FIRST ORDER DIFFERENCE EQUATIONS WITH MAXIMA AND NONLINEAR FUNCTIONAL BOUNDARY VALUE CONDITIONS

FERHAN M. ATICI[1], ALBERTO CABADA[2], AND JUAN B. FERREIRO[3]

Abstract. This paper is devoted to the existence of solutions for a problem of first order difference equations with maxima and with nonlinear functional boundary value conditions. Such boundary conditions include, among others, initial, periodic, antiperiodic and multipoint boundary value conditions, as particular cases.

AMS No: 39A10.
Keywords: Upper and lower solutions; Functional boundary value conditions

1. Introduction

Numerous processes in Physics, Engineering or Biology [11, 12] are modelled by an ordinary differential equation type

$$x'(t) = f(t, x(t), \max_{t-h \leq s \leq t} x(s)) \, , t \in [0, T].$$

These class of delay functional equations are known as equation with maxima, and on it the behavior of a solution in a time t depends on the maximum value reached by this solution in a previous interval $[t-h, t]$, where $h > 0$ is a delay parameter.

In practical situations to find the solution of the equation is very difficult, so we consider the analogous discrete of the differential equation that is used as model. This leads to the difference equations.

In particular, we study the following first order difference implicit equation with maxima

$$\Delta u_k = f\left(k, u_{k+1}, \max_{l \in \{k-h+1, \dots, k+1\}} u_l\right), \; k \in I, \tag{1.1}$$

$$u_k = \varphi(k, u), \; k \in I_h, \tag{1.2}$$

where $\Delta u_k = u_{k+1} - u_k$, $I = \{0, 1, \dots, T-1\}$, $I_h = \{-h+1, \dots, -1\}$, $f \in \mathcal{C}(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\varphi \in \mathcal{C}(I \times \mathbb{R}^{T+h}, \mathbb{R})$.

A solution of this equation will be an element $u = (u_{-h+1}, \dots, u_0, \dots, u_T)$ of $\mathbb{R}^{T+h}$, satisfying (1.1) and (1.2).

We look for solutions that satisfy nonlinear functional boundary conditions type

$$B(u_0, u) = 0,$$

with $B : \mathbb{R} \times \mathbb{R}^{T+h} \to \mathbb{R}$ a continuous function.

To be concise, the considered problem is the following.

$$\Delta u_k = f\left(k, u_{k+1}, (\phi u)_{k+1}\right), \; k \in I, \tag{1.3}$$

$$u_k = \varphi(k, u), \; k \in I_h, \tag{1.4}$$

$$B(u_0, u) = 0, \tag{1.5}$$

where function $\phi : \mathbb{R}^{T+h} \longrightarrow \mathbb{R}$ is defined by

$$(\phi u)_k := \max_{l \in \{k-h,\dots,k\}} u_l, \quad \text{for all} \quad k \in \{0, \dots, T\}. \tag{1.6}$$

Throughout this paper we denote by $J = \{-h+1, \dots, T\}$. Moreover we say that two functions x and y defined on a discrete interval K satisfy that $x \leq y$ on K if and only if $x_k \leq y_k$ for all $k \in K$ and we will denote

$$[x, y] = \{z : K \to \mathbb{R}; \quad x_k \leq z_k \leq y_k, \text{ for all } k \in K\}.$$

To deduce the existence results, we assume that there is a pair of related lower and upper solutions, that are given by the following definition.

Definition 1.1. We define the concept of *related lower and upper solutions* of problem (1.3) – (1.5) as a pair $\alpha = \{\alpha_{-h+1}, \dots, \alpha_0, \dots, \alpha_T\}$ and $\beta = \{\beta_{-h+1}, \dots, \beta_0, \dots, \beta_T\}$ of real sequences such that $\alpha_k \leq \beta_k$ for all $k \in J$,

$$\Delta \alpha_k \leq f(k, \alpha_{k+1}, (\phi\alpha)_{k+1}), \; k \in I,$$

$$\Delta \beta_k \geq f(k, \beta_{k+1}, (\phi\beta)_{k+1}), \; k \in I,$$

$$\alpha_k \leq \varphi(k, \alpha) \leq \varphi(k, \beta) \leq \beta_k, \; k \in I_h$$

and

$$B(\alpha_0, u) \leq 0 \leq B(\beta_0, u), \qquad \text{for all } u \in [\alpha, \beta]. \tag{1.7}$$

It is clear that when $B(\alpha_0, \cdot)$ and $B(\beta_0, \cdot)$ are nonincreasing then the equation (1.7) becomes

$$B(\alpha_0, \alpha) \leq 0 \leq B(\beta_0, \beta).$$

This is the case, for instance, of the periodic problem ($u_0 = u_T$)

$$B(x, y) = x - y_T$$

that we will refer as (PP), or the multi-point boundary conditions ($u_0 = \sum_{k=0}^{l} a_k \, u(j_k)$)

$$B(x, y) = x - \sum_{k=0}^{l} a_k \, y(j_k),$$

with $a_k \geq 0$ and $\{j_k\}_{k=0}^{n} \subset J$.

