepsilon > 0$, there exists a proper map $f_\varepsilon\colon X \longrightarrow E$ such that:*

(1) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$ for every $x \in X$,

(2) for any $\varepsilon > 0$ and $u \in E$ with $\|u\| \le \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution.

Then $S = f^{-1}(0)$ is an R_δ-set.

Theorem 1.11. *Let* $E = \mathcal{C}([0,a],\mathbb{R}^n)$, $X = \overline{B(0,r)}$ *a closed ball in* E, *and* $F\colon X \longrightarrow E$ *a compact map (that is, the image of bounded sets is relatively compact). Suppose that*

1. *there exists an* $x_0 \in \mathbb{R}^n$ *such that* $F(u)(0) = x_0$ *for every* $u \in X$,

2. *for every* $\varepsilon \in (0,a]$ *and for every* $u, v \in X$, *if* $u(t) = v(t)$ *for each* $t \in [0,\varepsilon]$, *then* $F(u)(t) = F(v)(t)$ *for each* $t \in [0,\varepsilon]$.

Then there exists a sequence $f_n\colon X \longrightarrow E$ *of proper maps satisfying* (1) *and* (2) *in Theorem* 1.10, *with respect to the map* $f(u) = u - F(u)$.

Corollary 1.12. *Under the hypotheses of Theorem* 1.11, $\operatorname{Fix}(F)$ *is an* R_δ*-set.*

Theorem 1.13 (Baire category theorem)**.** *Let* X *be a metric space. Then*

- *Every countable intersection of open dense sets is dense.*

- *If the countable union of closed sets has an interior point, then one of them has an interior point.*

The two previous conditions are in fact equivalent.

Definition 1.14. Let X be a real Banach space. We define

$$X^* := \{f\colon X \longrightarrow \mathbb{R} \mid f \text{ is linear and continuous}\}.$$

The space X^* is called the dual space of X.

Definition 1.15. Let X be a Banach space. The weak topology on X is the initial topology on X with respect to the family of functions X^*, that is, the coarsest topology on X such that each element of X^* is a continuous function.

We will use the following results from linear functional analysis.

Theorem 1.16. *Let* X *be a real Banach space and* $\{x_n\}_n$ *a sequence of elements in* X. *Then*

(i) *The sequence* $\{x_n\}_n$ *converges to* x *in the weak topology (or converges weakly to* x*) if and only if the sequence* $\{f(x_n)\}_n$ *converges to* $f(x)$ *for every* $f \in X^*$.

(ii) *If* X *is a Hilbert space, the sequence* $\{x_n\}_n$ *converges weakly to* x, *and* $\|x_n\|$ *converges to* $\|x\|$, *then the sequence* $\{x_n\}_n$ *converges (strongly) to* x.

(iii) *If* X *is a Hilbert space, then every closed and bounded subset of* X *is weakly relatively compact.*

(iv) *If T is a compact linear map between Banach spaces and $\{x_n\}_n$ is a sequence which converges weakly to x, then the sequence $\{Tx_n\}_n$ converges to Tx.*

Definition 1.17. Let X be a topological space, M an arbitrary subset of X, and $f\colon M \longrightarrow [-\infty, \infty]$ a map. We say that

- f is lower semicontinuous if the set $\{x \in M : f(x) \leq c\}$ is closed for all $c \in \mathbb{R}$.
- f is upper semicontinuous if $-f$ is lower semicontinuous.

Definition 1.18. Let X be a metric space, $M \subset X$, and $f\colon M \longrightarrow [-\infty, \infty]$ a map. We say that f is sequentially lower semicontinuous if, for any sequence $\{x_n\}_n$ convergent to $x \in M$, we have that

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

We note that for metric spaces X, lower semicontinuous and sequentially lower semicontinuous are equivalent.

Definition 1.19. Let X be a Banach space, $M \subset X$, and $f\colon M \longrightarrow [-\infty, \infty]$ a map. We say that

- f is weakly lower semicontinuous if for all $c \in \mathbb{R}$ we have that the set $\{u \in M : f(u) \leq c\}$ is weakly closed.
- f is weakly sequentially lower semicontinuous if for any sequence $\{x_n\}_n$, with $x_n \in M$, which converges weakly to an element $x \in M$, we have that

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Proposition 1.20. *Let X be a real Banach space, K a convex subset of X, and $f\colon K \longrightarrow \mathbb{R}$ a map. If f is a lower semicontinuous and convex function, then f is weakly lower semicontinuous.*

Definition 1.21. Let $f\colon M \subset X \longrightarrow \mathbb{R}$ be a map defined on M, a subset of a Banach space X. We say that f is coercive if for any sequence $\{x_n\}_n$ with $x_n \in M$ for each $n \in \mathbb{R}$ and $\|x_n\| \longrightarrow +\infty$, we have that $f(x_n) \longrightarrow +\infty$.

Definition 1.22. Let X be a real Banach space, M a nonempty subset of X, and $f\colon M \longrightarrow \mathbb{R}$ a map. We say that a sequence $\{x_n\}_n$ is minimizing if the sequence $\{f(x_n)\}_n$ converges to $\inf\{f(v) : v \in M\}$.

Theorem 1.23. *Let X be a real Banach space, M a nonempty subset of X, and $f\colon M \longrightarrow [-\infty, \infty]$ a map. Suppose that*

1. *X is a reflexive Banach space.*

2. *M is bounded and weakly sequentially closed (that is, for any sequence $\{x_n\}_n$ in M which converges weakly to an element $x \in X$, we have that $x \in M$).*

3. *f is weakly sequentially lower semicontinuous.*

Then there exists $x \in M$ such that $f(x) = \min\{f(v) : v \in M\}$.

Proposition 1.24. *Let X be a reflexive and real Banach space, M a closed and convex subset of X, and $f\colon M \longrightarrow \mathbb{R}$ a coercive and weakly sequentially lower semicontinuous map. There exists $x \in M$ such that $f(x) = \min\{f(v) : v \in M\}$.*

Corollary 1.25. *Let X be a reflexive and real Banach space, M a closed and convex subset of X, and $f\colon M \longrightarrow \mathbb{R}$ a convex, coercive, and lower semicontinuous map. Then there exists $x \in M$ such that $f(x) = \min\{f(v) : v \in M\}$. Moreover, if f is strictly convex, then the minimum is unique.*

1.2 Fixed point theorems

Fixed point theorems are a very useful tool in order to get existence, uniqueness, or multiplicity of solutions for several problems. In this section, we state some results in both finite and infinite dimensions. We start with Brouwer fixed point theorem.

Theorem 1.26 (Brouwer)**.** *Let B be a closed ball of $\mathbb{R}^n$ and $f\colon B \longrightarrow B$ a continuous map. Then f has at least a fixed point.*

We present next an extension of the well-known Bolzano theorem to higher dimensions. This result guarantees the existence of a zero of a function based on the sign behavior at the boundary, the same way Bolzano theorem does on a compact interval. It was formulated and proved in the 1880s (see [99, 100]) by Poincaré, and forgotten for a long period of time. It was rediscovered by Miranda in 1940 [93], and he showed that this result is equivalent to Theorem 1.26. Since then, this result has been known as the Poincaré–Miranda theorem. Its statement is as follows.

Theorem 1.27 (Poincaré–Miranda)**.** *Let $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ be two elements of $\mathbb{R}^n$ satisfying $a_i \le b_i$ for all $i \in \{1, \dots, n\}$,*

$$R = \prod_{i=1}^{n} [a_i, b_i],$$

and $f\colon R \longrightarrow \mathbb{R}^n$ a continuous map, $f = (f_1, \dots, f_n)$. If for every $i \in \{1, \dots, n\}$ we have that

- $f_i(x) \leq 0$ *for every* $x \in R$ *with* $x_i = a_i$,

- $f_i(x) \geq 0$ *for every* $x \in R$ *with* $x_i = b_i$,

then there exists $x \in R$ *such that* $f(x) = 0$*.*

There have been several generalizations of this result, with some applications. See for example [4, 63, 66, 87, 114, 118]. We will explain later in the section a generalization given in [114], which will be used in Chapter 2.

The following result is also similar to Brouwer fixed point theorem. It also gives us the existence of a fixed point, but in this case for a function defined on a closed starlike region with values in all $\mathbb{R}^n$.

Theorem 1.28 (Poincaré–Bohl)**.** *Let* D *be a closed and bounded region of* $\mathbb{R}^n$*, which is starlike with respect to the origin, and* $f\colon D \longrightarrow \mathbb{R}^n$ *a continuous map. If* $f(p) \neq \lambda p$ *for all* $\lambda > 1$ *and for all* $p \in \partial D$*, then* f *has at least one fixed point in* D*.*

We consider next the so-called Poincaré–Birkhoff fixed point theorem, which is also known as Poincaré's last geometric theorem. Consider an area-preserving homeomorphism of a planar circular annulus onto itself, such that

- the points of the inner boundary advance along the inner boundary in the clockwise sense,

- the points of the outer boundary advance along the outer boundary in the counter-clockwise sense.

Then this homeomorphism has at least two fixed points. This was conjectured by Poincaré in [101], and he also proved it in some special cases. In fact, he writes

> "Je n'ai jamais presenté au public un travail aussi inachevé ; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles quim'avaient engagé à l'entreprendre. [...]
>
> J'ai donc été amené à rechercher si ce théorème est vrai ou faux, mais j'ai rencontré des difficultés auxquelles je ne m'attendais pas. [...]
>
> Il semble que dans ces conditions, je devrais m'abstenir de toute publication tant que je n'aurai pas résolu la question ; mais après les inutiles efforts que j'ai faits pendant de longs mois, il m'a paru que le plus sage était de laisser le problème mûrir, en m'en reposant durant quelques années ; cela serait très bien si j'étais sûr de pouvoir le reprendre un jour ; mais à mon âge je ne puis en répondre."

The result was proved by Birkhoff in [12], with a mistake that he corrected later in [13]. Different proofs and versions of this result have been proved ever since, for example [29, 64, 65, 95].

We consider next a partial twist theorem. It was first proved in [102]. This result gives us the existence of only one point, however it avoids the hypothesis of "area-preserving".

Theorem 1.29. *Let Γ_- and Γ_+ be two closed and convex curves surrounding the origin,* $\mathrm{Int}(\Gamma_-)$ *and* $\mathrm{Int}(\Gamma_+)$ *the interior (in the sense of Jordan curve theorem) of Γ_- and Γ_+,* $\mathrm{Int}(\Gamma_-) \subset \mathrm{Int}(\Gamma_+)$*, $\mathcal{A}$ the annulus bounded by Γ_- and Γ_+, and $F\colon \overline{\mathrm{Int}(\Gamma_+)} \longrightarrow \mathbb{R}^2$ a continuous map. Take $U(0)$ a neighborhood of the origin and L a real orthogonal matrix with* $\det(L) = 1$*. We denote*

$$E = \{z \in \mathcal{A} : |F(z)| \leq |z|\},$$
$$J = \{z \in \mathcal{A} : F(z) \in \mathbb{R}^2 \setminus U(0) \text{ and } \langle Lz, F(z)\rangle = 0\}.$$

If for any curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ which connects Γ_- and Γ_+ we have that $\gamma([a,b]) \cap (J \cup E) \neq \emptyset$, then F has at least one fixed point.

We present a sketch of the idea of the proof. Define $Q = J \cup E$, which is a nonempty, bounded, and closed set, and denote

$$\Omega = \{z \in \mathbb{R}^2 : \text{ there exists a curve } \gamma \text{ joining } z \text{ and } 0$$
$$\text{which does not intersect } Q\}.$$

It can be proved that $\mathrm{Int}(\Gamma_-) \subset \Omega \subset \mathrm{Int}(\Gamma_+)$, that Ω is open, and that $\partial\Omega \subset Q$. Finally, if we suppose that F has no fixed points in $\overline{\mathrm{Int}(\Gamma_+)}$, then the function $G(z) = F(z) - z$ has no zeros in $\overline{\mathrm{Int}(\Gamma_+)}$. In particular, we can use $\deg_B(G, \Omega, 0)$ and prove that $\deg_B(G, \Omega, 0) \neq 0$ (see Section 1.3 for some results on degree theory). This implies the existence of a fixed point of F.

We present now an extension of the Brouwer fixed point theorem to Banach spaces.

Theorem 1.30 (Schauder)**.** *Let X be a Banach space and C a nonempty, convex, bounded, and closed subset of X. If $T\colon C \longrightarrow C$ is a continuous and compact map, then T has a fixed point.*

Finally, to end this section, we consider an extension of Poincaré–Miranda theorem (Theorem 1.27) to multivalued maps. This result was stated and proved on [114].

Theorem 1.31. *Let $M_1, \ldots, M_n$ be positive numbers and F a set-valued map from $M = [-M_1, M_1] \times \cdots \times [-M_n, M_n]$ to $\mathbb{R}^n$. Suppose that there exists:*

- *a Banach space X with* $\dim(X) \geq n$*,*

- *a linear, bounded, and surjective map* $\varphi\colon X \longrightarrow \mathbb{R}^n$*,*
- *an* R_δ*-map* ϕ *from* M *to* X *such that* $F = \varphi \circ \phi$*.*

If the following condition is satisfied,

$$\begin{gathered} \textit{for each } 1 \leq i \leq n, \ \textit{if } x \in M \ \textit{with } |x_i| = M_i \\ \textit{and } y \in F(x), \ \textit{then } x_i \cdot y_i \geq 0; \end{gathered} \tag{1.1}$$

then there exists $x \in M$ *such that* $0 \in F(x)$*.*

1.3 Degree theory

The topological degree is a tool which will give us information about solutions of equations. For example, consider the equation

$$f(x) = y, \quad x \in A, \tag{1.2}$$

with $f\colon X \longrightarrow Y$ a function (normally continuous at least), X and Y two spaces (Euclidean spaces, differentiable manifolds, Banach spaces, etc.), y an element of Y, and A a subset of X. Sometimes this equation can not be solved directly, nor we can obtain approximations for the solutions. This tool is a method to try to get information about the solutions.

The first definition of degree for maps between subsets of $\mathbb{R}^n$ was given by Brouwer [28]. We will construct the Brouwer degree using an approach introduced by Nagumo [94].

Definition 1.32. Let A be a nonempty, open, and bounded set of $\mathbb{R}^n$, $y \in \mathbb{R}^n$, and $f\colon \overline{A} \longrightarrow \mathbb{R}^n$ a map. We say that the triple (f, A, y) is admissible for the Brouwer degree if f is continuous and $y \notin f(\partial A)$.

In order to construct Brouwer degree, the idea is to define it first for smooth functions f and regular values y. Then it is approximated for continuous functions and arbitrary values y.

Definition 1.33. Consider (f, A, y) with f a $\mathcal{C}^\infty$ function from $\overline{A}$ to $\mathbb{R}^n$ and $y \notin f(\partial A)$ a regular value (the set $f^{-1}(y)$ does not contain critical points). Then we define the Brouwer degree of (f, A, y) as

$$\deg_B(f, A, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det(Df(x))),$$

where $Df(x)$ is the Jacobian matrix.

It can be proved that the set $f^{-1}(y)$ is a finite set (possibly empty) due to the Inverse Function Theorem, so the definition makes sense. Sard Theorem allows us to remove the assumption that y is a regular value.

Theorem 1.34 (Sard)**.** *Let $f\colon U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a $\mathcal{C}^k$ map, defined on U, an open subset of R^n. If $k > \max\{0, n-m\}$, then the set of critical values of f has Lebesgue measure 0 in $\mathbb{R}^m$.*

Take $f\colon \overline{A} \longrightarrow \mathbb{R}^n$ a $\mathcal{C}^\infty$ map and $y \notin f(\partial A)$. Sard theorem implies that any neighborhood of y contains regular values for f, so we can define

$$\deg_B(f, A, y) := \deg_B(f, A, y^*),$$

with y^* a regular value sufficiently close to y. It can be proved that $\deg_B(f, U, y)$ is well-defined, that is, any regular value y^* sufficiently close to y gives us the same degree.

Finally, we can also remove the smoothness of the function f by considering Weierstrass approximation theorem, that is, for a given continuous function f, we take g a $\mathcal{C}^\infty$ map close to f on $\overline{A}$, and we define

$$\deg_B(f, A, y) := \deg_B(g, A, y).$$

Once again, it can be proved that this is well-defined.

The following theorem contains some properties of the Brouwer degree.

Theorem 1.35. *The Brouwer degree satisfies the following properties:*

(1) *If A is an open subset of $\mathbb{R}^n$ and $y \in A$, then $\deg_B(\mathrm{Id}, A, y) = 1$.*

(2) *If $A_1, A_2 \subset A$ are open and disjoint, and $f(x) \neq y$ if $x \in \overline{A} \setminus (A_1 \cup A_2)$, then*
$$\deg_B(f, A, y) = \deg_B(f, A_1, y) + \deg_B(f, A_2, y).$$

(3) *Given $H\colon \overline{A} \times [0,1] \longrightarrow \mathbb{R}^n$ continuous such that $y \notin H(\partial A \times [0,1])$, we have that $\deg_B(H(\cdot, t), A, y)$ does not depend on $t \in [0,1]$.*

(4) *If $\deg_B(f, A, y) \neq 0$, then there exists $x \in A$ such that $f(x) = y$.*

Moreover, if we take

$$\Pi = \{(f, A, y) : A \subset \mathbb{R}^n,\ A \text{ open and bounded},\ f \in \mathcal{C}(\overline{A}, \mathbb{R}^n),\ y \notin f(\partial A)\},$$

then, it can be proved that there is only one map from Π to $\mathbb{R}$ which satisfies Properties (1)–(3) in Theorem 1.35. In order to extend the Brouwer degree to functions between infinite dimensional spaces, a generalization with the same conditions and approach is not possible. We would have several difficulties, for example, it is not clear what is the definition of the determinant for $Df(x)$, and Weierstrass approximation theorem does not hold.

In fact, there is no degree theory in infinite dimension for the whole family of continuous maps. We summarize here the construction of degree introduced by Leray and Schauder [89]. It concerns maps of the form $f = \operatorname{Id} - k$, with X a Banach space, $k\colon \overline{A} \longrightarrow X$ a continuous and compact map, A an open and bounded subset of X, and $y \in X \setminus f(\partial A)$.

The idea of the construction is as follows. Approximate k by maps k_ε ($\varepsilon > 0$) such that k_ε is contained on a finite dimensional subspace $X_\varepsilon \subset X$, with $y \in X_\varepsilon$. Then, it can be shown that $\deg_B((\operatorname{Id}_\varepsilon - k_\varepsilon)|_{X_\varepsilon}, A \cap X_\varepsilon, y)$ is defined and "stabilizes" for sufficiently small ε. Finally, we can define

$$\deg_{LS}(f, A, y) = \deg_{LS}(\operatorname{Id} - k, A, y) := \deg_B((\operatorname{Id}_\varepsilon - k_\varepsilon)|_{X_\varepsilon}, A \cap X_\varepsilon, y),$$

which is called the Leray–Schauder degree of f in A at y. As before, it can be proved that it is well-defined. Furthermore, this degree verifies the same properties considered in Theorem 1.35 for the Brouwer degree.

To finish this section, we will use the degree theory in order to study problems which can be written as

$$Lx = Nx \tag{1.3}$$

in an abstract space, with L a linear and noninvertible map. We briefly comment the ideas contained in [69].

Definition 1.36. Let X and Z be Banach spaces, and consider a linear map $L\colon \operatorname{Dom}(L) \subset X \longrightarrow Z$. We say that L is a Fredholm map of index 0 if $\operatorname{Im}(L)$ is a closed subset of Z, $\operatorname{Ker}(L)$ and $\operatorname{coker}(L)$ have finite dimension, and $\dim(\operatorname{Ker}(L)) = \dim(\operatorname{coker}(L)) = \operatorname{codim}(\operatorname{Im}(L))$.

If L is a Fredholm map of index 0, it can be proved, using Hahn–Banach theorem, that there exist two continuous projectors (that is, continuous, linear, and idempotent) $P\colon X \longrightarrow X$ and $Q\colon Z \longrightarrow Z$ such that

$$X \xrightarrow{\;P\;} \operatorname{Dom}(L) \xrightarrow{\;L\;} Z \xrightarrow{\;Q\;} Z$$

is an exact sequence, that is, $\operatorname{Im}(P) = \operatorname{Ker}(L)$ and $\operatorname{Im}(L) = \operatorname{Ker}(Q)$. Furthermore, it can be easily proved that

$$\operatorname{Ker}(Q) = \operatorname{Im}(\operatorname{Id} - Q) \quad \text{and} \quad \operatorname{Dom}(L) \cap \operatorname{Ker}(P) = (\operatorname{Id} - P)X.$$

It follows that the map

$$L|_{\operatorname{Dom}(L) \cap \operatorname{Ker}(P)}\colon (\operatorname{Id} - P)X \longrightarrow \operatorname{Im}(L)$$

is invertible. We denote the inverse of this map by K_P. Then, the following result holds.

Proposition 1.37. *Consider* $\Lambda\colon \operatorname{coker} L \longrightarrow \operatorname{Ker}(L)$ *any isomorphism and* $\pi\colon Z \longrightarrow \operatorname{coker}(L)$ *the canonical surjection. Then*

$$Lx = y \Longleftrightarrow (\operatorname{Id} - P)x = (\Lambda\pi + K_P(\operatorname{Id} - Q))y.$$

Based on this last result, and taking $y = Nx$, we get:

Proposition 1.38. *Consider* $\Lambda\colon \operatorname{coker} L \longrightarrow \operatorname{Ker}(L)$ *any isomorphism and* $\pi\colon Z \longrightarrow \operatorname{coker}(L)$ *the canonical surjection. If* Ω *is an open and bounded subset of* X*,* $N\colon \overline{\Omega} \longrightarrow Z$ *is continuous,* $\pi N(\overline{\Omega})$ *is bounded, and* $K_P(\operatorname{Id} - Q)N$ *is compact, then*

$$Lx = Nx \Longleftrightarrow (\operatorname{Id} - P)x = (\Lambda\pi + K_P(\operatorname{Id} - Q))Nx \Longleftrightarrow x = Mx,$$

with $M = P + (\Lambda\pi + K_P(\operatorname{Id} - Q))N$*.*

We have that M is a compact map. Therefore, if $0 \notin (L - N)(\operatorname{Dom}(L) \cap \partial\Omega)$, we have that the Leray–Schauder degree $\deg_{LS}(\operatorname{Id} - M, \Omega, 0)$ is well-defined. This implies that we can use some results of the Leray–Schauder degree theory in order to find solutions of (1.3).

It can be proved that $\deg_{LS}(\operatorname{Id} - M, \Omega, 0)$ depends only on L, N, Ω and Λ, and $|\deg_{LS}(\operatorname{Id} - M, \Omega, 0)|$ does not depend on Λ. We have:

Definition 1.39. Under the conditions of Proposition 1.38, we define the Coincidence Degree of L and N in Ω, denoted by $\deg_{CD}((L, N), \Omega)$, as

$$\deg_{CD}((L, N), \Omega) := \deg_{LS}(\operatorname{Id} - M, \Omega, 0),$$

with M defined as in Proposition 1.38, and Λ an orientation-preserving isomorphism.

We will give sufficient conditions for the existence of a solution to (1.3).

Definition 1.40. Let $N\colon X \longrightarrow Y$ be a continuous map between two normed spaces and Ω an open and bounded subset of X. We say that N is L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(\operatorname{Id} - Q)N\colon \overline{\Omega} \longrightarrow X$ is a compact map.

Furthermore, $\operatorname{Im}(Q)$ is isomorphic to $\operatorname{Ker}(L)$, so there exists an isomorphism $J\colon \operatorname{Im}(Q) \longrightarrow \operatorname{Ker}(L)$.

Theorem 1.41. *Let* X *and* Y *be two Banach spaces,* Ω *an open and bounded subset of* X*, and* $L\colon \operatorname{Dom}(L) \subset X \longrightarrow Y$ *a Fredholm map of index* 0*. Suppose that the map* $N\colon \overline{\Omega} \subset X \longrightarrow Y$ *is* L*-compact on* $\overline{\Omega}$ *and*

1. $Lx \neq \lambda Nx$ *for every* $x \in \partial\Omega \cap \operatorname{Dom}(L)$ *and* $\lambda \in (0, 1)$*,*
2. $QNx \neq 0$ *for every* $x \in \partial\Omega \cap \operatorname{Ker}(L)$*,*
3. $\deg_B(JQN, \Omega \cap \operatorname{Ker}(L), 0) \neq 0$*, with* $J\colon \operatorname{Im}(Q) \longrightarrow \operatorname{Ker}(L)$ *an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{Dom}(L) \cap \overline{\Omega}$*.*

1.4 Autonomous dynamical systems

In this section, we make a brief introduction to the theory of continuous semigroups and its properties. The mathematical object which plays a very important role for the study of the asymptotic behavior is the global attractor. We will introduce several concepts and results in order to get the existence of the global attractor, and then we will also see some results of the behavior of the attractors under perturbations of the semigroup.

We will consider that the phase space is a metric space (X, d), although in applications it is more usual to consider Hilbert or Banach spaces. For more information and the proofs, see [30, 40, 74, 104].

Definition 1.42. A semigroup is a family $\{\pi(t) : t \geq 0\}$ of continuous maps from X to itself such that

1. $\pi(0)x = x$ for all $x \in X$,
2. $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \geq 0$,
3. the map $(t, x) \longmapsto \pi(t)x$ is continuous from $[0, \infty) \times X$ to X.

From now on, let $\{\pi(t) : t \geq 0\}$ be a semigroup.

Definition 1.43. Let A and B be two subsets of X. We define the Hausdorff semidistance as

$$d_{\mathrm{H}}(A, B) := \sup\{\inf\{d(a, b) : b \in B\} : a \in A\}.$$

Definition 1.44. Let A be a subset of X and $r > 0$. The r-neighborhood of A is defined as the set

$$\mathcal{O}_r(A) := \{x \in X : d(x, A) < r\}.$$

Definition 1.45. Let A be a subset of X. We say that A is

- positively π-invariant if $\pi(t)A \subset A$ for all $t \geq 0$,
- negatively π-invariant if $A \subset \pi(t)A$ for all $t \geq 0$,
- π-invariant if it is both positively and negatively invariant.

Definition 1.46. Let A and B be two subsets of X. We say that A π-attracts B if

$$\lim_{t \to \infty} d_{\mathrm{H}}(\pi(t)B, A) = 0.$$

Definition 1.47. Let A be a subset of X. The ω-limit set of A is defined by

$$\omega(A) := \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \pi(s)A}.$$

In applications, the following characterization for the ω-limit set is very useful.

Proposition 1.48. *For any $A \subset X$, we have that $y \in \omega(A)$ if and only if there exist two sequences $\{t_n\}_n$ and $\{x_n\}_n$, with $t_n \longrightarrow +\infty$ and $x_n \in A$, such that $\pi(t_n)x_n \longrightarrow y$.*

Definition 1.49. Let $\psi \colon \mathbb{R} \longrightarrow X$ be a continuous function. We say that ψ is a global solution of $\{\pi(t) : t \geq 0\}$ through $x \in X$ if $\psi(0) = x$ and $\pi(t)\psi(s) = \psi(t+s)$ for every $t \geq 0$ and $s \in \mathbb{R}$.

For any ψ a global solution, its orbit is given by

$$\Gamma(\psi) = \bigcup_{t \in \mathbb{R}} \psi(t).$$

Therefore, we have that

Proposition 1.50. *A set A is π-invariant if and only if it is formed by a collection of orbits of global solutions.*

Definition 1.51. A set A is the global attractor for $\{\pi(t) : t \geq 0\}$ if

1. A is compact,
2. A is π-invariant,
3. A π-attracts each bounded subset of X.

Theorem 1.52. *If a semigroup $\{\pi(t) : t \geq 0\}$ has a global attractor A, then*

$$A = \{x \in X : \text{ there exists a bounded global solution through } x\}.$$

Theorem 1.53. *The global attractor A is unique, it is the minimal compact set that attracts bounded sets, and the maximal closed and bounded invariant set.*

The following definitions are fundamental to obtain the existence of the global attractor for a semigroup. See for example [104].

Definition 1.54. We say that a semigroup is dissipative if there exists a bounded subset B_0 of X with the following property: given B a bounded subset of X, there exists $t_0 = t_0(B) \geq 0$ such that $\pi(t)B \subset B_0$ for all $t \geq t_0$. The set B_0 is called an absorbing set.

Remark 1.55. There are different notions of this concept and others related, weakening some properties.

Definition 1.56. We say that a semigroup is asymptotically compact if given two sequences $\{t_n\}_n$ and $\{x_n\}_n$, with $t_n \longrightarrow \infty$ and $\{x_n\}_n$ bounded, then the sequence $\{\pi(t_n)x_n\}_n$ has a convergent subsequence.

With these two previous definitions, we can characterize semigroups which have global attractors.

Theorem 1.57. *A semigroup $\{\pi(t) : t \geq 0\}$ has a global attractor if and only if it is asymptotically compact and dissipative. Moreover, the global attractor is the ω-limit set of the absorbing set.*

To end this section, we consider the continuity of the attractors under perturbations of the semigroup. This is the first level in the study of perturbations of attractors for dynamical systems.

Definition 1.58. Let $\{A_\eta\}_{\eta\in[0,1]}$ be a family of subsets of X. We say that this family is

- upper semicontinuous at $\eta = 0$ if $\lim\limits_{\eta\to 0} d_{\mathrm{H}}(A_\eta, A_0) = 0$,
- lower semicontinuous at $\eta = 0$ if $\lim\limits_{\eta\to 0} d_{\mathrm{H}}(A_0, A_\eta) = 0$.

The upper and lower semicontinuity can be characterized in terms of sequences, which helps to prove several results.

Proposition 1.59. *Let $\{A_\eta\}_{\eta\in[0,1]}$ be a family of subsets of X. Then*

1. *$\{A_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$ if and only if for any sequence $\{\eta_n\}_n$ convergent to 0, any sequence $\{x_n\}_n$ with $x_n \in A_{\eta_n}$ has a convergent subsequence with limit in A_0,*
2. *$\{A_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$ if and only if for any sequence $\{\eta_n\}_n$ convergent to 0 and any $x_0 \in A_0$, there exists a sequence $\{x_n\}_n$, with $x_n \in A_{\eta_n}$, such that $\{x_n\}_n$ converges to x_0.*

Definition 1.60. Let $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of semigroups. We say that this family is continuous at $\eta = 0$ if $\pi_\eta(t)x \longrightarrow \pi_0(t)x$ as $\eta \longrightarrow 0$ uniformly in compact subsets of $[0,\infty) \times X$.

The next result gives us conditions under which we have upper semicontinuity of the global attractor.

Theorem 1.61. *Let $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of semigroups which is continuous at $\eta = 0$. Suppose that, for each $\eta \in [0,1]$, $\{\pi_\eta(t) : t \geq 0\}$ has a global attractor A_η, and the set*

$$\overline{\bigcup_{\eta\in[0,1]} A_\eta}$$

is compact. Then the family $\{A_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$.

Lower semicontinuity is not as common as the upper semicontinuity, and it is harder to verify in applications. It requires a finer study of local structures.

Definition 1.62. Let A be a π-invariant bounded set and $\delta > 0$. The unstable set of A is defined as

$$W^u(A) := \Big\{x \in X : \text{there exists a global solution } \psi \text{ through } x, \\ \text{and } \lim_{t\to-\infty} d(\psi(t), A) = 0\Big\},$$

and the δ-local unstable set of A is defined as

$$W^u_\delta(A) := \Big\{x \in X : \text{there exists a global solution } \psi \text{ through } x, \\ d(\psi(t), A) < \delta \text{ for all } t \geq 0, \text{ and } \lim_{t\to-\infty} d(\psi(t), A) = 0\Big\}.$$

Before giving an example of a result guaranteeing lower semicontinuity, we need to consider the set $\mathcal{E}_\eta$ as the set of stationary solutions (constant global solutions) of $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$.

Theorem 1.63. *Let $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of semigroups, continuous at $\eta = 0$, and $\{\pi_\eta(t) : t \geq 0\}$ has a global attractor A_η for each $\eta \in [0,1]$. Assume furthermore*

1. *There exists $p \in \mathbb{N}$ such that $\{x_1^{*,\eta}, \dots, x_p^{*,\eta}\} \subset \mathcal{E}_\eta$ for all $\eta \in [0,1]$.*
2. *The global attractor A_0 satisfies*

$$A_0 = \bigcup_{j=1}^{p} W^u\left(x_j^{*,0}\right).$$

3. *There exists $\delta > 0$ such that the family $\left\{W^u_\delta\left(x_j^{*,\eta}\right)\right\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$ for each $j \in \{1, \dots, p\}$.*

Then the family $\{A_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$.

1.5 Nonautonomous dynamical systems

There are different ways to introduce the concept of nonautonomous dynamical systems. We present the two most typical in this section: the skew product flow formulation and evolution processes. We will mainly focus on the evolution process formulation in this manuscript. The principal objective is to extend the concept of global attractor to the nonautonomous framework. Some initial and important results can be seen in [42] with uniform attractors. In this section, we will mainly consider the notion of pullback attractor, which is another possible generalization for evolution processes of the notion of global attractor seen in Section 1.4. We will state some results in order to obtain a characterization of the evolution processes which have a pullback attractor. Most of the results from this chapter come from [37].

Definition 1.64. Let X and Σ be two complete metric spaces and θ a semigroup in Σ, that is, $\theta = \{\theta_t : t \geq 0\}$. A cocycle is a map $\varphi\colon [0, +\infty) \times \Sigma \times X \longrightarrow X$ which satisfies

1. $\varphi(0, \sigma)x = x$ for all $x \in X$ and $\sigma \in \Sigma$,
2. $\varphi(t + s, \sigma) = \varphi(t, \theta_s\sigma)\varphi(s, \sigma)$ for all $t, s \geq 0$ and $\sigma \in \Sigma$,
3. the map $(t, \sigma, x) \longmapsto \varphi(t, \sigma)x$ is continuous.

In this context, the second condition is called cocycle property, and the semigroup θ is called the driving semigroup of φ.

From the previous definition, we can define an associated autonomous dynamical system on $\mathbb{X} := X \times \Sigma$. The semigroup is given by

$$\Pi(t)(x, \sigma) := (\varphi(t, \sigma)x, \theta_t\sigma).$$

It is called the skew product semiflow.

Definition 1.65. A family of sets $\hat{A} = \{A(\sigma)\}_{\sigma\in\Sigma}$ with $A(\sigma) \subset X$ for all $\sigma \in \Sigma$ is called a nonautonomous set. We say that it is open/closed/compact if $A(\sigma)$ is open/closed/compact for each $\sigma \in \Sigma$.

Definition 1.66. Let φ be a cocycle with θ as its driving semigroup, and $\hat{A}$ a nonautonomous set. We say that $\hat{A}$ is positively (negatively) φ-invariant if $\varphi(t, \sigma)A(\sigma) \subset (\supset)A(\theta_t\sigma)$ for all $t \geq 0$ and $\sigma \in \Sigma$.

There are different notions of "attractors" for nonautonomous dynamical systems. We have, for example, the following dynamical systems:

1. the driving semigroup θ,
2. the skew product semiflow Π,
3. the nonautonomous dynamical system given by the cocycle,
4. the evolution process given by $U_\sigma(t, s)x = \varphi(t - s, \theta_s\sigma)x$ for each $\sigma \in \Sigma$.

We can consider the following attractors:

1. the global attractor for the driving semigroup,

2. the global attractor for the skew-product semiflow,

3. the cocycle attractor for the cocycle,

4. the pullback attractor for the evolution process,

5. the uniform attractor.

More information on these notions and others related can be found, for example, in [24, 25, 38, 39, 41, 42, 90].

We will focus next on evolution processes. It is a particular case of the previous formulation, just taking $\Sigma = \mathbb{R}$ and $\theta_t s = t + s$. First, we will denote $\mathcal{P} = \{(t, s) \in \mathbb{R}^2 : t \geq s\}$. Check [30, 37, 85] for more information.

Definition 1.67. An evolution process in X is a family of continuous maps $\mathcal{U} = \{U(t, s) : (t, s) \in \mathcal{P}\}$, which satisfies the following conditions:

1. $U(t, t)x = x$ for all $x \in X$ and $t \in \mathbb{R}$,

2. $U(t, s) = U(t, \tau)U(\tau, s)$ for all $t \geq \tau \geq s$,

3. the map $(t, s, x) \longmapsto U(t, s)x$ is continuous from $\mathcal{P} \times X$ to X.

For any $(t, s) \in \mathcal{P}$, the operator $U(t, s)$ takes a state $x \in X$ at the initial time s, and it gives us the state $U(t, s)x$ at the final time t. Under appropriate and suitable conditions, the solutions of a nonautonomous differential equation $x'(t) = f(t, x(t))$ generate an evolution process.

The same way as before, we can define the notion of nonautonomous set.

Definition 1.68. A family of sets $\hat{A} = \{A(t)\}_{t\in\mathbb{R}}$ with $A(t) \subset X$ for all $t \in \mathbb{R}$ is called a nonautonomous set. We say that it is nonempty/open/closed/compact if $A(t)$ is nonempty/open/closed/compact for each $t \in \mathbb{R}$.

Definition 1.69. A collection $\mathfrak{D}$ of nonautonomous sets is called a universe in X if every nonautonomous set in $\mathfrak{D}$ is nonempty and $\mathfrak{D}$ is inclusion-closed, that is, if $\hat{A} \in \mathfrak{D}$ and $B(t) \subset A(t)$ for all $t \in \mathbb{R}$, then $\hat{B} \in \mathfrak{D}$.

From now on, let $\mathcal{U}$ be an evolution process and $\mathfrak{D}$ a universe.

Definition 1.70. Let $\hat{A}$ be a nonautonomous set. We say that $\hat{A}$ is

- positively $\mathcal{U}$-invariant if $U(t, s)A(s) \subset A(t)$ for all $t \geq s$,

- negatively $\mathcal{U}$-invariant if $A(t) \subset U(t, s)A(s)$ for all $t \geq s$,

- $\mathcal{U}$-invariant if $U(t, s)A(s) = A(t)$ for all $t \geq s$.

Definition 1.71. Let $\hat{A}$ and $\hat{B}$ be two nonautonomous sets. We will say that $\hat{A}$ pullback $\mathcal{U}$-attracts $\hat{B}$ if

$$\lim_{s\to-\infty} d_{\mathrm{H}}(U(t,s)B(s), A(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Note that the final time is fixed, and the initial time goes to $-\infty$. It is not the same as "going back in time".

Definition 1.72. Let $\hat{A}$ be a nonautonomous set and $t \in \mathbb{R}$. The pullback ω-limit set of $\hat{A}$ at time t is defined by

$$\omega(\hat{A}, t) := \bigcap_{\tau\leq t} \overline{\bigcup_{s\leq\tau} U(t,s)A(s)}.$$

The pullback ω-limit of $\hat{A}$ is the family $\omega(\hat{A}) := \{\omega(\hat{A}, t)\}_{t\in\mathbb{R}}$

The same way it happened for the autonomous case, we have a characterization by sequences.

Proposition 1.73. *Let $\hat{A}$ be a nonautonomous set and $t \in \mathbb{R}$. Then $x \in \omega(\hat{A}, t)$ if and only if there exist sequences $\{s_n\}_n$ and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$ and $x_n \in A(s_n)$, such that $U(t, s_n)x_n \longrightarrow x$.*

Definition 1.74. A function $\psi\colon \mathbb{R} \longrightarrow X$ is called a global solution of $\mathcal{U}$ if $U(t,s)\psi(s) = \psi(t)$ for all $t \geq s$.

Definition 1.75. A nonautonomous set $\hat{A}$ is called the pullback $\mathfrak{D}$-attractor if it satisfies:

(1) $\hat{A}$ is a compact family,

(2) $\hat{A}$ is $\mathcal{U}$-invariant,

(3) $\hat{A}$ pullback $\mathcal{U}$-attracts every element $\hat{D}$ of $\mathfrak{D}$,

(4) $\hat{A}$ is the minimal closed family satisfying (3), that is, if $\hat{B}$ is a closed family that pullback $\mathcal{U}$-attracts each family of $\mathfrak{D}$, then $\hat{A} \subset \hat{B}$.

The last condition is used in order to get the uniqueness of the pullback attractor. Another different approach to get the uniqueness is to assume that $\hat{A} \in \mathfrak{D}$.

Example 1.76. Consider the following ordinary differential equation

$$x'(t) = -x(t) + t + 1. \tag{1.4}$$

For any initial data $x(s) = x_s$, we have that the solution is given by

$$x(t) = (x_s - s)e^{-(t-s)} + t.$$

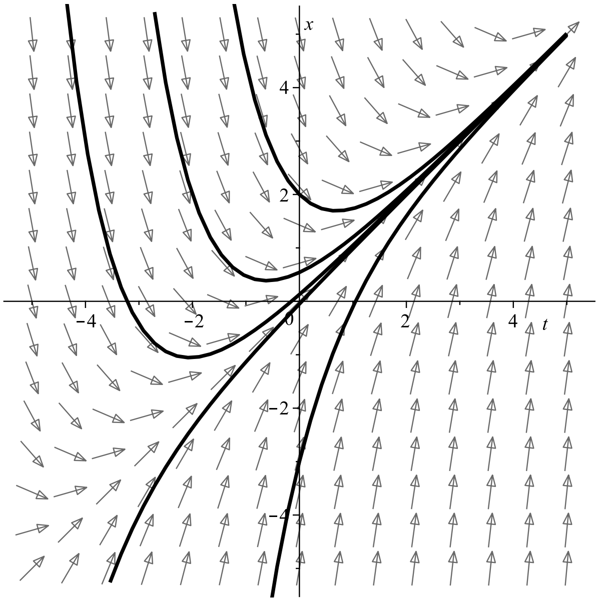


Figure 1.1: Phase portrait and some solutions of the differential equation (1.4) in Example 1.76.

If we fix s and x_s, we have that

$$\lim_{t\to\infty} x(t) = +\infty,$$

which implies that there is not a bounded attracting set as t goes to ∞. However, taking two different initial data x_s and y_s starting at the same time s, and x and y the solutions, we have that

$$x(t) - y(t) = (x_s - y_s)e^{-(t-s)},$$

and taking the limit as t goes to ∞ we get 0. This implies that two different solutions get closer and closer. The pullback attractor helps us understand this behavior. In Figure 1.1 we see a phase portrait and some solutions.

The pullback attractor is not the only mathematical object that is studied in order to obtain results about the asymptotic dynamics. In some situations, it can be useful, but it also has some disadvantages.

Definition 1.77. We say that an evolution process $\mathcal{U}$ is pullback $\mathfrak{D}$-dissipative if there exists $\hat{B}_0 \in \mathfrak{D}$ such that, for any $\hat{D} \in \mathfrak{D}$ and any $t \in \mathbb{R}$, there exists

$s_0 = s_0(\hat{D}, t) \leq t$ such that

$$s \leq s_0 \Longrightarrow U(t,s)D(s) \subset B_0(t).$$

The nonautonomous set $\hat{B}_0$ is called the pullback $\mathfrak{D}$-absorbing family.

Definition 1.78. We say that an evolution process $\mathcal{U}$ is pullback $\mathfrak{D}$-asymptotically compact if given $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and two sequences $\{s_n\}_n$ and $\{x_n\}_n$ with $s_n \longrightarrow -\infty$ and $x_n \in D(s_n)$, then the sequence $\{U(t, s_n)x_n\}_n$ has a convergent subsequence.

Theorem 1.79. *An evolution process $\mathcal{U}$ has a pullback $\mathfrak{D}$-attractor if and only if it is pullback $\mathfrak{D}$-asymptotically compact and pullback $\mathfrak{D}$-dissipative. Moreover, the pullback $\mathfrak{D}$-attractor $\hat{A}$ is given by $A(t) = \omega(\hat{B}_0, t)$ for each $t \in \mathbb{R}$, with $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family for $\mathcal{U}$.*

Definition 1.80. Let $\{\mathcal{U}_\eta\}_{\eta\in[0,1]}$ be a family of evolution processes. We say that this family is continuous at $\eta = 0$ if $U_\eta(t,s)x \longrightarrow U_0(t,s)x$ as $\eta \longrightarrow 0$, uniformly for (t,s,x) in compact subsets of $\mathcal{P} \times X$.

Definition 1.81. Let $\{\hat{A}_\eta\}_{\eta\in[0,1]}$ be a family of nonautonomous sets. We say that $\{\hat{A}_\eta\}_{\eta\in[0,1]}$ is

- upper semicontinuous at $\eta = 0$ if

$$\lim_{\eta\to 0} d_{\mathrm{H}}(A_\eta(t), A_0(t)) = 0 \quad \text{for all } t \in \mathbb{R},$$

- lower semicontinuous at $\eta = 0$ if

$$\lim_{\eta\to 0} d_{\mathrm{H}}(A_0(t), A_\eta(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Definition 1.82. Let $\hat{A}$ be a $\mathcal{U}$-invariant family, $\delta > 0$, and $t \in \mathbb{R}$. The unstable set of $\hat{A}$ at time t is defined as

$$W^u(\hat{A})(t) := \Big\{x \in X : \text{ there exists a global solution } \psi, \\ \psi(t) = x, \text{ and } \lim_{s\to-\infty} d(\psi(s), A(s)) = 0\Big\},$$

and the δ-local unstable set of $\hat{A}$ at time t is defined as

$$W^u_\delta(\hat{A})(t) := \Big\{x \in X : \text{ there exists a global solution } \psi, \ \psi(t) = x, \\ d(\psi(s), A) < \delta \text{ for all } s \leq t, \text{ and } \lim_{s\to-\infty} d(\psi(s), A(s)) = 0\Big\}.$$

Chapter 2

Impulsive Differential Equations

In this chapter, we focus on impulsive differential equations. We will consider different boundary value problems which are under the influence of impulsive action, and we will obtain several results regarding the existence of solutions, using different techniques seen in Chapter 1.

The chapter is organized as follows. We start with an introduction in Section 2.1. This section includes a description of these types of problems, some different classes, as well as some classical results regarding the existence of local solutions for impulsive differential equations. Then we focus on the two main classes which will be considered: impulses at fixed times and impulses at variable times. We highlight some similarities as well as some differences. Furthermore, we provide some definitions and results which will be used through the chapter. In Section 2.2 we focus our attention into impulsive boundary value problems with impulses at fixed times. First, we consider a class of first order problems with singularities, and we obtain the existence of periodic solutions. Then, we consider different classes of second order problems. Using critical point theory and fixed point theorems in both finite and infinite dimensions, we are able to get some existence results. Finally, in Section 2.3 we consider the case with impulses at variable times. In general, this situation is much more complicated. We obtain some existence results for periodic solutions of different classes of first and second order problems. The different existence results which were proved are illustrated with several examples through the chapter.

2.1 Introduction

Consider an evolution process described by

1. a differential equation
$$\frac{dx}{dt} = f(t, x), \tag{2.1}$$
with $f\colon \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ and Ω an open subset of $\mathbb{R}^n$,

2. two subsets M_t and N_t of Ω for each $t \in \mathbb{R}$,

3. an application $\mathcal{A}_t\colon M_t \longrightarrow N_t$ for each $t \in \mathbb{R}$.

As notation, let $x(t;t_0,x_0)$ be a solution of the differential equation (2.1) with initial data the point (t_0,x_0), that is, $x(t_0;t_0,x_0)=x_0$. The process would work as follows: Take an initial time t_0 with value x_0. Consider $x(t)=x(t;t_0,x_0)$. The process follows the curve $\{(t,x(t))\}$ until a time $t=t_1$, with $t_1>t_0$. On that time t_1, the point $(t,x(t))$ meets the set M_t for the first time. Then the point $(t_1,x(t_1))$ is transferred by the operator $\mathcal{A}_{t_1}$ to the point (t_1,x_1^+), with $x_1^+=\mathcal{A}_{t_1}x(t_1)$. The point x_1^+ belongs to N_{t_1}. Following that transfer, the process follows the curve $\{(t,x(t;t_1,x_1^+)\}$ until it meets again with M_t at a time $t_2>t_1$, when it will be transferred by $\mathcal{A}_{t_2}$ to a point x_2^+. The process follows this way recursively.

The instants of time t_j at which the integral curve of the differential equation (2.1) hits the sets M_t will be called the moments or times of impulsive effect.

For simplicity in the notation, we will use

$$x(t^+)=\lim_{s\to t^+}x(s),\qquad x(t^-)=\lim_{s\to t^-}x(s),\qquad \Delta x(t)=x(t^+)-x(t^-),$$

and we will assume that solutions are left-continuous, that is, $x(t)=x(t^-)$. We will consider the case where the impulsive effect occurs when a certain space-time relation like $\phi(t,x)=0$ is satisfied. Therefore, we could write the process as

$$\begin{cases} x'(t)=f(t,x(t)), & \phi(t,x(t))\neq 0,\\ \Delta x(t)=I(t,x(t)), & \phi(t,x(t))=0, \end{cases} \tag{2.2}$$

with $f\colon\mathbb{R}\times\Omega\longrightarrow\mathbb{R}^n$, $\phi\colon\mathbb{R}\times\Omega\longrightarrow\mathbb{R}$, and $I\colon\mathbb{R}\times\Omega\longrightarrow\Omega$.

In this particular case, for a fixed t we have

$$M_t=\{x\in\Omega:\phi(t,x)=0\},\qquad N_t=\Omega,$$
$$\mathcal{A}_t\colon x\in M_t\longrightarrow x+I(t,x)\in N_t.$$

We will assume that the equation $\phi(t,x)=0$ can be solved with respect to the variable t. Moreover, we will also assume that we have a countable number of solutions if we are working on an infinite interval of $\mathbb{R}$, and that we have a finite number of solutions if we are working on a finite interval. Therefore, our problems will be of the form

$$\begin{cases} x'(t)=f(t,x(t)), & t\neq\tau_j(x(t)),\\ \Delta x(t)=I_j(x(t)), & t=\tau_j(x(t)), \end{cases} \tag{2.3}$$

with $t=\tau_j(x)$ denoting the solutions of $\phi(t,x)=0$ and $\tau_j(x)<\tau_{j+1}(x)$ for every element x. We will denote by $\sigma_j=\{(t,x)\in\mathbb{R}\times\mathbb{R}^n : t=\tau_j(x)\}$.

Definition 2.1. A function $x\colon(t_0,t_0+\alpha)\longrightarrow\mathbb{R}^n$, with $\alpha>0$, is a (local) solution of (2.3) with initial condition $x(t_0^+)=x_0$ if

1. $x(t_0^+) = x_0$,

2. x is continuously differentiable and satisfies the differential equation $x'(t) = f(t, x(t))$ for $t \in (t_0, t_0 + \alpha)$ and $t \neq \tau_j(x(t))$,

3. if $t \in (t_0, t_0 + \alpha)$ and $t = \tau_j(x(t))$, then $x(t^+) = x(t) + I_j(x(t))$, x is left continuous at t, and there exists $\delta > 0$ such that if $s \in (t, t + \delta)$ then $s \neq \tau_k(x(s))$ for any k.

Note that t_0 could be equal to $\tau_j(x_0)$ for some j. If the function f is continuous, we have that the ordinary differential equation $x'(t) = f(t, x(t))$ has at least one solution. However, that is not the case here. Some additional conditions on τ_j and f are required. We write two theorems which guarantee the existence of a local solution.

Theorem 2.2. *Suppose that*

1. *f is continuous at $t \neq \tau_j(x)$, and, for each (t, x), there exists $g \in L^1_{loc}$ such that*

$$|f(s, y)| \leq g(s)$$

in a neighborhood of (t, x),

2. *for any $(\bar{t}, \bar{x})$ such that $\bar{t} = \tau_j(\bar{x})$, there exists $\delta > 0$ such that*

$$\left[t \in (\bar{t}, \bar{t} + \delta),\ |x - \bar{x}| < \delta\right] \Longrightarrow t \neq \tau_j(x).$$

Then for any (t_0, x_0), there exists $x\colon [t_0, t_0 + \alpha) \longrightarrow \mathbb{R}^n$ a local solution of (2.3) *with initial condition $x(t_0^+) = x_0$.*

Theorem 2.3. *Suppose that*

1. *the function f is continuous,*

2. *the functions τ_j are differentiable,*

3. *for any $(\bar{t}, \bar{x})$ such that $\bar{t} = \tau_j(\bar{x})$, there exists $\delta > 0$ such that*

$$\left[t \in (\bar{t}, \bar{t} + \delta),\ |x - \bar{x}| < \delta\right] \Longrightarrow D\tau_j(x) \cdot f(t, x) \neq 1.$$

Then for any (t_0, x_0), there exists $x\colon [t_0, t_0 + \alpha) \longrightarrow \mathbb{R}^n$ a local solution of (2.3) *with initial condition $x(t_0^+) = x_0$.*

These previous cases and results are given in a general framework. We will consider two main cases, depending on the functions τ_j: when all the functions τ_j are constants functions, and when at least one function τ_j is not constant.

If all the functions τ_j are constant, then we have differential equations with fixed moments of impulsive effect. These equations can be written as

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq t_j, \\ \Delta x(t) = I_j(x(t)), & t = t_j. \end{cases} \tag{2.4}$$

The functions τ_j in this case are given as $\tau_j(x) = t_j$ for every x. This implies that the moments of impulsive effect are fixed, and they can be given as a set $\{t_j\}$. Therefore, we know that the discontinuities of the solutions are going to be at $\{t_j\}$.

On the other hand, we have equations with impulses at variable times. This situation is more difficult than the previous one. The solutions are once again piecewise continuous functions, but the points of discontinuity depend on the solution, that is, two different solutions could have different times of discontinuity.

The main differences in the study of the two previous cases are the following:

- Space of functions where to look for a solution.
- Number of intersection points, that is, the number of solutions of $t = \tau_j(x)$.
- Noncontinuity of the intersection points.

We will explain these differences with more detail later.

Impulses at fixed times

Let $a, b \in \mathbb{R}$, $a < b$, and consider the interval $I = [a, b]$. Take $\{t_1, \ldots, t_q\} \subset [a, b]$, and we may suppose that $a = t_0 < t_1 < \cdots < t_q < t_{q+1} = b$. We define the spaces

$$\begin{aligned} \mathcal{PC}(a, b; \mathbb{R}^n; t_1, \ldots, t_q) = \{x \colon [a, b] \longrightarrow \mathbb{R}^n \mid\ & x \text{ is continuous} \\ & \text{for every } t \in [a, b] \text{ except } t_j, \text{ there exist} \\ & x(t_j^+),\ x(t_j^-), \text{ and } x(t_j) = x(t_j^-)\}, \end{aligned} \tag{2.5}$$

and, for $k \in \mathbb{N}$, the space

$$\begin{aligned} \mathcal{PC}^k(a, b; \mathbb{R}^n; t_1, \ldots, t_q) = \{x \colon [a, b] \longrightarrow \mathbb{R}^n \mid x^{(k)}(t) \text{ exists for all } t \in [a, b] \\ \text{except } t_j, \text{ there exists } x^{(k)}(t_j^+), x^{(k)}(t_j^-), \text{ and } x^{(k)}(t_j) = x^{(k)}(t_j^-)\}. \end{aligned}$$

They are Banach spaces with the supremum norm of the function (and its derivatives). A solution of (2.4) is a continuous function except on the times t_j, therefore we have that any solution belongs to spaces like the one considered in (2.5) (with different parameters). These spaces are the most usual space where to look for solutions. However, other spaces can be considered as well, like regulated functions, which will be explained later in more detail. Moreover, we have that each solution has a finite number of discontinuity points, which are $t_1, \ldots, t_q$.

We will state a couple of results which will be used in Section 2.2. We recall a version of the well-known Ascoli–Arzelà theorem for piecewise continuous functions. As in the continuous case, we need a notion of "equicontinuity" for this situation, in this case it is called (uniform) quasi equicontinuity.

Definition 2.4. Let S be a subset of $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$. We say that S is quasi-equicontinuous in $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x \in S$, $s,t \in (t_j, t_{j+1}]$ for some $j \in \{0,\ldots,q\}$, and $|s-t| < \delta$, then $|x(t) - x(s)| < \varepsilon$.

The notion of quasi equicontinuity of a subset of $\mathcal{PC}^k(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$ is analogous, the derivatives of x have to be considered as well.

Relatively compact sets in $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$ can be characterized by the following result of Ascoli–Arzelà-type.

Theorem 2.5. *Let S be a subset of $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$. S is relatively compact in $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$ if and only if S is bounded and quasi-equicontinuous in $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\ldots,t_q)$.*

Integral and differential inequalities play a very important role in the study of solutions of differential equations (both qualitative and quantitative). Some results about this topic and some generalizations can be found on [61]. The same could be said for impulsive integral and differential inequalities. The most well-known integral inequality is Grönwall's inequality, which is stated as follows:

Lemma 2.6. *Let $C \geq 0$ and $g, x \colon I \longrightarrow \mathbb{R}$ two nonnegative and continuous functions. If*

$$x(t) \leq C + \int_a^t g(s)x(s)\,ds, \quad t \in I,$$

then the following inequality holds:

$$x(t) \leq C \exp\left\{\int_a^t g(s)\,ds\right\}, \quad t \in I.$$

This inequality has a simple extension to the impulsive case. For example, see [88] for more information.

Lemma 2.7. *Let $C, \lambda_1, \ldots, \lambda_q$ be nonnegative numbers, $g \colon I \longrightarrow \mathbb{R}$ a nonnegative continuous function, and $x \in \mathcal{PC}(a,b;\mathbb{R};t_1,\ldots,t_q)$ and nonnegative.*

If

$$x(t) \leq C + \int_a^t g(s)x(s)\,ds + \sum_{t_j<t} \lambda_j x(t_j), \quad t \in I,$$

then the following inequality holds:

$$x(t) \leq C \prod_{t_j<t} (\lambda_j + 1) \exp\left\{\int_a^t g(s)\,ds\right\}, \quad t \in I.$$

Impulses at variable times

These systems are more complicated, as it was explained before. We start by considering the following example, taken from [88].

Example 2.8. Consider the system

$$\begin{cases} x'(t) = 0, & t \neq \tau_j(x(t)), \\ \Delta x(t) = (x(t))^2 \operatorname{sign}(x(t)) - x(t), & t = \tau_j(x(t)), \end{cases} \tag{2.6}$$

with $\tau_j(x) = x + 6j$ for $|x| < 3$ and $j \in \mathbb{N} \cup \{0\}$. Consider σ_j the "hypersurfaces"

$$\sigma_j = \{(t, x) : t = \tau_j(x)\}.$$

Taking $t_0 = 0$ as the initial time, we have the following situations (see also Figure 2.1).

- If $x_0 = 7/2$, then $x(t) = 7/2$ is a solution such that $t \neq \tau_j(x(t))$ for any $t > 0$ and any j, therefore it has no discontinuities.
- If $x_0 = \sqrt{2}$, then the solution experiences two jumps.
- If $x_0 = 1/2$, then the solution touches every hypersurface σ_j once, therefore we have an infinite number of jumps, and the solution converges to 0 as t goes to ∞.
- If $x_0 = -1$, then the solutions intersects σ_1 infinitely many times, and does not get past $t = 6$.

With this example we see that, for the same impulsive system, depending on the initial condition, we can have no times of impulsive effect, a finite number, or an infinite number.

For the purposes of our work, we will give later some hypotheses in order to avoid the so-called "beating" phenomena, that is, when a solution intersect with the same σ more than once. But first, we introduce several spaces of functions.

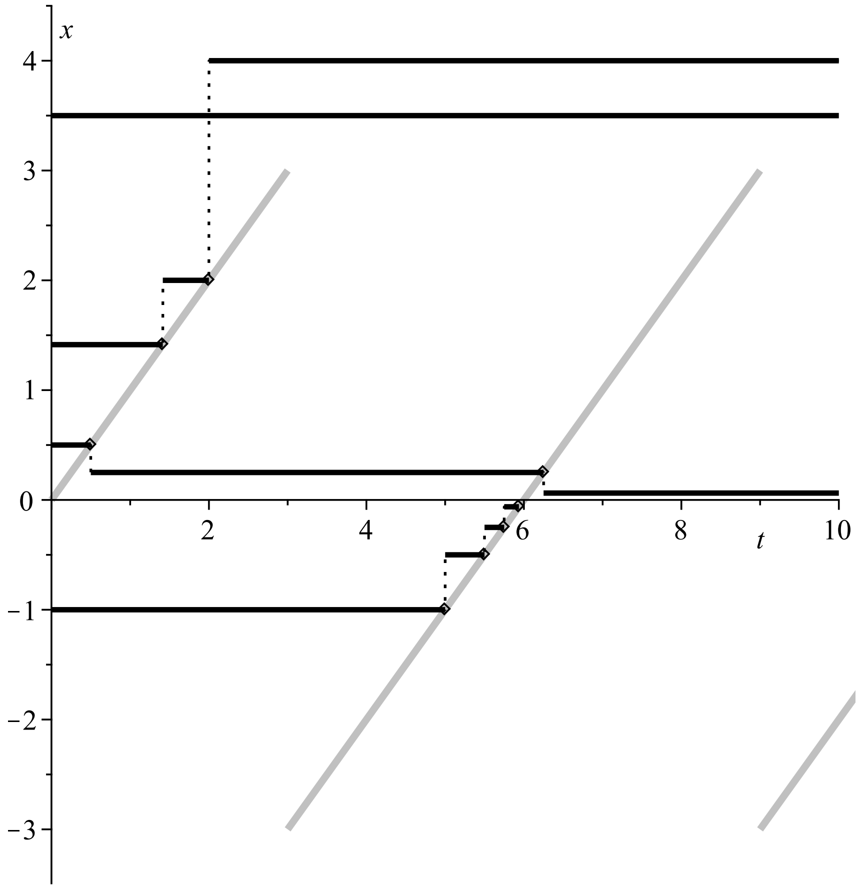


Figure 2.1: Some solutions of (2.6). The curves in gray represent σ_j and the curves in black represent the solutions of (2.6) with initial value x_0 equal to -1, $1/2$, $\sqrt{2}$, and $7/2$.

Definition 2.9. Let X be a Banach space and $x\colon [a,b] \longrightarrow X$ a function. We say that x is regulated if there exist $x(t^-)$ and $x(t^+)$ for every $t \in (a,b)$, and there exist $x(a^+)$ and $x(b^-)$.

We denote by $\operatorname{Reg}([a,b],X)$ the set of all regulated functions from $[a,b]$ to X. An equivalent condition for a function x to be regulated is the existence of a sequence of step functions which converge uniformly to x. The space $\operatorname{Reg}([a,b],X)$ is a Banach space by considering the supremum norm. Furthermore, as the domain of the functions is $[a,b]$, it can be proved that the set of discontinuities of any regulated function is, at most, countable.

The space $\mathcal{PC}(a,b;\mathbb{R}^n;t_1,\dots,t_q)$ is obviously a subset of $\operatorname{Reg}([a,b],\mathbb{R}^n)$. The space of right-continuous regulated functions has been used to study some impulsive functional differential equations (see for example [44–46]). Another interesting space of functions which can be useful are the functions of bounded variation.

In order to investigate neighborhoods of solutions, it can also be considered the following approach (see [2] for more information): let I_1 and I_2 be two intervals of $\mathbb{R}$ (finite or infinite), and $\theta_1 = \{\theta_i^1\}_i$ and $\theta_2 = \{\theta_i^2\}_i$ two strictly increasing sequences of real numbers, with the indices i ranging over a subset of $\mathbb{Z}$. Furthermore, suppose that if $j \in \{1,2\}$, then each θ_j is finite or infinite with $|\theta_i^j|$ going to ∞ as $|i|$ goes to ∞. Therefore, we have

Definition 2.10. Let $\varepsilon > 0$ and $x_j\colon I_j \longrightarrow \mathbb{R}^n$ be functions, $j \in \{1,2\}$, such that x_j is left continuous for all $t \in I_j$, continuous except on the elements of θ_j, and there exist $x_j((\theta_i^j)^+)$. We say that x_1 and x_2 are ε-equivalent if

1. the measure of the symmetric difference of I_1 and I_2 is less than ε, that is,

$$\operatorname{meas}((I_1 \setminus I_2) \cup (I_2 \setminus I_1)) < \varepsilon,$$

2. $|\theta_i^1 - \theta_i^2| < \varepsilon$ for every i,

3. if $t \in I_1 \cap I_2$ and $t \notin \widehat{[\theta_i^1,\theta_i^2]}$, then $\|x_1(t) - x_2(t)\| < \varepsilon$, where $\widehat{[a,b]}$ denotes the interval $[a,b]$ if $a \le b$ and the interval $[b,a]$ if $b < a$.

With this definition, it is possible to construct neighborhoods of a function.

Definition 2.11. Let x_1 and x_2 as in the previous definition and $\varepsilon > 0$. The function x_2 is said to be in the ε-neighborhood of x_1 if one of the following conditions hold:

- there exists an extension of x_2 to the interval I_1 such that this extension and x_1 are ε-equivalent,
- the restriction of x_2 to the interval I_1 and x_1 are ε-equivalent.

