MOLINO'S DESCRIPTION AND FOLIATED HOMOGENEITY

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Abstract. The topological Molino's description of equicontinuous foliated spaces, studied by the first author and Moreira Galicia, gives conditions to reduce their study to the particular case where the holonomy pseudogroup can be represented by a pseudogroup on some local group G generated by some of its local left translations (a G-foliated space). That description is sharpened in this paper by introducing a foliated action of a compact topological group on the resulting G-foliated space, like in the case of Riemannian foliations. Moreover a C^∞ version is also studied. The triviality of this compact group characterizes compact minimal G-foliated spaces, which are also characterized by their foliated homogeneity in the C^∞ case. We also give an example where the projection of the Molino's description is not a principal bundle, and another example of positive topological codimension where the foliated homogeneity cannot be checked by only comparing pairs of leaves—in the case of zero topological codimension, weak solenoids with this property were given by Fokkink and Oversteegen, and later by Dyer, Hurder and Lukina.

Contents

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1. Introduction

A description of certain compact minimal equicontinuous foliated spaces was given by the first author and Moreira Galicia [10]. It can be considered as a topological version of the Molino's description of Riemannian foliations on compact manifolds [37], in the minimal case. This gave another confirmation that equicontinuous foliated spaces should be considered as the topological Riemannian foliations, as asserted by Ghys [37, Appendix E]. That description reduces the study of such foliated spaces to the particular case of G-foliated spaces, which are the foliated spaces whose holonomy pseudogroup can be represented by a pseudogroup on some local group G generated by some of its local left translations. The classical example of G-foliated spaces are Lie foliations, which are used in the original Molino's theory to describe Riemannian foliations. According to the role played by Molino's theory in the study of Riemannian foliations, its topological version should have interesting applications; for instance, it was already used in [10] to study the growth of the leaves.

Dyer, Hurder and Lukina also gave an analogue of Molino's description for equicontinuous matchbox manifolds [18, 19], which is the case of compact connected minimal foliated spaces of topological codimension zero; i.e., with totally disconnected local transversals. The advantage of their construction is that it works without any additional condition, but their description is unique just when our hypotheses are fulfilled.

Our first goal is to show the following slight sharpening of the main result of the topological Molino's theory (Section 3). The terminology and notation used here are recalled in Section 2.

Theorem A (Cf. [10, Theorem A]). *Suppose that a foliated space* $X \equiv (X, \mathcal{F})$ *is compact, minimal, equicontinuous and strongly quasi-analytic, and the closure of its holonomy pseudogroup is also strongly quasi-analytic. Then there is a local group* G*, a compact topological group* H*, a compact minimal* G*-foliated space* $\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0)$*, a foliated map* $\hat{\pi}_0 : \widehat{X}_0 \to X$*, and a free foliated right* H*-action on* $\widehat{X}_0$ *such that the restrictions of* $\hat{\pi}_0$ *to the leaves of* $\widehat{X}_0$ *are the holonomy coverings of the leaves of* X*, and* $\hat{\pi}_0$ *induces a homeomorphism* $\widehat{X}_0/H \to X$*.*

Precisely, our new contribution in Theorem A is the existence of H satisfying the stated properties. If $\mathcal{H}$ is the representative of the holonomy pseudogroup of X on a space T induced by the choice of a good foliated atlas, and we fix some $u_0 \in T$, then H is the group of germs at u_0 of the maps g in the closure $\overline{\mathcal{H}}$ with $u_0 \in \operatorname{dom} g$ and $g(u_0) = u_0$. Following the construction of $\widehat{X}_0$ in [10], we get a compatible compact topology on H and a right foliated H-action on $\widehat{X}_0$ satisfying the statement of Theorem A.

We also show that the construction of $(G, H, \widehat{X}_0, \hat{\pi}_0)$ is independent of the choices involved up to an obvious equivalence relation (Proposition 3.1), and therefore $(G, H, \widehat{X}_0, \hat{\pi}_0)$ is called the Molino's description of X; in particular,

G is called the structural local group according to [37, 10], and H is called the discriminant group according to [18]. Under the hypothesis of Theorem A, we also prove the following additional properties:

- X is a G-foliated space for some local group G if and only if its discriminant group is trivial (Proposition 3.2).
- There is a subgroup in H isomorphic to the holonomy group of every leaf (Proposition 3.4).
- If X is C^∞, then its Molino's description becomes C^∞ in a unique obvious sense (Proposition 5.1).
- The map $\hat{\pi}_0$ may not be a fiber bundle (an example is given in Section 8.2). This is the only missing property when comparing with the Riemannian foliation case.

Our second goal is to characterize G-foliated spaces using a property called foliated homogeneity. A foliated space $X \equiv (X, \mathcal{F})$ is called foliated homogeneous if the group $\operatorname{Homeo}(X, \mathcal{F})$ of its foliated transformations acts transitively on itself (a foliated version of homogeneity). This notion was studied by Clark and Hurder in the case of matchbox manifolds [15], where homogeneity and foliated homogeneity are equivalent notions because $\operatorname{Homeo}(X) = \operatorname{Homeo}(X, \mathcal{F})$ since the leaves are the path connected components. Clark and Hurder have shown that a matchbox manifold is equicontinuous if and only if it is a weak solenoid (an inverse limit of a tower of covering maps between closed connected manifolds), and it is homogeneous if and only if it is a McCord solenoid (the covering maps can be chosen to be regular), also called strong solenoid. Since McCord solenoids are transversely modeled by left translations on profinite groups, they are particular cases of G-foliated spaces. For this reason, the mentioned Molino's description of Dyer, Hurder and Lukina is a procedure to construct McCord solenoids from weak solenoids. On the other hand, according to the original Molino's theory [37], among minimal Riemannian foliations on closed manifolds, the homogeneous ones are the Lie foliations (the G-foliations for Lie groups G). Thus, generalizing the case of matchbox manifolds and minimal Riemannian foliations on closed manifolds, it makes sense to ask whether any compact minimal foliated space is foliated homogeneous if and only if it is a G-foliated space. We give the following answers.

Theorem B. *If a foliated space X is compact, minimal and foliated homogeneous, then it satisfies hypotheses of Theorem A and is a G-foliated space for some local group G.*

Theorem C. *Suppose that a foliated space X is compact, minimal and C^∞. Then the following conditions are equivalent:*

(i) *X is C^∞ foliated homogeneous.*
(ii) *X is foliated homogeneous.*
(iii) *X satisfies the hypotheses of Theorem A and is a G-foliated space for some local group G.*

Here, a foliated space is said to be C^∞ when it has a foliated atlas whose changes of coordinates are C^∞ along the leaves, and their leafwise partial derivatives of arbitrary order are continuous (on the ambient space). Other related concepts are defined in the same way, like C^∞ foliated maps, C^∞ diffeomorphisms, (leafwise) tangent space, (leafwise) Riemannian metrics, (leafwise) Riemannian foliated spaces, etc. For C^∞ foliated spaces, the concept of C^∞ foliated homogeneity can be defined like foliated homogeneity using C^∞ foliated diffeomorphisms.

Theorem B follows with an adaptation of an argument of Clark and Hurder [15, Theorem 5.2], using that the canonical left action of $\operatorname{Homeo}(X, \mathcal{F})$ on X is micro-transitive by a theorem of Effros [20, 43].

To prove Theorem C, it is enough to show "(iii) ⇒ (i)" by Theorem B. Assuming (iii), we get the so-called structural right local transverse action, which has its own interest; for instance, it was introduced and used in [7] for Lie foliations. It is the unique "foliated right local action up to leafwise homotopies" of G on X, which corresponds to the local right translations on G via foliated charts (Proposition 6.6 and Section 6.3). Its construction uses a partition of unity subordinated to a foliated atlas and the leafwise center of mass for some (leafwise) Riemannian metric to merge the obvious right local transverse actions on the domains of foliated charts. The structural right local transverse action gives (i) because we always have leafwise homogeneity (Proposition 7.1).

In Theorem C, our proof of "(iii) ⇒ (i)" needs the C^∞ structure of X because we use the leafwise center of mass as an auxiliary tool. Of course, it could be possible to avoid the C^∞ condition and show "(iii) ⇒ (ii)" directly with other tools, but that procedure would certainly require more work.

Since there exist leaves without holonomy, and since the (differentiable) quasi-isometry type of the leaves is independent of the choice of a (leafwise) Riemannian metric on X, it follows that X is not foliated homogeneous if there is a leaf with holonomy, or if there is a pair of non-quasi-isometric leaves. The reciprocal statement is not true in general. Fokkink and Oversteegen [23, Theorem 35] constructed an example of a non-homogeneous weak solenoid all of whose leaves are simply connected, and therefore it has no holonomy, and its leaves are quasi-isometric to each other because weak solenoids are suspension foliated spaces. Dyer, Hurder and Lukina constructed more examples of such weak solenoids [19, Theorem 10.7]. In Section 8.3, we give an example of a compact foliated space X satisfying the conditions of Theorem A, which is not foliated homogeneous and has no holonomy, whose leaves are quasi-isometric to each other, and with locally connected local transversals (thus it is not a weak solenoid).

2. Preliminaries

See [39, Chapter II], [24] and [13, Chapter 11] for the needed preliminaries on foliated spaces and interesting examples, and [27, 28, 29] for the preliminaries on pseudogroups. We mainly follow [10, Sections 2 and 4A], which in turn follows [4, 5, 6]. Some ideas are also taken from [15, 9, 8]. The needed basic concepts and tools are recalled here for the reader's convenience, and a few new observations are also made.

In the whole paper, unless otherwise stated, spaces are assumed to be locally compact and Polish, and maps are assumed to be continuous. In particular, this applies to foliated spaces, topological groups, local groups and partial maps.

2.1. **Pseudogroups.** For spaces T and T', the notation $\phi : T \rightarrowtail T'$ is used for a partial map. We will only consider the case where its domain, $\operatorname{dom}\phi$, is open in T. The germ of ϕ at any $u \in \operatorname{dom}\phi$ will be denoted by $\boldsymbol{\gamma}(\phi, u)$. If ϕ is an open embedding, we may identify ϕ with the homeomorphism $\phi : \operatorname{dom}\phi \to \operatorname{im}\phi$ of an open subset of T to an open subset of T', whose inverse can be considered as a partial map with open domain, $\phi^{-1} : T' \rightarrowtail T$; in particular, when $T = T'$, such a ϕ is called a *local transformation* of T.

Given another space T'', let Φ and Ψ be families of partial maps $T \rightarrowtail T'$ and $T' \rightarrowtail T''$, respectively, with open domains. We use the notation $\Psi\Phi = \{\, \psi\phi \mid \phi \in \Psi,\ \psi \in \Psi \,\}$; in particular, $\Phi^n = \Phi\cdots\Phi$ (n times) if $T = T'$ and $n \in \mathbb{Z}^+$. If Φ consists of open embeddings, let $\Phi^{-1} = \{\, \phi^{-1} \mid \phi \in \Phi \,\}$.

Recall that a *pseudogroup* $\mathcal{H}$ on T is a family of local transformations of T that contains id_T, and is closed by the operations of composite, inversion, restriction to open sets and union. It is said that $\mathcal{H}$ is *generated* by $S \subset \mathcal{H}$ if $\mathcal{H}$ can be obtained from S using the above operations. By considering a pseudogroup as a direct generalization of a group of transformations, the basic dynamical concepts have obvious generalizations to pseudogroups, like *orbits*, *saturation*, (*topological*) *transitivity* and *minimality*. The orbit space is denoted by $T/\mathcal{H}$. The $\mathcal{H}$-saturation of any $A \subset T$ is denoted by $\mathcal{H}(A)$, and the orbit of any $u \in T$ by $\mathcal{H}(u)$. For any open $V \subset T$, the *restriction* $\mathcal{H}|_V := \{\, h \in \mathcal{H} \mid \operatorname{dom} h, \operatorname{im} h \subset V \,\}$ is a pseudogroup.

Given another pseudogroup $\mathcal{H}'$ on T', a *morphism* $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal collection of partial maps $T \rightarrowtail T'$ with open domain such that $\mathcal{H}'\Phi\mathcal{H} \subset \Phi$, $T = \bigcup_{\phi\in\Phi} \operatorname{dom}\phi$, and, for all $\phi, \psi \in \Phi$ and $u \in \operatorname{dom}\phi \cap \operatorname{dom}\psi$, there is some $h' \in \mathcal{H}'$ so that $\phi(u) \in \operatorname{dom} h'$ and $\boldsymbol{\gamma}(h'\phi, u) = \boldsymbol{\gamma}(\psi, u)$. Let Φ_0 be a family of partial maps $T \rightarrowtail T'$ with open domain such that $T = \mathcal{H}(\bigcup_{\phi\in\Phi} \operatorname{dom}\phi)$, and there is a subset S of generators of $\mathcal{H}$ such that, if $\phi, \psi \in \Phi_0$, $h \in S$ and $u \in \operatorname{dom}\phi \cap \operatorname{dom}\psi h$, then there is some $h' \in \mathcal{H}'$ so that $\phi(u) \in \operatorname{dom} h'$ and $\boldsymbol{\gamma}(h'\phi, u) = \boldsymbol{\gamma}(\psi h, u)$. Then there is a unique morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ containing Φ_0, which is said to be *generated* by Φ_0. For instance, id_T generates a morphism $\mathrm{id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ consisting of all possible unions of maps in $\mathcal{H}$; in particular, $\mathcal{H} \subset \mathrm{id}_{\mathcal{H}}$. For another pseudogroup $\mathcal{H}''$ on T'' and a morphism $\Psi : \mathcal{H}' \to \mathcal{H}''$, the family $\Psi\Phi$ generates a morphism $\mathcal{H} \to \mathcal{H}''$, which may be also denoted by $\Psi\Phi$ with some abuse of notation. In this way, the morphisms of pseudogroups form a category PsGr. There is a canonical functor Top → PsGr, assigning the pseudogroup generated by id_T, also denoted by T, to every topological space T, and assigning the morphism generated by ϕ, also denoted by ϕ, to every map $\phi : T \to T'$. A morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is an isomorphism of PsGr if and only if it is generated by a family Φ_0 of open embeddings such that Φ_0^{-1} generates a morphism $\mathcal{H}' \to \mathcal{H}$, which is the inverse Φ^{-1} in PsGr.

With the terminology of Haefliger [27, 28, 29], an *étalé morphism* $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal family of homeomorphisms of open subsets of T to open subsets of T' such that $\mathcal{H}'\Phi\mathcal{H} \subset \Phi$, $T = \bigcup_{\phi\in\Phi} \operatorname{dom}\phi$ and $\Phi\Phi^{-1} \subset \mathcal{H}'$. If moreover Φ^{-1} is an étalé morphism, then Φ is called an *equivalence*, and the pseudogroups $\mathcal{H}$ and $\mathcal{H}'$ are said to be *equivalent*. If Φ_0 is a family of homeomorphisms of open subsets of T to open subsets of T' such that $T = \mathcal{H}(\bigcup_{\phi\in\Phi} \operatorname{dom}\phi)$ and $\Phi_0\mathcal{H}\Phi_0^{-1} \subset \mathcal{H}'$, then there is a unique étalé morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ containing Φ_0, which is said to be *generated* by Φ_0. Any equivalence generates an isomorphism in PsGr, and, vice versa, any isomorphism in PsGr is generated by a unique equivalence. Hence isomorphism and equivalences are equivalent concepts. Equivalent pseudogroups are considered to have the same dynamics. For instance, $\mathcal{H}$ is equivalent to $\mathcal{H}|_V$ for any open $V \subset T$ that meets all $\mathcal{H}$-orbits. In fact, $\Phi : \mathcal{H} \to \mathcal{H}'$ is an equivalence if and only if $\mathcal{G} = \mathcal{H} \cup \mathcal{H}' \cup \Phi \cup \Phi^{-1}$ is a pseudogroup on $T \sqcup T'$ such that T and T' meet all $\mathcal{G}$-orbits, $\mathcal{G}|_T = \mathcal{H}$ and $\mathcal{G}|_{T'} = \mathcal{H}'$.

The germs $\boldsymbol{\gamma}(h, u)$, for $h \in \mathcal{H}$ and $u \in \operatorname{dom} h$, form a topological groupoid $\mathfrak{H}$, equipped with the sheaf topology and the operation induced by composite. Its unit subspace can be identified with T. In fact, $\mathfrak{H}$ is an étalé groupoid (the source and target maps, $s, t : \mathfrak{H} \to T$, are local homeomorphisms). Given $x \in T$, the group of elements of $\gamma \in \mathfrak{H}$ with $s(\gamma) = t(\gamma) = x$ is called the *germ group* of $\mathcal{H}$ at x.

Let us recall the following definitions of properties that $\mathcal{H}$ may have:

Compact generation: This means that there is a relatively compact open $U \subset T$, which meets all orbits, such that $\mathcal{H}|_U$ is generated by a finite set, $E = \{h_1, \dots, h_k\}$, and every h_i has an extension $\tilde{h}_i \in \mathcal{H}$ with $\overline{\operatorname{dom} h_i} \subset \operatorname{dom} \tilde{h}_i$. This E is called a *system of compact generation* of $\mathcal{H}$ on U.

(Strong) equicontinuity: This means that there are an open cover $\{T_i\}$ of T and a metric d_i inducing the topology of every T_i, and $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$ (S is symmetric and closed by composites[1]), such that, for every $\epsilon > 0$, there is some $\delta > 0$ so that

$$d_i(x, y) < \delta \Longrightarrow d_j(h(x), h(y)) < \epsilon$$

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[1] The term *pseudo$*$group* was used in [10] when these conditions are satisfied. This term was introduced in [36] for a family that moreover contains id_T and is also closed by restrictions to open subsets.

for all $h \in S$, indices i, j, and $x, y \in T_i \cap h^{-1}(T_j \cap \operatorname{im} h)$.

Strong quasi-analyticity: This means that $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$, such that, if any $h \in S$ is the identity on some non-empty open subset of its domain, then $h = \mathrm{id}_{\operatorname{dom} h}$.

Strong local freeness: This means that $\mathcal{H}$ is generated by some subset $S \subset \mathcal{H}$, with $S^2 \subset S = S^{-1}$, such that, if any $h \in S$ fixes some point in its domain, then $h = \mathrm{id}_{\operatorname{dom} h}$. Equivalently, this means that $\mathcal{H}$ is strongly quasi-analytic and all of its germ groups are trivial.

These properties are invariant by equivalences. If compact generation holds with some U, then it also holds with any other relatively compact open subset of T that meets all orbits. Let $\mathcal{P}$ denote any of the above last three properties. If $\mathcal{P}$ holds with S, then it also holds with its *localization*,

$$S_{\mathrm{loc}} = \{\, h|_O \mid h \in S,\ O \text{ is open in } \operatorname{dom} h \,\} .$$

Moreover we can add id_T to S if desired (obtaining $S^2 = S$). If $\mathcal{H}$ is compactly generated and satisfies $\mathcal{P}$, then, for every relatively compact open $U \subset T$ that meets all orbits, we can choose a system of compact generation E of $\mathcal{H}$ on U such that $\mathcal{H}|_U$ also satisfies $\mathcal{P}$ with $S = \bigcup_{n=1}^{\infty} E^n$. The following result lists some needed non-elementary properties.

Proposition 2.1 ([4, Proposition 8.9, and Theorems 11.1 and 12.1], [42] and [5, Theorems 3.3 and 5.2])**.** *Suppose that $\mathcal{H}$ is compactly generated, equicontinuous and strongly quasi-analytic. Then the following holds:*

(i) Assume that $\mathcal{H}$ satisfies the condition of compact generation with U, $E = \{h_1, \dots, h_k\}$ and $\tilde{h}_1, \dots, \tilde{h}_k$. For every $h = h_{i_n} \cdots h_{i_1} \in \bigcup_{n=1}^{\infty} E^n$, let $\tilde{h} = \tilde{h}_{i_n} \cdots \tilde{h}_{i_1}$. Then there is a finite family $\mathcal{V}$ of open subsets of T covering U such that, for any $h \in \bigcup_{n=1}^{\infty} E^n$ and $V \in \mathcal{V}$, we have $V \subset \operatorname{dom} \tilde{h}$ if $V \cap \operatorname{dom} h \neq \varnothing$.

(ii) Suppose that $\mathcal{H}$ satisfies the equicontinuity condition with a set S. Then $\overline{C(O,T) \cap S_{\mathrm{loc}}}$ consists of local transformations for all small enough open subsets $O \subset T$, where the closure is taken in the compact-open topology, and the pseudogroup $\overline{\mathcal{H}}$ generated by such transformations is equicontinuous. More precisely, $\overline{\mathcal{H}}$ satisfies the equicontinuity condition with the set $\overline{S}$ determined by the condition $C(O,T) \cap \overline{S} = \overline{C(O,T) \cap S_{\mathrm{loc}}}$ for all O as above.