It is important to note that in this case the dependence of function u on the values of negative integers is also considered.

Note that initial condition $u_0 = c_0$ is also covered. In this case we must define $B(x, y) = x - c_0$ and, as a consequence, $\alpha_0 \leq c_0 \leq \beta_0$.

It is clear that in the above situations the definition of a lower solution and an upper solution has no relationship so, in fact, they are not "related". This term has only sense when the nonincreasing character of $B(\alpha_0, \cdot)$ or $B(\beta_0, \cdot)$ does not hold.

If $B(\alpha_0, \cdot)$ and $B(\beta_0, \cdot)$ are nondecreasing, then equation (1.7) can be rewritten as

$$B(\alpha_0, \beta) \leq 0 \leq B(\beta_0, \alpha).$$

Under this formulation, in which both functions appear simultaneously, can be treated the anti - periodic boundary value conditions ($u_0 = -u_T$)

$$B(x, y) = x + y_T.$$

It is important to note that this type of boundary conditions covers nonlinear situations as

$$u_0 = \max_{k \in J_0} u_k, \text{with } J_0 \subset J,$$

or

$$u_0 = \min_{k \in J_1} u_k, \text{ with } J_1 \subset J,$$

or

$$u_0 = \sum_{k \in J_2} g_k(u_k), \ J_2 \subset J,$$

for suitable choices of functions g_k.

We remark that, as in the case of multipoint boundary value conditions, the dependence of function u on the values of negative integers is considered.

This paper is organized as follows. In Section 2, we prove the existence of at least one solution of problem (1.3) – (1.5) lying between α and β. In Section 3, by assuming some suitable monotonicity properties on function B, we prove the existence of extremal solutions of the considered problem. Moreover we present a monotone iterative technique that allows us to approximate the extremal solutions. Finally, in Section 4, we present some examples to illustrate the obtained results.

2. Existence of solutions

This section is devoted to prove the existence of solutions of a nonlinear first order boundary value problem in which functional dependence on the boundary conditions is allowed. We generalize the results given in [5] for the implicit non delayed case and nonfunctional boundary value conditions, in [3] for non delayed case and in [2] for periodic boundary value problems.

The obtained result is the following.

Theorem 2.1. *Suppose that there exist α and β a pair of related lower and upper solutions of problem (1.3) – (1.5). Assume also that $B \in C(\mathbb{R} \times \mathbb{R}^{T+h}, \mathbb{R})$, $f(k, \cdot, \cdot)$ is a continuous function in $[\alpha_{k+1}, \beta_{k+1}] \times [(\phi\alpha)_{k+1}, (\phi\beta)_{k+1}]$ for all $k \in I$, and $\varphi(k, \cdot)$ is a continuous function in $[\alpha, \beta]$ for every $k \in I_h$.*

If $f(k, x, \cdot)$ is nondecreasing for every $(k, x) \in I \times [\alpha_{k+1}, \beta_{k+1}]$ and $\varphi(k, \cdot)$ is also nondecreasing for every $k \in I_h$, then problem (1.3) – (1.5) has at least one solution $u \in [\alpha, \beta]$.

Proof. Consider the following modified problem:

$$\Delta u_k = f(k, p(k+1, u(k+1)), (\bar{\phi} u)_{k+1}),\ k \in I, \tag{2.1}$$

$$u_k = \varphi(k, \bar{u}),\ k \in I_h, \tag{2.2}$$

$$u_0 = p(0, u_0 - B(u_0, u)), \tag{2.3}$$

where $p(k, r) = \max\{\alpha_k, \min\{r, \beta_k\}\}$ for all $k \in J$ and $r \in \mathbb{R}$, and $\bar{u},\ \bar{\phi}\, u : \mathbb{R}^{T+h} \to \mathbb{R}$ are defined as $\bar{u}(k) = p(k, u(k))$ and $\bar{\phi} u(k) = \phi \circ \bar{u}(k)$ for all $k \in J$.

First we see that Problem (2.1) – (2.3) has a solution. Clearly u is a solution of (2.1) – (2.3) if and only if $u = col(u_{-h+1}, \ldots, u_0, \ldots, u_T)$ is a solution of the matrix equation

$$A\,u = F(u), \tag{2.4}$$

where $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1, & i = j, \\ -1, & h+1 \le i = j+1, \\ 0, & \text{otherwise}, \end{cases}$$

and $F(u)$ is the transpose of the vector

$$\left(\varphi(-h+1, \bar{u}), \ldots, \varphi(-1, \bar{u}), p(0, u_0 - B(u_0, u)), f(0, p(1, u_1), (\bar{\phi} u)_1), \ldots, f(T-1, p(T, u_T), (\bar{\phi} u)_T)\right).$$

Then, we rewrite (2.4) as the fixed point equation $u = A^{-1}F(u) \equiv H\,u$. Obviously, H is a continuous map from $\mathbb{R}^{T+h}$ to $\mathbb{R}^{T+h}$. By definition of p there exists $K > 0$ such that $\|Hu\|_\infty \le K$. Thus, Brower fixed point Theorem implies the existence of a fixed point of H and, in consequence, the existence of a solution of problem (2.1) – (2.3).