This definition implies that two ε-equivalent functions, when ε is small, have discontinuity points which are close. With these definitions, it is possible to construct a topology for the set of piecewise continuous functions.

Finally, in Section 2.3 we will use the space of functions $\mathcal{CJ}_q[a,b]$, first considered in [73]. We define

$$\mathcal{CJ}_q[a,b] := \mathcal{C}([a,b],\mathbb{R}^n) \times ((a,b)\times\mathbb{R}^n)^q.$$

It has the following interpretation: for each element $(x,(l_1,v_1),\dots,(l_q,v_q))$ in $\mathcal{CJ}_q[a,b]$, we consider the function $\tilde{x}$ given by

$$\tilde{x}(t) := \begin{cases} x(t), & a \le t \le l_{\pi(1)}, \\ x(t) + v_{\pi(1)} + \cdots + v_{\pi(j)}, & l_{\pi(j)} < t \le l_{\pi(j+1)},\ j \in \{1,\dots,q-1\}, \\ x(t) + v_{\pi(1)} + \cdots + v_{\pi(q)}, & l_{\pi(q)} < t \le b, \end{cases}$$

with π a permutation of $\{1,\dots,q\}$ such that $l_{\pi(j)} \le l_{\pi(j+1)}$. There is a correspondence between functions defined in $[a,b]$ which have q jumps and elements of $\mathcal{CJ}_q[a,b]$.

We explain several assumptions in order to avoid “beating” phenomena. We will consider the general system

$$\begin{cases} x'(t) = f(t,x(t)), & t \ne \tau_j(x(t)), \\ \Delta x(t) = I_j(x(t)), & t = \tau_j(x(t)). \end{cases}$$

We will assume that

(A1) $f\colon [a,b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and locally Lipschitz in the second variable, and all the solutions of $x'(t) = f(t,x(t))$ exist for all $t \in [a,b]$.

(A2) $I_j\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function for each $j \in \{1,\dots,q\}$.

(A3) $\tau_j\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, $\tau_j \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R})$, and $\tau_j(x) \in (a,b)$ for all $x\in\mathbb{R}^n$ for each $j \in \{1,\dots,q\}$.

Under these general hypotheses (A1)–(A3), the equations

$$t = \tau_j(x(t)) \tag{2.7}$$

could still have more than one solution. We impose the following additional conditions to avoid this situation.

(A4) Assume that the functions τ_j satisfy:

$$\tau_j(x) < \tau_{j+1}(x) \text{ for } j \in \{1,\dots,q-1\},\ x\in\mathbb{R}^n,$$
$$\tau_j(x+I_j(x)) \le \tau_j(x) < \tau_{j+1}(x+I_j(x)) \text{ for } j \in \{1,\dots,q-1\},\ x\in\mathbb{R}^n,$$
$$\tau_q(x+I_q(x)) \le \tau_q(x) \text{ for } x\in\mathbb{R}^n,$$
$$\exists\, M>0:\ \|D\tau_j(x)\| \le M \text{ for } x\in\mathbb{R}^n.$$

(A5) There exists $\alpha < 1$ with $D\tau_j(x) \cdot f(t,x) \le \alpha$ for every $(t,x) \in [a,b] \times \mathbb{R}^n$ and $j \in \{1,\dots,q\}$.

We have the following result.

Theorem 2.12. *If* (A1)–(A5) *hold and* $x\colon [a,b] \longrightarrow \mathbb{R}^n$ *is a solution of* (2.3)*, then* x *meets each surface* σ_j *exactly once.*

More information on impulsive differential equations and some of its applications can be found in [1, 7, 88, 98, 108, 113, 119].

2.2 Impulses at fixed times

In this section, we are going to look for solutions for different impulsive boundary value problems with impulses at fixed times. For the whole section, T is going to be a positive number and $t_1,\dots,t_q$ are going to denote the impulsive times, with $0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = T$.

We start with the following first order problem with singularities

$$x'(t) = -\frac{1}{(x(t))^\alpha} + e(t) \tag{2.8}$$

with impulses

$$\Delta x(t_j) = I_j(x(t_j)),\ j \in \{1,\dots,q\}. \tag{2.9}$$

Here, α is a positive real number, the functions $e, I_j : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and e is also a periodic function with period T. Problems with singularities have been widely studied and have multiple applications [116]. In particular, this problem was considered in [86].

Our objective is to look for positive, T-periodic solutions of (2.8)–(2.9). The differential equation (2.8) could have no periodic solutions (it suffices to take $e \equiv 0$, so all solutions are decreasing). However, with the introduction of impulses, the situation could change.

We are going to reduce (2.8)–(2.9) to a boundary value problem. However, we will consider a slightly more general situation. We take

$$\begin{cases} x'(t) = -g(x(t)) + e(t), & t \ne t_j, \\ \Delta x(t_j) = I_j(x(t_j)), & j \in \{1,\dots,q\}, \\ x(0) = x(T), \end{cases} \tag{2.10}$$

with $g\colon (0,\infty) \longrightarrow (0,\infty)$ a continuous and bijective function such that

$$\lim_{x\to 0^+} g(x) = +\infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = 0.$$

We will use Theorem 1.41 in order to prove the existence of solutions for (2.10).

We consider X as the Banach space given by

$$X = \{x \in \mathcal{PC}(0,T;\mathbb{R};t_1,\dots,t_q) : x(0) = x(T)\}$$

with the supremum norm, and the Banach space $Z = X \times \mathbb{R}^q$ with the usual product norm, that is,

$$\|z\|_Z = \|(x,a_1,\dots,a_q)\|_Z = \|x\|_X + |(a_1,\dots,a_q)|,$$

with $|\cdot|$ a norm of $\mathbb{R}^q$. We will not write the subscript in $\|\cdot\|_X$ nor in $\|\cdot\|_Z$.

We also consider the linear map $L\colon \operatorname{Dom}(L) \subset X \longrightarrow Z$ and the nonlinear map $N\colon \operatorname{Dom}(N) \subset X \longrightarrow Z$ given by

$$\begin{aligned} Lx(t) &= (x'(t), \Delta x(t_1), \dots, \Delta x(t_q)), \\ Nx(t) &= (-g(x(t)) + e(t), I_1(x(t_1)), \dots, I_q(x(t_q)))\,. \end{aligned}$$

We have that L is a linear map, $\operatorname{Dom}(L)$ is a subset of $\mathcal{PC}^1(0,T;\mathbb{R};t_1,\dots,t_q)\cap X$, and

$$\operatorname{Dom}(N) = \{x \in X : \min\{x(t) : t \in [0,T]\} > 0\}.$$

Proposition 2.13. *The linear map L is a Fredholm map of index* 0.

Proof. We need to prove that $\operatorname{Ker}(L)$ and $\operatorname{coker}(L)$ have the same finite dimension and that $\operatorname{Im}(L)$ is a closed subset of Z. It is easy to prove that

$$\begin{aligned} \operatorname{Ker}(L) &= \{x \in X : \exists\, c \in \mathbb{R} \text{ with } x(t) = c \text{ for every } t \in [0,T]\}, \\ \operatorname{Im}(L) &= \left\{ (x,a_1,\dots,a_q) \in Z : \int_0^T x(t)\,dt + \sum_{j=1}^q a_j = 0 \right\}. \end{aligned}$$

This implies that $\dim(\operatorname{Ker}(L)) = 1$, $\operatorname{codim}(\operatorname{Im}(L)) = 1$, and that $\operatorname{Im}(L)$ is a closed subset of Z. □

As a consequence, there exist two continuous projectors $P\colon X \longrightarrow X$ and $Q\colon Z \longrightarrow Z$ such that the following sequence is exact,

$$X \xrightarrow{\;P\;} \operatorname{Dom}(L) \xrightarrow{\;L\;} Z \xrightarrow{\;Q\;} Z. \tag{2.11}$$

In this case, we can take, for any $t \in [0,T]$,

$$Px(t) = \frac{1}{T}\int_0^T x(s)\,ds \tag{2.12}$$

and

$$Q(x,a_1,\dots,a_q)(t)=\left(\frac{1}{T}\left(\int_0^T x(s)\,ds+\sum_{j=1}^{q}a_j\right),0,\dots,0\right). \qquad (2.13)$$

Note that the function Px and the first component of $Q(x,a_1,\dots,a_q)$ are constant functions.

Proposition 2.14. *The maps P and Q defined as* (2.12) *and* (2.13) *are continuous projectors, and the sequence* (2.11) *is exact.*

Proof. It is obvious that P and Q are both linear maps. Furthermore, we have

$$\|Px\|=\left|\frac{1}{T}\int_0^T x(s)\,ds\right|\le\frac{1}{T}\,\|x\|\,T=\|x\|$$

and

$$\begin{aligned}(P(Px))(t)&=\frac{1}{T}\int_0^T(Px)(s)\,ds=\frac{1}{T}\int_0^T\frac{1}{T}\int_0^T x(u)\,du\,ds\\&=\frac{1}{T}\int_0^T x(u)\,du=(Px)(t)\end{aligned}$$

for any $t\in[0,T]$, so P is continuous and $P^2=P$. We prove next that $\mathrm{Im}(P)=\mathrm{Ker}(L)$.

- Take $y\in\mathrm{Im}(P)$, then there exists $x\in X$ such that

$$y(t)=\frac{1}{T}\int_0^T x(s)\,ds.$$

  As y is a constant function, then $y'(t)=0$ for all $t\in[0,T]$ and $\Delta y(t_j)=0$ for $j\in\{1,\dots,q\}$. Therefore, $Ly=(0,0,\dots,0)$, so $y\in\mathrm{Ker}(L)$.

- Take $y\in\mathrm{Ker}(L)$. Then $y'(t)=0$ for all $t\in[0,T]$ and $\Delta y(t_j)=0$ for every $j\in\{1,\dots,q\}$. This implies that y is a constant function. But then $Py=y$, so $y\in\mathrm{Im}(P)$.

We consider the map Q. Then

$$\begin{aligned}\|Q(x,a_1,\dots,a_q)\|&=\left|\int_0^T x(s)\,ds+\frac{1}{T}\sum_{j=1}^{q}a_j\right|\le\|x\|+\frac{1}{T}\sum_{j=1}^{q}|a_j|\\&\le\|x\|+\frac{1}{T}\,\alpha\,|a|\le\tilde{\alpha}\|(x,a_1,\dots,a_q)\|,\end{aligned}$$

where α comes from the fact that every norm in $\mathbb{R}^q$ is equivalent. The first coordinate of $Q(Q(x, a_1, \ldots, a_q))$ is

$$\begin{aligned}\frac{1}{T}\int_0^T \left(\frac{1}{T}\int_0^T x(u)\,du + \frac{1}{T}\sum_{j=1}^{q} a_j\right) ds + \sum_{j=1}^{q} 0 \\ = \frac{1}{T}\int_0^T x(u)\,du + \frac{1}{T}\sum_{j=1}^{q} a_j,\end{aligned}$$

which is equal to the first coordinate of $Q(x, a_1, \ldots, a_q)$. This implies that Q is continuous and $Q^2 = Q$. Finally, we check that $\operatorname{Im}(L) = \operatorname{Ker}(Q)$.

- Take $(y, a_1, \ldots, a_q) \in \operatorname{Im}(L)$. This implies that there exists $x \in X$ such that $Lx = (y, a_1, \ldots, a_q)$. Therefore, $y(t) = x'(t)$ and $\Delta x(t_j) = a_j$. The first coordinate of QLx is

$$\begin{aligned}&\frac{1}{T}\int_0^T x'(u)\,du + \frac{1}{T}\sum_{j=1}^{q} \Delta x(t_j) \\ &= \frac{1}{T}\left(-x(0) + \sum_{j=1}^{q} -\Delta x(t_j) + x(T) + \sum_{j=1}^{q} \Delta x(t_j)\right) \\ &= \frac{x(T) - x(0)}{T} = 0.\end{aligned}$$

  This implies that $QLx = (0, 0, \ldots, 0)$, so $(y, a_1, \ldots, a_q) \in \operatorname{Ker}(Q)$.

- Take $(y, a_1, \ldots, a_q) \in \operatorname{Ker}(Q)$. Then we have that

$$\frac{1}{T}\int_0^T y(s)\,ds + \frac{1}{T}\sum_{j=1}^{q} a_j = 0.$$

  Consider the function x defined as

$$x(t) = \int_0^t y(s)\,ds + \sum_{t_j < t} a_j.$$

  Then $x'(t) = y(t)$ and $\Delta x(t_j) = a_j$ for $j \in \{1, \ldots, q\}$. Therefore, we have that $x \in \mathcal{PC}(0, T; \mathbb{R}; t_1, \ldots, t_q)$. Moreover, $x(0) = 0$ and

$$x(T) = \int_0^T y(s)\,ds + \sum_{j=1}^{q} a_j = 0.$$

  Therefore, $x \in X$, so $Lx = (y, a_1, \ldots, a_q)$. □

This implies that the map $L_{|\mathrm{Dom}(L)\cap\mathrm{Ker}(P)}\colon\, \mathrm{Dom}(L)\cap\mathrm{Ker}(P)\longrightarrow \mathrm{Im}(L)$ is invertible, so we can consider its inverse $K_P\colon\, \mathrm{Im}(L)\longrightarrow \mathrm{Dom}(L)\cap\mathrm{Ker}(P)$, which is given by

$$\begin{aligned}K_P(x,a_1,\dots,a_q)(t) = {}& \int_0^t x(s)\,ds + \sum_{0<t_j<t} a_j \\ & - \frac{1}{T}\int_0^T\int_0^s x(u)\,du\,ds - \sum_{j=1}^q a_j\frac{T-t_j}{T}.\end{aligned}$$

Proposition 2.15. *The map K_P is continuous.*

Proof. It is clear that K_P is a linear map. Fix $(x,a_1,\dots,a_q)$. If $t\in[0,T]$,

$$\begin{aligned}&|K_P(x,a_1,\dots,a_q)(t)| \\ &\quad\le \int_0^t |x(s)|\,ds + \sum_{j=1}^q |a_j| + \frac{1}{T}\int_0^T\int_0^s |x(u)|\,du\,ds + \sum_{j=1}^q |a_j|\frac{T-t_j}{T} \\ &\quad\le \|x\|t + \frac{1}{T}\int_0^T s\|x\|\,ds + 2\sum_{j=1}^q|a_j| \le \|x\|t + \frac{\|x\|T}{2} + 2\sum_{j=1}^q |a_j| \\ &\quad\le \frac{\|x\|3T}{2} + 2\sum_{j=1}^q|a_j| \le \frac{3T}{2}\|x\| + 2\alpha|a| \le \tilde{\alpha}\|(x,a_1,\dots,a_q)\|.\end{aligned}$$

This implies that $\|K_P(x,a_1,\dots,a_q)\|\le\tilde{\alpha}\|(x,a_1,\dots,a_q)\|$, so we can conclude that K_P is continuous. □

We are going to look for solutions of (2.10) in open and bounded sets such as

$$\Omega_{c_1}^{c_2} := \{x\in X : c_1 < \min\{x(t) : t\in[0,T]\} \le \max\{x(t) : t\in[0,T]\} < c_2\},$$

with $0<c_1<c_2$. In order to apply Theorem 1.41, we need to prove that N is an L-compact on $\Omega_{c_1}^{c_2}$.

Proposition 2.16. *The map N is L-compact on $\Omega_{c_1}^{c_2}$.*

Proof. For simplicity, we will denote $\Omega=\Omega_{c_1}^{c_2}$. We divide the proof into two steps.

Step 1: $N(\overline{\Omega})$ and $QN(\overline{\Omega})$ are bounded subsets of Z.

We have that $Nx(t) = (-g(x(t))+e(t), I_1(x(t_1)),\dots,I_q(x(t_q)))$, so

$$|-g(x(t))+e(t)| \le |e(t)| + |g(x(t))| \le C_0,$$

because $g|_{[c_1,c_2]}$ is continuous and e is continuous and periodic. Moreover, the functions $I_j|_{[c_1,c_2]}$ are bounded, so we have that there exists $C_j > 0$ such that

$$|I_j(x(t_j))| \leq C_j.$$

We can conclude that $N(\overline{\Omega})$ and $QN(\overline{\Omega})$ are both bounded.

Step 2: If $\widetilde{\Omega}$ is a bounded subset of Z, then $K_P(\widetilde{\Omega})$ is relatively compact.

We are going to prove that $K_P(\widetilde{\Omega})$ is relatively compact.

- $\|K_P(x, a_1, \ldots, a_q)\| \leq \tilde{\alpha}\|(x, a_1, \ldots, a_q)\| \leq M$, because $\widetilde{\Omega}$ is bounded.
- As $\widetilde{\Omega}$ is bounded, then there exists $c > 0$ such that

$$\|(x, a_1, \ldots, a_q)\| \leq c,$$

so $\|x\| \leq c$. Fix $\varepsilon > 0$ and take $\delta < \varepsilon/c$. If $y \in K_P(\widetilde{\Omega})$, then there exists $(x, a_1, \ldots, a_q)$ such that $y = K_P(x, a_1, \ldots, a_q)$. For any $t, s \in (t_j, t_{j+1})$ with $|t - s| < \delta$ and $j \in \{0, \ldots, q\}$, we have

$$|y(t) - y(s)| = \left| \int_s^t x(u)\, du \right| \leq \|x\| \cdot |t - s| \leq c \cdot |t - s| < \varepsilon.$$

This implies that $K_P(\widetilde{\Omega})$ is a quasi-equicontinuous set.

Therefore, $K_P(\widetilde{\Omega})$ is a relatively compact set, by Theorem 2.5. □

We introduce the following hypotheses.

(B1) There exist $m_j, M_j \in \mathbb{R}$ such that $m_j \leq I_j(s) \leq M_j$ for all $s > 0$.

(B2) $0 < (m_1 + \cdots + m_q) + \int_0^T e(t)\, dt.$

(B3) If $\widetilde{M_j} = \max\{|M_j|, |m_j|\}$, then

$$g^{-1}\left(\frac{M_1 + \cdots + M_q}{T} + \frac{1}{T} \int_0^T e(t)\, dt \right) > \int_0^T e(t)\, dt + \int_0^T |e(t)|\, dt + \sum_{j=1}^{q} (M_j + \widetilde{M_j}).$$

Lemma 2.17. *Suppose that Hypotheses* (B1)–(B3) *hold. Then there exist two constants A_1 and A_2 such that $0 < A_2 \leq x(t) \leq A_1$ for all $t \in [0, T]$, with $\lambda \in (0, 1)$ and x a solution of $Lx = \lambda Nx$. Moreover, A_1 and A_2 do not depend on λ.*

Proof. Let x be a positive solution of $Lx = \lambda Nx$ for some $\lambda \in (0,1)$. Then

$$\begin{cases} x'(t) = -\lambda g(x(t)) + \lambda e(t), & t \in [0,T],\, t \neq t_j, \\ \Delta x(t_j) = \lambda I_j(x(t_j)), & j \in \{1,\dots,q\}. \end{cases}$$

Integrating over $[0,T]$ we obtain

$$\int_0^T x'(t)\,dt = -\lambda \int_0^T g(x(t))\,dt + \lambda \int_0^T e(t)\,dt. \tag{2.14}$$

The first integral is

$$\begin{aligned} \int_0^T x'(t)\,dt &= \sum_{j=1}^{q+1} \int_{t_{j-1}^+}^{t_j^-} x'(t)\,dt = \sum_{j=1}^{q+1} x(t_j^-) - x(t_{j-1}^+) \\ &= -x(0) - \sum_{j=1}^{q} (x(t_j^+) - x(t_j^-)) + x(T) \qquad (2.15)\\ &= -\lambda \sum_{j=1}^{q} I_j(x(t_j)). \end{aligned}$$

From Equations (2.14) and (2.15) we obtain

$$\begin{aligned} -\lambda \sum_{j=1}^{q} I_j(x(t_j)) &= -\lambda \int_0^T g(x(t))\,dt + \lambda \int_0^T e(t)\,dt, \\ \implies \int_0^T g(x(t))\,dt &= \sum_{j=1}^{p} I_j(x(t_j)) + \int_0^T e(t)\,dt. \qquad (2.16) \end{aligned}$$

Therefore, (2.16) implies that

$$0 < (m_1 + \dots + m_q) + \int_0^T e(t)\,dt \le \int_0^T g(x(t))\,dt \le (M_1 + \ldots + M_q) + \int_0^T e(t)\,dt,$$

by (B1) and (B2). Therefore, we get by the mean value theorem that there exist $\xi, \eta \in [0,T] \setminus \{t_1,\dots,t_q\}$ such that

$$\begin{aligned} Tg(x(\xi)) &\le (M_1 + \dots + M_q) + \int_0^T e(t)\,dt, \\ Tg(x(\eta)) &\ge (m_1 + \dots + m_q) + \int_0^T e(t)\,dt. \end{aligned}$$

Moreover,

$$x(\xi) \geq g^{-1}\left(\frac{M_1 + \cdots + M_q}{T} + \frac{1}{T}\int_0^T e(t)\,dt\right), \tag{2.17}$$

$$x(\eta) \leq g^{-1}\left(\frac{m_1 + \cdots + m_q}{T} + \frac{1}{T}\int_0^T e(t)\,dt\right). \tag{2.18}$$

Then, from (B1) and (2.16), we obtain

$$\begin{aligned}
\int_0^T |x'(t)|\,dt &= \int_0^T \lambda\,|-g(x(t)) + e(t)|\;dt = \lambda\int_0^T |e(t) - g(x(t))|\;dt \\
&\leq \int_0^T [|e(t)| + g(x(t))]\;dt \leq \int_0^T |e(t)|\,dt + \int_0^T g(x(t))\,dt \\
&= \int_0^T |e(t)|\,dt + \sum_{j=1}^{q} I_j(x(t_j)) + \int_0^T e(t)\,dt \\
&\leq (M_1 + \cdots + M_q) + \int_0^T e(t)\,dt + \int_0^T |e(t)|\,dt. \qquad (2.19)
\end{aligned}$$

We have that, for $t, s \in [0,T] \setminus \{t_1, \ldots, t_q\}$,

$$\int_s^t x'(u)\,du = x(t) - x(s) - \sum_{s<t_j<t} \Delta x(t_j)$$

if $t > s$. A similar result holds if $s < t$.

On the one hand, take $t \in [0,T] \setminus \{t_1, \ldots, t_q\}$, and suppose that $\eta < t$. Equations (2.18) and (2.19) imply that

$$\begin{aligned}
&\int_\eta^t x'(u)\,du = x(t) - x(\eta) - \sum_{\eta<t_j<t} \Delta x(t_j) \\
&\Longrightarrow x(t) = x(\eta) + \sum_{\eta<t_j<t} \Delta x(t_j) + \int_\eta^t x'(u)\,du \\
&\Longrightarrow |x(t)| \leq |x(\eta)| + \sum_{j=1}^{q} |\Delta x(t_j)| + \int_0^T |x'(u)|\,du.
\end{aligned}$$

This implies that

$$|x(t)| \leq |x(\eta)| + \sum_{j=1}^{q} |\Delta x(t_j)| + \int_0^T |x'(u)|\,du$$

$$
\begin{aligned}
&\leq g^{-1}\left(\frac{m_1+\cdots+m_q}{T}+\frac{1}{T}\int_0^T e(t)\,dt\right)+\left(\widetilde{M}_1+\cdots+\widetilde{M}_q\right)\\
&\qquad +(M_1+\cdots+M_q)+\int_0^T e(t)\,dt+\int_0^T |e(t)|\,dt\\
&= A_1<\infty.
\end{aligned}
$$

We have a similar result if $t<\eta$.

On the other hand, take $t\in[0,T]\setminus\{t_1,\dots,t_q\}$, and suppose $\xi<t$. We have that Equations (2.17), (2.19), and (B3) imply that

$$
\begin{aligned}
&\int_\xi^t x'(u)\,du = x(t)-x(\xi)-\sum_{\xi<t_j<t}\Delta x(t_j)\\
&\qquad \Longrightarrow x(t)=x(\xi)+\sum_{\xi<t_j<t}\Delta x(t_j)+\int_\xi^t x'(u)\,du.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
|x(t)| &\geq |x(\xi)|-\sum_{j=1}^{q}|\Delta x(t_j)|-\int_0^T |x'(u)|\,du\\
&\geq g^{-1}\left(\frac{M_1+\cdots+M_q}{T}+\frac{1}{T}\int_0^T e(t)\,dt\right)-\left(\widetilde{M}_1+\cdots+\widetilde{M}_q\right)\\
&\qquad -(M_1+\cdots+M_q)-\int_0^T e(t)\,dt-\int_0^T |e(t)|\,dt\\
&= A_2>0.
\end{aligned}
$$

We have a similar result if $t<\xi$. □

Theorem 2.18. *Suppose that the Hypotheses* (B1)–(B3) *are satisfied. Then the boundary value problem with impulses* (2.10) *has at least one solution.*

Proof. For simplicity, we denote $\Omega=\Omega_{A_2-\sigma_2}^{A_1+\sigma_1}$, that is,

$$
\Omega=\{x\in X:\min\{x(t):t\in[0,T]\}>A_2-\sigma_2,\ A_2-\sigma_2<\|x\|<A_1+\sigma_1\},
$$

with $0<\sigma_2<A_2$ and $\sigma_1>0$, so the elements of Ω are positive functions. We have already proved that N is L-compact in Ω (Proposition 2.16). Moreover, we have that

$$
Lx=\lambda Nx \Longrightarrow A_2\leq x(t)\leq A_1 \ \text{ for every } t\in[0,T]\Longrightarrow x\notin\partial\Omega,
$$

from Lemma 2.17. We consider $J\colon (b,0,\dots,0) \in \operatorname{Im}(Q) \longrightarrow b \in \operatorname{Ker}(L)$ an isomorphism. We have to prove that $QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker}(L)$ and $\deg_B(JQN, \Omega \cap \operatorname{Ker}(L), 0) \neq 0$.

Take $x \in \operatorname{Ker}(L)$ with $QNx = 0$. We have to prove that $x \notin \partial\Omega$.

$$QNx = 0 \Longrightarrow \frac{1}{T}\int_0^T \left[-g(x(t)) + e(t)\right]\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(x(t_j)) = 0, \tag{2.20}$$

$$x \in \operatorname{Ker}(L) \Longrightarrow x \text{ constant} \Longrightarrow x(t) = x(0) \ \text{ for } t \in [0,T]. \tag{2.21}$$

Therefore, Equations (2.20) and (2.21) imply that

$$g(x(0)) = \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(x(0)).$$

Moreover,

$$\frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(m_1 + \cdots + m_q) \leq g(x(0)),$$
$$g(x(0)) \leq \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(M_1 + \cdots + M_q).$$

Using the function g^{-1} we get

$$g^{-1}\left(\frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(M_1 + \cdots + M_q)\right) \leq x(0),$$
$$x(0) \leq g^{-1}\left(\frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(m_1 + \cdots + m_q)\right).$$

This implies that $A_2 - \sigma_2 < A_2 \leq x(t) \leq A_1 < A_1 + \sigma_1$, so $x \notin \partial\Omega$.

We identify $\operatorname{Ker}(L) \cap \Omega$ with the interval $(A_2 - \sigma_2, A_1 + \sigma_1)$ of $\mathbb{R}$. This implies that

$$\deg_B(JQN, \Omega \cap \operatorname{Ker}(L), 0) = \deg_B(\varphi, (a,b), 0),$$

with $(a,b) = (A_2 - \sigma_2, A_1 + \sigma_1)$ and $\varphi\colon [a,b] \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = -g(x) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(x).$$

We have that

$$
\begin{aligned}
\varphi(a) &= -g(a) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}\sum_{j=1}^{q} I_j(a) \\
&\leq -g(a) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(M_1 + \cdots + M_q) \\
&= -g(A_2 - \sigma_2) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(M_1 + \cdots + M_q) \\
&< -g(A_2) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(M_1 + \cdots + M_q) \\
&< 0,
\end{aligned}
$$

where the last inequality follows from Lemma 2.17. Moreover,

$$
\begin{aligned}
\varphi(b) &= -g(b) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}\sum_{j=1}^{q} I_j(b) \\
&\geq -g(b) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(m_1 + \cdots + m_q) \\
&= -g(A_1 + \sigma_1) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(m_1 + \cdots + m_q) \\
&> -g(A_1) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}(m_1 + \cdots + m_q) \\
&> 0.
\end{aligned}
$$

As a consequence, we have $\deg_B(\varphi,(a,b),0) \neq 0$. We use Theorem 1.41, so the problem (2.10) has a positive T-periodic solution. □

The same ideas of this result can be applied to slightly more general impulsive boundary value problems. For example, the following extension can be considered.

$$
\begin{cases}
x'(t) = f(x(t)) + e(t), & t \neq t_j, \\
\Delta x(t_j) = I_j(x(t_j)), & j \in \{1, \dots, q\}, \\
x(0) = x(T),
\end{cases}
\tag{2.22}
$$

with $f\colon (0,\infty) \longrightarrow (a,b)$ continuous, $a \in [-\infty, +\infty)$, and $b \in (-\infty, +\infty]$. We point out that a can be $-\infty$ or a real number, and b can be $+\infty$ or finite.

It is obvious that (2.22) includes (2.10), it suffices to take $f(x) = -g(x)$, $a = -\infty$, and $b = 0$. The definitions of the spaces and operators used to prove this result are the same as before, with the obvious changes between f and g. Moreover, the proofs of their properties are analogous.

We introduce the following hypotheses:

(C1) $\lim_{s\to 0^+} f(s) = a^+$ and $\lim_{s\to\infty} f(s) = b^-$.

(C2) There exist $m_k, M_k \in \mathbb{R}$ such that $m_j \le I_j(s) \le M_j$ for every $s > 0$.

(C3) $c_1, c_2 \in (a, b)$, with

$$c_1 = \frac{-m_1 - \cdots - m_q}{T} - \frac{1}{T}\int_0^T e(t)\,dt,$$
$$c_2 = \frac{-M_1 - \cdots - M_q}{T} - \frac{1}{T}\int_0^T e(t)\,dt.$$

(C4) For $\widetilde{M_j} = \max\{|M_j|, |m_j|\}$ and $r_2 = \inf\{s > 0 : f(s) \ge c_2\}$, it holds that

$$r_2 - 2(\widetilde{M_1} + \cdots + \widetilde{M_q}) - \int_0^T e(t) + |e(t)|\,dt > 0.$$

Some examples of functions f with the previous conditions include

$$f(x) = x^\beta - \frac{1}{x^\alpha}, \quad \text{with } \alpha, \beta > 0,$$
$$f(x) = \log(x),$$
$$f(x) = x \sin\left(\frac{20}{x}\right) - \frac{1}{x},$$
$$f(x) = -\frac{1}{x^\alpha} - \frac{1}{x^\beta}, \quad \text{with } \alpha, \beta > 0.$$

We illustrate hypotheses (C1) and (C3) in Figure 2.2.

The following result is analogous to Lemma 2.17. We include the full proof for completeness.

Lemma 2.19. *Suppose that Hypotheses* (C1)–(C4) *are satisfied. Then there exist two positive constants* A_1 *and* A_2 *such that* $A_2 \le x(t) \le A_1$ *for every* $t \in [0, T]$, *with* x *a solution of* $Lx = \lambda N x$, $\lambda \in (0, 1)$. *Moreover, these constants do not depend on* λ.

Proof. Take $x \in X$ with $\min\{x(t) : t \in [0, T]\} > 0$ and $\lambda \in (0, 1)$ such that

$$\begin{cases} x'(t) = \lambda f(x(t)) + \lambda e(t), & t \in [0, T],\ t \ne t_j, \\ \Delta x(t_j) = \lambda I_j(x(t_j)), & j \in \{1, \ldots, q\}. \end{cases}$$

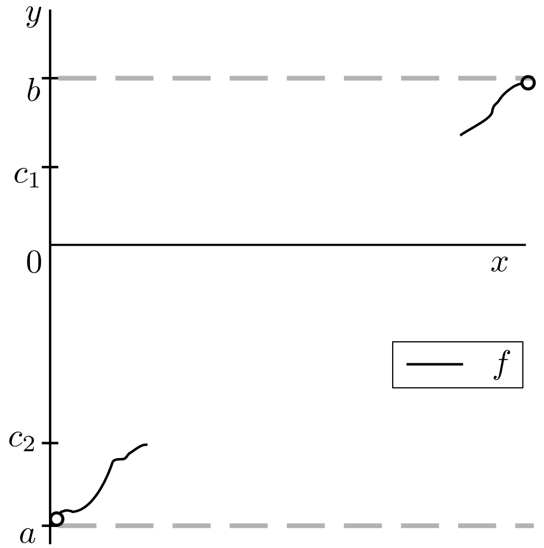


Figure 2.2: Example of a function f with hypotheses (C1) and (C3) satisfied. The zone where the function f is not plotted means that it can take any value in (a, b) as long as it keeps being continuous.

We have that

$$\int_0^T x'(t)\,dt = \lambda \int_0^T f(x(t))\,dt + \lambda \int_0^T e(t)\,dt. \tag{2.23}$$

Moreover, we can deduce

$$\begin{aligned}\int_0^T x'(t)\,dt &= \sum_{j=1}^{q+1} \int_{t_{j-1}^+}^{t_j^-} x'(t)\,dt \\ &= -x(0) - \sum_{j=1}^{q} (x(t_j^+) - x(t_j^-)) + x(T) \\ &= -\lambda \sum_{j=1}^{q} I_j(x(t_j)).\end{aligned} \tag{2.24}$$

We have that

$$-\int_0^T f(x(t))\,dt = \sum_{j=1}^{q} I_j(x(t_j)) + \int_0^T e(t)\,dt, \tag{2.25}$$

from Equations (2.23) and (2.24). We obtain from (2.25) that

$$\begin{aligned}(m_1+\cdots+m_q)+\int_0^T e(t)\,dt &\le \int_0^T -f(x(t))\,dt \\ &\le (M_1+\ldots+M_q)+\int_0^T e(t)\,dt,\end{aligned} \tag{2.26}$$

using Hypothesis (C2). There exist $\xi,\eta\in[0,T]\setminus\{t_1,\ldots,t_q\}$ such that

$$-Tf(x(\xi)) \le (M_1+\cdots+M_q)+\int_0^T e(t)\,dt,$$

$$-Tf(x(\eta)) \ge (m_1+\cdots+m_q)+\int_0^T e(t)\,dt.$$

This implies that $f(x(\xi))\ge c_2$ and $f(x(\eta))\le c_1$, so there exists $r_1,r_2>0$ with $x(\xi)\ge r_2$ and $x(\eta)\le r_1$. We use (C1) and (C3), and we can take r_2 as in Hypothesis (C4) and $r_1=\sup\{s>0: f(s)\le c_1\}$, for example.

For any $t,s\in[0,T]\setminus\{t_1,\ldots,t_q\}$, we have that

$$\int_s^t x'(u)\,du = x(t)-x(s)-\sum_{s<t_j<t}\Delta x(t_j)$$

if $t>s$. A similar result holds if $t<s$.

On the one hand, if $t\in[0,T]\setminus\{t_1,\ldots,t_q\}$ and $\eta<t$, we have

$$\begin{aligned}x(t) &= x(\eta)+\sum_{\eta<t_j<t}\Delta x(t_j)+\int_\eta^t x'(u)\,du \\ &\le r_1+\left(\widetilde{M}_1+\cdots+\widetilde{M}_q\right)+\int_0^T |x'(u)|\,du.\end{aligned}$$

We obtain a similar result if $t<\eta$.

On the other hand, if $t\in[0,T]\setminus\{t_1,\ldots,t_q\}$ and $\xi<t$, we have

$$\begin{aligned}x(t) &= x(\xi)+\sum_{\xi<t_j<t}\Delta x(t_j)+\int_\xi^t x'(u)\,du \\ &\ge r_2-\left(\widetilde{M}_1+\cdots+\widetilde{M}_q\right)-\int_0^T |x'(u)|\,du.\end{aligned}$$

We obtain a similar result if $t<\xi$.

Therefore, we have

$$\begin{aligned}\int_0^T |x'(t)|\,dt &= \int_0^T \lambda\left|f(x(t))+e(t)\right|\,dt \\ &\le \widetilde{M}_1+\cdots+\widetilde{M}_q+\int_0^T e(t)\,dt+\int_0^T |e(t)|\,dt,\end{aligned}$$

from (2.25), (2.26), and (C2). As a consequence, there exist A_1 and A_2 positive numbers such that $A_2 < x(t) < A_1$. Moreover,

$$A_2 = r_2 - 2(\widetilde{M}_1 + \cdots + \widetilde{M}_q) - \int_0^T e(t) + |e(t)|\, dt > 0,$$

by Hypothesis (C4). □

Theorem 2.20. *Suppose that the Hypotheses* (C1)–(C4) *are satisfied. Then the problem* (2.22) *has at least one solution.*

Proof. It is similar to the proof of Theorem 2.18. We prove the final part. We identify $\mathrm{Ker}(L) \cap \Omega$ with $(p, q) = (A_2 - \sigma_2, A_1 + \sigma_1) \subset \mathbb{R}$, so

$$\deg_B(JQN, \Omega \cap \mathrm{Ker}(L), 0) = \deg_B(\varphi, (p, q), 0),$$

with $\varphi \colon [p, q] \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = f(x) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(x).$$

First, we have that

$$\begin{aligned}\varphi(p) &= f(p) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(p)\\ &\le f(A_2 - \sigma_2) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(M_1 + \cdots + M_q)\\ &< 0,\end{aligned}$$

because $A_2 - \sigma_2 < r_2$. Analogously,

$$\begin{aligned}\varphi(q) &= f(q) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}\sum_{j=1}^{q} I_j(q)\\ &\ge f(A_1 + \sigma_1) + \frac{1}{T}\int_0^T e(t)\, dt + \frac{1}{T}(m_1 + \cdots + m_q)\\ &> 0,\end{aligned}$$

because $A_1 + \sigma_1 > r_1$. Then, we have that $\deg_B(\varphi, (p, q), 0) \neq 0$. Therefore, Theorem 1.41 implies that there exists $x \in \mathrm{Dom}(L) \cap \overline{\Omega}$ with $Lx = Nx$, that is, (2.22) has a positive T-periodic solution. □

We present several examples to illustrate Theorems 2.18 and 2.20.

Example 2.21. Consider the problem

$$\begin{cases} x'(t) = -\dfrac{1}{(x(t))^{0.05}}, \\ \Delta x(\pi/4) = e^{-x(\pi/4)} - 3/4, \\ \Delta x(3\pi/4) = 2 + \cos(x(3\pi/4))/2, \\ x(0) = x(\pi). \end{cases}$$

Take $m_1 = -3/4$, $M_1 = 1/4$, $\widetilde{M_1} = 3/4$, $m_2 = 3/2$, $M_2 = 5/2$, and $\widetilde{M_2} = 5/2$. This implies that $m_1 + m_2 = 3/4 > 0$ and

$$\left(\frac{\pi}{1/4 + 5/2}\right)^{1/0.05} \approx 14.33 > 6 = (1/4 + 3/4) + (5/2 + 5/2).$$

Theorem 2.18 implies that there exists a π-periodic solution.

Example 2.22. Consider

$$\begin{cases} x'(t) = -\dfrac{1000}{e^{x(t)} - 1} + |\cos(t)|, \\ \Delta x(t_1) = e^{-x(t_1)}, \\ x(0) = x(\pi). \end{cases}$$

We take $m_1 = 0$ and $M_1 = 1$, so we have that $\widetilde{M_1} = 1$. This implies that $m_1 + \int_0^\pi e(t)\,dt = 2 > 0$. Moreover,

$$g^{-1}(y) = \log\left(\frac{1000 + y}{y}\right).$$

Finally, we obtain that

$$g^{-1}\left(\frac{1}{\pi} + \frac{2}{\pi}\right) \approx 6.95 > 6 = 2 + 2 + (1 + 1).$$

Theorem 2.18 implies that there exists a π-periodic solution.

Example 2.23. Consider

$$\begin{cases} x'(t) = -\dfrac{1}{x(t)} + x(t)^2 + 0.2\cos(4t), \\ \Delta x(t_1) = 0.1\sin(x(t_1)), \\ x(0) = x(\pi). \end{cases}$$

We have that $f((0,\infty)) = \mathbb{R}$ and f is a strictly increasing function. Hypotheses (C1) and (C2) are verified with $m_1 = -0.1$ and $M_1 = 0.1$. The image of f is $\mathbb{R}$, so (C3) holds trivially. Finally,

$$r_2 \approx 0.98 > 0.6 = \int_0^\pi 0.2\cos(4t) + |0.2\cos(4t)| \; dt + 2 \cdot 0.1.$$

Hypothesis (C4) holds. Theorem 2.20 implies that there exists a positive solution of this problem.

Example 2.24. Consider

$$\begin{cases} x'(t) = -\dfrac{1}{(x(t))} + x(t)\sin\left(\dfrac{20}{x(t)}\right), \\ \Delta x(1) = \dfrac{2}{25}\arctan(x(1)), \\ x(0) = x(\pi). \end{cases}$$

We have that $f((0,\infty)) = (-\infty, 20)$ and f verifies (C1). Take $m_1 = -\pi/25$ and $M_1 = \pi/25$, because $m_1 \le 2\arctan(x(t_1))/25 \le M_1$. So (C2) is satisfied. Moreover, we have that $c_1 = 1/25$ and $c_2 = -1/25$, which belong to $(-\infty, 20)$. Therefore, (C3) holds. Finally,

$$r_2 \approx 0.98 > 0.25 \approx 2\pi/25.$$

So we have that (C4) holds. Theorem 2.20 implies that there exists a positive solution of this problem.

We continue this section with the following second order problem for impulsive differential equations:

$$\begin{cases} -u''(t) + a(t)u(t) = f(t, u(t), u'(t)), & t \neq t_j, \\ \Delta u'(t_j) = I_j(u(t_j)), & j \in \{1, \dots, q\}, \\ u(0) = u(T) = 0, \end{cases} \tag{2.27}$$

with the maps $f\colon [0,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $I_j : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, $a \in L^\infty(0,T)$, and $\operatorname{ess\,inf} a > -\lambda_1$, with $\lambda_1 = \pi^2/T^2$.

We begin by fixing some notation. For $[a,b]$ a bounded interval of $\mathbb{R}$, we will denote by $H^k(a,b)$ the Sobolev space formed by the functions u in $L^2(a,b)$ with weak derivatives up to order k also in $L^2(a,b)$. We will also use the Sobolev space $H_0^1(a,b)$, which can be characterized as the functions in $H^1(a,b)$ such that $u(a) = 0 = u(b)$. In this last space, we will consider the inner product

$$(u,v) := \int_a^b u'(t)v'(t)\,dt, \tag{2.28}$$

instead of the usual inner product that comes from $H^1(a,b)$

$$\int_a^b u'(t)v'(t)\,dt + \int_a^b u(t)v(t)\,dt.$$

In fact, these two inner products are equivalent in $H_0^1(a,b)$. We will denote by $\|\cdot\|$ the norm that comes from the inner product (2.28).

We will try to obtain the existence of solutions using variational methods. However, Equation (2.27) does not have a variational structure. This is because the derivative of the solution appears in f. Therefore, the construction of a functional which has as critical points the solutions of (2.27) is not obvious. In order to avoid this, we consider the following problem, which does not depend on the derivative: for each $w \in H_0^1(0,T)$,

$$\begin{cases} -u''(t) + a(t)u(t) = f(t, u(t), w'(t)), & t \neq t_j, \\ \Delta u'(t_j) = I_j(u(t_j)), & j \in \{1, \dots, q\}, \\ u(0) = u(T) = 0. \end{cases} \tag{2.29}$$

This problem does have a variational structure, and it can be solved using critical point theory and variational methods. This approach was used for PDEs in [54, 70, 91, 110]. The idea we propose is to find a unique solution $u_w \in H_0^1(0,T)$ of (2.29) for each $w \in H_0^1(0,T)$. Then, we use a fixed point theorem to show that there exists a solution for the complete problem (2.27). Other ideas include using an iterative technique in order to find a solution for the complete problem (2.27).

We have that $a \in L^\infty(0,T)$ and $\operatorname{ess\,inf} a > -\lambda_1$. This implies that

$$(u,v)_a = \int_0^T u'(t)v'(t)\,dt + \int_0^T a(t)u(t)v(t)\,dt$$

is an inner product in $H_0^1(0,T)$. We have that the norm $\|u\|_a = {(u,u)_a}^{1/2}$ given by this inner product is equivalent to the usual norm in $H_0^1(0,T)$. This is due to Poincaré inequality.

Proposition 2.25. *The norms $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent, that is, there exist $c_1, c_2 > 0$ such that*

$$c_1\|u\| \le \|u\|_a \le c_2\|u\| \quad \text{for every } u \in H_0^1(0,T).$$

We also recall the following well-known result:

Proposition 2.26. *Taking $c = \sqrt{T}$, we have that*

$$\|u\|_\infty \le c\,\|u\| \quad \text{for every } u \in H_0^1(0,T).$$

Proof. Take $u \in H_0^1(0,T)$. This implies that $u(0) = 0$. For any $t \in [0,T]$, we have that

$$\begin{aligned} |u(t)| &= \left| u(0) + \int_0^t u'(s)\,ds \right| = \left| \int_0^t u'(s)\,ds \right| \\ &\le \int_0^t |u'(s)|\,ds \le \int_0^T |u'(s)|\,ds. \end{aligned}$$

Hölder's inequality implies that

$$|u(t)| \le \sqrt{T}\left(\int_0^T |u'(s)|^2\,ds \right)^{1/2}.$$

As a consequence, we have that $\|u\|_\infty \le c\,\|u\|$. □

Proposition 2.27. *Let $\{f_n\}_n$ be a sequence in $L^2(0,T)$ which is convergent to f. Then there exists $\{f_{n_k}\}_k$ a subsequence such that $f_{n_k}(t)$ converges to $f(t)$ almost everywhere.*

Proposition 2.28. *Let $[a,b]$ be a bounded interval in $\mathbb{R}$ and $k \in \mathbb{N}$. Then the embeddings $H^{k+1}(a,b) \longrightarrow H^k(a,b)$ and $H^k(a,b) \longrightarrow \mathcal{C}^{k-1}([a,b])$ are compact.*

Now, for $w \in H_0^1(0,T)$, we consider the functional $\varphi_w \colon H_0^1(0,T) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \varphi_w(u) = \frac{1}{2}\int_0^T (u'(t))^2 + a(t)(u(t))^2\,dt \\ + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)\,ds - \int_0^T F_{w'(t)}(t,u(t))\,dt, \end{aligned}$$

where

$$F_\eta(t,u)=\int_0^u f(t,\xi,\eta)\,d\xi.$$

This functional belongs to $\mathcal{C}^1(H_0^1(0,T),\mathbb{R})$, and

$$\begin{aligned}\varphi'_w(u)\cdot v=&\int_0^T u'(t)v'(t)\,dt+\int_0^T a(t)u(t)v(t)\,dt\\&+\sum_{j=1}^q I_j(u(t_j))v(t_j)-\int_0^T f(t,u(t),w'(t))v(t)\,dt.\end{aligned}\tag{2.30}$$

Suppose that u is a solution of (2.29). Employing the same approach that it was used in [96, 115], we take $v\in H_0^1(0,T)$, we multiply the differential equation by v, and we integrate. Then, we have

$$-\int_0^T u''(t)v(t)\,dt+\int_0^T a(t)u(t)v(t)\,dt=\int_0^T f(t,u(t),w'(t))v(t)\,dt.$$

We split in subintervals and use integration by parts on the first integral, and we get

$$\begin{aligned}&\int_0^T u'(t)v'(t)\,dt+\int_0^T a(t)u(t)v(t)\,dt\\&\quad=\int_0^T f(t,u(t),w'(t))v(t)\,dt-\sum_{j=1}^q I_j(u(t_j))v(t_j).\end{aligned}$$

This implies that we can define the notion of weak solution for (2.29).

Definition 2.29. A function $u\in H_0^1(0,T)$ is a weak solution of (2.29) if

$$\begin{aligned}&\int_0^T u'(t)v'(t)\,dt+\int_0^T a(t)u(t)v(t)\,dt\\&\quad=\int_0^T f(t,u(t),w'(t))v(t)\,dt-\sum_{j=1}^q I_j(u(t_j))v(t_j)\end{aligned}$$

holds for every $v\in H_0^1(0,T)$.

Therefore, in view of (2.30) we have that the critical points of φ_w are the weak solutions of (2.29).

Our first objective is to guarantee the existence of critical points of the map φ_w. We introduce some initial hypotheses. Suppose that there exist $\mu\in\mathcal{C}([0,\infty),[0,\infty))$ and $\beta\in L^2(0,T)$, with $\beta(t)\ge 0$ a.e. $t\in[0,T]$, such that

$$|F_\eta(t,u)|\le\mu(|u|)\beta(t)\quad\text{and}\quad|f(t,u,\eta)|\le\mu(|u|)\beta(t).$$

We state the following conditions.

(D1) There exist $\gamma_0, \dots, \gamma_q, a_{0,\gamma_0}, \dots, a_{q,\gamma_q}, b_{0,\gamma_0}, \dots, b_{q,\gamma_q} \in \mathbb{R}$ with $\gamma_0, \dots, \gamma_q$ positive, and

$$F_\eta(t,u) \leq b_{0,\gamma_0} + a_{0,\gamma_0}|u|^{\gamma_0} \quad \text{and} \quad \int_0^u I_j(s)\,ds \geq b_{j,\gamma_j} + a_{j,\gamma_j}|u|^{\gamma_j}.$$

Define $D = \{j \in \{1,\dots,q\} : a_{j,\gamma_j} \leq 0\}$. Suppose that

(D1.1) Either $D \neq \emptyset$;

(D1.1.1) $\gamma_j < 2$ for every $j \in D$;

(D1.1.1.a) $a_{0,\gamma_0} \leq 0$;

(D1.1.1.b) $a_{0,\gamma_0} > 0$, $\gamma_0 = 2$, and $\dfrac{c_1}{2} - Tc^{\gamma_0}a_{0,\gamma_0} > 0$;

(D1.1.1.c) $a_{0,\gamma_0} > 0$, $\gamma_0 < 2$;

(D1.1.2) $\gamma_j \leq 2$ for $j \in D$, $\gamma_{j_1} = \cdots = \gamma_{j_q} = 2$, and $D_0 = \{j_1, \dots, j_q\}$;

(D1.1.2.a) $a_{0,\gamma_0} \leq 0$ and $\dfrac{c_1}{2} - \sum\limits_{\substack{j\in D_0\\ a_{j,2}<0}} c^2 a_{j,2} > 0$;

(D1.1.2.b) $a_{0,\gamma_0} > 0$, $\gamma_0 = 2$, and $\dfrac{c_1}{2} - \sum\limits_{\substack{j\in D_0\\ a_{j,2}<0}} c^2 a_{j,2} - c^2Ta_{0,2} > 0$;

(D1.1.2.c) $a_{0,\gamma_0} > 0$, $\gamma_0 < 2$, and $\dfrac{c_1}{2} - \sum\limits_{\substack{j\in D_0\\ a_{j,2}<0}} c^2 a_{j,2} > 0$;

(D1.2) Or $D = \emptyset$, that is, $a_{j,\gamma_j} \geq 0$ for every $j \in \{1,\dots,q\}$;

(D1.2.1) $a_{0,\gamma_0} \leq 0$;

(D1.2.2) $a_{0,\gamma_0} > 0$ and $\gamma_0 < 2$;

(D1.2.3) $a_{0,\gamma_0} > 0$, $\gamma_0 = 2$, and $\dfrac{c_1}{2} - c^2Ta_{0,2} > 0$.

(D2) The map $u \longmapsto \int_0^u I_j(s)\,ds$ is convex, the map $u \longmapsto F_\eta(t,u)$ is concave, and at least one is strict.

If one of the different cases in (D1) is satisfied, we have that $\varphi_w(u) \geq \alpha(\|u\|)$, with α independent of w, and such that $\alpha(s) \longrightarrow +\infty$ when $s \longrightarrow \infty$. The proof is very similar in all the cases, we will only check one case. Therefore, suppose that $D \neq \emptyset$, $\gamma_j < 2$ for every $j \in D$, and $a_{0,\gamma_0} \leq 0$, that is, suppose that (D1.1.1.a) holds. If $u \in H_0^1(0,T)$, we have that

$$\varphi_w(u) = \frac{1}{2}\int_0^T (u'(t))^2 + a(t)(u(t))^2\,dt$$
$$+ \sum_{j=1}^q \int_0^{u(t_j)} I_j(s)\,ds - \int_0^T F_{w'(t)}(t,u(t))\,dt$$

$$
\begin{aligned}
&\geq \frac{c_1}{2}\|u\|^2 + \sum_{j=1}^{q} b_{j,\gamma_j} + a_{j,\gamma_j}|u(t_j)|^{\gamma_j} - \int_0^T b_{0,\gamma_0} + a_{0,\gamma_0}|u(t)|^{\gamma_0}\, dt \\
&\geq \frac{c_1}{2}\|u\|^2 + \sum_{j=1}^{q} b_{j,\gamma_j} - b_{0,\gamma_0}T - \sum_{j\in D}|a_{j,\gamma_j}||u(t_j)|^{\gamma_j} \\
&\geq \frac{c_1}{2}\|u\|^2 + \sum_{j=1}^{q} b_{j,\gamma_j} - b_{0,\gamma_0}T - \sum_{j\in D}|a_{j,\gamma_j}|c^{\gamma_j}\|u\|^{\gamma_j} \\
&= \alpha(\|u\|).
\end{aligned}
$$

Therefore, we consider $\alpha\colon [0,\infty) \longrightarrow [0,\infty)$ as

$$
\alpha(s) = \frac{c_1}{2}s^2 + \sum_{j=1}^{q} b_{j,\gamma_j} - b_{0,\gamma_0}T - \sum_{j=1}^{q}|a_{j,\gamma_j}|c^{\gamma_j}s^{\gamma_j}.
$$

We have that the function α is continuous, and furthermore

$$
\lim_{s\to+\infty} \alpha(s) = \infty.
$$

Theorem 2.30. *Suppose that* (D2) *and one case of* (D1) *are satisfied. Then the map φ_w is strictly convex, sequentially weakly lower semicontinuous, and coercive.*

Proof. The map $x \longmapsto x^2$ is convex, (D2) is satisfied, and every norm is convex. This implies the strict convexity. The norm is sequentially weakly lower semicontinuous, and any weakly convergent sequence in $H_0^1(0,T)$ is uniformly convergent by Theorem 1.16, item (iv), and Proposition 2.28. Then φ_w is sequentially weakly lower semicontinuous. Finally, we have that $\varphi_w(u) \geq \alpha(\|u\|)$, which implies that φ_w is coercive. □

Corollary 2.31. *Under the hypotheses of Theorem* 2.30*, there exists a global minimum u_w of φ_w for every $w \in H_0^1(0,T)$, which is unique. Furthermore, there exists $M > 0$ such that $\|u_w\| \leq M$ for every $w \in H_0^1(0,T)$.*

Proof. It is a consequence of Corollary 1.25 and Theorem 2.30. We have that α does not depend on w, so M does not depend on w. □

We have proved that there is only one critical point u_w of the functional φ_w for each $w \in H_0^1(0,T)$. Then we can define the solution map as

$$
S\colon w \in H_0^1(0,T) \longrightarrow u_w \in H_0^1(0,T). \tag{2.31}
$$

It is easy to see that the solutions of Equation (2.27) are precisely the fixed points of the solution map S. Our next objective is to guarantee the existence of fixed points for the map S.

Lemma 2.32. *Suppose that Hypotheses* (D1) *and* (D2) *are satisfied. The map* S*, defined in* (2.31)*, is continuous and compact.*

Proof. Let $\{w_n\}_n$ a sequence in $H_0^1(0,T)$. We want to prove that if $\{w_n\}_n$ converges to w, then $\{Sw_n\}_n$ also converges to Sw, and that if $\{w_n\}_n$ is a bounded sequence, then there exists a convergent subsequence of $\{Sw_n\}_n$.

S continuous

Suppose that the sequence $\{w_n\}_n$ is convergent to an element $w \in H_0^1(0,T)$, and denote $u_n = S(w_n)$. We take any subsequence of $\{u_n\}_n$, which will be denoted the same. We have that $\|u_n\| \le M$, so there exists a subsequence $\{u_{n_k}\}_k$ which is weakly convergent to some $u \in H_0^1(0,T)$ (see Theorem 1.16, item (iii)). Therefore, by Proposition 2.28 and Theorem 1.16, item (iv), we have

$$u_{n_k} \longrightarrow u \text{ in } \mathcal{C}([0,T]).$$

We know that $w_{n_k} \longrightarrow w$ in $H_0^1(0,T)$, which in turn implies that $w'_{n_k} \longrightarrow w'$ in $L^2(0,T)$. Proposition 2.27 implies that there exists a subsequence $\{w_{n_{k_l}}\}_l$ such that

$$w'_{n_{k_l}}(t) \longrightarrow w'(t) \text{ a.e. } t \in [0,T].$$

We have that f is continuous, so we obtain

$$f(t, u_{n_{k_l}}(t), w'_{n_{k_l}}(t)) \longrightarrow f(t, u(t), w'(t)) \text{ a.e. } t \in [0,T].$$

Moreover, as all the functions I_j are continuous, we also have

$$I_j(u_{n_{k_l}}(t_j)) \longrightarrow I_j(u(t_j)).$$

We will prove that $\{u_{n_{k_l}}\}_l$ converges to u. Take $v \in H_0^1(0,T)$. Lebesgue's Dominated Convergence Theorem implies that

$$\begin{array}{ccccccccc}
(u_{n_{k_l}}, v) & + & \displaystyle\sum_{j=1}^{q} I_j(u_{n_{k_l}}(t_j))v(t_j) & - & \displaystyle\int_0^T f(t, u_{n_{k_l}}(t), w'_{n_{k_l}}(t))v(t)\,dt & = & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(u, v) & + & \displaystyle\sum_{j=1}^{q} I_j(u(t_j))v(t_j) & - & \displaystyle\int_0^T f(t, u(t), w'(t))v(t)\,dt & = & 0.
\end{array}$$

This implies that $Sw = u$, so the sequence $\{u_{n_{k_l}}\}_l$ converges weakly to Sw, as a consequence of Riesz representation theorem and Theorem 1.16, item (i). In particular, if we take $v = u_{n_{k_l}}$ we have that

$$\|u_{n_{k_l}}\| \longrightarrow \|u\|.$$

Therefore Theorem 1.16, item (ii), implies that $u_{n_{k_l}}$ converges to u in $H_0^1(0,T)$, that is, $Sw_{n_{k_l}}$ converges to Sw. We have proved that any subsequence of $\{u_n\}_n$ has a convergent subsequence to Sw. This implies that $\{u_n\}_n$ converges to Sw.

S compact

Suppose that the sequence $\{w_n\}_n$ is bounded. We have to prove that $\{Sw_n\}_n$ has a convergent subsequence. Take any $j \in \{0,\dots,q\}$ and $v \in H_0^1(0,T)$ with $v(t)=0$ for any $t \in [0,T] \setminus (t_j,t_{j+1})$. As Sw_n are weak solutions of (2.29), we have that

$$\int_{t_j}^{t_{j+1}} (Sw_n)'(t)v'(t) + a(t)(Sw_n)(t)v(t)\,dt - \int_{t_j}^{t_{j+1}} f(t,Sw_n(t),w_n'(t))v(t)\,dt$$

is equal to 0. Therefore, we have that

$$\int_{t_j}^{t_{j+1}} (Sw_n)'(t)v'(t) = \int_{t_j}^{t_{j+1}} [f(t,Sw_n(t),w_n'(t)) - a(t)(Sw_n)(t)]v(t)\,dt.$$

This implies that there exists $(Sw_n)'' \in L^2(t_j,t_{j+1})$. Then $Sw_n \in H^2(t_j,t_{j+1})$. We will check that Sw_n is a bounded sequence in $H^2(t_j,t_{j+1})$.

$$\begin{aligned}
\|Sw_n\|_{H^2}{}^2 &= \|Sw_n\|_{H^1}{}^2 + \|(Sw_n)''\|_{L^2}{}^2 \\
&\le M^2 + \int_{t_j}^{t_{j+1}} (a(t)Sw_n(t) - f(t,Sw_n(t),w_n'(t)))^2\,dt \\
&\le M^2 + \int_{t_j}^{t_{j+1}} (a(t)Sw_n(t))^2\,dt + \int_{t_j}^{t_{j+1}} (f(t,Sw_n(t),w_n'(t)))^2\,dt \\
&\quad + 2\int_{t_j}^{t_{j+1}} |a(t)f(t,Sw_n(t),w_n'(t))Sw_n(t)|\,dt \\
&\le M^2 + M_1{}^2 + \int_{t_j}^{t_{j+1}} (\mu(|Sw_n(t)|)\beta(t))^2\,dt \\
&\quad + 2\int_{t_j}^{t_{j+1}} |a(t)|\mu(|Sw_n(t)|)\beta(t)|Sw_n(t)|\,dt.
\end{aligned}$$

We also have that $\|Sw_n\|_\infty \le c\|Sw_n\| \le cM$. If we take

$$M_2 = \max\{\mu(s) : s \in [0,cM]\},$$

then we have that

$$\begin{aligned}
\|Sw_n\|_{H^2}{}^2 &\le \cdots \\
&\le M^2 + M_1{}^2 + \int_{t_j}^{t_{j+1}} (M_2\beta(t))^2\,dt + 2\int_{t_j}^{t_{j+1}} |a(t)|M_2cM\beta(t)\,dt \\
&< K < \infty.
\end{aligned}$$

This implies that the restriction of $\{Sw_n\}_n$ is a bounded sequence in $H^2(t_j, t_{j+1})$ for every $j \in \{0, \dots, q\}$. In order to finish the proof, we use a diagonal argument.

If $j = 0$, we have that $\{Sw_n\}_n$ is a bounded sequence in $H^2(0, t_1)$. This implies that there exists a subsequence $\{Sw_{n_{k_0}}\}_{k_0}$ which converges weakly by Theorem 1.16, item (iii). We denote u^0 the limit. Proposition 2.28 implies that the inclusion $H^2(0, t_1) \subset H^1(0, t_1)$ is compact, so $\{Sw_{n_{k_0}}\}_{k_0}$ converges to u^0 in $H^1(0, t_1)$.

If $j = 1$, we have that $\{Sw_{n_{k_0}}\}_{k_0}$ is a bounded sequence in $H^2(t_1, t_2)$. This implies that there exists a subsequence $\{Sw_{n_{k_1}}\}_{k_1}$ which converges weakly. We denote u^1 the limit. The inclusion $H^2(t_1, t_2) \subset H^1(t_1, t_2)$ is compact, so $Sw_{n_{k_1}} \longrightarrow u^1$ in $H^1(t_1, t_2)$.

We use the same argument for $j \in \{2, \dots, q\}$. Therefore, we end up with a subsequence $\{Sw_{n_k}\}_k$ of $\{Sw_n\}_n$ such that

$$
\begin{aligned}
Sw_{n_k}|_{(t_j, t_{j+1})} &\longrightarrow u^j \quad \text{weakly in } H^2(t_j, t_{j+1}), \\
Sw_{n_k}|_{(t_j, t_{j+1})} &\longrightarrow u^j \quad \text{in } H^1(t_j, t_{j+1}),
\end{aligned}
$$

for every $j \in \{0, \dots, q\}$. Furthermore, Proposition 2.28 implies that

$$
Sw_{n_k}|_{(t_j, t_{j+1})} \longrightarrow u^j \text{ in } \mathcal{C}([t_j, t_{j+1}])
$$

for every $j \in \{0, \dots, q\}$. We define u as the function

$$
u(t) = \begin{cases} u^0(t), & t \in [0, t_1], \\ u^1(t), & t \in (t_1, t_2], \\ \dots & \\ u^q(t), & t \in (t_q, T]. \end{cases}
$$

Moreover, we have that

$$
(u^j)' \in H^2(t_j, t_{j+1}) \subset \mathcal{C}^1([t_j, t_{j+1}]).
$$

This implies that there exists $(u^j)'(t_j^+)$. Furthermore, as the restriction of $\{Sw_{n_k}\}_k$ to (t_j, t_{j+1}) converges weakly to u^j in $H^2(t_j, t_{j+1})$, we have that $Sw_{n_k} \longrightarrow u^j$ in $\mathcal{C}^1([t_j, t_{j+1}])$, so

$$
(u^j)'(t_j^+) = \lim_{k \to \infty} (Sw_{n_k})'(t_j^+) \quad \text{and} \quad (u^j)'(t_j^-) = \lim_{k \to \infty} (Sw_{n_k})'(t_j^-).
$$

As a consequence, $\Delta u'(t_j) = I_j(u(t_j))$. Finally, $Sw_{n_k} \longrightarrow u$ in $H^1(0, T)$, and $u(0) = u(T)$, so $Sw_{n_k} \longrightarrow u$ in $H_0^1(0, T)$. As a consequence, $\{Sw_n\}_{n \in \mathbb{N}}$ has a convergent subsequence. □

Theorem 2.33. *If* (D1) *and* (D2) *hold, the map S has a fixed point.*

Proof. Corollary 2.31 implies that $\|Sw\| \leq M$ for all $w \in H_0^1(0,T)$. Therefore, the map $S\colon \overline{B(0,M)} \subset H_0^1(0,T) \longrightarrow \overline{B(0,M)}$ is a continuous and compact map. Theorem 1.30 guarantees the existence of a fixed point w. □

As a consequence, we have our main result.