(iii) The orbit closures are minimal sets, and therefore $\mathcal{H}$ is transitive if and only if it is minimal.

In Proposition 2.1-(ii), the pseudogroup $\overline{\mathcal{H}}$ is called the *closure* of $\mathcal{H}$.

2.2. Relation of pseudogroups with local groups and local actions.

The general definition of local group is rather involved [34], but, in the locally compact case, a *local group* G can be considered as neighborhood of the identity element e in some topological group [16, 17]. Two such neighborhoods in the same topological group define *equivalent* local groups;

thus it can be said that, up to equivalences, a local group is the "germ" of a topological group at the identity element. For the sake of simplicity, the family of open neighborhoods of e in G will be denoted by $\mathcal{N}(G,e)$. Given another local group G' with identity element e', a *local homomorphism* of G to G' is a partial map with open domain, $\sigma : G \rightarrowtail G'$, such that $e \in \operatorname{dom}\sigma$, $\sigma(e) = e'$, and $\sigma(gh) = \sigma(g)\sigma(h)$ for all $g,h \in \operatorname{dom}\sigma$ such that the products gh and $\sigma(g)\sigma(h)$ are defined with $gh \in \operatorname{dom}\sigma$. Two local homomorphisms of G to G' are *equivalent* when they have the same germ at e. If there is a local homomorphism $\tau : G' \rightarrowtail G$ such that $\tau\sigma$ and $\sigma\tau$ are equivalent to id_G and $\mathrm{id}_{G'}$, then σ is called a *local isomorphism*. The term *sublocal group* will be used for a subspace $H \subset G$ such that $(H \cap V)^2, (H \cap V)^{-1} \subset H$ for some $V \in \mathcal{N}(G,e)$; in particular, $e \in H$, but $H \cap V$ is not required to be closed in V (contrary to [26, Definition 2.10.]). A sublocal group becomes a local group with the induced structure, but it may not be locally compact, and the inclusion map of any sublocal group is a local homomorphism. A *right local action* of G on T is a partial map with open domain, $\chi : T \times G \rightarrowtail T$, where $T \times \{e\} \subset \operatorname{dom}\chi$ and $\chi(u,e) = u$ for all $u \in T$, and such that, for all $g,h \in G$ and $u \in T$, if the product gh is defined and $(u,g), (u,gh), (\chi(u,g),h) \in \operatorname{dom}\chi$, then $\chi(\chi(u,g),h) = \chi(u,gh)$. Two right local actions of G on T are *equivalent* when they agree around $T \times \{e\}$. If T is compact, we can assume $\operatorname{dom}\chi = T \times O$ for some $O \in \mathcal{N}(G,e)$. For any open $V \subset T$, the restriction $\chi : \chi^{-1}(V) \cap (V \times G) \to V$ is a right local action of G on V, called the restriction of χ to V. Given an open cover $\{T_i\}$ of T and a right local action χ_i of G on every T_i such that the restrictions of χ_i and χ_j to $T_i \cap T_j$ are equivalent, it is easy to check that there is a unique right local action of G on T, up to equivalences, whose restriction to every T_i is equivalent to χ_i.

Consider another right local action χ' of G' on T'. A partial map with open domain, $\phi : T \rightarrowtail T'$, is called *locally equivariant* if there is some open neighborhood Σ of $\operatorname{dom}\phi \times \{e\}$ in $\operatorname{dom}\chi \cap (\phi \times \mathrm{id}_G)^{-1}(\operatorname{dom}\chi')$ such that $\chi(\Sigma) \subset \operatorname{dom}\phi$ and $\phi\chi(u,g) = \chi'(\phi(u),g)$ for all $(u,g) \in \Sigma$. Note that composites, restrictions to open sets and unions of locally equivariant partial maps with open domain are locally equivariant, as well as their inverses whenever defined. A family of partial maps $T \rightarrowtail T'$ with open domain is called *locally equivariant* when all of its elements are locally equivariant.

Local anti-homomorphisms, *left local actions*, their *equivalences* and corresponding *locally equivariant* maps are similarly defined.

For instance, any finite dimensional metrizable locally compact local group is indeed locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group [34, Theorem 107] (corrected according to [26], or using [16, 17] and [38, Section IV.4.9]). As a concrete example, we can consider the product of any local Lie group and any countable family of finite groups. By Ado's theorem, the equivalence classes of local Lie groups and their local homomorphisms correspond one-to-one to finite dimensional real Lie algebras and their homomorphisms. A typical example of right local action of a local group G on itself is given by its local right translations, and any local left translation of G becomes locally equivariant.

Proposition 2.2 ([5, Theorems 3.3 and 5.2], [10, Lemma 2.36, Theorem 2.38 and Remark 21])**.** *The following holds:*

- *(i) Suppose that $\mathcal{H}$ is minimal, compactly generated, equicontinuous and strongly quasi-analytic. Then $\overline{\mathcal{H}}$ is strongly locally free if and only if $\mathcal{H}$ is equivalent to a pseudogroup on some local group G generated by the left local action by local left translations of a finitely generated dense sublocal group $\Gamma \subset G$.*
- *(ii) Let $\mathcal{G}$ and $\mathcal{G}'$ be the pseudogroups on local groups G and G' generated by the left local actions by local left translations of respective finitely generated dense sublocal groups Γ and Γ'. Let $\Phi : \mathcal{G} \to \mathcal{G}'$ be a morphism such that $\mathcal{G}(e) \mapsto \mathcal{G}'(e')$ by the induced map $G/\mathcal{G} \to G/\mathcal{G}'$. Then Φ is generated by a local homomorphism $G \rightarrowtail G'$ that restricts to a local homomorphism $\Gamma \rightarrowtail \Gamma'$.*

Proposition 2.3. *Let $\Phi : \mathcal{H} \to \mathcal{H}'$ be an equivalence between compactly generated pseudogroups. Let χ be a right local action of G on T such that $\mathcal{H}$ is locally equivariant. Then there is a unique right local action χ' of G on T', up to equivalences, such that Φ and $\mathcal{H}'$ are locally equivariant.*

Proof. Let E be a system of compact generation of $\mathcal{H}$ on a relatively compact open $U \subset T$, and let $\tilde{h}$ be an extension of every $h \in E$ with $\overline{\operatorname{dom} h} \subset \operatorname{dom} \tilde{h}$. There is a subset $\Phi_0 \subset \Phi$ such that $\{\, \operatorname{dom}\phi \times \operatorname{im}\phi \mid \phi \in \Phi_0 \,\}$ covers $U \times T'$, $\{\, \operatorname{im}\phi \mid \phi \in \Phi_0 \,\}$ is locally finite in T', and every $\phi \in \Phi_0$ has an extension $\tilde{\phi} \in \Phi$ with $\overline{\operatorname{dom}\phi} \subset \operatorname{dom}\tilde{\phi}$. Write $\{\phi_i\} = \{\, \phi h \mid h \in E,\ \phi \in \Phi_0 \,\}$, and let $\tilde{\phi}_i = \tilde{\phi}\tilde{h}$ if $\phi_i = \phi h$ for $h \in E$ and $\phi \in \Phi_0$. Moreover let $U_i = \operatorname{dom}\phi_i$, $U'_i = \operatorname{im}\phi_i$, $\widetilde{U}_i = \operatorname{dom}\tilde{\phi}_i$, $\widetilde{U}'_i = \operatorname{im}\tilde{\phi}_i$, $\widetilde{U}_{ij} = \tilde{\phi}_j^{-1}(\widetilde{U}'_i \cap \widetilde{U}'_j) = \operatorname{dom}\tilde{\phi}_i^{-1}\tilde{\phi}_j$ and $\widetilde{U}'_{ij} = \phi_j(\widetilde{U}_i \cap \widetilde{U}_j) = \operatorname{dom}\tilde{\phi}_i\tilde{\phi}_j^{-1}$. The following assertion is easy to check.

Claim 1. $\{\phi_i\}$ generates Φ and $\{\phi_i\phi_j^{-1}\}$ generates $\mathcal{H}'$.

Let $\Omega = \operatorname{dom}\chi$, and let Σ_{ij} be an open neighborhood of $\widetilde{U}_{ij} \times \{e\}$ in $\Omega \cap (\tilde{\phi}_i^{-1}\tilde{\phi}_j \times \mathrm{id}_G)^{-1}(\Omega)$ such that $\chi(\Sigma_{ij}) \subset \widetilde{U}_{ij}$ and $\tilde{\phi}_i^{-1}\tilde{\phi}_j\chi(u,g) = \chi(\tilde{\phi}_i^{-1}\tilde{\phi}_j(u),g)$ for all $(u,g) \in \Sigma_{ij}$. Let

$$\Omega'_0 = \{\, (u',g) \in T' \times G \mid u' \in \overline{U'_i} \cap \overline{U'_j} \Rightarrow (\tilde{\phi}_j^{-1}(u'),g) \in \Sigma_{ij},\ \forall i,j \,\}\,.$$

Claim 2. Ω'_0 is open in $T' \times G$.

Take some $(u',g) \in \Omega'_0$. Let $\mathfrak{I}$ be the set of indices i such that $u' \in \overline{U'_i}$, and let $\mathfrak{I}'$ be the set of pairs of indices, (i,j), such that $u' \in \overline{U'_i} \cap \overline{U'_j}$, which are finite sets because $\{U'_i\}$ is locally finite in T'. Then, using that $\overline{U'_i} \subset \widetilde{U}'_i$, every $\tilde{\phi}_i$ is a homeomorphism, and Σ_{ij} is an open neighborhood of $(\tilde{\phi}_j^{-1}(u'),g)$ in $\widetilde{U}_{ij} \times G$ for all $(i,j) \in \mathfrak{I}'$, it follows that there are open neighborhoods, V of u' in T' and P of g in G, such that $V \cap \overline{U'_i} = \varnothing$ if $i \notin \mathfrak{I}$, and $\tilde{\phi}_j^{-1}(V) \times P \subset \Sigma_{ij}$ for all $(i,j) \in \mathfrak{I}'$. Thus $V \times P \subset \Omega'_0$.

Claim 3. A map $\chi'_0 : \Omega'_0 \to T'$ is defined by $\chi'_0(u', g) = \tilde{\phi}_i\chi(\phi_i^{-1}(u'), g)$ if $u' \in U'_i$.

Let $(u', g) \in \Omega'_0$ such that $u' = \phi_i(u_i) = \phi_j(u_j)$ for some $u_i \in U_i$ and $u_j \in U_j$. We have $(u_j, g) \in \Sigma_{ij}$ because $(u', g) \in \Omega'_0$. Hence $\chi(u_j, g) \in \widetilde{U}_{ij}$ and $\tilde{\phi}_i^{-1}\tilde{\phi}_j\chi(u_j, g) = \chi(u_i, g)$, obtaining $\tilde{\phi}_j\chi(u_j, g) = \tilde{\phi}_i\chi(u_i, g)$. This shows that χ'_0 is well defined. Its continuity follows from the continuity of χ since the maps $\tilde{\phi}_i$ are homeomorphisms.

Let $\{V'_i\}$ be an open covering of T' with $\overline{V'_i} \subset U'_i$, and define $V_i = \phi_i^{-1}(V'_i)$. Let

$$\Omega' = \{\, (u', g) \in \Omega'_0 \mid u' \in \overline{V'_i} \Rightarrow \chi'_0(u', g) \in U'_i,\ \forall i \,\}\,, \qquad \chi' = \chi'_0|_{\Omega'}\,.$$

Claim 4. χ' is a local action of G on T'.

First, it is easy to check that Ω' is open in Ω'_0, and therefore Ω' is also open in $T' \times G$ by Claim 2.

Second, $T' \times \{e\} \subset \Omega'_0$ because $\widetilde{U}_{ij} \times \{e\} \subset \Sigma_{ij}$ for all i, j. Moreover, for all $u' \in T'$,

$$\chi'(u', e) = \tilde{\phi}_i\chi(\phi_i^{-1}(u'), e) = \tilde{\phi}_i\phi_i^{-1}(u') = u'\,.$$

Hence $T' \times \{e\} \subset \Omega'$ and $\chi'(u', e) = u'$ for all $u' \in T'$.

Third, assume that gh is defined and $(u', g), (u', gh), (\chi'(u', g), h) \in \Omega'$ for some $g, h \in G$ and $u' \in T'$. Then, for i with $u' \in V'_i$, we have $\chi'(u', g) \in U'_i$ and $(\phi_i^{-1}(u'), g), (\phi_i^{-1}(u'), gh), (\phi_i^{-1}\chi'(u', g), h) \in \Omega$, obtaining

$$\begin{aligned}\chi'(\chi'(u', g), h) &= \tilde{\phi}_i\chi(\phi_i^{-1}\chi'(u', g), h) = \tilde{\phi}_i\chi(\chi(\phi_i^{-1}(u'), g), h)\\ &= \tilde{\phi}_i\chi(\phi_i^{-1}(u'), gh) = \chi'(u', gh)\end{aligned}$$

because χ is a right local action. This completes the proof of Claim 4.

Obviously, all maps ϕ_i become locally equivariant by the definition of χ'_0 and χ'; indeed, up to equivalences, χ' is the unique local action satisfying this property because $\{U'_i\}$ covers T'. Therefore the maps $\phi_i\phi_j^{-1} : U'_{ij} \to U'_{ji}$ are also locally equivariant. So Φ and $\mathcal{H}'$ are locally equivariant by Claim 1 and because $\mathcal{H}$ is locally equivariant. □

Let χ be a right local action of G on T such that $\mathcal{H}$ is locally equivariant. Consider the following property that $(T, \mathcal{H}, \chi)$ may have:

$$\mathcal{H}(\chi(\{u\} \times P)) = T \quad \forall u \in T, \forall P \in \mathcal{N}(G, e) \mid \{u\} \times P \subset \operatorname{dom}\chi. \tag{1}$$

Lemma 2.4. *Property* (1) *is preserved by locally equivariant pseudogroup equivalences.*

Proof. Elementary. □

2.3. **Foliated spaces.** The notation introduced here will be used in Sections 3–7.

Let X be a space and $n \in \mathbb{Z}^{\geq 0}$. The main results of the paper will require X to be compact, but this condition is avoided for the basic concepts. Let $\mathcal{U}$ be a family consisting of pairs (U_i, ξ_i), called *foliated charts*, where $\{U_i\}$ is an open cover of X, and every ξ_i is a homeomorphism $U_i \to B_i \times T_i$ for some contractible open subset $B_i \subset \mathbb{R}^n$ and a space T_i. Every (U_i, ξ_i) induces a projection $p_i : U_i \to T_i$ whose fibers are called *plaques*. Assume that finite intersections of plaques are open in the plaques. Then the open subsets of the plaques form a base of a finer topology in X, becoming an n-manifold whose connected components are called *leaves*. In this case, it is said that $\mathcal{U}$ defines a *foliated structure* $\mathcal{F}$ of *dimension* n on X, $X \equiv (X, \mathcal{F})$ is called a *foliated space* (or *lamination*), and $\mathcal{U}$ is called a *foliated atlas*. Two foliated atlases define the same foliated structure if their union is a foliated atlas. The subspaces $\xi_i^{-1}(\{\mathbf{v}\} \times T_i) \subset X$, $\mathbf{v} \in B_i$, are called *local transversals* defined by the foliated chart (U_i, ξ_i). A *transversal* is a subspace $\Sigma \subset X$ where any point has a neighborhood that is a local transversal of some foliated chart. A transversal is called *global* if it meets all leaves.

A foliated space can be considered as a weak version of a regular dynamical system where the leaves play the role of the orbits. In this way, several basic dynamical concepts have obvious versions for foliated spaces, like *saturation*, (*topological*) *transitivity* and *minimality*. The partition of X into leaves is enough to describe $\mathcal{F}$. The leaf through a point x may be denoted by L_x, and the leaf space by $X/\mathcal{F}$. The saturation of a subset $A \subset X$ is denoted by $\mathcal{F}(A)$.

We can assume that the foliated atlas $\mathcal{U}$ is *regular*[2] in the sense that it satisfies the following properties [6, Definition 5.1] (see also [31, 13, 25]):

- there is another foliated atlas $\widetilde{\mathcal{U}} = \{\widetilde{U}_i, \tilde{\xi}_i\}$ of X, with $\tilde{\xi}_i : \widetilde{U}_i \to \widetilde{B}_i \times \widetilde{T}_i$ and distinguished submersions $\tilde{p}_i : U_i \to \widetilde{T}_i$, such that $\overline{U_i} \subset \widetilde{U}_i$, $\overline{B_i} \subset \widetilde{B}_i$, T_i is an open subspace of $\widetilde{T}_i$, and $\xi_i = \tilde{\xi}_i|_{U_i}$ (thus $p_i = \tilde{p}_i|_{U_i}$);
- $\{U_i\}$ is locally finite; and
- every plaque of (U_i, ξ_i) meets at most one plaque of (U_j, ξ_j).

By the last condition, there are homeomorphisms[3] $h_{ij} : p_j(U_i \cap U_j) \to p_i(U_i \cap U_j)$, the *elementary holonomy transformations*, such that $h_{ij} p_j = p_i$ on $U_i \cap U_j$, obtaining the *defining cocycle* $\{U_i, p_i, h_{ij}\}$; it describes $\mathcal{F}$ and satisfies the cocycle condition $h_{ik} = h_{ij} h_{jk}$ on $p_k(U_i \cap U_j \cap U_k)$. So the changes of coordinates $\xi_i \xi_j^{-1} : \xi_j(U_i \cap U_j) \to \xi_i(U_i \cap U_j)$ are of the form

$$\xi_i \xi_j^{-1}(\mathbf{v}, u) = (g_{ij}(\mathbf{v}, u), h_{ij}(u)) , \tag{2}$$

for some maps $g_{ij} : \xi_j(U_i \cap U_j) \to B_i$.

The "transverse dynamics" of X is described by its *holonomy pseudogroup*, which is (the equivalence class of) the pseudogroup $\mathcal{H}$ generated by the maps h_{ij} on $T := \bigsqcup_i T_i$. Its elements are called *holonomy transformations*. There is a canonical identity $X/\mathcal{F} \equiv T/\mathcal{H}$, where the $\mathcal{H}$-orbit that corresponds to a leaf L is $\bigcup_i p_i(L \cap U_i)$. Via this identity, $\mathcal{F}$-leaves and $\mathcal{H}$-orbits have corresponding dynamical concepts.

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[2] Regularity of the foliated atlas is used with another meaning in [15].

[3] This convention for the order of these subindices agrees with [15] and differs from [10]. The same kind of convention will be used in the local respresentations of foliated maps.

We can assume that $\widetilde{\mathcal{U}}$ is also regular, obtaining elementary holonomy transformations $\tilde{h}_{ij} : \tilde{p}_j(\widetilde{U}_i \cap \widetilde{U}_j) \to \tilde{p}_i(\widetilde{U}_i \cap \widetilde{U}_j)$, extending the maps h_{ij}, which generate another representative of the holonomy pseudogroup, $\widetilde{\mathcal{H}}$ on $\widetilde{T} := \bigsqcup_i \widetilde{T}_i$; T is an open subspace of $\widetilde{T}$ that meets all $\widetilde{\mathcal{H}}$-orbits, and $\mathcal{H} = \widetilde{\mathcal{H}}|_T$. Let $\sigma_i : T_i \to U_i$ and $\tilde{\sigma}_i : \widetilde{T}_i \to \widetilde{U}_i$ be the sections of every p_i and $\tilde{p}_i$ defined by fixing an element of B_i (thus $\sigma_i = \tilde{\sigma}_i|_{T_i}$). We can assume that the sets $\tilde{\sigma}_i(\widetilde{T}_i)$ are separated by open sets in X, and therefore $\bigcup_i \tilde{\sigma}_i : \widetilde{T} \to \bigcup_i \tilde{\sigma}_i(\widetilde{T}_i)$ and $\bigcup_i \sigma_i : T \to \bigcup_i \sigma_i(T_i)$ are homeomorphisms to complete transversals.

