



Mathematical analysis of a levitation model

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ABSTRACT

The aim of this paper is to study a model of electromagnetic levitation for a metallic rigid body. The model is constituted by the transient linear model of eddy currents under the hypothesis of axisymmetry, written in terms of a magnetic potential vector, coupled with an ODE which governs the vertical motion of the body. The electromagnetic model is a parabolic-elliptic PDE which parabolicity region is the position occupied by the body, which changes with time. Besides, Lorentz force appears in the RHS of the ODE. Thus, the model exhibits a coupling of geometrical nature. We establish the existence and uniqueness of solution of the coupled problem and we study its maximally defined solution. In particular, we prove that a blow-up of the velocity of the body cannot happen. Our techniques involve: a reformulation of the coupled problem as a causal differential equation, an adaptation of the theory about this kind of equations and a result of locally Lipschitz dependence of the magnetic potential vector with respect to the velocity of the body.

1. Introduction

In this article, we study mathematically a model of electromagnetic levitation. We consider a rigid cylindrical metallic body subjected to an electromagnetic field induced by a cylindrical coil. The electromagnetic field exerts on the body the Lorentz force, so the position of the body changes with time. This displacement in turn influences the value of the electromagnetic field.

The electromagnetic model considered is the transient linear model of eddy currents under the hypothesis of axisymmetry, posed in a “computational” domain (a bounded three-dimensional cylinder that contains the coil and the body and which boundary is far enough from them, introduced to pose the problem in a bounded domain). This model is a parabolic-elliptic PDE, in which the parabolicity region is that occupied by the metallic body and therefore it changes with time. A similar model was studied in Bermúdez et al. [1], but there the authors neglected in the expression of the current density, $\mathbf{J}$, the term that involves $\mathbf{v} \times \mathbf{B}$ while in the present article we take into account said term in the particular case in which the velocity is vertical and only depends on time. This is sufficient for our objective since due to axisymmetry, the resultant of the Lorentz force is vertical, so the body moves vertically under the action of this resultant and the gravitational force according to Newton’s second law. The levitation model studied is the coupling of both models. It is therefore the coupling of a parabolic-elliptic PDE with an ODE. It should be noted that this coupling is of a geometric type, since the velocity determines the position of the body and therefore the region of parabolicity of the PDE. Although the electromagnetic submodel is linear, the coupled problem is nonlinear.

We establish the existence and uniqueness of solution for the coupled problem and we also prove that the maximally defined solution does not present a blow-up of the velocity and consequently the maximally defined solution exists as long as the metallic body neither touches the coil nor the boundary of the “computational” domain. This seems physically reasonable. To do this, we reformulate the coupled problem as a causal differential equation for velocity. Since the theory of existence and uniqueness of

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solution developed in Logemann and Ryan [2] is not applicable as it is, we have made an adaptation of said theory. This adaptation together with a priori estimations and regularity results for the electromagnetic problem allows us to establish the existence and uniqueness of the solution of the coupled problem and to study the maximally defined solution.

The outline of the article is as follows. In Section 2 we present mathematically the problem, we state the most important results relating to the electromagnetic problem, we reformulate the coupled problem as a causal differential equation and we state the results for the coupled problem. In Section 3 we describe the transient model of eddy currents, we introduce the magnetic potential vector under axisymmetry conditions and we describe the model that governs the displacement of the body. In Section 4, we obtain a weak formulation of the electromagnetic problem and prove that it is well posed as well as some regularity results. In Section 5, we justify that the analysis of the reformulation of the coupled problem as a causal differential equation needs to adapt the framework of Logemann and Ryan [2], relative to the existence and uniqueness of solutions of causal differential equations. Section 6 is devoted to the development of this adaptation. In Section 7 we state and prove the main results of this article, that is, the existence and uniqueness of the solution of the coupled problem and the properties of the maximally defined solution. To do this, we prove that the hypotheses required in the theory exposed in Section 6 are satisfied, for which we establish a property of locally Lipschitz dependence of the magnetic potential vector with respect to the velocity of the body.