Theorem 2.34. *If* (D1) *and* (D2) *hold, then* (2.27) *admits a solution.*

We present one example applying Theorem 2.34.

Example 2.35. Consider

$$\begin{cases} -u''(t) + u(t) = -1 - u(t)^3[3 + \cos(u'(t))](t+1)^5, & t \neq t_1, \\ \Delta u'(t_1) = 10^6 u(t_1)^3 + 1, \\ u(0) = u(T) = 0, \end{cases}$$

with $t_1 = 0.3$ and $T = 1$. Thus, $f(t,u,\eta) = -1 - u^3(3 + \cos(\eta))(t+1)^5$ and $I(s) = 10^6 s^3 + 1$. In this case, we can take

$$b_{0,4} = -1, \quad a_{0,4} = -1/2, \quad a_{1,4} = \frac{10^6}{8} \quad \text{and} \quad b_{1,4} = -1.$$

So $D = \emptyset$ and $a_{0,4} < 0$. Therefore, we are in case (D1.2.1). Furthermore, (D2) is also satisfied because

$$u \longmapsto \int_0^u I(s)\, ds = \frac{10^6}{4} u^4 + u$$

is convex and

$$u \longmapsto F_\eta(t,u) = \frac{-(t+1)^5(3 + \cos(\eta))}{4} u^4$$

is strictly concave. Then, Theorem 2.34 guarantees the existence of a solution. In Figure 2.3 we can see a plot of the solution.

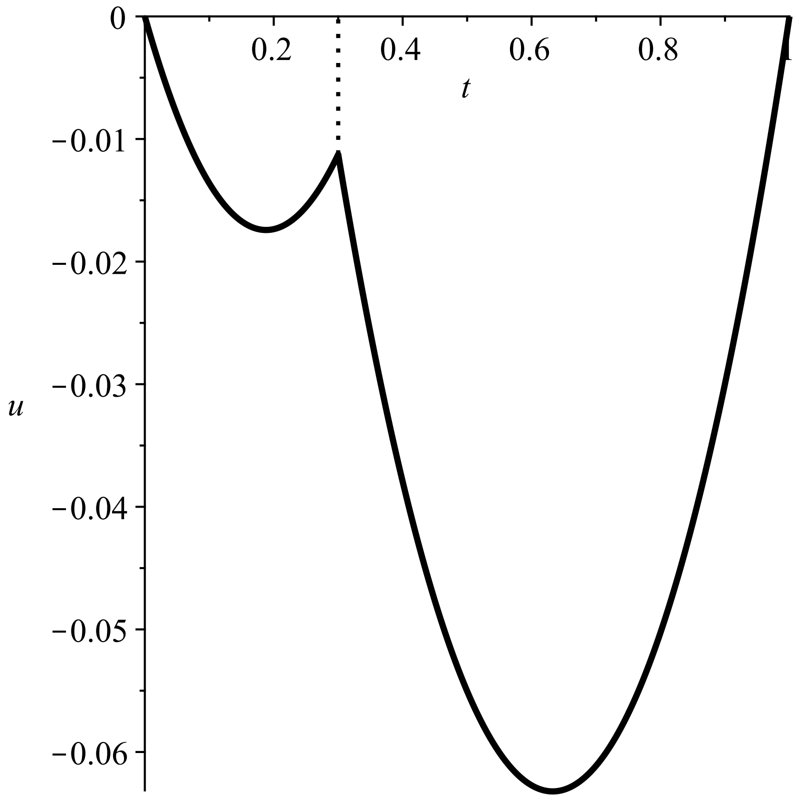


Figure 2.3: Solution of Example 2.35.

We finish this section by considering the general second order problem with impulses.

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \ldots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \ldots, q\}. \end{cases} \tag{2.32}$$

We will prove the existence of solutions for (2.32) with both Dirichlet and Neumann boundary conditions. Note that here the impulses are present in both the position and the derivative. We will denote by $X = \mathcal{PC}^1(0, T; \mathbb{R}^n; t_1, \ldots, t_q)$. Assume the following initial hypotheses:

(E1) The map $f\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and there exist $a_1, a_2, a_3 \in \mathcal{C}([0, T], [0, \infty))$ such that

$$|f(t, x, y)| \leq a_1(t)|x| + a_2(t)|y| + a_3(t) \quad \text{for } (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

(E2) For each $j \in \{1, \ldots, q\}$, the maps $I_j, J_j\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous.

We consider the following family of initial value problems

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \ldots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \ldots, q\}, \\ x(0) = c, \ x'(0) = d, \end{cases} \tag{2.33}$$

with $c, d \in \mathbb{R}^n$. Problem (2.33) is equivalent to finding a function x such that

$$x(t) = c + d\,t + \sum_{t_j < t} [I_j(x(t_j), x'(t_j)) + J_j(x(t_j), x'(t_j))\,t] + \int_0^t (t - s) f(s, x(s), x'(s))\,ds.$$

The existence of local solutions of (2.33) follows from the continuity of f and Theorem 2.2, for example.

Lemma 2.36. *If* (E1) *and* (E2) *are satisfied, and x is a local solution of* (2.33)*, then x can be extended to* $[0, T]$.

Proof. It is an application of Lemma 2.6 on each subinterval $[t_j, t_{j+1}]$. Take any $t \in [0, t_1]$, then, we have that

$$x(t) = c + \int_0^t x'(s)\,ds \qquad \text{and} \qquad x'(t) = d + \int_0^t x''(s)\,ds.$$

Therefore, we have that

$$\begin{aligned}|x'(t)| &\le |d| + \int_0^t a_1(s)|x(s)| + a_2(s)|x'(s)| + a_3(s)\,ds\\ &\le |d| + \int_0^t a_1(s)|c| + a_1(s)\int_0^s |x'(u)|\,du + a_2(s)|x'(s)| + a_3(s)\,ds\\ &\le |d| + \int_0^{t_1} |c|a_1(s) + a_3(s)\,ds\\ &\quad + \int_0^t (a_1(s)s + a_2(s))\sup\{|x'(u)| : u \in [0,s]\}\,ds.\end{aligned}$$

We define $y(t) := \sup\{|x'(u)| : u \in [0,t]\}$. We deduce

$$y(t) \le |d| + \int_0^{t_1} |c|a_1(s) + a_3(s)\,ds + \int_0^t (a_1(s)s + a_2(s))y(s)\,ds.$$

Lemma 2.6 implies that

$$y(t) \le K_1 \exp\left\{\int_0^t a_1(u)u + a_2(u)\,du\right\},$$

with K_1 defined as

$$K_1 := |d| + \int_0^{t_1} |c|a_1(s) + a_3(s)\,ds.$$

Therefore, we have

$$y(t) \le K_1 \exp\left\{\int_0^{t_1} a_1(u)u + a_2(u)\,du\right\} = M_1,$$

which implies that $|x'(t)| \le M_1$ for all $t \in [0,t_1]$. Then

$$|x(t)| \le |c| + \int_0^t |x'(s)|\,ds \le |c| + M_1 t \le |c| + M_1 t_1 = N_1.$$

Therefore, there exist $M_1, N_1 \ge 0$ such that $|x(t)| \le N_1$ and $|x'(t)| \le M_1$ for every $t \in [0,t_1]$. Moreover, the functions I_1 and J_1 are continuous, so there exist $P_1, Q_1 \ge 0$ such that

$$|I_1(z,w)| \le P_1 \quad\text{and}\quad |J_1(z,w)| \le Q_1, \qquad \text{for } |z| \le N_1,\ |w| \le M_1.$$

This implies that $|x(t_1^+)| \le N_1 + P_1$ and $|x'(t_1^+)| \le M_1 + Q_1$. Using the same approach as before for the interval $(t_1,t_2]$, we obtain that there exist $M_2, N_2 \ge 0$ such that

$$|x'(t)| \le M_2 \quad\text{and}\quad |x(t)| \le N_2, \qquad \text{for } t \in (t_1,t_2].$$

We continue this procedure, and we obtain that there exist $M, N \ge 0$ such that $|x'(t)| \le M$ and $|x(t)| \le N$ for all $t \in [0,T]$. □

We consider (2.32) with Dirichlet boundary conditions, that is,

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ x(0) = 0, \ x(T) = 0. \end{cases} \tag{2.34}$$

We assume

(E3) For every $i \in \{1, \dots, n\}$, there exists $m_i > 0$ such that

$$[(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \ |y_i| \geq m_i] \Longrightarrow y_i f_i(t, x, y) > 0.$$

(E4) For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, q\}$, there exists $m_i > 0$ such that

$$\begin{aligned} J_{j,i}(x, y) + y_i &\geq m_i, && \text{for } x, y \in \mathbb{R}^n, \ y_i \geq m_i; \\ J_{j,i}(x, y) + y_i &\leq -m_i, && \text{for } x, y \in \mathbb{R}^n, \ y_i \leq -m_i. \end{aligned}$$

(E5) For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, q\}$, there exists $m_i > 0$ such that

$$\operatorname{sign}(I_{j,i}(x, y)) = \operatorname{sign}(x_i) \quad \text{for } x, y \in \mathbb{R}^n.$$

Consider the family of initial value problems

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ x(0) = 0, \ x'(0) = d, \end{cases} \tag{2.35}$$

with $d \in \mathbb{R}^n$. This is a particular case of (2.33) with $c = 0$. We are going to prove that there exists $d \in \mathbb{R}^n$ such that there exists a solution x of (2.35) that verifies $x(T) = 0$.

We define the operator $A\colon (d, x) \in \mathbb{R}^n \times X \longrightarrow A_d(x) \in X$, with

$$\begin{aligned} [A_d(x)](t) = d\, t + \sum_{t_j < t} [I_j(x(t_j), x'(t_j)) + J_j(x(t_j), x'(t_j))\, t] \\ + \int_0^t (t - s) f(s, x(s), x'(s))\, ds. \end{aligned}$$

A standard application of Theorem 2.5 yields

Lemma 2.37. *The operator A is well-defined, continuous, and compact.*

We define the following operators:

$$\begin{aligned}
&\varphi\colon x \in X \longrightarrow x(T) \in \mathbb{R}^n,\\
&\phi\colon d \in \mathbb{R}^n \longrightarrow \mathrm{Fix}(A_d) \subset X,\\
&F\colon d \in \mathbb{R}^n \longrightarrow \{x(T) : x \in \mathrm{Fix}(A_d)\} \subset \mathbb{R}^n.
\end{aligned}$$

Obviously, we have that $F = \varphi \circ \phi$. Furthermore, ϕ is upper semicontinuous, with compact values, and an R_δ-map. The proof of this last result is very similar to the proofs of Theorem 1.11 applied to this particular case and Corollary 1.12 (see [71, page 160] for the proofs of Theorem 1.11 and Corollary 1.12).

This implies that we have that X is a Banach space with $\dim(X) \geq n$, φ a linear, bounded, and surjective map from X to $\mathbb{R}^n$, and ϕ an R_δ-map from $\mathbb{R}^n$ to X (in particular from any subset of $\mathbb{R}^n$).

Remark 2.38. Peano's existence theorem says that the initial value problem

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{2.36}$$

has local solutions if f is a continuous function. However, we do not have uniqueness of solution. Therefore, it is interesting to study the characteristics of the set $\mathrm{Sol}(x_0)$, the set of solutions of (2.36). For example, for dimension 1, Peano showed that the set $\{x(t) : x \in \mathrm{Sol}(x_0)\}$ is nonempty, compact, and connected in the real line, for t in a neighborhood of t_0. This was later generalized to any finite dimension n. Aronszajn [5] proved that the set $\mathrm{Sol}(x_0)$ is an R_δ-set in the Banach space of continuous functions. This result has been extended to other situations. See for example [5, 62, 68, 71–73, 121].

Theorem 2.39. *If* (E1)–(E5) *are satisfied, then problem* (2.34) *has at least one solution.*

Proof. For simplicity, for each $i \in \{1, \ldots, n\}$, we take the biggest $m_i > 0$ in Hypotheses (E3)–(E5). Take $d \in \mathbb{R}^n$ such that $|d_i| = m_i$ for some $i \in \{1, \ldots, n\}$, $x \in \mathrm{Fix}(A_d)$, and m_i from the Hypotheses (E3)–(E5). We have to prove that $d_i \cdot x_i(T) \geq 0$.

If $d_i = m_i$, suppose first that there exists $\delta \in (0, t_1)$ such that $x_i'(t) < d_i$ for all $t \in (0, \delta)$. Without loss of generality, we can assume $0 < x_i'(t)$. We consider the function $y(t) = (x_i'(t))^2$. We have that $x_i'(t) < d_i = x_i'(0)$ for all $t \in (0, \delta)$, so we can suppose that $y(t) < y(0)$ for all $t \in (0, \delta)$, by taking δ sufficiently small. This implies that $y'(0) \leq 0$. On the other hand,

$$y'(0) = 2 \cdot x_i'(0) \cdot x_i''(0) = 2 \cdot x_i'(0) \cdot f_i(0, x(0), x'(0)) > 0,$$

because $x_i'(0) = c_i = m_i$ and Hypothesis (E3) is satisfied. We have a contradiction, so there exists $\varepsilon \in (0, t_1)$ such that $x_i'(t) \geq d_i$ for all $t \in [0, \varepsilon]$. Suppose that there exists $t \in [\varepsilon, t_1)$ with $x_i'(t) < d_i$. This implies that we can define $t_* := \inf\{t \in [\varepsilon, t_1) : x_i'(t) < d_i\}$, and we have that

$$x_i'(t) \geq d_i \text{ for all } t \in [0, t_*] \quad \text{and} \quad x_i'(t_*) = d_i.$$

Then $y(0) \geq (d_i)^2 = y(t_*)$. The mean value theorem implies that there exists $t^* \in (0, t_*)$ with $y'(t^*) \leq 0$ (in fact, in this case, equal to 0). But

$$y'(t^*) = 2 \cdot x_i'(t^*) \cdot x_i''(t^*) = 2 \cdot x_i'(t^*) \cdot f_i(t^*, x(t^*), x'(t^*)) > 0,$$

because $x_i'(t^*) \geq d_i$ and (E3) holds. We arrive at a contradiction, and we can conclude that

$$x_i'(t) \geq d_i \text{ and } x_i(t) > 0 \quad \text{for all } t \in [0, t_1].$$

At time $t = t_1$, there is a jump in both position and its derivative. As a consequence of (E4) and (E5), we have that

$$\begin{aligned} x_i'(t_1^+) &= J_{1,i}(x(t_1), x'(t_1)) + x_i'(t_1) \geq m_i = d_i, \\ x_i(t_1^+) &= I_{1,i}(x(t_1), x'(t_1)) + x_i(t_1) \geq 0. \end{aligned}$$

We can prove that $x_i'(t) \geq d_i$ and $x_i(t) \geq 0$ for all $t \in [0, T]$, proceeding the same way as before.

If $d_i = -m_i$, the proof is analogous, so it can be proved that $x_i'(t) \leq d_i$ and $x_i(t) \leq 0$ for all $t \in [0, T]$.

Therefore, we have just proved that $d_i \cdot x_i(T) \geq 0$ for every $x \in \operatorname{Fix}(A_d)$ and $|d_i| = m_i$. This implies that Theorem 1.31 can be applied, so there exists $d \in \mathbb{R}^n$ and x a solution of (2.35) such that $x(T) = 0$, which implies that x is a solution of (2.34). □

Example 2.40. Consider the interval $[0, 1]$, a unique impulsive point $t_1 = 0.2$, and $n = 1$. Take the system

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_1, \\ \Delta x(t_1) = I(x(t_1), x'(t_1)), \\ \Delta x'(t_1) = J(x(t_1), x'(t_1)), \\ x(0) = 0, \ x(1) = 0, \end{cases} \tag{2.37}$$

with the functions f, I, and J defined as

$$\begin{aligned} f(t, x, y) &= \exp(t - |x|) + \cos(x)\sin(t) + y\cos(1/y), \\ I(x, y) &= x, \\ J(x, y) &= (2 + \sin(x))y^3. \end{aligned}$$

When $y = 0$ on f, we take the obvious limit. It is easy to check that Hypotheses (E1)–(E5) of Theorem 2.39 are satisfied. It suffices to take $m = 4$ to fulfill (E3), (E4), and (E5). Then, there exists a solution of (2.37). In Figure 2.4 we see a plot of the solution.

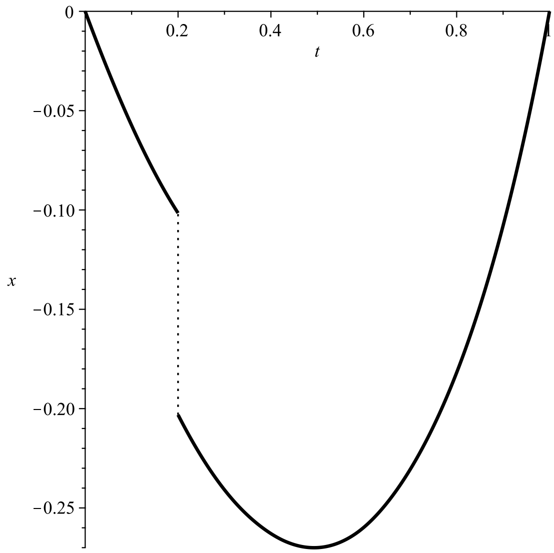


Figure 2.4: Solution of Example 2.40.

Finally, we consider (2.32) with Neumann boundary conditions, that is,

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ x'(0) = 0, \ x'(T) = 0. \end{cases} \tag{2.38}$$

Assume the following hypotheses:

(E6) For every $i \in \{1, \dots, n\}$, there exists $m_i > 0$ such that

$$[(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \ |x_i| \geq m_i] \Longrightarrow x_i \cdot f_i(t, x, y) \geq 0.$$

(E7) For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, q\}$,

$$\begin{array}{ll} I_{j,i}(x, y) + x_i \geq m_i, & \text{for } x, y \in \mathbb{R}^n, \ x_i \geq m_i; \\ I_{j,i}(x, y) + x_i \leq -m_i, & \text{for } x, y \in \mathbb{R}^n, \ x_i \leq -m_i. \end{array}$$

(E8) For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, q\}$,

$$\operatorname{sign}(J_{j,i}(x, y)) = \operatorname{sign}(y_i) \quad \text{for } x, y \in \mathbb{R}^n.$$

Consider the family of initial value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, \dots, q\}, \\ x(0) = c, \; x'(0) = 0, \end{cases} \tag{2.39}$$

with $c \in \mathbb{R}^n$. This is a particular case of (2.33) with $d = 0$. We are going to prove that there exists $c \in \mathbb{R}^n$ such that there exists a solution x of (2.39) that verifies $x'(T) = 0$.

We define the operator $A\colon (c, x) \in \mathbb{R}^n \times X \longrightarrow A_c(x) \in X$, with

$$\begin{aligned}[A_c(x)](t) = c + \sum_{t_j < t} [I_j(x(t_j), x'(t_j)) + J_j(x(t_j), x'(t_j))\, t] \\ + \int_0^t (t-s) f(s, x(s), x'(s))\, ds.\end{aligned}$$

Lemma 2.41. *The operator A is well-defined, continuous, and compact.*

We define the following operators:

$$\begin{aligned} &\varphi\colon x \in X \longrightarrow x'(T) \in \mathbb{R}^n, \\ &\phi\colon c \in \mathbb{R}^n \longrightarrow \operatorname{Fix}(A_c) \subset X, \\ &F\colon c \in \mathbb{R}^n \longrightarrow \{x'(T) : x \in \operatorname{Fix}(A_c)\} \subset \mathbb{R}^n. \end{aligned}$$

In order to apply Theorem 1.31, it suffices to check that Condition (1.1) is fulfilled, the other conditions are analogous to the Dirichlet case.

Theorem 2.42. *If* (E1), (E2), (E6)–(E8) *are satisfied, then problem* (2.38) *has at least one solution.*

Proof. For simplicity, for each $i \in \{1, \dots, n\}$, we take the biggest $m_i > 0$ in Hypotheses (E6)–(E8). Fix any $i \in \{1, \dots, n\}$. Take $c \in \mathbb{R}^n$ such that $|c_i| = m_i + 1$, with m_i from the hypotheses, and $x \in \operatorname{Fix}(A_c)$. We have to prove that $c_i \cdot x_i'(T) \geq 0$.

If $c_i = m_i + 1$, suppose first that there exists $t \in (0, t_1)$ such that $x_i'(t) < 0$. We define $t_* = \inf\{t \in (0, t_1) : x_i'(t) < 0\} \geq 0$. This implies that

$$x_i'(t_*) = 0 \text{ and } x_i(t) \geq c_i \quad \text{for} \quad t \in [0, t_*].$$

Note that the interval $[0, t_*]$ could be just one point, if $t_* = 0$. The function x_i' is continuous on the interval $(0, t_1)$, so there exist $t^* \in (t_*, t_1)$ such that

$$\int_{t_*}^{t^*} |x_i'(s)|\, ds \leq 1.$$

For any $t \in [t_*, t^*]$, we have that

$$\begin{aligned} x_i(t) = x_i(t_*) + \int_{t_*}^{t} x_i'(s)\, ds \geq c_i + \int_{t_*}^{t} x_i'(s)\, ds \geq c_i - \int_{t^*}^{t} |x_i'(s)|\, ds \\ \geq c_i - \int_{t_*}^{t^*} |x_i'(s)|\, ds \geq c_i - 1 = m_i + 1 - 1 = m_i. \end{aligned}$$

Therefore, we have that

$$x_i(t) \cdot f_i(t, x(t), x'(t)) = x_i(t) \cdot x_i''(t) \geq 0,$$

because $x_i(t) \geq m_i$ and Hypothesis (E6) holds. Then we get $x_i''(t) \geq 0$. As a consequence, x_i' is a nondecreasing function on the interval $[t_*, t^*]$, a contradiction with the definition of t_*. This implies that $x_i'(t) \geq 0$ for all $t \in [0, t_1]$. At $t = t_1$ there is a jump, so

$$\begin{aligned} x_i(t_1^+) &= I_{1,i}(x(t_1), x'(t_1)) + x_i(t_1) > m_i, \\ x_i'(t_1^+) &= J_{1,i}(x(t_1), x'(t_1)) + x_i'(t_1) \geq 0, \end{aligned}$$

using Hypotheses (E7) and (E8). We can prove the same way that $x_i'(t) \geq 0$ and $x_i(t) \geq m_i$ for all $t \in [0, T]$, so $x_i'(T) \geq 0$.

If $c_i = -m_i - 1$, the proof is analogous, and we get that $x_i'(t) \leq 0$ for all $t \in [0, T]$.

This implies that $c_i \cdot x_i'(T) \geq 0$ for every $x \in \mathrm{Fix}(A_c)$ and $|c_i| = m_i + 1$. Theorem 1.31 can be applied, so there exists $c \in \mathbb{R}^n$ and x a solution of (2.39) such that $x'(T) = 0$, which implies that x is a solution of (2.38). □

Example 2.43. Consider the interval $[0, 1]$, $t_1 = 0.2$, $t_2 = 0.75$, and $n = 1$. Consider the system

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j), x'(t_j)), & j \in \{1, 2\}, \\ \Delta x'(t_j) = J_j(x(t_j), x'(t_j)), & j \in \{1, 2\}, \\ x'(0) = 0, \; x'(1) = 0, \end{cases} \tag{2.40}$$

with the functions f, I_1, I_2, J_1 and J_2 defined as

$$\begin{aligned} f(t, x, y) &= (t^2 + 5) \arctan(y) \cos(x) + y^{1/3}, \\ I_1(x, y) &= x + 200 \cos(x), \\ I_2(x, y) &= -\sqrt[3]{x}, \\ J_1(x, y) &= \arctan(y), \\ J_2(x, y) &= y(\cos(y) + 11/10). \end{aligned}$$

It is easy to check that Hypotheses (E1), (E2), (E6)–(E8) of Theorem 2.42 are satisfied. It suffices to take $m = 1000$ to fulfill (E6), (E7), and (E8). Then, there exists a solution of (2.40).

2.3 Impulses at variable times

In this section, we look for solutions for impulsive boundary value problems with impulses at variable times. This situation is much more complicated than the results considered in the previous section. In fact, there are very few results for boundary value problems for state-dependent impulsive differential equations. Some of them are [8, 11, 67, 103]. We will obtain some existence results of periodic solutions. Once again, in this section, T is going to be a positive number.

As a first example, consider the following periodic boundary value problem.

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq \tau_j(x(t)), \\ \Delta x(t) = I_j(x(t)), & t = \tau_j(x(t)), \\ x(0) = x(T). \end{cases} \tag{2.41}$$

Assume the following hypotheses:

(F1) The function $f\colon [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and there exist a_1, a_2 two nonnegative continuous functions from $[0,T]$ such that

$$|f(t,x)| \leq a_1(t)|x| + a_2(t) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^n.$$

(F2) For each $j \in \{1, \dots, q\}$, the map $I_j\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous.

(F3) For every $i \in \{1, \dots, n\}$, there exists $m_i > 0$ such that if $(t,x) \in [0,T] \times \mathbb{R}^n$ and $|x_i| \geq m_i$, then $x_i \cdot f_i(t,x) > 0$.

(F4) For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, q\}$,

$$\begin{aligned} I_{j,i}(x) + x_i &\geq m_i, & &\text{for } x \in \mathbb{R}^n,\ x_i \geq m_i, \\ I_{j,i}(x) + x_i &\leq -m_i, & &\text{for } x \in \mathbb{R}^n,\ x_i \leq -m_i. \end{aligned}$$

Furthermore, we will also assume (A3)–(A5). As in Section 2.1, we denote by σ_j the "hypersurfaces"

$$\sigma_j = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t = \tau_j(x)\}.$$

Conditions (A3)–(A5) and (F1)–(F4) imply that, if x is any solution of (2.41), then x intersects each hypersurface σ_j exactly once. Consider the following family of initial value problems.

$$\begin{cases} x' = f(t, x), & t \neq \tau_j(x(t)), \\ \Delta x(t) = I_j(x(t)), & t = \tau_j(x(t)), \\ x(0) = c, \end{cases} \tag{2.42}$$

with $c \in \mathbb{R}^n$. We are going to prove, using Theorem 1.31, that there exists $c \in \mathbb{R}^n$ such that there exists a solution x of (2.42) that satisfies $x(T) = c = x(0)$.

In Section 2.1, the space of functions $\mathcal{CJ}_q([0,T])$ was introduced. The results in [73], applied to our particular case of differential equations and not inclusions, imply the following result.

Theorem 2.44. *Every solution of* (2.42) *intersects each* σ_j *only once, and the solution set, which will be denoted by* $\mathrm{Sol}(c)$*, is an* R_δ*-set in* $\mathcal{CJ}_q([0,T])$*.*

Therefore, we define the following maps

$$
\begin{aligned}
&\varphi\colon (x,(l_1,v_1),\ldots,(l_q,v_q)) \in \mathcal{CJ}_q([0,T]) \longrightarrow \tilde{x}(T) - \tilde{x}(0) \in \mathbb{R}^n,\\
&\phi\colon c \in \mathbb{R}^n \longrightarrow \mathrm{Sol}(c) \subset \mathcal{CJ}_q([0,T]),\\
&F\colon c \in \mathbb{R}^n \longrightarrow (\varphi \circ \phi)(c) \subset \mathbb{R}^n.
\end{aligned}
$$

Equivalently, the map φ can be written as

$$\varphi((x,(l_1,v_1),\ldots,(l_q,v_q))) = x(T) + \sum_{j=1}^{q} v_q - x(0).$$

We have that φ is a linear, bounded, and surjective map and that ϕ is an R_δ-map (by Theorem 2.44).

Theorem 2.45. *If* (A3)–(A5) *and* (F1)–(F4) *are satisfied, then the problem* (2.41) *has at least one solution.*

Proof. In order to apply Theorem 1.31, it only remains to check that Condition (1.1) is satisfied. Fix any $i \in \{1,\ldots,n\}$, take $c \in \mathbb{R}^n$ such that $|c_i| = m_i$, and $(x,(l_1,v_1),\ldots,(l_q,v_q)) \in \mathrm{Sol}(c)$. We want to prove that $c_i(\tilde{x}_i(T) - \tilde{x}_i(0)) \geq 0$.

We consider first the case $c_i = m_i$. We will prove that $\tilde{x}_i(t) \geq c_i$ for all $t \in [0,1]$. We define the auxiliary function $y(t) = (x_i(t))^2$, which will be used through the proof. Suppose that there exists $\delta \in (0,l_1)$ such that $x_i(t) < m_i$ for all $t \in (0,\delta)$. Then we have that $y(t) < y(0)$ for every $t \in (0,\delta)$. This implies that $y'(0) \leq 0$. However, we have that

$$y'(t) = 2 \cdot x_i(t) \cdot x_i'(t) = 2 \cdot x_i(t) \cdot f_i(t,x(t)) > 0,$$

by Hypothesis (F3). We obtain a contradiction. This implies that there exists $\varepsilon > 0$ such that

$$x_i(t) \geq m_i \quad \text{for } t \in [0,\varepsilon].$$

Suppose that there exists $t \in [\varepsilon, l_1]$ with $x_i(t) < m_i$. We define

$$t_* := \inf\{t \in [\varepsilon,l_1] : x_i(t) < m_i\}.$$

Therefore, we have that $x_i(t_*) = m_i$, which implies that $y(0) = (m_i)^2 = y(t_*)$. Then, there exists $\bar{t} \in (0, t_*)$ such that $y'(\bar{t}) = 0$. Once again, we have

$$y'(\bar{t}) = 2 \cdot x_i(\bar{t}) \cdot x_i'(\bar{t}) = 2 \cdot x_i \cdot (\bar{t}) \cdot f_i(\bar{t}, x(\bar{t})) > 0,$$

which is a contradiction. As a consequence, we have that $x_i(t) \geq m_i$ for all $t \in [0, l_1]$. Moreover, we have that

$$\tilde{x}_i(l_1^+) = x_i(l_1) + v_{1,i} \quad \text{and} \quad v_1 = I_1(\tilde{x}(l_1)) = I_1(x(l_1)).$$

By Hypothesis (F4) we have $\tilde{x}_i(l_1^+) \geq m_i$. A similar argument as before implies that

$$\tilde{x}_i(t) \geq m_i \quad \text{for } t \in (l_1, l_2],$$

by taking $t_* = \inf\{t \in (l_1, l_2] : x_i(t) < m_i\}$ and obtaining a contradiction with (F3). Therefore, we will obtain that $\tilde{x}_i(t) \geq m_i$ for all $t \in [0, T]$. In particular,

$$\tilde{x}_i(T) \geq m_i = c_i = \tilde{x}_i(0).$$

Then $c_i(\tilde{x}_i(T) - \tilde{x}_i(0)) \geq 0$.

We consider the case $c_i = -m_i$. The argument is very similar. First, it is proved that there exists $\varepsilon > 0$ such that $x_i(t) \leq -m_i$ for all $t \in [0, \varepsilon]$. Then, if there exists $t \in [\varepsilon, l_1]$ such that $x_i(t) > -m_i$, we take

$$t_* = \inf\{t \in [\varepsilon, l_1] : x_i(t) > -m_i\},$$

so $y(0) = y(t_*)$, which implies that there exists $\bar{t}$ with $y'(\bar{t}) = 0$, a contradiction. Then Hypothesis (F4) implies that $\tilde{x}_i(l_1^+) \leq -m_i$. As a consequence, we can prove that $\tilde{x}_i(t) \leq -m_i$ for all $t \in [0, T]$.

Finally, this implies that we have that

$$c_i(\tilde{x}_i(T) - \tilde{x}_i(0)) \geq 0,$$

so Condition (1.1) is satisfied. Theorem 1.31 implies that there exists $c \in \mathbb{R}^n$ and $(x, (l_1, v_1), \ldots, (l_q, v_q)) \in \mathrm{Sol}(c)$ such that $\tilde{x}(T) - \tilde{x}(0) = 0$, therefore $\tilde{x}$ is a solution of (2.41). □

Example 2.46. Consider the interval $[0, 1]$ and the system

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq \tau(x), \\ \Delta x(t) = I(x(t)), & t = \tau(x), \\ x(0) = x(1), \end{cases} \tag{2.43}$$

with

$$\begin{aligned} f(t, x) &= (\cos(\pi t)^2 \cos(x) + 2)x + \sin(2\pi t) + 1, \\ \tau(x) &= \frac{1}{2} + \frac{2}{5 \cdot 2^{x^2} + 1}, \\ I(x) &= \mathrm{sign}(x) x^2. \end{aligned}$$

It is easy to check that Hypotheses (A3)–(A5) and (F1)–(F4) of Theorem 2.45 are satisfied. Then there exists a solution of (2.43). In Figure 2.5 we see a plot of the solution on the interval $[0, 1]$. As $f(0, x) = f(1, x)$, if we extend f to a periodic function in the first variable, and we consider $\tau_j(x) := \tau(x) + j$, $I_j(x) = I(x)$ for every $j \in \mathbb{Z}$, we have that the solution can be extended to $\mathbb{R}$. In Figure 2.6 we see a plot of this extension.

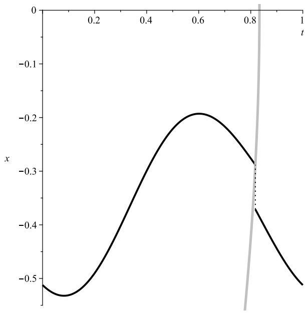


Figure 2.5: Periodic solution of (2.43) in Example 2.46. The black curve represents the solution on the interval $[0, 1]$, while the gray curve represents the hypersurface $\{(t, x) \in \mathbb{R}^2 : t = \tau(x)\}$.

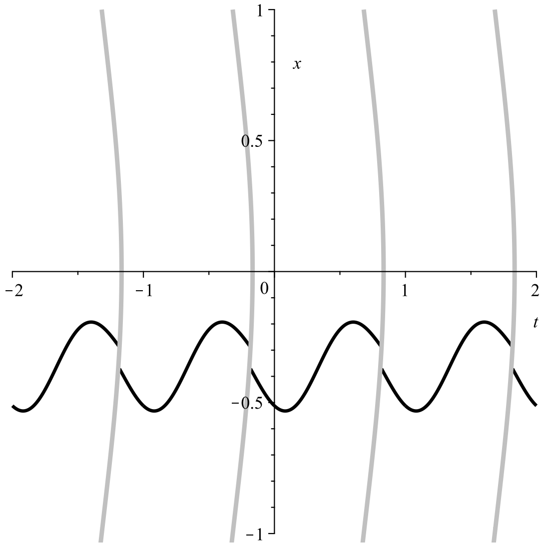


Figure 2.6: Extended periodic solution of (2.43) in Example 2.46.

To finish the section, we consider the following second order differential equation with state-dependent impulses:

$$\begin{cases} x''(t) + g(x(t)) = p(t, x(t), x'(t)), & t \neq \tau_j(x(t), x'(t)), \\ x(t^+) = x(t) + I_j(x(t), x'(t)), & \\ x'(t^+) = x'(t) + J_j(x(t), x'(t)), & t = \tau_j(x(t), x'(t)), \end{cases} \tag{2.44}$$

where $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $p\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is bounded, continuous, and T-periodic in the first variable, and $I_j, J_j, \tau_j\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ are also continuous maps.

Our aim is to obtain the existence of periodic solutions for (2.44) via the study of the Poincaré map and the use of some fixed point theorems on finite dimensional spaces.

In order to study problem (2.44), we reduce it to a boundary value problem, and we look for solutions in the interval $[0, T]$ which coincide at its initial and final times, that is,

$$\begin{cases} x''(t) + g(x(t)) = p(t, x(t), x'(t)), & t \neq \tau_j(x(t), x'(t)), \\ x(t^+) = x(t) + I_j(x(t), x'(t)), & \\ x'(t^+) = x'(t) + J_j(x(t), x'(t)), & t = \tau_j(x(t), x'(t)), \\ x(0) = x(T), \ x'(0) = x'(T). & \end{cases} \tag{2.45}$$

If we can find a solution of (2.45), we can extend it to $\mathbb{R}$, supposing that the maps τ_j, I_j and J_j satisfy

$$\tau_{j+q} = \tau_j + T, \quad I_{j+q} = I_j + T, \quad \text{and} \quad J_{j+q} = J_i + T$$

for some $q \in \mathbb{N}$ and for all $j \in \mathbb{Z}$.

For simplicity, we will consider the case with just one "hypersurface" in $[0, T]$. Moreover, we will denote the map τ by γ, because τ is going to be used for the time-map. Therefore, we write (2.45) as the following equivalent system of first order

$$\begin{cases} x'(t) = y(t), & \\ y'(t) = -g(x(t)) + p(t, x(t), y(t)), & t \neq \gamma(x(t), y(t)), \\ x(t^+) = x(t) + I_1(x(t), y(t)), & \\ y(t^+) = y(t) + J_1(x(t), y(t)), & t = \gamma(x(t), y(t)), \\ x(0) = x(T), \ y(0) = y(T). & \end{cases} \tag{2.46}$$

We assume (A1)–(A5) considered in Section 2.1. This implies that every solution of the differential system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -g(x(t)) + p(t, x(t), y(t)), \end{aligned}$$

intersects the surface $t = \gamma(x, y)$ only once (see Theorem 2.12). For every initial data $u_0 = (x_0, y_0)$, we consider t_{u_0} that time of impulsive effect. Furthermore,

Lemma 2.47. *The map $\mathit{\Gamma}\colon u_0 \in \mathbb{R}^2 \longmapsto t_{u_0} \in (0, T)$ is continuous.*

Let $u(\cdot; u_0)$ denote the unique solution of

$$\begin{cases} x'(t) = y(t), & \multirow{2}{*}{$t \neq \gamma(x(t), y(t)),$} \\ y'(t) = -g(x(t)) + p(t, x(t), y(t)), & \\ x(t^+) = x(t) + I_1(x(t), y(t)), & \multirow{2}{*}{$t = \gamma(x(t), y(t)),$} \\ y(t^+) = y(t) + J_1(x(t), y(t)), & \\ x(0) = x_0, \ y(0) = y_0. \end{cases}$$

As a consequence, we consider the map

$$P\colon u_0 \in \mathbb{R}^2 \longrightarrow u(T; u_0) \in \mathbb{R}^2, \tag{2.47}$$

and we can prove the following result.

Lemma 2.48. *The map P is continuous.*

Proof. We define the functions

$$\begin{aligned} &f_1\colon x \in \mathbb{R}^2 \longmapsto (x, x) \in \mathbb{R}^2 \times \mathbb{R}^2, \\ &f_2\colon (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto (t_x, y) \in [0, T] \times \mathbb{R}^2, \\ &f_3\colon (t, x) \in [0, T] \times \mathbb{R}^2 \longmapsto (t, u(t; 0, x)) \in [0, T] \times \mathbb{R}^2, \\ &f_4\colon (t, x) \in [0, T] \times \mathbb{R}^2 \longmapsto (t, x + I(x)) \in [0, T] \times \mathbb{R}^2, \\ &f_5\colon (t, x) \in [0, T] \times \mathbb{R}^2 \longmapsto u(T; t, x) \in \mathbb{R}^2. \end{aligned}$$

Each of these functions is continuous, and $P(u_0) = (f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(u_0)$. This implies that P is a continuous map. □

Our aim is to apply Theorems 1.28 and 1.29 to obtain fixed points of the map P, and thus solutions of (2.45). In order to do that, we will need the following definitions.

Definition 2.49. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous map. We say that F has the property of partial boundedness if there is a bounded set $D \subset \mathbb{R}^2$, a convex cone, and a curve $\Gamma\colon \lambda \in [0, \infty) \longrightarrow (x(\lambda), y(\lambda)) \in \mathbb{R}^2$, contained in the cone, such that

$$\lim_{\lambda \to \infty} (|x(\lambda)| + |y(\lambda)|) = +\infty \quad \text{and} \quad (F \circ \Gamma)([0, \infty)) \subset D.$$

An example of a function with this property can be seen in Figure 2.7.

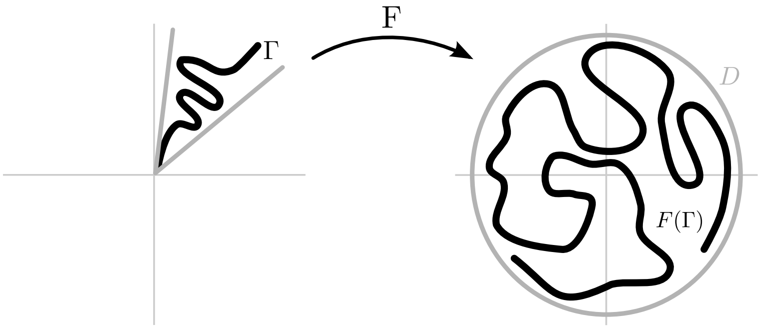


Figure 2.7: Example of a function with the partial boundedness property.

Definition 2.50. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous map, suppose that $|F(x,y)| \to \infty$ as $|(x,y)| \to \infty$, and let (Υ, Ξ) be F in polar coordinates, that is, Υ represents the modulus and Ξ represents the angle. We will say that F satisfies property (NR) if

$$\begin{aligned}&\text{there exists } R_0 > 0 \text{ such that } r > R_0 \text{ and } \theta \in \mathbb{R}\\ &\text{implies that } -\pi \le \Xi(r,\theta) - \theta \le 0.\end{aligned} \tag{NR}$$

We start by considering problem (2.46) without impulses, that is,

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -g(x(t)) + p(t, x(t), y(t)). \end{cases} \tag{2.48}$$

These type of systems (sometimes called of Duffing-type) have been widely studied, see for example [78, 80]. We consider (2.48) under the assumption that g and p are both locally Lipschitz and

$$\lim_{x\to+\infty} g(x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -\infty. \tag{g_∞}$$

The obvious examples of functions g which satisfy this condition (g_∞) are the functions $g(x) = x^{2n-1}$, with $n \in \mathbb{N}$. Furthermore, if p is a bounded function, then we can show that solutions are globally defined on $\mathbb{R}$ (see [60]).

The so-called time map plays a very important role to prove existence results in various boundary value problems related to ODEs, see for example [52, 97]. It arises from a phase-plane analysis of the autonomous differential equation

$$x''(t) + g(x(t)) = 0. \tag{2.49}$$

This is system (2.48) for $p \equiv 0$.

We define the function G as

$$G(x) = \int_0^x g(s)\, ds.$$

Condition (g_∞) implies that $G(x) \longrightarrow \infty$ when $|x| \longrightarrow \infty$. Furthermore, the function G is continuous. This implies that G is bounded from below, and therefore it has an absolute minimum. The first integral of (2.49) is the function

$$V(x,y) = \frac{y^2}{2} + G(x).$$

We can check that for c large enough, given a solution $(x(t), y(t))$ with orbit $V(x,y) = c$, then it is a periodic solution. Furthermore, for $\zeta > 0$ large enough and $x(0) = \zeta$, $y(0) = 0$, the least period of this solution, which is going to be denoted by $\tau(\zeta)$, is given by the expression

$$\tau(\zeta) = \sqrt{2} \int_{h(\zeta)}^{\zeta} \frac{1}{\sqrt{c - G(y)}}\, dy, \tag{2.50}$$

with $h(\zeta)$ a negative number such that $G(h(\zeta)) = G(\zeta)$ and $c = G(\zeta)$. This expression is not very hard to obtain. It is obtained, for example, in a similar case, in [77, Chapter 12]. For ζ large enough, then the map $\tau(\zeta)$ is continuous, and it is called the time-map. We are also going to consider the maps

$$\tau^+(\zeta) = \sqrt{2} \int_{0}^{\zeta} \frac{1}{\sqrt{c - G(y)}}\, dy,$$
$$\tau^-(\zeta) = \sqrt{2} \int_{h(\zeta)}^{0} \frac{1}{\sqrt{c - G(y)}}\, dy.$$

Then we have $\tau(\zeta) = \tau^-(\zeta) + \tau^+(\zeta)$. More information about these maps can be found, for example, in [52, 59, 60].

Lemma 2.51. *For each $R_1 > 0$, there is $R_2 > R_1$ such that*

- $|(x_0, y_0)| \leq R_1 \Longrightarrow |(x(t;0,(x_0,y_0)), y(t;0,(x_0,y_0)))| \leq R_2 \quad \forall\, t \in [0,T]$,
- $|(x_0, y_0)| \geq R_2 \Longrightarrow |(x(t;0,(x_0,y_0)), y(t;0,(x_0,y_0)))| \geq R_1 \quad \forall\, t \in [0,T]$.

The proof of this result is straightforward and can be found in [60].

Using the previous lemma, we know that for a given initial condition (x_0, y_0), with ${x_0}^2 + {y_0}^2$ sufficiently large, the solution $(x(t), y(t))$ satisfies

$$x(t)^2 + y(t)^2 \neq 0 \quad \text{for every } t \in [0,T].$$

This implies that we can consider polar coordinates on this problem, if we are far enough from the origin, that is, we consider the system

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{(g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta))\cos\theta}{r}, \\ r' = r\cos\theta\sin\theta - (g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta))\sin\theta. \end{cases} \tag{2.51}$$

For an initial condition (r_0, θ_0), take $(r(t; 0, r_0, \theta_0), \theta(t; 0, r_0, \theta_0))$ the solution of (2.51) verifying the initial condition (r_0, θ_0) at time $t = 0$. If the initial time is clear, we write $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$.

The proof of the following result is a consequence of results that can be found in [60].

Lemma 2.52. *Suppose* (g_∞) *is satisfied. Then there exists* $d > 0$ *sufficiently large such that*

$$r_0 > d \Longrightarrow \frac{d}{dt}\theta(t; r_0, \theta_0) < 0 \quad \textit{for every } \theta_0 \in \mathbb{R}.$$

This lemma tells us that the solutions of (2.48) turn clockwise around the origin.

Definition 2.53. Take Z_+ as the set of nonnegative integers. For $r \geq d$ and $t \in (0, T]$, let $n_*(r, t)$ and $n^*(r, t)$ be

$$n_*(r,t) = \max\left\{ n \in Z_+ : n \leq \inf\left\{ \frac{|\theta(t; r, \theta_0) - \theta_0|}{2\pi} : \theta_0 \in \mathbb{R} \right\}\right\},$$

$$n^*(r,t) = \min\left\{ n \in Z_+ : n \geq \sup\left\{ \frac{|\theta(t; r, \theta_0) - \theta_0|}{2\pi} : \theta_0 \in \mathbb{R} \right\}\right\}.$$

These two nonnegative numbers have the following interpretation: any solution $(x(t), y(t))$ with $\sqrt{x(0)^2 + y(0)^2} = r$ makes at least $n_*(r, t)$, and at most $n^*(r, t)$, turns around the origin on the interval $[0, t]$.

The behavior of the maps τ, τ^+ and τ^- is very important. We consider the following situations:

$$\lim_{\zeta \to +\infty} \tau(\zeta) = 0, \tag{τ_0}$$

$$\lim_{\zeta \to +\infty} \tau^+(\zeta) = \infty, \tag{τ_∞^+}$$

$$\lim_{\zeta \to +\infty} \tau^-(\zeta) = \infty. \tag{τ_∞^-}$$

As an example, we have the following results, which can be found, for example, in [59, Chapter 10].

Theorem 2.54. *Let* $g \colon \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function such that* g *is superlinear at infinity, that is,*

$$\lim_{|x| \to \infty} \frac{g(x)}{x} = +\infty.$$

Then (g_∞) *and* (τ_0) *are satisfied.*

Theorem 2.55. *Let $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that*

$$\lim_{|x| \to \infty} g'(x) = +\infty.$$

Then (g_∞) and (τ_0) are satisfied.

Theorem 2.56. *Let $g \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function verifying (g_∞) and*

$$\lim_{x \to +\infty} \frac{g(x)}{x} = 0.$$

Then (τ_∞^+) is satisfied.

An analogous result holds for the limit at $-\infty$ and (τ_∞^-). We present for completeness a sketch of the proofs. The main idea of the proof of these theorems is to consider polar coordinates. Then $\tau(\zeta)$ can be expressed as

$$\tau(\xi) = \int_0^{2\pi} \frac{1}{\sin^2\theta + g(r\cos\theta)\cos^2\theta/(r\cos\theta)}. \tag{2.52}$$

Fix $\varepsilon > 0$ sufficiently small. Take $A, B > 0$ sufficiently large such that $A/B < \varepsilon$. Then we consider

$$C_1 = \{(x, y) \in \mathbb{R}^2 : |x| \leq A, y > B\}, \qquad C_2 = \{(x, y) \in \mathbb{R}^2 : x \geq A\},$$
$$C_3 = \{(x, y) \in \mathbb{R}^2 : |x| \leq A, y < -B\}, \qquad C_4 = \{(x, y) \in \mathbb{R}^2 : x \leq -A\},$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4.$$

We have, for sufficiently large ζ, that the closed orbits that are periodic solutions are contained in C. Take $[t_j, t_{j+1}]$ the interval of time such that the solutions stays in C_j. Therefore, we have that

$$\tau(\zeta) = (t_2 - t_1) + \cdots + (t_5 - t_4) = t_5 - t_1.$$

It can be proved easily that $t_2 - t_1 < 2A/B$ and $t_4 - t_3 < 2A/B$. On the one hand, if g is superlinear at infinity, we have that $t_3 - t_2$ and $t_5 - t_4$ is sufficiently small, because $g(x)/x$ is large for $|x|$ large. On the other hand, if g is sublinear at infinity, we have that $t_3 - t_2$ and $t_5 - t_4$ is large, because $g(x)/x$ is positive and small for large x.

These previous results show that conditions (τ_0), (τ_∞^+), and (τ_∞^-) are satisfied by functions g with superlinear or sublinear behavior at infinity.

The following two lemmas highlight the difference between conditions (τ_0) and (τ_∞^+) together with (τ_∞^-). The common idea of their proofs is to consider a set of the form

$$D = \left\{(a, b) \in \mathbb{R}^2 : |(a, b)| \geq R\right\},$$

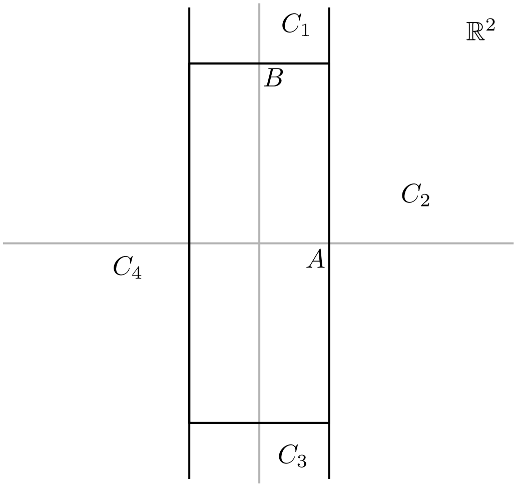


Figure 2.8: Division of C into four regions.

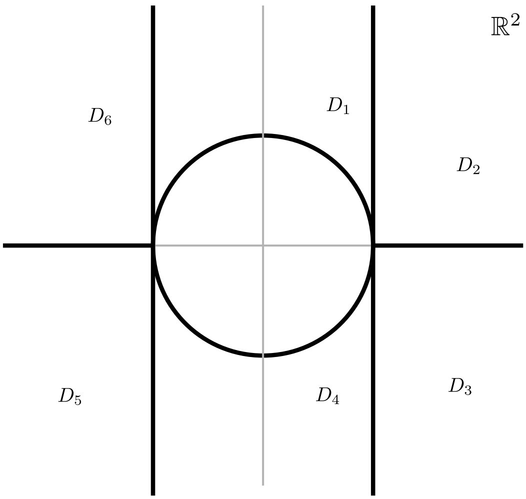


Figure 2.9: Division of D into six regions.

for some R big enough. This set D can be divided into six regions (see Figure 2.9). Furthermore, (g_∞) implies that any solution of (2.48) moves from D_i to D_j following the cycle

$$D_1 \longrightarrow D_2 \longrightarrow D_3 \longrightarrow D_4 \longrightarrow D_5 \longrightarrow D_6 \longrightarrow D_1.$$

Lemma 2.57. *If* (g_∞) *and* (τ_0) *are satisfied, then*

$$\lim_{r\to\infty} n_*(r,t) = +\infty.$$

The idea of the proof of Lemma 2.57 can be found in Section 5 of [60]. We make here a sketch for completeness.

The main objective is to prove the following claim: for every $\varepsilon \in (0,1)$, there exists $R(\varepsilon)$ sufficiently large such that

$$r \geq R(\varepsilon) \Longrightarrow n_*(r,t) \geq \left\lfloor \frac{t}{6\varepsilon} - 3 \right\rfloor,$$

with $\lfloor\cdot\rfloor$ representing the floor function. In order to prove this claim, for a fixed ε, we can find $R_0(\varepsilon) \leq R_1(\varepsilon) \leq R_2(\varepsilon)$ sufficiently large using (g_∞), (τ_0), and Lemma 2.51 verifying some technical conditions. Suppose that $|u(0)| \geq R_2(\varepsilon)$ and $|u(s)| \geq R_1(\varepsilon)$ for every $s \in [0,t]$. We take

$$D = \left\{(a,b) \in \mathbb{R}^2 : |(a,b)| \geq R_1(\varepsilon)\right\},$$

and we consider the sets $D_1, \ldots, D_6$ as in the Figure 2.9.

We have not proved that a solution crosses from D_i to D_j. Theoretically, a solution could stay on some D_i for every $s \in [0,t]$. However, it can be proved that

$$\frac{|\theta(t) - \theta(0)|}{2\pi} \geq n_k - 2$$

for any $k \in \{1, \ldots, 6\}$, with n_k the number of different subintervals of $[0,t]$ such that $u(s) \in D_k$. The next step is to find upper bounds for the length of each subinterval. It can be proved that this length is always less or equal than ε, so the claim is proved taking $R(\varepsilon) := R_2(\varepsilon)$. Therefore, $n_*(r,t) \longrightarrow \infty$ as $\varepsilon \longrightarrow 0^+$.

Lemma 2.58. *If the conditions* (g_∞), (τ_∞^+), *and* (τ_∞^-) *are satisfied, a solution satisfies* $r(t;z_0) \geq R_0$, *and* $\theta(t_2;z_0) - \theta(t_1;z_0) = -2\pi$, *then* $t_2 - t_1$ *is arbitrarily large provided* R_0 *is large enough.*

This is basically [59, Lemma 10.5]. We make here a sketch of the proof for completeness. If the solution rotates once around the origin (which is precisely what the condition $\theta(t_2;z_0) - \theta(t_1;z_0) = -2\pi$ says), then there is an interval

$[s_1, s_2]$ such that the solution stays in D_2 for all times on $[s_1, s_2]$. An analogous argument can be done for the sets D_3, D_5, and D_6. Then we have that

$$\begin{aligned} x(t) > c_0,\ x'(t) = y(t) > 0 \quad \text{for every } t \in (s_1, s_2), \\ x(s_1) = c_0,\ x(s_2) = x_2 > c_0,\ y(s_2) = 0, \end{aligned}$$

for some $c_0 \geq R_0$. Then, it can be proved that

$$y(t)y'(t) \geq -g(x(t))x'(t) + mx'(t),$$

with $m = \min\{p(t, x, y) : t \in [0, T], x, y \in \mathbb{R}\}$. After using the mean value theorem and integrating on $[t, s_2]$ with $t \in (s_1, s_2)$, we have that

$$y(t)^2 \leq 2[G(x_2) - G(x(t))].$$

Moreover, $y(t)$ is positive, so we have that

$$0 < \frac{y(t)}{\sqrt{2(G(x_2) - G(x(t)))}} \leq 1.$$

Therefore, we get

$$\frac{\tau^+(x_2) - \tau^-(c_0)}{2} \leq s_2 - s_1.$$

As c_0 is fixed and $x_2 > R_0$, (τ^+_∞) implies that, for R_0 sufficiently large, $s_2 - s_1$ can be large enough.

The proof for D_3, D_5, and D_6 is similar, taking into account the signs of $x(t)$ and $y(t)$.

Lemma 2.59. *Suppose* (g_∞) *and* (τ_0) *are satisfied, and* $t \in (0, T]$*. Then*

$$\forall\, N \in \mathbb{N},\ \exists\, \rho > 0 : r_0 > \rho \Longrightarrow \theta(t; r_0, \theta_0) - \theta_0 < -2N\pi \quad \forall\, \theta_0 \in \mathbb{R}.$$

Proof. Let $N \in \mathbb{N}$ be fixed. Lemma 2.57 implies that

$$\lim_{r \to \infty} n_*(r, t) = +\infty.$$

There exists $\rho > 0$ such that $r \geq \rho \Longrightarrow n_*(r, t) \geq N$. Without loss of generality, we may assume $\rho > d$, with d coming from Lemma 2.52. Then

$$\begin{aligned} N \leq n_*(r, t) \leq \frac{|\theta(t; r, \theta_0) - \theta_0|}{2\pi} \quad \forall\, \theta_0 \in \mathbb{R} \\ \Longrightarrow \theta_0 - \theta(t; r, \theta_0) \geq 2\pi N \quad \forall\, \theta_0 \in \mathbb{R}. \end{aligned}$$

The result follows from this last inequality. □

We state and prove the existence of periodic solutions for the impulsive system (2.46). The previous results, which were about the nonimpulsive case, give us the necessary framework to state and prove the existence of periodic solutions of (2.44).

Theorem 2.60. *Suppose* (A1)–(A5), (g_∞), *and* (τ_0) *are satisfied, and let*

$$\phi\colon (x,y) \in \mathbb{R}^2 \longrightarrow (x + I_1(x,y), y + J_1(x,y)) \in \mathbb{R}^2.$$

If ϕ *has the property of partial boundedness (see Definition* 2.49*), then* P *has at least one fixed point, that is, there exists at least one* T*-periodic solution of* (2.44).

Proof. There exist D a bounded subset $\mathbb{R}^2$, a convex cone, and a curve Γ starting at the origin, $\Gamma\colon \lambda \in [0,\infty) \longrightarrow (x(\lambda), y(\lambda)) \in \mathbb{R}^2$, contained in the cone, such that

$$\lim_{\lambda\to\infty} (|x(\lambda)| + |y(\lambda)|) = +\infty \quad \text{and} \quad (\phi \circ \Gamma)([0,\infty)) \subset D.$$

Without loss of generality, we may assume that D is a closed set, hence D is compact. The function f_5, as defined in the proof of Lemma 2.48, is also continuous. There exists $M_D > 0$ such that

$$|f_5(t,x)| \leq M_D \quad \text{for all } (t,x) \in [\gamma_-, \gamma_+] \times D,$$

which is a compact set. Take R_1 and R_2 sufficiently large, with $R_2 > R_1 > M_D$, such that

$$\theta(\gamma_+; R_1, \theta_0) - \theta(0; R_1, \theta_0) > -a,$$
$$\theta(\gamma_-; R_2, \theta_0) - \theta(0; R_2, \theta_0) < -a - 4\pi,$$

for some $a > 0$ and all $\theta_0 \in \mathbb{R}$. This follows from Lemma 2.59 and the fact that $0 \leq n_*(r,t) \leq n^*(r,t)$. We restrict ourselves to initial arguments on $[0, 2\pi]$. Then we have that

$$\begin{aligned}\theta(t_{(R_2,\theta_0)}; R_2, \theta_0) - \theta(t_{(R_1,\theta_1)}; R_1, \theta_1) &\leq \theta(\gamma_-; R_2, \theta_0) - \theta(\gamma_+; R_1, \theta_1)\\ &< -a - 4\pi + a + (\theta_0 - \theta_1) \leq -4\pi + 2\pi = -2\pi \end{aligned} \tag{2.53}$$

for $\theta_0, \theta_1 \in [0, 2\pi)$, with $t_{(R_j,\theta_i)}$ the unique impulsive point.

Take $C_i = \{z \in \mathbb{R}^2 : |z| = R_i\}$, $i \in \{1,2\}$. In order to use Theorem 1.29, let $\beta\colon I \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ a curve connecting C_1 and C_2, with z_1 and z_2 its initial and final points. The curve

$$\tilde{\beta}\colon t \in I \longmapsto P_2(f_3(f_2(f_1(\beta(t))))) \in \mathbb{R}^2$$

makes at least one turn around the origin because of (2.53), where the map $P_2\colon [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denotes the projection (see Figure 2.10). So the curves

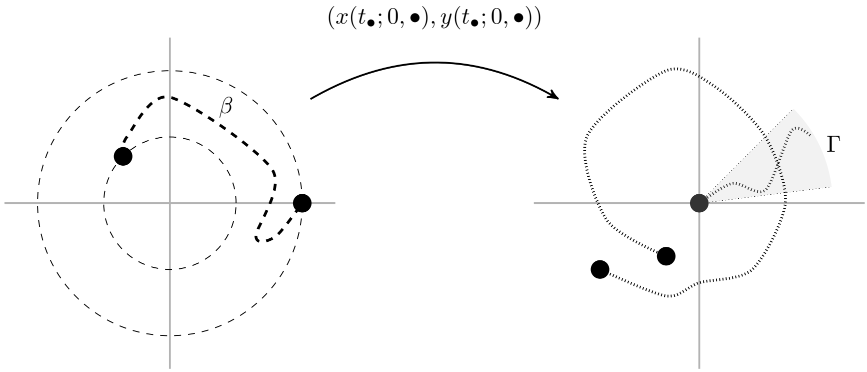


Figure 2.10: Intersection of the curves $\tilde{\beta}$ and Γ.

$\tilde{\beta}$ and Γ intersect at least in one point, because the curve Γ is inside a convex cone. Let z be that point. Then $\phi(z) \in D$. Furthermore, we take t^* such that $\tilde{\beta}(t^*) = z$, and define $w = \beta(t^*)$. Then we have that

$$\begin{aligned}
&f_4(f_3(f_2(f_1(w)))) = (t_w, \phi(z)),\\
&|f_5(t,x)| \le M_D \quad \text{for every } (t,x) \in [\gamma_-, \gamma_+] \times D,\\
&(t_w, \phi(z)) \in [\gamma_-, \gamma_+] \times D,\\
&M_D < R_1 \le |w| \le R_2.
\end{aligned}$$

This implies that $|P(w)| \le M_D < |w|$. Therefore, the map P satisfies the hypotheses of Theorem 1.29. We can guarantee that the map $P\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has at least one fixed point, so there exists a solution of (2.46), and thus a T-periodic solution of (2.44). □

Example 2.61. Consider the following problem

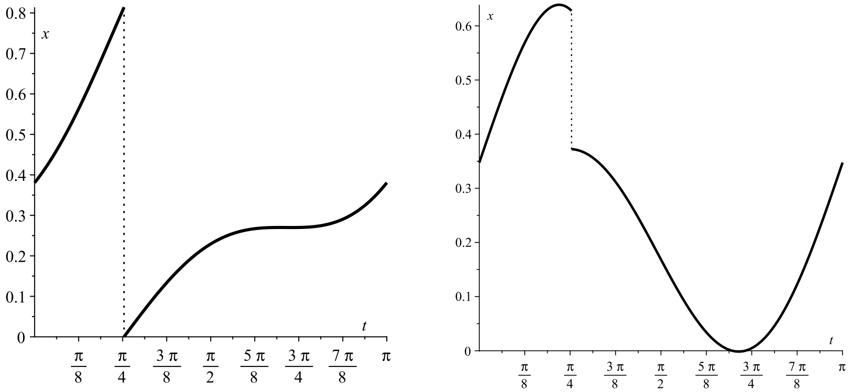
$$\begin{cases} x''(t) + x^3(t) = (\arctan(x(t)) + \cos(2t))/2, & t \neq \gamma(x(t), x'(t)),\\ x(t^+) = x(t) + I_1(x(t), x'(t)), & \\ x'(t^+) = x'(t) + J_1(x(t), x'(t)), & t = \gamma(x(t), x'(t)), \end{cases}$$

with $T = \pi$, $I_1(x,y) = -x$,

$$J_1(x,y) = \begin{cases} \dfrac{-|x|y}{2}, & |x| \le 1,\\ \dfrac{-y}{2}, & |x| \ge 1, \end{cases}$$

and

$$\gamma(x,y) = \frac{1}{2} + \arctan\left(\frac{1}{4}x^4 + \frac{1}{2}y^2\right).$$



(a) Plot of the solution.