Given a finite sequence of indices, $\mathcal{I} = (i_0, \dots, i_\alpha)$, let $h_{\mathcal{I}} = h_{i_\alpha i_{\alpha-1}} \cdots h_{i_1 i_0}$ if $\alpha > 0$, and $h_{\mathcal{I}} = \mathrm{id}_{T_{i_0}}$ if $\alpha = 0$. If $\operatorname{dom} h_{\mathcal{I}} \neq \varnothing$, then $\mathcal{I}$ is called *admissible*. Let $c : I := [0,1] \to X$ be a path from x to y, which is *leafwise* in the sense that $c(I)$ is contained in some leaf L. Let us say that c is ($\mathcal{U}$-) *covered* by $\mathcal{I}$ if there is a partition of I, $0 = t_0 < t_1 < \dots < t_{\alpha+1} = 1$, such that $c([t_k, t_{k+1}]) \subset U_{i_k}$ for all $k = 0, \dots, \alpha$. In this case, $u := p_{i_0}(x) \in \operatorname{dom} h_{\mathcal{I}}$ and $h_{\mathcal{I}}(u) = p_{i_\alpha}(y)$. If $\mathcal{I} = (i_0, \dots, i_\alpha)$ and $\mathcal{J} = (j_0, \dots, j_\beta)$ cover c and c', respectively, with $j_0 = i_\alpha$, then $\mathcal{I}\mathcal{J} := (i_0, \dots, i_\alpha = j_0, \dots, j_\beta)$ and $\mathcal{I}^{-1} := (i_\alpha, \dots, i_0)$ cover cc' and c^{-1}, respectively, and we have $h_{\mathcal{I}\mathcal{J}} = h_{\mathcal{J}} h_{\mathcal{I}}$ and $h_{\mathcal{I}^{-1}} = h_{\mathcal{I}}^{-1}$. By using $\widetilde{\mathcal{U}}$, we can similarly define $\tilde{h}_{\mathcal{I}}$, which is an extension of $h_{\mathcal{I}}$. Recall that, for another admissible sequence $\mathcal{J} = (j_0, \dots, j_\beta)$ with $j_0 = i_0$ and $j_\beta = i_\alpha$, covering another path c' from x to y in L, if c and c' are endpoint-homotopic in L, then $u \in \operatorname{dom} h_{\mathcal{J}}$ and $\boldsymbol{\gamma}(h_{\mathcal{I}}, u) = \boldsymbol{\gamma}(h_{\mathcal{J}}, u)$. Any leafwise path is covered by some admissible sequence, and, vice versa, for all $\mathcal{I} = (i_0, \dots, i_\alpha)$, $x \in U_{i_0}$ and $y \in U_{i_\alpha}$ with $p_{i_0}(x) \in \operatorname{dom} h_{\mathcal{I}}$ and $h_{\mathcal{I}} p_{i_0}(x) = p_{i_\alpha}(y)$, there is some leafwise path from x to y covered by $\mathcal{I}$.

The *holonomy group* of a leaf L at a point $x \in L \cap U_i$ is the germ group,

$$\operatorname{Hol}(L, x) = \{ \boldsymbol{\gamma}(h, u) \mid h \in \mathcal{H},\ u \in \operatorname{dom} h,\ h(u) = u \} ,$$

where $u = p_i(x)$. It depends only on L up to conjugation by germs of holonomy transformations. The *holonomy homomorphism*, $\mathrm{hol} : \pi_1(L, x) \to \operatorname{Hol}(L, x)$, is given by $\mathrm{hol}([c]) = \boldsymbol{\gamma}(h_{\mathcal{I}}^{-1}, u)$ if c is covered by $\mathcal{I} = (i_0, \dots, i_\alpha)$ with $i_0 = i_\alpha = i$. This homomorphism is well defined and onto according to the previous observations, and it induces a regular covering $\widetilde{L}^{\mathrm{hol}}$ of L, the *holonomy covering*. We will consider the canonical right action of $\operatorname{Hol}(L, x)$ on $\widetilde{L}^{\mathrm{hol}}$ by covering transformations. A leaf is said to be *without holonomy* if its holonomy group is trivial, and X is called *without holonomy* when all leaves have no holonomy. The union of leaves without holonomy is a dense G_δ in X, and therefore Borel and residual [30, 22]. A path connected subset of a leaf, $D \subset L$, is said to be *without holonomy* if the composite

$$\pi_1(D, x) \longrightarrow \pi_1(L, x) \xrightarrow{\mathrm{hol}} \operatorname{Hol}(L, x)$$

is trivial for some (and therefore all) $x \in D$.

It is said that X is (*strongly*) *equicontinuous*, *strongly quasi-analytic* or *strongly locally free* if $\mathcal{H}$ satisfies these properties; thus X is strongly locally free just when it is strongly quasi-analytic and has no holonomy. In the definition of these conditions for $\mathcal{H}$, by refining $\mathcal{U}$ if necessary, we can assume that the metrics d_i are defined on the sets T_i, and we can take

$$S = \{\, h_{\mathcal{I}} \mid \mathcal{I} \text{ is an admissible sequence}\,\}$$

if desired. For a local group G, we say that X is a *G-foliated space* if $\mathcal{H}$ is equivalent to a pseudogroup generated by some local left translations on G.

If X is compact, then $\mathcal{U}$ is finite and T is relatively compact in $\widetilde{T}$, obtaining that $\widetilde{\mathcal{H}}$ satisfies the definition of compact generation with the generators h_{ij} of $\widetilde{\mathcal{H}}|_T = \mathcal{H}$ and their extensions $\tilde{h}_{ij}$. So $\mathcal{H}$ is also compactly generated. If moreover $\mathcal{F}$ is equicontinuous, then the properties of Propositions 2.1 and 2.2 apply to $\mathcal{H}$; in particular, the leaf closures are minimal sets, and therefore X is transitive if and only if it is minimal.

Foliated spaces with boundary can be defined in a similar way, adapting the definition of manifold with boundary: every B_i would be a contractible open set in the half space $H^n \equiv \mathbb{R}^{n-1} \times [0,\infty)$. The *boundary* of X, $\partial X = \bigcup_i \xi_i^{-1}(\partial B_i \times T_i)$, becomes a foliated space without boundary. The basic concepts recalled here about foliated spaces have direct extensions to foliated spaces with boundary.

Any open $U \subset X$ becomes a foliated space with the *restriction* $\mathcal{F}|_U$, defined by all possible foliated charts of $\mathcal{F}$ with domain in U. Let $X' = (X', \mathcal{F}')$ be another foliated space with $\dim \mathcal{F}' = n'$. Then $X \times X'$ has a *product* foliated structure of dimension $n+n'$, $\mathcal{F} \times \mathcal{F}'$, defined by the foliated charts that are products of foliated charts of $\mathcal{F}$ and $\mathcal{F}'$. Thus the leaves of $X \times X'$ are products of leaves of X and X'. Any connected (second countable) manifold can be considered a foliated space with one leaf, and any space can be considered as a foliated space whose leaves are the points. Like in the case of foliations, a typical example of foliated space can be obtained by *suspension* of an action of the fundamental group of a manifold on a space (see Section 8.1).

Let $X' \equiv (X', \mathcal{F}')$ be another foliated space, let $\mathcal{U}' = \{U'_a, \xi'_a\}$ be a regular foliated atlas of X', where $\xi'_a : U'_a \to B'_a \times T'_a$, giving rise to a defining cocycle $\{U'_a, p'_a, h'_{ab}\}$, and the corresponding representative of the holonomy pseudogroup, $\mathcal{H}'$ on $T' := \bigsqcup_a T_a$ generated by $\{h'_{ab}\}$. A map $\phi : X \to X'$ is called *foliated* when it maps leaves to leaves. Then every local representation $\xi'_a \phi \xi_i^{-1} : \xi_i(U_i \cap \phi^{-1}(U'_a)) \to B'_a \times T'_a$ is of the form

$$\xi'_a \phi \xi_i^{-1}(\mathbf{v}, u) = (\phi^1_{ai}(\mathbf{v}, u), \phi^2_{ai}(u)) \tag{3}$$

for some maps $\phi^1_{ai} : \xi_i(U_i \cap \phi^{-1}(U'_a)) \to B'_a$ and $\phi^2_{ai} : p_i(U_i \cap \phi^{-1}(U'_a)) \to T'_a$. The maps ϕ^2_{ai} generate a morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ [8, 9], which is said to be *induced* by ϕ.

An action of a group on X is called *foliated* when it is given by foliated homeomorphisms. A homotopy H between foliated maps $\phi, \psi : X \to X'$ is said to be *leafwise* if it is a foliated map $X \times I \to X'$, where $X \times I$ is endowed with the foliated structure with leaves $L \times I$, for leaves L of X; in particular, every path $H(x, \cdot) : I \to X'$ ($x \in X$) is leafwise. In this case, ϕ and ψ induce the same morphism $\mathcal{H} \to \mathcal{H}'$ [9, Proposition 6.1]. A *leafwise* isotopy has a similar definition.

Let $V \subset \mathbb{R}^n \times Y$ be an open subset, and let $r \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. A map $g : V \to \mathbb{R}^{n'}$ is called (*differentiable of class*) C^r when, for any integer $0 \leq k \leq r$ (it is enough to take $k = r$ if $r < \infty$), all partial derivatives of g up to order k with respect to the coordinates of $\mathbb{R}^n$ are defined and continuous on V. A change of coordinates $\xi_j \xi_i^{-1}$ is called C^r when the map g_{ij} in (2) is C^r. If all changes of coordinates are C^r, then $\mathcal{U}$ defines a C^r *structure* on X, which becomes a C^r foliated space. In this case, $\mathcal{U}$ and its foliated charts are called C^r. Two such foliated atlases of X define the same C^r structure if their union also defines a C^r structure. The leaves of C^r foliated spaces canonically become C^r manifolds. Many concepts of C^r manifolds have strightforward generalizations to C^r foliated spaces, like C^r *foliated maps*, C^r *foliated diffeomorphisms*, C^r *foliated embeddings*, C^r *foliated actions*, C^r *leafwise homotopies/diffeotopies*, C^r *vector bundles*, C^r *sections*, the (*leafwise*) *tangent bundle* TX (or $T\mathcal{F}$), the (*leafwise*) *tangent map* $T\phi : TX \to TX'$ of a C^r foliated map $\phi : X \to X'$, (*leafwise*) *Riemannian metrics*, etc. For instance, a foliated map $\phi : X \to X'$ is C^r when, for all local representations $\xi'_a \phi \xi_i^{-1}$, the maps ϕ^1_{ai} of (3) are C^r.

Any C^r foliated space has a C^r partition of unity subordinated to any open cover [39, Proposition 2.8]. A version of the Reeb's stability theorem holds for C^2 foliated spaces [4, Proposition 1.7].

Recall that a subset A in a Riemannian manifold M is called *convex* when, for all $x, y \in A$, there is a unique minimizing geodesic segment from x to y in M that lies in A (see e.g. [14, Section IX.6]). For example, sufficiently small balls are convex. If X is C^∞, given any C^∞ Riemannian metric on X, we can choose $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ so that the plaques of their charts are convex balls in the leaves. This follows from the relation between the convexity and injectivity radii [14, Theorem IX.6.1], and the continuity of the injectivity radius on closed manifolds [21, 41]—the case of closed manifolds easily yields local lower bounds of the injectivity radius on arbitrary manifolds, valid for all metrics that are close enough to a given metric in the weak C^∞ topology.

2.4. **Spaces of foliated maps.** Suppose that X and X' are C^r for some $r \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. We use the following notation[4] for sets of maps $X \to X'$:

- $C^r(X, \mathcal{F}; X', \mathcal{F}')$ is the set of C^r foliated maps.
- $\operatorname{Diffeo}^r(X, \mathcal{F}; X', \mathcal{F}')$ (or $\operatorname{Diffeo}^r(X, \mathcal{F})$ if $X = X'$) is the set of C^r foliated diffeomorphisms.
- $\operatorname{Emb}^r(X, \mathcal{F}; X', \mathcal{F}')$ is the set of C^r foliated embeddings.
- $\operatorname{Prop}^r(X, \mathcal{F}; X', \mathcal{F}')$ is the set of proper C^r foliated maps.
- $\operatorname{Homeo}(X, \mathcal{F}; X', \mathcal{F}')$ (or $\operatorname{Homeo}(X, \mathcal{F})$ if $X = X'$) is the set of foliated homeomorphisms.

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[4] The foliated structures are added to this notation to avoid ambiguity.

If $r = 0$ or it is clear that $r = \infty$, then r is removed from the above notation. $\operatorname{Homeo}(X, \mathcal{F})$ is a subgroup of the group of homeomorphisms, $\operatorname{Homeo}(X)$.

Let us define two foliated versions of the weak/strong C^r topology. In the first version, consider any $\phi \in C^r(X, \mathcal{F}; X', \mathcal{F}')$, locally finite families of foliated charts, $\mathcal{U} = \{U_i, \xi_i\}$ of X and $\mathcal{U}' = \{U'_a, \xi'_a\}$ of X', a family of compact subsets of X, $\mathcal{K} = \{K_i\}$, so that $K_i \subset U_i$ and $f(K_i) \subset U'_{a_i}$ for all i and corresponding indices a_i, a family $\mathcal{E} = \{\epsilon_i\}$ of positive numbers, and any integer $0 \leq k \leq r$ (it is enough to take $k = r$ if $r < \infty$). Then let $\mathcal{N}^k_{\mathrm{F}}(\phi, \mathcal{U}, \mathcal{U}', \mathcal{K}, \mathcal{E})$ be the set of foliated maps $\psi : X \to X'$ such that $\psi(K_i) \subset U'_{a_i}$ and

$$\left| \frac{\partial^\alpha (\phi^1_{a_i i} - \psi^1_{a_i i})}{\partial \mathbf{v}^\alpha} (\mathbf{v}, u) \right| < \epsilon_i \, ,$$

for all i, $(\mathbf{v}, u) \in \xi_i(K_i)$ and multi-indices α with $|\alpha| \leq k$, where $\phi^1_{a_i i}$ and $\psi^1_{a_i i}$ are given by (3). All possible sets $\mathcal{N}^k_{\mathrm{F}}(\phi, \mathcal{U}, \mathcal{U}', \mathcal{K}, \mathcal{E})$ form a base of open sets in a topology on $C^r(X, \mathcal{F}; X', \mathcal{F}')$, called the *strong foliated C^r topology*. The *weak foliated C^r topology* is similarly defined by using finite families of indices i. The subindex "WF/SF" will be added to the notation to indicate that the weak/strong foliated C^r topology in a family of C^r foliated maps. Note that $C_{\mathrm{WF}}(X, \mathcal{F}; X', \mathcal{F}')$ has the compact-open topology. Of course both topologies coincide when X is compact, and only the subindex "F" will be added in this case.

If X is compact, then the group of homeomorphisms, $\operatorname{Homeo}(X)$, is a Polish topological group with the compact-open topology [11, Theorem 3]. Moreover $\operatorname{Homeo}(X, \mathcal{F})$ is a closed subgroup of $\operatorname{Homeo}(X)$, and therefore it is also a Polish topological group.

Some important results on spaces of C^r maps between manifolds have straightforward generalizations to C^r foliated spaces, like the following.

Proposition 2.5. *The following properties hold:*

- *(i) The injectivity/surjectivity of the restrictions of the tangent map to the fibers defines an open subset of* $C^r_{\mathrm{SF}}(X, \mathcal{F}; X', \mathcal{F}')$ *for* $1 \leq r \leq \infty$.
- *(ii)* $\operatorname{Prop}^r(X, \mathcal{F}; X', \mathcal{F}')$ *is open in* $C^r_{\mathrm{SF}}(X, \mathcal{F}; X', \mathcal{F}')$ *for* $0 \leq r \leq \infty$.

Proof. Adapt the proofs of [32, Theorems 2.1.1 and 2.1.2]. □

For general C^r foliated maps $X \to X'$, $r \geq 1$, the injectivity/surjectivity of the restrictions of their tangent maps to the fibers does not have any consequence on their transverse behavior, given by the induced morphisms $\mathcal{H} \to \mathcal{H}'$. Thus the foliated immersions/submersions or foliated local homeomorphisms cannot be described using only the tangent map. So conditions on the induced morphisms $\mathcal{H} \to \mathcal{H}'$ must be added to extend some deeper results. For this reason, we use a second version of weak/strong C^r topology introduced in [8], which is finer than the weak/strong foliated C^r topology. The *strong plaquewise C^r topology* has a base of open sets $\mathcal{N}^k_{\mathrm{P}}(\phi, \mathcal{U}, \mathcal{U}', \mathcal{K}, \mathcal{E})$, defined by adding the condition $p'_{a_i}\phi = p'_{a_i}\psi$ on every K_i to the above definition of $\mathcal{N}^k_{\mathrm{F}}(\phi,\mathcal{U},\mathcal{U}',\mathcal{K},\mathcal{E})$; using (3), this extra condition can be also written as $\phi^2_{a_i i} = \psi^2_{a_i i}$ on $p_i(K_i)$ for all i. The *weak plaquewise C^r topology* is similarly defined by requiring the conditions only for finite families of indices i. The subindex "WP/SP" will be added to the notation to indicate that the weak/strong plaquewise C^r topology is considered in a family of C^r foliated maps. Note that, if two foliated maps are close enough in $C^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$, then they induce the same morphism $\mathcal{H} \to \mathcal{H}'$; in fact, they are leafwisely homotopic if $r = \infty$, as follows by taking basic open sets $\mathcal{N}^k_{\mathrm{P}}(\phi,\mathcal{U},\mathcal{U}',\mathcal{K},\mathcal{E})$ as above where the plaques of the foliated charts in $\mathcal{U}'$ are convex balls in the leaves for a given Riemannian metric on X', and then using geodesic segments to define homotopies.

With the strong plaquewise C^r topology, we can continue the direct extensions of results about spaces of C^r maps between manifolds.

Proposition 2.6. *The following properties hold:*

- *(i)* $\mathrm{Emb}^r(X,\mathcal{F};X',\mathcal{F}')$ *is open in* $C^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$ *for* $1 \le r \le \infty$.
- *(ii) For* $1 \le r \le \infty$, *the set of closed* C^r *foliated embeddings is open in* $C^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$.
- *(iii)* $\mathrm{Diffeo}^r(X,\mathcal{F};X',\mathcal{F}')$ *is open in* $C^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$ *for* $1 \le r \le \infty$.
- *(iv)* $C^s(X,\mathcal{F};X',\mathcal{F}')$ *is dense in* $C^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$ *for* $0 \le r < s \le \infty$.
- *(v)* $\mathrm{Diffeo}^s(X,\mathcal{F};X',\mathcal{F}')$ *is dense in* $\mathrm{Diffeo}^r_{\mathrm{SP}}(X,\mathcal{F};X',\mathcal{F}')$ *for* $1 \le r < s \le \infty$.
- *(vi) If* $1 \le r < \infty$, *any* C^r *foliated space is* C^r *diffeomorphic to a* C^∞ *foliated space.*
- *(vii) If* $1 \le r < s \le \infty$, *two* C^s *foliated spaces are* C^s *diffeomorphic if and only if they are* C^r *diffeomorphic.*

Proof. Adapt the proofs of [32, Theorems 2.1.4, 2.1.6, 2.2.6, 2.2.7, 2.2.9 and 2.2.10, and Corollary 2.1.6]. □

Like in the case of manifolds, it easily follows from Proposition 2.6-(iv) that, for $0 \le r < s \le \infty$, if there is a C^r leafwise homotopy between C^s foliated maps, then there is a C^s leafwise homotopy between them.

The above openness statements are stronger with the strong foliated C^r topology, whereas the denseness statements are stronger for the strong plaquewise C^r topology. There is no version of Proposition 2.6-(i) with the strong foliated C^r topology (for instance, consider the case of compact spaces foliated by points). However we can prove a weaker form of that statement by using certain subspaces $C^r_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ defined as follows. A foliated map $\phi : X \to X'$ is called a *transverse embedding* (respectively, *transverse equivalence*) if the induced morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is generated by embeddings (respectively, Φ is an isomorphism). Observe that $\mathcal{F}'(\phi(X)) = X'$ if ϕ is a transverse equivalence. A subset $\mathcal{M} \subset C(X,\mathcal{F};X',\mathcal{F}')$ of transverse embeddings (respectively, transverse equivalences) is called *uniform* if there are some foliated atlases, $\mathcal{U}$ of X and $\mathcal{U}'$ of X' like in Section 2.3, such that, for all $\phi \in \mathcal{M}$, the maps ϕ^2_{ai} in (3) are embeddings (respectively, open embeddings). Note that, if these properties hold with $\mathcal{U}$ and $\mathcal{U}'$, then they hold with all finer atlases. For example, $\operatorname{Emb}(X,\mathcal{F};X',\mathcal{F}')$ consists of uniform transverse embeddings, and $\operatorname{Homeo}(X,\mathcal{F};X',\mathcal{F}')$ consists of uniform transverse equivalences.