Let u be one of such solutions. Suppose that $u \not\geq \alpha$ in J.

From the definition of p, we have that $\bar{u} \in [\alpha, \beta]$. Now the nondecreasing character of $\varphi(k, \cdot)$ implies that $\alpha_k \le u_k \le \beta_k$ for all $k \in I_h$.

Let $j_0 = \min\{j \in I \text{ such that } \alpha_j > u_j\} \ge 1$. Obviously $\alpha_{j_0-1} \le u_{j_0-1}$, in consequence, since $p(k, u(k)) \ge \alpha_k$ for all $k \in J$ and f is nondecreasing in the third variable, from the definition of ϕ and $\bar{\phi}$ we have that

$$\Delta u_{j_0-1} = f(j_0 - 1, \alpha_{j_0}, (\bar{\phi} u)_{j_0}) \ge f(j_0 - 1, \alpha_{j_0}, (\phi\alpha)_{j_0}) \ge \Delta\alpha_{j_0-1}.$$

Thus, $0 > u_{j_0} - \alpha_{j_0} \ge u_{j_0-1} - \alpha_{j_0-1} \ge 0$ and we attain a contradiction.

Reasoning similarly with β, we conclude that $u \in [\alpha, \beta]$.

If $u_0 - B(u_0, u) < \alpha_0$, by definition of function p, we have that $u_0 = \alpha_0$, and then, since $u \in [\alpha, \beta]$, from condition (1.7), we arrive at

$$\alpha_0 > \alpha_0 - B(\alpha_0, u) \ge \alpha_0,$$

which is a contradiction.

The other inequality $u_0 - B(u_0, u) \le \beta_0$ holds similarly.

Thus, every solution u of (2.1) – (2.3) is a solution of (1.3) – (1.5) and it belongs to $[\alpha, \beta]$. □

As a direct consequence of this result we can prove the following ones

Corollary 2.1. Suppose that there exist $\alpha \leq \beta$ on J satisfying the following inequalities

$$\Delta\alpha_k - f(k, \alpha_{k+1}, (\phi\alpha)_{k+1}) \leq 0 \leq \Delta\beta_k - f(k, \beta_{k+1}, (\phi\beta)_{k+1}), \quad k \in I$$

and

$$\alpha_k \leq c_k \leq \beta_k,\ k \in \{-h+1, \ldots, 0\}.$$

If $f(k, \cdot, \cdot)$ is a continuous function in $[\alpha_{k+1}, \beta_{k+1}] \times [(\phi\alpha)_{k+1}, (\phi\beta)_{k+1}]$ such that $f(k, x, \cdot)$ is nondecreasing for every $(k, x) \in I \times [\alpha_{k+1}, \beta_{k+1}]$, then the initial value problem

$$\begin{aligned} \Delta u_k &= f(k, u_{k+1}, (\phi u)_{k+1}),\ k \in I, \\ u_k &= c_k,\ k \in \{-h+1, \ldots, 0\}, \end{aligned}$$

has at least one solution $u \in [\alpha, \beta]$.

Corollary 2.2. Suppose that there exist $\alpha \leq \beta$ on J satisfying the following inequalities

$$\Delta\alpha_k - f(k, \alpha_{k+1}, (\phi\alpha)_{k+1}) \leq 0 \leq \Delta\beta_k - f(k, \beta_{k+1}, (\phi\beta)_{k+1}),\ k \in I,$$

$$\alpha_k \leq \varphi(k, \alpha) \leq \varphi(k, \beta) \leq \beta_k,\ k \in I_h,$$

$$\alpha_0 \leq \alpha_T \quad \text{and} \quad \beta_0 \geq \beta_T.$$

If $f(k, \cdot, \cdot)$ is a continuous function in $[\alpha_{k+1}, \beta_{k+1}] \times [(\phi\alpha)_{k+1}, (\phi\beta)_{k+1}]$ such that $f(k, x, \cdot)$ is nondecreasing for every $(k, x) \in I \times [\alpha_{k+1}, \beta_{k+1}]$ and $\varphi(k, \cdot)$ is a continuous function in $[\alpha, \beta]$ such that $\varphi(k, \cdot)$ is nondecreasing for every $k \in I$, then the periodic problem

$$\begin{aligned} \Delta u_k &= f(k, u_{k+1}, (\phi u)_{k+1}),\ k \in I, \\ u_k &= \varphi(k, u),\ k \in I_h, \\ u_0 &= u_T, \end{aligned}$$

has at least one solution $u \in [\alpha, \beta]$.