(b) Plot of the derivative.

Figure 2.11: Solution of Example 2.61.

Let us check the hypotheses of Theorem 2.60.

- $g(x) = x^3$ is superlinear at infinity, so Theorem 2.54 implies (g_∞) and (τ_0).
- $p(t, x, y) = (\arctan(x) + \sin(2t))/2$ is continuous, locally Lipschitz, and bounded.
- Hypotheses (A1)–(A3) are straightforward.
- Hypothesis (A4) is satisfied, as it is easy to check that

$$\gamma(x + I_1(x, y), y + J_1(x, y)) \leq \gamma(x, y) \quad \text{for } x, y \in \mathbb{R}.$$

- Hypothesis (A5) is satisfied, since

$$\begin{aligned} D\gamma(x, y) \cdot f(t, x, y) &= D_1\gamma(x, y) \cdot y + D_2\gamma(x, y) \cdot (-g(x) + p(t, x, y)) \\ &= \frac{8y(\arctan(x) + \cos(2t))}{x^8 + 4x^4y^2 + 4y^4 + 16} \leq 0.7 < 1. \end{aligned}$$

- The map $(x, y) \longmapsto (x + I_1(x, y), y + J_1(x, y))$ satisfies the partial boundedness property, by taking the curve $\Gamma \colon \lambda \in [0, \infty) \longrightarrow (\lambda, 0) \in \mathbb{R}^2$.

Theorem 2.60 can be applied, so there exists a π-periodic solution of this problem. On Figure 2.11 we represent the approximate solution and the derivative.

We prove now the second result for the existence of periodic solutions for (2.46).

Theorem 2.62. *Suppose* (A1)–(A5), (g_∞), (τ_∞^+), *and* (τ_∞^-) *are satisfied and let*

$$\phi\colon (x,y)\in\mathbb{R}^2 \longrightarrow (x+I_1(x,y), y+J_1(x,y))\in\mathbb{R}^2.$$

If ϕ satisfies Property (NR), *then P has at least one fixed point, that is, there exists at least one T-periodic solution of* (2.44).

Proof. Take R_0, R_1, R_2 large enough such that

$$\begin{aligned}
&r(t;0,r_0,\theta_0)\geq R_1, \quad \text{for all } r_0>R_0,\ t\in[0,\gamma_+],\ \theta_0\in\mathbb{R},\\
&-\pi/2<\theta(\gamma_+;0,r_0,\theta_0)-\theta_0<0, \quad \text{for all } r_0>R_0,\ \theta_0\in\mathbb{R},\\
&\Upsilon(r_0,\theta_0)\geq R_2, \quad \text{for all } r_0\geq R_1,\ \theta_0\in\mathbb{R},\\
&-\pi\leq\Xi(r_0,\theta_0)-\theta_0\leq 0, \quad \text{for all } r_0>R_1,\ \theta_0\in\mathbb{R},\\
&-\pi/2<\theta(T;t,r_0,\theta_0)-\theta_0<0, \quad \text{for all } r_0\geq R_2,\ t\in[\gamma_-,\gamma_+],\ \theta_0\in\mathbb{R}.
\end{aligned}$$

This is possible by Lemmas 2.51 and 2.58 and Property (NR) of ϕ.

We are using polar coordinates on the system without impulses. On the impulsive case, we have that, for an initial condition z_0 with polar coordinates (r_0,θ_0),

$$\begin{aligned}
&t_1=t_{z_0},\ r_1=r(t_1;0,r_0,\theta_0),\ \theta_1=\theta(t_1;0,r_0,\theta_0),\\
&r_1^+=\Upsilon(r_1,\theta_1),\ \theta_1^+=\Xi(r_1,\theta_1),\\
&r_2=r(T;t_1,r_1^+,\theta_1^+),\ \theta_2=\theta(T;t_1,r_1^+,\theta_1^+).
\end{aligned}$$

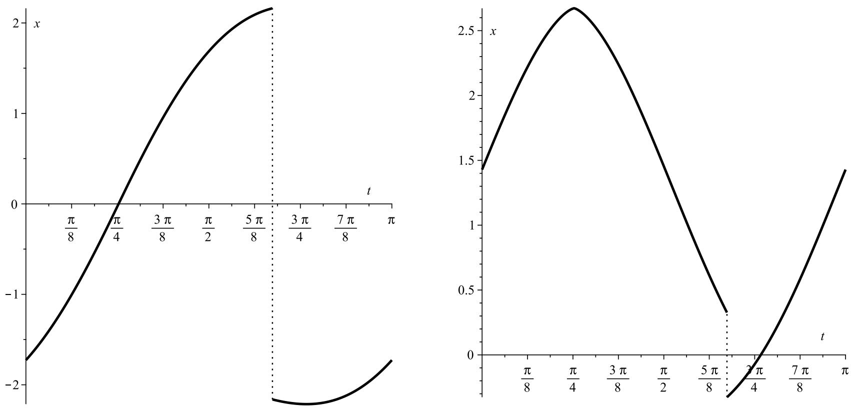
Then, for the impulsive system, the values at time T are given by (r_2,θ_2), that is, the Poincaré map in polar coordinates for $r\geq R_0$ is precisely (r_2,θ_2). Furthermore, consider the disk $D=B(0,R_0)$. Then, for any $z_0\in\partial D$, if (r_0,θ_0) are the polar coordinates of z_0, we have that,

$$\begin{aligned}
&-\pi/2<\theta(t_1;0,r_0,\theta_0)-\theta_0<0,\\
&-\pi\leq\Xi(r_1,\theta_1)-\theta_1\leq 0,\\
&-\pi/2<\theta(T;t_1,r_1^+,\theta_1^+)-\theta_1^+<0.
\end{aligned}$$

This implies that $-2\pi<\theta(T;t_1,r_1^+,\theta_1^+)-\theta_0<0$, that is, $-2\pi<\theta_2-\theta_0<0$. Therefore, we have that the Poincaré map satisfies the hypotheses of Theorem 1.28, so there exists at least one T-periodic solution of (2.44). □

Example 2.63. Consider the following problem

$$\begin{cases}
x''(t)+x(t)^{1/3}=\cos(2t), & t\neq\gamma(x(t),x'(t)),\\
x(t^+)=x(t)+I_1(x(t),x'(t)), & \\
x'(t^+)=x'(t)+J_1(x(t),x'(t)), & t=\gamma(x(t),x'(t)),
\end{cases}$$



(a) Plot of the solution. (b) Plot of the derivative.

Figure 2.12: Solution of Example 2.63.

with $T = \pi$, $I_1(x, y) = -2x$, $J_1(x, y) = -2y$, and

$$\gamma(x, y) = 2 + \exp\left(-\frac{3}{4}x^{4/3} - \frac{1}{2}y^2\right).$$

The hypotheses of Theorem 2.62 hold, as we have that $\phi(x, y) = (-x, -y)$, which implies that $\Xi(r, \theta) - \theta = -\pi$. Therefore, there exists a T-periodic solution. In Figure 2.12 we represent the approximate solution and the derivative of this problem.

Chapter 3

Impulsive Dynamical Systems

This chapter focuses on the theory of impulsive dynamical systems. This theory describes models in which a continuous evolution is abruptly interrupted by sudden changes of state, which, in applications, can be interpreted as either jumps of state or forced corrections to the evolution law, in order to prevent unwanted results. We will mainly focus on the study of impulsive nonautonomous dynamical systems, in particular impulsive evolution processes, and the existence of attractors, along with some properties.

This chapter is divided as follows. We start with an introduction in Section 3.1, which includes a summary of results for the autonomous framework. In Section 3.2 we develop the theory of impulsive evolution processes, together with some properties. In Section 3.3, we present the concepts of pullback semiattractors and pullback attractors for impulsive evolution processes, and we obtain results to ensure their existence, developing two different but related approaches (namely, the nonautonomous tube conditions in Subsection 3.3.1 and other conditions like Condition (T) in Subsection 3.3.2). In Section 3.4, we begin the study of perturbations of attractors in the impulsive framework. We obtain some results regarding the upper semicontinuity (nonexplosion of solutions), and a weaker version of the lower semicontinuity (nonimplosion of solutions). In Section 3.5, we extend some results given in Sections 3.2 and 3.3 to the multivalued situation, defining the notion of impulsive generalized processes. In Section 3.6, we study nonautonomous dynamical systems using cocycles. Here, in contrast with previous works, we will assume that the impulses occur in the driving semigroup and not directly in the cocycle. Finally, in Section 3.7, we will give applications and examples of the results obtained in the previous sections. These applications include, among others, a nonautonomous integrate-and-fire model, an impulsive and two-dimensional Navier–Stokes equation, an impulsive multivalued reaction-diffusion equation, and cascade systems.

3.1 Introduction

Some early results about the theory of impulsive dynamical systems can be seen in the early 1970s, by Russian physicist V. Rozko [105–107]. Later we can highlight some results by Kaul [81–83] and Ciesielski [47–50]. In recent years, specially in the case of impulsive autonomous dynamical systems, there have been several results on attractors and its properties [14, 16, 19–23, 53, 56–58].

We start with a continuous semigroup, as in Definition 1.42. For each $A \subset X$ and $J \subset [0, +\infty)$, we define the set

$$F(A, J) := \{x \in X : \pi(t)x \in A \text{ for some } t \in J\} = \bigcup_{t \in J} \pi(t)^{-1}(A).$$

Definition 3.1. An impulsive dynamical system, which will be denoted by (π, X, M, I), consists in a semigroup π in a metric space X, a nonempty, closed subset $M \subset X$ such that for every $x \in M$ there exists $\varepsilon = \varepsilon(x) > 0$ satisfying

$$\bigcup_{t \in (0,\varepsilon)} \{\pi(t)z\} \cap M = \emptyset, \tag{3.1}$$

and a continuous function $I \colon M \longrightarrow X$.

The set M will be called the impulsive set and the function I the impulse or impulsive function. The role of I will be explained in Definition 3.4.

Remark 3.2. Condition (3.1) is based on [53], and it is different from the usual condition for impulsive dynamical systems (see for example [19, Definition 1.2] or [23, Definition 2.7]).

Consider the set

$$M^+(x) = \left(\bigcup_{t>0} \pi(t)x\right) \cap M \qquad \text{for each } x \in X. \tag{3.2}$$

It can be proved that, if $M^+(x) \neq \emptyset$, then there exists a unique $s > 0$ such that $\pi(s)x \in M$ and $\pi(t)x \notin M$ for $0 < t < s$. The idea of the proof can be seen in Proposition 3.28, which is a generalization of this result. As a consequence, we can define the function $\phi \colon X \longrightarrow (0, \infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ \infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

If $M^+(x) \neq \emptyset$, then $\phi(x)$ denotes the smallest positive time such that the trajectory of x meets M, that is, if $\pi(t)x \in M$, then $\phi(x) \leq t$. This map is called the impact time map of π. We can give its first property.

Proposition 3.3. *The impact time map ϕ is lower semicontinuous in $X \setminus M$.*

Additional conditions (some of which will be explained later) are required in order to prove the upper semicontinuity.

Definition 3.4. Let $x \in X$. The impulsive semitrajectory of x, defined in an interval $[0, \ell(x))$, with $\ell(x) \in (0, \infty]$, and denoted by $\tilde{\pi}(\cdot)x$, is given by:

Step 1: If $M^+(x) = \emptyset$, then $\phi(x) = \infty$. Then we define $\tilde{\pi}(t)x = \pi(t)x$ for all $t \geq 0$. This implies that, in this situation, the process ends. If $M^+(x) \neq \emptyset$, then we denote $x_0^+ = x$, $s_0 = t_0 = \phi(x_0^+) < \infty$, $x_1^+ = I(\pi(s_0)x_0^+)$, and define $\tilde{\pi}(\cdot)x$ in $[0, t_0]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & 0 \leq t < t_0, \\ x_1^+, & t = t_0. \end{cases}$$

From this definition we go to Step 2.

Step 2: If $M^+(x_1^+) = \emptyset$, then $\phi(x_1^+) = \infty$, and we define $\tilde{\pi}(t)x = \pi(t - t_0)x_1^+$ for all $t \geqslant t_0$. In this case, the process ends here. However, if $M^+(x_1^+) \neq \emptyset$, then we denote $s_1 = \phi(x_1^+) < \infty$, $t_1 = s_1 + t_0$, $x_2^+ = I(\pi(s_1)x_1^+)$, and define $\tilde{\pi}(\cdot)x$ in $[t_0, t_1]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - t_0)x_1^+, & t_0 \leq t < t_1, \\ x_2^+, & t = t_1, \end{cases}$$

and, in this case, we move to the next step.

Inductive Step: Assume that $\tilde{\pi}(\cdot)x$ is defined in the interval $[t_{n-1}, t_n]$, and that $\tilde{\pi}(t_n)x = x_{n+1}^+$ (setting $t_{-1} = 0$), where $s_n = \phi(x_n^+)$ and $t_n = s_n + t_{n-1}$. If $M^+(x_{n+1}^+) = \emptyset$, then $\phi(x_{n+1}^+) = \infty$, and we define $\tilde{\pi}(t)x = T(t - t_n)x_{n+1}^+$ for $t \geq t_n$. In this case, the process ends here. However, if $M^+(x_{n+1}^+) \neq \emptyset$, then we denote $s_{n+1} = \phi(x_{n+1}^+) < \infty$, $t_{n+1} = s_{n+1} + t_n$, $x_{n+2}^+ = I(\pi(s_{n+1})x_{n+1}^+)$, and define $\tilde{\pi}(\cdot)x$ in $[t_n, t_{n+1}]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - t_n)x_{n+1}^+, & t_n \leq t < t_{n+1}, \\ x_{n+2}^+, & t = t_{n+1}. \end{cases}$$

Conclusion Step: This process can end after a finite number of steps, or continue indefinitely. The first case happens when $M^+(x_n^+) = \emptyset$ for some $n \in \mathbb{N}$. The second case happens when $M^+(x_n^+) \neq \emptyset$ for all $n \in \mathbb{N}$. In the first case, $\tilde{\pi}(\cdot)x$ is defined in $[0, \infty)$ (and $\ell(x) = \infty$), but in the second case it is defined in the interval $[0, \ell(x))$, with

$$\ell(x) = \lim_{n \to \infty} t_n = \sum_{i=0}^{\infty} s_i,$$

and it can be either finite or infinity.
The times $\{t_n\}_n$ will be called the jump times of the impulsive dynamical system (π, X, M, I) at x.

From the definition we have the following useful results.

Proposition 3.5. *Let (π, X, M, I) be an impulsive dynamical system such that $I(M) \cap M = \emptyset$. Then $\tilde{\pi}(t)x \notin M$ for any $x \in X$ and $t \in (0, \ell(x))$.*

Proposition 3.6. *Let (π, X, M, I) be an impulsive dynamical system. Then*

1. *$\tilde{\pi}(0)x = x$ for every $x \in X$,*
2. *$\tilde{\pi}(t+s)z = \tilde{\pi}(t)\tilde{\pi}(s)z$ for each $t, s \in [0, \ell(x))$ with $t + s \in [0, \ell(x))$.*

In order to study global attractors for autonomous dynamical systems, we need to assume that all impulsive trajectories exist for all $t \geq 0$, that is,

$$\ell(x) = \infty \quad \text{for all } x \in X. \tag{3.3}$$

In order to simplify the exposition, we will assume that

$$\text{There exists } \xi > 0 \text{ such that } \phi(x) \geq 2\xi \text{ for all } x \in I(M). \tag{$\mathrm{H_{aut}}$}$$

It is easy to see that Condition ($\mathrm{H_{aut}}$) implies (3.3). However, we are also going to need this condition in some proofs. From now on, when referring to an impulsive dynamical system (π, X, M, I), we will implicitly assume that Condition ($\mathrm{H_{aut}}$) holds.

Definitions 1.45, 1.46, 1.47, 1.54, and 1.56 for continuous semigroups can be extended for impulsive semigroups $\tilde{\pi}$, we just replace π by $\tilde{\pi}$. They concern the concepts of invariance, attraction, ω-limit set, asymptotic compactness, and dissipativeness. However, the fact that impulsive trajectories are not continuous implies that the definition of global attractor for an impulsive dynamical system (π, X, M, I) needs to be changed with respect to the continuous case (see Definition 1.51).

Definition 3.7. Let $A \subset X$ and (π, X, M, I) an impulsive dynamical system. We say that A is a semiattractor for (π, X, M, I) if it is compact and $\tilde{\pi}$-attracts every bounded subset of X. Moreover, we say that A is a global attractor for (π, X, M, I) if the set $A \setminus M$ is $\tilde{\pi}$-invariant.

Remark 3.8. For impulsive dynamical systems, this definition of global attractor does not imply uniqueness. In fact, if A_1 and A_2 are two global attractors for (π, X, M, I), then $A_1 \setminus M = A_2 \setminus M$.

Definition 3.9. Let (π, X, M, I) be an impulsive dynamical system and $x \in X$. We say that $\psi \colon \mathbb{R} \longrightarrow X$ is a global solution through x if $\psi(0) = x$ and $\tilde{\pi}(t)\psi(s) = \psi(t+s)$ for $t \geq 0$ and $s \in \mathbb{R}$.

Theorem 3.10. *Let (π, X, M, I) be an impulsive dynamical system and A a global attractor. Then*

$$A \setminus M = \{x \in X : \text{ there exists } \psi \text{ a bounded global solution through } x\}.$$

The following result is very similar to the one we have on the continuous case, see Theorem 1.57.

Theorem 3.11. *An impulsive dynamical system* (π, X, M, I) *has a semiattractor if and only if* $\tilde{\pi}$ *is asymptotically compact and dissipative. The semiattractor* A*, if it exists, will be given by* $A = \tilde{\omega}(B_0)$*, with* B_0 *an absorbing set.*

The following example (taken from [23, Example 3.11]) shows that a semiattractor may not be a global attractor.

Example 3.12. Consider the differential system

$$\begin{cases} x'(t) = x(t)(1 - x(t))(1 + x(t)), \\ y'(t) = -y(t), \end{cases}$$

and the semigroup π in $\mathbb{R}^2$ given by the solutions. Take $M = \{1/2\} \times [-1, 0]$ and $I(1/2, y) = (3/4, y)$ for any $y \in [-1, 0]$. We can take $B_0 = [-2, 2] \times [-2, 2]$ as an absorbing set, so the impulsive semigroup $\tilde{\pi}$ is dissipative and asymptotically compact, as B_0 is a compact set. Then $A = [-1, 1] \times \{0\}$ is the $\tilde{\omega}$-limit of B_0, it $\tilde{\pi}$-attracts all bounded sets, and it is compact. By Definition 3.7, it is a semiattractor. However, neither A nor $A \setminus M$ are $\tilde{\pi}$-negatively invariant.

To obtain a global attractor, as in [23], we require additional conditions. We present two different approaches, in line with what we will do for impulsive evolution processes. Both of the approaches ask for the semigroup π to behave nicely near M.

The first approach considered is the so-called tube conditions. First, we need an additional condition together with (3.1), which is:

$$\forall\, x \in M,\ \exists\, \varepsilon = \varepsilon(x) > 0 : F(x, (0, \varepsilon)) \cap M = \emptyset. \tag{3.4}$$

However, this new condition is still not enough for the invariance.

Definition 3.13. Let $\{\pi(t) : t \geq 0\}$ be a semigroup, $x \in X$, and S a closed subset of X such that $x \in S$. We say that S is a section through x if there exist $\lambda > 0$ and L a closed subset of X such that:

1. $F(L, \lambda) = S$,
2. $F(L, [0, 2\lambda])$ contains a neighborhood of x,
3. $F(L, \nu) \cap F(L, \zeta) = \emptyset$ if $0 \leq \nu < \zeta \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is said to be a λ-tube and the set L is a bar.

We illustrate a tube in Figure 3.1.

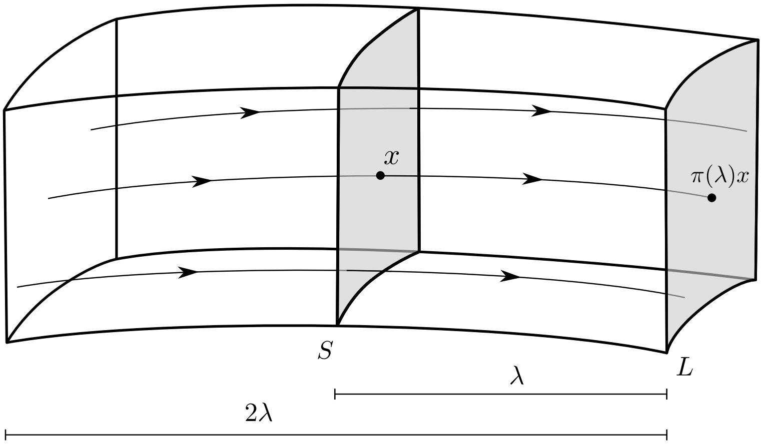


Figure 3.1: Example of a tube.

Definition 3.14. Let (π, X, M, I) be an impulsive dynamical system and $x \in X$. We say that x satisfies

- the tube condition ($\mathrm{TC}_{\mathrm{aut}}$) if there exists a λ-section S through x such that $S \subset M \cap F(L, [0, 2\lambda])$,
- the strong tube condition ($\mathrm{STC}_{\mathrm{aut}}$) if there exists a λ-section S through x such that $S = M \cap F(L, [0, 2\lambda])$,
- the special strong tube condition ($\mathrm{SSTC}_{\mathrm{aut}}$) if it satisfies ($\mathrm{STC}_{\mathrm{aut}}$) and

$$F(L, [0, \lambda]) \cap I(M) = \emptyset.$$

We have seen that the impact time map is lower semicontinuous at $X \setminus M$ (see Proposition 3.3). The tube condition helps us prove that the impact time map is upper semicontinuous on X.

Proposition 3.15. *Let (π, X, M, I) be an impulsive dynamical system such that every point in M satisfies ($\mathrm{TC}_{\mathrm{aut}}$). Then, the impact time map $\phi \colon X \longrightarrow (0, \infty]$ is upper semicontinuous.*

We present next some results about the behavior of the impulsive trajectories near M and some convergence results.

Proposition 3.16. *Let (π, X, M, I) be an impulsive dynamical system, $y \in M$ satisfying ($\mathrm{SSTC}_{\mathrm{aut}}$) with a λ-tube $F(L, [0, 2\lambda])$, and $I(M) \cap M = \emptyset$. Then $\tilde{\pi}(t)X \cap F(L, [0, \lambda]) = \emptyset$ for $t > \lambda$.*

Proposition 3.17. *Let (π, X, M, I) be an impulsive dynamical system with $I(M) \cap M = \emptyset$, every element of M satisfies* (STC$_{\text{aut}}$)*, $x \notin M$, $\{x_n\}_n$ a sequence convergent to x, and $t \geq 0$. Then there exists a sequence $\{\eta_n\}_n$ of nonnegative numbers such that $\tilde{\pi}(t + \eta_n)x_n \longrightarrow \tilde{\pi}(t)x$.*

Proposition 3.18. *Let (π, X, M, I) be an impulsive dynamical system such that every element of M satisfies* (STC$_{\text{aut}}$)*, $x \notin M$, and $\{x_n\}_n$ a sequence in $X \setminus M$ convergent to x. Then, for any sequence $\{\alpha_n\}_n$ of nonnegative numbers convergent to* 0*, we have $\tilde{\pi}(\alpha_n)x_n \longrightarrow x$.*

Proposition 3.19. *Let (π, X, M, I) be an impulsive dynamical system, $x \in M$ satisfying* (SSTC$_{\text{aut}}$)*, $F(L, [0, 2\lambda])$ the λ-tube, and $\{x_n\}_n$ a sequence in $F(L, (\lambda, 2\lambda])$ convergent to x. Then there exist a subsequence $\{x_{n_k}\}_k$ and a sequence of positive numbers $\{\varepsilon_k\}_k$ convergent to* 0 *such that $\pi(\varepsilon_k)x_{n_k} \in M$, $\phi(x_{n_k}) = \varepsilon_k$, and $\pi(\varepsilon_k)x_{n_k}$ converges to x.*

These last results allow us to prove that the set $\tilde{\omega}(B) \setminus M$ is $\tilde{\pi}$-invariant for any nonempty bounded subset B of X, as long as $\tilde{\omega}(B)$ is compact and $\tilde{\pi}$-attracts B. In particular, this works for the semiattractor given by Theorem 3.11. As a consequence, we get:

Theorem 3.20. *Let (π, X, M, I) be an impulsive dynamical system such that $\tilde{\pi}$ is asymptotically compact and dissipative. Suppose also that $I(M) \cap M = \emptyset$ and every point in M satisfies* (SSTC$_{\text{aut}}$)*. Then (π, X, M, I) has a global attractor $\mathcal{A}$.*

We present next the second approach to obtain the existence of a global attractor. First, we do not need Condition (3.4), which was needed in the previous approach. The main condition we need is the following one:

$$\begin{cases} \text{Let } t > 0,\ x \in M, \text{ and } \{z_n\}_n \text{ a convergent sequence in } X \text{ such} \\ \text{that } \pi(t)z_n \text{ converges to } x. \text{ Then there exist a subsequence} \\ \{z_{n_k}\}_k \text{ of } \{z_n\}_n \text{ and a sequence } \{\alpha_k\}_k \text{ convergent to 0 such} \\ \text{that } \pi(t+\alpha_k)z_{n_k} \in M. \end{cases} \tag{T$_{\text{aut}}$}$$

This condition was introduced in [23] in the context of generalized semiflows. As it was explained for the tube condition (TC$_{\text{aut}}$), it can be proved that Condition (T$_{\text{aut}}$) implies that the impact time map is upper semicontinuous on X.

Proposition 3.21. *Let (π, X, M, I) be an impulsive dynamical system such that every point in M satisfies* (T$_{\text{aut}}$)*. Then the impact time map $\phi \colon X \longrightarrow (0, \infty]$ is upper semicontinuous.*

With this Condition (T$_{\text{aut}}$), together with Condition (H$_{\text{aut}}$) as before, and $I(M) \cap M = \emptyset$, it is possible to prove the existence of a global attractor.

Theorem 3.22. *Let (π, X, M, I) be an impulsive dynamical system such that $\tilde{\pi}$ is asymptotically compact and dissipative. If $I(M) \cap M = \emptyset$ and Conditions* ($\mathrm{H_{aut}}$) *and* ($\mathrm{T_{aut}}$) *are satisfied, then (π, X, M, I) has a global attractor A, which is the ω-limit of an absorbing set.*

We finish this section by noting that there is a relation between the tube conditions ($\mathrm{TC_{aut}}$) and Condition ($\mathrm{T_{aut}}$). In fact, it can be proved that

Proposition 3.23. *Let $x \in X$ and (π, X, M, I) an impulsive dynamical system. If x satisfies the tube condition* ($\mathrm{TC_{aut}}$)*, then x satisfies Condition* ($\mathrm{T_{aut}}$).

3.2 Impulsive evolution processes

In this section, we present the theory of impulsive evolution processes and several characteristics of these type of processes.

We take (X, d) a metric space and $\mathcal{U}$ an evolution process in X, as in Definition 1.67. In order to define an impulsive evolution process, we need some new definitions. In the autonomous case, we required the set M to be a closed set and the map I to be continuous. The first idea would be to consider a closed nonautonomous set, as in Definition 1.68. However, this is not enough for our purposes. We need the following definition.

Definition 3.24. Let $\hat{D} = \{D(t)\}_{t\in\mathbb{R}}$ be a nonautonomous set. $\hat{D}$ is called collectively closed if, given two sequences $\{t_n\}_n$ and $\{x_n\}_n$, with $t_n \longrightarrow t$, $x_n \in D(t_n)$, and $x_n \longrightarrow x$, then $x \in D(t)$. Also, the family $\hat{D}$ is called collectively compact if, given two sequences $\{t_n\}_n$ and $\{x_n\}_n$, with $t_n \longrightarrow t$ and $x_n \in D(t_n)$, then the sequence $\{x_n\}_n$ has a convergent subsequence with limit in $D(t)$.

Definition 3.25. Let $\hat{D}$ be a collectively closed family of sets and f a family of functions, $f = \{f_t \colon D(t) \longrightarrow X\}_{t\in\mathbb{R}}$. We say that f is collectively continuous if, for any sequences $\{t_n\}_n$ and $\{x_n\}_n$, with $t_n \longrightarrow t$, $x_n \longrightarrow x$, and $x_n \in D(t_n)$, we have that $f_{t_n}(x_n) \longrightarrow f_t(x)$.

We define next the concept of an impulsive evolution process.

Definition 3.26. An impulsive evolution process, which we will denote by

$$\tilde{\mathcal{U}} = (\mathcal{U}, X, \hat{M}, I),$$

is formed by an evolution process $\mathcal{U}$ in a metric space X, a collectively closed family $\hat{M}$, and $I = \{I_t \colon M(t) \longrightarrow X\}_{t\in\mathbb{R}}$, a collectively continuous family of functions. Furthermore, the collectively closed family $\hat{M}$ satisfies that, for every $x \in M(s)$, there exists $\varepsilon = \varepsilon(x,s) > 0$ such that

$$\bigcup_{r\in(0,\varepsilon)} \big(\{U(s+r,s)x\} \cap M(s+r)\big) = \emptyset. \tag{3.5}$$

The family $\hat{M}$ is often called the impulsive family, each $M(t)$ is called the impulsive set at time t, the family of functions $\hat{I}$ is called the impulse function, and each individual function I_t is called the impulse function at time t. The role of I will become clear in Definition 3.31.

Remark 3.27. Condition (3.5) is different from others, which also include backwards transversality. However, it will be enough for most of our results.

For each $x \in X$ and $s \in \mathbb{R}$, we define the set

$$\mathcal{M}(x,s) = \bigcup_{r>0} \big(\{U(r+s,s)x\} \cap M(r+s)\big). \tag{3.6}$$

This definition allows us to prove the following result.

Proposition 3.28. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process, $x \in X$, $s \in \mathbb{R}$, and $\mathcal{M}(x,s) \neq \emptyset$. Then there exists a unique $\tau > 0$ such that $U(s+\tau,s)x \in M(s+\tau)$ and $U(s+r,s)x \notin M(s+r)$ for all $0 < r < \tau$.*

Proof. We take $\tau = \inf\{r > 0 \colon U(s+r,s)x \in M(s+r)\}$. If we prove that τ is positive, then we would be finished. Suppose that $\tau = 0$. Then there exists a sequence of positive numbers $\{r_n\}_n$ convergent to 0 such that

$$U(r_n+s,s)z \in M(r_n+s).$$

But this contradicts (3.5). Therefore, we are finished. ☐

This result allows us to define the impact time map for impulsive evolution processes.

Definition 3.29. Let $\tilde{\mathcal{U}}$ be an impulsive evolution process. We define the impact time map of $\tilde{\mathcal{U}}$ as the function $\phi\colon X \times \mathbb{R} \longrightarrow (0,\infty]$ given by

$$\phi(x,s) = \begin{cases} \tau, & \text{if } M^+(x,s) \neq \emptyset, \\ \infty, & \text{if } M^+(x,s) = \emptyset, \end{cases}$$

where τ was given in Proposition 3.28.

This definition implies that if $t > 0$ and $U(s+t,s)x \in M(s+t)$, then the impact time map satisfies $\phi(x,s) \leq t$.

Proposition 3.30. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process and ϕ its impact time map. Then ϕ is lower semicontinuous in the set*

$$\mathbb{H} = \bigcup_{t\in\mathbb{R}} \big(X \setminus M(t)\big) \times \{t\},$$

which is contained in $X \times \mathbb{R}$.

Proof. Let $\{(x_n, s_n)\}_n$ be any sequence in $\mathbb{H}$ convergent to a point $(x, s) \in \mathbb{H}$. We want to prove that

$$\phi(x, s) \leq \liminf_{n\to\infty} \phi(x_n, s_n).$$

If $\liminf_{n\to\infty} \phi(x_n, s_n) = \infty$, then there is nothing to prove. Therefore, we assume that

$$\liminf_{n\to\infty} \phi(x_n, s_n) = \alpha \qquad \text{for some } \alpha \in [0, \infty).$$

We can assume, without loss of generality, that $\phi(x_n, s_n) \longrightarrow \alpha$. We have that

$$\begin{aligned}
&U(s_n + \phi(x_n, s_n), s_n)x_n \in M(s_n + \phi(x_n, s_n)),\\
&s_n + \phi(x_n, s_n) \longrightarrow s + \alpha,\\
&U(s_n + \phi(x_n, s_n), s_n)x_n \longrightarrow U(s + \alpha, s)x.
\end{aligned}$$

The fact that $\hat{M}$ is collectively closed implies that $U(s + \alpha, s)x \in M(s + \alpha)$, so $0 < \phi(x, s) \leq \alpha$, which implies that $\alpha > 0$ and $\phi(x, s) \leq \alpha$. □

We can define now the impulsive semitrajectories. We will see the importance of the collection of functions I, which had not been used until now.

Definition 3.31. Let $\tilde{\mathcal{U}}$ be an impulsive evolution process, $x \in X$, and $s \in \mathbb{R}$. The impulsive trajectory of $x \in X$ starting at time s, which is denoted by $\tilde{U}(\cdot, s)z$ and defined in an interval $[s, \ell(x, s))$, is given by:

Step 1: If $\mathcal{M}(x, s) = \emptyset$, then $\phi(x, s) = \infty$. We then define $\tilde{U}(t, s)x = U(t, s)x$ for every $t \geq s$ and $\ell(x, s) = \infty$. The process ends here in this case. If $\mathcal{M}(x, s) \neq \emptyset$, we denote $x_0^+ := x$, $s_0 := s$, $s_1 := s_0 + \phi(x_0^+, s_0)$, and $x_1^+ := I_{s_1}(U(s_1, s_0)x_0^+)$. Finally, we define $\tilde{U}(\cdot, s)x$ in $[s_0, s_1]$ by

$$\tilde{U}(t, s)x = \begin{cases} U(t, s_0)x_0^+, & s_0 \leq t < s_1,\\ x_1^+, & t = s_1.\end{cases}$$

In this situation, we go to Step 2.

Step 2: If $\mathcal{M}(x_1^+, s_1) = \emptyset$, then $\phi(x_1^+, s_1) = \infty$. In this case, we define, as in the previous step, $\tilde{U}(t, s)x = U(t, s_1)x_1^+$ for all $t \geq s_1$, and we get $\ell(x, s) = \infty$.

In this situation, the process ends here. However, if $\mathcal{M}(x_1^+, s_1) \neq \emptyset$, we have $\phi(x_1^+, s_1) < \infty$. We define $s_2 := s_1 + \phi(x_1^+, s_1)$ and $x_2^+ := I_{s_2}(U(s_2, s_1)x_1^+)$. Finally, $\tilde{U}(\cdot, s)x$ on $[s_1, s_2]$ is given by

$$\tilde{U}(t,s)x = \begin{cases} U(t, s_1)x_1^+, & s_1 \leq t < s_2, \\ x_2^+, & t = s_2, \end{cases}$$

We then move to the next step.

Inductive Step: Suppose that $\tilde{U}(\cdot, s)x$ is already defined in the interval $[s_{n-1}, s_n]$ and $\tilde{U}(s_n, s)x = x_n^+$, with $s_n = s_{n-1} + \phi(x_{n-1}^+, s_{n-1})$. Once again, two different situations arise. If $\mathcal{M}(x_n^+, s_n) = \emptyset$, then we have that $\phi(x_n^+, s_n) = \infty$. We can define $\tilde{U}(t,s)x = U(t, s_n)x_n^+$ for $t \geq s_n$, and we get $\ell(x, s) = \infty$. In this situation, the process ends here. However, if $\mathcal{M}(x_n^+, s_n) \neq \emptyset$, we set $s_{n+1} := s_n + \phi(x_n^+, s_n)$, $x_{n+1}^+ = I_{s_{n+1}}(U(s_{n+1}, s_n)x_n^+)$, and finally we define $\tilde{U}(\cdot, s)x$ in $[s_n, s_{n+1}]$ by

$$\tilde{U}(t,s)x = \begin{cases} U(t, s_n)x_n^+, & s_n \leq t < s_{n+1}, \\ x_{n+1}^+, & t = s_{n+1}. \end{cases}$$

Conclusion Step: The process that has been described can end after a finite number of steps, or continue indefinitely. The first case happens when $\mathcal{M}(x_n^+, s_n) = \emptyset$ for some $n \in \mathbb{N}$. Then, we have that $\ell(x, s) = \infty$ and $\tilde{U}(\cdot, s)x$ is defined on $[s, \infty)$. The second case happens if $\mathcal{M}(x_n^+, s_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\tilde{U}(\cdot, s)x$ is defined in the interval $[s, \ell(x, s))$, with

$$\ell(x, s) = \lim_{n \to \infty} s_n,$$

and $\ell(x, s)$ can be finite number or infinity.
The times $\{s_n\}$ are called the jump times of the impulsive evolution process $\tilde{\mathcal{U}}$ at (x, s).

From this definition, we get the following results.

Proposition 3.32. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process such that*

$$I_\tau(M(\tau)) \cap M(\tau) = \emptyset \text{ for all } \tau \in \mathbb{R}. \tag{I}$$

Then, for each $x \in X$ and $t \in (s, \ell(x, s))$, we have that $\tilde{U}(t, s)x \notin M(t)$.

Lemma 3.33. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process, $x \in X$, and $s \in \mathbb{R}$ with $\phi(x, s) < \infty$. If we denote $s_1 = s + \phi(x, s)$ and $x_1^+ = \tilde{U}(s_1, s)x$, we have:*

(i) if $s < \tau < s_1$, then

$$s_1 = \tau + \phi(U(\tau, s)x, \tau) \qquad \text{and} \qquad \tilde{U}(s_1, s)x = \tilde{U}(s_1, \tau)\tilde{U}(\tau, s)x,$$

(ii) *if* $s < \tau < t < s_1$, *then*

$$t < \tau + \phi(U(\tau, s)x, \tau) \qquad \text{and} \qquad \tilde{U}(t, s)x = \tilde{U}(t, \tau)\tilde{U}(\tau, s)x,$$

(iii) *if* $s_1 < t \leq s_1 + \phi(x_1^+, s_1)$, *then* $\tilde{U}(t, s)x = \tilde{U}(t, s_1)\tilde{U}(s_1, s)x$.

Proof. The proof of this result is not very difficult. It just uses Definition 3.31 of the impulsive trajectories.

(i) Denote $r = \phi(U(\tau, s)x, \tau) + \tau$. We have that

$$U(r, s)x = U(r, \tau)U(\tau, s)x \in M(r),$$

by definition of r. Then $r \geq s_1$. For any $t \in (\tau, r)$, we have

$$U(t, s)x = U(t, \tau)U(\tau, s)x \notin M(t).$$

As a consequence, $s_1 \geq r$. We can conclude that $r = s_1$. We denote $x_\tau = \tilde{U}(\tau, s)x = U(\tau, s)x$. We have that

$$\begin{aligned} \tilde{U}(s_1, \tau)x_\tau &= I_{s_1}(U(s_1, \tau)x_\tau) = I_{s_1}(U(s_1, \tau)U(\tau, s)x) \\ &= I_{s_1}(U(s_1, s)x) = \tilde{U}(s_1, s)x. \end{aligned}$$

This implies that $\tilde{U}(s_1, s)x = \tilde{U}(s_1, \tau)x_\tau = \tilde{U}(s_1, \tau)\tilde{U}(\tau, s)x$.

(ii) Denote $x_\tau = U(\tau, s)x$. By hypothesis, $\tau \in (s, s_1)$, so we have that $\tilde{U}(\tau, s)x = x_\tau$. We obtain $\phi(x_\tau, \tau) + \tau = \phi(x, s) + s$, from item (i). We also have that $\tau < t < s + \phi(x, s)$, so we obtain $\tau < t < \tau + \phi(x_\tau, \tau)$. As a consequence, $\tilde{U}(t, \tau)x_\tau = U(t, \tau)x_\tau$. Finally,

$$\begin{aligned} \tilde{U}(t, s)x = U(t, s)x &= U(t, \tau)U(\tau, s)x = U(t, \tau)x_\tau \\ &= \tilde{U}(t, \tau)x_\tau = \tilde{U}(t, \tau)\tilde{U}(\tau, s)x. \end{aligned}$$

(iii) For $t \in (s_1, s_1 + \phi(x_1^+, s_1))$, we have

$$\tilde{U}(t, s)x = U(t, s_1)\tilde{U}(s_1, s)x = \tilde{U}(t, s_1)\tilde{U}(s_1, s)x,$$

from Definition 3.31. If $t = s_1 + \phi(x_1^+, s_1)$ we have

$$\tilde{U}(t, s)x = I_{s_1}(U(t, s_1)x_1^+) = \tilde{U}(t, s_1)x_1^+ = \tilde{U}(t, s_1)\tilde{U}(s_1, s)x. \qquad \square$$

As a consequence, we can deduce the following result.

Proposition 3.34. *Let* $\tilde{\mathcal{U}}$ *be an impulsive evolution process. Then*

1. $\tilde{U}(t, t)x = x$ *for every* $x \in X$ *and* $t \in \mathbb{R}$,

2. $\tilde{U}(t,s)x = \tilde{U}(t,\tau)\tilde{U}(\tau,s)x$ *for* $s \leq \tau \leq t < \ell(x,s)$.

We are concerned about the asymptotic behavior of solutions. We assume that the impulsive trajectories exist for all $t \geq s$, that is, we will assume that

$$\ell(x,s) = \infty \text{ for all } x \in X \text{ and } s \in \mathbb{R}. \tag{3.7}$$

With this condition, the definitions of $\tilde{\mathcal{U}}$-invariance and pullback $\tilde{\mathcal{U}}$-attraction for an impulsive evolution process $\tilde{\mathcal{U}}$ are analogous to the ones for $\mathcal{U}$, just replacing U by $\tilde{U}$ (see Definitions 1.70 and 1.71).

3.3 Pullback attractors for impulsive evolution processes

In this section, we will study the existence of pullback attractors for impulsive evolution processes. We start with some general and partial results. Later, in Subsections 3.3.1 and 3.3.2, we focus in proving the invariance property (c) of Definition 3.35.

From now on, we will assume that $\tilde{\mathcal{U}}$ is an impulsive evolution process satisfying (3.7) and that $\mathfrak{D}$ is a universe in X (see Definition 1.69). We start with the definition of pullback $\mathfrak{D}$-semiattractor and pullback $\mathfrak{D}$-attractor.

Definition 3.35. We say that a family $\hat{A} \in \mathfrak{D}$ of X is a pullback $\mathfrak{D}$-semiattractor for $\tilde{\mathcal{U}}$ if:

(a) $\hat{A}$ is compact,

(b) $\hat{A}$ pullback $\tilde{\mathcal{U}}$-attracts each $\hat{D} \in \mathfrak{D}$.

When $\hat{A}$ also satisfies:

(c) the family $\hat{A} \setminus \hat{M} = \{A(t) \setminus M(t)\}_{t\in\mathbb{R}}$ is $\tilde{\mathcal{U}}$-invariant,

we say that $\hat{A}$ is a pullback $\mathfrak{D}$-attractor for $\tilde{\mathcal{U}}$.

Proposition 3.36. *Let* $\hat{A}_1$ *and* $\hat{A}_2$ *be two pullback* $\mathfrak{D}$*-attractors for* $\tilde{\mathcal{U}}$*. Then* $\hat{A}_1 \setminus \hat{M} = \hat{A}_2 \setminus \hat{M}$*.*

Proof. We have that $\hat{A}_1$ is a pullback $\mathfrak{D}$-attractor and $\hat{A}_2 \in \mathfrak{D}$. As $\mathfrak{D}$ is a universe, then $\hat{A}_2 \setminus \hat{M} \in \mathfrak{D}$. The invariance of $\hat{A}_2 \setminus \hat{M} \in \mathfrak{D}$ implies that

$$d_{\mathrm{H}}(A_2(t) \setminus M(t), A_1(t)) = d_{\mathrm{H}}(\tilde{U}(t,s)(A_2(s) \setminus M(s)), A_1(t))$$

for every $t \in \mathbb{R}$ and $s \leq t$. Taking the limit as s goes to $-\infty$ we get

$$d_{\mathrm{H}}(A_2(t) \setminus M(t), A_1(t)) = 0.$$

This implies that $A_2(t) \setminus M(t) \subset A_1(t)$, so $A_2(t) \setminus M(t) \subset A_1(t) \setminus M(t)$. If we interchange $\hat{A}_1$ and $\hat{A}_2$, we obtain that $A_1(t) \setminus M(t) \subset A_2(t) \setminus M(t)$. We get the desired result. $\square$

Definition 3.37. A function $\psi\colon \mathbb{R} \longrightarrow X$ is a global solution of $\tilde{\mathcal{U}}$ if

$$\tilde{U}(t,s)\psi(s) = \psi(t) \quad \text{for } t \geq s.$$

Proposition 3.38. *Let $\tilde{\mathcal{U}}$ satisfy Condition* (I) *and $\hat{A}$ a pullback $\mathfrak{D}$-attractor for $\tilde{\mathcal{U}}$. For each $t \in \mathbb{R}$ we define*

$$\Xi(t) = \big\{\psi(t) : \psi \text{ is a global solution of } \tilde{\mathcal{U}} \text{ with } \hat{\psi} = \{\psi(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}\big\}.$$

Then $A(t) \setminus M(t) = \Xi(t)$ for each $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. For one side, take $x \in \Xi(t)$. Then there exists ψ a global solution of $\tilde{\mathcal{U}}$ with $\hat{\psi} = \{\psi(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ and $x = \psi(t)$. Condition (I) implies that $\psi(t) \notin M(t)$ for all $t \in \mathbb{R}$ (see Proposition 3.32). We know that $\hat{\psi} \in \mathfrak{D}$, ψ is a global solution of $\tilde{\mathcal{U}}$ and $\hat{A}$ is a pullback $\mathfrak{D}$-attractor. This implies that

$$d_{\mathrm{H}}(\psi(t), A(t)) = d_{\mathrm{H}}(\tilde{U}(t,s)\psi(s), A(t)) \longrightarrow 0$$

as s goes to $-\infty$. Therefore, $\psi(t) \in A(t)$, because $A(t)$ is compact and thus closed. Hence, $\psi(t) \in A(t) \setminus M(t)$.

For the other side, take $x \in A(t) \setminus M(t)$. We have that the family $\hat{A} \setminus \hat{M}$ is $\tilde{\mathcal{U}}$-invariant, so $x \in \tilde{U}(t, t-1)(A(t-1) \setminus M(t-1))$. As a consequence, there exists $x_{-1} \in A(t-1) \setminus M(t-1)$ such that $x = \tilde{U}(t, t-1)x_{-1}$. With this approach, we construct a sequence $\{x_{-n}\}_n$ with $x_{-n} \in A(t-n) \setminus M(t-n)$ and $\tilde{U}(t-n+1, t-n)x_{-n} = x_{-n+1}$. We define a function $\psi\colon \mathbb{R} \longrightarrow X$ as

$$\psi(s) = \begin{cases} \tilde{U}(s, t-n)x_{-n}, & s \in [t-n, t-n+1],\ n \in \mathbb{N}, \\ \tilde{U}(s,t)x, & s \in [t, \infty). \end{cases}$$

The function ψ is a global solution of $\tilde{\mathcal{U}}$ by construction. Moreover, we have that $\psi(s) \in A(s) \setminus M(s) \subset A(s)$ for every $s \in \mathbb{R}$. The nonautonomous set $\hat{\psi} = \{\psi(s)\}_{s\in\mathbb{R}}$ belongs to $\mathfrak{D}$, because $\mathfrak{D}$ is a universe and $\hat{A} \in \mathfrak{D}$. As a consequence, $x = \psi(t) \in \Xi(t)$. $\square$

We note that a pullback $\mathfrak{D}$-semiattractor may not be a pullback $\mathfrak{D}$-attractor (see Example 3.12 for the autonomous case).

The pullback ω-limit is very important in the theory of pullback attractors for continuous evolution processes. For the impulsive case, it is also very important. However, the fact that trajectories are not continuous requires us to change its definition, in order to obtain good properties.

Definition 3.39. The impulsive pullback ω-limit set of a nonempty family $\hat{D}$ at time $t \in \mathbb{R}$ is denoted by $\tilde{\omega}(\hat{D}, t)$, and it is defined as the set of elements $x \in X$ such that there exist sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ for each $n \in \mathbb{N}$, such that

$$x = \lim_{n \to \infty} \tilde{U}(t + \varepsilon_n, s_n) x_n.$$

The impulsive pullback ω-limit of $\hat{D}$ is the family $\tilde{\omega}(\hat{D}) = \{\tilde{\omega}(\hat{D}, t)\}_{t \in \mathbb{R}}$.

Definition 3.40. An impulsive evolution process $\tilde{\mathcal{U}}$ will be called pullback $\mathfrak{D}$-asymptotically compact if for all $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ for each $n \in \mathbb{N}$, then the sequence $\{\tilde{U}(t + \varepsilon_n, s_n) x_n\}_n$ has a convergent subsequence in X.

Definition 3.41. An impulsive evolution process $\tilde{\mathcal{U}}$ will be called pullback $\mathfrak{D}$-dissipative if there exists $\hat{B}_0 \in \mathfrak{D}$ collectively closed such that, for all $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and sequences $\{s_n\}_n$, and $\{\varepsilon_n\}_n$, with $s_n \longrightarrow -\infty$ and $\varepsilon_n \longrightarrow 0$, there exists $n_0 = n_0(\hat{D}, t) \in \mathbb{N}$ such that

$$\tilde{U}(t + \varepsilon_n, s_n) D(s_n) \subset B_0(t + \varepsilon_n) \quad \text{for all } n \geq n_0.$$

Any family $\hat{B}_0$ with this property is a pullback $\mathfrak{D}$-absorbing family.

We continue by analyzing some results to guarantee these two conditions and also some consequences of these definitions.

Proposition 3.42. *Let $\hat{B}_0 \in \mathfrak{D}$ be a nonempty and collectively closed family. This family is a pullback $\mathfrak{D}$-absorbing family for $\tilde{\mathcal{U}}$ if and only if for every $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and every function $\varepsilon \colon \mathbb{R} \longrightarrow \mathbb{R}$ with $\varepsilon(s) \longrightarrow 0$ as $s \longrightarrow -\infty$ and $t + \varepsilon(s) \geq s$ for all $s \leq t$, there exists $s_0 = s_0(\hat{D}, t, \varepsilon) \leq t$ such that*

$$s \leq s_0 \Longrightarrow \tilde{U}(t + \varepsilon(s), s) D(s) \subset B_0(t + \varepsilon(s)).$$

Proof. Take $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family for $\tilde{\mathcal{U}}$. Suppose that the result is false. Therefore, there exists $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and a function $\varepsilon \colon \mathbb{R} \longrightarrow \mathbb{R}$ with $\varepsilon(s) \longrightarrow 0$ as $s \longrightarrow -\infty$ and $t + \varepsilon(s) \geq s$ for all $s \leq t$, such that the conclusion is not true. This implies that there exist sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ such that $\tilde{U}(t + \varepsilon_n, s_n) x_n \notin B_0(t + \varepsilon_n)$ for all $n \in \mathbb{N}$. But $\hat{B}_0$ was a pullback $\mathfrak{D}$-absorbing family for $\tilde{\mathcal{U}}$, so we get a contradiction. The other implication is straightforward. □

Proposition 3.43. *Let $\hat{B}_0 \in \mathfrak{D}$ be a nonempty family which is collectively closed and positively $\tilde{\mathcal{U}}$-invariant. Suppose that for each $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{R}$, there exists $s_0 = s_0(\hat{D}, t) \leq t$ such that*

$$s \leq s_0 \Longrightarrow \tilde{U}(t, s) D(s) \subset B_0(t).$$

Then $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative.

Proof. Take $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and sequences $\{s_n\}_n$ and $\{\varepsilon_n\}_n$ with $s_n \longrightarrow -\infty$ and $\varepsilon_n \longrightarrow 0$. We want to prove that there exists $n_0 = n_0(\hat{D}, t) \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \tilde{U}(t + \varepsilon_n, s_n)D(s_n) \subset B_0(t + \varepsilon_n).$$

From the hypothesis, we know that there exists $s_0 = s_0(\hat{D}, t-1) \leq t-1$ such that $\tilde{U}(t-1, s)D(s) \subset B_0(t-1)$ for all $s \leq s_0$. This implies that we can take $n_0 \in \mathbb{N}$ such that $s_n \leq s_0$ and $|\varepsilon_n| < 1$ for all $n \geq n_0$. We have

$$\begin{aligned}\tilde{U}(t + \varepsilon_n, s_n)D(s_n) &= \tilde{U}(t + \varepsilon_n, t-1)\tilde{U}(t-1, s_n)D(s_n) \\ &\subset \tilde{U}(t + \varepsilon_n, t-1)B_0(t-1) \subset B_0(t + \varepsilon_n).\end{aligned}$$

The positive $\tilde{\mathcal{U}}$-invariance of $\hat{B}_0$ implies the last inclusion. As a consequence, $\hat{B}_0$ is a pullback $\mathfrak{D}$-absorbing family, and thus $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative. ☐

Proposition 3.44. *Let $\tilde{\mathcal{U}}$ be a pullback $\mathfrak{D}$-dissipative impulsive evolution process with $\hat{B}_0$ a collectively compact pullback $\mathfrak{D}$-absorbing family. Then $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact.*

Proof. Let $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ such that $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ for each $n \in \mathbb{N}$. We want to prove that $\{\tilde{U}(t + \varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence. The impulsive evolution process $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative, which implies that there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \tilde{U}(t + \varepsilon_n, s_n)x_n \in B_0(t + \varepsilon_n).$$

We have that $t + \varepsilon_n \longrightarrow t$ and $\hat{B}_0$ was collectively compact, so the sequence $\{\tilde{U}(t + \varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence to a point in $B_0(t)$. ☐

Proposition 3.45. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which is pullback $\mathfrak{D}$-asymptotically compact and $\hat{D} \in \mathfrak{D}$. Then the impulsive pullback ω-limit $\tilde{\omega}(\hat{D})$ is a nonempty family in X, collectively compact, and it pullback $\tilde{\mathcal{U}}$-attracts $\hat{D}$.*

Proof. The family $\tilde{\omega}(\hat{D})$ is nonempty because $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact. In order to prove the collective compactness, we need to show that given two sequences $\{t_n\}_n$ and $\{y_n\}_n$ with $t_n \longrightarrow t$ and $y_n \in \tilde{\omega}(\hat{D}, t_n)$, then the sequence $\{y_n\}_n$ has a convergent subsequence with limit in $\tilde{\omega}(\hat{D}, t)$. As $y_n \in \tilde{\omega}(\hat{D}, t_n)$, there exists $s_n \leq t_n - n$, $|\varepsilon_n| \leq 1/n$ and $x_n \in D(s_n)$ such that

$$d(y_n, \tilde{U}(t_n + \varepsilon_n, s_n)x_n) \leq \frac{1}{n}.$$

$\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact, so the sequence

$$\{\tilde{U}(t_n + \varepsilon_n, s_n)x_n\}_n = \{\tilde{U}(t + t_n - t + \varepsilon_n, s_n)x_n\}_n$$

has a convergent subsequence. The subsequence will be denoted the same, and let $x \in X$ be the limit, which belongs to $\tilde{\omega}(\hat{D}, t)$, by definition. Finally, the sequence $\{y_n\}_n$ converges to x, and we are finished.

We have yet to prove the pullback $\tilde{\mathcal{U}}$-attraction. Suppose that the result is false. Then, by definition, there exist $\varepsilon > 0$, $t \in \mathbb{R}$, and two sequences $\{s_n\}_n$ and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$ and $x_n \in D(s_n)$ for each $n \in \mathbb{N}$, such that

$$d_{\mathrm{H}}(\tilde{U}(t, s_n)x_n, \tilde{\omega}(\hat{D}, t)) \geq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

As $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact, there exists $x \in X$ such that

$$x = \lim_{n \to \infty} \tilde{U}(t, s_n)x_n,$$

up to a subsequence. Then $x \in \tilde{\omega}(\hat{D}, t)$, and we get a contradiction. □

Proposition 3.46. *Let $\tilde{\mathcal{U}}$ be a pullback $\mathfrak{D}$-dissipative impulsive evolution process, $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family, and $\hat{D} \in \mathfrak{D}$. Then, we have $\tilde{\omega}(\hat{D}) \subset \hat{B}_0$.*

Proof. Let $t \in \mathbb{R}$ and $x \in \tilde{\omega}(\hat{D}, t)$. We have to prove that $x \in B_0(t)$. As $x \in \tilde{\omega}(\hat{D}, t)$, there exist sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$, such that

$$x = \lim_{n \to \infty} \tilde{U}(t + \varepsilon_n, s_n)x_n.$$

The family $\hat{B}_0$ is a pullback $\mathfrak{D}$-absorbing family, so there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \tilde{U}(t + \varepsilon_n, s_n)x_n \in \tilde{U}(t + \varepsilon_n, s_n)D(s_n) \subset B_0(t + \varepsilon_n).$$

The family $\hat{B}_0$ is collectively closed, so we have that $x \in B_0(t)$. □

Theorem 3.47. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which is pullback $\mathfrak{D}$-asymptotically compact and pullback $\mathfrak{D}$-dissipative. Then $\tilde{\mathcal{U}}$ has a pullback $\mathfrak{D}$-semiattractor $\hat{A} \in \mathfrak{D}$. Moreover, $\hat{A}$ is collectively compact.*

Proof. Let $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. Define $\hat{A} = \tilde{\omega}(\hat{B}_0)$. We are going to prove that $\hat{A}$ is a pullback $\mathfrak{D}$-semiattractor for $\tilde{\mathcal{U}}$. We have that $\hat{B}_0 \in \mathfrak{D}$. Proposition 3.45 implies that $\hat{A}$ is a nonempty family in X, it is collectively compact and it pullback $\tilde{\mathcal{U}}$-attracts $\hat{B}_0$. This implies that $A(t)$ is compact for all $t \in \mathbb{R}$. Furthermore, Proposition 3.46 implies that $\hat{A} \subset \hat{B}_0$. As a consequence, we have that $\hat{A} \in \mathfrak{D}$.

Finally, we have to prove that $\hat{A}$ pullback $\tilde{\mathcal{U}}$-attracts every $\hat{D} \in \mathfrak{D}$. Fix $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and $\varepsilon > 0$. We have to prove that there exists $s_1 \leq t$ such that

$$s \leq s_1 \Longrightarrow d_{\mathrm{H}}(\tilde{U}(t, s)D(s), A(t)) < \varepsilon.$$

The family $\hat{A}$ pullback $\tilde{\mathcal{U}}$-attracts $\hat{B}_0$, so there exists $s_0 \leq t$ such that

$$s \leq s_0 \Longrightarrow d_{\mathrm{H}}(\tilde{U}(t,s)B_0(s), A(t)) < \varepsilon.$$

Using Proposition 3.42 with $\varepsilon(s) = 0$, there exists $s_1 \leq s_0$ such that

$$s \leq s_1 \Longrightarrow \tilde{U}(s_0, s)D(s) \subset B_0(s_0).$$

Therefore, for $s \leq s_1$ we have

$$\tilde{U}(t,s)D(s) = \tilde{U}(t,s_0)\tilde{U}(s_0,s)D(s) \subset \tilde{U}(t,s_0)B_0(s_0).$$

As a consequence,

$$s \leq s_1 \Longrightarrow d_{\mathrm{H}}(\tilde{U}(t,s)D(s), A(t)) \leq d_{\mathrm{H}}(\tilde{U}(t,s_0)B_0(s_0), A(t)) < \varepsilon.$$

This proves that

$$\lim_{s \to -\infty} d_{\mathrm{H}}(\tilde{U}(t,s)D(t), A(t)) = 0.$$

Then $\hat{A}$ is a pullback $\mathfrak{D}$-semiattractor for $\tilde{\mathcal{U}}$. □

We present some necessary conditions for the existence of a pullback $\mathfrak{D}$-semiattractor for an impulsive evolution process $\tilde{\mathcal{U}}$. First, we need a condition, which will also be used to prove the existence of a pullback $\mathfrak{D}$-attractor.

$$\begin{gathered}\text{There exists } \xi > 0 \text{ such that } \phi(z,s) \geq 2\xi \\ \text{for all } s \in \mathbb{R} \text{ and } z \in I_s(M(s)).\end{gathered} \tag{H}$$

In particular, this Condition (H) implies (3.7).

Proposition 3.48. *If $\tilde{\mathcal{U}}$ has a pullback $\mathfrak{D}$-semiattractor $\hat{A}$ and satisfies Conditions* (H) *and* (I)*, then $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact.*

Proof. Let $\hat{A}$ be a pullback $\mathfrak{D}$-semiattractor for $\tilde{\mathcal{U}}$. Take $t \in \mathbb{R}$, $\hat{D} \in \mathfrak{D}$, and three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ such that $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ for each $n \in \mathbb{N}$. We want to prove that $\{\tilde{U}(t+\varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence. First, we assume that $\varepsilon_n \leq \xi/4$ for all $n \in \mathbb{N}$. The family $\hat{A}$ pullback $\tilde{\mathcal{U}}$-attracts $\hat{D}$, so

$$d_{\mathrm{H}}(\tilde{U}(t-\xi, s_n)x_n, A(t-\xi)) \leq d_{\mathrm{H}}(\tilde{U}(t-\xi, s_n)D(s_n), A(t-\xi)) \longrightarrow 0$$

as n goes to ∞. We can find subsequences $\{s_{n_k}\}_k$, $\{x_{n_k}\}_k$, and a sequence $\{y_k\}_k$ in $A(t-\xi)$ such that

$$d_{\mathrm{H}}(\tilde{U}(t-\xi, s_{n_k})x_{n_k}, y_k) \leq \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

The set $A(t-\xi)$ is compact and $\{y_k\}_k$ is a sequence in $A(t-\xi)$, so there exists a convergent subsequence of $\{y_k\}_k$ to an element $y \in A(t-\xi)$. The convergent subsequence will be denoted the same. As a consequence, we have that the sequence $\{\tilde{U}(t-\xi, s_{n_k})x_{n_k}\}_k$ converges to y. We relabel the sequence, and assume $z_n = \tilde{U}(t-\xi, s_n)x_n \longrightarrow y$. Note that Condition (I) and Proposition 3.32 imply that $z_n \notin M(t-\xi)$. Consider the partial trajectory defined by $\varphi_n\colon u \in [t-\xi, t+\varepsilon_n] \longrightarrow \tilde{U}(u, t-\xi)z_n \in X$. We distinguish two cases.

Case 1: Up to a subsequence, denoted the same, there are no jump times of $\tilde{\mathcal{U}}$ at $(z_n, t-\xi)$ in the interval $[t-\xi, t+\varepsilon_n]$.

In this case, we have that

$$\tilde{U}(t+\varepsilon_n, t-\xi)z_n = U(t+\varepsilon_n, t-\xi)z_n \longrightarrow U(t, t-\xi)y.$$

We have that $\tilde{U}(t+\varepsilon_n, t-\xi)z_n = \tilde{U}(t+\varepsilon_n, s_n)x_n$, which implies that the sequence $\{\tilde{U}(t+\varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence, and we are finished.

Case 2: Up to a subsequence, denoted the same, there is at least one jump time of $(z_n, t-\xi)$ in the interval $[t-\xi, t+\varepsilon_n]$.

In this case, Condition (H) implies that there is only one jump time in that interval. We denote it by τ_n, and it belongs to $(t-\xi, t+\varepsilon_n]$. Then we have

$$\begin{aligned}
\tilde{U}(t+\varepsilon_n, s_n)x_n &= U(t+\varepsilon_n, \tau_n)\tilde{U}(\tau_n, t-\xi)z_n, \\
\tilde{U}(\tau_n, t-\xi)z_n &= I_{\tau_n}(U(\tau_n, t-\xi)z_n),
\end{aligned}$$

as $U(\tau_n, t-\xi)z_n \in M(\tau_n)$. We can assume that $\tau_n \longrightarrow \tau$ for some $\tau \in [t-\xi, t]$, by taking subsequences, if necessary. Then, by definition of $\hat{M}$ and I, we have that

$$\begin{aligned}
&U(\tau_n, t-\xi)z_n \longrightarrow U(\tau, t-\xi)y \in M(\tau), \\
&\tilde{U}(\tau_n, t-\xi)z_n) = I_{\tau_n}(U(\tau_n, t-\xi)z_n) \longrightarrow I_\tau(U(\tau, t-\xi)y).
\end{aligned}$$

Therefore, we obtain that $\tilde{U}(t+\varepsilon_n, s_n)x_n$ converges to $U(t,\tau)I_\tau(U(\tau, t-\xi)y)$. $\square$

Proposition 3.49. *Let $\hat{A}$ be a pullback $\mathfrak{D}$-semiattractor of $\tilde{\mathcal{U}}$ which is collectively compact. For $r > 0$, define*

$$\hat{A}_r = \{A_r(t)\}_{t\in\mathbb{R}} = \left\{\overline{\mathcal{O}_r(A(t))}\right\}_{t\in\mathbb{R}},$$

If $\hat{A}_r \in \mathfrak{D}$ for some $r > 0$, then for each $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{R}$, there exists $s_0 = s_0(D,t) \le t$ such that

$$\tilde{U}(t,s)D(s) \subset A_r(t) \quad \text{for all } s \le s_0.$$

Moreover, if the family $\hat{A}_r$ is also positively $\tilde{\mathcal{U}}$-invariant, then $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative.