2. Statement of the problem and main results

The main objective of this article is to mathematically study the levitation model presented below. This model, which will be obtained in Section 3, consists of the PDE

$$\sigma \frac{\partial A}{\partial t} - \frac{\partial}{\partial r} \left[\frac{1}{\mu r} \frac{\partial(rA)}{\partial r} \right] - \frac{\partial}{\partial z} \left[\frac{1}{\mu} \frac{\partial A}{\partial z} \right] + \sigma v \frac{\partial A}{\partial z} = J_S \quad \text{in } (0, T) \times \Omega, \quad (2.1)$$

supplemented with the boundary condition

$$A = 0 \text{ on } (0, T) \times \Gamma_D \quad (2.2)$$

and the initial condition

$$A(0) = A^0 \text{ in } \Omega_0, \quad (2.3)$$

coupled with the ODE

$$m \frac{d^2 u}{dt^2} = -mg + f(t), \quad (2.4)$$

with initial conditions

$$u(0) = 0, \quad \frac{du}{dt}(0) = v_0, \quad (2.5)$$

where

$$f(t) = -2\pi \int_{\Omega_t} \sigma \left(\frac{\partial A}{\partial t} + v \frac{\partial A}{\partial z} \right) \frac{\partial A}{\partial z} r dr dz, \quad (2.6)$$

$v = \frac{du}{dt}$ and $\Omega_t = \Omega_0 + u(t)e_z$.

Eqs. (2.1)–(2.3) constitute the electromagnetic model. The unknown function $A = A(t, r, z)$ is the azimuthal component of the magnetic potential vector, r and z are cylindrical coordinates, $\mu > 0$ is a constant and $v = v(t)$ stands for the velocity of the metallic rigid body. The domain $\Omega := (0, r_{\max}) \times (z_{\min}, z_{\max})$, $\Gamma_D := \partial\Omega \setminus (\{0\} \times [z_{\min}, z_{\max}])$, Ω_0 is an open set such that $\Omega_0 \subset \Omega$ and $T > 0$ is the final time up to which the source term J_S is defined. We assume that $T < +\infty$. Function σ is positive only in $\bar{\Omega}_t$, the meridian section of the region occupied by the metallic body at instant t . More precisely

$$\sigma(t, r, z) = \begin{cases} \hat{\sigma}(r, z - u(t)) & \text{if } (r, z) \in \bar{\Omega}_t \\ 0 & \text{otherwise} \end{cases} \quad (2.7)$$

where $\hat{\sigma}$ is a given measurable function satisfying (4.6). Because of this, Eq. (2.1) is a parabolic-elliptic PDE whose parabolicity region is Ω_t , hence time-dependent.

The mechanical model consists of Eqs. (2.4) and (2.5), where $u(t)$ is the vertical displacement of the body at time t , and $m > 0$, $g > 0$ and $v_0 \in \mathbb{R}$ are constants. The term $f(t)$ stands for the Lorentz force and is given by (2.6), hence it depends on A . More details about the model are given in Section 3.

We write $\hat{\Omega} = \Omega_0$ (since we will take Ω_0 as reference domain and also to conform to some notations of Bermúdez et al. [1]) and we assume the geometric constraints on $\hat{\Omega}$ labelled (i), (ii) and (iii) in Section 4.

Now we present the most important results of this paper. We first address those related to the electromagnetic problem. In this problem, the velocity $v \in C([0, T])$ is assumed to be a given data. As said in the introduction, this case is not covered by Bermúdez et al. [1]. Note that the vertical displacement of the body is given by $u = Pv$, where $(Pv)(t) = \int_0^t v(\tau) d\tau$. We assume that

$$\underline{u} < (Pv)(t) < \bar{u} \quad \forall t \in [0, T] \quad (2.8)$$

where $\underline{u} < 0 < \bar{u}$ are suitable constants which are chosen so that the above constraint ensures that the body remains in Ω and does not touch the coil. The precise conditions satisfied by $\underline{u}$ and $\bar{u}$ are given by (4.2).

Notations for functional spaces, bilinear forms and the set Q are detailed in [Section 4](#). (For the sake of notational compactness, here and thereafter ∂_t and ∂_z will be often used to denote the derivative with respect to t and z respectively.)