This result gives us an alternative existence result to the one given in [2, Theorem 3.3], where the existence of solutions is ensured whenever $\varphi(k, u) = u_0$ and f satisfies the inequalities

$$\begin{aligned} f(k, \alpha_{k+1}, (\phi\alpha)_{k+1}) + M\alpha_{k+1} + N(\phi\alpha)_{k+1} &\leq f(k, x_{k+1}, (\phi x)_{k+1}) + Mx_{k+1} + N(\phi x)_{k+1} \\ &\leq f(k, \beta_{k+1}, (\phi\beta)_{k+1}) + M\beta_{k+1} + N(\phi\beta)_{k+1}, \end{aligned}$$

for some $N < M$ such that either $(N + M)\,T < 1$ or $N\,T\,(1 + M)^{-1-h} < 1$.

Corollary 2.3. Suppose that there exist $\alpha \leq \beta$ on J satisfying the following inequalities

$$\Delta\alpha_k - f(k, \alpha_{k+1}, (\phi\alpha)_{k+1}) \leq 0 \leq \Delta\beta_k - f(k, \beta_{k+1}, (\phi\beta)_{k+1}),\ k \in I,$$

$$\alpha_k \leq \varphi(k, \alpha) \leq \varphi(k, \beta) \leq \beta_k,\ k \in I_h,$$

$$\alpha_0 \leq -\beta_T \quad \text{and} \quad \beta_0 \geq -\alpha_T.$$

If $f(k,\cdot,\cdot)$ is a continuous function in $[\alpha_{k+1},\beta_{k+1}]\times[(\phi\alpha)_{k+1},(\phi\beta)_{k+1}]$ such that $f(k,x,\cdot)$ is nondecreasing for every $(k,x)\in I\times[\alpha_{k+1},\beta_{k+1}]$ and $\varphi(k,\cdot)$ is a continuous function in $[\alpha,\beta]$ such that $\varphi(k,\cdot)$ is nondecreasing for every $k\in I_h$, then the anti – periodic problem

$$\begin{aligned}\Delta u_k &= f(k,u_{k+1},(\phi u)_{k+1}),\ k\in I,\\ u_k &= \varphi(k,u),\ k\in I_h,\\ u_0 &= -u_T,\end{aligned}$$

has at least one solution $u\in[\alpha,\beta]$.

Whenever $\varphi(k,u_0)=u_0$, this last result is [1, Corollary 2.1], for the non delayed case in which f does no depend on the third variable.

3. Extremal solutions

In this section we improve the existence result given in the previous section in the particular case of $B(x,\cdot)$ was a nonincreasing function for each $x\in[\alpha_0,\beta_0]$. To be concise, we prove that, if such monotonicity property holds and function φ is defined in $I\times\mathbb{R}$, under the hypotheses of Theorem 2.1, problem (1.3) – (1.5) has the maximal and the minimal solutions lying between α and β. Here, we say that x^* is the maximal solution in $[\alpha,\beta]$ if any other solution $y\in[\alpha,\beta]$ satisfies that $y\le x^*$. The concept of minimal solution is analogous by reversing the inequality. We refer to the maximal and minimal solutions as extremal solutions.

As we have noted before, this result is applicable to initial, periodic and multi-point boundary value problems.

The monotonicity condition imposed in B cannot be relaxed. To see this, it is enough to think about the following anti - periodic boundary value problem

$$\Delta u_k = g(u_{k+1}),\ \ k\in I,\quad u_k=u_0=-u_T,\ \ k\in I_h,\ T\text{ odd},\tag{3.1}$$

with $g:\mathbb{R}\to\mathbb{R}$ defined as

$$g(x)=\begin{cases}4-2x, & \text{if } x>1\\ 2\,x, & \text{if } |x|\le 1\\ -2x-4, & \text{if } x<-1.\end{cases}$$

It is obvious that $\alpha\equiv-2$ and $\beta\equiv 2$ are a pair of related lower and upper solutions of (3.1). So, Theorem 2.1 ensures the existence of at least one solution $u\in[-2,2]$.

Let u be a solution of the equation

$$\Delta u_k = g(u_{k+1}),\ \ k\in I.$$

From the fact that function $f(x)\equiv x-g(x)$ satisfies the following conditions:

- f is strictly increasing in $(-\infty,-1]\cup[1,+\infty)$,
- $f(x)=-x$ for all $x\in[-1,1]$,
- f is one to one in $(-\infty,-5/3)\cup(5/3,+\infty)$,
- $f[-5/3,5/3]=[-1,1]$.

We conclude that the equation $u_0 = f(x)$, with $u_0 \in (-1,1)$ has exactly three solutions, but only one belongs to $(-1,1)$ which is equals to $-u_0$. The other ones are $x_1 = (u_0+4)/3 > 1$ and $x_2 = (u_0-4)/3 < -1$. It is not difficult to verify that the solutions that start at x_1 and x_2 are strictly monotone.

On the other hand we have that $f(x) = \pm 1$ if and only if $u_1 = \mp 1$ and $v_1 = \mp 5/3$ and the solution that start at v_1 is strictly monotone.

If $u_0 \in (1,2]$ then $u_1 = (u_0+4)/3 > 1$ and, by recurrence we conclude that $u_k > 1$ for all $k \in J$, that is, u cannot be a solution of (3.1).