Proof. Fix $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{R}$. We assume that $\hat{A}_r \in \mathfrak{D}$ for some $r > 0$. The family $\hat{A}$ is pullback $\mathfrak{D}$-attracting, so, by definition, there exists $s_0 = s_0(\hat{D}, t) \leq t$ such that

$$s \leq s_0 \Longrightarrow d_{\mathrm{H}}(\tilde{U}(t,s)D(s), A(t)) < r.$$

This implies that

$$s \leq s_0 \Longrightarrow \tilde{U}(t,s)D(s) \subset \mathcal{O}_r(A(t)) \subset A_r(t),$$

and the first part is proved.

In order to prove the second part, we are going to use Proposition 3.43. It is enough to prove that $\hat{A}_r$ is collectively closed, since we are assuming that $\hat{A}_r$ is positively $\tilde{\mathcal{U}}$-invariant. Take two sequences $\{t_n\}_n$ and $\{x_n\}_n$ such that $t_n \longrightarrow t$ and $x_n \in A_r(t_n)$ with $x_n \longrightarrow x$. We have to prove that $x \in A_r(t)$. For any $n \in \mathbb{N}$, there exist $y_n \in \mathcal{O}_r(A(t_n))$ and $z_n \in A(t_n)$ such that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{and} \quad d(y_n, z_n) < r.$$

The family $\hat{A}$ is collectively compact, $z_n \in A(t_n)$, and $t_n \longrightarrow t$, so we have that $\{z_n\}_n$ has a convergent subsequence in $A(t)$. We denote the subsequence the same, and $z \in A(t)$ the limit, so we may assume $z_n \longrightarrow z \in A(t)$. This implies that

$$\begin{aligned} d(x,z) &\leq d(x,x_n) + d(x_n,y_n) + d(y_n,z_n) + d(z_n,z) \\ &< d(x,x_n) + \frac{1}{n} + r + d(z_n,z). \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we get $d(x,z) \leq r$. Finally, as $z \in A(t)$, this implies that $x \in \overline{\mathcal{O}_r(A(t))} = \hat{A}_r$. ☐

In the next two subsections, we will obtain conditions in order to get pullback attractors.

3.3.1 Tube conditions

We start by defining, for every $s \in \mathbb{R}$, $r \geq 0$, and family $\hat{D}$, the set

$$F(\hat{D}, r, s) = \{x \in X : U(r+s, s)x \in D(r+s)\}. \tag{3.8}$$

For simplicity of notation, we shall write $F(D, r, s)$ if $D(t) = D$ for all $t \in \mathbb{R}$, and $F(x, r, s)$ if $D(t) = \{x\}$ for all $t \in \mathbb{R}$.

First, we need an additional condition aside from (3.5), which is:

$$\forall\, x \in M(s),\ \exists\, \varepsilon = \varepsilon(x,s) > 0 : \bigcup_{r\in(0,\varepsilon)} \big(F(x,r,s-r) \cap M(s-r)\big) = \emptyset \tag{3.9}$$

Definition 3.50. Let $\mathcal{U}$ be a continuous evolution process, $x \in X$, $s_0 \in \mathbb{R}$, $\lambda > 0$, and $\hat{S}$ a closed family in X with $x \in S(s_0)$. We say that $\hat{S}$ is a λ-section through x at s_0 if there exists $\hat{L}$, a collectively closed family in X, such that

(i) $F(\hat{L}, \lambda, s) = S(s)$ for all $s \leq s_0 + \lambda$;

(ii) the set $\bigcup_{t\in[0,2\lambda]} F(\hat{L}, t, s_0)$ contains a neighborhood of x in X;

(iii) $F(\hat{L}, \mu, s) \cap F(\hat{L}, \nu, s) = \emptyset$ for all $s \leq s_0 + \lambda$ and $0 \leq \mu < \nu \leq 2\lambda$.

In this situation, we say that $\hat{L}$ is a λ-bar and $\bigcup_{t\in[0,2\lambda]} F(\hat{L}, t, s_0)$ is a λ-tube.

Sometimes we will remove the λ when its value is not needed explicitly. A reason to do this is the following result.

Proposition 3.51. *Let $\hat{S}$ be a λ-section through x at s_0 and $\mu \in (0, \lambda)$. Then $\hat{S}$ is a μ-section through x at s_0. The μ-bar $\hat{L}_\mu$ is given by $L_\mu(t) = F(\hat{L}, \lambda - \mu, t)$ for each $t \in \mathbb{R}$.*

Proof. The family $\hat{L}_\mu$ is collectively closed because $\hat{L}$ is collectively closed. First, we prove that

$$F(\hat{L}_\mu, r, t) = F(\hat{L}, \lambda - \mu + r, t) \quad \text{for all } r \geq 0 \text{ and } t \in \mathbb{R}. \tag{3.10}$$

By definition, we have

$$F(\hat{L}_\mu, r, t) = \{x \in X \colon U(t + r, t)x \in L_\mu(t + r)\},$$
$$L_\mu(t + r) = \{z \in X \colon U(t + \lambda - \mu + r, t + r)z \in L(t + \lambda - \mu + r)\}.$$

We deduce the following

$$\begin{aligned} x \in F(\hat{L}_\mu, r, t) &\iff U(t + r, t)x \in L_\mu(t + r) \\ &\iff U(t + \lambda - \mu + r, t + r)U(t + r, t)x \in L(t + \lambda - \mu + r) \\ &\iff U(t + \lambda - \mu + r, t)x \in L(t + \lambda - \mu + r) \\ &\iff x \in F(\hat{L}, \lambda - \mu + r, t). \end{aligned}$$

This implies that Equation (3.10) is true. We take $s \leq s_0 + \mu$. We use Equation (3.10) with $t = s$ and $r = \mu$. We get that

$$F(\hat{L}_\mu, \mu, s) = F(\hat{L}, \lambda, s) = S(s),$$

and thus we have proved (i) in Definition 3.50. Take $0 \leq \eta < \nu \leq 2\mu$ and s as before. Then

$$F(\hat{L}_\mu, \eta, s) \cap F(\hat{L}_\mu, \nu, s) = F(\hat{L}, \eta + \lambda - \mu, s) \cap F(\hat{L}, \nu + \lambda - \mu, s) = \emptyset.$$

This proves (iii). The proof of (ii) is more difficult. Suppose that it is not true, that is, the set

$$\bigcup_{t \in [0,2\mu]} F(\hat{L}_\mu, t, s_0)$$

does not contain a neighborhood of x. However, as $\hat{S}$ is a λ-section through x at s_0, we have that there exists a sequence $\{x_n\}_n$, convergent to x, such that

$$x_n \notin \bigcup_{t \in [0,2\mu]} F(\hat{L}_\mu, t, s_0) \quad \text{and} \quad x_n \in \bigcup_{t \in [0,2\lambda]} F(\hat{L}, t, s_0).$$

Equation (3.10) implies that

$$\bigcup_{t \in [0,2\mu]} F(\hat{L}_\mu, t, s_0) = \bigcup_{t \in [0,2\mu]} F(\hat{L}, \lambda - \mu + t, s_0) = \bigcup_{t \in [\lambda-\mu,\lambda+\mu]} F(\hat{L}, t, s_0).$$

As a consequence, we get

$$x_n \in \bigcup_{t \in [0,\lambda-\mu) \cup (\lambda+\mu, 2\lambda]} F(\hat{L}, t, s_0) \quad \text{for each } n \in \mathbb{N}.$$

Then, for every $n \in \mathbb{N}$, there exists $t_n \in [0, \lambda - \mu) \cup (\lambda + \mu, 2\lambda]$ such that $x_n \in F(\hat{L}, t_n, s_0)$. We may assume that the sequence $\{t_n\}_n$ converges to some $t \in [0, \lambda - \mu] \cup [\lambda + \mu, 2\lambda]$. This implies that

$$U(t_n + s_0, s_0)x_n \in L(t_n + s_0) \;\text{ and }\; U(t_n + s_0, s_0)x_n \longrightarrow U(t + s_0, s_0)x.$$

The family $\hat{L}$ is collectively closed, so $U(t + s_0, s_0)x \in L(t + s_0)$. This implies that $x \in F(\hat{L}, t, s_0)$. But $x \in S(s_0) = F(\hat{L}, \lambda, s_0)$, and item (iii) implies that $t = \lambda$. Thus, we arrive at a contradiction. □

We introduce next the tube conditions for impulsive evolution processes.

Definition 3.52. Let $\tilde{\mathcal{U}}$ be an impulsive evolution process, $s_0 \in \mathbb{R}$, and an element $x \in M(s_0)$. We say that x satisfies:

- the tube condition (TC) if there exists a λ-section $\hat{S}$ through x at s_0 with

$$s \leq s_0 + \lambda \Longrightarrow S(s) \subset M(s) \cap \bigcup_{t \in [0,2\lambda]} F(\hat{L}, t, s);$$

- the strong tube condition (STC) if there exists a λ-section $\hat{S}$ through x at s_0 such that

$$s \leq s_0 + \lambda \Longrightarrow S(s) = M(s) \cap \bigcup_{t \in [0,2\lambda]} F(\hat{L}, t, s);$$

- the special strong tube condition (SSTC) if it satisfies (STC) and

$$s \leq s_0 + \lambda \Longrightarrow I_s(M(s)) \cap \bigcup_{t \in [0,\lambda)} F(\hat{L}, t, s) = \emptyset.$$

Furthermore, if for every $s_0 \in \mathbb{R}$ and $x \in M(s_0)$ one of these conditions is satisfied, we will say that the impulsive evolution process $\tilde{\mathcal{U}}$ satisfies that condition.

First, we are going to prove that we are able to obtain an upper semicontinuity property for the impact time map. However, we are not able to prove it for both variables, we do it for every fixed time s.

Proposition 3.53. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process that satisfies* (TC) *and $s \in \mathbb{R}$. Then the map $\phi(\cdot, s)\colon X \longrightarrow (0, \infty]$ is upper semicontinuous.*

Proof. Take $x \in X$ such that $\phi(x, s) \in (0, \infty)$. If $\phi(x, s) = \infty$, there is nothing to prove. Fix $\varepsilon > 0$. We will prove that there exists V a neighborhood of x such that

$$v \in V \Longrightarrow \phi(v, s) < \phi(x, s) + \varepsilon.$$

For simplicity of notation, we set $s_1 = s + \phi(x, s)$ and $y = U(s_1, s)x \in M(s_1)$. Moreover, $U(t, s)x \notin M(t)$ for every $t \in (s, s_1)$. As $y \in M(s_1)$ and $\tilde{\mathcal{U}}$ satisfies (TC), there exists $\hat{S}$ a λ-section through y at s_1 with a λ-bar $\hat{L}$. By Proposition 3.51, we may assume that $\lambda < \phi(x, s)$ and $\lambda < \varepsilon$. We consider W a neighborhood of y such that

$$y \in W \subset \bigcup_{t \in [0,2\lambda]} F(\hat{L}, t, s_1).$$

The map $U(s_1, s)\colon X \longrightarrow X$ is continuous, so there exists V a neighborhood of x such that $U(s_1, s)V \subset W$. This implies that

$$U(s_1, s)v \in \bigcup_{t \in [0,2\lambda]} F(\hat{L}, t, s_1) \quad \text{for every } v \in V.$$

Then, for any $v \in V$, there exists $t_v \in [0, 2\lambda]$ such that $U(s_1, s)v \in F(\hat{L}, t_v, s_1)$. In other words,

$$U(s_1 + t_v, s_1)U(s_1, s)v \in L(s_1 + t_v) \Longleftrightarrow U(s_1 + t_v, s)v \in L(s_1 + t_v).$$

This implies that $U(s_1 + t_v - \lambda, s)v \in F(\hat{L}, \lambda, s_1 + t_v - \lambda) = S(s_1 + t_v - \lambda)$. But Condition (TC) implies that $U(s_1 + t_v - \lambda, s)v \in M(s_1 + t_v - \lambda)$, because $s_1 + t_v - \lambda \leq s_1 + \lambda$. As a consequence,

$$\phi(v, s) \leq \phi(x, s) + t_v - \lambda \leq \phi(x, s) + \lambda < \phi(x, s) + \varepsilon.$$

Therefore, there exists V a neighborhood of x such that

$$v \in V \Longrightarrow \phi(v, s) < \phi(x, s) + \varepsilon. \qquad \square$$

As a consequence of Proposition 3.30 and this last result, we obtain

Corollary 3.54. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process satisfying* (TC) *and $s \in \mathbb{R}$. Then the map $\phi(\cdot, s)\colon X \setminus M(s) \longrightarrow (0, \infty]$ is continuous.*

We begin our quest to prove the existence of pullback $\mathfrak{D}$-attractors with a few auxiliary results.

Proposition 3.55. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process satisfying Condition* (I) *and* (SSTC)*, $t \in \mathbb{R}$, and $y \in M(t)$. If $\hat{S}$ is a λ-section through y at t with a λ-bar $\hat{L}$, then*

$$\tilde{U}(t, s)X \cap \bigcup_{r \in [0, \lambda]} F(\hat{L}, r, t) = \emptyset \qquad \text{for all } s < t - \lambda.$$

Proof. Suppose that the result is false. Then there exists $x \in X$ and $\mu \in [0, \lambda]$ such that $z = \tilde{U}(t, s)x \in F(\hat{L}, \mu, t)$ for some $s < t - \lambda$. Therefore, we have that

$$U(t + \mu, t)z \in L(t + \mu).$$

First, we will see that $\mu \neq \lambda$. If $\mu = \lambda$, then $U(t+\lambda, t)z \in L(t+\lambda)$, which implies that $z \in F(\hat{L}, \lambda, t) = S(t) \subset M(t)$. This is a contradiction with Condition (I) and Proposition 3.32, since $z = \tilde{U}(t, s)x \notin M(t)$. Thus, we have that $\mu \in [0, \lambda)$.

If $t - s < \phi(x, s)$, then we have that $z = \tilde{U}(t, s)x = U(t, s)x$. We define $w = U(t - (\lambda - \mu), s)x$. This implies that

$$\begin{aligned} U(t + \mu, s)x &= U(t + \mu, t - (\lambda - \mu))U(t - (\lambda - \mu), s)x \\ &= U(t + \mu, t - (\lambda - \mu))w \in L(t + \mu). \end{aligned}$$

As a consequence, $w \in F(\hat{L}, \lambda, t - (\lambda - \mu)) = S(t - (\lambda - \mu)) \subset M(t - (\lambda - \mu))$. This implies that $\phi(x, s) \leq t - (\lambda - \mu) - s < t - s$, a contradiction.

If $t - s \geq \phi(x, s)$, we consider t_1 and x^+ such that

$$\begin{aligned} &z = \tilde{U}(t, s)x = U(t, t_1)x^+, \\ &x^+ \in I_{t_1}(M(t_1)), \\ &t \in [t_1, t_1 + \phi(x^+, t_1)). \end{aligned}$$

Moreover, we also have that

$$U(t+\mu, t_1)x^+ = U(t+\mu, t)U(t, t_1)x^+ = U(t+\mu, t)z \in L(t+\mu).$$

We have to distinguish two different cases.

Case 1: $t \geq t_1 + \lambda - \mu$.
This implies that $U(t+\mu-\lambda, t_1)x^+ \in F(\hat{L}, \lambda, t-(\lambda-\mu))$, that is,

$$U(t+\mu-\lambda, t_1)x^+ \in S(t-(\lambda-\mu)) \subset M(t-(\lambda-\mu)).$$

Therefore, we have $\phi(x^+, t_1) \leq (t+\mu-\lambda) - t_1 < t - t_1 < \phi(x^+, t_1)$, a contradiction.

Case 2: $t < t_1 + \lambda - \mu$.
This implies that $U(t_1 + (t+\mu-t_1), t_1)x^+ \in L(t+\mu)$, so by definition we get that $x^+ \in F(\hat{L}, t+\mu-t_1, t_1)$. Finally, we know that $t+\mu-t_1 < \lambda$, so we obtain that

$$x^+ \in \bigcup_{r\in[0,\lambda)} F(\hat{L}, r, t_1).$$

However, $x^+ \in I_{t_1}(M(t_1))$ and $t_1 \leq t$, and this contradicts (SSTC). $\square$

The previous proposition tells us that if the elapsed time is longer than λ, then we can not be on

$$\bigcup_{r\in[0,\lambda]} F(\hat{L}, r, t).$$

Proposition 3.56. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process satisfying* (TC)*, $s \in \mathbb{R}$, $t \geq s$, $x \in X \setminus M(s)$, and $\{x_n\}_n$ a convergent sequence to x. Then there exists a sequence $\{\eta_n\}_n$ convergent to* 0 *such that $\tilde{U}(t+\eta_n, s)z_n \longrightarrow \tilde{U}(t,s)x$.*

Proof. Suppose first that $\phi(x,s) = \infty$. The set $X \setminus M(s)$ is open, $x \in X \setminus M(s)$, and the sequence $\{x_n\}_n$ converges to x. This implies that we can assume that $x_n \in X \setminus M(s)$ for every $n \in \mathbb{N}$. Therefore, $\phi(x_n, s) \longrightarrow \phi(x,s)$, by Corollary 3.54. As a consequence, we can obtain that $\phi(x_n, s) > t - s$ for n sufficiently large, so

$$\tilde{U}(t,s)x_n = U(t,s)x_n \longrightarrow U(t,s)x = \tilde{U}(t,s)x.$$

Taking $\eta_n = 0$ for all $n \in \mathbb{N}$ we finish this part.

On the other hand, if $\phi(x,s) < \infty$, we can assume that $\phi(x_n, s) < \infty$ for all $n \in \mathbb{N}$. To simplify notation, we take $s_1 = s + \phi(x,s)$. We can consider three different cases.

Case 1: $s \le t < s_1$.

We take $\varepsilon \in (0, s_1 - t)$. Proposition 3.30 implies that there exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, then $\phi(x,s) - \varepsilon < \phi(x_n, s)$. This implies that $t < s + \phi(x_n, s)$. Taking $\eta_n = 0$, the proof is analogous as the case $\phi(x,s) = \infty$.

Case 2: $t = s_1$.

In this case, we have $\tilde{U}(t,s)x = \tilde{U}(s_1,s)x = I_{s_1}(U(s_1,s)x)$. Moreover, we also have that $U(s_1,s)x_n \longrightarrow U(s_1,s)x$, and

$$\tilde{U}(s + \phi(x_n,s), s)x_n = I_{s+\phi(x_n,s)}(U(s + \phi(x_n,s), s)x_n).$$

This last value is convergent to $I_{s_1}(U(s_1,s)x) = \tilde{U}(s_1,s)x$ by Corollary 3.54. Therefore, we define $\eta_n = \phi(x_n,s) - \phi(x,s)$ for each $n \in \mathbb{N}$. We will prove that this is the sequence we are looking for. By Corollary 3.54, we can conclude that $\eta_n \longrightarrow 0$, and

$$\tilde{U}(t + \eta_n, s)x_n = \tilde{U}(s_1 + \eta_n, s)x_n = \tilde{U}(s + \phi(x_n,s), s)x_n,$$

which converges to $\tilde{U}(s_1,s)x = \tilde{U}(t,s)x$.

Case 3: $t > s_1$.

We denote $s_0 = s$, and consider $s_1, \dots, s_m$ the jump times of

$$\varphi_n \colon r \in [s,t] \longrightarrow \tilde{U}(r,s)x \in X,$$

which is the partial impulsive semitrajectory at x starting at time s. This implies that $t = s_m + r_1$, with $r_1 \ge 0$. We take $x_0^+ = x$ and define, for $i \in \{0, \dots, m-1\}$,

$$\begin{aligned} s_{i+1} &= s_i + \phi(x_i^+, s_i), \\ x_{i+1}^+ &= I_{s_{i+1}}(U(s_{i+1}, s_i)x_i^+) = \tilde{U}(s_{i+1}, s_i)x_i^+. \end{aligned}$$

Case 2 implies that there exists a sequence $\{\eta_n\}_n$ convergent to 0 such that $\tilde{U}(s_1 + \eta_n, s)x_n \longrightarrow \tilde{U}(s_1,s)x$. We can assume, from Proposition 3.30, that

$$\frac{\phi(x_1^+, s_1)}{2} < \phi(\tilde{U}(s_1 + \eta_n, s)x_n, s_1 + \eta_n) \quad \text{for every } n \in \mathbb{N}.$$

We define $\delta = \min\{s_{i+1} - s_i \colon i = 0, \cdots, m-1\}$, and a as the positive number given by

$$2a = \begin{cases} \min\{r_1, \delta\}, & \text{if } r_1 > 0, \\ \delta, & \text{if } r_1 = 0. \end{cases}$$

Without loss of generality, we suppose that $|\eta_n| \le a$ for every $n \in \mathbb{N}$. This implies that

$$a \le \frac{s_2 - s_1}{2} < \phi(\tilde{U}(s_1 + \eta_n, s)x_n, s_1 + \eta_n),$$

and, as a consequence,

$$\tilde{U}(s_1 + a, s_1 + \eta_n)\tilde{U}(s_1 + \eta_n, s)x_n = U(s_1 + a, s_1 + \eta_n)\tilde{U}(s_1 + \eta_n, s_0)x_n.$$

This converges to $U(s_1 + a, s_1)\tilde{U}(s_1, s)x = \tilde{U}(s_1 + a, s_1)\tilde{U}(s_1, s)x$.

If $m = 1$, we have that $r_1 > 0$, as $r_1 = 0$ is Case 2. Then we have that $a < r_1 < \phi(\tilde{U}(s_1, s)x, s_1)$. We continue as in Case 1, and we obtain

$$\begin{aligned}\tilde{U}(t, s)x_n &= \tilde{U}(s_1 + r_1, s_1 + a)\tilde{U}(s_1 + a, s)x_n \\ &= U(s_1 + r_1, s_1 + a)\tilde{U}(s_1 + a, s)x_n\end{aligned}$$

for n sufficiently large. This is convergent to

$$U(s_1 + r_1, s_1 + a)\tilde{U}(s_1 + a, s)x = \tilde{U}(t, s)x.$$

If $m = 2$, we denote $y_n = \tilde{U}(s_1 + a, s_0)x_n$ and $y = \tilde{U}(s_1 + a, s_0)x$. We get that $a + \phi(y, s_1 + a) = \phi(\tilde{U}(s_1, s_0)x, s_1)$, by Lemma 3.33. This implies that

$$s_2 = s_1 + \phi(\tilde{U}(s_1, s_0)x, s_1) = s_1 + a + \phi(y, s_1 + a).$$

Then, there exists a sequence $\{\eta_n\}_n$, which converges to 0, such that we have $\tilde{U}(s_2 + \eta_n, s_1 + a)y_n \longrightarrow \tilde{U}(s_2, s_1 + a)y$. As a consequence, we obtain $\tilde{U}(s_2 + \eta_n, s)x_n \longrightarrow \tilde{U}(s_2, s)x$. As before, if $r_1 = 0$, we are done, and if $r_1 > 0$, then we follow Case 1 to obtain $\tilde{U}(t, s)z_n \longrightarrow \tilde{U}(t, s)x$.

If $m > 2$, we proceed as before in order to get a sequence $\{\eta_n\}_n$ convergent to 0 with $\tilde{U}(s_m + \eta_n, s)x_n \longrightarrow \tilde{U}(s_m, s)x$. One more time, if $r_1 = 0$, we are done, and if $r_1 > 0$, we follow Case 1. □

Lemma 3.57. *Let $s \in \mathbb{R}$, $x \notin M(s)$, and $\{\alpha_n\}_n$, $\{\beta_n\}_n$ two sequences convergent to 0 such that $\beta_n \leq \alpha_n$. Then, for any sequence $\{x_n\}_n$ convergent to x with $x_n \notin M(s + \beta_n)$ for each $n \in \mathbb{N}$, we have that $\tilde{U}(s + \alpha_n, s + \beta_n)x_n \longrightarrow x$.*

Proof. We know that ϕ is lower semicontinuous at (x, s), by using Proposition 3.30. Then, we have that

$$0 \leq \alpha_n - \beta_n < \frac{\phi(x, s)}{2} < \phi(x_n, s + \beta_n)$$

for n sufficiently large. This implies that

$$\begin{aligned}\tilde{U}(s + \alpha_n, s + \beta_n)x_n &= \tilde{U}(s + \beta_n + (\alpha_n - \beta_n), s + \beta_n)x_n \\ &= U(s + \beta_n + (\alpha_n - \beta_n), s + \beta_n)x_n,\end{aligned}$$

which converges to $U(s, s)x = x$. □

With this result, we are able to prove the following.

Corollary 3.58. *With the hypotheses of Proposition* 3.56*, we can assume that the sequence* $\{\eta_n\}_n$ *is nonnegative.*

Proof. We have that $y_n = \tilde{U}(t+\eta_n, s)x_n \longrightarrow y = \tilde{U}(t,s)x$ for some sequence $\{\eta_n\}_n$ convergent to 0, by Proposition 3.56. By Lemma 3.57, we can take $\alpha_n = \eta_n + |\eta_n|$ and $\beta_n = \eta_n$, so we obtain

$$\tilde{U}(t+\eta_n+|\eta_n|, s)x_n = \tilde{U}(t+\eta_n+|\eta_n|, t+\eta_n)y_n \longrightarrow y = \tilde{U}(t,s)x. \qquad \square$$

Lemma 3.59. *Let* $\tilde{\mathcal{U}}$ *be an impulsive evolution process satisfying* (STC)*,* $s \in \mathbb{R}$*, and* $x \in M(s)$*. Suppose that* $\{x_n\}_n$ *is a convergent sequence to* x *such that*

$$x_n \in \bigcup_{t\in(\lambda,2\lambda]} F(\hat{L}, t, s).$$

Then there exist a subsequence $\{x_{n_k}\}_k$ *such that, if* $\varepsilon_k = \phi(x_{n_k}, s)$*, we have*

$$y_k = U(s+\varepsilon_k, s)x_{n_k} \in M(s+\varepsilon_k) \quad \text{and} \quad y_k \longrightarrow x.$$

Proof. We know that

$$x_n \in \bigcup_{t\in(\lambda,2\lambda]} F(\hat{L}, t, s)$$

for every $n \in \mathbb{N}$, so there exists $r_n \in (\lambda, 2\lambda]$ such that $x_n \in F(\hat{L}, r_n, s)$. This implies that $U(s+r_n, s)x_n \in L(s+r_n)$. We assume that the sequence $\{r_n\}_n$ is convergent to a number $r \in [\lambda, 2\lambda]$, by relabeling the sequence, if necessary. Then $U(s+r_n, s)x_n \longrightarrow U(s+r, s)x \in L(s+r)$. As a consequence, we have that $x \in F(\hat{L}, r, s)$. Finally, we know that $x \in S(s) = F(\hat{L}, \lambda, s)$ by (STC), which implies that $r = \lambda$, by definition of section through x at s.

We define $\delta_n = r_n - \lambda$ and $z_n = U(s+\delta_n, s)x_n$. We know that $\{\delta_n\}_n$ converges to 0 and that $z_n \longrightarrow U(s,s)x = x$. Furthermore, we have that

$$U(s+\delta_n+\lambda, s+\delta_n)z_n = U(s+r_n, s)x_n \in L(s+r_n).$$

As a consequence, we get that

$$z_n \in F(\hat{L}, \lambda, s+\delta_n) = S(s+\delta_n) \subset M(s+\delta_n).$$

We are going to prove that $\delta_n = \phi(x_n, s)$ for all $n \in \mathbb{N}$, which would imply our result by taking $y_n := z_n$. Suppose that it is false. Then there exist $n \in \mathbb{N}$ and $t_0 \in (0, \delta_n)$ with $w_n = U(s+t_0, s)x_n \in M(s+t_0)$. We have that

$$\begin{aligned} U(s+t_0+(r_n-t_0), s+t_0)w_n &= U(s+r_n, s+t_0)w_n \\ &= U(s+r_n, s+t_0)U(s+t_0, s)x_n \\ &= U(s+r_n, s)x_n \in L(s+r_n). \end{aligned}$$

This implies that $w_n \in F(\hat{L}, r_n - t_0, s + t_0)$. We also have that

$$w_n \in M(s+t_0) \cap \bigcup_{t\in[0,2\lambda]} F(\hat{L}, t, s+t_0) = S(s+t_0).$$

Then, we get $w_n \in S(s+t_0) = F(\hat{L}, \lambda, s+t_0)$. Finally, this implies that $r_n - t_0 = \lambda$, so $t_0 = r_n - \lambda = \delta_n$, which is a contradiction with $t_0 \in (0, \delta_n)$. $\square$

Now we have all the necessary tools to prove an invariance property for impulsive pullback ω-limits.

Proposition 3.60. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which satisfies Condition* (I) *and* (TC)*. Suppose also that it is pullback $\mathfrak{D}$-asymptotically compact. For any $\hat{D} \in \mathfrak{D}$, the family $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is positively $\tilde{\mathcal{U}}$-invariant.*

Proof. Proposition 3.45 implies that the family $\tilde{\omega}(\hat{D})$ is nonempty and collectively compact. In order to prove that $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is positively $\tilde{U}$-invariant, we fix $s \in \mathbb{R}$, $t > s$, and $x \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. We need to prove that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

We know that $x \in \tilde{\omega}(\hat{D}, s)$, so, by definition, there exist three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$, such that

$$x = \lim_{n\to\infty} \tilde{U}(s+\varepsilon_n, s_n)x_n.$$

We denote, for simplicity, $y_n = \tilde{U}(s+\varepsilon_n, s_n)x_n$. Condition (I) implies that $y_n \notin M(s+\varepsilon_n)$. Furthermore, $x \notin M(s)$, so Proposition 3.30 implies that

$$\phi(x,s) \le \liminf_{n\to\infty} \phi(y_n, s+\varepsilon_n).$$

As a consequence, for n sufficiently large, we know that

$$0 < \frac{\phi(x,s)}{2} < \phi(y_n, s+\varepsilon_n).$$

We take $\alpha = \min\{t-s, \phi(x,s)/2\} > 0$. We assume that $|\varepsilon_n| \le \alpha$, as $\varepsilon_n \longrightarrow 0$. Then $\tilde{U}(s+\alpha, s+\varepsilon_n)y_n = U(s+\alpha, s+\varepsilon_n)y_n$, which implies that

$$\tilde{U}(s+\alpha, s+\varepsilon_n)y_n \longrightarrow U(s+\alpha, s)x = \tilde{U}(s+\alpha, s)x.$$

Corollary 3.58 tells us that there exists a sequence $\{\eta_n\}_n$ of nonnegative numbers convergent to 0 such that

$$\tilde{U}(t+\eta_n, s+\alpha)\tilde{U}(s+\alpha, s+\varepsilon_n)y_n \longrightarrow \tilde{U}(t, s+\alpha)\tilde{U}(s+\alpha, s)x = \tilde{U}(t,s)x.$$

This implies that $\tilde{U}(t+\eta_n, s_n)x_n \longrightarrow \tilde{U}(t,s)x$. Therefore, $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D}, t)$. Finally, Proposition 3.32 implies that $\tilde{U}(t,s)x \notin M(t)$, so we can conclude that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$, and we are finished. $\square$

The negative $\tilde{\mathcal{U}}$-invariance is more difficult, and we need stronger hypotheses to prove it.

Proposition 3.61. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process satisfying Conditions* (I), (H), *and* (SSTC). *Suppose also that it is pullback $\mathfrak{D}$-asymptotically compact. For any $\hat{D} \in \mathfrak{D}$, the family $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is negatively $\tilde{\mathcal{U}}$-invariant.*

Proof. We fix $s \in \mathbb{R}$, $t > s$, and $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$. We have to prove that there exists $y \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$ such that $\tilde{U}(t, s)y = x$.

As $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$, by definition there exist $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ such that

$$\tilde{U}(t + \varepsilon_n, s_n)x_n \longrightarrow x.$$

We will assume that $s_n \leq s$ for every $n \in \mathbb{N}$ as $s_n \longrightarrow -\infty$. For simplicity, we denote $y_n = \tilde{U}(s, s_n)x_n$. We know that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact, which implies that $\{y_n\}_n$ has a convergent subsequence (which will be denoted the same) to a point $y \in X$, which, by definition, will belong to $y \in \tilde{\omega}(\hat{B}, s)$. We have to distinguish two cases.

Case 1: $y \in M(s)$.

Using (SSTC), we know that

$$\bigcup_{r \in [0, 2\lambda]} F(\hat{L}, r, s)$$

contains a neighborhood of y. As $y_n \longrightarrow y$, we may assume that y_n belongs to this set and that $|s - s_n| > \lambda$ for all $n \in \mathbb{N}$. Proposition 3.55 implies that

$$y_n \in \bigcup_{u \in (\lambda, 2\lambda]} F(\hat{L}, r, s)$$

for each $n \in \mathbb{N}$. Lemma 3.59 implies that, up to a subsequence of $\{y_n\}_n$, still denoted the same, and taking $\delta_n = \phi(y_n, s)$, we have that $\delta_n \longrightarrow 0$, $z_n = U(s + \delta_n, s)y_n \longrightarrow y$, and $z_n \in M(s + \delta_n)$ for every $n \in \mathbb{N}$. This implies that

$$z_n^+ = \tilde{U}(s + \delta_n, s_n)x_n = \tilde{U}(s + \delta_n, s)y_n = I_{s+\delta_n}(z_n) \longrightarrow I_s(y) =: z.$$

Moreover, we also get that $z \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. We define $a = \min\{t - s, \xi\} > 0$, with ξ from Condition (H). As $\delta_n \longrightarrow 0$, we assume that $\delta_n \in (0, a)$. This implies that

$$v_n := \tilde{U}(s + a, s + \delta_n)z_n^+ = U(s + a, s + \delta_n)z_n^+$$

converges to $U(s+a,s)z = \tilde{U}(s+a,s)z =: v$. Corollary 3.58 implies that there exists a sequence $\{\eta_n\}_n$ of nonnegative numbers convergent to 0 such that

$$\tilde{U}(t+\eta_n, s+a)v_n \longrightarrow \tilde{U}(t,s+a)v.$$

This implies that $\tilde{U}(t+\eta_n, s_n)x_n \longrightarrow \tilde{U}(t,s)z$, and therefore we obtain that $\tilde{U}(t,s)z \in \tilde{\omega}(\hat{D},t) \setminus M(t)$.

Subcase 1: Up to a subsequence, $\eta_n \leq \varepsilon_n$.
We have that

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = \tilde{U}(t+\varepsilon_n, t+\eta_n)\tilde{U}(t+\eta_n, s_n)x_n.$$

By Lemma 3.57 we have that $x = \tilde{U}(t,s)z$.

Subcase 2: Up to a subsequence, $\eta_n > \varepsilon_n$.
We have that

$$\tilde{U}(t+\eta_n, s_n)x_n = \tilde{U}(t+\eta_n, t+\varepsilon_n)\tilde{U}(t+\varepsilon_n, s_n)x_n.$$

By Lemma 3.57 we have that $x = \tilde{U}(t,s)z$.

Case 2: $y \notin M(s)$.

We know that $y \in \tilde{\omega}(\hat{D},s) \setminus M(s)$. Corollary 3.58 implies that there exists a sequence $\{\eta_n\}_n$ of nonnegative numbers convergent to 0 with

$$\tilde{U}(t+\eta_n, s)y_n \longrightarrow \tilde{U}(t,s)y.$$

Using the same ideas as in subcases of Case 1, we get $\tilde{U}(t,s)y = x$. □

As a consequence, we obtain the following result.

Theorem 3.62. *Let $\tilde{\mathcal{U}}$ be a pullback $\mathfrak{D}$-asymptotically compact and pullback $\mathfrak{D}$-dissipative impulsive evolution process. If it also satisfies Conditions* (I), (H), *and* (SSTC), *then $\tilde{\mathcal{U}}$ has a collectively closed pullback $\mathfrak{D}$-attractor $\hat{A}$.*

Proof. Take $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. Theorem 3.47 implies that the family $\hat{A} = \tilde{\omega}(\hat{B}_0) \in \mathfrak{D}$ is a pullback $\mathfrak{D}$-semiattractor, which is also collectively closed. Finally, we have that $\hat{A} \setminus \hat{M}$ is $\tilde{\mathcal{U}}$-invariant by Propositions 3.60 and 3.61. As a consequence, we have that $\hat{A}$ is a pullback $\mathfrak{D}$-attractor. □

3.3.2 New conditions

In this subsection, we will obtain conditions for the $\tilde{\mathcal{U}}$-invariance of $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ for any family $\hat{D} \in \mathfrak{D}$. These conditions will be different to the ones considered in Subsection 3.3.1.

Before stating the main results of this subsection, we note that, in this subsection, we will not need Condition (3.5), as it was needed in Subsection 3.3.1. In our to prove our results, we are going to need:

$$\begin{cases} \text{Let } s \in \mathbb{R},\ t > s,\ x \in M(t), \text{ and } \{z_n\}_n \text{ a convergent sequence such} \\ \text{that } \{U(t,s)z_n\}_n \text{ converges to } x. \text{ Then there exist a subsequence} \\ \{z_{n_k}\}_k \text{ of } \{z_n\}_n, \text{ and a sequence } \{\alpha_k\}_k \text{ with } \alpha_k \longrightarrow 0 \\ \text{and } t + \alpha_k \geq s, \text{ such that } U(t+\alpha_k, s)z_{n_k} \in M(t+\alpha_k). \end{cases} \tag{T}$$

First, as it was done in Subsection 3.3.1, we are going to prove an upper semicontinuity property for the impact time map.

Proposition 3.63. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process that satisfies Condition* (T) *and $s \in \mathbb{R}$. Then the map $\phi(\cdot, s)\colon X \longrightarrow (0,\infty]$ is upper semi-continuous.*

Proof. Take $x \in X$. We take $\{x_n\}_n$ a sequence convergent to x. We want to prove that

$$\limsup_{n\to\infty} \phi(x_n, s) \leq \phi(x, s).$$

If $\phi(x,s) = \infty$, then there is nothing to prove. Suppose that $\phi(x,s) \in (0,\infty)$. Define

$$\beta := \limsup_{n\to\infty} \phi(x_n, s),$$

so there exists a subsequence of $\{\phi(x_n, s)\}_n$ which converges to β. We denote it the same, so we assume that $\phi(x_n, s) \longrightarrow \beta$. We have that

$$\begin{aligned} &U(s+\phi(x,s), s)x_n \longrightarrow U(s+\phi(x,s), s)x, \\ &U(s+\phi(x,s), s)x \in M(s+\phi(x,s)). \end{aligned}$$

Condition (T) implies that there exist a subsequence $\{x_{n_k}\}_k$ and a sequence $\{\alpha_k\}_k$, with $\{\alpha_k\}_k$ convergent to 0, $s + \phi(x,s) + \alpha_k > s$, and

$$U(s+\phi(x,s)+\alpha_k, s)x_{n_k} \in M(s+\phi(x,s)+\alpha_k).$$

Then we have that

$$s + \phi(x_{n_k}, s) \leq s + \phi(x,s) + \alpha_k \longrightarrow s + \phi(x,s).$$

This implies that $\beta \leq \phi(x,s)$. □

Theorem 3.64. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which is pullback $\mathfrak{D}$-asymptotically compact and which satisfies Conditions* (I), (T), *and* (H). *For any $s \in \mathbb{R}$, $t \geq s$ with $t - s \in [0, \xi]$, and $\hat{D} \in \mathfrak{D}$, we have that*

$$\tilde{\omega}(\hat{D}, t) \setminus M(t) \subset \tilde{U}(t, s)\left(\tilde{\omega}(\hat{D}, s) \setminus M(s)\right).$$

Proof. First, we assume that $t > s$. The case $t = s$ follows easily from Proposition 3.34. Take $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$. We have to prove that there exists $z \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$ such that $\tilde{U}(t, s)z = x$.

As $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$, by definition, we know that there exist sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ with $s_n \longrightarrow 0$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ such that

$$\lim_{n\to\infty} \tilde{U}(t + \varepsilon_n, s_n)x_n = x.$$

We suppose that $|\varepsilon_n| \leq \xi/4$ and $s_n \leq t - 2\xi$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we consider the partial impulsive semitrajectory

$$\varphi_n \colon u \in [s_n, t + \xi/4] \longrightarrow \tilde{U}(u, s_n)x_n \in X.$$

The map φ_n has a finite number $N_n \geq 0$ of jump times of $\mathcal{U}$ at (x_n, s_n). We denote by τ_n the last jump time. In the case that $N_n = 0$ we take $\tau_n = s_n$. The proof will be split in three different cases.

Case 1: Up to a subsequence, which will be denoted the same, there exists $\varepsilon \in (0, \xi/2)$ with $\tau_n < s - \varepsilon/2$ for all $n \in \mathbb{N}$.

For $u \in [s - \varepsilon/2, t + \xi/4]$, we have that

$$\begin{aligned} \tilde{U}(u, s_n)x_n &= \tilde{U}(u, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n \\ &= U(u, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n, \end{aligned} \tag{3.11}$$

by the definition of τ_n, because $\tau_n < s - \varepsilon$. The sequence $\{\tilde{U}(s - \varepsilon/2, s_n)x_n\}_n$ has a convergent subsequence, as $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact. We assume that

$$y_n := \tilde{U}(s - \varepsilon/2, s_n)x_n \longrightarrow y.$$

We will prove that $U(u, s - \varepsilon/2)y \notin M(u)$ for any $u \in (s - \varepsilon/2, t + \xi/4)$. Suppose that $U(u, s - \varepsilon/2)y \in M(u)$ for some $u \in (s - \varepsilon/2, t + \xi/4)$, that is, $\phi(y, s - \varepsilon/2) < t + \xi/4 - (s - \varepsilon/2)$. Then, we have that the sequence $\{y_n\}_n$ converges to y and

$$U(u, s - \varepsilon/2)y_n \longrightarrow U(u, s - \varepsilon/2)y.$$

Condition (T) implies that there exist a subsequence $\{y_{n_k}\}_k$ and $\{\alpha_k\}_k$ convergent to 0 such that $u + \alpha_k \geq s - \varepsilon/2$ and $U(u + \alpha_k, s - \varepsilon/2)y_{n_k} \in M(u + \alpha_k)$ for all $k \in \mathbb{N}$. But this is a contradiction, because $\tau_{n_k} < s - \varepsilon$. We also have that

$$\tilde{U}(s, s_n)x_n = U(s, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n = U(s, s - \varepsilon/2)y_n,$$

where we have used Equation (3.11). This implies that we get

$$\tilde{U}(s, s_n)x_n \longrightarrow U(s, s - \varepsilon/2)y = z.$$

As a consequence, we have that $z \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. Finally, we are going to prove that $x = \tilde{U}(t, s)z$. Once again, by Equation (3.11),

$$\tilde{U}(t + \varepsilon_n, s_n)x_n = \tilde{U}(t + \varepsilon_n, s)\tilde{U}(s, s_n)x_n = U(t + \varepsilon_n, s)\tilde{U}(s, s_n)x_n,$$

which implies that $x = U(t, s)z$. Moreover, we also know that $z \notin M(s)$ and $U(u, s)z \notin M(u)$ for every $u \in (s, t]$, so $x = U(t, s)z = \tilde{U}(t, s)z$.

Case 2: Up to a subsequence, which will be denoted the same, there exists $\varepsilon \in (0, \xi/2)$ with $s + \varepsilon < \tau_n$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we have that $\tau_n \in (s + \varepsilon, t + \xi/4]$. Up to a subsequence, still denoted the same, we have that $\tau_n \longrightarrow \bar{\tau} \in [s + \varepsilon, t + \xi/4]$. We define $v_n = \tilde{U}(t - 3\xi/2, s_n)x_n$. The sequence $\{v_n\}_n$ has a convergent subsequence, using that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact. We still denote the sequence by v_n, and the limit will be denoted by $v \in X$, so we assume $v_n \longrightarrow v$. Condition (H) and $|(t - 3\xi/2) - (t + \xi/4)| = 7\xi/4 < 2\xi$ imply that τ_n is the unique jump time of $\mathcal{U}$ at (x_n, s_n) in $[s - \varepsilon, t + \xi/4]$. As a consequence, for $u \in [t - 3\xi/2, \tau_n)$, we have that

$$\tilde{U}(u, t - 3\xi/2)v_n = U(u, t - 3\xi/2)v_n.$$

Furthermore, we also have that

$$U(\tau_n, t - 3\xi/2)v_n \in M(\tau_n) \quad \text{and} \quad U(\tau_n, t - 3\xi/2)v_n \longrightarrow U(\bar{\tau}, t - 3\xi/2)v.$$

This implies that we have $U(\bar{\tau}, t - 3\xi/2)v \in M(\bar{\tau})$. Finally,

$$w_n := \tilde{U}(\tau_n, t - 3\xi/2)v_n = I_{\tau_n}(U(\tau_n, t - 3\xi/2)v_n),$$

which converges to $w := I_{\bar{\tau}}(U(\bar{\tau}, t - 3\xi/2)v) \in I_{\bar{\tau}}(M(\bar{\tau}))$. Once again, Condition (H) implies that $\tilde{U}(u, t - 3\xi/2)v = U(u, \bar{\tau})w$ for every $u \in [\bar{\tau}, t + \xi/4]$, because $|t + \xi/4 - \bar{\tau}| < 2\xi$.

First, we are going to prove that $U(u, t - 3\xi/2)v \notin M(u)$ for $u \in (t - 3\xi/2, \bar{\tau})$. Suppose it is false, that is, assume that there exists $u \in (t - 3\xi/2, \bar{\tau})$ such that $U(u, t - 3\xi/2)v \in M(u)$. We know that

$$v_n \longrightarrow v \quad \text{and} \quad U(u, t - 3\xi/2)v_n \longrightarrow U(u, t - 3\xi/2)v.$$

Condition (T) says that there exists a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$ and a sequence $\{\alpha_k\}_k$ convergent to 0, with $u + \alpha_k \geq t - 3\xi/2$, such that we have $U(u + \alpha_k, t - 3\xi/2)v_{n_k} \in M(r + \alpha_k)$. We take $\delta > 0$ with $u < \bar{\tau} - \delta < \bar{\tau}$. For large k, we have $u+\alpha_k \le \bar{\tau}-\delta < \tau_{n_k}$. But this contradicts the fact that τ_{n_k} was the only jump time in the interval $[t-3\xi/2, t+\xi/4]$. As a consequence,

$$U(u,t-3\xi/2)v \notin M(u) \quad \text{ for } u \in (t-3\xi/2, \bar{\tau}).$$

Therefore, we can define $z=\tilde{U}(s,t-3\xi/2)v$, which does not belong to $M(s)$. Then

$$\tilde{U}(s,s_n)x_n = \tilde{U}(s,t-3\xi/2)v_n = U(s,t-3\xi/2)v_n$$

converges to

$$U(s,t-3\xi/2)v = \tilde{U}(s,t-3\xi/2)v = z,$$

so $z \in \tilde{\omega}(\hat{D}, s)$. Finally, we prove that $x=\tilde{U}(t,s)z$. We know that

$$\tilde{U}(t,s)z = \tilde{U}(t,s)\tilde{U}(s,t-3\xi/2)v = \tilde{U}(t,t-3\xi/2)v,$$

and we will prove that $x=\tilde{U}(t,t-3\xi/2)v$. We have

$$\tilde{U}(u,t-3\xi/2)v = \begin{cases} U(u,t-3\xi/2)v, & u \in [t-3\xi/2, \bar{\tau}), \\ U(u,\bar{\tau})w, & u \in [\bar{\tau}, t+\xi/4]. \end{cases}$$

We distinguish two subcases:

Subcase 1: Up to a subsequence, denoted the same, $\tau_n \le t+\varepsilon_n$.
In this subcase $\bar{\tau} \le t$ and

$$\begin{aligned} \tilde{U}(t+\varepsilon_n, s_n)x_n &= \tilde{U}(t+\varepsilon_n, \tau_n)\tilde{U}(\tau_n, t-3\xi/2)v_n \\ &= U(t+\varepsilon_n, \tau_n)\tilde{U}(\tau_n, t-3\xi/2)v_n. \end{aligned}$$

Taking limits, we obtain $x = U(t,\bar{\tau})w = \tilde{U}(t,t-3\xi/2)v$.

Subcase 2: Up to a subsequence, denoted the same, $\tau_n > t+\varepsilon_n$.
This time we have $\bar{\tau} \ge t$ and

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = \tilde{U}(t+\varepsilon_n, t-3\xi/2)v_n = U(t+\varepsilon_n, t-3\xi/2)v_n.$$

Taking limits, we have $x = U(t,t-3\xi/2)v$. If $\bar{\tau}=t$, we would have that $x=U(t,t-3\xi/2)v \in M(t)$, a contradiction, as $x \notin M(t)$. This implies that $\bar{\tau} > t$. As a consequence, $U(t,t-3\xi/2)v = \tilde{U}(t,t-3\xi/2)v$.

Case 3: $\tau_n \longrightarrow s$.

For each $n \in \mathbb{N}$, we define

$$z_n := \tilde{U}(\tau_n, s_n)x_n \in I_{\tau_n}(M(\tau_n)) \quad \text{ and } \quad v_n := \tilde{U}(t-3\xi/2, s_n)x_n.$$

Condition (H) implies that there is only one jump time for φ_n inside the interval $[t-3\xi/2, t+\xi/4]$, which is τ_n. Then we have

$$\tilde{U}(u,s_n)x_n := \begin{cases} U(u,t-3\xi/2)v_n, & u \in [t-3\xi/2,\tau_n), \\ U(u,\tau_n)z_n, & u \in [\tau_n, t+\xi/4]. \end{cases} \tag{3.12}$$

This implies that $z_n = I_{\tau_n}(U(\tau_n, t-3\xi/2)v_n)$. The impulsive evolution process is pullback $\mathfrak{D}$-asymptotically compact, so the sequences $\{v_n\}_n$ and $\{z_n\}_n$ have convergent subsequences, which will be denoted the same. Therefore, we will assume that $v_n \longrightarrow v$ and $z_n \longrightarrow z$ for some $v, z \in X$. First, we know that $U(\tau_n, t-3\xi/2)v_n \in M(\tau_n)$. The fact that $\hat{M}$ is collectively closed implies that $U(s,t-3\xi/2)v \in M(s)$. Moreover,

$$I_{\tau_n}(U(\tau_n,t-3\xi/2)v_n) \longrightarrow I_s(U(s,t-3\xi/2)v),$$

which implies $z = I_s(U(s,t-3\xi/2)v)$. Condition (H) implies that $\phi(z,s) \geq 2\xi$, so $\tilde{U}(u,s)z = U(u,s)z$ for all $u \in [s, s+2\xi]$. Finally, by Equation (3.12), we have

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n,\tau_n)z_n,$$

which implies that $x = U(t,s) = \tilde{U}(t,s)$. This finishes the proof. □

Theorem 3.65. *Under the hypotheses of Theorem* 3.64*,* $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ *is negatively* $\tilde{\mathcal{U}}$*-invariant.*

Proof. Fix $s \in \mathbb{R}$ and $t \geq s$. The case $t = s$ is trivial, so we assume $t > s$. We take $n \in \mathbb{N}$ such that

$$0 < T := \frac{t-s}{n} \leq \xi.$$

Theorem 3.64 implies that, for $m \in \{0, \ldots, n-1\}$, $\tilde{\omega}(\hat{D}, t-mT) \setminus M(t-mT)$ is contained in

$$\tilde{U}(t-mT, t-(m+1)T)\left(\tilde{\omega}(\hat{D}, t-(m+1)T) \setminus M(t-(m+1)T)\right).$$

Moreover, from Proposition 3.34, we have that

$$\tilde{U}(t,s) = \tilde{U}(t,t-T)\cdots\tilde{U}(t-(n-1)T,s),$$

which implies that $\tilde{\omega}(\hat{D},t) \setminus M(t) \subset \tilde{U}(t,s)\left(\tilde{\omega}(\hat{D},s) \setminus M(s)\right)$. □

We have proved the negative invariance of $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ for any $\hat{D}$. We are going to see that for a pullback $\mathfrak{D}$-semiattractor, negative invariance implies positive invariance.

Proposition 3.66. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which satisfies Condition* (I)*. If $\hat{A}$ is a pullback $\mathfrak{D}$-semiattractor and $\hat{A} \setminus \hat{M}$ is negatively $\tilde{\mathcal{U}}$-invariant, then $\hat{A} \setminus \hat{M}$ is $\tilde{\mathcal{U}}$-invariant.*

Proof. We will write $B(t) = A(t) \setminus M(t)$ for all $t \in \mathbb{R}$ for simplicity. As $\hat{A} \in \mathfrak{D}$, we also have that $\hat{B} \in \mathfrak{D}$. We will prove that $\tilde{U}(t,s)B(s) \subset B(t)$ for all $t \geq s$. If $t = s$, then there is nothing to prove, so we assume that $t > s$. The family $\hat{B}$ is negatively $\tilde{\mathcal{U}}$-invariant, so

$$B(s) \subset \tilde{U}(s,v)B(v) \text{ for every } v \leq s.$$

If we apply $\tilde{U}(t,s)$ we will get, as a consequence,

$$d_{\mathrm{H}}(\tilde{U}(t,s)B(s), A(t)) \leq d_{\mathrm{H}}(\tilde{U}(t,v)B(v), A(t)) \longrightarrow 0$$

as $v \longrightarrow -\infty$, by the pullback $\mathfrak{D}$-attraction. This implies that

$$\tilde{U}(t,s)B(s) \subset \overline{A(t)} = A(t),$$

because $A(t)$ is closed (as it is compact). Finally, Proposition 3.32 tells us that $\tilde{U}(t,s)X \cap M(t) = \emptyset$, so we deduce that $\tilde{U}(t,s)\left(A(s) \setminus M(s)\right) \subset A(t) \setminus M(t)$. ☐

We can conclude the following.

Corollary 3.67. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which is pullback $\mathfrak{D}$-dissipative and pullback $\mathfrak{D}$-asymptotically compact. If Conditions* (I)*,* (T)*, and* (H) *are satisfied, then $\tilde{\mathcal{U}}$ has a pullback $\mathfrak{D}$-attractor.*

However, we are also able to prove the positive invariance for $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ for any $\hat{D}$.

Theorem 3.68. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process which is pullback $\mathfrak{D}$-asymptotically compact and satisfies Conditions* (H)*,* (I)*, and* (T)*. If $\hat{D} \in \mathfrak{D}$, $s \in \mathbb{R}$, and $0 < t - s \leq \xi$, then we have*

$$\tilde{U}(t,s)\big(\tilde{\omega}(\hat{D},s) \setminus M(s)\big) \subset \tilde{\omega}(\hat{D},t) \setminus M(t).$$

Proof. Take $x \in \tilde{\omega}(\hat{D},s) \setminus M(s)$. We will prove that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D},t) \setminus M(t)$. In fact, by Condition (I) and Proposition 3.32, we only need to prove that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D},t)$, as $\tilde{U}(t,s)x \notin M(t)$. By definition, as $x \in \tilde{\omega}(\hat{D},s)$, there exist $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$ such that

$$\tilde{U}(s+\varepsilon_n, s_n)x_n \longrightarrow x.$$

We suppose that $s + \varepsilon_n < t$ for all $n \in \mathbb{N}$, and consider

$$\varphi_n \colon u \in [s_n, t + \xi/4] \longrightarrow \tilde{U}(u, s_n)x_n \in X.$$

This map has a finite number $N_n \geq 0$ of jump times of $\mathcal{U}$ at (x_n, s_n). We denote by τ_n the last jump time. In the case that $N_n = 0$, we take $\tau_n = s_n$. We split the proof into three cases.

Case 1: Up to a subsequence, still denoted by n, there exists $0 < \varepsilon < \xi/2$ such that $\tau_n < s - \varepsilon$.

Suppose that $\varepsilon_n > -\varepsilon/2$ for all $n \in \mathbb{N}$. $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact, so the sequence $\{y_n\}_n$, with $y_n := \tilde{U}(s - \varepsilon/2, s_n)x_n$, has a convergent subsequence. We denote the subsequence the same, so we assume that $y_n \longrightarrow y$. We know that τ_n was the last jump time in $[s_n, s + 5\xi/4]$ and $\tau_n < s - \varepsilon/2$, so we get

$$\begin{aligned}\tilde{U}(s + \varepsilon_n, s_n)x_n &= \tilde{U}(s + \varepsilon_n, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n \\ &= U(s + \varepsilon_n, s - \varepsilon/2)y_n.\end{aligned}$$

This implies that $U(s, s - \varepsilon/2)y = x$, taking limits. Moreover, we obtain that

$$\tilde{U}(s, s_n)x_n = \tilde{U}(s, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n = U(s, s - \varepsilon/2)y_n,$$

which also converges to x. Finally, we also have that

$$\tilde{U}(u, s_n)x_n = \tilde{U}(u, s)\tilde{U}(s, s_n)x_n = U(u, s)\tilde{U}(s, s_n)x_n$$

for each $u \in (s, s + \xi)$, so $\tilde{U}(u, s_n)x_n \longrightarrow U(u, s)x$. Therefore, we have that $\tilde{U}(t, s_n)x_n \longrightarrow U(t, s)x$. To end this case, we will prove that $\tilde{U}(t, s)x = U(t, s)x$. If this is not true, there exists $u \in (s, t]$ with $U(u, s)x \in M(u)$. Condition (T) implies that there exist a subsequence $\{\tilde{U}(s, s_{n_k})x_{n_k}\}_k$ and a sequence $\{\alpha_k\}_k$, convergent to 0, with $u + \alpha_k \geq s$ and $U(u + \alpha_k, s)\tilde{U}(s, s_{n_k})x_{n_k} \in M(u + \alpha_k)$ for all k. Then $u + \alpha_k$ is a jump time, and this is a contradiction with τ_{n_k} being the last jump time. As a consequence, $\tilde{U}(t, s)x = U(t, s)x$, so

$$\tilde{U}(t, s_n)x_n \longrightarrow \tilde{U}(t, s)x.$$

This implies that $\tilde{U}(t, s)x \in \tilde{\omega}(\hat{D}, t)$.

Case 2: Up to a subsequence, still denoted by n, there exists $0 < \varepsilon < \xi/2$ such that $\tau_n > s + \varepsilon$.

We suppose that $\varepsilon_n \in (-\varepsilon/2, \varepsilon)$. We define $y_n := \tilde{U}(s - \varepsilon/2, s_n)x_n$. $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact, so $\{y_n\}_n$ has a convergent subsequence. We denote it the same, so we assume that $y_n \longrightarrow y \in X$ for some $y \in X$. We know that τ_n is the unique jump time in the interval $[s - \xi/2, s + 5\xi/4]$, and $\tau_n > s + \varepsilon > s + \varepsilon_n$. Therefore, we have

$$\begin{aligned}\tilde{U}(s + \varepsilon_n, s_n)x_n &= \tilde{U}(s + \varepsilon_n, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n \\ &= U(s + \varepsilon_n, s - \varepsilon/2)y_n.\end{aligned}$$

Taking limits, we have that $x = U(s, s - \varepsilon/2)y$. Moreover,

$$\tilde{U}(s, s_n)x_n = \tilde{U}(s, s - \varepsilon/2)\tilde{U}(s - \varepsilon/2, s_n)x_n = U(s, s - \varepsilon/2)y_n,$$

which also converges to $U(s, s-\varepsilon/2)y = x$. We split the proof of this case into three different subcases.

Subcase 1: Up to a subsequence, denoted the same, there exists $\delta > 0$ such that $\tau_n > t + \delta$.

For every $u \in (s, t]$, we have

$$\tilde{U}(u, s_n)x_n = \tilde{U}(u, s)\tilde{U}(s, s_n)x_n = U(u, s)\tilde{U}(s, s_n)x_n \longrightarrow U(u, s)x.$$

Taking $u = t$, we obtain $\tilde{U}(t, s_n)x_n \longrightarrow U(t, s)x$. We are going to prove $\tilde{U}(t, s)x = U(t, s)x$. If it is not true, there exists $u \in (s, t]$ with $U(u, s)x \in M(u)$. Condition (T), as before, implies that there exist a subsequence $\{\tilde{U}(s, s_{n_k})x_{n_k}\}_k$ of $\{\tilde{U}(s, s_n)x_n\}_n$ and a sequence $\{\alpha_k\}_k$ convergent to 0, such that $u + \alpha_k \geq s$ and $U(u + \alpha_k, s)\tilde{U}(s, s_{n_k})x_{n_k} \in M(u + \alpha_k)$ for every k. For k sufficiently large, $s \leq u + \alpha_k < t + \delta < \tau_{n_k}$, a contradiction with τ_{n_k} being the unique jump time in the interval $[s - \xi/2, s + 5\xi/4]$. Consequently, we have that $\tilde{U}(t, s_n)x_n \longrightarrow \tilde{U}(t, s)x$, which implies $\tilde{U}(t, s)x \in \tilde{\omega}(\hat{D}, t)$.

Subcase 2: Up to a subsequence, denoted the same, there exists $\delta > 0$ such that $\tau_n < t - \delta$.

We have $s + \varepsilon < \tau_n < t - \delta$. Taking a subsequence if necessary, we assume that $\{\tau_n\}_n$ is convergent and $\bar{\tau} \in [s + \varepsilon, t - \delta]$ its limit. Furthermore, as τ_n is the only jump time in $[s - \xi/2, s + 5\xi/4]$, we have that

$$U(\tau_n, s)\tilde{U}(s, s_n)x_n \in M(\tau_n),$$

so we get $U(\bar{\tau}, s)x \in M(\bar{\tau})$. Moreover, if $u \in (t - \delta, t]$, we have

$$\begin{aligned}\tilde{U}(u, s_n)x_n &= \tilde{U}(u, \tau_n)\tilde{U}(\tau_n, s)\tilde{U}(s, s_n)x_n \\ &= U(u, \tau_n)I_{\tau_n}\big(U(\tau_n, s)\tilde{U}(s, s_n)x_n\big),\end{aligned}$$

which converges to $U(u, \bar{\tau})I_{\bar{\tau}}\big(U(\bar{\tau}, s)x\big)$. Then

$$\tilde{U}(t, s_n)x_n \longrightarrow U(t, \bar{\tau})I_{\bar{\tau}}\big(U(\bar{\tau}, s)x\big).$$

We will prove that $U(t, \bar{\tau})I_{\bar{\tau}}\big(U(\bar{\tau}, s)x\big)$ is equal to $\tilde{U}(t, s)x$. We first prove that $U(u, s)x \notin M(u)$ for some $u \in (s, \bar{\tau}]$. If this is false, then there exists $u \in (s, \bar{\tau})$ such that $U(u, s)x \in M(u)$. Condition (T) would imply a contradiction, because we would have that $u + \alpha_k$ is a jump time and $u + \alpha_k < \tau_{n_k}$ for k sufficiently large (similarly to before). We know that $U(\bar{\tau}, s)x \in M(\bar{\tau})$. Condition (H) and $|t - \bar{\tau}| < 2\xi$ imply that

$$\tilde{U}(t, s)x = \tilde{U}(t, \bar{\tau})\tilde{U}(\bar{\tau}, s)x = U(t, \bar{\tau})I_{\bar{\tau}}(U(\bar{\tau}, s)x).$$

As a consequence we get that $\tilde{U}(t,s_n)x_n \longrightarrow \tilde{U}(t,s)x$, which implies that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D},t)$.

Subcase 3: $\tau_n \longrightarrow t$.
We obtain that $U(t,s)x \in M(t)$, the same way as in Subcase 2. Then we have that

$$\tilde{U}(\tau_n,s_n)x_n = \tilde{U}(\tau_n,s)\tilde{U}(s,s_n)x_n = I_{\tau_n}\big(U(\tau_n,s)\tilde{U}(s,s_n)x_n\big),$$

which converges to $I_t(U(t,s)x)$. We will prove that $\tilde{U}(t,s)x = I_t\left(U(t,s)x\right)$. If this is not true, then there exists $u \in (s,t)$ such that $U(u,s)x \in M(u)$. Again, with Condition (T), we obtain a contradiction. Then we obtain that $\tilde{U}(t,s)x = I_t\left(U(t,s)x\right)$. Finally,

$$\tilde{U}(t+(\tau_n-t),s_n)x_n \longrightarrow \tilde{U}(t,s)x,$$

which implies that $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D},t)$.