Proposition 2.7. *For $1 \le r \le \infty$, let $\mathcal{M} \subset C^r_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ be a uniform subspace of transverse embeddings. Then $\operatorname{Emb}^r(X,\mathcal{F};X',\mathcal{F}') \cap \mathcal{M}$ is open in $\mathcal{M}$.*

Proof. It is enough to prove the case $r = 1$. For any $\phi \in \operatorname{Emb}^1(X,\mathcal{F};X',\mathcal{F}') \cap \mathcal{M}$, consider a basic open set $\mathcal{N}_1 := \mathcal{N}^1_{\mathrm{F}}(\phi,\mathcal{U},\mathcal{U}',\mathcal{K},\mathcal{E})$ in $C^1_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ as above. We can assume that $\mathcal{K}$ (and therefore $\mathcal{U}$) covers X, and $\mathcal{U}'$ covers X'. After refinements, we can choose $\mathcal{U}$, $\mathcal{U}'$ and $\mathcal{K}$ such that the maps ψ^2_{ai} are embeddings for all $\psi \in \mathcal{M}$, and the interiors $V_i := \mathring{K}_i$ cover X. Take an open cover $\{W_i\}$ of X with $\overline{W_i} \subset V_i$ for all i. By [32, Lemma 1.3], we can choose $\mathcal{E}$ such that the maps $\psi : p_i^{-1}(u) \cap V_i \to p'^{-1}_{a_i}(\psi^2_{a_i i}(u))$ are C^1 embeddings for $u \in p_i(V_i)$ and $\psi \in \mathcal{N}_1$. Hence $\psi : V_i \to X'$ is a C^1 foliated embedding for all $\psi \in \mathcal{N}_1 \cap \mathcal{M}$.

Now, we adapt the final part of the proof of [32, Theorem 1.4] as follows. Since ϕ is an embedding, we get disjoint open subsets $V'_i, W'_i \subset X'$ for every i such that $\phi(\overline{W_i}) \subset W'_i$ and $\phi(X \smallsetminus V_i) \subset V'_i$. Then it is easy to find a neighborhood $\mathcal{N}_0$ of ϕ in $C_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ so that $\psi(\overline{W_i}) \subset W'_i$ and $\psi(X \smallsetminus V_i)) \subset V'_i$ for all $\psi \in \mathcal{N}_0$. We finally obtain $\mathcal{N}_0 \cap \mathcal{N}_1 \cap \mathcal{M} \subset \operatorname{Emb}^1(X,\mathcal{F};X',\mathcal{F}')$. □

Proposition 2.8. *For $1 \le r \le \infty$, let $\mathcal{M} \subset C^r_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ be a uniform subspace of transverse equivalences. Then $\operatorname{Diffeo}^r(X,\mathcal{F};X',\mathcal{F}') \cap \mathcal{M}$ is open in $\mathcal{M}$.*

Proof. We adapt the proofs of [32, Corollary 1.6 and Theorem 1.6]. The set

$$\mathcal{M}' = \{\, \phi \in \operatorname{Prop}^r(X,\mathcal{F};X',\mathcal{F}') \mid T_x\phi \text{ is surjective } \forall x \in X \,\}$$

is closed in $C^r_{\mathrm{SF}}(X,\mathcal{F};X',\mathcal{F}')$ by Proposition 2.5-(i),(ii). On the other hand, $\operatorname{Emb}^r(X,\mathcal{F};X',\mathcal{F}') \cap \mathcal{M}$ is open in $\mathcal{M}$ by Proposition 2.7. Thus the result follows because $\operatorname{Emb}^r(X,\mathcal{F};X',\mathcal{F}') \cap \mathcal{M}' = \operatorname{Diffeo}^r(X,\mathcal{F};X',\mathcal{F}')$. □

According to Proposition 2.6-(vi),(vii), we will only consider either (C^0) foliated spaces or C^∞ foliated spaces from now on.

Proposition 2.9. *Let $\phi : X \to X'$ be a foliated map. Suppose that X' is equipped with a C^∞ structure. Then there is at most one C^∞ structure on X such that ϕ is C^∞ and $T_x\phi$ is an isomorphism for all $x \in X$.*

Proof. Consider two C^∞ structures on X, and take C^∞ foliated charts, $\xi_1 : U_1 \to B_1 \times T_1$ of the first C^∞ structure on X, $\xi_2 : U_2 \to B_2 \times T_2$ of the second C^∞ structure on X, and $\xi' : U' \to B' \times T'$ of the C^∞ structure on X'. We can assume that $U_2 \subset U_1$ and $\phi(U_1) \subset U'$. Then

$$\begin{aligned}
\xi'\phi\xi_1^{-1}(\mathbf{v}_1, u_1) &= (g_1'(\mathbf{v}_1, u_1), h_1'(u_1)) \,, \\
\xi'\phi\xi_2^{-1}(\mathbf{v}_2, u_2) &= (g_2'(\mathbf{v}_2, u_2), h_2'(u_2)) \,, \\
\xi_1\xi_2^{-1}(\mathbf{v}_2, u_2) &= (g_{12}(\mathbf{v}_2, u_2), h_{12}(u_2)) \,,
\end{aligned}$$

for $(\mathbf{v}_k, u_k) \in B_k \times T_k$, $k = 1, 2$, where $g_k' : B_k \times T_k \to B'$ has partial derivatives of arbitrary order with respect to $\mathbf{v}_k$, continuous on $B_k \times T_k$, and $g_{12} : B_2 \times T_2 \to B_1$ is continuous. Moreover the differential map of g_1' with respect to $\mathbf{v}_1$ is an isomorphism at any point. Therefore, by the inverse function theorem, we can assume that $g_1'(\cdot, u_1) : B_1 \to g_1'(B_1 \times \{u_1\})$ is a C^∞ diffeomorphism for all $u_1 \in T_1$. Its inverse function is denoted by $\bar{g}_1'(\cdot, u_1) : g_1'(B_1 \times \{u_1\}) \to B_1$. For any small ball $B_0' \subset B'$, let $T_{10} \subset T_1$ be the open subset that consists of the points $u_1 \in T_1$ such that $B_0' \subset g_1'(B_1 \times \{u_1\})$. It also follows from the inverse function theorem that the partial derivatives of arbitrary order of $\bar{g}_1'(\cdot, u_1) : B_0' \to B_1$ depend continuously on u_1. Since

$$g_{12}(\mathbf{v}_2, u_2) = \bar{g}_1(g_2(\mathbf{v}_2, u_2), h_{12}(u_2))$$

on $B_2 \times h_{21}(T_{10})$, the function $g_{12} : B_2 \times h_{21}(T_{10}) \to B_1$ has partial derivatives of arbitrary order with respect to $\mathbf{v}_2$, continuous on $B_2 \times h_{21}(T_{10})$. □

2.5. **Center of mass.** In Section 6.2, we will use the center of mass of a mass distribution on a Riemannian manifold M [35], [14, Section IX.7].

Let $\Omega \subset M$ be a compact submanifold with boundary with $\dim \Omega = \dim M$. For $0 \le r \le \infty$, let $\mathcal{C}(\Omega)$ be the set of functions $f \in C^{r+2}(\Omega)$ such that $\operatorname{grad} f$ is an outward pointing vector field on $\partial\Omega$ and $\operatorname{Hess} f$ is positive definite on the interior $\mathring{\Omega}$ of Ω. Note that $\mathcal{C}(\Omega)$ is open in the Banach space $C^{r+2}(\Omega)$ with the norm $\| \ \|_{C^{r+2},\Omega,g}$, and therefore it is a C^∞ Banach manifold. Moreover $\mathcal{C}(\Omega)$ is preserved by the operations of sum and product by positive numbers. Any $f \in \mathcal{C}(\Omega)$ attains its minimum value at a unique point $\mathbf{m}_\Omega(f) \in \mathring{\Omega}$, defining a function $\mathbf{m}_\Omega : \mathcal{C}(\Omega) \to \mathring{\Omega}$.

Lemma 2.10 ([3, Lemma 10.1 and Remark 11-(ii)])**.** *The map* $\mathbf{m}_\Omega$ *is* C^r.

Suppose that M is connected and complete. Let (A, μ) be a probability space, B a convex open ball of radius $r > 0$ in M, and $f : A \to B$ a measurable map, which is called a *mass distribution* on B. Consider the C^∞ function $P_{f,\mu} : B \to \mathbb{R}$ defined by

$$P_{f,\mu}(x) = \frac{1}{2} \int_A d(x, f(a))^2 \, \mu(a) \,.$$

Proposition 2.11 (H. Karcher [35, Theorem 1.2])**.** *We have the following:*

- *(i)* $\operatorname{grad} P_{f,\mu}$ *is an outward pointing vector field on the boundary* $\partial\overline{B}$.
- *(ii) If* $\delta > 0$ *is an upper bound for the sectional curvatures of* M *in* B, *and* $2r < \pi/2\sqrt{\delta}$, *then* $\operatorname{Hess} P_{f,\mu}$ *is positive definite on* B.

If the hypotheses of Proposition 2.11 are satisfied, then $P_{f,\mu} \in \mathcal{C}(\overline{B})$, and therefore $\mathcal{C}_{f,\mu} := \mathbf{m}_{\overline{B}}(P_{f,\mu}) \in B$ is defined and called the *center of mass* of f (with respect to μ). This point is independent of the choice of B satisfying the above conditions. The following is a consequence of Lemma 2.10.

Corollary 2.12 ([3, Corollary 10.3]; cf. [35, Corollary 1.6])**.** *The following properties hold:*

(i) $\mathcal{C}_{f,\mu}$ depends continuously on f and the metric tensor of M.

(ii) If A is the Borel σ-algebra of a metric space, then $\mathcal{C}_{f,\mu}$ depends continuously on μ in the weak-$\ast$ topology.

Consider the following particular case. Let N be a C^∞ manifold, $\phi = (\phi_1,\dots,\phi_k) : N \to M^k$ a C^∞ map, and $\lambda = (\lambda_1,\dots,\lambda_k)$ a finite C^∞ partition of unity of N. For every $x \in N$, consider the probability measure $\mu_{\phi,\lambda,x} = \sum_{i=1}^k \lambda_i(x)\,\delta_{\phi_i(x)}$, where δ_y denotes the Dirac mass at every $y \in M$. Suppose that, for all $x \in N$, the points $\phi_1(x),\dots,\phi_k(x)$ lie in a ball B_x of M satisfying the conditions of Proposition 2.11. Then we can define center of mass $\mathcal{C}_{\phi,\lambda,x}$ of id_{B_x} with respect to $\mu_{\phi,\lambda,x}$, which is independent of the choice of B_x. The following sharpening of Corollary 2.12 also follows from Lemma 2.10.

Corollary 2.13. *The map $N \to M$, $x \mapsto \mathcal{C}_{\phi,\lambda,x}$, is C^∞.*

3. MOLINO'S DESCRIPTION

Consider the notation of Section 2.3 in the rest of the paper.

Proof of Theorem A. Most of the properties stated in this theorem were already proved in [10, Theorem A]. It only remains to prove the part concerning H. For this purpose, we have to recall the construction of G, $\widehat{X}_0$, $\widehat{\mathcal{F}}_0$ and $\hat{\pi}_0$. We can assume that X satisfies the conditions of equicontinuity and strong quasi-analyticity with the same set S, and that $\overline{\mathcal{H}}$ satisfies the conditions of equicontinuity and strong quasi-analyticity with the induced set $\overline{S}$. Let $\overline{S}_{\text{c-o}}$ be the space $\overline{S}$ with the restriction of the compact-open topology on the set of partial maps $T \rightarrowtail T$ with open domain [1]. Consider the subspace

$$\overline{S}_{\text{c-o}} * T = \{\, (g,u) \in \overline{S} \times T \mid u \in \operatorname{dom} g \,\} \subset \overline{S}_{\text{c-o}} \times T\ ,$$

and equip the set $\widehat{T}$ of all germs of maps in $\overline{S}$ (or $\overline{\mathcal{H}}$) with the final topology induced by the germ map $\gamma : \overline{S}_{\text{c-o}} * T \to \widehat{T}$ (this is not the restriction of the sheaf topology). Consider the restrictions $s,t : \widehat{T} \to T$ of the sourse and target maps. The space $\widehat{T}$ is locally compact and Polish, and $\hat{\pi} := (s,t) : \widehat{T} \to T \times T$ is continuous and proper.

Fix some point $u_0 \in T_{i_0} \subset T$. Then the subspace $\widehat{T}_0 := s^{-1}(u_0) \subset \widehat{T}$ is locally compact and Polish. This definition is different from the one given in [10, Section 3D], where $\widehat{T}_0 = t^{-1}(u_0)$ was considered. This change can be made because the inversion of local transformations defines a homeomorphism of $\overline{S}_{\text{c-o}}$ [10, Proposition 3.1], and therefore the germ inversion defines a homeomorphism of $\widehat{T}$, which becomes a topological groupoid by [1, Proposition 10]. The rest of definitions and arguments of [10, Sections 3D–3G] must be changed accordingly. For instance, take $\hat{\pi}_0 = t : \widehat{T}_0 \to T$ (instead of $\hat{\pi}_0 = s$, used in [10]), which is open, continuous and proper, and its fibers are homeomorphic to each other [10, Section 3D]. We have $\widehat{T}_0 \equiv \bigsqcup_i \widehat{T}_{i,0}$, where $\widehat{T}_{i,0} = \hat{\pi}_0^{-1}(T_i)$.

Note that $H := \hat{\pi}_0^{-1}(u_0) = \hat{\pi}^{-1}(u_0, u_0)$ becomes a compact Polish group since $\widehat{T}$ is a topological groupoid. Moreover the germ product defines a continuous free right action of H on $\widehat{T}_0$ whose orbits are clearly equal to the fibers of $\hat{\pi}_0 : \widehat{T}_0 \to T$. Thus this map induces a continuous bijection $\widehat{T}_0/H \to T$. In fact this bijection is a homeomorphism, as easily follows by using also that H is compact, $\widehat{T}_0$ is locally compact, and T is Hausdorff.

For any $h \in \mathcal{H}$, define $\hat{h} : \hat{\pi}_0^{-1}(\operatorname{dom} h) \to \hat{\pi}_0^{-1}(\operatorname{im} h)$ by $\hat{h}(\boldsymbol{\gamma}(g, u_0)) = \boldsymbol{\gamma}(hg, u_0)$ for $g \in \overline{S}$ with $u_0 \in \operatorname{dom} g$ and $g(u_0) \in \operatorname{dom} h$ (instead of $\hat{h}(\boldsymbol{\gamma}(g, u)) = \boldsymbol{\gamma}(gh^{-1}, h(u))$ for $u \in \operatorname{dom} g \cap \operatorname{dom} h$ with $g(u) = u_0$, used in [10]). The maps $\hat{h}$ are local transformations of $\widehat{T}_0$ satisfying $h\hat{\pi}_0 = \hat{\pi}_0\hat{h}$, $\widehat{\mathrm{id}_T} = \mathrm{id}_{\widehat{T}_0}$, $\widehat{hh'} = \hat{h}\widehat{h'}$ and $\hat{h}^{-1} = \widehat{h^{-1}}$ [10, Sections 3E]. Moreover it is easy to see that every $\hat{h}$ is H-equivariant (note that $\operatorname{dom}\hat{h}$ and $\operatorname{im}\hat{h}$ are H-invariant). Let $\widehat{\mathcal{H}}_0$ be the pseudogroup on $\widehat{T}_0$ generated by $\widehat{S}_0 = \{\, \hat{h} \mid h \in S \,\}$. There is a local group G and some dense finitely generated sublocal group $\Gamma \subset G$ such that $\widehat{\mathcal{H}}_0$ is equivalent to the pseudogroup generated by the local action of Γ on G by local left translations [10, Proposition 3.41]—this was proved by checking that $\widehat{\mathcal{H}}_0$ is compactly generated, equicontinuous and strongly locally free, and its closure is also strongly locally free, and then applying Proposition 2.2-(i). Furthermore $\hat{\pi}_0$ generates a morphism $\widehat{\mathcal{H}}_0 \to \mathcal{H}$.

Let $\check{U}_{i,0} = U_i \times \widehat{T}_{i,0} \times \{i\} \equiv U_i \times \widehat{T}_{i,0}$, equipped with the product topology, and consider the topological sum

$$\check{X}_0 := \bigsqcup_i (U_i \times \widehat{T}_{i,0}) = \bigcup_i \check{U}_{i,0} \ ,$$

and the closed subspaces

$$\widetilde{U}_{i,0} := \{\, (x, \gamma, i) \in \check{U}_{i,0} \mid p_i(x) = \hat{\pi}_0(\gamma) \,\} \subset \check{U}_{i,0} \ , \quad \widetilde{X}_0 := \bigcup_i \widetilde{U}_{i,0} \subset \check{X}_0 \ .$$

Note that $\widetilde{X}_0$ is the topological sum of the spaces $\widetilde{U}_{i,0}$. Consider the equivalence relation "$\sim$" on $\widetilde{X}_0$ defined by $(x, \gamma, i) \sim (y, \delta, j)$ if $x = y$ and $\gamma = \widehat{h_{ji}}(\delta)$. Let $\widehat{X}_0$ be the corresponding quotient space, let $q : \widetilde{X}_0 \to \widehat{X}_0$ be the quotient map, let $[x, \gamma, i] = q(x, \gamma, i)$, let $\widehat{U}_{i,0} = q(\widetilde{U}_{i,0})$, and let $\tilde{p}_{i,0} : \widetilde{U}_{i,0} \to \widehat{T}_{i,0}$ denote the restriction of $\check{p}_{i,0} : \check{U}_{i,0} \equiv U_i \times \widehat{T}_{i,0} \to \widehat{T}_{i,0}$, which induces a map $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$. Moreover a map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([x, \gamma, i]) = x$. Observe that $\widehat{U}_{i,0} = \hat{\pi}_0^{-1}(U_i)$. Then $\widehat{X}_0$ is compact and Polish, $\{\widehat{U}_{i,0}, \hat{p}_{i,0}, \widehat{h_{ij}}\}$ is a defining cocycle of a minimal foliated structure $\widehat{\mathcal{F}}_0$ on $\widehat{X}_0$, $\hat{\pi}_0$ is continuous and open, the fibers of $\hat{\pi}_0$ are homeomorphic to each other, and the restriction of $\hat{\pi}_0$ to the leaves of $\widehat{X}_0$ are the holonomy coverings of the leaves of X [10, Section 4B]. In the proof of these properties, it was used that every restriction $q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is a homeomorphism.

Since every $\widehat{T}_{i,0}$ is H-invariant, we get an induced free right action of H on every $\check{U}_{i,0} \equiv U_i \times \widehat{T}_{i,0}$, acting as the identity on the factor U_i, yielding a right H-action on $\check{X}_0$ by union. This restricts to a free right action of H on $\widetilde{X}_0$, preserving every $\widetilde{U}_{i,0}$, because the H-orbits in $\widehat{T}_0$ are equal to the fibers $\hat{\pi}_0 : \widehat{T}_0 \to T$. Since moreover every $\widehat{h_{ij}}$ is H-equivariant, we get an induced right action on $\widehat{X}_0$, given by $[x, \gamma, i] \cdot \sigma = [x, \gamma\sigma, i]$ for $[x, \gamma, i] \in \widehat{X}_0$ and $\sigma \in H$. This action is also free because every restriction $q : \widetilde{U}_{i,0} \to \widehat{U}_{i,0}$ is a homeomorphism, and it is easy to see that its orbits equal the fibers of $\hat{\pi}_0 : \widehat{X}_0 \to X$. Finally note that every map $\hat{p}_{i,0} : \widehat{U}_{i,0} \to \widehat{T}_{i,0}$ is H-equivariant, and therefore H acts on $\widehat{X}_0$ by foliated transformations. □

In the rest of this section, assume that X satisfies the hypotheses of Theorem A. Consider structures $(G, H, \widehat{X}_0, \hat{\pi}_0)$ satisfying the conditions of its statement, where $\widehat{X}_0$ is considered as a foliated space and H-space. If desired, we may also add a finitely generated dense sublocal group $\Gamma \subset G$ to the notation, $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$, so that the holonomy pseudogroup of $\widehat{X}_0$ is represented by the pseudogroup generated by the left local action of Γ on G by local left translations. It is said that two such structures, $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ and $(G', \Gamma', H', \widehat{X}'_0, \hat{\pi}'_0)$, are *equivalent* if there are a local isomorphism $\psi : G \rightarrowtail G'$ that restricts to a local isomorphism $\Gamma \rightarrowtail \Gamma'$, an isomorphism $\chi : H \to H'$, and a foliated χ-equivariant homeomorphism $\phi : \widehat{X}_0 \to \widehat{X}'_0$ such that $\hat{\pi}_0 = \hat{\pi}'_0 \phi$ (the condition on Γ and Γ' is omitted if Γ and Γ' are not considered). In this case, (ψ, χ, ϕ) is called an *equivalence*. This notion of equivalence is natural because it clearly means that the descriptions of the foliated space X given by $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ and $(G', \Gamma', H', \widehat{X}'_0, \hat{\pi}'_0)$ are essentially the same, giving rise to equivalent invariants of X. For instance, G, Γ and H have the same algebraic and topological properties as G', Γ' and H', and $\hat{\pi}_0$ is a principal bundle projection if and only is $\hat{\pi}'_0$ is also a principal bundle projection.