The weak formulation of the electromagnetic problem (2.1)–(2.3) reads:

Problem 1. Given $J_S \in L^2(0, T; \mathcal{V}')$ and $A^0 \in L_r^2(\Omega_0)$, find $A \in L^2(0, T; \mathcal{V})$ with $\partial_t A \in L_r^2(Q)$, such that

$$\int_{\Omega_t} \sigma(\partial_t A(t)) Z r dr dz + a(A(t), Z) + b(t, A(t), Z) = \langle J_S(t), Z \rangle$$

$$\forall Z \in \mathcal{V}, \quad \text{a.e. } t \in [0, T],$$

$$A(0) = A^0 \quad \text{in } \Omega_0.$$

Remark 2.1. For $A \in L^2(0, T; \mathcal{V})$ and $\partial_t A \in L_r^2(Q)$, it follows that $A \in H_r^1(Q)$. Thus, because of the trace result from Bermúdez et al. [1, Lemma A.1], we have that $A|_{\{0\} \times \Omega_0} \in L_r^2(\{0\} \times \Omega_0) \simeq L_r^2(\Omega_0)$. Hence, the initial condition in [Problem 1](#) makes sense.

Regarding the existence and uniqueness of solution, we have the following results.

Theorem 2.1. [Problem 1](#) has at most one solution A and the following a priori estimate holds:

$$\sup_{0 \leq t \leq T} \left[\int_{\Omega_t} \sigma |A(t)|^2 r dr dz \right]^{1/2} + \|A\|_{L^2(0, T; \mathcal{V})} \quad (2.9)$$

$$\leq C_1(\alpha) \left[\left(\int_{\Omega_0} \sigma(0) |A^0|^2 r dr dz \right)^{1/2} + \|J_S\|_{L^2(0, T; \mathcal{V}')} \right].$$

Theorem 2.2. If $J_S \in H^1(0, T; \mathcal{V}')$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$, then there exists a solution to [Problem 1](#) which satisfies $A \in L^\infty(0, T; \mathcal{V})$ and

$$\|\partial_t A\|_{L_r^2(Q)} + \|A\|_{L^\infty(0, T; \mathcal{V})} \leq C_2(\underline{\sigma}, \bar{\sigma}, M, \alpha, \Omega_0)(1 + \|v\|_{C([0, T])}) \quad (2.10)$$

$$\times \left[\|A^0\|_{\tilde{H}_r^1(\Omega_0)} + \|J_S\|_{C([0, T]; \mathcal{V}')} + \|J_S\|_{L^2(0, T; \mathcal{V}')} + \|\partial_t J_S\|_{L^2(0, T; \mathcal{V}')} \right].$$

Arguing as in Bermúdez et al. [1, Remark 5.1], we deduce a first result of spatial regularity. More concretely:

Remark 2.2. If $J_S \in H^1(0, T; \mathcal{V}') \cap L^2(0, T; L_r^2(\Omega))$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$, then the solution to [Problem 1](#) satisfies $A \in L^2(0, T; \tilde{H}_r^2(\Omega))$ and the following estimate holds:

$$\|A\|_{L^2(0, T; \tilde{H}_r^2(\Omega))} \leq C_{reg} \mu \left[\|J_S\|_{L^2(0, T; L_r^2(\Omega))} + \|\sigma \partial_t A\|_{L_r^2(Q)} + \bar{\sigma} \|v\|_{C([0, T])} \|A\|_{L^2(0, T; \mathcal{V})} \right], \quad (2.11)$$

where C_{reg} is the constant appearing in the regularity result stated in Gopalakrishnan and Pasciak [8, Theorem 4.1] (with the unit square replaced by Ω).

Now we address the time regularity. (Here and in the sequel χ_Q stands for the characteristic function of Q . Similar notation will be used with analogous meaning for other measurable sets.)