Case 3: $\tau_n \longrightarrow s$.
Take $\varepsilon \in (0,\xi/2)$. As in Case 2, we can assume that, if $y_n := \tilde{U}(s-\varepsilon/2,s_n)x_n$, then $\{y_n\}_n$ converges to some $y \in X$, and that $s-\varepsilon/2 < \tau_n < s+\varepsilon$.

Subcase 1: Up to a subsequence, denoted the same, $s+\varepsilon_n < \tau_n$.
We get that

$$\begin{aligned}\tilde{U}(s+\varepsilon_n,s_n)x_n &= \tilde{U}(s+\varepsilon_n,s-\varepsilon/2)\tilde{U}(s-\varepsilon/2,s_n)x_n\\ &= U(s+\varepsilon_n,s-\varepsilon/2)y_n,\end{aligned}$$

which converges to $U(s,s-\varepsilon/2)y = x$. Moreover, τ_n was the unique jump time in the interval $[s-\xi/2,s+5\xi/4]$, so we have that

$$U(\tau_n,s-\varepsilon/2)y_n = U(\tau_n,s-\varepsilon/2)\tilde{U}(s-\varepsilon/2,s_n)x_n \in M(\tau_n).$$

Finally, we know that $\tau_n \longrightarrow s$ and $\hat{M}$ is collectively closed. This implies that $x = U(s,s-\varepsilon/2)y \in M(s)$, a contradiction. Consequently, this subcase is not possible.

Subcase 2: There exists a subsequence of τ_n (still denoted by n) such that $s+\varepsilon_n \geq \tau_n$.
We have that

$$\begin{aligned}\tilde{U}(s+\varepsilon_n,s_n)x_n &= \tilde{U}(s+\varepsilon_n,\tau_n)\tilde{U}(\tau_n,s-\varepsilon/2)y_n\\ &= U(s+\varepsilon_n,\tau_n)I_{\tau_n}\left(U(\tau_n,s-\varepsilon/2)y_n\right),\end{aligned}$$

which converges to $I_s(U(s, s-\varepsilon/2)y) = x$. So $x \in I_s(M(s))$. Condition (H) implies that $\tilde{U}(t,s)x = U(t,s)x$, because $t - s \leq\ll 2\xi$. Finally,

$$\tilde{U}(t, s_n)x_n = \tilde{U}(t, s+\varepsilon_n)\tilde{U}(s+\varepsilon_n, s_n)x_n = U(t, s+\varepsilon_n)\tilde{U}(s+\varepsilon_n, s_n)x_n,$$

which converges to $U(t,s) = \tilde{U}(t,s)x$. Consequently, $\tilde{U}(t,s)x \in \tilde{\omega}(\hat{D}, t)$. □

Theorem 3.69 (Positive invariance). *Let $\tilde{\mathcal{U}}$ be satisfying Conditions* (H), (I), *and* (T), *pullback $\mathfrak{D}$-asymptotically compact, and $\hat{D} \in \mathfrak{D}$. Then the family $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is positively $\tilde{\mathcal{U}}$-invariant.*

Proof. Fix $s \in \mathbb{R}$ and $t \geq s$. The case $t = s$ is trivial, so we assume $t > s$. We take $n \in \mathbb{N}$ such that

$$0 < T := \frac{t-s}{n} \leq \xi.$$

Theorem 3.68 implies that, for $m \in \{1, \dots, n\}$,

$$\begin{aligned}\tilde{U}(s+mT, s+(m-1)T)\big(\tilde{\omega}(\hat{D}, s+(m-1)T) \setminus M(s+(m-1)T)\big)\\ \subset \tilde{\omega}(\hat{D}, s+mT) \setminus M(s+mT)\end{aligned}$$

Moreover, from Proposition 3.34, we have that

$$\tilde{U}(t,s) = \tilde{U}(t, s+(n-1)T)\cdots\tilde{U}(s+T, s),$$

which implies that $\tilde{U}(t,s)\big(\tilde{\omega}(\hat{D}, s) \setminus M(s)\big) \subset \tilde{\omega}(\hat{D}, t) \setminus M(t)$. □

The previous results give us the following:

Theorem 3.70 (Invariance). *If an impulsive evolution process $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact and satisfies Conditions* (H), (I), *and* (T), *then for any $\hat{D} \in \mathfrak{D}$, the set $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is $\tilde{\mathcal{U}}$-invariant.*

As a consequence, we obtain Corollary 3.67 again. With the following condition, the previous results are also true:

$$\begin{cases}\text{Let } s \in \mathbb{R},\ x \in X \setminus M(s), \text{ and } \{x_n\}_n \text{ a sequence convergent}\\ \text{to } x. \text{ Then } \liminf\limits_{n\to\infty} \phi(x_n, s) \leq \phi(x,s).\end{cases} \tag{NT}$$

Proposition 3.63 implies that if an impulsive evolution process $\tilde{\mathcal{U}}$ satisfies Condition (T), then it also satisfies Condition (NT). The proofs of Theorems 3.64 and 3.68 with Condition (NT) instead of Condition (T) are very similar (note, for example, that the proof of Case 1 in Theorem 3.64 is basically the same with Condition (NT) instead of Condition (T))

We end this section with a result relating the nonautonomous tube conditions, seen in Subsection 3.3.1, and Condition (T), seen in this subsection. We will show that (TC) implies Condition (T).

Theorem 3.71. *If an impulsive evolution process satisfies* (TC)*, then Condition* (T) *holds.*

Proof. Fix $s \in \mathbb{R}$, $t > s$, $x \in M(t)$, and $\{z_n\}_n$ a convergent sequence such that

$$U(t,s)z_n \longrightarrow x.$$

We have to prove that there exists a subsequence $\{z_{n_k}\}_k$ and a sequence $\{\alpha_k\}_k$ convergent to 0, such that $t + \alpha_k \geq s$ and $U(t+\alpha_k, s)z_{n_k} \in M(t+\alpha_k)$. We take z the limit of the sequence $\{z_n\}_n$, so we have that $U(t,s)z = x$. We have that $x \in M(t)$, so Condition (3.5) implies that there exists $\varepsilon > 0$ such that

$$U(t+r,t)x \notin M(t+r) \qquad \text{for } r \in (0,\varepsilon).$$

As $\tilde{U}$ satisfies (TC), we know that there exist $\lambda > 0$ and $\hat{S}$ a λ-section through x at t. We assume that $\lambda < \varepsilon$, by Proposition 3.51. Furthermore, we have that there exists a neighborhood of x in X contained in

$$\bigcup_{r\in[0,2\lambda]} F(\hat{L}, r, t).$$

The sequence $\{U(t,s)z_n\}_n$ converges to x, so we have

$$U(t,s)z_n \in \bigcup_{r\in[0,2\lambda]} F(\hat{L}, r, t)$$

for n sufficiently large. This implies that there exists $r_n \in [0,2\lambda]$ such that $U(t+r_n, s)z_n = U(t+r_n, t)U(t,s)z_n \in L(t+r_n)$. Relabeling the sequence, if necessary, we can assume that $\{r_n\}_n$ converges to a number $r \in [0,2\lambda]$. We consider $\alpha_n = r_n - \lambda$. We have that $U(t+r_n,s)z_n \in L(t+r_n)$. The family $\hat{L}$ is collectively closed, so $U(t+r,s)z \in L(t+r)$. We also know that $x \in S(t) \cap M(t)$. Then (TC) implies that

$$x \in F(\hat{L}, \lambda, t).$$

However, $U(t+r,t)x = U(t+r,s)z \in L(t+r)$. As a consequence, $x \in F(\hat{L}, r, t)$, that is, $r = \lambda$. Therefore, we have proved that the sequence $\{\alpha_n\}_n$ converges to 0. Moreover, we can assume that $t + \alpha_n > s$ for every $n \in \mathbb{N}$. Finally,

$$U(t+\alpha_n, s)z_n \in F(\hat{L}, \lambda, t+\alpha_n) = S(t+\alpha_n) \subset M(t+\alpha_n). \qquad \square$$

3.4 Continuity of attractors

In this section, we study perturbations of the pullback attractors for impulsive evolution processes. We obtain some results on the upper semicontinuity and weak version of the lower semicontinuity.

From now on, consider a family $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ of impulsive evolution processes $\tilde{\mathcal{U}}_\eta = (\mathcal{U}_\eta, X, \hat{M}^\eta, I^\eta)$ for each $\eta \in [0,1]$. Recall from Definition 1.80 that we say that a family of continuous evolution processes $\{\mathcal{U}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$ if $U_\eta(t,s)x \longrightarrow U_0(t,s)x$ as $\eta \longrightarrow 0$, uniformly for (t,s,x) in compact subsets of $\mathcal{P} \times X$.

Definition 3.72. Let $\{\hat{D}^\eta\}_{\eta\in[0,1]}$ be a collection of nonautonomous sets of X and $\{I^\eta\}_{\eta\in[0,1]} = \{\{I_t^\eta \colon D^\eta(t) \longrightarrow X\}_{t\in\mathbb{R}}\}_{\eta\in[0,1]}$ a collection of families of functions.

(1) We say that $\{\hat{D}^\eta\}_{\eta\in[0,1]}$ is collectively closed at $\eta = 0$, if for any sequences $\{\eta_k\}_k$ convergent to 0, $\{t_k\}_k$ convergent to t, and $\{x_k\}_k$ with $x_k \in D^{\eta_k}(t_k)$ and convergent to x, we have that $x \in D^0(t)$.

(2) We say that $\{I^\eta\}_{\eta\in[0,1]}$ is collectively continuous at $\eta = 0$, if for any sequences $\{\eta_k\}_k$ convergent to 0, $\{t_k\}_k$ convergent to t, and $\{x_k\}_k$ with $x_k \in D^{\eta_k}(t_k)$ and convergent to x, we have that $I_{t_k}^{\eta_k}(x_k)$ converges to $I_t^0(x)$.

Remark 3.73. Definitions 3.24 and 3.72 have a similar name, but we are talking about two different things.

From now on, we assume that we have a family of impulsive evolution processes $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ such that:

- $\tilde{\mathcal{U}}_\eta$ satisfies (3.7) and (I) for each $\eta \in [0,1]$,
- the associated family of continuous evolution processes $\{\mathcal{U}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$,
- the collection of impulsive sets $\{M^\eta\}_{\eta\in[0,1]}$ is collectively closed at $\eta = 0$,
- the collection of impulse functions $\{I^\eta\}_{\eta\in[0,1]}$ is collectively continuous at $\eta = 0$,
- $\tilde{\mathcal{U}}_0$ satisfies (3.9).

Lemma 3.74. *Let $s \in \mathbb{R}$, $x_0 \notin M^0(s)$, and $\{x_k\}_k$ and $\{\eta_k\}_k$ two sequences convergent to x_0 and 0, respectively. Then:*

(a) if $\{s_k\}_k$ is a sequence convergent to s, then $x_k \notin M^{\eta_k}(s_k)$ for k sufficiently large;

(b) if $\{s_k\}_k$ is a sequence convergent to s, then

$$\liminf_{k\to\infty} \phi_{\eta_k}(x_k, s_k) \geq \phi_0(x_0, s),$$

with ϕ_η the impact time map of $\tilde{\mathcal{U}}_\eta$ for every $\eta \in [0,1]$;

(c) *if* $\{\alpha_k\}_k$ *and* $\{\beta_k\}_k$ *are two sequences convergent to* 0 *such that* $\beta_k \leq \alpha_k$ *then*

$$\tilde{U}_{\eta_k}(s+\alpha_k, s+\beta_k)x_k \longrightarrow x_0.$$

Proof. We fix $s \in \mathbb{R}$, $x_0 \notin M^0(s)$, and two sequences $\{x_k\}_k$ and $\{\eta_k\}_k$ which are convergent to x_0 and 0.

(a) It follows from the collective closedness of $\{\hat{M}^\eta\}_{\eta\in[0,1]}$ at $\eta = 0$, because if $x_k \in M^{\eta_k}(s_k)$ for a subsequence, then we would have $x_0 \in M^0(s)$.

(b) We take

$$\alpha := \liminf_{k\to\infty} \phi_{\eta_k}(x_k, s_k),$$

and we suppose, by relabeling the sequence, that $\phi_{\eta_k}(x_k, s_k) \longrightarrow \alpha$. It follows that

$$U_{\eta_k}(s_k + \phi_{\eta_k}(x_k, s_k), s_k)x_k \in M^{\eta_k}(s_k + \phi_{\eta_k}(x_k, s_k)).$$

Taking limits, we have that $U_0(s+\alpha, s)x \in M^0(s+\alpha)$, so $\phi_0(x, s) \leq \alpha$.

(c) Using (b), we have that

$$0 \leq \alpha_k - \beta_k < \frac{\phi_0(x_0, s)}{2} < \phi_{\eta_k}(x_k, s+\beta_k),$$

for k large. As a consequence,

$$\begin{aligned}\tilde{U}_{\eta_k}(s+\alpha_k, s+\beta_k)x_k &= \tilde{U}_{\eta_k}(s+\beta_k+(\alpha_k-\beta_k), s+\beta_k)x_k \\ &= U_{\eta_k}(s+\beta_k+(\alpha_k-\beta_k), s+\beta_k)x_k,\end{aligned}$$

which converges to $U_0(s,s)x_0 = x_0$. □

In order to prove our results, we are going to need the following condition, which we have called Condition Collective (T).

$$\left\{\begin{array}{l}\text{Let } s\in\mathbb{R},\ t>s,\ x\in M^0(t), \text{ two convergent sequences } \{x_n\}_n \text{ and} \\ \{\eta_n\}_n \text{ such that } \{\eta_n\}_n \text{ converges to } 0, \text{ and } U_{\eta_n}(t,s)x_n \longrightarrow x. \\ \text{Then there exist subsequences } \{x_{n_k}\}_k \text{ and } \{\eta_{n_k}\}_k, \text{ and a} \\ \text{sequence } \{\alpha_k\}_k \text{ with } \alpha_k \longrightarrow 0 \text{ and } t+\alpha_k \geq s \text{ such that} \\ U_{\eta_{n_k}}(t+\alpha_k, s)x_{n_k} \in M^{\eta_{n_k}}(t+\alpha_k).\end{array}\right. \qquad \text{(CT)}$$

Lemma 3.75. *Let* $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ *be a family of impulsive evolution processes satisfying Condition* (CT). *If* $x_0 \in X$, $s \in \mathbb{R}$, *and* $\{\eta_k\}_k$ *and* $\{x_k\}_k$ *are two sequences convergent to* 0 *and* x_0, *respectively, then*

$$\limsup_{k\to\infty} \phi_{\eta_k}(x_k, s) \leq \phi_0(x_0, s).$$

Proof. Assume that $\phi_0(x_0,s) < \infty$, because if $\phi_0(x_0,s) = \infty$, the result is obvious. Denote $t = \phi_0(x_0,s)$, then $U_0(s+t,s)x_0 \in M^0(t+s)$. Furthermore, $U_{\eta_k}(t+s,s)x_k \longrightarrow U_0(t+s,s)x_0$. Denote also

$$\beta := \limsup_{k\to\infty} \phi_{\eta_k}(x_k,s),$$

and we suppose, by relabeling the sequence, that $\phi_{\eta_k}(x_k,s) \longrightarrow \beta$. We use Condition (CT), and, up to a subsequence, there exist a sequence $\{\alpha_k\}_k$, convergent to 0, such that $s+t+\alpha_k \geq s$ and

$$U_{\eta_k}(s+t+\alpha_k,s)x_k \in M^{\eta_k}(s+t+\alpha_k).$$

As a consequence,

$$\phi_{\eta_k}(x_k,s) \leq t+\alpha_k = \phi_0(x_0,s)+\alpha_k.$$

Taking limits, we obtain $\beta \leq \phi_0(x_0,s)$. □

From Lemmas 3.74 and 3.75 we obtain

Theorem 3.76. *Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes which satisfies Condition* (CT)*, and fix $s \in \mathbb{R}$, $x_0 \notin M^0(s)$, and $\{x_k\}_k$ and $\{\eta_k\}_k$ two sequences convergent to x_0 and 0, respectively. Then*

$$\lim_{k\to\infty} \phi_{\eta_k}(x_k,s) = \phi_0(x_0,s).$$

Proposition 3.77. *Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes which satisfies Condition* (CT)*. Take $s \in \mathbb{R}$, $t \geq s$, $x_0 \notin M^0(s)$, and two sequences $\{x_k\}_k$ and $\{\eta_k\}_k$ convergent to x_0 and 0. Then there exists a sequence $\{\varepsilon_k\}_k$ convergent to 0 such that $\tilde{U}_{\eta_k}(t+\varepsilon_k,s)x_k \longrightarrow \tilde{U}_0(t,s)x_0$.*

Proof. We have to distinguish several cases depending on the value of $\phi_0(x_0,s)$. Lemma 3.74, item (b) implies that, if $\phi_0(x_0,s) = \infty$, then we have that $\phi_{\eta_k}(x_k,s) > t-s$ for k sufficiently large. This implies that

$$\tilde{U}_{\eta_k}(t,s)x_k = U_{\eta_k}(t,s)x_k \longrightarrow U_0(t,s)x_0 = \tilde{U}_0(t,s)x_0.$$

Taking $\varepsilon_k = 0$ we are finished.

On the other hand, assume that $\phi_0(x_0,s) < \infty$. Lemma 3.75 implies that $\phi_{\eta_k}(x_k,s) < \infty$ for k sufficiently large. We distinguish three different cases.

Case 1: $0 \leq t-s < \phi_0(x_0,s)$.

From Lemma 3.74, item (b), we have $t-s < \phi_{\eta_k}(x_k,s)$ for large k. We take $\varepsilon_k = 0$, therefore

$$\tilde{U}_{\eta_k}(t+\varepsilon_k,s)x_k = \tilde{U}_{\eta_k}(t,s)x_k = U_{\eta_k}(t,s)x_k,$$

which converges to $U_0(t,s)x_0 = \tilde{U}_0(t,s)x_0$.

Case 2: $t - s = \phi_0(x_0, s)$.

We take $t_k = s + \phi_{\eta_k}(x_k, s)$. Theorem 3.76 implies that the sequence $\{t_k\}_k$ converges to t. Then

$$\tilde{U}_{\eta_k}(t_k, x)x_k = I^{\eta_k}_{t_k}(U_{\eta_k}(t_k, s)x_k) \longrightarrow I^0_t(U_0(t, s)x_0) = \tilde{U}_0(t, s)x_0.$$

Finally, we take $\varepsilon_k = \phi_{\eta_k}(x_k, s) - \phi_0(x_0, s) = t_k - t$, and we get that $\varepsilon_k \longrightarrow 0$ and

$$\tilde{U}_{\eta_k}(t + \varepsilon_k, s)x_k = \tilde{U}_{\eta_k}(t_k, s)x_k \longrightarrow \tilde{U}_0(t, s)x_0.$$

Case 3: $t - s > \phi_0(x_0, s)$.

We denote $s_0 = s$, and we consider $s_1, \ldots, s_m$ the jump times of $\tilde{\mathcal{U}}_0$ at (x_0, s) in the interval $[s, t]$. This implies that $t = s_m + r$ for $r \geq 0$. We define

$$\delta = \min\{s_{i+1} - s_i \colon i = 0, \ldots, m-1\},$$

and, from this definition of δ, we also consider

$$2a = \begin{cases} \min\{r, \delta\}, & r > 0, \\ \delta, & r = 0. \end{cases}$$

We take $x_0^+ = x_0$ and define, for $i \in \{0, \ldots, m-1\}$,

$$\begin{aligned} s_{i+1} &= s_i + \phi_0(x_i^+, s_i), \\ x_{i+1} &= U_0(s_{i+1}, s_i)x_i^+, \\ x_{i+1}^+ &= I^0_{s_{i+1}}(x_{i+1}). \end{aligned}$$

Case 2 implies that there exists a sequence $\{\gamma_k\}_k$ convergent to 0 such that

$$\tilde{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k \longrightarrow \tilde{U}_0(s_1, s)x_0.$$

Moreover, we can suppose that $|\gamma_k| \leq a$. Condition (I) and Lemma 3.74, item (b), imply that

$$\frac{\phi_0(\tilde{U}_0(s_1, s_0)x, s_1)}{2} < \phi_{\eta_k}(\tilde{U}_{\eta_k}(s_1 + \varepsilon_k, s_0)x_k, s_1 + \varepsilon_k) \quad \text{for every } k \in \mathbb{N}.$$

As a consequence,

$$a \leq \frac{s_2 - s_1}{2} = \frac{\phi_0(\tilde{U}_0(s_1, s_0)x, s_1)}{2} < \phi_{\eta_k}(\tilde{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k, s_1 + \gamma_k).$$

Therefore, we obtain

$$\begin{aligned} \tilde{U}_{\eta_k}(s_1 + a, s_0)x_k &= \tilde{U}_{\eta_k}(s_1 + a, s_1 + \gamma_k)\tilde{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k \\ &= U_{\eta_k}(s_1 + a, s_1 + \gamma_k)\tilde{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k, \end{aligned} \tag{3.13}$$

which converges to $U_0(s_1+a,s_1)\tilde{U}_0(s_1,s)x_0 = \tilde{U}_0(s_1+a,s_1)x_0$. We also have, by Lemma 3.33, that

$$\begin{aligned}\phi_0(\tilde{U}_0(s_1+a,s)x_0, s_1+a) &= \phi_0(U_0(s_1+a,s_1)\tilde{U}_0(s_1,s)x_0, s_1+a)\\ &= \phi_0(\tilde{U}_0(s_1,s)x_0,s_1)-a.\end{aligned} \tag{3.14}$$

If $m=1$, we have that $0<r<\phi_0(\tilde{U}_0(s_1,s)x_0,s_1)$, which together with Equation (3.14), imply that $0<r-a<\phi_0(\tilde{U}(s_1+a,s_0)x_0,s_1+a)$. We continue as in Case 1, and we have that

$$\tilde{U}_{\eta_k}(s_1+r,s_0)x_k = \tilde{U}_{\eta_k}(s_1+r,s_1+a)\tilde{U}_{\eta_k}(s_1+a,s_0)x_k,$$

which is convergent to

$$\tilde{U}_0(s_1+r,s_1+a)\tilde{U}_0(s_1+a,s_0)x_0 = \tilde{U}_0(t,s)x_0$$

If $m>1$, we have that

$$s_2 = s_1+\phi_0(\tilde{U}_0(s_1,s_0)x,s_1) = s_1+a+\phi_0(\tilde{U}_0(s_1+a,s)x_0,s_1+a).$$

Equation (3.13), Condition (I), and Case 2 imply that there exists a sequence $\{\varepsilon_k\}_k$ convergent to 0 such that

$$\tilde{U}_{\eta_k}(s_2+\varepsilon_k,s_0)x_k = \tilde{U}_{\eta_k}(s_2+\varepsilon_k,s_1+a)\tilde{U}_{\eta_k}(s_1+a,s_0)x_k,$$

which converges to $\tilde{U}_0(s_2,s_0)x$.

If $m=2$ and $r=0$, then we have finished. If $m=2$ and $r>0$, then we follow the steps laid out at the beginning of Case 3. If $m>2$, we use the same steps as before to get a sequence $\{\varepsilon_k\}_k$ convergent to 0 such that

$$\tilde{U}(s_m+\varepsilon_k,s)x_n \longrightarrow \tilde{U}(s_m,s)x_0.$$

One more time, if $r=0$, we are finished, and if $r>0$, then we are able to use Case 1. ☐

Corollary 3.78. *With the hypotheses of Proposition* 3.77*, we can take the sequence* $\{\varepsilon_k\}_k$ *of nonnegative numbers.*

Proof. Proposition 3.77 implies that there exists a sequence $\{\varepsilon_k\}_k$ convergent to 0 such that

$$y_k := \tilde{U}_{\eta_k}(t+\varepsilon_k,s)x_k \longrightarrow \tilde{U}_0(t,s)x_0 =: y_0.$$

We define $\alpha_k=\varepsilon_k+|\varepsilon_k|$ and $\beta_k=\varepsilon_k$, and we consider the sequences $\{\alpha_k\}_k$ and $\{\beta_k\}_k$. We know that $y_0\notin M^0(s)$ by Condition (I). Lemma 3.74, item (c), implies that

$$\tilde{U}_{\eta_k}(t+\alpha_k,s)x_k = \tilde{U}_{\eta_k}(t+\alpha_k,t+\beta_k)y_k,$$

which converges to $y_0=\tilde{U}_0(t,s)x_0$. Finally, the sequence $\{\alpha_k\}_k$ is the sequence we are looking for, because $\alpha_k\geq 0$ for every $k\in\mathbb{N}$. ☐

We start by defining the upper semicontinuity for pullback attractors for impulsive evolution processes. Note that it is different from Definition 1.81.

Definition 3.79. Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes with a pullback $\mathfrak{D}$-attractor $\hat{A}_\eta$ for every $\eta \in [0,1]$. This family is upper semicontinuous at $\eta = 0$ if

$$\lim_{\eta\to 0} d_{\mathrm{H}}(A_\eta(t) \setminus M^\eta(t), A_0(t)) = 0 \quad \text{for every } t \in \mathbb{R}.$$

We have already defined Condition (CT), and we have used it to prove some results about the impact time maps. In order to prove that the family of pullback $\mathfrak{D}$-attractors is upper semicontinuous at $\eta = 0$, we also need the following condition:

$$\exists\, \xi > 0 : \phi_\eta(x,s) \geq 2\xi \quad \text{for every } x \in I_s^\eta(M^\eta(s)),\ s \in \mathbb{R} \text{ and } \eta \in [0,1]. \tag{CH}$$

The first result is a technical lemma.

Lemma 3.80. *Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes with a pullback $\mathfrak{D}$-attractor for every $\eta \in [0,1]$, and satisfying* (CT), (CH), *and such that*

$$\bigcup_{\eta\in[0,1]} A_\eta(t) \text{ is relatively compact for all } t \in \mathbb{R}$$

If $s \in \mathbb{R}$, $\{\eta_k\}_k$, and $\{x_k\}_k$ are two sequences such that $\{\eta_k\}_k$ converges to 0, $x_k \in A_{\eta_k}(s) \setminus M^{\eta_k}(s)$, and $\{x_k\}_k$ converges to $x_0 \in M^0(s)$, then we have that $\phi_{\eta_k}(x_k, s) \longrightarrow 0$.

Proof. We know that $x_k \in A_{\eta_k}(s) \setminus M^{\eta_k}(s)$, so Proposition 3.38 implies that there exists ψ_k a global solution of $\tilde{\mathcal{U}}_{\eta_k}$ with $\psi_k(s) = x_k$ and $\hat{\psi}_k \in \mathfrak{D}$. We define s_k as the last jump time of ψ_k in the interval $[s-1, s+\xi/2]$, if it exists. If it does not exist, we take $s_k = s - 1$. We split to proof in two different cases.

Case 1: Up to a subsequence, there exists $0 < \varepsilon < \min\{\xi/2, 1/2\}$ such that $|s_k - s| \geq 2\varepsilon$.

We claim that there can not be jump times in the interval $[s-\varepsilon, s+\varepsilon]$. Indeed,

- If $s_k < s - 2\varepsilon$, then there are no jump times in $[s-\varepsilon, s+\varepsilon]$, because s_k was the last jump time in $[s-1, s+\xi/2]$, and $[s-\varepsilon, s+\varepsilon] \subset [s-1, s+\xi/2]$.
- If $s_k > s + 2\varepsilon$ and $\bar{s}$ is a jump time in $[s-\varepsilon, s+\varepsilon]$, then we have that $\phi_{\eta_k}(y, \bar{s}) = s_k - \bar{s}$ for some $y \in I_{\bar{s}}^{\eta_k}(M^{\eta_k}(\bar{s}))$. But this implies that

  $$\phi_{\eta_k}(y, \bar{s}) = s_k - \bar{s} \leq s + \xi/2 - (s - \xi/2) = \xi < 2\xi,$$

  a contradiction with Condition (CH).

We have that

$$\begin{aligned} x_k &= \tilde{U}_{\eta_k}(s, s-1)\psi_k(s-1) \\ &= \tilde{U}_{\eta_k}(s, s-\varepsilon/2)\tilde{U}_{\eta_k}(s-\varepsilon/2, s-1)\psi_k(s-1) \\ &= U_{\eta_k}(s, s-\varepsilon/2)\tilde{U}_{\eta_k}(s-\varepsilon/2, s-1)\psi_k(s-1). \end{aligned}$$

We define $y_k := \tilde{U}_{\eta_k}(s-\varepsilon/2, s-1)\psi_k(s-1)$. Therefore, the sequence $\{y_k\}_k$ satisfies

$$y_k \in \bigcup_{\eta\in[0,1]} A_\eta(s-\varepsilon/2),$$

which is relatively compact. As a consequence, the sequence $\{y_k\}_k$ has a convergent subsequence. Therefore, we assume that $y_k \longrightarrow y_0$. We will use Condition (CT) to prove that this case is not possible. We have that the sequence $\{y_k\}_k$ converges to y_0 and that $U_{\eta_k}(s, s-\varepsilon/2)y_k = x_k$, which converges to $x_0 \in M^0(s)$. Then Condition (CT) implies that there exist a subsequence of $\{y_k\}_k$, a subsequence of $\{\eta_k\}_k$ (both of which will be denoted the same), and a sequence $\{\alpha_k\}_k$ convergent to 0, such that $s+\alpha_k \geq s-\varepsilon/2$ and

$$U_{\eta_k}(s+\alpha_k, s-\varepsilon/2)y_k \in M^{\eta_k}(s+\alpha_k).$$

This contradicts the fact that there are no jump times in $[s-\varepsilon, s+\varepsilon]$ for k large. As a consequence, Case 1 can not happen.

Case 2: $s_k \longrightarrow s$.

Take $0 < \varepsilon < \min\{\xi/2, 1/2\}$ and k sufficiently large. Then s_k is the only jump time in the interval $(s-\varepsilon, s+\varepsilon)$, because of Condition (CH). We will split the proof into two subcases, either $s_k \leq s$ or $s_k > s$, up to subsequences. We will prove that the subcase $s_k \leq s$ can not happen, and therefore we will have proved that $s_k > s$.

Suppose that $s_k \leq s$ and assume that $s-\varepsilon/2 < s_k \leq s$. As in Case 1, the sequence $\{y_k\}_k$ with $y_k := \tilde{U}_{\eta_k}(s-\varepsilon/2, s-1)\psi_k(s-1)$ has a convergent subsequence. We can suppose that $y_k \longrightarrow y_0$ for some $y_0 \in X$. Then

$$\begin{aligned} x_k &= \tilde{U}_{\eta_k}(s, s-1)\psi_k(s-1) \\ &= \tilde{U}_{\eta_k}(s, s_k)\tilde{U}_{\eta_k}(s_k, s-\varepsilon/2)\tilde{U}_{\eta_k}(s-\varepsilon/2, s-1)\psi_k(s-1) \\ &= U_{\eta_k}(s, s_k)\tilde{U}_{\eta_k}(s_k, s-\varepsilon/2)y_k \\ &= U_{\eta_k}(s, s_k)I^{\eta_k}_{s_k}(U_{\eta_k}(s_k, s-\varepsilon/2)y_k). \end{aligned}$$

Taking limits, we have that $x_0 = I^0_s(U_0(s, s-\varepsilon/2)y_0)$. But this implies that $x_0 \in M^0(s) \cap I^0_s(M^0(s))$, a contradiction with Condition (I).

As a consequence, the only possibility is that the sequence $\{s_k\}_k$ converges to s and $s_k > s$ for k sufficiently large. We know that $x_k = \tilde{U}_{\eta_k}(s, s-1)\psi_k(s-1)$

and that s_k was the only jump time in $(s-\varepsilon, s+\varepsilon)$. Therefore, we can conclude that $U_{\eta_k}(s_k, s)x_k \in M^{\eta_k}(s_k)$, so

$$\phi_{\eta_k}(x_k, s) = s_k - s \longrightarrow 0. \qquad \square$$

We now prove the upper semicontinuity for a collection of pullback attractors of a family of impulsive evolution processes.

Theorem 3.81. *Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes satisfying* (CT) *and* (CH)*, $\tilde{\mathcal{U}}_\eta$ has a pullback $\mathfrak{D}$-attractor $\hat{A}_\eta$ for each $\eta \in [0,1]$, and $\hat{A}_0$ is collectively closed. Furthermore, assume that for each $t \in \mathbb{R}$, there exists $\gamma_t > 0$ such that*

$$D(t) = \overline{\bigcup_{\eta\in[0,\gamma_t]} \bigcup_{s\in[t-\gamma_t, t+\gamma_t]} A_\eta(s) \setminus M^\eta(s)} \text{ is compact} \tag{3.15}$$

and $\hat{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$. Then the family $\{\hat{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$, in the sense of Definition 3.79*.*

Proof. We fix $t \in \mathbb{R}$ and sequences $\{\eta_k\}_k$ and $\{x_k\}_k$, with $\{\eta_k\}_k$ convergent to 0 and $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$. In view to Definition 3.79 and an analogous result as Proposition 1.59, if we prove that the sequence $\{x_k\}_k$ has a convergent subsequence with limit in $A_0(t)$, then we will be finished.

As $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$, we have that

$$x_k \in \bigcup_{\eta\in[0,\gamma_t]} A_\eta(t) \setminus M^\eta(t)$$

for large k, so there exists a convergent subsequence. We denote the convergent subsequence the same, so we assume $x_k \longrightarrow x_0$. We will prove that $x_0 \in A_0(t)$, and we will be done.

Again, as $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$, Proposition 3.38 implies that there exists a global solution ψ_k of $\tilde{\mathcal{U}}_{\eta_k}$, with $\psi_k(t) = x_k$ and $\hat{\psi}_k \in \mathfrak{D}$. For k large, we have that

$$\psi_k(t-1) \in \bigcup_{\eta\in[0,\gamma_{t-1}]} A_\eta(t-1) \setminus M^\eta(t-1).$$

Therefore, we have a convergent subsequence of $\{\psi(t-1)\}_k$. In other words, we have $N_1 \subset \mathbb{N}$, N_1 infinite, and $x_{-1} \in X$ such that

$$\lim_{\substack{k\to\infty \\ k\in N_1}} \psi_k(t-1) = x_{-1}.$$

Consider now the sequence $\{\psi_k(t-2)\}_{k\in N_1}$. For $k \in N_1$ large, we have

$$\psi_k(t-2) \in \bigcup_{\eta\in[0,\gamma_{t-2}]} A_\eta(t-2) \setminus M^\eta(t-2).$$

Then we have that there exist $N_2 \subset N_1$ an infinite subset, and $x_{-2} \in X$ such that

$$\lim_{\substack{k\to\infty \\ k\in N_2}} \psi_k(t-2) = x_{-2}.$$

We construct by induction a subsequence of $\{\psi_k\}_k$ and a sequence of elements $\{x_{-m}\}_m$ such that

$$\lim_{k\to\infty} \psi_k(t-m) = x_{-m} \quad \text{for } \ m \in \mathbb{N}.$$

We distinguish two cases.

Case 1: $x_0 \notin M^0(t)$.

We will construct ψ_0 a global solution of $\tilde{\mathcal{U}}_0$ with $\psi(t) = x_0$ and $\hat{\psi}_0 \in \mathfrak{D}$. Then Proposition 3.38 would imply that $x_0 \in A_0(t) \setminus M^0(t)$, and we would be finished.

Subcase 1: $x_{-m_j} \notin M^0(t - m_j)$ for all $j \in \mathbb{N}$ for a sequence $\{m_j\}_j$ of positive numbers which is strictly increasing.

Fix $j \in \mathbb{N}$. Corollary 3.78 implies that we have a sequence $\{\varepsilon_k^j\}_k$ of nonnegative numbers which converges to 0 such that

$$\lim_{k\to\infty} \tilde{U}_{\eta_k}(t - m_j + \varepsilon_k^j, t - m_{j+1})\psi_k(t - m_{j+1}) = \tilde{U}_0(t - m_j, t - m_{j+1})x_{-m_{j+1}}.$$

We know that

$$\tilde{U}_{\eta_k}(t - m_j + \varepsilon_k^j, t - m_{j+1})\psi_k(t - m_{j+1}) = \tilde{U}_{\eta_k}(t - m_j + \varepsilon_k^j, t - m_j)\psi_k(t - m_j),$$

therefore, Lemma 3.74, item (c), implies that

$$\lim_{k\to\infty} \tilde{U}_{\eta_k}(t - m_j + \varepsilon_k^j, t - m_{j+1})\psi_k(t - m_{j+1}) = x_{-m_j}.$$

As a consequence, we get $\tilde{U}_0(t - m_j, t - m_{j+1})x_{-m_{j+1}} = x_{-m_j}$. We define $\psi_0 \colon \mathbb{R} \longrightarrow X$ by

$$\psi_0(r) = \begin{cases} \tilde{U}_0(r,t)x_0, & r \geq t. \\ \tilde{U}_0(r, t - m_j)x_{-m_j}, & r \in [t - m_j, t - m_{j-1}), \ j \in \mathbb{N}, \end{cases}$$

with $m_0 = 0$. By construction, ψ_0 is a global solution and $\psi_0(t) = x_0$. We will prove that $\hat{\psi}_0 \in \mathfrak{D}$.

Take $r \geq t$. There exists a sequence $\{\varepsilon_k\}_k$, by Corollary 3.78, such that $\varepsilon_k \geq 0$, $\varepsilon_k \longrightarrow 0$, and

$$\tilde{U}_{\eta_k}(r + \varepsilon_k, t)x_k \longrightarrow \tilde{U}_0(r,t)x_0 = \psi_0(r).$$

We know that $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$ and that $\hat{A}_{\eta_k} \setminus \hat{M}^{\eta_k}$ is $\tilde{\mathcal{U}}_{\eta_k}$-invariant. Therefore,

$$\tilde{U}_{\eta_k}(r+\varepsilon_k, t)x_k \in A_{\eta_k}(r+\varepsilon_k) \setminus M^{\eta_k}(r+\varepsilon_k).$$

As a consequence, we obtain that

$$\psi_0(r) \in \overline{\bigcup_{\eta\in[0,\gamma_r]} \bigcup_{s\in[r-\gamma_r, r+\gamma_r]} A_\eta(s) \setminus M^\eta(s)} = D(r).$$

If $r < t$, then there exists $j \in \mathbb{N}$ such that $r \in [t-m_j, t-m_{j-1})$. Once again, there exists a sequence $\{\varepsilon_k\}_k$, by Corollary 3.78, such that $\varepsilon_k \geq 0$, $\varepsilon_k \longrightarrow 0$, and

$$\tilde{U}_{\eta_k}(r+\varepsilon_k, t-m_j)\psi_k(t-m_j) \longrightarrow \tilde{U}_0(r, t-m_j)x_{-m_j} = \psi_0(r).$$

As before, we obtain that $\psi_0(r) \in D(r)$. Then we have proved that $\hat{\psi}_0 \subset \hat{D}$. As $\hat{D} \in \mathfrak{D}$ and $\mathfrak{D}$ is a universe, we get $\hat{\psi}_0 \in \mathfrak{D}$, and we finish the proof of this Subcase.

Subcase 2: $x_{-m} \in M^0(t-m)$ for all $m \in \mathbb{N}$ sufficiently large.
We know, by construction, that $\psi_k(t-m) \longrightarrow x_{-m}$. We also know that $\psi_k(t-m) \in A_{\eta_k}(t-m) \setminus M^{\eta_k}(t-m)$ and $(x_0)_{-m} \in M^0(t-m)$. Lemma 3.80 implies that $s_{k,m} := \phi_{\eta_k}(\psi_k(t-m), t-m) \longrightarrow 0$, by taking subsequences, if necessary. Fix m sufficiently large and $\beta \in (0, \min\{\xi/2, 1/4\})$. We define

$$w_{k,m} = U_{\eta_k}(t-m+s_{k,m}, t-m)\psi_k(t-m)$$

By definition, we have that $w_{k,m} \in M^{\eta_k}(t-m+s_{k,m})$. Moreover,

$$\lim_{k\to\infty} w_{k,m} = \lim_{k\to\infty} U_{\eta_k}(t-m+s_{k,m}, t-m)\psi_k(t-m) = x_{-m},$$

which belongs to $M^0(t-m)$. Therefore, we have that

$$I^{\eta_k}_{t_{k,m}}(w_{k,m}) \longrightarrow I^0_{t-m}(x_{-m}) \notin M^0(t-m),$$

taking $t_{k,m} = t-m+s_{k,m}$, for simplicity. Lemma 3.74, item (b), implies that

$$s_{k,m} < \beta < \xi \leq \frac{\phi_0(I^0_{t-m}(x_{-m}), t-m)}{2} \leq \phi_{\eta_k}(I^{\eta_k}_{t_k}(w_{k,m}), t-m+s_{k,m})$$

for k sufficiently large. This implies that

$$\begin{aligned}
\psi_k(t-m+\beta) &= \tilde{U}_{\eta_k}(t-m+\beta, t-m+s_{k,m})\psi_k(t-m+s_{k,m}) \\
&= \tilde{U}_{\eta_k}(t-m+\beta, t-m+s_{k,m})I^{\eta_k}_{t_k}(w_{k,m}) \\
&= U_{\eta_k}(t-m+\beta, t-m+s_{k,m})I^{\eta_k}_{t_k}(w_{k,m}),
\end{aligned}$$

which converges to

$$U_0(t-m+\beta,t-m)I^0_{t-m}(x_{-m}) = \tilde{U}_0(t-m+\beta,t-m)I^0_{t-m}(x_{-m}).$$

This element does not belong to $M^0(t-m+\beta)$. For each m sufficiently large, we have the sequence $\{\psi_k(t-m+\beta\}_k$ which converges to an element not in $M^0(t-m+\beta)$. Then, we repeat the construction at the start of the proof, instead of $\psi_k(t-m)$ consider $\psi_k(t-m+\beta)$. At that point, we will follow Subcase 1 and get a global solution ψ_0 of $\tilde{\mathcal{U}}_0$ with $\psi_0(t) = x_0$ and $\hat{\psi}_0 \in \mathfrak{D}$.

Case 2: $x_0 \in M^0(t)$.
We have that $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$ and $x_k \longrightarrow x_0$. Lemma 3.80 implies that $\phi_{\eta_k}(x_k,t) \longrightarrow 0$. We suppose that

$$0 < \phi_{\eta_k}(x_k,t) < \frac{\xi}{4}.$$

Moreover, Equation (3.9) implies that there exists $\varepsilon = \varepsilon(x_0,t) > 0$ such that

$$\bigcup_{r\in(0,\varepsilon)} F_0(x_0,r,t-r) \cap M^0(t-r) = \emptyset. \tag{3.16}$$

We take $m_0 \in \mathbb{N}$ such that $1/m_0 < \min\{\varepsilon, \xi/2\}$, we fix $m \geq m_0$, and we define

$$w_{k,m} := \psi_k(t-1/m) \in A_{\eta_k}(t-1/m) \setminus M^{\eta_k}(t-1/m).$$

We know that

$$\bigcup_{\eta\in[0,\gamma_{t-1/m}]} A_\eta(t-1/m) \setminus M^\eta(t-1/m)$$

is relatively compact, so we can assume that $w_{k,m} \longrightarrow w^0_m$ as $k \longrightarrow \infty$.

Claim: there exists $k_1 \in \mathbb{N}$ such that

$$s_{k,m} := \phi_{\eta_k}(w_{k,m}, t-1/m) > 1/m$$

for $k \geq k_1$.

If the claim is false, then there exists a subsequence (denoted the same) such that $s_{k,m} \leq 1/m$. Therefore, we suppose that the sequence is convergent (by relabeling if necessary), that is, $s_{k,m} \longrightarrow \alpha \in [0,1/m]$. Then

$$z_{k,m} := U_{\eta_k}(t-1/m+s_{k,m}, t-1/m)w_{k,m} \longrightarrow U_0(t-1/m+\alpha, t-1/m)w^0_m =: z^0_m.$$

We have that $z_{k,m} \in M^{\eta_k}(t-1/m+s_{k,m})$. As a consequence, we can obtain that $z^0_m \in M^0(t-1/m+\alpha)$. Moreover,

$$v_{k,m} := I^{\eta_k}_{t-1/m+s_{k,m}}(z_{k,m}) \longrightarrow I^0_{t-1/m+\alpha}(z^0_m) =: v^0_m.$$

Condition (CH) implies that $\phi_0(v_m^0, t-1/m+\alpha) \geq 2\xi$. Lemma 3.74 implies that, for k sufficiently large, we have

$$\frac{1}{m} < \xi \leq \frac{\phi_0(v_m^0, t-1/m+\alpha)}{2} \leq \phi_{\eta_k}(v_{k,m}, t-1/m+s_{k,m}).$$

This implies that

$$\begin{aligned}
&U_{\eta_k}(t+\phi_{\eta_k}(x_k,t), t-1/m+s_{k,m})v_{k,m} \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)U_{\eta_k}(t,t-1/m+s_{k,m})v_{k,m} \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)U_{\eta_k}(t,t-\tfrac{1}{m}+s_{k,m})\tilde{U}_{\eta_k}(t-\tfrac{1}{m}+s_{k,m},t-\tfrac{1}{m})w_{k,m} \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)\tilde{U}_{\eta_k}(t,t-1/m)w_{k,m} \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)\tilde{U}_{\eta_k}(t,t-1/m)\psi_k(t-1/m) \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)\psi_k(t) \\
&\quad = U_{\eta_k}(t+\phi_{\eta_k}(x_k,t),t)x_k,
\end{aligned}$$

which belongs to $M^{\eta_k}(t+\phi_{\eta_k}(x_k,t))$. In the third equality we have used Condition (CH), $1/m - s_{k,m} < 1/m < 2\xi$, and

$$\tilde{U}_{\eta_k}(t-1/m+s_{k,m}, t-1/m)w_{k,m} \in I^{\eta_k}_{t-1/m+s_{k,m}}(M^{\eta_k}(t-1/m+s_{k,m})).$$

As a consequence, we have that

$$\begin{aligned}
\phi_{\eta_k}(v_{k,m}, t-1/m+s_{k,m}) &\leq \phi_{\eta_k}(x_k,t) + 1/m - \phi_{\eta_k}(w_{k,m}, t-1/m) \\
&< \phi_{\eta_k}(x_k,t) + 1/m < \frac{\xi}{2} + \frac{\xi}{2} = \xi,
\end{aligned}$$

a contradiction with Condition (CH). This implies that the claim is proved.

Then, for k sufficiently large (greater than k_1), we have

$$\begin{aligned}
U_{\eta_k}(t,t-1/m)w_{k,m} &= \tilde{U}_{\eta_k}(t,t-1/m)w_{k,m} \\
&= \tilde{U}_{\eta_k}(t,t-1/m)\psi_k(t-1/m) \\
&= \psi_k(t) = x_k.
\end{aligned} \tag{3.17}$$

We have that $U_{\eta_k}(t,t-1/m)w_{k,m} \notin M^{\eta_k}(t)$ by the claim. Taking limits in (3.17), we get that

$$U_0(t,t-1/m)w_m^0 = x_0.$$

We have that $w_m^0 \notin M^0(t-1/m)$, using Equation (3.16). We use Case 1 to obtain that $w_m^0 \in A_0(t-1/m) \setminus M^0(t-1/m)$, because we have

$$w_{k,m} \in A_{\eta_k}(t-1/m) \setminus M^{\eta_k}(t-1/m) \quad \text{and} \quad w_{k,m} \longrightarrow w_0.$$

Everything we have proved so far was valid for any $m \geq m_0$. Consider the sequence $\{w_m^0\}_m$. For m sufficiently large, we have that $w_m^0 \in D(t)$, which is compact. Therefore, $\{w_m^0\}_m$ has a convergent subsequence. By relabeling if necessary, we assume that $w_m^0 \longrightarrow w^0$ for some $w^0 \in X$. But we know that $U_0(t, t-1/m)w_m^0 = x_0$ for all m sufficiently large. Taking limits, we have that $w^0 = x_0$, therefore $w_m^0 \longrightarrow x_0$. Finally, we have that $\hat{A}_0$ is collectively closed by hypothesis, and $w_m^0 \in A_0(t-1/m)$. Therefore, $x_0 \in A_0(t)$. □

Remark 3.82. Considering the universe $\mathfrak{D}_b$ of union bounded families, which consists of families $\hat{D} = \{D(t)\}_{t\in\mathbb{R}}$ such that

$$\bigcup_{t\in\mathbb{R}} D(t) \text{ is bounded in } X,$$

Condition (3.15) of Theorem 3.81 can be replaced by

$$\overline{\bigcup_{\eta\in[0,1]} \bigcup_{t\in\mathbb{R}} A(t) \setminus M(t)} \quad \text{is compact.}$$

The lower semicontinuity, although in theory it appears to be very similar, is much more complicated than the upper semicontinuity.

Definition 3.83. Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes which has a pullback $\mathfrak{D}$-attractor $\hat{A}_\eta$ for each $\eta \in [0,1]$. The family $\{\hat{A}_\eta\}_{\eta\in[0,1]}$ is said to be weak-lower semicontinuous at $\eta = 0$, if for all $t \in \mathbb{R}$, $x_0 \in A_0(t)$, and any sequence $\{\eta_k\}_k$ of nonnegative numbers convergent to 0, there exist two sequences $\{\varepsilon_k\}_k$ and $\{x_k\}_k$, with $\{\varepsilon_k\}_k$ nonnegative and convergent to 0, and $x_k \in A_{\eta_k}(t+\varepsilon_k) \setminus M^{\eta_k}(t+\varepsilon_k)$, such that $x_k \longrightarrow x_0$.

Remark 3.84. The lower semicontinuity at $\eta = 0$ should be

$$d_{\mathrm{H}}(A_0(t), A_\eta(t) \setminus M^\eta(t)) = 0 \quad \text{for all } t \in \mathbb{R}$$

for impulsive evolution processes, similarly as it was considered in the upper semicontinuity. According to Proposition 1.59, this notion would be equivalent to: for any $t \in \mathbb{R}$, $x_0 \in A_0(t)$, and $\{\eta_k\}_k$ a sequence convergent to 0, there exists a sequence $\{x_k\}_k$ such that $x_k \in A_{\eta_k}(t) \setminus M^{\eta_k}(t)$ and $x_k \longrightarrow x_0$.

We start by defining the unstable sets, as in Definition 1.82.

Definition 3.85. Let $\tilde{\mathcal{U}}$ an impulsive evolution process, $\mathfrak{D}$ a universe in X, $\delta > 0$, and $\hat{E} \in \mathfrak{D}$ a $\tilde{\mathcal{U}}$-invariant family. Then we define:

(a) the unstable set of $\hat{E}$ at time $t \in \mathbb{R}$ as

$$W^u(\hat{E})(t) = \Big\{\psi(t) \mid \psi : \mathbb{R} \longrightarrow X \text{ is a global solution of } \tilde{\mathcal{U}} \text{ with } \lim_{s\to-\infty} d(\psi(s), E(s)) = 0\Big\};$$

(b) the unstable family of $\hat{E}$ as

$$W^u(\hat{E}) = \{W^u(\hat{E})(t)\}_{t\in\mathbb{R}};$$

(c) the (δ-)local unstable set of $\hat{E}$ at time t as

$$\begin{aligned} W^u_\delta(\hat{E})(t) = \{\psi(t) \mid \xi\colon \mathbb{R} \longrightarrow X \text{ is a global solution of } \tilde{\mathcal{U}} \text{ with} \\ d(\psi(s), E(s)) < \delta \text{ for all } s \leq t \text{ and} \\ \lim_{s\to-\infty} d(\psi(s), E(s)) = 0\Big\}; \end{aligned}$$

(d) the (δ-)local unstable family of $\hat{E}$ as

$$W^u_\delta(\hat{E}) = \{W^u_\delta(\hat{E})(t)\}_{t\in\mathbb{R}}.$$

Definition 3.86. Let $\mathfrak{D}$ be a universe. A family $\hat{E} = \{E(t)\}_{t\in\mathbb{R}}$ is said to be backwards in $\mathfrak{D}$ if there exist $\tau \in \mathbb{R}$ and $\hat{D} \in \mathfrak{D}$, such that $E(t) \subset D(t)$ for every $t \leq \tau$.

Furthermore, let $\tilde{\mathcal{U}}$ be an impulsive evolution process. A global solution ψ of $\tilde{\mathcal{U}}$ is said to be backwards in $\mathfrak{D}$ if the family $\hat{\psi}$ is backwards in $\mathfrak{D}$.

Lemma 3.87. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process and $\hat{A}$ a pullback $\mathfrak{D}$-attractor. If $\tilde{\mathcal{U}}$ satisfies Condition* (I) *and $\hat{E}$ is a $\tilde{\mathcal{U}}$-invariant family backwards in $\mathfrak{D}$, then $E(t) \subset A(t)$ for all $t \in \mathbb{R}$. Moreover, when the invariant set is $\hat{\psi}$, with ψ a backwards in $\mathfrak{D}$ global solution, then $\psi(t) \in A(t) \setminus M(t)$ for all $t \in \mathbb{R}$.*

Proof. Take $t \in \mathbb{R}$. The family $\hat{E}$ is backwards in $\mathfrak{D}$, which implies that there exist $\tau \in \mathbb{R}$ and $\hat{D} \in \mathfrak{D}$ with $E(s) \subset D(s)$ for every $s \leq \tau$. Take $s \leq \min\{t, \tau\}$, then, by the invariance of $\hat{E}$, we get

$$E(t) = \tilde{U}(t, s)E(s) \subset \tilde{U}(t, s)D(s).$$

The family $\hat{A}$ is a pullback $\mathfrak{D}$-attractor, so we obtain

$$d_{\mathrm{H}}(E(t), A(t)) \leq d_{\mathrm{H}}(\tilde{U}(t, s)D(s), A(t)) \longrightarrow 0$$

as $s \longrightarrow -\infty$. This implies that $E(t) \subset \overline{A(t)} = A(t)$ because $A(t)$ is compact, hence closed. Therefore, we conclude that $E(t) \subset A(t)$ for any $t \in \mathbb{R}$.

If $\hat{E} = \hat{\psi}$, Condition (I) and Proposition 3.32 imply that $\psi(t) \notin M(t)$, therefore $\psi(t) \in A(t) \setminus M(t)$. $\square$

For the rest of this section, we will consider that $\mathfrak{D}$ is a universe which satisfies:

$$\text{given } \hat{D} \in \mathfrak{D}, \text{ there exists } \varepsilon > 0 \text{ such that } \{\mathcal{O}_\varepsilon(D(t))\}_{t\in\mathbb{R}} \in \mathfrak{D}. \tag{3.18}$$

Proposition 3.88. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process and $\hat{A}$ a pullback $\mathfrak{D}$-attractor. If $\tilde{\mathcal{U}}$ satisfies Condition* (I) *and $\hat{E}$ is a $\tilde{\mathcal{U}}$-invariant family which is backwards in $\mathfrak{D}$, then $W^u(\hat{E})(t) \subset A(t) \setminus M(t)$ for every $t \in \mathbb{R}$.*

Proof. Fix any $t \in \mathbb{R}$ and take $x \in W^u(\hat{E}(t)$. Then there exists ψ a global solution of $\tilde{\mathcal{U}}$ such that

$$\lim_{s \to -\infty} d(\psi(s), E(s)) = 0. \tag{3.19}$$

The family $\hat{E}$ is backwards in $\mathfrak{D}$, which implies that there exist $\tau \in \mathbb{R}$ and $\hat{D} \in \mathfrak{D}$ with $E(s) \subset D(s)$ for all $s \leq \tau$. Equation (3.18) implies that there exists $\varepsilon > 0$ such that $\{\mathcal{O}_\varepsilon(D(t))\}_{t \in \mathbb{R}} \in \mathfrak{D}$. Equation (3.19) implies that there exists $s_0 \leq \tau$ such that

$$s \leq s_0 \Longrightarrow d(\psi(s), E(s)) < \varepsilon.$$

Therefore, we get that

$$s \leq s_0 \Longrightarrow \psi(t) \in \mathcal{O}_\varepsilon(E(t)) \subset \mathcal{O}_\varepsilon(D(t)).$$

As $\{\mathcal{O}_\varepsilon(D(t))\}_{t \in \mathbb{R}} \in \mathfrak{D}$, this implies that the global solution ψ is backwards in $\mathfrak{D}$. Lemma 3.87 implies that $\psi(t) \in A(t) \setminus M(t)$. □

Corollary 3.89. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process and $\hat{A}$ a pullback $\mathfrak{D}$-attractor. Suppose that $\tilde{\mathcal{U}}$ satisfies* (3.7) *and Condition* (I). *Let $\mathfrak{B}$ be the collection of the global solutions which are backwards in $\mathfrak{D}$. Then*

$$A(t) \setminus M(t) = \bigcup_{\psi \in \mathfrak{B}} W^u(\hat{\psi})(t) \quad \textit{for every } t \in \mathbb{R}. \tag{3.20}$$

Proof. Fix $t \in \mathbb{R}$. Proposition 3.88 implies that

$$\bigcup_{\psi \in \mathfrak{B}} W^u(\hat{\psi})(t) \subset A(t) \setminus M(t).$$

If $x \in A(t) \setminus M(t)$, then Proposition 3.38 implies that we have a global solution ψ with $\psi(t) = x$ and $\hat{\psi} \in \mathfrak{D}$. Therefore, we have that $\hat{\psi}$ is backwards in $\mathfrak{D}$, so $x = \psi(t) \in W^u(\hat{\psi})(t)$. □

This implies that the set $A(t) \setminus M(t)$ can be described from the global solutions which are backwards in $\mathfrak{D}$ and their unstable sets.

Definition 3.90. Let ψ_1, ψ_2 be two global solutions of an impulsive evolution process $\tilde{\mathcal{U}}$ which satisfies (3.7). ψ_1 and ψ_2 are backwards-separated (or separated in the past) if

$$\limsup_{s \to -\infty} d(\psi_1(s), \psi_2(s)) > 0.$$

If ψ_1 and ψ_2 are not backwards-separated (that is, they "coincide at $-\infty$"), then their unstable sets are the same, that is,

$$W^u(\hat{\psi}_1)(t) = W^u(\hat{\psi}_2)(t) \quad \text{for every } t \in \mathbb{R}.$$

From this definition we get the following result:

Corollary 3.91. *Let $\tilde{\mathcal{U}}$ be an impulsive evolution process and $\hat{A}$ a pullback $\mathfrak{D}$-attractor. Suppose that $\tilde{\mathcal{U}}$ satisfies* (3.7) *and Condition* (I)*. Let $\mathfrak{B}_s$ be the collection of the global solutions which are backwards in $\mathfrak{D}$ and backwards-separated two-by-two. Then*

$$A(t) \setminus M(t) = \bigcup_{\psi \in \mathfrak{B}_s} W^u(\hat{\psi})(t) \qquad \textit{for every } t \in \mathbb{R}. \tag{3.21}$$

Theorem 3.92. *Let $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ be a family of impulsive evolution processes such that $\tilde{\mathcal{U}}_\eta$ has a pullback $\mathfrak{D}$-attractor $\hat{A}_\eta$ for each $\eta \in [0,1]$. Suppose that $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ satisfies* (CT)*. Furthermore, suppose that*

(i) there is a sequence of backwards in $\mathfrak{D}$ global solutions $\{\psi_{j,0}\}_j$ of $\tilde{\mathcal{U}}_0$ such that

$$A_0(t) = \overline{\bigcup_{j\in\mathbb{N}} W^u(\hat{\psi}_{j,0})(t)};$$

(ii) for every $j \in \mathbb{N}$, there exist a family $\{\psi_{j,\eta}\}_\eta$ of global solutions which are backwards in $\mathfrak{D}$ and $t_j \in \mathbb{R}$ such that

$$\lim_{\eta\to 0} \sup\{d(\psi_{j,\eta}(t), \psi_{j,0}(t)) \colon t \le t_j\} = 0;$$

(iii) for each $j \in \mathbb{N}$, there are $\delta_j > 0$ and $t_j \in \mathbb{R}$ such that

$$\lim_{\eta\to 0} \sup\{d_H(W^u_{\delta_j}(\hat{\psi}_{j,0})(t), W^u_{\delta_j}(\hat{\psi}_{j,\eta})(t)) \colon t \le t_j\} = 0.$$

Then $\{\hat{A}_\eta\}_{\eta\in[0,1]}$ is weak-lower semicontinuous at $\eta = 0$, in the sense of Definition 3.83*.*

Proof. Let $t \in \mathbb{R}$, $x_0 \in A_0(t)$, and a sequence $\{\eta_k\}_k$ of nonnegative numbers convergent to 0. We have to prove that there exist two sequences $\{\varepsilon_k\}_k$ and $\{x_k\}_k$ such that $\varepsilon_k \ge 0$, $\{\varepsilon_k\}_k$ convergent to 0, $x_k \in A_{\eta_k}(t+\varepsilon_k) \setminus M^{\eta_k}(t+\varepsilon_k)$, and $x_k \longrightarrow x_0$.

Fix $\varepsilon > 0$. We have that $x_0 \in A_0(t)$. Item (i) implies that exists $j \in \mathbb{N}$ and $x_\varepsilon \in W^u(\hat{\psi}_{j,0})(t)$ with $d(x, x_\varepsilon) < \varepsilon/2$. By the definition of unstable set, take ψ a global solution of $\tilde{\mathcal{U}}_0$ with $\psi(t) = x_\varepsilon$, and

$$\lim_{s\to-\infty} d(\psi(s), \psi_{j,0}(s)) = 0. \tag{3.22}$$

Items (ii) and (iii) imply that there exist a sequence $\{\psi_{j,\eta_k}\}_k$ of backwards in $\mathfrak{D}$ global solutions, $t_j \in \mathbb{R}$, and $\delta_j > 0$ such that

$$\lim_{k\to\infty} \sup\{d(\psi_{j,\eta_k}(t), \psi_{j,0}(t)) \colon t \leq t_j\} = 0,$$
$$\lim_{k\to\infty} \sup\{d_{\mathrm{H}}(W^u_{\delta_j}(\hat{\psi}_{j,0})(t), W^u_{\delta_j}(\hat{\psi}_{j,\eta_k})(t)) \colon t \leq t_j\} = 0.$$

We know that ψ is a global solution satisfying (3.22), which implies that there exists $s < \min\{t, t_j\}$ such that

$$z := \psi(s) \in W^u_{\delta_j}(\hat{\xi}_{j,0})(s).$$

Item (iii) implies that there exists a sequence $\{z_k\}_k$ such that

$$z_k \in W^u_{\delta_j}(\widehat{\psi}_{j,\eta_k})(t-\tau) \subset A_{\eta_k}(t-\tau) \quad \text{and} \quad z_k \longrightarrow z.$$

Furthermore, $z := \psi(s)$ implies that $z \notin M(s)$, by Condition (I) and Proposition 3.32. Therefore, Corollary 3.78 implies that there exists a sequence of nonnegative numbers $\{\varepsilon_k\}_k$ such that

$$x_k := \tilde{U}_{\eta_k}(t+\varepsilon_k, s)z_k \longrightarrow \tilde{U}_0(t,s)z.$$

But we know that $\tilde{U}_0(t,s)z = \tilde{U}_0(t,s)\psi(s) = \psi(t) = x_\varepsilon$. We also have that $z_k \in W^u_{\delta_j}(\widehat{\psi}_{j,\eta_k})(s)$, so there exists ψ_k a backwards in $\mathfrak{D}$ global solution with $\psi_k(s) = z_k$. Therefore,

$$x_k = \psi_k(t+\varepsilon_k) \in A_{\eta_k}(t+\varepsilon_k) \setminus M^{\eta_k}(t+\varepsilon_k).$$

Moreover, we know that $x_k \longrightarrow x_\varepsilon$, so there exists $K \in \mathbb{N}$ such that if $k \geq K$, then $d(x_k, x_\varepsilon) < \varepsilon/2$. As a consequence,

$$d(x_k, x) \leq d(x_k, x_\varepsilon) + d(x_\varepsilon, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

Remark 3.93. The study of lower semicontinuity for continuous evolution processes is hard. There are several papers with perturbations of autonomous problems (see, for instance, [35, 36, 75, 76]). However, nonautonomous perturbations of nonautonomous problems is even more difficult. The difficulty is to check the analogous to Conditions (i)–(iii) of Theorem 3.92.

3.5 Multivalued impulsive evolution processes

In this section, we extend the theory developed in Sections 3.2 and 3.3 to the multivalued situation. This implies that both the evolution process $\mathcal{U}$ and the family of functions I are going to be multivalued functions. These dynamical systems include some reaction-diffusion equations or differential inclusions, specially when the uniqueness of solutions can not be proved. Some results can be found on [9, 10, 31, 32, 51, 109, 111].