Proposition 3.1 (Cf. [10, Propositions 3.43, 4.12 and 4.13])**.** *All structures $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ constructed in the proof of Theorem A are equivalent.*

Proof. We have to prove that the equivalence class of $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ is independent of the choices of u_0, S and $\{U_i, p_i, h_{ij}\}$. Most of this is already proved in [10, Propositions 3.43, 4.12 and 4.13]. We only have to check what concerns H.

To begin with, take another point of $u_1 \in T_{i_1} \subset T$, and let $\widehat{T}_1$, $\hat{\pi}_1$, $\widehat{S}_1$, $\widehat{\mathcal{H}}_1$, G_1, Γ_1 and H_1 be constructed like $\widehat{T}_0$, $\hat{\pi}_0$, $\widehat{S}_0$, $\widehat{\mathcal{H}}_0$, $G_0 := G$, $\Gamma_0 := \Gamma$ and $H_0 := H$ by using u_1 instead of u_0. Now, for each $h \in \mathcal{H}$, let us use the notation $\hat{h}_0 := \hat{h} \in \widehat{\mathcal{H}}_0$, and let $\hat{h}_1 : \hat{\pi}_1^{-1}(\operatorname{dom} h) \to \hat{\pi}_1^{-1}(\operatorname{im} h)$ be the map in $\widehat{\mathcal{H}}_1$ defined like $\hat{h}$. In particular, the maps $(\widehat{h_{ij}})_1$ are defined like the maps $(\widehat{h_{ij}})_0 := \widehat{h_{ij}}$. There is some $f_0 \in \overline{S}$ such that $u_0 \in \operatorname{dom} f_0$ and

$f_0(u_0) = u_1$. Let $\theta : \widehat{T}_0 \to \widehat{T}_1$ be defined by $\theta(\boldsymbol{\gamma}(f, u_0)) = \boldsymbol{\gamma}(f f_0^{-1}, u_1)$ (instead of $\theta(\boldsymbol{\gamma}(f, x)) = \boldsymbol{\gamma}(f_0 f, x)$, like in [10]). This map is a homeomorphism, and satisfies $\hat{\pi}_0 = \hat{\pi}_1\theta$, $\operatorname{dom}\hat{h}_1 = \theta(\operatorname{dom}\hat{h}_0)$ and $\hat{h}_1\theta = \theta\hat{h}_0$ for all $h \in S$, obtaining that θ generates an equivalence $\Theta : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}_1$ [10, Proposition 3.42]. For $k = 0, 1$, let $\mathcal{G}_k$ be the pseudogroup on G_k generated by local left translations by elements of Γ_k. Via equivalences $\widehat{\mathcal{H}}_k \to \mathcal{G}_k$, Θ corresponds to an equivalence $\Theta' : \mathcal{G}_0 \to \mathcal{G}_1$. Since the local right translations of G_1 generate equivalences of $\mathcal{G}_1$, we can assume that the orbits of the identity elements correspond by the induced map $G_0/\mathcal{G}_0 \to G_1/\mathcal{G}_1$. By Proposition 2.2-(ii), it follows that Θ' is generated by a local isomorphism $\psi : G_0 \rightarrowtail G_1$ that restricts to a local isomorphism $\Gamma \rightarrowtail \Gamma'$. On the other hand, the conjugation mapping, $\boldsymbol{\gamma}(f, u_0) \mapsto \boldsymbol{\gamma}(f_0 f f_0^{-1}, u_1)$, defines an isomorphism $\chi : H_0 \to H_1$ so that θ is χ-equivariant.

Now, define $\widehat{X}_1 \equiv (\widehat{X}_1, \widehat{\mathcal{F}}_1)$, $[x, \gamma, i]_1$ and $\hat{\pi}_1 : \widehat{X}_1 \to X$ like $\widehat{X}_0 \equiv (\widehat{X}_0, \widehat{\mathcal{F}}_0)$, $[x, \gamma, i]_0 \coloneqq [x, \gamma, i]$ and $\hat{\pi}_0 : \widehat{X}_0 \to X$, using $\widehat{T}_1$, $\hat{\pi}_1 : \widehat{T}_1 \to T$ and the maps $(\widehat{h_{ij}})_1$ instead of $\widehat{T}_0$, $\hat{\pi}_0 : \widehat{T}_0 \to T$ and the maps $(\widehat{h_{ij}})_0$. According to [10, Proposition 4.12], a foliated homeomorphism $\phi : \widehat{X}_0 \to \widehat{X}_1$ is defined by $\phi([x, \gamma, i]_0) = [x, \theta(\gamma), i]_1$, which satisfies $\hat{\pi}_0 = \hat{\pi}_1\phi$ and induces the equivalence $\Theta : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}_1$. Moreover ϕ is χ-equivariant: for all $[x, \gamma, i]_0 \in \widehat{X}_0$ and $\sigma \in H_0$,

$$\begin{aligned}\phi([x, \gamma, i]_0 \cdot \sigma) &= \phi([x, \gamma\sigma, i]_0) = [x, \theta(\gamma\sigma), i]_1\\ &= [x, \theta(\gamma)\chi(\sigma), i]_1 = [x, \theta(\gamma), i]_1 \cdot \chi(\sigma)\ .\end{aligned}$$

All choices of S define the same space $\widehat{T}_0$ by [10, Propositions 3.43], giving rise to the same Molino's description.

To prove the independence of $\{U_i, p_i, h_{ij}\}$, it is enough to consider the case where $\{U_i, p_i, h_{ij}\}$ refines another defining cocycle $\{U'_a, p'_a, h'_{ab}\}$. Let $\mathcal{H}'$ be the corresponding representative of the holonomy pseudogroup on $T' = \bigsqcup_a T'_a$. If $U_i \subset U'_{a_i}$, there is an induced open embedding $\phi_i : T_i \to T'_{a_i}$. These maps generate an equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$. In fact, $h'_{a_i a_j}\phi_j = \phi_i h_{ij}$. Let $u'_0 = \phi_{i_0}(u_0) \in T'_{a_{i_0}} \subset T'$, and let $S' \subset \mathcal{H}'$ be a generating subset such that $S'^2 \subset S' = S'^{-1}$. We can also use $\{U'_a, p'_a, h'_{ab}\}$, u'_0 and S' to define $\widehat{T}'_0$, $\hat{\pi}'_0 : \widehat{T}'_0 \to T'$ and $\widehat{\mathcal{H}}'_0$ like $\widehat{T}_0$, $\hat{\pi}_0 : \widehat{T}_0 \to T$ and $\widehat{\mathcal{H}}_0$; in particular, the generators $\widehat{h'_{ab}}$ of $\widehat{\mathcal{H}}'_0$ are defined like the generators $\widehat{h_{ij}}$ of $\widehat{\mathcal{H}}_0$. We get open embeddings $\hat{\phi}_{i,0} : \widehat{T}_{i,0} \to \widehat{T}'_{a_i,0}$ defined by $\hat{\phi}_{i,0}(\boldsymbol{\gamma}(g, u_0)) = \boldsymbol{\gamma}(\phi_i g \phi_{i_0}^{-1}, u'_0)$, which generate an equivalence $\widehat{\Phi}_0 : \widehat{\mathcal{H}}_0 \to \widehat{\mathcal{H}}'_0$ (this is a corrected version of [10, Proposition 3.44]). Let $(G', \Gamma', H', \widehat{X}'_0, \hat{\pi}'_0)$ be the Molino's description defined with $\widehat{T}'_0$, $\hat{\pi}'_0 : \widehat{T}'_0 \to T'$ and the maps $\widehat{h'_{ab}}$. Let us use the notation $[x, \gamma', a]'$ for the element of $\widehat{X}'_0$ represented by a tern (x, γ', a). Let $\mathcal{G}$ and $\mathcal{G}'$ be the pseudogroups on G and G' generated by the local left translations by elements of Γ and Γ'. Via equivalences $\mathcal{H} \to \mathcal{G}$ and $\mathcal{H}' \to \mathcal{G}'$, $\widehat{\Phi}_0$ corresponds to an equivalence $\widehat{\Phi}'_0 : \mathcal{G} \to \mathcal{G}'$. As above, we can assume that the orbits of the identity elements correspond by the induced map $G/\mathcal{G} \to G'/\mathcal{G}'$, and therefore, according to Proposition 2.2-(ii), $\widehat{\Phi}'_0$ is generated by a local isomorphism $\psi : G \rightarrowtail G'$ that restricts to a local isomorphism $\Gamma \rightarrowtail \Gamma'$. Moreover $\hat{\phi}_{i_0,0}$ restricts to an isomorphism $\chi : H \to H'$ so that any map in $\widehat{\Phi}_0$ is χ-equivariant. Finally, a canonical foliated homeomorphism $\phi : \widehat{X}_0 \to \widehat{X}'_0$ is well defined by $\phi([x,\gamma,i]) = [x,\hat{\phi}_{i,0}(\gamma),a_i]'$ [10, Proposition 4.13]. It is easy to check that ϕ is H-equivariant. □

By Proposition 3.1, the equivalence class of any structure $(G,\Gamma,H,\widehat{X}_0,\hat{\pi}_0)$ constructed in the proof of Theorem A can be called the *Molino's description* of X. According to the discussion of [10, Section 1.E], these structures are kind of a topological interpretation of the original Molino's description in the case of a Riemannian foliation. That similarity can be indeed realized as an equivalence between the original Molino's description and ours in that case. According to Molino's terminology, the local isomorphism class of G is called the *structural local group* [10], and, with the terminology of [18, 19], $\widehat{X}_0$ will be called the *Molino space* and H the *discriminant group*.

Proposition 3.2. *X is a G-foliated space for some local group G if and only if its discriminant group is trivial.*

Proof. The "if" part of the statement is directly given by Theorem A. To prove the "only if" part, assume X is a G-foliated space for some local group G. Thus $\overline{\mathcal{H}}$ is strongly locally free, obtaining that $H = \{e\}$ according to the definition of H given in the proof of Theorem A. □

For every $\hat{x} \in \widehat{X}_0$, let $\widehat{L}_{\hat{x}}$ denote the leaf of $\widehat{X}_0$ through $\hat{x}$, and consider the identity $\widetilde{L}^{\mathrm{hol}}_x \equiv \widehat{L}_{\hat{x}}$ given by Theorem A.

Lemma 3.3. *For $x \in X$ and $\hat{x} \equiv [x,\gamma,i] \in \hat{\pi}_0^{-1}(x)$, let $c : I \to X$ be a leafwise path from x to some point y, and let $\hat{c}$ be the unique lift of c to $\widetilde{L}^{\mathrm{hol}}_x \equiv \widehat{L}_{\hat{x}}$ beginning at $\hat{x}$. Then $\hat{c}(1) \equiv [y,\delta\gamma,j_\beta]$, where $\delta = \boldsymbol{\gamma}(h_{\mathcal{J}},p_i(x))$ for any $\mathcal{J} = (j_0,\dots,j_\beta)$ covering c with $j_0 = i$.*

Proof. Take a partition $0 = t_0 < t_1 < \dots < t_{\beta+1} = 1$ of I such that $c([t_k,t_{k+1}]) \subset U_{j_k}$ for $k = 0,\dots,\beta$. For $s \in I$, the path $c_s(t) := c(st)$ in L is covered by $\mathcal{J}_s := (j_0,\dots,j_{\beta_s})$, where $\beta_s = \min\{\, k \in \{0,\dots,\beta\} \mid t_{k+1} \ge s \,\}$, and let $\delta_s = \boldsymbol{\gamma}(h_{\mathcal{J}_s},p_i(x))$. Then it is easy to see that $\hat{c}(s) = [c(s),\delta_s\gamma,j_{\beta_s}]$. □

Fix some point $x_0 \in p_{i_0}^{-1}(u_0) \subset U_{i_0} \subset X$.

Proposition 3.4. *For $\hat{x}_0 \in \hat{\pi}_0^{-1}(x_0)$, we have*

$$\operatorname{Hol}(L_{x_0},x_0) = \{\, \gamma \in H \mid \widehat{L}_{\hat{x}_0} \cdot \gamma = \widehat{L}_{\hat{x}_0} \,\}\,, \tag{4}$$

and the map $\widetilde{L}^{\mathrm{hol}}_{x_0} \equiv \widehat{L}_{\hat{x}_0} \hookrightarrow \widehat{X}_0$ becomes equivariant with respect to the homomorphism $\operatorname{Hol}(L_{x_0},x_0) \hookrightarrow H$.

Proof. Consider the notation of the proof of Theorem A. Observe that $\operatorname{Hol}(L_{x_0},x_0)$ is a subgroup of H:

$$\begin{aligned}\operatorname{Hol}(L_{x_0},x_0) &= \{\,\boldsymbol{\gamma}(h,u_0) \mid h\in\mathcal{H},\ u_0\in\operatorname{dom}h,\ h(u_0)=u_0\,\}\\ &\subset H = \{\,\boldsymbol{\gamma}(g,u_0)\mid g\in\overline{\mathcal{H}},\ u_0\in\operatorname{dom}g,\ g(u_0)=u_0\,\}\ .\end{aligned}$$

If $\gamma = \operatorname{hol}([c])\in\operatorname{Hol}(L_{x_0},x_0)$ for some $[c]\in\pi_1(L_{x_0},x_0)$, then $\gamma=\boldsymbol{\gamma}(h_{\mathcal{I}}^{-1},u_0)$ for some $\mathcal{I}=(i_0,i_1,\dots,i_\alpha)$ covering c with $i_\alpha=i_0$. For any $y\in L_{x_0}$ and $\hat y\in\widehat{L}_{\hat x_0}\cap\hat\pi_0^{-1}(y)$, we have $\hat y\equiv[y,\delta,i]$, where $\delta=\boldsymbol{\gamma}(h_{\mathcal{J}},u_0)$ for some admissible sequence $\mathcal{J}=(j_0,\dots,j_\beta)$, with $j_0=i_0$ and $j_\beta=i$, which covers a leafwise path $c_y: I\to X$ from x_0 to y. Then $\hat y\cdot\gamma$ is the final point of the lift to $\widetilde{L}_x^{\mathrm{hol}}\equiv\widehat{L}_{\hat x_0}$, beginning at $\hat y$, of the loop $c_y^{-1}cc_y$ in L_{x_0}, based at y. Thus

$$\hat y\cdot\gamma\equiv[y,\delta\gamma\delta^{-1}\delta,i_0]=[y,\delta\gamma,i_0]=[y,\delta,i_0]\cdot\gamma\ ,$$

where the identity between these elements of $\widetilde{L}_x^{\mathrm{hol}}$ and $\widehat{L}_{\hat x_0}$ is given by Lemma 3.3, applied to $\hat y$ and $c_y^{-1}cc_y$, because $\mathcal{J}^{-1}\mathcal{I}\mathcal{J}$ is defined and covers $c_y^{-1}cc_y$, and $h_{\mathcal{J}^{-1}\mathcal{I}\mathcal{J}}=h_{\mathcal{J}}h_{\mathcal{I}}h_{\mathcal{J}}^{-1}$. This proves the inclusion "$\subset$" in (4) and the equivariance of $\widetilde{L}_{x_0}^{\mathrm{hol}}\equiv\widehat{L}_{\hat x_0}\hookrightarrow\widehat{X}_0$. On the other hand, since the right H-action on $\widehat{X}_0$ is free, foliated and preserves every $\hat\pi_0$-fiber, any element of the right hand side of (4) defines a covering transformation of the restriction $\hat\pi_0:\widehat{L}_{\hat x_0}\to L_{x_0}$, showing the inclusion "$\supset$" in (4). □

According to the proof of Proposition 3.1, it follows from Proposition 3.4 that, for all $x\in X$ and $\hat x\in\hat\pi_0^{-1}(x)$, there is an isomorphism

$$\operatorname{Hol}(L_x,x)\cong\{\,\gamma\in H\mid\widehat{L}_{\hat x}\cdot\gamma=\widehat{L}_{\hat x}\,\}$$

so that the map $\widetilde{L}_x^{\mathrm{hol}}\equiv\widehat{L}_{\hat x}\hookrightarrow\widehat{X}_0$ becomes equivariant with respect to the induced injective homomorphism $\operatorname{Hol}(L_x,x)\to H$. Nevertheless this isomorphism is not canonical in general.

4. FOLIATED HOMOGENEOUS FOLIATED SPACES

The foliated space X is called *foliated homogeneous* when the canonical left action of $\operatorname{Homeo}(X,\mathcal{F})$ on X is transitive. Similarly, if X is C^∞, it is called C^∞ *foliated homogeneous* when the canonical left action of $\operatorname{Diffeo}(X,\mathcal{F})$ on X is transitive. A priory, C^∞ foliated homogeneity is stronger than foliated homogeneity, but we will see that indeed they are equivalent conditions for compact minimal C^∞ foliated spaces (Section 7).

Take any complete metric d inducing the topology of X, and let D be the induced complete metric on $\operatorname{Homeo}(X)$ defined by

$$D(\phi,\psi)=\sup_{x\in X}d(\phi(x),\psi(x))+\sup_{x\in X}d(\phi^{-1}(x),\psi^{-1}(x))\ .$$

In this way, $\operatorname{Homeo}(X)$ becomes a completely metrizable topological group, and its canonical left action on X is continuous. Moreover it is easy to check that $\operatorname{Homeo}(X,\mathcal{F})$ is closed in $\operatorname{Homeo}(X)$, and therefore $\operatorname{Homeo}(X,\mathcal{F})$ is also a completely metrizable topological group.

Suppose that X is compact. Then D induces the compact-open topology on $\mathrm{Homeo}(X)$, as follows from [11, Theorem 3], obtaining that $\mathrm{Homeo}(X)$ is also second countable. So $\mathrm{Homeo}(X)$ is a Polish group, and $\mathrm{Homeo}(X,\mathcal{F})$ a Polish subgroup. Therefore, by a theorem of Effros [20, 43], if X is foliated homogeneous, then the canonical left action of $\mathrm{Homeo}(X,\mathcal{F})$ on X is *micro-transitive*; i.e., for all $x\in X$ and any neighbourhood $\mathcal{N}$ of id_X in $\mathrm{Homeo}(X,\mathcal{F})$, the set $\mathcal{N}\cdot x$ is a neighborhood of x in X.

Proof of Theorem B. Clark and Hurder have proved that any C^∞ homogeneous matchbox manifold is equicontinuous [15, Theorem 5.2]. Indeed, their argument applies to any compact minimal foliated homogeneous foliated space. Moreover the C^∞ structure is not used in that result. Thus the conditions of our statement are enough to get that $(X,\mathcal{F})$ is equicontinuous.

The rest of the proof uses the same main tool as in [15, Theorem 5.2], the indicated theorem of Effros.

Let us prove that $\mathcal{H}$ is strongly locally free. Since $\{U_i\}$ is finite, there is some $\epsilon>0$ such that $d(\overline{U_i},X\smallsetminus\widetilde{U}_i)<\epsilon$ for all i. Since the action of $\mathrm{Homeo}(X,\mathcal{F})$ on X is micro-transitive, there is some $\delta>0$ such that, for all $x,y\in X$ with $d(x,y)<\delta$, there exists some $\phi\in\mathrm{Homeo}(X,\mathcal{F})$ so that $D(\phi,\mathrm{id}_X)<\epsilon$ and $\phi(x)=y$.

Since every T_i has compact closure in $\widetilde{T}_i$, we easily get a finite open cover $\{T_{ia}\}$ of T_i such that the d-diameter of every $\sigma_i(T_{ia})$ is smaller than δ. Let $U_{ia}=\xi_i^{-1}(B_i\times T_{ia})$, $\xi_{ia}=\xi_i|_{U_{ia}}$, $\widetilde{U}_{ia}=\widetilde{U}_i$ and $\tilde{\xi}_{ia}=\tilde{\xi}_i$. By using $\{U_{ia},\xi_{ia}\}$ and $\{\widetilde{U}_{ia},\tilde{\xi}_{ia}\}$, varying i and a, instead of $\{U_i,\xi_i\}$ and $\{\widetilde{U}_i,\tilde{\xi}_i\}$, it follows that we can assume that the d-diameter of every $\sigma_i(T_i)$ is smaller than δ.