In an analogous way one can verify that if $u_0 \in [-2,-1)$ then the solutions are strictly decreasing.

As a consequence the unique solutions of problem (3.1) in $[-2,2]$ satisfy that $u_0 \in [-1,1]$. In this case, one can verify that $u_k = (-1)^k u_0$ are the unique solutions of problem (3.1), so we deduce that this problem has no extremal solutions in $[\alpha,\beta]$.

One can see sufficient conditions that ensure the uniqueness of solutions of non delayed equations and develop iterative methods of approximate it in [1, 3]. Existence results under different definitions of lower and upper solutions for this type of equations are given in [8].

Now, we prove the existence of extremal solutions.

Theorem 3.1. *Under the hypothesis of Theorem 2.1, if $B(u_0,\cdot)$ is a nonincreasing function for each $u_0 \in [\alpha_0,\beta_0]$ and $\varphi(k,u) \equiv \varphi(k,u_0)$ then problem (1.3) – (1.5) has extremal solutions lying between α and β.*

Proof. From Theorem 2.1 we know that problem (1.3) – (1.5) has at least one solution in $[\alpha,\beta]$.

Let x_1, x_2 be two solutions of problem (1.3) – (1.5) in $[\alpha,\beta]$ and define, for each $k \in J$, functions $\gamma(k) = \max\{x_1(k), x_2(k)\}$ and $\delta(k) = \min\{x_1(k), x_2(k)\}$.

In an analogous way to [5, Theorem 2.2], it is not difficult to verify that (γ,β) and (α,δ) are two pairs of related lower and upper solutions of problem (1.3) – (1.5). As a consequence, by using Theorem 2.1 again, we have that there exist $u_1 \in [\gamma,\beta]$ and $u_2 \in [\alpha,\delta]$ two solutions of this problem. So the set of solutions in $[\alpha,\beta]$ is directed and, see [7] for details, it has extremal solutions in $[\alpha,\beta]$. □

Now, we are in a position to approximate the extremal solutions of problem (1.3) – (1.5), as follows.

Theorem 3.2. *Assume that the hypotheses of Theorem 3.1 are satisfied and also that there exists a constant $m > 0$ such that*

$$(H)\ \ f(k,y,z) + m\, y \le f(k,x,z) + m\, x,\ k \in I,\ \alpha_{k+1} \le y \le x \le \beta_{k+1},\ z \in [(\phi\alpha)_{k+1}, (\phi\beta)_{k+1}].$$

Then there exist two monotone sequences in $\mathbb{R}^{T+h}$, $\{\gamma_n\}$ and $\{\delta_n\}$ such that $\alpha = \gamma_0 \le \gamma_n \le \delta_n \le \delta_0 = \beta$ for every $n \in \mathbb{N}$, which converge (componentwise) to the minimal and the maximal solutions of (1.3) – (1.5) in $[\alpha,\beta]$, respectively.

Proof. Let $\eta \in [\alpha,\beta]$. Define $G_\eta(k,x,y) = f(k,\eta_{k+1},(\phi\eta)_{k+1}) + m(\eta_{k+1}-x)$, $\varphi_\eta(k,x) = \varphi(k,\eta_0)$ and $B_\eta(x,y) = B(x,\eta)$.

We consider the following problem:

$$(P_\eta)\begin{cases} \Delta u_k &= G_\eta(k,u_{k+1},(\phi u)_{k+1});\ k\in I,\\ u_k &= \varphi_\eta(k,u_0),\ k\in I_h,\\ 0 &= B_\eta(u_0,u). \end{cases}$$

We have that for each $\eta\in[\alpha,\beta]$ and $k\in I$ function $G_\eta(k,\cdot,\cdot)$ is continuous and $G_\eta(k,x,\cdot)$ is nondecreasing. Furthermore

$$G_\eta(k,\alpha_{k+1},(\phi\alpha)_{k+1}) - f(k,\alpha_{k+1},(\phi\alpha)_{k+1}) = f(k,\eta_{k+1},(\phi\eta)_{k+1}) + m(\eta_{k+1}-\alpha_{k+1}) -$$
$$f(k,\alpha_{k+1},(\phi\alpha)_{k+1}) \ge f(k,\eta_{k+1},(\phi\alpha)_{k+1}) + m(\eta_{k+1}-\alpha_{k+1}) - f(k,\alpha_{k+1},(\phi\alpha)_{k+1}) \ge 0.$$

In consequence

$$\Delta\alpha_k \le G_\eta(k,\alpha_{k+1},(\phi\alpha)_{k+1})\ k\in I.$$

In the same way, we deduce that

$$\Delta\beta_k \ge G_\eta(k,\beta_{k+1},(\phi\beta)_{k+1})\ \ k\in I.$$

Furthermore

$$\alpha_k \le \varphi(k,\alpha_0) \le \varphi_\eta(k,\alpha) = \varphi(k,\eta_0) = \varphi_\eta(k,\beta) \le \varphi(k,\beta_0) \le \beta_k, \qquad k\in I_h,$$

and

$$B_\eta(\alpha_0,\alpha) = B(\alpha_0,\eta) \le B(\alpha_0,\alpha) \le 0 \le B(\beta_0,\beta) \le B(\beta_0,\eta) = B_\eta(\beta_0,\beta).$$

Let ξ be the minimal solution of problem (1.3) – (1.5) in $[\alpha,\beta]$. Such a solution exists by Theorem 3.1. Clearly ξ is a solution of (P_ξ).