We begin by recalling the definition of generalized process

Definition 3.94. A generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t\in\mathbb{R}}$ in X is a family of sets $\mathscr{G}(t)$ consisting of functions $\varphi\colon [t, +\infty) \longrightarrow X$ satisfying:

(G1) (Existence) For each $t \in \mathbb{R}$ and $x \in X$, there exists at least one $\varphi \in \mathscr{G}(t)$ such that $\varphi(t) = x$.

(G2) (Translation) If $\varphi \in \mathscr{G}(t)$ and $s \geq 0$, then the map $\varphi^{+s} \in \mathscr{G}(t+s)$, with $\varphi^{+s} = \varphi|_{[t+s,+\infty)}$.

(G3) (Upper semicontinuity with respect to initial data) If $\{\varphi_n\}_n \subset \mathscr{G}(s)$ and $\varphi_n(s) \longrightarrow x$, then there exist a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and $\varphi \in \mathscr{G}(s)$ with $\varphi(s) = x$ such that $\varphi_{n_k}(t) \longrightarrow \varphi(t)$ as $k \longrightarrow \infty$ for each $t \geq s$.

We will assume that

(G4) (Continuity) Every map $\varphi\colon [\tau, +\infty) \longrightarrow X$ in $\mathscr{G}(\tau)$ is continuous.

Definition 3.95. We say that a generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t\in\mathbb{R}}$ is exact (or strict) if it satisfies the following condition:

(G5) (Concatenation) If $\varphi \in \mathscr{G}(\tau)$, $\psi \in \mathscr{G}(r)$, and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$, then $\theta \in \mathscr{G}(\tau)$, with θ defined as

$$\theta(t) := \begin{cases} \varphi(t), & t \in [\tau, s], \\ \psi(t), & t > s. \end{cases}$$

Definition 3.96. Let $\mathscr{G}$ be a generalized process. A multivalued process $\{U(t,s)\}_{t\geq s}$ is a family of multivalued operators $U(t,s)\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ defined as

$$U(t,s)D := \{\varphi(t) : \varphi \in \mathscr{G}(s), \varphi(s) \in D\}.$$

This multivalued process satisfies:

1. $U(t,t)x = x$ for all $t \in \mathbb{R}$ and $x \in X$,

2. $U(t,s)x \subset U(t,\tau)\left(U(\tau,s)x\right)$ for all $s \leq \tau \leq t$ and $x \in X$.

Furthermore, if we have an exact generalized process, then on the second property we have an equality, that is,

$$U(t,s) = U(t,\tau)U(\tau,s).$$

We recall the following result, which is useful for our purposes. The proof can be seen in [9, Theorem 2.2], for the autonomous case. We include the adaptation to generalized processes for completeness.

Proposition 3.97. *Let $\mathscr{G}$ be an exact generalized process, $s \in \mathbb{R}$, and $\{\varphi_n\}_n$, φ elements of $\mathscr{G}(s)$ such that $\varphi_n(t)$ converges to $\varphi(t)$ for all $t > s$. Then $\varphi_n(t)$ converges to $\varphi(t)$ uniformly for t in compact subsets of (s, ∞). In particular, we have the following property:*

$$\begin{cases} \text{If } \{\varphi_n\}_n \subset \mathscr{G}(s) \text{ and } \varphi_n(s) \longrightarrow x, \text{ then there exist a subsequence} \\ \{\varphi_{n_k}\}_k \text{ and } \varphi \in \mathscr{G}(s) \text{ with } \varphi(s) = x \text{ and } \varphi_{n_k}(t) \longrightarrow \varphi(t) \\ \text{uniformly for } t \text{ in compact subsets of } (s, \infty). \end{cases} \tag{3.23}$$

Proof. Take $[a, b]$ a nonempty, closed, and bounded interval of $(s, +\infty)$, $\varepsilon > 0$, and $k \in \mathbb{N}$. We define

$$S_{k,\varepsilon} := \{t \in [a, b] : n \geq k \Longrightarrow d(\varphi_n(t), \varphi(t)) \leq \varepsilon\}.$$

As φ and every function φ_n is continuous, we get that $S_{k,\varepsilon}$ is closed. Furthermore, we have that

$$\bigcup_{k\in\mathbb{N}} S_{k,\varepsilon} = [a, b].$$

Baire Category theorem (Theorem 1.13) implies that there exists $k \in \mathbb{N}$ such that $S_{k,\varepsilon}$ contains an open interval. We define the following set:

$$S_\varepsilon := \{t_0 \in (s, +\infty) \mid \exists\, \delta > 0,\ N \in \mathbb{N} \text{ such that } \\ n \geq N,\ |t - t_0| < \delta \Longrightarrow d(\varphi_n(t), \varphi(t)) \leq \varepsilon\}.$$

We will prove that S_ε is a nonempty, open, and dense subset of $(s, +\infty)$.

- Nonempty and dense: Take (a, b) a nonempty, open, and bounded set. We will prove that $(a, b) \cap S_\varepsilon \neq \emptyset$. The construction at the beginning of the proof implies that

  $$[a, b] = \bigcup_{k\in\mathbb{N}} S_{k,\varepsilon}.$$

  Theorem 1.13 (Baire Category) implies that there exist $c, d \in [a, b]$, with $c < d$, and $k \in \mathbb{N}$ such that $(c, d) \subset S_{k,\varepsilon}$. This implies that

  $$\frac{c + d}{2} \in S_\varepsilon \cap (a, b).$$

  This implies that for every open set we have that its intersection with S_ε is nonempty, therefore S_ε is nonempty and dense.

- Open: Take $t_0 \in S_\varepsilon$. Then there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$[n \geq N,\ |t - t_0| < \delta] \Longrightarrow d(\varphi_n(t), \varphi(t)) \leq \varepsilon.$$

For any $t^* \in (t_0 - \delta/2, t_0 + \delta/2)$, taking $\delta/2$ we have that

$$[n \geq N,\ |t - t^*| < \delta/2] \Longrightarrow d(\varphi_n(t), \varphi(t)) \leq \varepsilon.$$

Therefore, we have that $(t_0 - \delta/2, t_0 + \delta/2) \subset S_\varepsilon$.

Consider the set

$$K = \bigcap_{n=1}^{\infty} S_{1/n}.$$

Once again, Baire Category theorem (Theorem 1.13) implies that K is dense in $(s, +\infty)$, because each $S_{1/n}$ is open and dense. Suppose that we do not have uniform convergence on compact subsets of $(s, +\infty)$. Then there exists $\{t_n\}_n$ a convergent sequence to t, with $t > s$, such that $\{\varphi_n(t_n)\}_n$ does not converge to $\varphi(t)$. Then we can suppose that there exists $\delta > 0$ such that

$$d(\varphi_n(t_n), \varphi(t)) > \delta \quad \text{for every } n \in \mathbb{N}. \tag{3.24}$$

Take $s_0 \in K$, $s_0 < t$, and consider $\psi_n = {\varphi_n}^{+t_n+s_0-t-s}$. We have that

$$\psi_n(s) = \varphi_n(s + t_n + s_0 - t - s) = \varphi_n(t_n - t + s_0) \longrightarrow \varphi(s_0).$$

Therefore, (G3) implies that there exist a subsequence $\{\psi_{n_k}\}_k$ and $\psi \in \mathscr{G}(s)$ such that

$$\psi_{n_k}(\tau) \longrightarrow \psi(\tau) \quad \text{for } \tau \geq s.$$

Take $\tau \geq 0$ such that $s_0 + \tau \in K$. Then we have that

$$\psi_n(s + \tau) = \varphi_n(s + \tau + t_n + s_0 - t - s) = \varphi_n(s_0 + \tau + t_n - t),$$

which converges to $\varphi(s_0 + \tau)$. But we also know that $\psi_{n_k}(s + \tau) \longrightarrow \psi(s + \tau)$. Therefore, we deduce that $\varphi(s_0 + \tau) = \psi(s + \tau)$ if $s_0 + \tau \in K$. As K is dense and φ, ψ are continuous, we have that $\varphi(s_0 + \tau) = \psi(s + \tau)$ for all $\tau \geq 0$. Then

$$\varphi_{n_k}(t_{n_k}) = \psi_{n_k}(t + s - s_0 + t_{n_k} - t + s_0 - s) = \psi_{n_k}(s + t - s_0),$$

which converges to $\psi(s + t - s_0) = \varphi(s_0 + t - s_0) = \varphi(t)$, a contradiction with (3.24). □

This result, in general, is not valid for compact subsets of $[s, \infty)$. Examples can be found in [43] or in [9, Section 6.2].

The notion of invariance and pullback attraction are analogous to the continuous case.

Definition 3.98. An impulsive generalized process $(\mathscr{G}, \hat{M}, I)$ consists of a generalized process $\mathscr{G}$, a collectively closed family of sets $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ such that for every $s \in \mathbb{R}$, $x \in M(s)$, and $\varphi \in \mathscr{G}(s)$ with $\varphi(s) = x$, we have

$$\exists\, \varepsilon = \varepsilon(\varphi, s) > 0 \text{ such that } \bigcup_{r\in(0,\varepsilon)} \{\varphi(s + r)\} \cap M(s + r) = \emptyset, \tag{3.25}$$

and collection of collectively upper semicontinuous multifunctions which are compact-valued $I = \{I_t\colon M(t) \longrightarrow \mathscr{P}(X)\}_{t\in\mathbb{R}}$, that is:

$$\begin{array}{l}\text{if } \{t_n\}_n,\ \{x_n\}_n, \text{ and } \{y_n\}_n \text{ are three sequences such that } t_n \longrightarrow t,\\ x_n \longrightarrow x, \text{ and } y_n \in I_{t_n}(x_n), \text{ then there exists a convergent}\\ \text{subsequence of } \{y_n\}_n \text{ with limit in } I_t(x).\end{array} \tag{3.26}$$

Remark 3.99. Equation (3.26) follows from Proposition 1.9.

Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. For each $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$, we define the impact time map by

$$\phi(\varphi, s) := \inf\{t > 0 : \varphi(s + t) \in M(s + t)\}, \tag{3.27}$$

and we denote $\phi(\varphi, s) = \infty$ if $\varphi(s + t) \notin M(s + t)$ for all $t > 0$.

Proposition 3.100. *The map $\phi(\varphi, s) > 0$ for all $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$.*

Proof. Fix $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$. If $\varphi(s) \in M(s)$, then $\phi(\varphi, s) \geq \varepsilon$, with ε given by (3.25). If $\varphi(s) \notin M(s)$ and $\phi(\varphi, s) = 0$, then there exists a sequence $\{r_n\}_n$ of positive numbers convergent to 0, such that $\varphi(s + r_n) \in M(s + r_n)$. As φ is continuous and $\hat{M}$ is collectively closed, then $\varphi(s) \in M(s)$, a contradiction. ☐

Remark 3.101. If $\phi(\varphi, s) \neq \infty$, then $\varphi(s + \phi(\varphi, s)) \in M(s + \phi(\varphi, s))$.

Definition 3.102. Given $s \in \mathbb{R}$, a map $\tilde{\varphi}\colon [s, \omega) \longrightarrow X$, with $\omega \in (s, +\infty)$, will be called an impulsive trajectory of $(\mathscr{G}, \hat{M}, I)$ if there exists a division of $[s, \omega)$ into a family of subintervals

$$[s, \omega) = [t_0, t_1) \cup [t_1, t_2) \cup \cdots$$

with $t_0 = s$, $t_k < t_{k+1}$, and the union could be finite or not finite. Furthermore, for each k, there exists $\varphi_k \in \mathscr{G}(t_k)$ satisfying:

(i) $\phi(\varphi_k, t_k) = \infty$ or $\phi(\varphi_k, t_k) = t_{k+1} - t_k$,

(ii) $\tilde{\varphi}(t) = \varphi_k(t)$ for $t \in [t_k, t_{k+1})$,

(iii) if $\phi(\varphi_k, t_k) \neq \infty$, then $\tilde{\varphi}(t_{k+1}) \in I_{t_{k+1}}(\varphi_k(t_{k+1}))$.

The times t_k will be called jump times of $\tilde{\varphi}$, the family of impulsive trajectories starting at s will be denoted by $\tilde{\mathscr{G}}(s)$ and we will also denote $\tilde{\mathscr{G}} = \{\tilde{\mathscr{G}}(s)\}_{s\in\mathbb{R}}$.

Proposition 3.103. *For each $s \in \mathbb{R}$ and $x \in X$, there exists $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, defined in $[s, \omega)$, with $\tilde{\varphi}(s) = x$.*

Proof. By definition of impulsive trajectory and (G1), there exists $\varphi_0 \in \mathscr{G}(s)$ with $\varphi_0(s) = x$. If $\phi(\varphi_0, s) = \infty$, then $\tilde{\varphi}(t) = \varphi_0(t)$ for all $t \geq s$. On the other hand, if $\phi(\varphi_0, s) \neq \infty$, then $\phi(\varphi_0, s) > 0$. Denote $t_1 := s + \phi(\varphi_0, s)$. We have that $\varphi_0(t_1) \in M(t_1)$. Take $x_1 \in I_{t_1}(\varphi_0(t_1))$ and $\varphi_1 \in \mathscr{G}(t_1)$ with $\varphi_1(t_1) = x_1$. If $\phi(\varphi_1, t_1) = \infty$, then we define

$$\tilde{\varphi}(t) = \begin{cases} \varphi_0(t), & s \leq t < t_1, \\ \varphi_1(t), & t_1 \leq t. \end{cases}$$

If $\phi(\varphi_1, t_1)$ is finite, we denote $t_2 = t_1 + \phi(\varphi_1, t_1)$, and then $\varphi_1(t_2) \in M(t_2)$. Take $x_2 \in I_{t_2}(\varphi_1(t_2))$ and $\varphi_2 \in \mathscr{G}(t_2)$ with $\varphi_2(t_2) = x_2$. We continue analogously. □

Proposition 3.104. *Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process such that*

$$I_\tau(M(\tau)) \cap M(\tau) = \emptyset \text{ for all } \tau \in \mathbb{R}. \tag{I-Mult}$$

Then for each $s \in \mathbb{R}$, $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, and $t \in (s, \omega)$, we have $\tilde{\varphi}(t) \notin M(t)$.

From now on we will assume:

$$\text{Every impulsive trajectory is defined on } [s, +\infty). \tag{3.28}$$

From the definition of impulsive trajectories, we can define a new family of multivalued maps $\{\tilde{U}(t,s)\}_{t\geq s}$, given by $\tilde{U}(t,s) \colon \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ and defined as

$$\tilde{U}(t,s)D := \{\tilde{\varphi}(t) : \tilde{\varphi} \in \tilde{\mathscr{G}}(s),\ \tilde{\varphi}(s) \in D\}.$$

Lemma 3.105. *Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. Then*

1. *$\tilde{\mathscr{G}}$ satisfies* (G2) *and* (G5),
2. *$\tilde{U}(t,s) = \tilde{U}(t,\tau)\tilde{U}(\tau,s)$ for any $s \leq \tau \leq t$.*

The definitions of invariance and pullback attraction for $\tilde{U}$ are analogous, just replace U by $\tilde{U}$, as well as pullback $\mathfrak{D}$-semiattractor and pullback $\mathfrak{D}$-attractor. We present a few definitions that are very similar to the single-valued case.

Definition 3.106. Let $\hat{D}$ be a family of sets. The impulsive pullback ω-limit set of $\hat{D}$ at time $t \in \mathbb{R}$, denoted by $\tilde{\omega}(\hat{D}, t)$, is defined as the set of elements $x \in X$ such that there exist three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{\tilde{\varphi}_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$, and $\tilde{\varphi}_n(s_n) \in D(s_n)$ for each $n \in \mathbb{N}$, such that $\tilde{\varphi}_n(t + \varepsilon_n) \longrightarrow x$. The impulsive pullback ω-limit of $\hat{D}$ is the family $\tilde{\omega}(\hat{D}) = \{\tilde{\omega}(\hat{D}, t)\}_{t\in\mathbb{R}}$.

Definition 3.107. We say that $\tilde{\mathscr{G}}$ is pullback $\mathfrak{D}$-asymptotically compact if for each $D \in \mathfrak{D}$, $t \in \mathbb{R}$, and sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{\tilde{\varphi}_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$, and $\tilde{\varphi}_n(s_n) \in D(s_n)$, then the sequence $\{\tilde{\varphi}_n(t+\varepsilon_n)\}_n$ has a convergent subsequence.

Definition 3.108. We say that $\tilde{\mathscr{G}}$ is pullback $\mathfrak{D}$-dissipative if there exists $\hat{B}_0 \in \mathfrak{D}$ collectively closed such that for all $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and sequences $\{s_n\}_n$ and $\{\varepsilon_n\}_n$, with $s_n \longrightarrow -\infty$ and $\varepsilon_n \longrightarrow 0$, there exists $n_0 = n_0(\hat{D}, t) \in \mathbb{N}$ such that, if $n \geq n_0$, $\tilde{\varphi} \in \tilde{\mathscr{G}}(s_n)$, and $\tilde{\varphi}(s_n) \in D(s_n)$, then $\tilde{\varphi}(t+\varepsilon_n) \in B_0(t+\varepsilon_n)$. The family $\hat{B}_0$ is called pullback a $\mathfrak{D}$-absorbing family.

We present some properties, once again, analogous with the definitions laid out in Section 3.3. We do not include their proofs, as they are very similar.

Proposition 3.109. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process, $\hat{D} \in \mathfrak{D}$, and $t \in \mathbb{R}$. Then the impulsive pullback ω-limit of $\hat{D}$, $\tilde{\omega}(\hat{D})$, is nonempty, collectively compact, and pullback attracts $\hat{D}$.*

Proposition 3.110. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-dissipative impulsive generalized process with $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. Then, for any $\hat{D} \in \mathfrak{D}$, we have that $\tilde{\omega}(\hat{D}) \subset \hat{B}_0$.*

Later, we will prove the existence of a pullback $\mathfrak{D}$-attractor. We will need the following condition, which is a generalization of Condition (H):

$$\begin{cases} \text{There exists } \xi > 0 \text{ such that } \phi(\varphi, s) \geq 2\xi \text{ for all } s \in \mathbb{R} \\ \text{and } \varphi \in \tilde{\mathscr{G}}(s) \text{ with } \varphi(s) \in I_s(M(s)). \end{cases} \tag{H-Mult}$$

Remark 3.111. Condition (H-Mult) implies (3.28), that is, all impulsive trajectories are defined until $+\infty$. It also implies that if $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ and $t_1 < t_2$ are two different jump times of $\tilde{\varphi}$, then $t_2 - t_1 \geq 2\xi$.

We prove a result guaranteeing the existence of pullback $\mathfrak{D}$-semiattractor.

Theorem 3.112. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process and pullback $\mathfrak{D}$-dissipative. Then there exists a pullback $\mathfrak{D}$-semiattractor, given by the impulsive pullback ω-limit of the pullback $\mathfrak{D}$-absorbing family.*

Proof. Take $\hat{A} = \tilde{\omega}(\hat{B}_0)$, with $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. The family $\hat{A}$ pullback $\mathfrak{D}$-attracts $\hat{B}_0$ and $\hat{A} \subset \hat{B}_0$, by Proposition 3.110. Then $\hat{A} \in \mathfrak{D}$ because $\hat{B}_0 \in \mathfrak{D}$ and $\mathfrak{D}$ is a universe. Furthermore, the family $\hat{A}$ is collectively compact, by Proposition 3.109, so $A(t)$ is compact for all $t \in \mathbb{R}$. It remains to prove that $\hat{A}$ pullback attracts every $\hat{D} \in \mathfrak{D}$.

Fix $t \in \mathbb{R}$, $\hat{D} \in \mathfrak{D}$ and $\varepsilon > 0$. We want to prove that there exists $r \leq t$ such that if

$$\big[s \leq r,\ \tilde{\varphi} \in \tilde{\mathscr{G}}(s) \text{ with } \tilde{\varphi}(s) \in D(s)\big] \Longrightarrow d(\tilde{\varphi}(t), A(t)) < \varepsilon.$$

We know that $\hat{A}$ pullback attracts $\hat{B}_0$, so there exists $s_0 \leq t$ such that

$$\big[s \leq s_0,\ \tilde{\varphi} \in \tilde{\mathscr{G}}(s) \text{ with } \tilde{\varphi}(s) \in B_0(s)\big] \Longrightarrow d(\tilde{\varphi}(t), A(t)) < \varepsilon.$$

By pullback $\mathfrak{D}$-dissipativity, there exists $s_1 \leq s_0$ such that if $s \leq s_1$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in D(s)$, then $\tilde{\varphi}(s) \in B_0(s)$. Finally, take $r := s_1$. If $s \leq r$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in D(s)$, then we know that $\tilde{\varphi}(s_0) \in B_0(s_0)$, so $\tilde{\varphi}_{|[s_0,\infty)} \in \tilde{\mathscr{G}}(s_0)$ and $\tilde{\varphi}(s_0) \in B_0(s_0)$. This implies that $d(\tilde{\varphi}(t), A(t)) < \varepsilon$, because $\hat{A}$ pullback attracts $\hat{B}_0$. □

Next, we find conditions to prove the invariance of the pullback $\mathfrak{D}$-semi-attractor we have just obtained. First, we need a condition closely related to (G3) in Definition 3.94 and to Condition (3.23) in Proposition 3.97, but a little stronger.

(G3′) If $\tau_n \longrightarrow \tau$, $\varphi_n \in \mathscr{G}(\tau_n)$, and $\varphi_n(\tau_n) \longrightarrow x$, then there is a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and $\varphi \in \mathscr{G}(\tau)$, with $\varphi(\tau) = x$ and

$$\lim_{k\longrightarrow\infty} \sup_{t\in J} d(\varphi_{n_k}(t), \varphi(t)) = 0 \quad \text{for } J \text{ compact}.$$

However, as it was seen in Section 3.3, this condition alone is not going to be enough. We also require an adaptation of Conditions (T) or (NT).

$$\begin{cases} \text{Fix } s \in \mathbb{R},\ x \in X \setminus M(s),\ \{\varphi_n\}_n \text{ a sequence in } \mathscr{G}(s), \\ \text{and } \varphi \in \mathscr{G}(s) \text{ such that } \varphi(s) = x \text{ and } \varphi_n(t) \longrightarrow \varphi(t) \\ \text{for each } t \geq s. \text{ Then } \liminf\limits_{n\to\infty} \phi(\varphi_n, s) \leq \phi(\varphi, s). \end{cases} \tag{NT-Mult}$$

We will prove the negative invariance for $\tilde{\omega}(\hat{D}) \setminus \hat{M}$. We will write the complete proof to highlight the places where we use (G3′) and (NT-Mult).

Theorem 3.113. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions* (H-Mult)*,* (I-Mult)*, and* (NT-Mult)*. If $t > s$ and $t - s \in (0, \xi]$, then*

$$\tilde{\omega}(\hat{D}, t) \setminus M(t) \subset \tilde{U}(t, s)\big(\tilde{\omega}(\hat{D}, s) \setminus M(s)\big)$$

for any $\hat{D} \in \mathfrak{D}$.

Proof. Take $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$. We want to prove that there exists $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s)$, $\tilde{\varphi}(s) \notin M(s)$, and $\tilde{\varphi}(t) = x$.

As $x \in \tilde{\omega}(\hat{D}, t)$, there exist three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{\tilde{\varphi}_n\}_n$, with $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$, and $\tilde{\varphi}_n(s_n) \in D(s_n)$, such that we have that $\tilde{\varphi}_n(t + \varepsilon_n) \longrightarrow x$. Each impulsive trajectory $\tilde{\varphi}_n$, which is defined on $[s_n, +\infty)$, has $N_n \geq 0$ jump times. We consider τ_n the last jump time on the interval $[s_n, t + \xi/4]$. If there are no jump times in that interval, we take $\tau_n = s_n$. We will split the proof into three different cases.

Case 1: Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $\tau_n < s - \varepsilon$.

We have that there exist $\psi_n \in \mathscr{G}(s - \varepsilon/2)$ such that $\tilde{\varphi}_n(r) = \psi_n(r)$ for every $r \in [s - \varepsilon/2, t + \xi/4]$, because $\tau_n < s - \varepsilon$ and τ_n was the last jump time of $\tilde{\varphi}_n$ in $[s_n, t + \xi/4]$. The sequence $\{\tilde{\varphi}_n(s - \varepsilon/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we can assume $\tilde{\varphi}_n(s - \varepsilon/2) \longrightarrow y$, or equivalently, $\psi_n(s - \varepsilon/2) \longrightarrow y$. By the definition of generalized process, in particular (G3), there exist a subsequence, still denoted the same, and $\psi \in \mathscr{G}(s - \varepsilon/2)$ such that $\psi(s - \varepsilon/2) = y$ and $\psi_n(r) \longrightarrow \psi(r)$ for each $r \geq s - \varepsilon/2$. We claim that $\psi(r) \notin M(r)$ for $r \in [s - \varepsilon/2, t]$. Suppose that $\psi(r) \in M(r)$ for some $r \in [s - \varepsilon/2, t]$. This implies that $\phi(\psi, s - \varepsilon/2) \leq t - (s - \varepsilon/2)$. But $\tilde{\varphi}_n$ has no jump times on $[s - \varepsilon/2, t + \xi/4]$, so $(t + \xi/4) - (s - \varepsilon/2) \leq \phi(\psi_n, s - \varepsilon/2)$. Then Condition (NT-Mult) would imply a contradiction, as

$$(t + \xi/4) - (s - \varepsilon/2) \leq \phi(\psi, s - \varepsilon/2) \leq t - (s - \varepsilon/2).$$

Therefore, we have that $\tilde{\varphi}_n(t + \varepsilon_n) = \psi_n(t + \varepsilon_n)$ for n sufficiently large, so $x = \psi(t)$. Consider $\tilde{\theta} \in \tilde{\mathscr{G}}(t)$ such that $\tilde{\theta}(t) = x$, and define $\tilde{\psi} \in \tilde{\mathscr{G}}(s)$ as

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \leq r \leq t, \\ \tilde{\theta}(r), & t \leq r. \end{cases}$$

We have that $\tilde{\varphi}(s) = \psi(s) \notin M(s)$ and $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s)$, because $\tilde{\varphi}_n(s) = \psi_n(s)$ and $\psi_n(s) \longrightarrow \psi(s)$. This implies that $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$, and $\tilde{\varphi}(t) = x$.

Case 2: Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $s + \varepsilon < \tau_n$.

We have that $\tau_n \in (s + \varepsilon, t + \xi/4]$, so we may assume that the sequence $\{\tau_n\}_n$ converges to $\bar{\tau} \in [s + \varepsilon, t + \xi/4]$. The sequence $\{\tilde{\varphi}_n(t - 3\xi/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we may assume $\tilde{\varphi}_n(t - 3\xi/2) \longrightarrow v$. Note that $t - 3\xi/2 < s$. We have that τ_n is the only jump time in $[t - 3\xi/2, t + \xi/4]$, because $|(t + \xi/4) - (t - 3\xi/2)| = 7\xi/4 < 2\xi$ and Condition (H-Mult) applies (see Remark 3.111). This implies that there exist $\psi_n \in \mathscr{G}(t - 3\xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & t - 3\xi/2 \leq r < \tau_n, \\ \theta_n(r), & \tau_n \leq r \leq t + \xi/4. \end{cases}$$

By definition of generalized process, there exist a subsequence (still denoted the same) and $\psi \in \mathscr{G}(t - 3\xi/2)$ such that $\psi(t - 3\xi/2) = v$ and $\psi_n(r) \longrightarrow \psi(r)$ for $r \geq t - 3\xi/2$. This implies that $\psi_n(\tau_n) \longrightarrow \psi(\bar{\tau})$ by Proposition 3.97. The fact that $\hat{M}$ is collectively closed implies that $\psi(\bar{\tau}) \in M(\bar{\tau})$, because $\psi_n(\tau_n) \in M(\tau_n)$. We also have that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity with compact values of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$ (denoted the same) and $y \in I_{\bar{\tau}}(\psi(\bar{\tau}))$, such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to y. In particular, this implies that $y \in \tilde{\omega}(\hat{D}, \bar{\tau})$, because $\tilde{\varphi}_n(\tau_n) = \tilde{\varphi}_n(\bar{\tau} + (\tau_n - \bar{\tau}))$. Moreover, using Condition (G3′), there exist a subsequence, denoted the same, and $\theta \in \mathscr{G}(\bar{\tau})$, such that

$$\lim_{n\to\infty} \sup_{r\in J} d(\theta_n(r), \theta(r)) = 0 \quad \text{for each } J \text{ compact.}$$

This implies that $\theta_n(\tau_n)$ converges to $\theta(\bar{\tau})$, so $\theta(\bar{\tau}) = y$. We will prove that $\psi(r) \notin M(r)$ for every $r \in (t - 3\xi/2, \bar{\tau})$. If there exists $r \in [t - 3\xi/2, \bar{\tau})$ such that $\psi(r) \in M(r)$, then

$$\phi(\psi, t - 3\xi/2) \leq r - (t - 3\xi/2) < \bar{\tau} - (t - 3\xi/2).$$

But τ_n was the only jump time of $\tilde{\varphi}_n$ on $[t - 3\xi/2, t + \xi/4]$, which implies that $\phi(\psi_n, t - 3\xi/2) = \tau_n - (t - 3\xi/2)$. But then we get a contradiction, because Condition (NT-Mult) implies

$$\bar{\tau} - (t - 3\xi/2) \leq \phi(\psi, t - 3\xi/2) < \bar{\tau} - (t - 3\xi/2).$$

First, take $z = \psi(s) \in \tilde{\omega}(\hat{D}, s)$ because $\tilde{\varphi}_n(s) = \psi_n(s)$ and $\psi_n(s)$ converges to $\psi(s)$. But we have proved that $z \notin M(s)$. Consider $\tilde{\alpha} \in \tilde{\mathscr{G}}(t + \xi/4)$ with $\tilde{\alpha}(t + \xi/4) = \theta(t + \xi/4)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \leq r < \bar{\tau}, \\ \theta(r), & \bar{\tau} \leq r \leq t + \xi/4, \\ \tilde{\alpha}(r), & t + \xi/4 \leq r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. We want to prove that $\tilde{\varphi}(t) = x$.

Subcase 1: For a subsequence (denoted the same), $\tau_n \leq t + \varepsilon_n$.

This implies that $\bar{\tau} \leq t$. Then we have that $\tilde{\varphi}_n(t + \varepsilon_n) = \theta_n(t + \varepsilon_n) \longrightarrow \theta(t)$, so $x = \theta(t) = \tilde{\varphi}(t)$.

Subcase 2: For a subsequence (denoted the same), $t + \varepsilon_n < \tau_n$.

This implies that $t \leq \bar{\tau}$. On the one hand, if $t = \bar{\tau}$, then $\tilde{\varphi}_n(t+\varepsilon_n) = \psi_n(t+\varepsilon_n)$, which converges to $\psi(t) = \psi(\bar{\tau}) \in M(\bar{\tau})$ by Proposition 3.97, therefore we have $x = \psi(\bar{\tau}) \in M(t)$, a contradiction. This implies that t cannot be equal to $\bar{\tau}$ in this Subcase. On the other hand, if $t < \bar{\tau}$, then $\tilde{\varphi}_n(t + \varepsilon_n) = \psi_n(t + \varepsilon_n)$, which converges to $\psi(t) = \tilde{\varphi}(t)$, so $x = \tilde{\varphi}(t)$.

Case 3: τ_n converges to s.

We have that τ_n is the only jump time of $\tilde{\varphi}_n$ in $(t-3\xi/2, t+\xi/4)$, for n sufficiently large. Then there exist $\psi_n \in \mathscr{G}(t-3\xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & t-3\xi/2 \leq r < \tau_n, \\ \theta_n(r), & \tau_n \leq r \leq t+\xi/4. \end{cases}$$

The sequence $\{\tilde{\varphi}_n(t-3\xi/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we may assume $\tilde{\varphi}_n(t-3\xi/2) \longrightarrow v$, with $v \in X$. By definition of generalized process, there exist a subsequence (still denoted the same) and $\psi \in \mathscr{G}(t-3\xi/2)$ such that $\psi(t-3\xi/2) = v$ and $\psi_n(r) \longrightarrow \psi(r)$ for $r \geq t-3\xi/2$. This implies that $\psi(s) \in M(s)$, because $\psi_n(\tau_n) \in M(\tau_n)$, $\hat{M}$ is collectively closed, and Proposition 3.97 applies. Furthermore, we have that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity with compact values of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$, denoted the same, and $y \in I_s(\psi(s))$, such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to y, so $y \in \tilde{\omega}(\hat{D}, s)$. Condition (G3$'$) implies that there exist a subsequence, denoted the same, and $\theta \in \mathscr{G}(s)$ such that

$$\lim_{n\to\infty} \sup_{r\in J} d(\theta_n(r), \theta(r)) = 0 \quad \text{for each } J \text{ compact.}$$

This implies that $\theta_n(\tau_n)$ converges to $\theta(s) = y \in I_s(\psi(s))$, so $y \notin M(s)$ by Condition (I-Mult). Furthermore, $\tilde{\varphi}_n(\tau_n) = \tilde{\varphi}_n(s + (\tau_n - s))$ converges to y, so $y \in \tilde{\omega}(\hat{D}, s)$. Using the same arguments as before, we have that $\theta(r) \notin M(r)$ for $r \in [s, t+\xi/4]$, because $\theta(s) = y \in I_s(\psi(s))$ and Condition (H-Mult) applies. Take $\tilde{\alpha} \in \tilde{\mathscr{G}}(t+\xi/4)$ with $\tilde{\alpha}(t+\xi/4) = \theta(t+\xi/4)$, and define

$$\tilde{\varphi}(r) = \begin{cases} \theta(r), & s \leq r \leq t+\xi/4, \\ \tilde{\alpha}(r), & t+\xi/4 \leq r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ and $\tilde{\varphi}(s) = y \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. Finally,

$$\tilde{\varphi}_n(t+\varepsilon_n) = \theta_n(t+\varepsilon_n) \longrightarrow \theta(t) = \tilde{\varphi}(t),$$

which implies that $x = \tilde{\varphi}(t)$. □

Theorem 3.114. *Under the hypotheses of Theorem* 3.113, $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ *is negatively* $\tilde{\mathcal{U}}$*-invariant.*

We have just proved that $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is negatively invariant. With similar proofs as in Subsection 3.3.2 and Theorem 3.113 we also have that

Theorem 3.115. *Let* $\tilde{\mathscr{G}}$ *be a pullback* $\mathfrak{D}$*-asymptotically compact impulsive generalized process and pullback* $\mathfrak{D}$*-dissipative,* $\hat{A}$ *the pullback* $\mathfrak{D}$*-semiattractor given by Theorem* 3.112*. If* $\hat{A} \setminus \hat{M}$ *negatively invariant and Condition* (I-Mult) *is satisfied, then* $\hat{A} \setminus \hat{M}$ *is also positively invariant.*

Corollary 3.116. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process and pullback $\mathfrak{D}$-dissipative, $\hat{A}$ a pullback $\mathfrak{D}$-semiattractor such that $\hat{A}\setminus\hat{M}$ is negatively invariant, and satisfying Condition* (I-Mult)*. Then $\hat{A}$ is a pullback $\mathfrak{D}$-attractor.*

Theorem 3.117. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions* (H-Mult)*,* (I-Mult)*, and* (NT-Mult)*. Then $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is positively invariant for any $\hat{D} \in \mathfrak{D}$.*

Corollary 3.118. *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions* (H-Mult)*,* (I-Mult)*, and* (NT-Mult)*. Then $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is invariant for any $\hat{D} \in \mathfrak{D}$.*

3.6 Impulses in driving semigroups

In this section, we will study impulsive nonautonomous dynamical systems using cocycles. We assume that the impulses occur in the driving semigroup and not directly in the cocycle.

Consider a nonautonomous dynamical system as in Definition 1.64. In this framework, the theory of impulsive nonautonomous dynamical systems considered, for example, in [15, 17, 18], follows the following path: take two metric spaces X and Σ, a cocycle φ, and a driving semigroup θ. Then the skew product semigroup Π on $X \times \Sigma$ is well-defined. Consider M a subset of X the impulsive set and $I\colon M \longrightarrow X$ the impulse function. The process works as follows: given an initial data (x, σ), consider the map $t \longmapsto \varphi(t, \sigma, x)$. If this trajectory touches M at time t_0, then the process goes to $(I(\varphi(t_0, \sigma, x)), \theta(t)\sigma)$ and it starts there from this point forward.

Here we will consider a different approach. The impulsive set M will be considered in Σ and not in X. The motivation is that sometimes it is useful to "make corrections" on the driving semigroup to "control" the cocycle.

Let $\varphi\colon [0, \infty) \times \Sigma \times X \longrightarrow X$ a cocycle and $\theta\colon \Sigma \longrightarrow \Sigma$ its driving semigroup. For simplicity, we will denote $\theta_t\sigma := \theta(t)\sigma$. Take $M \subset X$ and $I\colon M \longrightarrow \Sigma$ a continuous function, and consider the impulsive dynamical system (θ, X, M, I). Therefore, the set M must satisfy (3.1) (with θ instead of π). We will assume that $\tilde{\theta}$ satisfies Equation (3.3).

We define $\mathbb{M} := X \times M$ and $\mathbb{I}\colon \mathbb{M} \longrightarrow X \times \Sigma$ given by $\mathbb{I}(x, \sigma) = (x, I(\sigma))$. This implies that $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ is an impulsive dynamical system.

Definition 3.119. Let $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ the impulsive dynamical systems considered before and $(t, \sigma) \in [0, \infty) \times \Sigma$. We define $\varphi_c(t, \sigma)\colon X \longrightarrow X$ by

$$\varphi_c(t, \sigma)x = \pi_X(\tilde{\Pi}(t)(x, \sigma)),$$

with $\pi_X : X \times \Sigma \longrightarrow X$ the projection on X. The map $\varphi_c : [0,\infty) \times \Sigma \times X \longrightarrow X$ is called the coupled impulsive cocycle.

This definition implies that $\tilde{\Pi}(t)(x,\sigma) = (\varphi_c(t,\sigma)x, \tilde{\theta}_t\sigma)$ for all $t \geq 0$, $\sigma \in \Sigma$, and $x \in X$.

The coupled impulsive cocycle can also be given in terms of φ: if $\phi_\theta(\sigma) = \infty$, then $\varphi_c(t,\sigma)x = \varphi(t,\sigma)x$ for all $t \geq 0$. If $\phi_\theta(\sigma) < \infty$, denote $\sigma_0 = \sigma$, $x_0 = x$, and $s_0 = t_0 = \phi_\theta(\sigma)$. Then $\varphi_c(t,\sigma)x = \varphi(t,\sigma)x$ for $t \in [0,t_0]$. Furthermore, denote $\sigma_1 = \tilde{\theta}_{s_0}(\sigma_0)$ and $x_1 = \varphi(s_0,\sigma)x$. Next, if $\phi_\theta(\sigma_1) = \infty$, then we have $\varphi_c(t,\sigma)x = \varphi(t-t_0,\sigma_1)x_1$ for all $t \geq t_0$. If $\phi_\theta(\sigma_1) < \infty$, then denote $s_1 = \phi_\theta(\sigma_1)$ and $t_1 = t_0 + s_1$. Then $\varphi_c(t,\sigma)x = \varphi(t-t_0,\sigma_1)x_1$ for $t \in [t_0,t_1]$. Next, we denote $\sigma_2 = \tilde{\theta}_{s_1}(\sigma_1)$ and $x_2 = \varphi(t_1 - t_0, \sigma_1)x_1$.

Inductively, we follow this process. Either it ends after a finite number of steps (the case if $\phi_\theta(\sigma_n) = \infty$ for some $n \in \mathbb{N}$) or it continues indefinitely ($\phi_\theta(\sigma_n) < \infty$ for all $n \in \mathbb{N}$). In this last case, as $\tilde{\theta}$ satisfies (3.3), we know there exist sequences $\{t_n\}_n$, $\{\sigma_n\}_n$, and $\{x_n\}_n$ such that $\{t_n\}_n$ is increasing and going to $+\infty$, $\sigma_n \in I(M)$, and $x_n \in X$ for all $n \in \mathbb{N}$, and

$$\varphi_c(t,\sigma)x = \varphi(t - t_{n-1}, \sigma_n)x_n$$

From properties of $\tilde{\Pi}$, we get that φ_c satisfies:

1. $\varphi_c(0,\sigma)x = x$ for all $x \in X$ and $\sigma \in \Sigma$;
2. $\varphi_c(t+s,\sigma) = \varphi_c(t,\tilde{\theta}_s\sigma)\varphi_c(s,\sigma)$, for all $t,s \geq 0$ and $\sigma \in \Sigma$.

These two conditions are the first two condition in Definition 1.64 for a cocycle. However, the map φ_c is not continuous because $\tilde{\theta}$ is not continuous. But we do have the following result:

Proposition 3.120. *Let $\sigma \in \Sigma$ be fixed. The map*

$$(t,x) \in [0,\infty) \times X \longrightarrow \varphi_c(t,\sigma)x \in X \tag{3.29}$$

is continuous.

Proof. Set $s = \phi_\theta(\sigma)$. If $s = \infty$, then $\varphi_c(t,\sigma)x = \varphi(t,\sigma)x$ for all $t \geq 0$ and $x \in X$. We have that φ is a continuous map, so we get that the map (3.29) is also continuous.

Assume next that $s < \infty$. Take two sequences $\{t_n\}_n$ and $\{x_n\}_n$ convergent to t and x, respectively.

Case 1: $t < s$.

We know that $\{t_n\}_n$ converges to t, so we can suppose that $t_n < s$ for every $n \in \mathbb{N}$. Then we have that

$$\varphi_c(t_n,\sigma)x_n = \varphi(t_n,\sigma)x_n \longrightarrow \varphi(t,\sigma)x = \varphi_c(t,\sigma)x,$$

because φ is continuous.

Case 2: $t = s$.
In this case, we have that, if $\alpha_n = t_n - s$, then the sequence $\{\alpha_n\}_n$ converges to 0. We may assume that

$$|\alpha_n| < \phi_\theta(\tilde{\theta}_s\sigma) \quad \text{for every } n \in \mathbb{N}.$$

Up to subsequences, we can assume that $\alpha_n < 0$ for all $n \in \mathbb{N}$, or $\alpha_n \geq 0$ for all $n \in \mathbb{N}$. If $\alpha_n < 0$ for all $n \in \mathbb{N}$, then $t_n < s$ and

$$\varphi_c(t_n, \sigma)x_n = \varphi(t_n, \sigma)x_n \longrightarrow \varphi(s, \sigma)x = \varphi_c(s, \sigma)x.$$

If $\alpha_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} \varphi_c(t_n, \sigma)x_n &= \varphi_c(\alpha_n + s, \sigma)x_n \\ &= \varphi_c(\alpha_n, \tilde{\theta}_s\sigma)\varphi_c(s, \sigma)x_n \\ &= \varphi(\alpha_n, \tilde{\theta}_s\sigma)\varphi(s, \sigma)x_n, \end{aligned}$$

because $\alpha_n < \phi_\theta(\tilde{\theta}_s\sigma)$ and $\varphi_c(s, \sigma)x_n = \varphi(s, \sigma)x_n$. Then we have that

$$\varphi_c(t_n, \sigma)x_n \longrightarrow \varphi(s, \sigma)x = \varphi_c(s, \sigma)x.$$

Case 3: $t > s$.
It follows from an argument by induction on the number of impulses of $\tilde{\theta}(\sigma)$ in the interval $[0, t]$. □

In the following definitions, we are going to consider that the driving semigroup and the cocycle are not necessarily continuous.

Definition 3.121. Let X and Σ be two complete metric spaces, φ a cocycle with θ the driving semigroup. We say that φ is uniformly asymptotically compact if for all sequences $\{x_n\}_n$, $\{\sigma_n\}_n$, and $\{t_n\}_n$ such that $x_n \in X$, $\sigma_n \in \Sigma$, both $\{x_n\}_n$ and $\{\sigma_n\}_n$ bounded, and $t_n \longrightarrow \infty$, then the sequence $\{\varphi(t_n, \sigma_n)x_n\}_n$ has a convergent subsequence.

Definition 3.122. Let X and Σ be two complete metric spaces, φ a cocycle with θ the driving semigroup. We say that φ is uniformly dissipative if there exists $B_0 \subset X$ bounded such that, for all $B \subset X$ bounded, for all $\Gamma \subset \Sigma$ bounded, there exists $t_0 = t_0(B, \Gamma) \geq 0$ such that

$$t \geq t_0, \ \sigma \in \Gamma \Longrightarrow \varphi(t, \sigma)B \subset B_0.$$

The bounded set B_0 is called a uniformly absorbing set for the cocycle φ.

Definition 3.123. Let X and Σ be two complete metric spaces, φ a cocycle with θ the driving semigroup. A set $A \subset X$ is called the uniform attractor for φ if

(a) A is compact,

(b) for each $B \subset X$ bounded and $\Gamma \subset \Sigma$ bounded, then

$$\lim_{t\to\infty} \sup_{\sigma\in\Gamma} d_{\mathrm{H}}(\varphi(t,\sigma)B, A) = 0,$$

(c) A is the minimal closed set satisfying the previous condition, that is, if C is closed and satisfies (b), then $A \subset C$.

Furthermore, Theorem 1.79 is also valid for evolution processes in which the third condition of the definition, namely, the continuity of U from $\mathcal{P} \times X$ to X, is not satisfied.

Remark 3.124. If the impulsive dynamical system (θ, Σ, M, I) satisfies $(\mathrm{H_{aut}})$ and/or $(\mathrm{T_{aut}})$, then $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ also satisfies $(\mathrm{H_{aut}})$ and/or $(\mathrm{T_{aut}})$.

Theorem 3.125. *Let (θ, Σ, M, I) be an impulsive dynamical system asymptotically compact, dissipative, and satisfying $I(M) \cap M = \emptyset$ and Conditions $(\mathrm{H_{aut}})$ and $(\mathrm{T_{aut}})$. Assume also that φ_c is uniformly dissipative and uniformly asymptotically compact. Then $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ has a global attractor $\mathbb{A}$ in $X \times \Sigma$ and $A = \pi_X(\mathbb{A})$ is the uniform attractor of φ_c.*

Proof. Let B_0 a uniformly absorbing set for φ_c and Γ_0 an absorbing set for $\tilde{\theta}$. This implies that $B_0 \times \Gamma_0$ is an absorbing set for $\tilde{\Pi}$. Therefore, $\tilde{\Pi}$ is dissipative. Take a bounded sequence $\{(x_n, \sigma_n)\}_n$ in $X \times \Sigma$ and $\{t_n\}_n$ a sequence going to ∞. As φ_c is uniformly asymptotically compact, we have that $\{\varphi_c(t_n, \sigma_n)x_n\}_n$ has a convergent subsequence, which we denote the same. Furthermore, $\tilde{\theta}$ is asymptotically compact, so $\{\tilde{\theta}_{t_n}\sigma_n\}_n$ has a convergent subsequence, which we denote the same again. Then $\{\tilde{\Pi}(t_n)(x_n, \sigma_n)\}_n$ is convergent. As a consequence, $\tilde{\Pi}$ is asymptotically compact. Finally, the definition of $\mathbb{M}$ and $\mathbb{I}$ and Remark 3.124 imply that $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ satisfies $\mathbb{I}(\mathbb{M}) \cap \mathbb{M} = \emptyset$ and Conditions $(\mathrm{H_{aut}})$ and $(\mathrm{T_{aut}})$. Then Theorem 3.22 implies that $(\Pi, X \times \Sigma, \mathbb{M}, \mathbb{I})$ has a global attractor $\mathbb{A}$, and $A = \pi_X(\mathbb{A})$ is the uniform attractor of φ_c. □

Suppose that (θ, Σ, M, I) is an impulsive dynamical system asymptotically compact, dissipative, and satisfying $I(M) \cap M = \emptyset$, and Conditions $(\mathrm{H_{aut}})$ and $(\mathrm{T_{aut}})$. Then Theorem 3.22 implies that (θ, Σ, M, I) has a global attractor Ξ. Theorem 3.10 implies that

$$\Xi \setminus M = \{\psi(0) \colon \psi \text{ is a bounded global solution of } \tilde{\theta}\}. \tag{3.30}$$

We consider the restriction of $\tilde{\theta}$ to $\Xi \setminus M$. For a bounded global solution ψ, we consider the evolution process

$$U_\psi(t,s) := \varphi_c(t-s, \psi(s)), \quad t \geq s. \tag{3.31}$$

Theorem 3.126. *Let φ_c be uniformly asymptotically compact and uniformly dissipative, then U_ψ has a pullback attractor $\hat{A}_\psi$, where ψ is any bounded global solution.*

Proof. Take a bounded global solution ψ. We will prove that U_ψ is pullback asymptotically compact and pullback dissipative.

Fix $t \in \mathbb{R}$, $\{s_n\}_n$ and $\{x_n\}_n$ two sequences such that $s_n \longrightarrow -\infty$ and $\{x_n\}_n$ bounded in X. Then

$$U_\psi(t, s_n)x_n = \varphi_c(t - s_n, \psi(s_n))x_n.$$

We know that φ_c is uniformly asymptotically compact, $t - s_n \longrightarrow \infty$, and $\{\psi(s_n)\}_n$ is bounded. This implies that the sequence $\{\varphi(t - s_n, \psi(s_n))x_n\}_n$ has convergent subsequence, so $\{U_\psi(t, s_n)x_n\}_n$ has a convergent subsequence. Therefore, U_ψ is pullback asymptotically compact.

We fix B_0 a uniformly absorbing set for φ_c. Take B any bounded set and $t \in \mathbb{R}$. We have that φ_c is uniformly dissipative and $\psi(\mathbb{R})$ is bounded, so there exists $t_0 = t_0(B, \psi(\mathbb{R})) \geq 0$ such that

$$[t - s \geq t_0,\ \psi(s) \in \Xi \setminus M] \Longrightarrow \varphi(t - s, \psi(s))B \subset B_0.$$

But $t - s \geq t_0$ implies that $s \leq t - t_0$, so taking $s_0 := t - t_0$ we have that

$$s \leq s_0 \Longrightarrow U_\psi(t, s)B \subset B_0,$$

which implies that U_ψ is pullback dissipative. As a consequence, there exists a pullback attractor $\hat{A}$ for U_ψ by Theorem 1.79. $\square$

Take X and Y two Banach space and consider the following system

$$\begin{cases} x' = f(x, y),\ t > 0, \\ y' = g(y),\ t > 0, \\ (x(0), y(0)) = (x_0, y_0) \in X \times Y. \end{cases} \tag{3.32}$$

This type of systems are known as cascade systems. For more information, check [3, 79]. We suppose that we have a unique solution $\Pi(\cdot)(x_0, y_0)$ for every initial data (x_0, y_0), the solution is defined for all $t \geq 0$, and the map

$$(t, x_0, y_0) \in [0, \infty) \times X \times Y \longrightarrow \Pi(t)(x_0, y_0) \in X \times Y$$

is continuous. Consider

$$\begin{cases} \dot{y} = g(y),\ t > 0, \\ y(0) = y_0 \in Y. \end{cases}$$

We have a semigroup θ on Y and

$$\begin{cases} \dot{x} = f(x, \theta_t y_0), \ t > 0, \\ x(0) = x_0 \in X. \end{cases}$$

Let $\varphi(\cdot, y_0)x_0$ be the solution to this problem. Therefore, we have that $\Pi(t)(x_0, y_0) = (\varphi(t, y_0)x_0, \theta_t y_0)$, and φ is a cocycle.

Take M an impulsive set and $I\colon M \longrightarrow Y$ an impulse function for θ in Y. Therefore, we have that (θ, Y, M, I) is an impulsive dynamical system. If it satisfies (3.3), then $\tilde{\theta}$ is an impulsive semigroup, and we obtain the impulsive semigroup $\tilde{\Pi}$ in $X \times Y$ and its associated coupled impulsive cocycle φ_c.

Furthermore, if the hypotheses of Theorem 3.22 are satisfied for the impulsive dynamical system (θ, Y, M, I), then we have a global attractor Ξ. Moreover, if φ_c is uniformly asymptotically compact and uniformly dissipative, Theorem 3.126 says the there exists a pullback attractor $\hat{A}_\psi$ for the associated evolution process given by Equation (3.31), with ψ a bounded global solution of $\tilde{\theta}$ in $\Xi \setminus M$. This associated evolution process U_ψ can be seen as the solution of the problem

$$x'(t) = f(x, \psi(t)), \ t \geq s,$$

in some “weak sense”.

3.7 Applications

In this section, we will some application of the results obtained through the chapter. First, we will present three results that help us prove asymptotical compactness or dissipativeness.

Proposition 3.127. *Let (π, X, M, I) an impulsive dynamical system, with X a Banach space and $I(M)$ bounded. Suppose that there exists $k > 0$ such that, for every bounded subset B, there exists $h_B\colon [0, \infty) \longrightarrow [0, \infty)$ a bounded function such that*

$$\begin{aligned} &\lim_{t \to \infty} h_B(t) = 0, \\ &\sup_{x \in B} \|\pi(t)x\|^2 \leq h_B(t) + k \quad \text{for all } t \geq 0. \end{aligned} \tag{3.33}$$

Then $\tilde{\pi}$ is dissipative.

Proof. We take $K = \{\pi(t)x \colon x \in I(M) \text{ and } 0 \leq t < \phi(x)\}$, a bounded set because $I(M)$ is bounded and (3.33) applies.

First, we are going to prove that $\tilde{\pi}(K) \subset K$ for all $t \geq 0$. Take any $t \geq 0$ and $z \in K$. This implies that there exist $x \in I(M)$ and $s \in [0, \phi(x))$ such that $z = \pi(s)x$. We have that

$$\tilde{\pi}(t)z = \tilde{\pi}(t)\pi(s)x = \tilde{\pi}(t)\tilde{\pi}(s)x = \tilde{\pi}(t+s)x.$$

Case 1: $t + s < \phi(x)$.

$$\tilde{\pi}(t)z = \tilde{\pi}(t+s)x = \pi(t+s)x \in K.$$

Case 2: $t + s = \phi(x)$.

$$\tilde{\pi}(t)z = \tilde{\pi}(t+s)x = I(\pi(t+s)x) \in I(M) \subset K.$$

Case 3: $t + s > \phi(x)$.

This implies that there exist $x^+ \in I(M)$ and $r \in [0, \phi(x^+)]$ with $\tilde{\pi}(t)z = \tilde{\pi}(r)x^+$. Then, if $r \in [0, \phi(x^+))$, this implies that

$$\tilde{\pi}(t)z = \tilde{\pi}(r)x^+ = \pi(r)x^+ \in K.$$

If $r = \phi(x^+)$, then

$$\tilde{\pi}(t)z = \tilde{\pi}(r)x^+ = I(\pi(r)x^+) \in I(M) \subset K.$$

Therefore we conclude that $\tilde{\pi}(t)K \subset K$.

We define $B_0 = \{x \in X : \|x\|^2 \leq 2k\} \cup K$, which is bounded. Take $B \subset X$ an arbitrary bounded set and $t_0 = t_0(B)$ such that $h_B(t) < k$ if $t \geq t_0$ (it is possible to find such t_0 by (3.33)). We denote

$$B_1 = \{x \in B : \phi(x) \leq t_0\} \quad \text{and} \quad B_2 = \{x \in B : \phi(x) > t_0\}.$$

Let $t \geq t_0$ and $x \in B_1$. Then $\phi(x) \leq t$, and this implies

$$\tilde{\pi}(t)x = \tilde{\pi}(t - \phi(x))\tilde{\pi}(\phi(x))x.$$

But $\tilde{\pi}(\phi(x))x \in I(M) \subset K$, and therefore

$$\tilde{\pi}(t)x \in \tilde{\pi}(t - \phi(x))K \subset K \subset B_0.$$

Let $t \geq t_0$ and $x \in B_2$. If $\phi(x) \leq t$, then with an analogous argument as before, we get that $\tilde{\pi}(t)x \in K \subset B_0$. If $\phi(x) > t$, then $\tilde{\pi}(t)x = \pi(t)x$. Furthermore, we have that

$$\|\pi(t)x\|^2 \leq h_B(t) + k < k + k = 2k,$$

so $\tilde{\pi}(t)x \in \{y \in X : \|y\|^2 \leq 2k\} \subset B_0$. □

Proposition 3.128. *Let $\tilde{\mathcal{U}}$ be a pullback $\mathfrak{D}$-dissipative impulsive evolution process which satisfies Condition* (H)*. Suppose that $\hat{B}_0$ is a pullback $\mathfrak{D}$-absorbing family with $B_0(t)$ bounded for every $t \in \mathbb{R}$, and $U(t,s)\colon X \longrightarrow X$ a compact map for every $t > s$. Then we have that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact.*

Proof. We take $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ such that $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$. We have to prove that the sequence $\{\tilde{U}(t+\varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence.

For each $n \in \mathbb{N}$ we consider the partial impulsive semitrajectory

$$\varphi_n\colon u \in [s_n, t+\xi/4] \longrightarrow \tilde{U}(u, s_n)x_n \in X.$$

We denote by τ_n the last jump time of φ_n (if there is no jump time, take $\tau_n = s_n$). We assume that $\varepsilon_n \leq \xi/4$ for all $n \in \mathbb{N}$. We will split the proof in three different cases.

Case 1: Up to a subsequence, which will be denoted the same, there exists $\delta > 0$ such that $\tau_n \leq t - \delta$.

We can assume that $\varepsilon_n > -\delta/2$ and that $y_n = \tilde{U}(t-\delta, s_n)x_n \in B_0(t-\delta)$, because $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative. We know that $U(t-\delta/2, t-\delta)$ is compact, so we have that $\{U(t-\delta/2, t-\delta)y_n\}_n$ has a convergent subsequence, because $B_0(t-\delta)$ is a bounded set. We will denote the convergent subsequence the same, so we will write

$$U(t-\delta/2, t-\delta)y_n \longrightarrow y.$$

As τ_n was the last jump time, we have that

$$\tilde{U}(u, s_n)x_n = U(u, t-\delta)y_n, \quad \text{for } u \in [t-\delta, t+\xi/4].$$

This implies that

$$\begin{aligned}\tilde{U}(t+\varepsilon_n, s_n)x_n &= U(t+\varepsilon_n, t-\delta)\tilde{U}(t-\delta, s_n)x_n \\ &= U(t+\varepsilon_n, t-\delta/2)U(t-\delta/2, t-\delta)\tilde{U}(t-\delta, s_n)x_n \\ &= U(t+\varepsilon_n, t-\delta/2)U(t-\delta/2, t-\delta)y_n,\end{aligned}$$

which converges to $U(t, t-\delta/2)y$.

Case 2: Up to a subsequence, which will be denoted the same, there exists $\delta > 0$ such that $t+\delta < \tau_n \leq t+\xi/4$.

We can assume that $-\xi/4 < \varepsilon_n < \delta$ and $y_n = \tilde{U}(t-\xi/2, s_n)x_n \in B_0(t-\xi/2)$, because $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative. As $t+\delta < \tau_n < t+\xi/4$, Condition (H) implies that there is only one jump time in $[t-\xi, t+\xi/4]$, which is precisely

τ_n. This implies that $\tilde{U}(t-\xi/4, t-\xi/2)y_n = U(t-\xi/4, t-\xi/2)y_n$, and by hypothesis, $U(t-\xi/4, t-\xi/2)$ is a compact map. As $B_0(t-\xi/2)$ is bounded, then $\{U(t-\xi/4, t-\xi/2)y_n\}_n$ has a convergent subsequence, which will be denoted the same, so

$$U(t-\xi/4, t-\xi/2)y_n \longrightarrow y.$$

This implies that

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, t-\xi/4)\tilde{U}(t-\xi/4, s_n)x_n,$$

which converges to $U(t, t-\xi/4)y$.

Case 3: The sequence $\{\tau_n\}_n$ converges to t.
In this case, we separate the proof into two subcases:

Subcase 1: Up to a subsequence, still denoted the same, we have $\tau_n \leq t+\varepsilon_n$. We can assume that $t-\xi/4 < \tau_n$ and $y_n = \tilde{U}(t-\xi/2, s_n)x_n \in B_0(t-\xi/2)$. As $t-\xi/4 < \tau_n < t+\xi/4$, Condition (H) implies that there is only one jump time in $[t-\xi, t+\xi/4]$, which is precisely τ_n. Then

$$\tilde{U}(t-\xi/4, s_n)x_n = U(t-\xi/4, t-\xi/2)y_n.$$

Therefore, the sequence $\{U(t-\xi/4, t-\xi/2)y_n\}_n$ has a convergent subsequence (denoted the same) to some y. Then

$$\tilde{U}(\tau_n, s_n)x_n = I_{\tau_n}(U(\tau_n, t-\xi/4)\tilde{U}(t-\xi/4, s_n)x_n),$$

which converges to $I_t(U(t, t-\xi/4)y)$. This implies that

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, \tau_n)\tilde{U}(\tau_n, s_n)x_n.$$

This last sequence converges to $I_t(U(t, t-\xi/2)y)$, and we are finished.

Subcase 2: Up to a subsequence, still denoted the same, we have $t+\varepsilon_n < \tau_n$. We can assume that $\varepsilon_n > -\xi/4$ and $y_n = \tilde{U}(t-\xi/2, s_n)x_n \in B_0(t-\xi/2)$. Condition (H) implies that there is only one jump time in $[t-\xi, t+\xi/4]$, which is τ_n. We have that

$$\tilde{U}(t-\xi/4, s_n)x_n = U(t-\xi/4, t-\xi/2)y_n.$$

The sequence $\{U(t-\xi/4, t-\xi/2)y_n\}_n$ has a convergent subsequence, by compactness of the map $U(t-\xi/4, t-\xi/2)$. The convergent subsequence will be denoted the same, and let y be the limit. Finally,

$$\tilde{U}(t+\varepsilon_n, s_n)x_n)U(t+\varepsilon_n, t-\xi/4)U(t-\xi/4, t-\xi/2)y_n,$$

which converges to $U(t, t-\xi/2)y$. □

Lemma 3.129. *Let $\mathcal{U}$ be a pullback $\mathfrak{D}$-asymptotically compact evolution process (according to Definition 1.78). Suppose that $\{I_t(M(t))\}_{t\in\mathbb{R}} \in \mathfrak{D}$ and*

$$\bigcup_{s\leq t} I_s(M(s))$$

is relatively compact in X for all $t \in \mathbb{R}$. Then $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-asymptotically compact.

Proof. We take $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and three sequences $\{s_n\}_n$, $\{\varepsilon_n\}_n$, and $\{x_n\}_n$ such that $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$, and $x_n \in D(s_n)$. We want to prove that the sequence $\{\tilde{U}(t+\varepsilon_n, s_n)x_n\}_n$ has a convergent subsequence.

For each $n \in \mathbb{N}$ we consider the partial impulsive semitrajectory

$$\varphi_n : u \in [s_n, t+\varepsilon_n] \longrightarrow \tilde{U}(u, s_n)x_n \in X.$$

We split the proof in two different cases.

Case 1: up to a subsequence, still denoted the same, there are no jump times of φ_n.

In this situation we have that $\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, s_n)x_n$. We suppose that $\varepsilon_n > -1$. We know that $\mathcal{U}$ is $\mathfrak{D}$-asymptotically compact, which implies that the sequence $\{U(t-1, s_n)x_n\}_n$ has a convergent subsequence (which will be denoted the same) to a point $y \in X$. Therefore,

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, t-1)U(t-1, s_n)x_n \longrightarrow U(t, t-1)y.$$

Case 2: there exists $n_0 \in \mathbb{N}$ such that φ_n has at least one jump time for all $n \geq n_0$.

Without loss of generality, we can assume that φ_n has at least one jump time for all $n \in \mathbb{N}$ and $\varepsilon_n < 1$. We denote by τ_n the last jump time of φ_n. Therefore, for any $n \in \mathbb{N}$, we have

$$y_n := \tilde{U}(\tau_n, s_n)x_n \in I_{\tau_n}(M(\tau_n)) \subset \bigcup_{s\leq t+1} I_s(M(s)),$$

which we know is relatively compact. Therefore, we have that $\{y_n\}_n$ has a convergent subsequence (which we will denote the same) to a point $y \in X$. We split the proof of this case in two subcases.

Subcase 1: $\{\tau_n\}$ is a bounded sequence.

Up to a subsequence, which will be denoted the same, we assume that $\{\tau_n\}_n$ is convergent to a number t_0. Then

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, \tau_n)\tilde{U}(\tau_n, s_n)x_n = U(t+\varepsilon_n, \tau_n)y_n,$$

which converges to $U(t,t_0)y$.

Subcase 2: $\{\tau_n\}$ is not a bounded sequence.