Take S equal to the family of the maps $h_{\mathcal{I}}$ for admissible sequences $\mathcal{I}$. Suppose that some $h_{\mathcal{I}}\in S$ fixes a point $u\in\operatorname{dom}h_{\mathcal{I}}$. Thus $\mathcal{I}=(i_0,\dots,i_\alpha)$ with $i_\alpha=i_0$. Let $x=\sigma_{i_0}(u)\in U_{i_0}$ and let $c:I\to X$ be a leafwise loop in L_x based at x and $\mathcal{U}$-covered by $\mathcal{I}$. Take any point $v\in\operatorname{dom}h_{\mathcal{I}}$, and let $y=\sigma_{i_0}(v)\in U_{i_0}$. Since the d-diameter of $\sigma_{i_0}(T_{i_0})$ is smaller than δ, according to our application of the Effros theorem, there is some $\phi\in\mathrm{Homeo}(X,\mathcal{F})$ with $\phi(x)=y$ and $d(c(t),\phi c(t))<\epsilon$ for all $t\in I$. Hence the leafwise path $\phi c:I\to X$ is $\widetilde{\mathcal{U}}$-covered by $\mathcal{I}$. It follows that $\tilde{h}_{\mathcal{I}}(v)=p_{i_0}\phi c(1)=p_{i_0}\phi(x)=p_{i_0}(y)=v$, obtaining $h_{\mathcal{I}}(v)=v$. This shows that $h_{\mathcal{I}}=\mathrm{id}_{\operatorname{dom}h_{\mathcal{I}}}$, and therefore $\mathcal{H}$ satisfies the condition of being strongly locally free with this S.

$\mathcal{H}$ is strongly quasi-analytic because it is strongly locally free, and therefore the hypotheses of Theorem A are satisfied. In particular, the closure $\overline{\mathcal{H}}$ is defined and generated by the set $\overline{S}$ induced by the above S.

Now, let us sharpen the above argument to prove that $\overline{\mathcal{H}}$ is also strongly locally free, and therefore $(X,\mathcal{F})$ is a G-foliated space for some local group G by Proposition 2.2-(i). For any $g\in\overline{S}$ with $O=\operatorname{dom}g$, there is a sequence of admissible sequences, $\mathcal{I}_k=(i_{k,0},\dots,i_{k,\alpha_k})$, such that $O\subset\operatorname{dom}h_{\mathcal{I}_k}$ for all k and $g=\lim_k h_{\mathcal{I}_k}|_O$ in the compact-open topology. Thus $i_0:=i_{k,0}$ is independent of k. Suppose that $g(u)=u$ for some $u\in O$, which means that $u'_k:=h_{\mathcal{I}_k}(u)\to u$ as $k\to\infty$. So we can assume that $i_{k,\alpha_k}=i_0$ for all k. Let

$x = \sigma_{i_0}(u) \in U_{i_0}$ and $x'_k = \sigma_{i_0}(u'_k) \in U_{i_0}$. We get $x'_k = \sigma_{i_0}(u'_k) \to \sigma_{i_0}(u) = x$ because $u'_k \to u$. For every k, there exits a leafwise path c_k, $\mathcal{U}$-covered by $\mathcal{I}_k$, with $c_k(0) = x$ and $c_k(1) = x'_k$. For any $v \in O$, we have $v'_k := h_{\mathcal{I}_k}(v) \to g(v)$, and let $y = \sigma_{i_0}(v) \in U_{i_0}$. As before, there is some $\phi \in \operatorname{Homeo}(X, \mathcal{F})$ such that $\phi(x) = y$ and $d(c_k(t), \phi c_k(t)) < \epsilon$ for all $t \in I$, and let $y'_k := \phi c_k(1) = \phi(x'_k)$. Hence the leafwise path ϕc_k is $\widetilde{\mathcal{U}}$-covered by $\mathcal{I}_k$, obtaining $p_{i_0}(y'_k) = \tilde{h}_{\mathcal{I}_k}(v) = h_{\mathcal{I}_k}(v) = v'_k$. Thus $v'_k \to v$ because $y'_k = \phi(x'_k) \to \phi(x) = y$ and $p_{i_0}(y) = v$. So $g(v) = v$, showing $g = \mathrm{id}_O$. Therefore $\overline{\mathcal{H}}$ satisfies the condition of being strongly locally free with $\overline{S}$. □

5. C^∞ Molino's description

In this section, suppose that X is C^∞ and satisfies the hypotheses of Theorem A, and let $(G, H, \widehat{X}_0, \hat\pi_0)$ represent its Molino's description.

Proposition 5.1. *$\widehat{X}_0$ has a unique C^∞ structure so that $\hat\pi_0$ is C^∞ and $T\hat\pi_0 : T\widehat{\mathcal{F}}_0 \to T\mathcal{F}$ restricts to isomorphisms between the fibers. Moreover the foliated H-action is also C^∞.*

Proof. Consider the notation of the proof of Theorem A, assuming that the C^∞ structure of X is defined by $\mathcal{U}$. For every i, let $\hat\xi_{i,0} : \widehat{U}_{i,0} \to B_i \times \widehat{T}_{i,0}$ be the composite of homeomorphisms,

$$\begin{aligned} \widehat{U}_{i,0} \xrightarrow{q^{-1}} \widetilde{U}_{i,0} &\equiv \{\, (x,\gamma) \in U_i \times \widehat{T}_{i,0} \mid p_i(x) = \hat\pi_0(\gamma) \,\} \\ &\longrightarrow \{\, (\mathbf{v}, u, \gamma) \in B_i \times T_i \times \widehat{T}_{i,0} \mid u = \hat\pi_0(\gamma) \,\} \equiv B_i \times \widehat{T}_{i,0} \;, \end{aligned}$$

where the second map is the restriction of $\xi_i \times \mathrm{id} : \check{U}_{i,0} \to B_i \times T_i \times \widehat{T}_{i,0}$. Thus $\hat\xi_{i,0}([x,\gamma,i]) = (\mathbf{v}, \gamma)$ for $[x,\gamma,i] \in \widehat{U}_{i,0}$, where $\mathbf{v} \in B_i$ is determined by $\xi_i(x) = (\mathbf{v}, \hat\pi_0(\gamma))$. Then $\widehat{\mathcal{U}} = \{\widehat{U}_{i,0}, \hat\xi_{i,0}\}$ is a foliated atlas of $\widehat{X}_0$, whose changes of coordinates are of the form

$$\hat\xi_{j,0}\hat\xi_{i,0}^{-1}(\mathbf{v},\gamma) = (g_{ij}(\mathbf{v}, \hat\pi_0(\gamma)), \widehat{h_{ij}}(\gamma)) \;,$$

for $(\mathbf{v},\gamma) \in \hat\xi_{i,0}(\widehat{U}_{i,0} \cap \widehat{U}_{j,0})$. Hence this atlas defines a C^∞ structure of $\widehat{X}_0$.

The foliated map $\hat\pi_0 : \widehat{X}_0 \to X$ and the foliated H-action on $\widehat{X}_0$ are C^∞ because $\widehat{U}_{i,0} = \hat\pi_0^{-1}(U_i)$ is H-invariant, and

$$\xi_i \hat\pi_0 \hat\xi_{i,0}^{-1}(\mathbf{v},\gamma) = (\mathbf{v}, \hat\pi_0(\gamma)) \;, \quad \hat\xi_{i,0}(\hat\xi_{i,0}^{-1}(\mathbf{v},\gamma)\cdot\sigma) = (\mathbf{v}, \gamma\cdot\sigma) \;,$$

for $(\mathbf{v},\gamma) \in B_i \times \widehat{T}_{i,0}$ and $\sigma \in H$. It also follows that $T\hat\pi_0$ restricts to isomorphisms between the fibers.

By Proposition 2.9 applied to $\hat\pi_0$, the C^∞ structure on $\widehat{X}_0$ is determined by the condition that $\hat\pi_0$ is C^∞ and $T\hat\pi_0$ restricts to isomorphisms between the fibers. □

If $\widehat{X}_0$ is equipped with the unique C^∞ structure given by Proposition 5.1, then $(G, H, \widehat{X}_0, \hat\pi_0)$ is called the C^∞ *Molino's description* of X.

6. Right local transverse actions

6.1. **Topological right local transverse actions.** The foliated homeomorphisms leafwisely homotopic to the identity form a normal subgroup $\operatorname{Homeo}_0(X,\mathcal{F})$ of $\operatorname{Homeo}(X,\mathcal{F})$, obtaining the (possibly non-Hausdorff) topological group

$$\overline{\operatorname{Homeo}}(X,\mathcal{F})=\operatorname{Homeo}(X,\mathcal{F})/\operatorname{Homeo}_0(X,\mathcal{F})\ .$$

Suppose that X is compact for the sake of simplicity. Then a *right local transverse action* of a local group G on X can be defined as a map $\phi:X\times O\to X$, for some $O\in\mathcal{N}(G,e)$, such that $\phi^g:=\phi(\cdot,g)\in\operatorname{Homeo}(X,\mathcal{F})$ for all $g\in O$, and $O\to\overline{\operatorname{Homeo}}(X,\mathcal{F})$, $g\mapsto[\phi^g]$, is a local anti-homomorphism of G to $\overline{\operatorname{Homeo}}(X,\mathcal{F})$. Two right local transverse actions, $\phi:X\times O\to X$ and $\psi:X\times P\to X$, are declared to be *equivalent* if there is some $Q\in\mathcal{N}(G,e)$ such that $Q\subset O\cap P$ and the restrictions $\phi,\psi:X\times Q\to X$ are leafwise homotopic with respect to the foliated structure on $X\times Q$ with leaves $L\times\{g\}$, for leaves L of X and points $g\in Q$.

Lemma 6.1. *If G is locally contractible, then the equivalence class of ϕ is determined by the induced local anti-homomorphism of G to $\overline{\operatorname{Homeo}}(X,\mathcal{F})$.*

Proof. Let $\psi:X\times P\to X$ be another right local transverse action inducing the same local anti-homomorphism of G to $\overline{\operatorname{Homeo}}(X,\mathcal{F})$ as ϕ. Thus there is some $Q\in\mathcal{N}(G,e)$ such that $Q\subset O\cap P$ and ϕ^g is leafwisely homotopic to ψ^g for all $g\in Q$. Since G is locally contractible, we can suppose that there is a homotopy $E:Q\times I\to Q$ of the constant map $\operatorname{const}_{g_0}$ to id_Q, for some point $g_0\in Q$. By choosing Q small enough and using $g_0^{-1}Q$ instead of Q, we can also assume that $g_0=e$. Let $g_t=E(g,t)$ for $g\in Q$ and $t\in I$. Given any leafwise homotopy $H:X\times I\to X$ of ϕ^e to ψ^e, the map $F:X\times Q\times I\to X$, defined by

$$F(x,g,t)=\psi^{g_t}(\psi^e)^{-1}\phi^{g_t^{-1}g}(\phi^e)^{-1}H(x,t),$$

is a leafwise homotopy between the restrictions $\phi,\psi:X\times Q\to X$, where $X\times Q$ is foliated with leaves $L\times\{g\}$, for leaves L of X and points $g\in Q$. This follows by using that $(\psi^e)^{-1}$, $(\phi^e)^{-1}$ and $H(\cdot,t)$ are leafwisely homotopic to id_X, and ψ^{g_t} and $\phi^{g_t}\phi^{g_t^{-1}g}$ are leafwisely homotopic to ϕ^{g_t} and ϕ^g, respectively. ☐

According to Lemma 6.1, when G is locally contractible, a right local transverse action of G on X could be defined as a local anti-homomorphism G to $\overline{\operatorname{Homeo}}(X,\mathcal{F})$, given by a map $O\to\overline{\operatorname{Homeo}}(X,\mathcal{F})$, $g\mapsto[\phi^g]$, for some $O\in\mathcal{N}(G,e)$ and some foliated map $\phi:X\times O\to X$ with $\phi^g\in\operatorname{Homeo}(X,\mathcal{F})$ for all $g\in O$. This corresponds to the definition of right transverse action of Lie groups on foliated manifolds given in [7]. But it seems impossible to extend Lemma 6.1 to arbitrary local groups, which motivates our more involved definition.

Lemma 6.2. *We can assume $\phi^e=\operatorname{id}_X$.*

Proof. Consider the foliated structure on $X \times O$ with leaves $L \times \{g\}$, for leaves L of X and points $g \in O$. The foliated map $\psi : X \times O \to X$, defined by $\psi^g := \phi^g(\phi^e)^{-1}$, satisfies the stated conditions. In fact, if $H : X \times I \to X$ is a leafwise homotopy of $(\phi^e)^{-1}$ to id_X, then $F : X \times O \times I \to X$, defined by $F(\cdot, g, t) = \phi^g H(\cdot, t)$, is a leafwise homotopy of ϕ to ψ. □

From now on, suppose that $\phi^e = \mathrm{id}_X$ according to Lemma 6.2. Then, since X is compact, there is some $O' \in \mathcal{N}(G, e)$ such that $O' \subset O$ and $\phi(U_i \times O') \subset \widetilde{U}_i$ for all i. The foliated restrictions $\phi : U_i \times O' \to \widetilde{U}_i$ induce maps $\bar{\phi} : T_i \times O' \to \widetilde{T}_i$, and let $\bar{\phi} : T \times O' \to \widetilde{T}$ denote their union. Then the restriction $\bar{\phi} : \Omega := \bar{\phi}^{-1}(T) \to T$ is a right local action of G on T, which will be said to be *induced* by ϕ.

Lemma 6.3. *$\mathcal{H}$ is locally equivariant (with respect to $\bar{\phi} : \Omega \to T$).*

Proof. It is enough to prove that the maps h_{ij} are locally equivariant. Let $u \in p_j(U_i \cap U_j)$ and $g \in O'$, and take any $x \in U_i \cap U_j$ such that $p_j(x) = u$. We have $h_{ij}(u) = p_i(x)$, $\phi(x, g) \in \widetilde{U}_i \cap \widetilde{U}_j$ and $\bar{\phi}(u, g) = p_j\phi(x, g)$, yielding

$$\tilde{h}_{ij}\bar{\phi}(u, g) = p_i\phi(x, g) = \bar{\phi}(p_i(x), g) = \bar{\phi}(h_{ij}(u), g)\ .$$

So $h_{ij}\bar{\phi}(u, g) = \bar{\phi}(h_{ij}(u), g)$ for all (u, g) in $(p_j(U_i \cap U_j) \times O') \cap \bar{\phi}^{-1}(T_i)$, which is an open neighborhood of $p_j(U_i \cap U_j) \times \{e\}$ in Ω. □

Lemma 6.4. *If X has no holonomy, then the equivalence class of ϕ determines the equivalence class of $\bar{\phi} : \Omega \to T$.*

Proof. Suppose that ϕ is equivalent to another right transverse local action $\psi : X \times P \to X$ with $\psi^e = \mathrm{id}_X$. Take some $P' \in \mathcal{N}(G, e)$ such that $P' \subset P$ and $\psi(U_i \times P') \subset \widetilde{U}_i$ for all i. As above, consider the map $\bar{\psi} : T \times P' \to \widetilde{T}$ induced by the foliated restrictions $\psi : U_i \times P' \to \widetilde{U}_i$, whose restriction $\bar{\psi} : \Sigma := \bar{\psi}^{-1}(T) \to T$ is a right local action of G on T. For some $Q \in \mathcal{N}(G, e)$ with $Q \subset O' \cap P'$, there is a leafwise homotopy $H : X \times Q \times I \to X$ between the foliated restrictions $\phi, \psi : X \times Q \to X$, where $X \times Q$ is foliated as before.

Claim 5. We have $\bar{\phi} = \bar{\psi}$ on $T \times Q'$ for some $Q' \in \mathcal{N}(G, e)$ with $Q' \subset Q$.

By absurdity, suppose that this assertion is not true. Then $\tilde{p}_{i_k}\phi^{g_k}(x_k) \neq \tilde{p}_{i_k}\psi^{g_k}(x_k)$ for some sequences, of indices i_k, of points $x_k \in U_{i_k}$, and $g_k \to e$ in G. Since X is compact, we can assume that $i_k = i$ for all k, and $x_k \to x$ in X; thus $x \in \overline{U_i} \subset \widetilde{U}_i$. Consider the leafwise paths $c_k = H(x_k, g_k, \cdot)$ and $c = H(x, e, \cdot)$. Note that $c_k \to c$ in the compact-open topology, and c is a loop in L_x based at x because $\phi^e = \psi^e = \mathrm{id}_X$. Let $\mathcal{J} = (j_0, \dots, j_\alpha)$ be an admissible sequence $\widetilde{\mathcal{U}}$-covering c with $j_0 = j_\alpha = i$. Hence $\mathcal{J}$ also $\widetilde{\mathcal{U}}$-covers c_k for k large enough, obtaining that $\tilde{p}_i\phi^{g_k}(x_k) \in \operatorname{dom}\tilde{h}_{\mathcal{J}}$ and $\tilde{h}_{\mathcal{J}}\tilde{p}_i\phi^{g_k}(x_k) = \tilde{p}_i\psi^{g_k}(x_k)$ for k large enough. Since $\tilde{p}_i\phi^{g_k}(x_k) \to \tilde{p}_i(x)$ in T_i and $\tilde{h}_{\mathcal{J}}$ is the identity on some neighborhood of $\tilde{p}_i(x)$ because X has no holonomy, it follows that $\tilde{h}_{\mathcal{J}}\tilde{p}_i\phi^{g_k}(x_k) = \tilde{p}_i\phi^{g_k}(x_k)$ for k large enough, yielding $\tilde{p}_i\phi^{g_k}(x_k) = \tilde{p}_i\psi^{g_k}(x_k)$ for k large enough, a contradiction.

By Claim 5, we get $\bar\phi = \bar\psi$ on $\Omega \cap \Sigma \cap (T \times Q')$, showing that the right local actions $\bar\phi : \Omega \to T$ and $\bar\psi : \Sigma \to T$ are equivalent. □

6.2. C^∞ **right local transverse actions.** From now on, assume that X is C^∞, and consider also the (possibly non-Hausdorff) topological group

$$\overline{\operatorname{Diffeo}}(X,\mathcal{F}) = \operatorname{Diffeo}(X,\mathcal{F}) / \operatorname{Diffeo}_0(X,\mathcal{F}) \;,$$

where $\operatorname{Diffeo}_0(X,\mathcal{F})$ is the normal subgroup of $\operatorname{Diffeo}(X,\mathcal{F})$ consisting of the foliated diffeomorphisms that are leafwisely homotopic to id_X; i.e., $\operatorname{Diffeo}_0(X,\mathcal{F}) = \operatorname{Diffeo}(X,\mathcal{F}) \cap \operatorname{Homeo}_0(X,\mathcal{F})$. It is said that the right local transverse action $\phi : X \times O \to X$ is C^∞ if it is C^∞ as foliated map, where $X \times O$ is foliated with leaves $L \times \{g\}$, for leaves L of X and points $g \in G$, and moreover $\phi^g \in \operatorname{Diffeo}(X,\mathcal{F})$ for all $g \in O$, and $O \to \overline{\operatorname{Diffeo}}(X,\mathcal{F})$, $g \mapsto [\phi^g]$, is a local anti-homomorphism of G to $\overline{\operatorname{Diffeo}}(X,\mathcal{F})$. A C^∞ *equivalence* between two C^∞ right local transverse actions is defined like in the case of right local transverse actions. Suppose also that ϕ is C^∞ from now on, and consider the induced right local action $\bar\phi : \Omega \to T$ defined in Section 6.1.