Furthermore, for all $\eta\le\xi$, it is satisfied that

$$\Delta\xi_k \ge G_\eta(k,\xi_{k+1},(\phi\xi)_{k+1}),\ k\in I;\quad \xi_k\ge\varphi_\eta(k,\xi),\ k\in I_h;\quad B_\eta(\xi_0,\xi)\ge 0.$$

Thus, from Theorem 3.1 we have that for each $\eta\in[\alpha,\xi]$, problem (P_η) has extremal solutions on $[\alpha,\xi]$.

Now, we define γ_1 as the minimal solution in $[\alpha,\beta]$ of problem (P_α).

By recurrence, we define γ_{n+1} as the minimal solution in $[\gamma_n,\beta]$ of problem (P_{γ_n}). By construction this sequence is nondecreasing in J, in consequence there exists $\psi\in\mathbb{R}^{T+1}$ such that

$$\psi(k) = \lim_{n\to\infty}\gamma_n(k)\ \ k\in J.$$

Clearly

$$\Delta\psi_k = f(k,\psi_{k+1},(\phi\psi)_{k+1})\ \ k\in J.$$

The continuity of B and φ implies that ψ is a solution of (1.3) – (1.5).

Now, using that $\psi \in [\gamma_n, \xi]$ for all $n \in \mathbb{N}$, we deduce that ψ is the minimal solution of problem (1.3) – (1.5) in $[\alpha, \beta]$.

Defining δ_{n+1} as the maximal solution in $[\alpha, \delta_n]$ of problem (P_{δ_n}) we approximate the maximal solution in $[\alpha, \beta]$ of problem (1.3) – (1.5). ☐

As we have seen in the proof, the nondecreasing assumption on function f is fundamental to ensure the validity of the result, in fact such a condition cannot be dropped as we can see in the following example in which function f is decreasing in the third variable and, despite it satisfies condition (H) for all $m > 0$, there is no solution lying between a pair of given lower and upper solutions.

Example 3.1. Let $h = 2$ be fixed. Consider problem

$$\Delta u_k = -2(\phi u)_{k+1},\ k \in \{0,1\}, \quad u_{-1} = u_0 = u_2.$$

It is not difficult to verify that it has a unique solution, given by the constant function zero. Now, defining $\alpha = \{-1,-1,-1,-1\}$ and $\beta = \{1,1,-1,0\}$, we have a lower and an upper solution of this problem respectively, such that $\alpha \le \beta$ on J and for which there is no solution in $[\alpha, \beta]$. ☐

Remark 3.1. As we have noted previously, the obtained results for problem (1.3) – (1.5) are still valid for the periodic problem (PP). In fact, we deduce, by using Theorem 3.2, existence and approximation results for the periodic problem (PP) under the assumption (H) and the nondecreasing character of function f in the third variable. If we consider the periodic problem (PP) in the case of $\varphi(k,u_0) = u_0$ for all $k \in I_h$, analogous results are obtained in [2, Theorem 3.1] by assuming in this case condition

$$f(k, x_{k+1}, (\phi x)_{k+1}) + M x_{k+1} + N(\phi x)_{k+1} \le f(k, y_{k+1}, (\phi y)_{k+1}) + M y_{k+1} + N(\phi y)_{k+1}.$$

for some $N < M$ such that either $(N + M)\,T < 1$ or $N\,T\,(1+M)^{-1-h} < 1$, whenever $x \le y$.

When f does not depend on the third variable, Theorem 3.2 applied to the periodic problem (PP) with $\varphi(k,u_0) = u_0$ for all $k \in I_h$ has been proven in [6].

To finish this section, we give an existence result for the periodic problem (PP), in which condition (H) is replaced by a stronger one, in this case without assuming the existence of a pair of lower and upper solutions. The existence result is the following.

Theorem 3.3. *Assume that $\varphi(k,\cdot)$ is a continuous function in $\mathbb{R}$ for all $k \in I_h$ that is nondecreasing. Assume also that $f(k,\cdot,\cdot)$ is a continuous function in $\mathbb{R}\times\mathbb{R}$ for all $k \in I$ and that there exists $M > 0$ such that $f(k,\xi,\tau) + M\,\xi$ is nonincreasing in ξ, nondecreasing in τ and that*

$$\inf_{x \le 0} f(k,0,x) > -\infty \ \text{and} \ \sup_{x \ge 0} f(k,0,x) < \infty, \qquad \text{for all } k \in I.$$

Then the periodic boundary value problem (with $\varphi(k,u) \equiv \varphi(k,u_0)$) has at least one solution.