Up to a subsequence, denoted the same, we may assume that $\tau_n \longrightarrow -\infty$. We may also assume that $\tau_n < t-1$ for all $n \in \mathbb{N}$. We know that $\{I_s(M(s))\}_{s\in\mathbb{R}} \in \mathfrak{D}$, $\tau_n \longrightarrow -\infty$, and $y_n \in I_{\tau_n}(M(\tau_n))$. The pullback $\mathfrak{D}$-asymptotic compactness of $\mathcal{U}$ implies that the sequence $\{U(t-1,\tau_n)y_n\}_n$ has a convergent subsequence, which will be denoted the same, to $z \in X$. As a consequence,

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, t-1)U(t-1,\tau_n)y_n,$$

which converges to $U(t,t-1)z$. □

Integrate-and-fire model

We will consider an integrate-and-fire neurons model as a first application. Similar models were discussed in [23, 26, 84]. This model will be given by

$$u'(t) = -\gamma(t)u(t) + S(t), \tag{3.34}$$

with the following condition:

$$\text{if } u(t) = \theta(t) \text{ then } u(t) \text{ resets to } u_r(t) < \theta(t).$$

These models describe, see [84], leaky, current-clamped membranes.

We will consider the impulsive evolution process associated with this model. We assume:

(H1) $\gamma\colon \mathbb{R} \longrightarrow (0,\infty)$ is continuous;

Variable	Description
$u(t)$	State variable (membrane potential)
$\gamma(t)$	Dissipation
$S(t)$	Stimulus applied
$\theta(t)$	Firing threshold
$u_r(t)$	Reset state

Table 3.1: Description of variables of the integrate-and-fire neuron model.

(H2) $S\colon \mathbb{R} \longrightarrow (0,\infty)$ is continuous,

$$k(t) := \int_0^\infty S(t-u)\exp\left\{-\int_{t-u}^t \gamma(v)\,dv\right\} du < \infty \text{ for each } t \in \mathbb{R},$$

$$\lim_{s\to-\infty} k(s)\exp\left\{-\int_s^t \gamma(v)dv\right\} = 0 \text{ for each } t \in \mathbb{R},$$

and

$$\lim_{\tau\to 0^+}\sup_{t\in\mathbb{R}} \int_0^\tau S(t-u)\exp\left\{-\int_{t-u}^t \gamma(v)\,dv\right\} du = 0;$$

(H3) $\theta\colon \mathbb{R} \longrightarrow (0,\infty)$ is continuously differentiable,

$$\lim_{s\to-\infty} \theta(s)\exp\left\{-\int_s^t \gamma(v)\,dv\right\} = 0 \text{ for each } t \in \mathbb{R} \tag{3.35}$$

and

$$\theta'(t) + \gamma(t)\theta(t) - S(t) \neq 0 \quad \text{for all } t \in \mathbb{R}; \tag{3.36}$$

(H4) $u_r\colon \mathbb{R} \longrightarrow [0,\infty)$ is continuous and there exist $a, \delta > 0$ such that, for all $\eta \in [0,\delta]$ and all $s \in \mathbb{R}$, we have $u_r(s) + a < \theta(s+\eta)$.

Remark 3.130. Conditions (H2) and (3.35) are satisfied, for instance, when γ has a positive infimum and S and θ are bounded.

We can solve the initial value problem

$$\begin{cases} u'(t) = -\gamma(t)u(t) + S(t), & \text{for } t > s, \\ u(s) = x_0 \in \mathbb{R}, \end{cases}$$

and get the evolution process $\mathcal{U}$, with $U(t,s)$ given by

$$U(t,s)x_0 = x_0\exp\left\{-\int_s^t \gamma(v)\,dv\right\} + \int_s^t S(x)\exp\left\{-\int_x^t \gamma(v)\,dv\right\} dx, \tag{3.37}$$

for $t > s$ and $x_0 \in \mathbb{R}$. We denote $M(t) = \{\theta(t)\}$ and $I_t(\theta(t)) = u_r(t)$ for each $t \in \mathbb{R}$. Then we have that $(U, \mathbb{R}, \hat{M}, I)$ is an impulsive evolution process, as $\hat{M}$ and I are collectively closed and collectively continuous, respectively. The only condition we need to check is (3.5). But this condition follows from Equation (3.36).

We will prove next that this impulsive evolution process has a pullback $\mathfrak{D}$-attractor for some universe $\mathfrak{D}$.

Proposition 3.131. *The impulsive evolution process* $\tilde{\mathcal{U}}$ *satisfies Condition* (H).

Proof. Take α such that we have

$$\int_0^\eta S(s+\eta-u)\exp\left\{-\int_{s+\eta-u}^{s+\eta}\gamma(v)\,dv\right\}du \le a \text{ for all } s\in\mathbb{R},$$

for every $\eta \in [0,\alpha]$, by property (H2). This implies that

$$\begin{aligned} U(s+\eta,s)u_r(s) &= u_r(s)\exp\left\{-\int_s^{s+\eta}\gamma(v)\,dv\right\} \\ &\qquad + \int_s^{s+\eta} S(x)\exp\left\{-\int_x^{s+\eta}\gamma(v)\,dv\right\}dx \\ &\le u_r(s) + \int_0^\eta S(s+\eta-u)\exp\left\{-\int_{s+\eta-u}^{s+\eta}\gamma(v)\,dv\right\}du \\ &\le u_r(s)+a, \end{aligned}$$

for all $s\in\mathbb{R}$. If $0\le\eta\le\min\{\delta,\alpha\}$, with $\delta>0$ given by (H4), we get

$$U(s+\eta,s)u_r(s)\le u_r(s)+a<\theta(s+\eta).$$

Therefore, $\phi(u_r(s),s)>\min\{\delta,\alpha\}>0$ for all $s\in\mathbb{R}$. As a consequence, we take $\xi=\min\{\delta,\alpha\}/2>0$, so Condition (H) is true. □

We prove that $\tilde{\mathcal{U}}$ satisfies Condition (T).

Proposition 3.132. *The impulsive evolution process* $\tilde{\mathcal{U}}$ *satisfies Condition* (T).

Proof. We fix $s\in\mathbb{R}$, $t>s$, and $\{z_n\}_n$ a convergent sequence to z such that $U(t,s)z_n \longrightarrow \theta(t)$. We want to prove that there exist $\{z_{n_k}\}_k$ a subsequence of $\{z_n\}_n$ and a sequence $\{\alpha_k\}_k$ such that $t+\alpha_k\ge s$, $\{\alpha_k\}_k$ convergent to 0, and $U(t+\alpha_k,s)z_{n_k}=\theta(t+\alpha_k)$.

The sequence $\{U(t,s)z_n\}_n$ converges to $\theta(t)$, so $U(t,s)z=\theta(t)$. Assumption (H3) implies that

$$\theta'(t) > -\gamma(t)\theta(t)+S(t) \quad \text{for all } t\in\mathbb{R}$$
$$\text{or}$$
$$\theta'(t) < -\gamma(t)\theta(t)+S(t) \quad \text{for all } t\in\mathbb{R}.$$

We suppose that $\theta'(t)>-\gamma(t)\theta(t)+S(t)$, and we define $f(r)=U(r,s)z-\theta(r)$ for $r\ge s$. This map is differentiable, $f(t)=0$, and

$$f'(r)=\frac{d}{dr}(U(r,s)z-\theta(r)) = -\gamma(r)U(r,s)z+S(r)-\theta'(r).$$

This implies that

$$f'(t) = -\gamma(t)U(t,s)z+S(t)-\theta'(t) = -\gamma(t)\theta(t)+S(t)-\theta'(t)<0.$$

As a consequence, there exists $\varepsilon_0 > 0$ with $t - s > \varepsilon_0$ and

$$U(t-\varepsilon, s)z > \theta(t-\varepsilon) \quad \text{and} \quad U(t+\varepsilon, s)z < \theta(t+\varepsilon),$$

for every $\varepsilon \in (0, \varepsilon_0]$. This implies that

$$U(t-\varepsilon_0, s)z_n > \theta(t-\varepsilon_0) \quad \text{and} \quad U(t+\varepsilon_0, s)z_n < \theta(t+\varepsilon_0).$$

Therefore, there exists $\alpha_n \in [-\varepsilon_0, \varepsilon_0]$ such that $U(t+\alpha_n, s)z_n = \theta(t+\alpha_n)$. If we prove that, up to a subsequence, $\{\alpha_n\}_n$ converges to 0, then we are finished. As $\alpha_n \in [-\varepsilon_0, \varepsilon_0]$, up to a subsequence (denoted the same), there exists $\bar{\alpha} \in [-\varepsilon_0, \varepsilon_0]$ such that $\alpha_n \longrightarrow \bar{\alpha}$. The family $\hat{M}$ is collectively closed, so

$$U(t+\bar{\alpha}, s)z = \theta(t+\bar{\alpha}).$$

We also know that $U(t+\eta, s)z \neq \theta(t+\eta)$ for all $\eta \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$. This implies that $\bar{\alpha} = 0$. □

We consider the set $\mathcal{R}$ of functions $w\colon \mathbb{R} \longrightarrow (0, \infty)$ that satisfy

$$\lim_{s \to -\infty} w(s) \exp\left\{-\int_s^t \gamma(v)\, dv\right\} = 0 \quad \text{for each } t \in \mathbb{R}.$$

Then, we consider the universe $\mathfrak{D}$ of families $\hat{D} = \{D(t)\}_{t\in\mathbb{R}}$ such that $D(t) \subset \mathbb{R}$ and there exists $\omega_{\hat{D}} \in \mathcal{R}$ with

$$D(t) \subset [-\omega_{\hat{D}}(t), \omega_{\hat{D}}(t)] \quad \text{for all } t \in \mathbb{R}.$$

Proposition 3.133. *The impulsive evolution process $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}$-dissipative and pullback $\mathfrak{D}$-asymptotically compact.*

Proof. Take $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, $s < t$, and $x \in D(s)$. Then

$$\begin{aligned}
|U(t,s)x| &\le |x| \exp\left\{-\int_s^t \gamma(v)\, dv\right\} \\
&\qquad + \int_s^t S(x) \exp\left\{-\int_x^t \gamma(v)\, dv\right\} dx \\
&= |x| \exp\left\{-\int_s^t \gamma(v)\, dv\right\} \\
&\qquad + \int_0^{t-s} S(t-u) \exp\left\{-\int_{t-u}^t \gamma(v)\, dv\right\} du \\
&\le \omega_{\hat{D}}(s) \exp\left\{-\int_s^t \gamma(v)\, dv\right\} \\
&\qquad + \int_0^{t-s} S(t-u) \exp\left\{-\int_{t-u}^t \gamma(v)\, dv\right\} du.
\end{aligned}$$

For every $t \in \mathbb{R}$ we define

$$\omega(t) = 2\max\left\{\theta(t),\ \int_0^\infty S(t-u)\exp\left\{-\int_{t-u}^t \gamma(v)\,dv\right\}du\right\} \text{ for each } t \in \mathbb{R},$$

and $B_0(t) = [-\omega(t), \omega(t)]$. We have that $\hat{B}_0$ is collectively closed and that $\hat{B}_0 \in \mathfrak{D}$. We take any $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, and two sequences $\{\varepsilon_n\}_n$ and $\{s_n\}_n$ such that $\varepsilon_n \longrightarrow 0$ and $s_n \longrightarrow -\infty$. For any $n \in \mathbb{N}$ and $x \in D(s_n)$, we have two different cases.

Case 1: There are no jump times of $\tilde{\mathcal{U}}$ at (x, s_n).
This implies that

$$\tilde{U}(t+\varepsilon_n, s_n)x = U(t+\varepsilon_n, s_n)x.$$

In this case, we have that

$$\begin{aligned}|U(t+\varepsilon_n, s_n)x| &\leq \omega_{\hat{D}}(s_n)\exp\left\{-\int_{s_n}^{t+\varepsilon_n}\gamma(v)\,dv\right\} \\ &\quad + \int_{-\varepsilon_n}^{t-s_n} S(t-u)\exp\left\{-\int_{t-u}^{t+\varepsilon_n}\gamma(v)\,dv\right\}du\end{aligned}$$

This last term does not depend on x, and we know that it converges to

$$\int_0^\infty S(t-u)\exp\left\{-\int_{t-u}^t \gamma(v)\,dv\right\}du.$$

Case 2: There is at least one jump time of $\tilde{\mathcal{U}}$ at (x, s_n).
Then we take $\tau \in [s_n, t+\varepsilon_n]$ the last jump time. Therefore, by definition of $\hat{M}$ and I,

$$\tilde{U}(t+\varepsilon_n, s_n)x_n = U(t+\varepsilon_n, \tau)u_r(\tau)$$

But $U(t+\varepsilon_n, \tau)u_r(\tau) \in [0, \theta(t+\varepsilon_n))$, because τ was the last jump time in the interval $[s_n, t+\varepsilon_n]$. Then,

$$\tilde{U}(t+\varepsilon_n, s_n)x \in B_0(t+\varepsilon_n).$$

As a consequence of the two cases, and because $x \in D(s_n)$ was arbitrary, this implies that there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \tilde{U}(t+\varepsilon_n, s_n)D(s_n) \subset B_0(t+\varepsilon_n).$$

Then we have proved that $\tilde{U}$ is pullback $\mathfrak{D}$-dissipative.

Furthermore, as each $B_0(t) \subset \mathbb{R}$ is compact and $U(t,s)$ is a compact map for every $t > s$, then Proposition 3.128 implies that $\tilde{U}$ is pullback $\mathfrak{D}$-asymptotically compact. □

In order to prove the pullback $\mathfrak{D}$-asymptotical compactness, we could also use the fact that $\mathcal{U}$ is asymptotically compact, $\{I_t(M(t))\}_{t\in\mathbb{R}} \in \mathfrak{D}$, and

$$\bigcup_{s\leq t} I_s(M(s))$$

is relatively compact in $\mathbb{R}$ for every $t \in \mathbb{R}$. Then we apply Lemma 3.129, and we get the result.

With all these results, the impulsive evolution process $\tilde{\mathcal{U}}$ has a collectively compact pullback $\mathfrak{D}$-attractor $\hat{A} \in \mathfrak{D}$, by Corollary 3.67, since it clearly satisfies Condition (I).

Impulsive Navier–Stokes equation

We are going to consider a two-dimensional impulsive Navier–Stokes equation.

Let $\Omega \subset \mathbb{R}^2$ be open with smooth boundary $\partial\Omega$. We assume that the Poincaré inequality holds, that is, there exists $\lambda_1 > 0$ such that

$$\lambda_1 \int_\Omega |\varphi(x)|^2 dx \leq \int_\Omega |\nabla\varphi(x)|^2\, dx \quad \text{for all } \varphi \in H_0^1(\Omega).$$

We consider the two-dimensional nonautonomous Navier–Stokes problem, given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u = f(t) - \nabla p, & \text{in } (s,+\infty)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (s,+\infty)\times\Omega, \\ u = 0 & \text{on } (s,+\infty)\times\partial\Omega, \\ u(s,x) = u_0(x) & \text{for } x\in\Omega. \end{cases}$$

with $\nu > 0$, f the forcing term, and p the pressure.

We take the following spaces:

$$\begin{aligned} &\mathcal{V} = \{u \in (\mathcal{C}_0^\infty(\Omega))^2 \colon \operatorname{div} u = 0\}, \\ &H, \text{ the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2, \\ &V, \text{ the closure of } \mathcal{V} \text{ in } (H_0^1(\Omega))^2, \end{aligned}$$

with the inner products defined as

$$(u,v) = \sum_{j=1}^{2} \int_\Omega u_j v_j\, dx \quad \text{and} \quad ((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\, dx,$$

in H and V. We also consider $|\cdot|$ and $\|\cdot\|$ the norms from those inner products. Finally, we consider $\langle\cdot,\cdot\rangle$ the duality between V and V^*, $\|\cdot\|_*$ the norm of V^*, and b the trilinear form

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

Assume that $u_0 \in H$ and $f \in L^2_{loc}(\mathbb{R}; V^*)$. For each $s \in \mathbb{R}$, consider the problem

$$\begin{cases} u \in L^2(s,T;V) \cap L^\infty(s,T;H), & \text{for } T > s, \\ \dfrac{d}{dt}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) = \langle f(t), v\rangle, & \text{for all } v \in V, \\ u(s) = u_0. \end{cases}$$

It has a unique solution $u(\cdot; s, u_0)$, which is defined in the interval $[s,\infty)$. The solution is in $\mathcal{C}^0([\tau,\infty); H)$. Define $U(t,s)u_0 := u(t,s;u_0)$, which is an evolution process in H, which will be referred to as $\mathcal{U}$.

Let $\sigma = \nu\lambda_1$, we suppose that $f \in L^2_{loc}(\mathbb{R}; V^*)$ satisfies

$$\int_{-\infty}^{t} e^{\sigma\zeta} \|f(\zeta)\|_*^2 d\zeta < \infty \quad \text{for every } t \in \mathbb{R}. \tag{3.38}$$

We define $\mathcal{R}_\sigma$ the set of functions $r \colon \mathbb{R} \longrightarrow (0,\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0,$$

and consider the universe $\mathfrak{D}_\sigma$ of the families $\hat{D}$ of subsets of H such that

$$D(t) \subset \{v \in H \colon |v| \le r_{\hat{D}}(t)\} \quad \text{for each } t \in \mathbb{R},$$

for some $r_{\hat{D}} \in \mathcal{R}_\sigma$.

Following the same ideas of [34, Theorem 17], we get:

- The evolution process $\mathcal{U}$ is pullback $\mathfrak{D}_\sigma$-asymptotically compact;
- For every $u_0 \in H$ and $t \ge s$,

$$|U(t,s)u_0|^2 \le e^{-\sigma(t-s)}|u_0|^2 + \frac{e^{-\sigma t}}{\nu} \int_s^t e^{\sigma\zeta}\|f(\zeta)\|_*^2 d\zeta. \tag{3.39}$$

- Take

$$R_\sigma(t) := \left(\frac{2e^{-\sigma t}}{\nu} \int_{-\infty}^{t} e^{\sigma\zeta} \|f(\zeta)\|_*^2 d\zeta \right)^{1/2}$$

  and $B_0(t) = \{v \in H \colon |v| \le R_\sigma(t)\}$ for every $t \in \mathbb{R}$. We get that $R_\sigma \in \mathcal{R}_\sigma$, which implies that $\hat{B}_0 \in \mathfrak{D}_\sigma$. Furthermore, for every $\hat{D} \in \mathfrak{D}_\sigma$ and $t \in \mathbb{R}$, there exists $s_0 = s_0(\hat{D}, t) \le t$ such that

$$s \le s_0 \Longrightarrow U(t,s)D(s) \subset B_0(t).$$

We consider $\hat{M}$ an arbitrary family in H such that it is collectively closed and satisfies (3.5), and a family of functions $I = \{I_t : M(t) \longrightarrow H\}_{t\in\mathbb{R}}$ such that I is collectively continuous. This implies that $\tilde{\mathcal{U}} = (\mathcal{U}, H, \hat{M}, I)$ is an impulsive evolution process. We suppose that

- $\tilde{\mathcal{U}}$ satisfies Conditions (I), (T), and (H);
- $|I_t(v)| \leq 2^{-1/2} R_\sigma(t)$ for every $v \in M(t)$;
- for every $t \in \mathbb{R}$, the set
$$\bigcup_{s\leq t} I_s(M(s))$$
is relatively compact.

Lemma 3.129 implies that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}_\sigma$-asymptotically compact. We will prove that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}_\sigma$-dissipative.

Let $\tau \in \mathbb{R}$, $|x| \leq 2^{-1/2} R_\sigma(\tau)$ and $t \geq \tau$. Then Equation (3.39) implies that

$$|U(t,\tau)x|^2 \leq e^{-\sigma(t-\tau)}|x|^2 + \frac{(R_\sigma(t))^2}{2} \leq (R_\sigma(t))^2, \tag{3.40}$$

that is, $U(t,\tau)z \in B_0(t)$.

Fix $\hat{D} \in \mathfrak{D}_\sigma$, $t \in \mathbb{R}$, and $|r| \leq 1$. Take $s_0 := s_0(\hat{D}, t-1) \leq t-1$ such that

$$s \leq s_0 \Longrightarrow e^{\sigma s}(r_{\hat{D}}(s))^2 \leq e^{\sigma(t-1)}\frac{(R_\sigma(t-1))^2}{2}. \tag{3.41}$$

Take $s \leq s_0$ and $v \in D(s)$. We have two options:

- There are no jump times of $\tilde{\mathcal{U}}$ at (v, s)

  Then we have
  $$\tilde{U}(t+r, s)u = U(t+r, s)u.$$
  Equations (3.39) and (3.41) imply that
  $$\begin{aligned} |\tilde{U}(t+r,s)v|^2 &\leq e^{-\sigma(t+r-s)}(r_{\hat{D}}(s))^2 + \frac{(R_\sigma(t+r))^2}{2} \\ &\leq e^{-\sigma(t+r)}e^{\sigma(t-1)}\frac{(R_\sigma(t-1))^2}{2} + \frac{(R_\sigma(t+r))^2}{2} \\ &\leq \frac{(R_\sigma(t+r))^2}{2} + \frac{(R_\sigma(t+r))^2}{2} \\ &= (R_\sigma(t+r))^2, \end{aligned}$$
  because of the definition of R_σ. As a consequence,
  $$s \leq s_0 \Longrightarrow \tilde{U}(t+r,s)v \in B_0(t+r).$$

- There is at least one jump time of $\tilde{\mathcal{U}}$ at (v, s).

  Take τ the last one in the interval $[s, t+r]$. Then

  $$\tilde{U}(t+r, s)v = U(t+r, \tau)\tilde{U}(\tau, s)v = U(t+r, \tau)w,$$

  with $w \in I_\tau(M(\tau))$. We have that $|w| \leq 2^{-1/2}R_\sigma(\tau)$. Therefore, Equation (3.40) implies that

  $$|\tilde{U}(t+r, s)v|^2 = |U(t+r, \tau)w|^2 \leq (R_\sigma(t+r))^2.$$

As a consequence of these two options, it can be easily proved that $\tilde{\mathcal{U}}$ is pullback $\mathfrak{D}_\sigma$-dissipative with $\hat{B}_0$ a pullback $\mathfrak{D}_\sigma$-absorbing family. Corollary 3.67 implies that there exists a pullback $\mathfrak{D}_\sigma$-attractor $\hat{A} \in \mathfrak{D}_\sigma$.

Perturbations of the integrate-and-fire model

We consider for each $\eta \in [0, 1]$ the problem

$$u'(t) = -\gamma_\eta(t)u(t) + S_\eta(t), \tag{3.42}$$

with the condition

$$\text{if } u(t) = \theta_\eta(t) \text{ then } u(t) \text{ resets to } u_{r,\eta}(t) < \theta_\eta(t). \tag{3.43}$$

We suppose

(I1) $\gamma_\eta \colon \mathbb{R} \longrightarrow (0, \infty)$ is a continuous map for each $\eta \in [0, 1]$ such that

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} |\gamma_\eta(t) - \gamma_0(t)| = 0,$$
$$\gamma_0^- = \inf_{t \in \mathbb{R}} \gamma_0(t) > 0, \quad \text{and} \quad \gamma_0^+ = \sup_{t \in \mathbb{R}} \gamma_0(t) < \infty;$$

(I2) $S_\eta \colon \mathbb{R} \longrightarrow (0, \infty)$ is a continuous map for each $\eta \in [0, 1]$ such that

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} |S_\eta(t) - S_0(t)| = 0,$$
$$\text{and}$$
$$S_0^+ = \sup_{t \in \mathbb{R}} S_0(t) < \infty;$$

(I3) $\theta_\eta \colon \mathbb{R} \longrightarrow (0, \infty)$ is a $\mathcal{C}^1$ map for each $\eta \in [0, 1]$ such that

$$\lim_{\eta \to 0} \sup_{t \in \mathbb{R}} \big(|\theta_\eta(t) - \theta_0(t)| + |\theta_\eta'(t) - \theta_0'(t)|\big) = 0.$$

Also, suppose that

$$\theta_0^+ = \sup_{t\in\mathbb{R}} \theta_0(t) < \infty,$$

and that there exists $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ we have

$$|\theta_0'(t) + \gamma_0(t)\theta_0(t) - S_0(t)| > \varepsilon;$$

(I4) $u_{r,\eta}\colon \mathbb{R} \longrightarrow [0,\infty)$ is a continuous map for each $\eta \in [0,1]$ such that

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} |u_{r,\eta}(t) - u_{r,0}(t)| = 0,$$

and there exist $a, \delta > 0$ such that for all $s \in \mathbb{R}$ and $\mu \in [0,\delta]$ we have

$$u_{r,0}(s) + a < \theta_0(s+\mu).$$

For each $\eta \in [0,1]$ the initial value problem

$$\begin{cases} u'(t) = -\gamma_\eta(t)u(t) + S_\eta(t), & \text{for } t > s, \\ u(s) = x \in \mathbb{R}, \end{cases}$$

can be solved explicitly, and we obtain U_η, similarly as in Equation (3.37), that is,

$$\begin{aligned} U_\eta(t,s)x_0 = x_0 \exp&\left\{-\int_s^t \gamma_\eta(v)\,dv\right\} \\ &+ \int_s^t S_\eta(x)\exp\left\{-\int_x^t \gamma_\eta(v)\,dv\right\}\,dx, \end{aligned} \tag{3.44}$$

for $t > s$ and $z \in \mathbb{R}$. Define $M_\eta(t) = \{\theta_\eta(t)\}$ and $I_t^\eta(\theta_\eta(t)) = u_{r,\eta}(t)$ for each $t \in \mathbb{R}$ and $\eta \in [0,1]$.

Using Conditions (I1)–(I4), there exists $\eta_1 \in (0,1]$ such that, for all $\eta \in [0,\eta_1]$ we have

$$\begin{cases} \sup_{t\in\mathbb{R}} |\gamma_\eta(t) - \gamma_0(t)| < \dfrac{\gamma_0^-}{2}, & \sup_{t\in\mathbb{R}} |S_\eta(t) - S_0(t)| < \dfrac{S_0^+}{2}, \\ \sup_{t\in\mathbb{R}} |\theta_\eta(t) - \theta_0(t)| < \min\left\{\dfrac{\theta_0^+}{2}, \dfrac{a}{4}\right\}, & \sup_{t\in\mathbb{R}} |u_{r,\eta}(t) - u_{r,0}(t)| < \dfrac{a}{2}, \\ \sup_{t\in\mathbb{R}} \left|\theta_\eta'(t) + \gamma_\eta(t)\theta_\eta(t) - S_\eta(t) - \theta_0'(t) - \gamma_0(t)\theta_0(t) + S_0(t)\right| < \dfrac{\varepsilon}{2}. & \end{cases} \tag{3.45}$$

Proposition 3.134. *Suppose that* (I1)–(I4) *holds and* η_1 *is as in* (3.45). *Then* $\tilde{\mathcal{U}}_\eta = (\mathcal{U}_\eta, \mathbb{R}, \hat{M}_\eta, I^\eta)$ *satisfies the Conditions* (H1)–(H4) *for each* $\eta \in [0,\eta_1]$.

This implies that we have $\tilde{\mathcal{U}}_\eta = (\mathcal{U}_\eta, \mathbb{R}, \hat{M}_\eta, I^\eta)$ an impulsive evolution process. Furthermore, Conditions (I), (H), and (T) are also satisfied for every $\eta \in [0, \eta_1]$.

Consider the universe of union bounded families $\mathfrak{D}_b$ (see Remark 3.82).

Proposition 3.135. *The impulsive evolution process $\tilde{\mathcal{U}}_\eta$ is pullback $\mathfrak{D}_b$-dissipative.*

Proof. Take $\hat{D} \in \mathfrak{D}_b$. Then we have that there exists $M > 0$ such that

$$\bigcup_{t \in \mathbb{R}} D(t) \subset [-M, M].$$

For $x \in D(s)$ and $t \geq s$, we have

$$|U(t,s)x| \leq M \exp\left\{-\frac{\gamma_0^-}{2}(t-s)\right\} + \frac{3S_0^+}{\gamma_0^-}\left(1 - \exp\left\{-\frac{\gamma_0^-}{2}(t-s)\right\}\right).$$

This last term converges to

$$\frac{3S_0^+}{\gamma_0^-}$$

as $s \longrightarrow -\infty$. Therefore, taking

$$\omega = 2 \max\left\{\theta_0^+, \frac{3S_0^+}{\gamma_0^-}\right\},$$

the interval $[-\omega, \omega]$ is pullback $\mathfrak{D}_b$-absorbing family. □

The fact that $\tilde{\mathcal{U}}_\eta$ is pullback $\mathfrak{D}_b$-asymptotically compact follows from the fact that we are working on a finite dimensional space. This implies that we have:

Corollary 3.136. *Assume that* (I1)–(I4) *hold, η_1 is as in* (3.45) *and ω as in the proof of Proposition* 3.135. *For each $\eta \in [0, \eta_1]$, the impulsive evolution process $\tilde{\mathcal{U}}_\eta = (\mathcal{U}_\eta, \mathbb{R}, \hat{M}_\eta, I^\eta)$ has a collectively compact pullback $\mathfrak{D}_b$-attractor $\hat{A}_\eta$, with $\hat{A}_\eta(t) \subset [-\omega, \omega]$ for each $t \in \mathbb{R}$.*

We verify that the conditions of Theorem 3.81 hold.

Proposition 3.137. *There exists $\xi > 0$ such that*

$$\phi_\eta(u_{r,\eta}(s), s) \geq 2\xi \quad \text{for all } s \in \mathbb{R} \text{ and } \eta \in [0, \eta_1].$$

Proof. We take $\alpha > 0$ such that, for $\lambda \in [0, \alpha]$, we have

$$\int_0^\lambda S_\eta(s + \lambda - u) \exp\left\{-\int_{s+\lambda-u}^{s+\lambda} \gamma_\eta(v)\, dv\right\} du \leq \frac{3S_0^+\lambda}{2} \leq \frac{a}{4}$$

for every $s \in \mathbb{R}$ and $\eta \in [0, \eta_1]$. This implies that

$$\begin{aligned} U_\eta(s+\lambda, s)u_{r,\eta}(s) &= u_{r,\eta}(s) \exp\left\{-\int_s^{s+\lambda} \gamma_\eta(v)\, dv\right\} \\ &\quad + \int_s^{s+\lambda} S_\eta(x) \exp\left\{-\int_x^{s+\lambda} \gamma_\eta(v)\, dv\right\} dx \\ &\leq u_{r,\eta}(s) + \int_0^{\lambda} S_\eta(s+\lambda-u) \exp\left\{-\int_{s+\lambda-u}^{s+\lambda} \gamma_\eta(v)\, dv\right\} du \\ &\leq u_{r,\eta}(s) + \frac{a}{4}, \end{aligned}$$

for every $s \in \mathbb{R}$. As a consequence, if $\lambda \in [0, \min\{\delta, \alpha\}]$, we have

$$U_\eta(s+\lambda, s)u_{r,\eta}(s) < \theta_\eta(s+\lambda).$$

Therefore, $\phi(u_{r,\eta}(s), s) > \min\{\delta, \alpha\} > 0$ for all $s \in \mathbb{R}$. Finally, taking

$$\xi = \frac{\min\{\delta, \alpha\}}{2}$$

we finish the proof. □

The families $\{\hat{M}_\eta\}_{\eta\in[0,1]}$ and $\{I^\eta\}_{\eta\in[0,1]}$ are collectively closed at $\eta = 0$ and collectively continuous at $\eta = 0$, respectively, because for any sequences $\{\eta_k\}_n$ and $\{t_k\}_k$ convergent to 0 and t, we have that

$$\theta_{\eta_k}(t_k) \longrightarrow \theta_0(t) \quad \text{and} \quad u_{r,\eta_k}(t_k) \longrightarrow u_{r,0}(t),$$

Moreover, the family of evolution processes $\{\mathcal{U}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$.

Hence, to apply Theorem 3.81, it only remains to show that Condition (CT) holds.

Proposition 3.138. *The family* $\{\tilde{\mathcal{U}}_\eta\}_{\eta\in[0,1]}$ *satisfies Condition* (CT).

Proof. We take $s \in \mathbb{R}$, $t > s$, $\{\eta_n\}$ a sequence convergent to 0, and $\{z_n\}_n$ a convergent sequence to z such that $U_{\eta_n}(t, s)z_n \longrightarrow \theta_0(t)$ (therefore we have that $U_0(t, s)z = \theta_0(t)$). We have to prove that there exist two subsequences, $\{\eta_{n_k}\}_k$ of $\{\eta_n\}_n$ and $\{z_{j_k}\}_k$ of $\{z_n\}_k$, and a sequence $\{\alpha_k\}_k$ such that $t + \alpha_k \geq s$, $\alpha_k \longrightarrow 0$, and $U_{\eta_{n_k}}(t + \alpha_k, s)z_{n_k} = \theta_{\eta_{n_k}}(t + \alpha_k)$ for each k.

Condition (I3) implies that

$$\theta_0'(t) + \gamma_0(t)\theta_0(t) - S_0(t) > 0 \quad \text{for every } t \in \mathbb{R},$$

or

$$\theta_0'(t) + \gamma_0(t)\theta_0(t) - S_0(t) < 0 \quad \text{for every } t \in \mathbb{R}.$$

We assume that $\theta_0'(t) + \gamma_0(t)\theta_0(t) - S_0(t) > 0$ for every $t \in \mathbb{R}$ (the proof in the other case is analogous). We define the map $f(r) = U_0(r,s)z - \theta_0(r)$ for $r \geq s$. This implies that f is differentiable, $f(t) = 0$ and

$$f'(r) = \frac{d}{dr}(U_0(r,s)z - \theta_0(r)) = -\gamma_0(r)U_0(r,s)z + S_0(r) - \theta_0'(r).$$

This implies that

$$f'(t) = -\gamma_0(t)U_0(t,s)z + S_0(t) - \theta_0'(t) = -\gamma_0(t)\theta_0(t) + S_0(t) - \theta_0'(t) < 0.$$

As a consequence there exists $\lambda_0 > 0$ with $t - s > \lambda_0$ and

$$U_0(t-\lambda,s)z > \theta_0(t-\lambda) \quad \text{ and } \quad U_0(t+\lambda,s)z < \theta_0(t+\lambda),$$

for all $\lambda \in (0, \lambda_0]$. Therefore, for n sufficiently large we have

$$U_{\eta_n}(t-\lambda_0,s)z_n > \theta_{\eta_n}(t-\lambda_0) \text{ and } U_{\eta_n}(t+\lambda_0,s)z_n < \theta_{\eta_n}(t+\lambda_0).$$

This implies that there is $\alpha_n \in [-\lambda_0, \lambda_0]$ such that $U_{\eta_n}(t+\alpha_n,s)z_n = \theta_{\eta_n}(t+\alpha_n)$. Finally, we will prove that $\{\alpha_n\}_n$ converges to 0 up to a subsequence. We know that $\alpha_n \in [\lambda_0, \lambda_0]$, so there exists a convergent subsequence, which will be denoted the same, such that $\alpha_n \longrightarrow \bar{\alpha}$. The family $\{\hat{M}_\eta\}_{\eta\in[0,1]}$ is collectively closed at $\eta = 0$, which implies that $U_0(t+\bar{\alpha},s)z = \theta_0(t+\bar{\alpha})$. But we know that $U_0(t+\lambda,s)z \neq \theta_0(t+\lambda)$ for every $\lambda \in [-\lambda_0,\lambda_0] \setminus \{0\}$. As a consequence, we get that $\bar{\alpha} = 0$. □

All the hypotheses of Theorem 3.81 are satisfied, so the family $\{\hat{A}_\eta\}_\eta$ of pullback $\mathfrak{D}_b$-attractors is upper semicontinuous at $\eta = 0$.

Multivalued situation

Let $F\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ be given by

$$F(t,x) = \begin{cases} -|\cos(t)|x, & |x| > 1, \\ x\left(|\cos(t)| - 1/2\right) - 1/2, & -1 < x < 0, \\ x\left(|\cos(t)| - 1/2\right) + 1/2, & 0 < x < 1, \\ [-|\cos(t)|, |\cos(t)|], & x \in \{-1,1\}, \\ [-1/2, 1/2], & x = 0; \end{cases}$$

and consider the ordinary differential inclusion

$$x'(t) \in F(t,x(t))$$

The solutions of the differential inclusion are absolutely continuous functions. Given an initial data (τ, x_τ), we will say that $x\colon [\tau,+\infty) \longrightarrow \mathbb{R}$ is a solution with initial data (τ, x_τ) if x is absolutely continuous, it satisfies the inclusion for almost every $t \geq \tau$, and $x(\tau) = x_\tau$.

We have uniqueness of solution if $x_\tau \neq 0$. If $x_\tau \notin \{-1, 0, 1\}$, then the solution is given by the solution of the differential equation until it reaches 1 or -1 (depending on the sign of the initial condition). When it reaches 1 or -1, the solution of the differential inclusion stays at that point. If $|x_\tau| = 1$, then the unique solution is the constant function $x(t) = x_\tau$ for all $t \geq \tau$.

If $x_\tau = 0$ then we have infinite many solutions. For any $T > \tau$, we have the following solutions:

$$\begin{cases} x(t) = 0 \quad \text{for all } t \geq \tau, \\ x(t) = \begin{cases} 0, & \tau \leq t \leq T, \\ \alpha(t), & T \leq t \leq T^*, \\ 1, & T^* \leq t. \end{cases} \\ x(t) = \begin{cases} 0, & \tau \leq t \leq T, \\ \beta(t), & T \leq t \leq T^*, \\ 1, & T^* \leq t. \end{cases} \end{cases}$$

where α denotes the solution of $x' = x(|\cos(t)| - 1/2) + 1/2$ with initial data $x(T) = 0$ and T^* is the time that α reaches 1 (respectively for β and the equation defined for values between -1 and 0 and the time it reaches -1).

We have an exact generalized process. For any $t \in \mathbb{R}$, let

$$M_t = \left\{ \frac{6 + \arctan(t)}{4} \right\}, \qquad I_t(x) = \{5 + \sin(t), 3\}.$$

We take $\mathfrak{D}$ the universe of all time-dependent families $\hat{D}$ such that there exists a bounded set D with $D(t) \subset D$ for all $t \in \mathbb{R}$. It is easy to see that all conditions of an impulsive generalized process are satisfied, it is pullback $\mathfrak{D}$-asymptotically compact, and pullback $\mathfrak{D}$-dissipative. Furthermore, Conditions (H-Mult), (I-Mult), and (NT-Mult) are also fulfilled, so we there exists a pullback $\mathfrak{D}$-attractor. It is not hard to see that the pullback $\mathfrak{D}$-attractor is given by

$$A(t) = [-1, 1] \cup \left[\frac{6 + \arctan(t)}{4}, 5 + \sin(t) \right].$$

Consider the nonautonomous differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (\tau,+\infty)\times(0,1), \\ u(0,t) = 0 = u(1,t), \\ u(\tau,x) = u_\tau(x), \end{cases} \tag{3.46}$$

where $b\colon \mathbb{R} \longrightarrow [0,\infty)$, $\omega\colon \mathbb{R} \longrightarrow [0,\infty)$ are continuous functions with

$$0 < b_0 \le b(t) \le b_1, \quad 0 \le \omega_0 \le \omega(t) \le \omega_1,$$

and H_0 is the Heaviside function, that is,

$$H_0(u) = \begin{cases} -1, & u < 0, \\ [-1,1], & u = 0, \\ 1, & u > 0. \end{cases}$$

This problem and similar ones have been studied, for example, in [6, 33, 117], where some results on the structure of the attractor were obtained.

We say that a continuous map $u\colon [\tau,+\infty) \longrightarrow L^2(0,1)$ is a strong solution of (3.46) if

1. $u(\tau) = u_\tau$,
2. For any $\delta > 0$ and $T > t+\delta$, u is absolutely continuous on $[t+\delta, T]$ and $u(t) \in H^2(0,1) \cap H_0^1(0,1)$ for almost all $t \in (\tau, T)$,
3. there exists $r\colon [\tau,+\infty) \longrightarrow L^2(0,1)$ such that:
    - $r(t) \in L^2(0,1)$,
    - $r(t)(x) \in b(t)H_0(u(t,x)) + \omega(t)u(t,x)$ for almost all $x \in (0,1)$,
    - $r \in L^2(\tau,T;L^2(0,1))$ for any $T > \tau$,
    - $\dfrac{du}{dt} - \Delta u = r(t)$, for almost all $t \in (\tau,+\infty)$.

Theorem 3.139 (Theorem 1 in [33])**.** *For any $u_\tau \in L^2(0,1)$, problem* (3.46) *has at least one strong solution.*

It is also proved that we have a continuous and exact generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t\in\mathbb{R}}$.

Let $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ a collectively closed family of sets with $M(t) \subset L^2(0,1)$ for all $t \in \mathbb{R}$ and $I = \{I_t\colon M(t) \longrightarrow \mathscr{P}(X)\}_{t\in\mathbb{R}}$ a collection of collectively upper semicontinuous multifunctions which are compact-valued such that $(\mathscr{G}, \hat{M}, I)$ is an impulsive generalized process. We will assume that $\hat{M}$ and I satisfy Conditions (H-Mult), (I-Mult), and (NT-Mult) and that the impulsive generalized process is pullback $\mathfrak{D}$-dissipative and pullback $\mathfrak{D}$-asymptotically compact, and it satisfies (G3$'$). Then we can say that there exists a pullback $\mathfrak{D}$-attractor $\hat{A}$. For example, if we assume that $\|I_t(u)\|^2 \leq C$ for some $C > 0$ and for all $t \in \mathbb{R}$ and $u \in M(t)$ we would have the pullback $\mathfrak{D}$-dissipativeness and the pullback $\mathfrak{D}$-asymptotically compactness.

Impulses in driving semigroups

The following system is similar to a problem considered in [112]. Take $\Omega \subset \mathbb{R}^3$ a bounded domain with smooth boundary $\partial\Omega$, and we consider

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = g(y(t)), & (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ \dfrac{dy}{dt} = f(y), & t > 0, \\ y(0) = y_0 \in \mathbb{R}, \end{cases}$$

with $g \colon \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function and $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ globally Lipschitz. Suppose that

$$\begin{cases} \dfrac{dy}{dt} = f(y), & t > 0, \\ y(0) = y_0, \end{cases}$$

has a unique solution $y(\cdot, y_0) \colon [0,\infty) \longrightarrow \mathbb{R}$ and that

$$(t, y_0) \in [0,\infty) \times \mathbb{R} \longrightarrow \theta(t)y_0 = y(t, y_0) \in \mathbb{R}$$

is a continuous semigroup in $\mathbb{R}$. Take $M \subset \mathbb{R}$ and $I \colon M \longrightarrow \mathbb{R}$ such that $(\theta, \mathbb{R}, M, I)$ is an impulsive dynamical system satisfying $(\mathrm{H}_{\mathrm{aut}})$ and which has a global attractor Ξ. We consider the restriction of $\tilde{\theta}$ to $\Sigma = \Xi$.

Lemma 3.140. *Define $g_\psi(t) = g(\psi(t))$, with ψ a global solution in Ξ. Then the map g_ψ is measurable, and there exists $k \geq 0$, which is independent of ψ, such that*

$$|g_\psi(t)| \leq k \quad \text{for all } t \in \mathbb{R}. \tag{3.47}$$

Moreover, $g_\psi \in L^2_{loc}(\mathbb{R})$, and each g_ψ is translation bounded, that is,

$$\sup_{h \in \mathbb{R}} \int_h^{h+1} |g_\psi(s)|^2 \, ds \leq k^2.$$

Proof. Any global solution ψ is a piecewise continuous function, so the composition $g \circ \psi$ is measurable because g is continuous. Moreover, Ξ is a compact set and g is continuous, so the rest of the results follow. □

Following [112, Theorems 3.1 and 3.4], for every global solution ψ in Ξ, $s \in \mathbb{R}$, $T > 0$, and $p \geq 2$, the problem

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = g(\psi(t)), & (t,x) \in (s, s+T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (s, s+T) \times \partial\Omega, \\ u(s,x) = u_0(x), & x \in \Omega, \end{cases}$$

has a unique solution $u(\cdot, s, u_0, \psi)$, which belongs to

$$\mathcal{C}([s, s+T], L^2(\Omega)) \cap L^2(s, s+T; H_0^1(\Omega)) \cap L^p(s, s+T; L^p(\Omega)).$$

Moreover, for every $u_0, v_0 \in L^2(\Omega)$ and $t \in [s, s+T]$, we get

$$\|u(t, s, u_0, \psi) - v(t, s, v_0, \psi)\|_{L^2(\Omega)} \leq \|u_0 - v_0\|_{L^2(\Omega)}.$$

Finally, there exists B_0 such that for every bounded subset B of $L^2(\Omega)$, there exists $t_0 \geq 0$ such that

$$\varphi_c(t-s, \psi(s))B \subset B_0 \quad \text{for } t - s \geq t_0.$$

Therefore, the couple impulsive cocycle φ_c is uniformly dissipative and uniformly asymptotically compact. Then, it has a uniform attractor A in $L^2(\Omega)$. Finally, the associated evolution process U_ψ considered in (3.31) has a pullback attractor A_ψ.

Results and Conclusions

In this manuscript, we have studied both differential equations and dynamical systems with impulses.

In the second chapter, we focus on the theory of impulsive differential equations. After an introduction which includes an analysis of properties of impulsive differential equations, we obtain several existence results of boundary value problems for impulsive differential equations. In particular, we can consider Theorems 2.18, 2.20, 2.34, 2.39, and 2.42 for the case of impulses at fixed times, and Theorems 2.45, 2.60, and 2.62 for the case of impulses at variable times.

In the third chapter, we develop the theory of impulsive dynamical systems. We work mainly in the nonautonomous case. First, we introduce the notion of impulsive evolutions processes (see Definitions 3.26 and 3.31). Then, we study the existence of a pullback $\mathfrak{D}$-attractor (see Definition 3.35) under the assumptions of pullback $\mathfrak{D}$-asymptotical compactness, pullback $\mathfrak{D}$-dissipativeness, and some extra conditions (see Subsection 3.3.1 for the nonautonomous tube conditions and Subsection 3.3.2 for Conditions (T) and (NT)). Therefore, we obtain Theorems 3.62 and 3.70.

Then, we study the continuity of attractors of impulsive evolution processes, obtaining a result about the upper semicontinuity (Theorem 3.81) and a weak version of lower semicontinuity (Theorem 3.92). We consider next the multivalued case, extending the results of Sections 3.2 and 3.3. Finally, we study the case of impulses in the driving semigroup.

Resumo

A presente tese de doutoramento, titulada *Impulses in Differential Equations and Dynamical Systems*, ten como obxectivo realizar un estudo sobre certos sistemas que presentan un comportamento que se caracteriza pola aparición de cambios abruptos no seu estado en determinados instantes de tempo. Estes cambios repentinos adoitan ocorrer cando o estado do sistema alcanza un determinado conxunto. Neste traballo asúmese que estas rápidas perturbacións en forma de cambios abruptos son instantáneas, é dicir, que actúan en forma de pulos, debido a que, en xeral, adoitan ter unha duración insignificante e desprezable, en comparación co resto do proceso.

É ben coñecido que tanto as ecuacións diferenciais como os sistemas dinámicos son unha das ferramentas fundamentais á hora de modelar unha gran cantidade de fenómenos que xorden na natureza, na tecnoloxía, en enxeñarías e en diversas ciencias. Este é un dos motivos da súa elevada importancia e do seu grande interese.

Desta forma, o obxecto de estudo deste traballo serán as ecuacións diferenciais con pulos e os sistemas dinámicos con pulos. Por exemplo, multitude de fenómenos biolóxicos que involucran limiares, como pode ser a dinámica de poboacións, certos modelos en medicina, en farmacoloxía, en robótica ou en economía, teñen ás veces cambios instantáneos. Estes cambios repentinos poden ser debidos a distintos tipos de desastres naturais, a actuación de forzas externa ou un crac bolsista, por poñer algúns exemplos. Polo tanto, as ecuacións diferenciais con pulos e os sistemas dinámicos con pulos son unhas ferramentas matemáticas útiles para intentar describir a evolución deste tipo de fenómenos.

O principal obxectivo deste manuscrito é afondar no coñecemento das ecuacións diferenciais con pulos e dos sistemas dinámicos con pulos. Por unha parte, obtéñense varios resultados de existencia de solución relativos a problemas de fronteira para este tipo de ecuacións diferenciais, empregando distintas e variadas técnicas. Por outro lado, realízase un estudo sobre o comportamento asintótico deste tipo de sistemas dinámicos. En particular, estudarase o caso non autónomo e utilizarase maioritariamente a teoría dos procesos de evolución.

A continuación preséntase un breve resumo dos tres capítulos polos que está composta esta tese de doutoramento.

Capítulo 1

O primeiro capítulo é unha pequena colección de conceptos e resultados preliminares, co obxectivo de crear un traballo autocontido. Estes resultados serán utilizados ao longo do resto do traballo.

Comézase cunha breve introdución a algúns conceptos topolóxicos, así como algúns resultados básicos de análise funcional e da teoría dos puntos críticos e dos métodos variacionais. A continuación expóñense varios resultados da teoría de puntos fixos, maioritariamente resultados en dimensión finita, pero tamén se falará dalgún en dimensión infinita, así como outros en funcións multívocas. Posteriormente faise unha breve introdución a algúns resultados da teoría do grao e tamén ao que se coñece como grao de coincidencia. Estes conceptos e resultados mencionados ata agora serán utilizados fundamentalmente no segundo capítulo.

Posteriormente, faise unha pequena introdución aos sistemas dinámicos continuos, tanto no caso autónomo como no caso non autónomo, co fin de estudar o comportamento asintótico destes sistemas. Para os sistemas dinámicos autónomos introdúcese o concepto de semigrupo e algunha das súas propiedades. O obxecto matemático que xoga un papel fundamental no estudo asintótico deste tipo de sistemas é o que se coñece como atractor global. No último medio século estudáronse moitas propiedades de semigrupos en multitude de espazos (sobre todo en espazos de Banach). E o concepto de atractor global axuda a entender moitas das propiedades cualitativas das solucións destes sistemas. Posteriormente danse algúns resultados garantindo a existencia deste obxecto, así como outros relativos ao seu comportamento baixo a influencia de perturbacións, falando da semicontinuidade superior e da semicontinuidade inferior.

Para o caso de sistemas dinámicos non autónomos, introduciranse os dous puntos de vista máis utilizados, os sistemas que veñen dados por un cociclo e un semigrupo, e os procesos de evolución. Centrarémonos fundamentalmente no caso dos procesos de evolución, que serán os utilizados maioritariamente ao longo do manuscrito. A análise das propiedades de procesos de evolución é moito máis recente que o caso autónomo, e hai distintos tipos de posibilidades para estudar o comportamento asintótico. En particular, neste traballo considerarase a noción de atractor pullback. O estudo deste tipo de atractores proporciona moita información sobre o comportamento asintótico de moitos modelos en diferentes ciencias, aínda que ten algunhas desvantaxes, motivo polo que non é a única noción de atractor posible para os sistemas non autónomos. Verase a definición do atractor pullback, así como algún resultado que garanta a súa existencia e algunha das súas propiedades. Os resultados desta última parte serán utilizados fundamentalmente durante o terceiro capítulo.

Capítulo 2

O segundo capítulo está dedicado ao estudo das ecuacións diferenciais con pulos. Realízase en primeiro lugar unha breve introdución a este tipo de ecuacións diferenciais e vendo algunha das súas peculiaridades. Preséntanse algunhas similitudes e diferenzas entre os dous casos fundamentais que se tratarán no manuscrito: cando os instantes de pulo están prefixados, e cando son variables e dependen de cada solución. En particular analizaranse certos problemas e dificultades que poden xurdir, sobre todo no caso de instantes variables, que en xeral é máis complicado. Analízanse algunhas opcións a considerar como espazos de funcións nos que buscar solucións e outro tipo de fenómenos non desexables que poden e adoitan aparecer neste tipo de ecuacións. Como espazo de funcións, por exemplo, falarase do espazo de funcións con descontinuidades en instantes prefixados, das funcións reguladas, e tamén se verá a construción dun espazo ad-hoc para este tipo de problemas, que será utilizado máis adiante.

A continuación pásase a expoñer algúns resultados relativos á existencia de solucións para algúns problemas de fronteira para ecuacións diferenciais con pulos. Nunha primeira parte considerarase o caso onde os pulos ocorren en instantes fixados previamente. Estúdanse problemas tanto de primeira como de segunda orde, e obtéñense fundamentalmente resultados relativos á existencia de solución. En particular, en primeiro lugar considerarase un problema de primeira orde que incorpora singularidades e pulos, baseándonos como modelo inicial na ecuación diferencial

$$\begin{cases} x'(t) = -\dfrac{1}{(x(t))^\alpha} + e(t), & t \neq t_j, \\ \Delta x(t_j) = I_j(x(t_j)), & j \in \{1, \dots, q\}, \end{cases}$$

onde α é un número positivo, e trátase dunha función continua e T-periódica, os puntos t_j son coñecidos e cumpren $0 < t_1 < \cdots < t_q < T$ e as funcións I_j tamén son continuas. O obxectivo consiste en atopar condicións para que este problema teña solucións periódicas, é dicir, $x(0) = x(T)$. A continuación consideraranse distintas xeneralizacións da ecuación diferencial involucrada, para incluír outros casos máis xerais, podendo incorporar por exemplo singularidades máis complicadas, pero tamén para outros tipos de funcións non lineais. Posteriormente estúdanse varios problemas de segunda orde. Nun primeiro momento intentarase aplicar técnicas variacionais para intentar garantir a existencia de solución. Esta técnica consiste en obter unha relación entre as solucións do problema diferencial con pulos e os puntos críticos dunha certa función real asociada definida nun espazo de funcións, en xeral nun espazo de Banach ou Hilbert. A continuación a idea é utilizar resultados relativos á existencia de puntos críticos. En particular aplicaranse técnicas variacionais a algúns problemas de segunda orde que non teñen unha estrutura variacional aparente, por exemplo o caso de problemas onde a derivada aparece no termo non lineal e con pulos na derivada como

$$\begin{cases} -x''(t) + a(t)x(t) = f(t, x(t), x'(t)), & t \neq t_j, \\ \Delta x'(t_j) = I_j(x(t_j)), & j \in \{1, \dots, q\}, \\ x(0) = 0 = x(T), \end{cases}$$

con f e I_j funcións continuas e a unha función en L^∞. Neste caso particular danse condicións para garantir a existencia de solucións, utilizando na demostración unha mestura de técnicas variacionais e da teoría de puntos críticos con algúns resultados de puntos fixos. Para rematar a sección, considéranse outros problemas de fronteira de segunda orde e tamén se estuda a existencia de solucións. Realízase unha análise da estrutura do conxunto de solucións para problemas de valor inicial asociados, en particular vendo que o conxunto das solucións é un conxunto R_δ. A continuación, utilizando algún resultado de punto fixo en dimensión finita, pero no caso de funcións multívocas, lógrase probar a existencia de solucións baixo diversas hipóteses.

Na seguinte sección continúase a considerar problemas de fronteira para ecuacións diferenciais con pulos, pero centrándose agora no caso onde os pulos ocorren en instantes que dependen da solución. Como xa foi comentado, este tipo de problema é máis complicado de estudar que o anterior. Comézase analizando un espazo de funcións, xa comentado na primeira sección do capítulo, e vendo algunhas propiedades do conxunto das solucións para os problemas de valor inicial asociados, en particular o feito de que o conxunto das solucións tamén vai ser un conxunto R_δ neste novo espazo de funcións, ao igual que sucedía no caso no que pulos ocorrían en instantes prefixados. A continuación, utilizando unha técnica completamente análoga á utilizada no final da sección anterior, considérase un problema periódico de primeira orde da forma

$$\begin{cases} x'(t) = f(t, x(t)), & t \neq \tau_j(x(t)), \\ \Delta x(t) = I_j(x(t)), & t = \tau_j(x(t)), \end{cases}$$

e obtense a existencia de solucións. Posteriormente, e para rematar esta sección e o capítulo, centrarémonos na busca de solucións periódicas para un problema da forma

$$x''(t) + g(x(t)) = p(t, x(t), x'(t)),$$

sendo g unha función continua e p unha función periódica, continua e limitada. Ao igual que o caso anterior, os pulos dependerán das solucións, e polo tanto produciranse en instantes de tempo variables. No caso particular que será estudado, os pulos ocorrerán nos instantes que resolvan as ecuacións

$$t = \tau_j(x(t), x'(t)), \ j \in \{1, \dots, q\},$$

da mesma forma que antes. Para estudar este problema, analízase a ecuación diferencial, e en particular a aplicación coñecida como time-map, que está relacionada co caso autónomo desta mesma ecuación diferencial. Esta aplicación será fundamental á hora de intentar probar a existencia de solucións periódicas. Analizaranse diversas propiedades do time-map, que dependerán fundamentalmente das características da función g. Dependendo de certas propiedades de g, por exemplo, como é o seu comportamento en infinito, probarase a existencia de solucións periódicas para este problema de distintas formas. Para obter estes resultados faise necesario un estudo dalgunhas propiedades cualitativas da aplicación de Poincaré que ten asociada este problema.

Para todos os distintos resultados de existencia de solucións probados neste capítulo, póñense diferentes exemplos de casos particulares onde as hipóteses son verificadas, e polo tanto onde se poden aplicar os resultados. Todos estes exemplos están repartidos ao longo do capítulo, así como unha comprobación rápida de que se verifican as hipóteses pedidas.

Capítulo 3

No terceiro e derradeiro capitulo, considéranse os sistemas dinámicos con pulos, centrándonos en particular no caso non autónomo. Comézase cunha pequena introdución aos sistemas dinámicos autónomos con pulos, dando unha colección de definicións así como algúns resultados recentes relativos ao estudo de atractores e certas propiedades, en especial o caso do atractor global para este tipo de sistemas, que ten unha definición similar á do caso continuo, pero non igual. A continuación pásase ao caso non autónomo, nun primeiro momento no marco dos procesos de evolución. Defínense as traxectorias impulsivas, os procesos de evolucións impulsivos e vense uns resultados básicos sobre a chamada "aplicación de tempo de impacto". Tamén se intenta facer unha adaptación dalgúns resultados teóricos do caso continuo a esta nova situación, facendo as modificacións necesarias en moitos casos.

Posteriormente pasa a estudarse a existencia do atractor pullback para este tipo de procesos de evolución. En primeiro lugar dáse unha definición do atractor pullback, remarcando as diferenzas coa definición habitual no caso continuo. Nesta nova situación, por exemplo, non se vai poder garantir a unicidade do atractor pullback, aínda que si que será posible obter un resultado de unicidade salvo unha pequena parte. A continuación considéranse os pullback ω-límites para este tipo de problemas e os conceptos de pullback asintoticamente compacto e pullback disipativo, que tamén contan con definicións diferentes ás do caso continuo. Cabe destacar que no caso continuo, se o proceso de evolución é pullback asintoticamente compacto e pullback disipativo, entón está garantida a existencia do atractor pullback. De feito trátase dunha equivalencia. Non obstante, nos procesos de evolución con pulos, este resultado non é certo. Polo tanto faise necesaria unha análise das distintas relacións que hai entre estes conceptos para os procesos de evolución con pulos. En particular, conseguir probar que o atractor pullback ten a propiedade da invariancia non é doado, e faise necesario engadir algunhas hipóteses máis. Nun primeiro momento defínense as "condicións de tubo" no marco dos procesos de evolución. Condicións similares xa foran consideradas no caso autónomo para obter certas propiedades da "aplicación de tempo de impacto", e ademais para probar que o candidato a atractor global era invariante. As "condicións de tubo" definidas aquí permiten obter as propiedades análogas para os procesos de evolución, e ademais tense que conxuntos similares aos pullback ω-límites son invariantes. Desta forma tamén o será o atractor pullback. Posteriormente, utilizando técnicas algo diferentes considéranse outras condicións, algo máis débiles que as "condicións de tubo", que permiten obter resultados moi similares. Todas as condicións pedidas para probar que o atractor pullback é invariante están baseadas no comportamento do proceso de evolución preto do conxunto no cal que se producen os pulos, o que non adoita ser doado de comprobar na práctica. Ademais, e a pesar de todo, as condicións pedidas non lograr ser óptimas, xa que é posible atopar atractores pullback sen que se cumpran ningunhas destas diferentes hipóteses para garantir a existencia.

A seguinte sección está dedicada ao estudo das perturbacións do atractor pullback en procesos de evolución con pulos. Dito doutro modo, trátase de intentar probar que se se considera un problema parecido entón o atractor tamén debería ser parecido e ademais terá propiedades similares. Centrarémonos no caso da semicontinuidade superior e da semicontinuidade inferior, que tamén son coñecidas como as propiedades de "non explosión" e "non implosión" dos atractores, respectivamente. A semicontinuidade superior quere dicir que o atractor do problema perturbado non pode converterse repentinamente en algo moito máis grande que o atractor do problema non perturbado, mentres que a semicontinuidade inferior indica que o atractor do problema perturbado non pode converterse repentinamente en algo moito máis pequeno que o atractor do problema non perturbado. Unha vez máis, haberá que realizar as adaptacións pertinentes a partir do caso continuo. Daranse condicións teóricas para garantir a semicontinuidade superior, que adoita ser a máis habitual e tamén máis sinxela de probar. Por outro lado, obtense unha versión "débil" da semicontinuidade inferior, xa que non foi posible lograr a semicontinuidade inferior. O motivo foron as descontinuidades que presentan os procesos de evolución con pulos.

Na quinta sección do capítulo estenderanse os resultados da segunda e terceira sección do capítulo, que foron vistos para procesos de evolución definidos univocamente, ao caso de procesos de evolución multívocos, que obviamente presenta maior dificultade. Nesta sección emprégase a noción máis habitual de proceso xeneralizado para construír as traxectorias, e ademais a aplicación pulo tamén pode ser multívoca. Porén, a adaptación das hipóteses da segunda e terceira seccións non son suficientes para lograr os resultados desexados, e faise necesario engadir hipóteses adicionais na mesma definición do proceso xeneralizado. A sección remata obtendo resultados análogos aos obtidos na segunda e terceira seccións sobre atractores pullback, como son a existencia do atractor pullback e certas propiedades dos pullback ω-límites.

Na sexta sección considérase outro tipo de sistemas dinámicos non autónomos, que non teñen que ser procesos de evolución. Utilízase a formulación con un semigrupo e un cociclo, que xa fora comentada brevemente no primeiro capítulo. Esta formulación permite obter outro tipo de resultados sobre o comportamento asintótico dos sistemas non autónomos. Ata o de agora, nesta formulación xa fora estudado o caso no que os pulos se producían no cociclo. Nesta sección considérase o caso no que os pulos se producen no semigrupo, e non no cociclo. Esta situación podería ter aplicacións para, dalgunha forma, poder facer correccións no semigrupo, e desta maneira intentar controlar o comportamento do cociclo, que habitualmente é o obxecto que máis interesa estudar. Ademais, considérase un proceso de evolución asociado a partir do semigrupo e do cociclo, e estúdanse distintas nocións de atractor para este tipo de problemas, como o atractor pullback dese proceso de evolución asociado ou o atractor uniforme. Finalmente, obtense un resultado de existencia para estes atractores e vese a relación que existe entre eles.

Para concluír, na última sección do capítulo proporciónanse diferentes exemplos onde se aplican os distintos resultados teóricos que foron obtidos nas seccións anteriores. A pesar do carácter práctico desta sección, comézase coa demostración de tres resultados que serven de axuda para lograr probar a compacidade asintótica e a disipatividade nos sistemas dinámicos con pulos. Estas propiedades son obtidas a partir de propiedades do sistema continuo asociado, e das aplicacións que son responsables dos pulos. Os resultados son útiles nas aplicacións. A continuación estúdase un modelo neuronal de tipo “integrate-and-fire”, obtendo a existencia do atractor pullback nun universo. Posteriormente considérase unha ecuación de Navier-Stokes autónoma en dimensión 2 con pulos dependendo do tempo, obtendo tamén a existencia do atractor pullback. Despois obtense un resultado relativo á semicontinuidade superior para o mesmo modelo neuronal comentado anteriormente, con algunhas simplificacións. Ademais, aplícanse os resultados relativos aos procesos de evolución multívocos a unha inclusión diferencial de dimensión 1 e a un modelo reacción-difusion non autónomo, que involucra á función de Heaviside. Finalmente, e como aplicación dos resultados relativos aos sistemas dinámicos non autónomos involucrando o cociclo e o semigrupo, considérase un sistema en forma de fervenza. Este sistema está formado por unha ecuación da calor e unha ecuación diferencial ordinaria de dimensión 1, que será o lugar onde se producen os pulos, e dalgunha forma, “controla” á ecuación da calor.

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Many systems present a dynamic behavior that is characterized by the fact that at certain times they undergo a sudden change throughout their evolution. This situation is going to be studied in this memory. On the one hand, different techniques will be used to study some boundary value problems for impulsive differential equations. This type of differential equations presents new and unexpected behaviors even in simple cases. On the other hand, the asymptotic behavior and attractors for dynamical systems with impulses will also be studied, mainly the case of evolution processes.