Lemma 6.5. *The C^∞ equivalence class of ϕ is determined by the equivalence class of $\bar\phi : \Omega \to T$.*

Proof. Let $\psi : X \times P \to X$ be another C^∞ right local transverse action of G on X with $\psi^e = \mathrm{id}_X$. Take some $P' \in \mathcal{N}(G,e)$ such that $P' \subset P$ and $\psi(U_i \times P') \subset \widetilde{U}_i$ for all i. Like in Section 6.1, let $\bar\psi : T \times P' \to \widetilde{T}$ be induced by the foliated restrictions $\psi : U_i \times P' \to \widetilde{U}_i$, and consider the right local action $\bar\psi : \Sigma := \bar\psi^{-1}(T) \to T$. Suppose that $\bar\psi = \bar\phi$ on some open neighborhood Θ of $T \times \{e\}$ in $\Omega \cap \Sigma$. So $\tilde p_i \phi(x,g) = \tilde p_i \psi(x,g)$ for all i and $(x,g) \in U_i \times (O \cap P)$ with $(p_i(x),g) \in \Theta$. Since X is compact, the open neighborhood of $X \times \{e\}$ in $X \times (O \cap P)$,

$$\bigcup_i \{\, (x,g) \in U_i \times (O \cap P) \mid (p_i(x),g) \in \Theta \,\} \;,$$

contains $X \times Q$ for some $Q \in \mathcal{N}(G,e)$. Hence $\phi(x,g)$ and $\psi(x,g)$ lie in the same plaque of some $\widetilde{U}_i$ for all $(x,g) \in X \times Q$. We can further assume that the plaques of the foliated charts in $\widetilde{\mathcal{U}}$ are convex for some choice of a Riemannian metric on X, obtaining a C^∞ leafwise homotopy between the foliated restrictions $\phi, \psi : X \times Q \to X$ by using geodesic segments in the leaves, where $X \times Q$ is foliated with leaves $L \times \{g\}$, for leaves L of X and points $g \in Q$. Therefore ϕ and ψ are C^∞ equivalent. □

Proposition 6.6. *If X is without holonomy, then the assignment of the induced right local action defines a bijection of the set of C^∞ equivalence classes of C^∞ right local transverse actions of G on X to the set of equivalence classes of right local actions of G on T satisfying that $\mathcal{H}$ is locally equivariant.*

Proof. By Lemmas 6.3, 6.4 and 6.5, it only remains to prove that, if $\mathcal{H}$ is locally equivariant with respect to a right local action $\chi : \Sigma \to T$ of G on T, then χ is induced by some C^∞ right local transverse action of G on X.

By Proposition 2.3, $\widetilde{\mathcal{H}}$ is locally equivariant with respect to some right local action $\tilde{\chi} : \widetilde{\Sigma} \to \widetilde{T}$ of G on $\widetilde{T}$, whose restriction to T is equivalent to χ. Since T is relatively compact in $\widetilde{T}$, there is some $P \in \mathcal{N}(G,e)$ such that $P \subset O$, $\overline{T} \times P \subset \widetilde{\Sigma}$ and $\tilde{\chi}(\overline{T_i} \times P) \subset \widetilde{T}_i$ for all i. Then, for $x \in \overline{U_i} \subset \widetilde{U}_i$ with $\tilde{\xi}_i(x) = (\mathbf{v}, u)$ and $g \in P$, the point $\phi_i(x,g) := \tilde{\xi}_i^{-1}(\mathbf{v}, \tilde{\chi}(u,g)) \in \widetilde{U}_i$ is well defined because $u = \tilde{p}_i(x) \in \tilde{p}_i(\overline{U_i}) = \overline{T_i}$.

Claim 6. There is some $Q \in \mathcal{N}(G,e)$ such that $Q \subset P$ and, if $x \in U_i \cap U_j$ and $g \in Q$, then $\phi_i(x,g), \phi_j(x,g) \in \widetilde{U}_i$ and $\tilde{p}_i\phi_i(x,g) = \tilde{p}_i\phi_j(x,g)$.

By absurdity, suppose that this assertion is not true. So $\tilde{p}_{i_k}\phi_{i_k}(x_k,g_k) \neq \tilde{p}_{i_k}\phi_{j_k}(x_k,g_k)$ for some sequences, of indices i_k, j_k, of points $x_k \in U_{i_k} \cap U_{j_k}$, and $g_k \to e$ in P. Since X is compact, we can assume that $i_k = i$ and $j_k = j$ for all k, and $x_k \to x$ in X. Thus $x \in \overline{U_i} \cap \overline{U_j} \subset \widetilde{U}_i \cap \widetilde{U}_j$, $\phi_i(x,g_k) \in \widetilde{U}_i$ and $\phi_j(x,g_k) \in \widetilde{U}_j$. Let $u_k = \tilde{p}_j(x_k)$ and $u = \tilde{p}_j(x)$. Since $\widetilde{\mathcal{H}}$ is locally equivariant, there are some open neighborhood W of u in $\operatorname{dom}\tilde{h}_{ij}$ and $Q \in \mathcal{N}(G,e)$ such that $Q \subset P$, $W \times Q, \tilde{h}_{ij}(W) \times Q \subset \widetilde{\Sigma}$, $\tilde{\chi}(W \times Q) \subset \operatorname{dom}\tilde{h}_{ij}$ and $\tilde{\chi}(\tilde{h}_{ij}(w),g) = \tilde{h}_{ij}\tilde{\chi}(w,g)$ for all $(w,g) \in W \times Q$. Take some open neighborhood N of x in X so that $\overline{N} \subset \widetilde{U}_i \cap \widetilde{U}_j$ and $\tilde{p}_j(\overline{N}) \subset W$. We can choose Q such that $\phi_j(\overline{N} \times Q) \subset \widetilde{U}_i \cap \widetilde{U}_j$, and therefore

$$\tilde{p}_j\phi_j\big(\overline{N} \times Q\big) = \tilde{\chi}\big(\tilde{p}_j\big(\overline{N}\big) \times Q\big) \subset \operatorname{dom}\tilde{h}_{ij}\ .$$

For k large enough, we have $(x_k,g_k) \in N \times Q$, obtaining

$$\tilde{p}_i\phi_i(x_k,g_k) = \tilde{\chi}(\tilde{h}_{ij}(u_k),g_k) = \tilde{h}_{ij}\tilde{\chi}(u_k,g_k) = \tilde{p}_i\phi_j(x_k,g_k)\ ,$$

a contradiction that proves Claim 6.

Given any Riemannian metric on X, we can assume that the plaques of every (U_i,ξ_i) and $(\widetilde{U}_i,\tilde{\xi}_i)$ are convex balls of diameter $< \pi/2\sqrt{\delta}$, where $\delta > 0$ is an upper bound for the sectional curvature of the leaves

Consider the open neighborhood Q of e in P given by Claim 6, and let $\{\lambda_i\}$ be a C^∞ partition of unity of X subordinated to $\{U_i\}$. For all $(x,g) \in X \times Q$, a probability measure on X is well defined by $\mu_{x,g} = \sum_i \lambda_i(x)\,\delta_{\phi_i(x,g)}$, where δ_y denotes the Dirac mass at every $y \in X$. By Claim 6, if $x \in \operatorname{supp}\lambda_i$, then $\mu_{x,g}$ is supported in the plaque $\tilde{p}_i^{-1}(\chi(p_i(x),g))$ of $(\widetilde{U}_i,\tilde{\xi}_i)$. Then, by Corollary 2.13, a C^∞ foliated map $\phi : X \times Q \to X$ is defined by taking $\phi(x,g)$ equal to the center of mass of $\mu_{x,g}$ in the common leaf through the points $\phi_i(x,g)$, where $X \times Q$ is foliated with leaves $L \times \{g\}$, for leaves L of X and points $g \in Q$. Let $\phi^g = \phi(\cdot,g) : X \to X$ for $g \in Q$. Note that $\phi^g(U_i) \subset \widetilde{U}_i$, and $\phi^e = \mathrm{id}_X$ because $\phi_i(x,e) = x$ for $x \in \overline{U_i}$.

Claim 7. There exists some $Q' \in \mathcal{N}(G,e)$ such that $Q'^2 \subset Q$ and there is a C^∞ leafwise homotopy of ϕ^{gh} to $\phi^h\phi^g$ for all $g,h \in Q'$.

Since X is compact, there is $Q' \in \mathcal{N}(G,e)$ such that $Q'^2 \subset Q$ and

$$\phi_j((\operatorname{supp} f_i \cap \operatorname{supp} f_j) \times Q') \subset U_i$$

for all i,j. Then, for all $x \in \operatorname{supp} f_i \cap \operatorname{supp} f_j$ and $g \in Q'$, the points $\phi_i(x,g)$ and $\phi_j(x,g)$ are in the plaque $p_i^{-1}(\chi(p_i(x),g))$ of (U_i,ξ_i) by Claim 6. Therefore $\phi(x,g) \in p_i^{-1}(\chi(p_i(x),g))$ according to Corollary 2.13. Applying again Claim 6 in a similar way, we get that $\phi(\phi(x,g),h)$ is in the plaque of $(\widetilde{U}_i,\tilde{\xi}_i)$ over $\tilde{\chi}(\chi(p_i(x),g),h) = \tilde{\chi}(p_i(x),gh)$ for all $h \in Q'$. On the other hand, since $gh \in Q'^2 \subset Q$, the same kind of argument shows that $\phi(x,gh)$ is in the plaque of $(\widetilde{U}_i,\tilde{\xi}_i)$ over $\tilde{\chi}(p_i(x),gh)$. Thus $\phi(\phi(x,g),h)$ and $\phi(x,gh)$ are in the same plaque of $(\widetilde{U}_i,\tilde{\xi}_i)$. Since these plaques are convex, we can use geodesic segments to construct a C^∞ leafwise homotopy between the foliated maps $\phi^h\phi^g$ and ϕ^{gh} for all $g,h \in Q'$.

Claim 8. There is some $Q'' \in \mathcal{N}(G,e)$ such that $Q'' \subset Q'$ and $\phi^g \in \operatorname{Diffeo}(X,\mathcal{F})$ for all $g \in Q''$.

For all $g \in Q'$, every restricted foliated map $\phi^g : U_i \to \widetilde{U}_i$ induces the open embedding $\tilde{\chi}^g : T_i \to \widetilde{T}_i$; i.e., $\{\,\phi^g \mid g \in Q'\,\}$ is a uniform family of transverse equivalences. Hence, since $\phi^e = \mathrm{id}_X$ and $g \mapsto \phi^g$ is continuous in the C^∞ foliated topology, it follows from Proposition 2.8 that there is some $Q'' \in \mathcal{N}(G,e)$ such that $\phi^g \in \operatorname{Diffeo}(X,\mathcal{F})$ for all $g \in Q''$.

From Claims 7 and 8, and since $\phi^e = \mathrm{id}_X$, we get that $\phi : X \times Q \to X$ is a C^∞ right transverse local action of G on X. The induced right local action of G on T is equivalent to χ because every $\phi^g : U_i \to \widetilde{U}_i$ induces $\tilde{\chi}^g : T_i \to \widetilde{T}_i$. □

Consider the following property that $(X,\mathcal{F},\phi)$ may have:

$$\mathcal{F}(\phi(\{x\} \times P)) = X \quad \forall x \in X, \forall P \in \mathcal{N}(G,e) \mid P \subset O. \tag{5}$$

Lemma 6.7. *Property* (5) *is invariant by equivalences of right transverse local actions.*

Proof. Elementary. □

Lemma 6.8. $(X,\mathcal{F},\phi)$ *satisfies* (5) *if and only if* $(T,\mathcal{H},\bar{\phi})$ *satisfies* (1).

Proof. Elementary. □

6.3. **Structural right transverse local action.** Now, suppose that X is a C^∞ compact minimal G-foliated space. Fix any equivalence Ψ of $\mathcal{H}$ to the pseudogroup $\mathcal{G}$ on G generated by local left translations with respect to some finitely generated dense sublocal group $\Gamma \subset G$. The local multiplication $\mu : G \times G \rightarrowtail G$ is a right local action of G on G so that $\mathcal{G}$ becomes locally equivariant. By Proposition 2.3, there is a unique right local action $\chi : T \times G \rightarrowtail T$, up to equivalences, such that $\mathcal{H}$ and Ψ become locally equivariant. According to Proposition 6.6, there is a unique right local transverse action $\phi : X \times O \to X$ of G on X inducing χ, up to equivalences, (whose equivalence class is) called the *structural right transverse local action*.

7. C^∞ G-FOLIATED SPACES ARE C^∞ FOLIATED HOMOGENEOUS

Suppose that X is compact and C^∞. Then the following result guarantees certain leafwise homogeneity.

Proposition 7.1. *Let L be the leaf of X, let D be a relatively compact regular domain without holonomy in L, and let $c : I \to D$ be any C^∞ path. Then, for any open neighborhood U of $c(I)$ in X, there is some C^∞ leafwise diffeotopy $\phi : X \times I \to X$ supported in U with $\phi(c(0), \cdot) = c$.*

Proof. Let E be a relatively compact open subset of L such that $c(I) \subset E$ and $\overline{E} \subset D \cap U$. By the homogeneity of L, there is a diffeotopy $\psi : L \times I \to L$ supported in E so that $\psi(\cdot, 0) = \mathrm{id}_X$ and $\psi(c(0), \cdot) = c$. Let Σ be a local transversal of X through x. By the Reeb's stability theorem for C^∞ foliated spaces [4, Proposition 1.7], there is a C^∞ foliated embedding $h : D \times \Sigma \to X$ that can be identified with the identity on $D \times \{x\} \equiv D$ and $\{x\} \times \Sigma \equiv \Sigma$. Write $h^{-1} = (h', h'') : \operatorname{im} h \to D \times \Sigma$. Take a compactly supported continuous function $f : \Sigma \to I$ with $h(\overline{E} \times \operatorname{supp} f) \subset U$ and $f(x) = 1$. Then the statement is satisfied with the C^∞ foliated diffeotopy $\phi : X \times I \to X$ defined by

$$\phi(x,t) = \begin{cases} h(\psi(h'(x), fh''(x)), h''(x)) & \text{if } x \in \operatorname{im} h \\ x & \text{otherwise}. \end{cases} \quad \square$$

Corollary 7.2. *If there is a C^∞ right transverse local action of G on X satisfying* (5)*, then X is C^∞ foliated homogeneous.*

Proof. Apply (5) and Proposition 7.1. □

Proof Theorem C. By Theorem B, it is enough to prove "(iii) ⇒ (i)." With the notation of Section 6.3, $(G, \mathcal{G}, \mu)$ satisfies (1) because

$$\mu((\Gamma \times \mu(\{g\} \times Q)) \cap \operatorname{dom} \mu) = G$$

for all $g \in G$ and $Q \in \mathcal{N}(G, e)$ with $\{g\} \times Q \subset \operatorname{dom} \mu$. So $(T, \mathcal{H}, \chi)$ also satisfies (1) by Lemma 2.4, and therefore $(X, \mathcal{F}, \phi)$ satisfies (5) by Lemma 6.8. Thus X is C^∞ foliated homogeneous by Corollary 7.2 □

8. EXAMPLES AND OPEN PROBLEMS

8.1. Molino's description of equicontinuous suspensions.

Let T be a compact space with a transitive left action of a compact topological group G, which is *quasi-analytic* in the sense that any $g \in G$ is the identity element $e \in G$ if it acts as the identity on some non-empty open set, and let $H \subset G$ be the isotropy group at some fixed point $u_0 \in T$. Moreover let $\Gamma \subset G$ be a dense subgroup isomorphic to $\pi_1(M)/\pi_1(L)$ for some regular covering L of some closed connected manifold M. Thus we have a right Γ-action on L by covering transformations, and a left Γ-action on T defined by the G-action. The induced diagonal Γ-action on $L \times T$, given by $(y, u) \cdot \gamma = (y \cdot \gamma, \gamma^{-1} \cdot u)$, is properly discontinuous and foliated, where $L \times T$ is foliated with leaves $L \times \{u\}$, for $u \in T$. The corresponding foliated quotient space, $L \times_\Gamma T$, is called the *suspension* of the Γ-action on T, and the quotient projection is a foliated covering map $L\times T\to L\times_\Gamma T$. The element in $L\times_\Gamma T$ defined by any $(y,u)\in L\times T$ will be denoted by $[y,u]$. Moreover the covering projection $\theta:L\to M$ induces a fiber bundle projection $\rho:L\times_\Gamma T\to M$, $\rho([y,u])=\theta(y)$, with typical fiber T; in particular, $L\times_\Gamma T$ is compact. Note that the fibers of ρ are transverse to the leaves; i.e., $\rho:L\times_\Gamma T\to M$ is a flat bundle. Any flat bundle with compact total space is given by a suspension.

Let us use the notation $X\equiv(X,\mathcal{F})$ for $L\times_\Gamma T$. Let $\mathcal{V}=\{V_i,\zeta_i\}$ be an atlas of M, with $\zeta_i:V_i\to B_i$ for some contractible open subset $B_i\subset\mathbb{R}^n$. Thus the flat bundle $\rho:X\to M$ is trivial over every V_i; i.e., there are homeomorphisms $\psi_i:U_i:=\rho^{-1}(V_i)\to V_i\times T$ such that $\rho:U_i\to V_i$ corresponds to the first factor projection $V_i\times T\to V_i$ and the leaves of $\mathcal{F}|_{U_i}$ correspond to the fibers of the second factor projection $V_i\times T\to T$. We get an induced foliated atlas $\mathcal{U}=\{U_i,\xi_i\}$ of X, where $\xi_i=(\zeta_i\times\mathrm{id}_T)\psi_i:U_i\to B_i\times T_i'$ with $T_i'\equiv T$. Assuming obvious conditions on $\mathcal{V}$, we get that $\mathcal{U}$ is regular. Then $\mathcal{U}$ induces a representative $\mathcal{H}'$ of the holonomy pseudogroup of X on $T'=\bigsqcup_i T_i'$. For any fixed index i_0, since $T_{i_0}'\equiv T$ meets all $\mathcal{H}'$-orbits, by restricting $\mathcal{H}'$ to T_{i_0}', we get a pseudogroup $\mathcal{H}$ on T equivalent to $\mathcal{H}'$, which is generated by the Γ-action on T. Thus X is minimal, equicontinuous and strongly quasi-analytic (take $S=\Gamma$ to check the last two properties for $\mathcal{H}$). Moreover $\overline{\mathcal{H}}$ is generated by the G-action on T, and therefore $\overline{\mathcal{H}}$ is also strongly quasi-analytic. So X satisfies the conditions of Theorem A.

Fix some $u_0\in T\equiv T_{i_0}'$, and consider the associated space $\widehat{T}_0'$ with the pseudugroup $\widehat{\mathcal{H}}_0'$, and the associated representative of the Molino's description, $(G',H',\widehat{X}_0'\equiv(\widehat{X}_0',\widehat{\mathcal{F}}_0'),\hat{\pi}_0')$, constructed like in the proof of Theorem A. Then $\widehat{T}_0:=\widehat{T}_{i_0,0}'$ meets all $\widehat{\mathcal{H}}_0'$-orbits, obtaining that $\widehat{\mathcal{H}}_0'$ is equivalent to its restriction $\widehat{\mathcal{H}}_0:=\widehat{\mathcal{H}}_0'|_{\widehat{T}_0}$. Thus $\widehat{T}_0=\{\,\gamma(g,u_0)\mid g\in G\,\}$ has the final topology induced by the map $G\to\widehat{T}_0$, $g\mapsto\gamma(g,u_0)$. This map is a continuous bijection, and therefore it is a homeomorphism because G is compact and $\widehat{T}_0$ is Hausdorff. So $\widehat{T}_0\equiv G$, $\widehat{\mathcal{H}}$ is generated by the action of G on itself by left translations, G' is locally isomorphic to G, and $\hat{\pi}_0:\widehat{T}_0\equiv G\to T$ is the orbit map $g\mapsto g\cdot u_0$. The composite $\rho\hat{\pi}_0':\widehat{X}_0'\to M$ is a fiber bundle with typical fiber $\widehat{T}_0\equiv G$, and $(\widehat{X}_0',\rho\hat{\pi}_0',\widehat{\mathcal{F}}_0')$ is also a flat bundle. Thus there is a foliated homeomorphism of $\widehat{X}_0'$ to $\widehat{X}_0\equiv(\widehat{X}_0,\widehat{\mathcal{F}}_0):=L\times_\Gamma G$. Moreover

$$H'\equiv H:=\{\,h\in G\mid h\cdot u_0=u_0\,\}\;,$$

the right H'-action on $\widehat{X}_0'$ corresponds to the right H-action on $\widehat{X}_0$ given by $[y,g]\cdot h=[y,gh]$, and the map $\hat{\pi}_0':\widehat{X}_0'\to X$ corresponds to the map $\hat{\pi}_0:\widehat{X}_0\to X$ defined by $\hat{\pi}_0([y,g])=[y,g{\cdot}u_0]$, which is induced by the foliated map $\mathrm{id}_L\times\hat{\pi}_0:L\times G\to L\times T$. Thus $(G,H,\widehat{X}_0,\hat{\pi}_0)$ is another representative of the Molino's description, which will be used in the next examples.