Proof. Choose $L \ge 0$ such that $L \ge \sup\{f(k,0,x) : k \in I,\ x \ge 0\}$.

Let u be a solution of

$$\Delta u_k + M u_{k+1} = L,\ k \in I, \quad u_k = \varphi(k,u_0),\ k \in I_h, \quad u_0 = u_T.$$

By using the fact that the Green's function related with operator $\Delta u_k + M u_{k+1}$ on the space $\{u : \{0, \dots, T\} \to \mathbb{R}; u_0 = u_T\}$ exists and is nonnegative (see [6]), this problem has a unique solution β that is nonnegative on $\{0, \dots, T\}$.

From the fact that $f(k, \xi, \tau) + M\xi$ is nonincreasing in ξ, we have

$$\triangle\beta_k + M\beta_{k+1} = L \geq f(k, 0, (\phi\beta)_{k+1}) \geq f(k, \beta_{k+1}, (\phi\beta)_{k+1}) + M\beta_{k+1}, \ k \in I.$$

In consequence β is an upper solution of (PP).

To find the lower solution, we consider the following unique solution α (which is nonpositive on $\{0, \dots, T\}$) of the periodic boundary value problem,

$$\Delta u_k + M u_{k+1} = R, \ k \in I, \quad u_k = \varphi(k, u_0), \ k \in I_h, \quad u_0 = u_T.$$

Where $R \leq 0$ is such that $R \leq \inf\{f(k, 0, x) : k \in I, \ x \leq 0\}$.

As in the previous case, we verify that α is a lower solution of (PP).

Hence $\alpha_k \leq 0 \leq \beta_k$ for k in I and $\beta_k = \varphi(k, \beta_0) \geq \varphi(k, \alpha_0) = \alpha_k$ on I_h. So, by Theorem 2.1, we deduce the existence of at least one solution in $[\alpha, \beta]$. □

4. Examples

Example 4.1. Let $J_1 = \{0, 1\} \subset I = \{0, 1, 2\}$, m, n, $l \in \mathbb{N}$ odds and $\varphi_k \leq 0$ for all $k \in I_h$. Consider the following problem

$$\begin{aligned} \Delta u_k &= \tfrac{1-(-1)^{k+1}}{2} - u_{k+1} + ((\phi u)_{k+1})^{1/n}, \quad k \in I, \\ u_k &= u_0 + \varphi_k, \ k \in I_h, \quad u_0 = (\min_{k \in J_1}\{u_k^l\})^m. \end{aligned}$$

One can easily verify that $\alpha = \{\varphi_{-h+1}, \dots, \varphi_{-1}, 0, 1, -1, 0\}$ and $\beta = \{\varphi_{-h+1}, \dots, \varphi_{-1}, 0, 2, 3, 4\}$ are lower and upper solutions of the given problem, respectively. Theorem 2.1 guarantees that there exists at least one solution of the problem between α and β. □

For the next example, let f be a real valued continuous and nondecreasing function such that $f(0) = 0$.

Example 4.2. Let $m \in \mathbb{N}$ odd, $n \in \mathbb{N}$, $l \in \{1, \cdots, T\}$, $h = 1$ and $C > 2$. Consider the following problem

$$\begin{aligned} \Delta u_k &= 1 + f(((\phi u)_{k+1})^m - u_{k+1}^n), \quad k \in I, \\ u_0 &= u_l / C. \end{aligned}$$

For any given real number a such that $a > 2\,l/(C-2)$, we see that $\alpha = \{0, 0, ..., 0\}$ is a lower solution and $\beta = \{a, 2a, 2(a+1), ..., 2(a+T)\}$ is an upper solution of the given two – point boundary value problem. Then from Theorem 2.1 we can ensure the existence of at least one solution of the problem between α and β. □

As we have seen in section 3, the monotonicity assumptions in third variable of function f are sufficient conditions that cannot be avoided in general. However, in the following example we present a problem in which it is exposed that such hypotheses are not necessary.

Example 4.3. Let $n \in \mathbb{N}$ be odd, $n \geq 3$, $\varphi_k \in \mathbb{R}$ such that $\varphi_k \leq 0$. Consider the following problem

$$\begin{aligned} \Delta u_k &= u_{k+1} - ((\phi u)_{k+1})^n, \quad k \in I, \\ u_k &= u_0 + \varphi_k, \quad k \in I_h, \\ u_0 &= u_T. \end{aligned}$$

If we choose $\alpha = \{-2 + \varphi_{-h+1}, \ldots, -2 + \varphi_{-1}, -1, ..., -1\}$ and $\beta = \{2, 2, ..., 2\}$ are a pair of related lower and upper solutions for the problem, and it is easy to verify that f does not satisfy the hypothesis of Theorem 2.1, however $u = \{\varphi_{-h+1}, \ldots, \varphi_{-1}, 0, ..., 0\}$ is a solution for that problem lying between α and β. □

Note that the previous example give us a solvable problem for which the conditions of Theorem 3.3 do not hold. So the conditions imposed in such a result are not necessary.