If M is C^∞, its C^∞ structure can be lifted to a C^∞ structure on L, which in turn can be lifted to $L\times T$, which finally give rise to a C^∞ structure on X so that the projection $\rho:X\to M$ is C^∞ and $T\rho$ has isomorphic restrictions to the fibers. This can be similarly applied to $\widehat{X}_0$, obtaining the C^∞ structure given by Proposition 5.1. The same procedure can be applied to any Riemannian metric on M, obtaining induced Riemannian metrics on X and $\widehat{X}_0$ so that the projections $\rho : X \to M$ and $\hat{\pi}_0 : \widehat{X}_0 \to X$ have locally isometric restrictions to the leaves.

The following result is well known. A proof is included for completeness.

Proposition 8.1. *The following properties are equivalent:*

(i) The Γ*-action on* T *has no fixed points.*
(ii) $\Gamma \cap gHg^{-1} = \{e\}$ *for all* $g \in G$.
(iii) The canonical foliated projection $L \times T \to X$ *restricts to homeomorphisms between the leaves.*

Proof. Let us prove "(i) ⇔ (ii)". Given any $\gamma \in \Gamma$ and $u \in T$, take some $g \in G$ such that $u = g \cdot u_0$. Then

$$\begin{aligned} \gamma u = u &\Leftrightarrow \gamma g \cdot u_0 = g \cdot u_0 \Leftrightarrow g^{-1}\gamma g \cdot u_0 = u_0 \\ &\Leftrightarrow g^{-1}\gamma g \in H \Leftrightarrow \gamma \in \Gamma \cap gHg^{-1} = \{e\} \Leftrightarrow \gamma = e\ . \end{aligned}$$

Let us prove "(i) ⇔ (iii)". For all $y, y' \in L$ and $u \in T$, we have $[y,u] = [y',u]$ if and only if there is some $\gamma \in \Gamma$ such that $(y',u) = (y \cdot \gamma, \gamma^{-1} \cdot u)$, which means $\gamma = e$ and $y' = y$. □

When the conditions of Proposition 8.1 are satisfied, X is strongly locally free (in particular, it has no holonomy), and all leaves are homeomorphic to L. If moreover M is C^∞/Riemannian, then $L \times T \to X$ restricts to diffeomorphisms/isometries between the leaves, obtaining that all leaves are diffeomorphic/isometric to L.

8.2. **The map $\hat{\pi}_0 : \widehat{X}_0 \to X$ may not be a principal bundle.** Consider the canonical inclusion $\mathrm{SO}(2) \subset \mathrm{SO}(3)$, and the canonical transitive analytic action of $\mathrm{SO}(3)$ on the sphere $S^2 \equiv \mathrm{SO}(3)/\mathrm{SO}(2)$. We get an induced transitive quasi-analytic left action of the compact topological group $G := \mathrm{SO}(3)^{\mathbb{N}}$ on the compact space $T := (S^2)^{\mathbb{N}}$. Fix $u_0 \in S^2$ whose isotropy group is $\mathrm{SO}(2)$, and let $\bar{u}_0 = (u_0, u_0, \dots) \in T$. The orbit map $\mathrm{SO}(3) \to S^2$, $g \mapsto g \cdot u_0$, is a non-trivial principal $\mathrm{SO}(2)$-bundle, and therefore it has no global sections. Then, using the arguments of the first and second examples of [40, Section 1], it easily follows that the orbit map $G \to T$, $(g_i) \mapsto (g_i) \cdot \bar{u}_0 = (g_i \cdot u_0)$, has no local sections. Since G is second countable, connected, compact and non-abelian, it contains a dense subgroup Γ isomorphic to the fundamental group of the closed oriented surface Σ_2 of genus 2 [12, Corollary 8.3]. Let L be the universal covering of Σ_2, which is diffeomorphic to the plane. Consider the corresponding suspension foliated space, $X = L \times_\Gamma T$, which satisfies the conditions of Theorem A, and the corresponding Molino's description $(G, H, \widehat{X}_0, \hat{\pi}_0)$ constructed in Section 8.1, where $\widehat{X}_0 = L \times_\Gamma G$, $H = \mathrm{SO}(2)^{\mathbb{N}}$, the right H-action on $\widehat{X}_0$ is given by $[y,g] \cdot h = [y, gh]$, and the map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by $\hat{\pi}_0([y,g]) = [y, g \cdot u_0]$.

Proposition 8.2. *The map $\hat{\pi}_0 : \widehat{X}_0 \to X$ has no local sections, and therefore it cannot be a principal H-bundle.*

Proof. Since $\hat{\pi}_0 : \widehat{X}_0 \to X$ is induced by $\mathrm{id}_L \times \hat{\pi}_0 : L \times G \to L \times T$, any local section of $\hat{\pi}_0$ with small enough domain defines a local section of $\hat{\pi}_0 : G \to T$. But this map has no local sections. □

8.3. **Foliated homogeneity may not be told by the leaves.**

Proposition 8.3. *If X is foliated homogeneous, then it is without holonomy, and all of its leaves are homeomorphic one another. If moreover X is C^∞ (respectively, compact and Riemannian), then all of its leaves are diffeomorphic (respectively, quasi-isometrically diffeomorphic) to each other.*

Proof. Elementary, using that there always exist leaves without holonomy in the first assertion, and using that the differentiable quasi-isometry class of the leaves is independent of the choice of the Riemannian metric on X in the last assertion (see e.g. [6, Proposition 10.5]). □

Let us exhibit an example where the reciprocal of Proposition 8.3 does not hold. To begin with, let G_1 and G_2 be second countable, connected compact topological groups, and let $G = G_1 \times G_2$. Assume that G_1 is non-abelian. Let us use the notation $g = (g_1, g_2)$ for the elements of G; in particular, we use $e = (e_1, e_2)$ for the identity element.

Proposition 8.4. *There exists a subset $\mathcal{P} \subset G \times G$, which is both residual and of full Haar measure, such that, for all $(g,h) \in \mathcal{P}$, the subgroup $\langle g,h \rangle$ is dense in G and freely generated by g and h, and $\langle g,h \rangle \cap (\{e_1\} \times G_2) = \{e\}$.*

Proof. By [12, Proposition 8.2], there are subsets, $\mathcal{O} \subset G \times G$ and $\mathcal{O}_1 \subset G_1 \times G_1$, which are residual and of full Haar measure, such that, for all $(g,h) \in \mathcal{O}$ and $(a,b) \in \mathcal{O}_1$, the subgroup $\langle g,h \rangle$ (respectively, $\langle a,b \rangle$) is dense in G (respectively, G_1) and freely generated by g and h (respectively, a and b). Then the statement is satisfied with

$$\mathcal{P} = \mathcal{O} \cap \{\, (g,h) \in G \times G \mid (g_1,h_1) \in \mathcal{O}_1 \,\} . \quad \square$$

Take $G_2 = \mathrm{SO}(3)$, and consider $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ and $S^2 \equiv \mathrm{SO}(3)/\mathrm{SO}(2)$ like in Section 8.2. By Proposition 8.4, G has a dense subgroup Γ freely generated by two elements such that $\Gamma \cap (\{e_1\} \times \mathrm{SO}(3)) = \{e\}$. Hence the first factor projection $G_1 \times \mathrm{SO}(3) \to G_1$ restricts to an injection $\Gamma \to G_1$, and Γ does not meet any conjugate of $\{e_1\} \times \mathrm{SO}(2)$ in G (all of them are contained in $\{e_1\} \times \mathrm{SO}(3)$). Consider the canonical left action of G and Γ on $T := G_1 \times S^2 \equiv G/(\{e_1\} \times \mathrm{SO}(2))$. There is a regular covering L of the closed oriented surface of genus two, Σ_2, whose group of covering transformations is isomorphic to Γ. Consider the corresponding suspension foliated space, $X = L \times_\Gamma T$, which satisfies the conditions of Theorem A, and the corresponding Molino's description $(G, H, \widehat{X}_0, \hat{\pi}_0)$ constructed in Section 8.1, where $\widehat{X}_0 = L \times_\Gamma G$, $H = \mathrm{SO}(2)$, the right H-action on $\widehat{X}_0$ is given by $[y,g] \cdot h = [y, gh]$, and the map $\hat{\pi}_0 : \widehat{X}_0 \to X$ is defined by

$\hat{\pi}_0([y,g]) = [y, g{\cdot}u_0]$. We can equip Σ_2 with C^∞ and Riemannian structures, and consider the induced C^∞ and Riemannian structures on X and $\widehat{X}_0$.

Since $H \neq \{e\}$, X is not foliated homogeneous by Theorem C (or Theorem B and Proposition 3.2). However this cannot be seen by comparing any pair of leaves since all of them are isometric to L, and X has no holonomy by "(ii) ⇔ (iii)" in Proposition 8.1.

This argument cannot produce matchbox manifolds because Proposition 8.4 requires G to be connected to apply [12, Proposition 8.2]. Examples with totally disconnected local transversals are given in [23, Theorem 35] and [19, Theorem 10.7].

8.4. **Inverse limits of minimal Lie foliations.** This example was suggested by S. Hurder. Let $(X,\mathcal{G})$ be the McCord solenoid defined as the projective limit of a tower of non-trivial regular coverings between closed connected manifolds,

$$\cdots \to M_k \xrightarrow{\phi_k} M_{k-1} \to \cdots \to M_0 \ .$$

Let $\Gamma_k = \pi_1(M_k)$, and consider the induced tower of homomorphisms between finite groups,

$$\cdots \to \Gamma_0/\Gamma_k \to \Gamma_0/\Gamma_{k-1} \to \cdots \to \Gamma_0/\Gamma_1 \ ,$$

whose inverse limit K contains a canonical dense copy of Γ_0. Then $(X,\mathcal{G})$ can be also described as the suspension foliated space $\widetilde{M_0} \times_{\Gamma_0} K$, where $\widetilde{M_0}$ is the universal covering of M_0. We get induced maps $\psi_k : X \to M_k$, whose restrictions to the leaves are covering maps. Suppose that M_0 is equipped with a minimal Lie G_0-foliation $\mathcal{F}_0$, for some simply connected Lie group G_0. Then every M_k can be endowed with the minimal Lie G_0-foliation $\mathcal{F}_k := (\phi_1 \cdots \phi_k)^* \mathcal{F}_0$. On every $\mathcal{G}$-leaf M, consider the pull-back of $\mathcal{F}_0$ by $\psi_0 : M \to M_0$. These foliations on all leaves of $\mathcal{G}$ can be combined to form a foliated structure $\mathcal{F}$ on X, which is a "Lie G_0-subfoliated structure" of $\mathcal{G}$ in an obvious sense. We can write $\mathcal{F} = \psi_0^* \mathcal{F}_0$, which equals $\psi_k^* \mathcal{F}_k$ for all k. Extending the notation of suspensions, we can also write $(X,\mathcal{F}) = (\widetilde{M_0}, \widetilde{\mathcal{F}_0}) \times_{\Gamma_0} K$, where $\widetilde{\mathcal{F}_0}$ is the lift of $\mathcal{F}_0$. It easily follows that $(X,\mathcal{F})$ is a minimal G-foliated space for $G = G_0 \times K$.

8.5. **Open problems.**

8.5.1. *Strong quasi-analyticity of* $\overline{\mathcal{H}}$. This problem was proposed in [10]. It is really unknown to the authors if the strong quasi-analyticity of $\overline{\mathcal{H}}$ is needed in Theorem A. More precisely, assuming that $\mathcal{H}$ is a minimal compactly generated equicontinuous strongly quasi-analytic pseudogroup, is $\overline{\mathcal{H}}$ strongly quasi-analytic? If minimality is not assumed, then counterexamples can be easily given. But the minimal case seems to be an interesting open problem. Among the wild matchbox solenoilds of [33] there might be counterexamples.

8.5.2. *Functorality, universality and uniqueness of the Molino's description.* It would be desirable to have a uniqueness of the Molino's description stronger than Proposition 3.1, stating that not only the structures $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ constructed in the proof of Theorem A, but also all possible structures $(G, \Gamma, H, \widehat{X}_0, \hat{\pi}_0)$ satisfying the conditions of its statement are equivalent. This would follow by showing a universality property, which in turn would follow by exhibiting its functoriality with respect to some kind of foliated maps. Since the definition of $\widehat{X}_0$ uses germs of maps in $\overline{\mathcal{H}}$, the functoriality of Molino's description could be achieved by showing that foliated maps between equicontinuous foliated spaces induce morphisms between the closures of their holonomy pseudogroups. This would be an extension of the case of Riemannian foliations, solved in [9, 8]. Such functiorality, universality and uniqueness of the Molino's description is not even proved in the Riemannian foliation case. A direct consequence would be that H is finite if and only if X is a *virtually foliated homogeneous* foliated space (a finite fold covering of X is foliated homogeneous as foliated space).

8.5.3. *How large is the class of inverse limits of minimal Lie foliations?* Since any metrizable locally compact local group of finite topological dimension is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group [34, Theorem 107], it was asked by S. Hurder whether any compact minimal foliated homogeneous foliated space of finite "topological codimension" can be realized as inverse limit of minimal Lie foliations, like in Section 8.4. This would generalize the results of [15] (see also [2]), where an affirmative answer is given for homogeneous matchbox manifolds (the case of codimension zero). If this is true, using also the Molino's description, it could be possible to prove that any equicontinuous foliated space satisfying the conditions of Theorem A is an inverse limit of Riemannian foliations.

8.5.4. *Molino's descriptions without assuming strong quasi-analyticity.* This problem arises from the Molino spaces constructed by Dyer, Hurder and Lukina in [19] for equicontinuous matchbox manifolds, where strong quasi-analyticity is not needed. Their Molino spaces are also foliated homogeneous, and their leaves cover the leaves of the original matchbox, but they may not be unique. Thus the following question makes sense. Does there exist this kind of Molino spaces for arbitrary compact minimal equicontinuous foliated spaces?

References

[1] A. Abd-Allah and R. Brown, *A compact-open topology on partial maps with open domain*, J. London Math. Soc. (2) **21** (1980), 480–486. MR 577723

[2] F. Alcalde Cuesta, Á. Lozano Rojo, and M. Macho Stadler, *Transversely Cantor laminations as inverse limits*, Proc. Amer. Math. Soc. **139** (2011), 2615–2630. MR 2784831

[3] J.A. Álvarez López, R. Barral Lijó, and A. Candel, *A universal Riemannian foliated space*, Topology Appl. **198** (2016), 47–85. MR 3433188

[4] J.A. Álvarez López and A. Candel, *Equicontinuous foliated spaces*, Math. Z. **263** (2009), no. 4, 725–774. MR 2551597

[5] ———, *Topological description of Riemannian foliations with dense leaves*, Pacific J. Math. **248** (2010), 257–276. MR 2741247

[6] ———, *Generic coarse geometry of leaves*, Lecture Notes in Mathematics, vol. 2223, Springer, Heidelberg-New York, 2018. MR 3822768

[7] J.A. Álvarez López and Y.A. Kordyukov, *Lefschetz distribution of Lie foliations*, C^*-algebras and elliptic theory II. Selected papers of the international conference, Będlewo, Poland, January 2006, Trends Math., Birkhäuser, Basel, 2008, pp. 1–40. MR 2408134

[8] J.A. Álvarez López and X.M. Masa, *Morphisms of pseudogroups and foliated maps*, Foliations 2005. Proceedings of the international conference, University of Łodź, Łodź, Poland, June 13–24, 2005, World Sci. Publ., Hackensack, NJ, 2006, pp. 1–19. MR 2284773

[9] ———, *Morphisms between complete Riemannian pseudogroups*, Topology Appl. **155** (2008), 544–603. MR 2388956

[10] J.A. Álvarez López and M.F. Moreira Galicia, *Topological Molino's theory*, Pacific J. Math. **280** (2016), 257–314. MR 3453564

[11] R. Arens, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593–610. MR 0019916

[12] E. Breuillard, T. Gelander, J. Souto, and P. Storm, *Dense embeddings of surface groups*, Geom. Topol. **10** (2006), 1373–1389. MR 2255501

[13] A. Candel and L. Conlon, *Foliations. I*, Graduate Studies in Mathematics, vol. 23, American Mathematical Society, Providence, RI, 2000. MR 1732868

[14] I. Chavel, *Riemannian geometry. A modern introduction*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006. MR 2229062

[15] A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, Trans. Amer. Math. Soc. **365** (2013), 3151–3191. MR 3034462

[16] L. van den Dries and I. Goldbring, *Globalizing locally compact local groups*, J. Lie Theory **20** (2010), 519–524. MR 2743102

[17] ———, *Erratum to "Globalizing locally compact local groups"*, J. Lie Theory **22** (2012), 489–490. MR 2976929

[18] J. Dyer, S. Hurder, and O. Lukina, *The discriminant invariant of Cantor group actions*, Topology Appl. **208** (2016), 64–92. MR 3506971

[19] ———, *Molino theory for matchbox manifolds*, Pacific J. Math. **289** (2017), no. 1, 91–151. MR 3652457

[20] E.G. Effros, *Transformation groups and* C^**-algebras*, Ann. of Math. (2) **81** (1965), 38–55. MR 0174987

[21] P.E. Ehrlich, *Continuity properties of the injectivity radius function*, Compositio Math. **29** (1974), 151–178. MR 0417977

[22] D.B.A. Epstein, K.C. Millett, and D. Tischler, *Leaves without holonomy*, J. London Math. Soc. (2) **16** (1977), no. 3, 548–552. MR 0464259

[23] R. Fokkink and L. Oversteegen, *Homogeneous weak solenoids*, Trans. Amer. Math. Soc. **354** (2002), no. 9, 3743–3755. MR 1911519

[24] É. Ghys, *Laminations par surfaces de Riemann*, Panoramas & Synthèses **8** (2000), 49–95. MR 1760843

[25] C. Godbillon, *Feuilletages*, Progress in Mathematics, vol. 98, Birkhäuser Verlag, Basel, 1991, Études géométriques. With a preface by G. Reeb. MR 1120547

[26] I. Goldbring, *Hilbert's fifth problem for local groups*, Ann. of Math. (2) **172** (2010), no. 2, 1269–1314. MR 2680491

[27] A Haefliger, *Some remarks on foliations with minimal leaves*, J. Differential Geom. **15** (1980), 269–284. MR 614370

[28] A. Haefliger, *Pseudogroups of local isometries*, Differential geometry (Santiago de Compostela, 1984), Res. Notes in Math., vol. 131, Pitman, Boston, MA, 1985, pp. 174–197. MR 864868

[29] ______, *Leaf closures in Riemannian foliations*, A fête of topology, Academic Press, Boston, MA, 1988, pp. 3–32. MR 928394

[30] G. Hector, *Feuilletages en cylindres*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., vol. 597, Springer, Berlin, 1977, pp. 252–270. MR 0451260

[31] G. Hector and U. Hirsch, *Introduction to the geometry of foliations. Part A*, second ed., Aspects of Mathematics, vol. 1, Friedr. Vieweg & Sohn, Braunschweig, 1986, Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy. MR 881799

[32] M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, no. 33, Springer-Verlag, New York-Heidelberg, 1976. MR 0448362

[33] S. Hurder and O. Lukina, *Wild solenoids*, arXiv:1702.03032, 2017.

[34] R. Jacoby, *Some theorems on the structure of locally compact local groups*, Ann. of Math. (2) **66** (1957), 36–69. MR 0089997

[35] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. **30** (1977), no. 5, 509–541. MR 0442975

[36] S. Matsumoto, *The unique ergodicity of equicontinuous laminations*, Hokkaido Math. J. **39** (2010), 389–403. MR 2743829

[37] P. Molino, *Riemannian foliations*, Progress in Mathematics, no. 73, Birkhäuser Boston, Inc., Boston, MA, 1988, Translated from the French by G. Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu. MR 932463

[38] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955. MR 0073104

[39] C.C. Moore and C. Schochet, *Global analysis on foliated spaces*, Mathematical Sciences Research Institute Publications, vol. 9, Springer-Verlag, New York, 1988, With appendices by S. Hurder, Moore, Schochet and Robert J. Zimmer. MR 0918974

[40] A. Ramsay, *Local product structure for group actions*, Ergodic Theory Dynam. Systems **11** (1991), 209–217. MR 1101091

[41] T. Sakai, *On continuity of injectivity radius function*, Math. J. Okayama Univ. **25** (1983), 91–97. MR 0701970

[42] C. Tarquini, *Feuilletages conformes*, Ann. Inst. Fourier **54** (2004), no. 2, 453–480. MR 2073842

[43] J. van Mill, *A note on the Effros theorem*, Amer. Math. Monthly **111** (2004), 801–806. MR 2104051

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