Due to the Theorem 2.1, we have existence of solution of problem (1.3) – (1.5) if there exists a pair of related lower and upper solutions for that problem. And it is not always easy to find them. In next example, we want to point out that the given additional sufficient conditions on f guarantee that the corresponding solutions of the explicit equations serve as upper and lower solutions.

Example 4.4. Consider the problem

$$\Delta u_k = f(k, u_{k+1}, ((\phi u)_{k+1}), \quad k \in I, \tag{4.1}$$

$$u_k = u_0 = 0 \quad k \in I_h. \tag{4.2}$$

Suppose that f satisfies the following inequalities

$$f(k, x, y) \leq (x + y)/2 + A \quad \forall x, y > 0, \tag{4.3}$$

$$f(k, x, y) \geq -(x + y)/2 - B \quad \forall x, y < 0, \tag{4.4}$$

for some $A, B > 0$.

Consider now the following problem:

$$\begin{aligned} \Delta u_k &= (u_{k+1} + (\phi u)_{k+1})/2 + A, \quad k \in I, \\ u_k &= u_0 = 0, \quad k \in I_h. \end{aligned}$$

It is easy to verify that the unique solution of this problem is strictly increasing. In consequence it is strictly positive on $\{0, \ldots, T\}$. So, the inequality (4.3) implies that such a solution is an upper solution of problem (4.1) – (4.2).

By using the inequality (4.4), we have that the unique solution of problem

$$\begin{aligned} \Delta u_k &= -(u_{k+1} + (\phi u)_{k+1})/2 - B, \quad k \in I, \\ u_k &= u_0 = 0, \quad k \in I_h, \end{aligned}$$

gives us a lower solution. Moreover it is strictly decreasing and negative on $\{0, \ldots, T\}$.

As a consequence, we have constructed a pair of lower and upper solutions of problem (4.1) – (4.2), that are well ordered in J. The existence of solution of the problem (4.1) – (4.2) follows from Theorem 2.1.

References

[1] R. P. Agarwal, A. Cabada, V. Otero–Espinar and S. Dontha, Existence and uniqueness of solutions for anti–periodic difference equations, *Archiv.Inequal. Appl.*, **2** (2004), 4, 397–411.

[2] F. Atici, A. Cabada and J. B. Ferreiro Existence and comparison results for first order periodic implicit difference equations with maxima. *In honor of Professor Lynn Erbe. J. Difference Equ. Appl.* **8** (2002), 4, 357–369.

[3] A. Cabada The Method of Lower and Upper Solutions for Periodic and Anti – Periodic Difference Equation *Electronic Trans. Num. Anal.* (To appear).

[4] A. Cabada and J.B. Ferreiro and E. Liz "Comparison results for first order difference equations with maxima." *Proceedings of the XVII Congress of Differential Equations and Applications/VII Congress of Applied Mathematics.* (2001) (In spanish).

[5] A. Cabada, V. Otero-Espinar, R. L. Pouso, Existence and Approximation of solutions for Discontinuous First Order Difference Problems with Nonlinear Functional Boundary Conditions in the Presence of Lower and Upper Solutions. *Computers & Math. Appl.* **39**, (2000), 21 – 33.

[6] A. Cabada and V. Otero-Espinar, Optimal existence results for nth order periodic boundary value difference equations. *J. Math. Anal. Appl.* **247**, (2000), 67 – 86.

[7] J. A. Cid, On extremal fixed points in Schauder's theorem with applications to differential equations. *Bull. Belg. Math. Soc. Simon Stevin,* **11** (2004), 1, 15–20.

[8] D. Franco, D. O'Regan and J. Perán, Upper and lower solution theory for first and second order difference equations, *Dyn. Syst. Appl.* **13** (2004), 2, 273–282.

[9] E. Liz and J.B. Ferreiro, A note on the global stability of generalized difference equations. *Appl. Math. Lett.* **15**, (2002), 655 – 659.

[10] S. Mohamad and K. Gopalsamy. "Continuous and discrete Halanay-type inequalities." *Bull. Aus. Math. Soc.* **61** *(2000), 371 – 385.*

[11] M. Pinto and S. Trofimchuk. "Stability and existence of multiple solutions for a quasilinear differential equation with maxima."*Proc. Roy. Soc. Edinburgh Sec. A, 130 (2000), 1103 – 1118.*

[12] H.K. Xu and E. Liz. "Boundary value problems for differential equations with maxima." *Nonlinear Studies, 3 (1996), 231 – 241.*

[1]Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101

E-mail address: [1]`ferhan.atici@wku.edu`

[2]Dep. Análise Matemática, Fac. Matemáticas, Univ. Santiago de, Compostela, 15784, Santiago de Compostela, Spain

E-mail address: [2]`cabada@usc.es`

[3]Dep. Matemática Aplicada, E. Politécnica Superior de Lugo, Univ., Santiago de Compostela, 27002, Lugo, Spain

E-mail address: [3] `jbosco@lugo.usc.es`