Theorem 2.3. Assume $v \in W^{1,1}(0, T)$, $J_S \in H^1(0, T; \mathcal{V}') \cap L^2(0, T; L_r^2(\Omega))$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$. Then the solution to [Problem 1](#) satisfies $\sqrt{t} \partial_t A \in L^2(0, T; \mathcal{V})$ and $\chi_Q \sqrt{t} \partial_t A \in L^\infty(0, T; L_r^2(\Omega))$. Moreover, the following estimate holds:

$$\|\sqrt{t} \partial_t A\|_{L^2(0, T; \mathcal{V})} + \|\chi_Q \sqrt{t} \partial_t A\|_{L^\infty(0, T; L_r^2(\Omega))}$$

$$\leq C_3(\underline{\sigma}, \bar{\sigma}, M, \alpha, C_{reg}, \mu, \Omega_0) \exp \left(\frac{\bar{\sigma}}{2} \int_0^T |v'(t)| dt \right)$$

$$\times \left[(1 + \|v\|_{C([0, T])}^3)(1 + T) + (1 + \|v\|_{C([0, T])}) \sqrt{T} \sqrt{\int_0^T |v'(t)| dt} \right]$$

$$\times \left[\|A^0\|_{\tilde{H}_r^1(\Omega_0)} + \|J_S\|_{C([0, T]; \mathcal{V}')} + \|\partial_t J_S\|_{L^2(0, T; \mathcal{V}')} + \|J_S\|_{L^2(0, T; L_r^2(\Omega))} \right] \quad (2.12)$$

The obtention of the weak formulation and the proofs of the above theorems are given in [Section 4](#). There we also include a result about the behavior of $A(t)$ as $t \rightarrow 0^+$ ([Theorem 4.1](#)), two additional spatial regularity results ([Theorems 4.2](#) and [4.3](#)) and an estimate that is needed to study the levitation problem ([Theorem 4.4](#)).

Now we present the approach we use to study levitation problem. We reformulate this coupled problem as a causal differential equation for the velocity.

From here until the end of this section $T_f > 0$ denotes the final time up to which the source term J_S is defined, and we assume that $T_f < +\infty$, $A^0 \in \tilde{H}_r^1(\Omega_0)$ and $J_S \in H^1(0, T_f; L_r^2(\Omega_S))$. (Thus, for the coupled problem T_f has the meaning of previous notation T .)

For all $T \in (0, T_f)$, we introduce the solution operator S_T as follows: we define

$$\text{dom}(S_T) := \{v \in C([0, T]); \underline{u} < (Pv)(t) < \bar{u} \quad \forall t \in [0, T]\} \quad (2.13)$$

and the operator $S_T : v \in \text{dom}(S_T) \mapsto A$, where A is the solution of the electromagnetic problem (that is, [Problem 1](#)) corresponding to the velocity $v = v_{e_z}$, (recall that $u = Pv$ is the vertical displacement of the metallic body). We now introduce the operator $\mathcal{F}$ defined by

$$(\mathcal{F}(v))(t) = -g + \frac{1}{m} f(t) \quad (2.14)$$

where f is the net Lorentz force corresponding to the azimuthal component $A = S_T(v)$ of the magnetic vector potential, namely,

$$f(t) = -2\pi \int_{\Omega_t(Pv)} (\sigma(Pv))(t) \left(\frac{\partial A}{\partial t}(t) + v(t) \frac{\partial A}{\partial z}(t) \right) \frac{\partial A}{\partial z}(t) r \, dr \, dz \quad (2.15)$$

where $\Omega_t(Pv) = \Omega_0 + (Pv)(t)e_z$ and, accordingly to (2.7), $(\sigma(Pv))(t)$ is the function defined in Ω by

$$(\sigma(Pv))(t)(r, z) = \begin{cases} \hat{\sigma}(r, z - (Pv)(t)) & \text{if } (r, z) \in \overline{\Omega_t(Pv)} \\ 0 & \text{otherwise.} \end{cases}$$

We rewrite the coupled problem as an initial value problem for a causal differential equation as follows:

$$\frac{dv}{dt} = (\mathcal{F}(v))(t), \quad 0 \leq t < T_f \quad (2.16)$$

$$v(0) = v_0 \quad (2.17)$$

$$\underline{u} < (Pv)(t) < \bar{u}, \quad 0 \leq t < T_f \quad (2.18)$$

Causal differential equations, also called Volterra functional differential equations or sometimes abstract Volterra integro-differential equations, constitute a general class of differential equations which encompass, as special cases, ordinary differential equations, delay differential equations, integro-differential equations and hysteretic differential equations. Standard references for functional equations with causal operators are Corduneanu [3,4], Gripenberg et al. [5].

In [2], the authors study the existence, uniqueness and continuation of the solution of causal differential equations in the framework of absolutely continuous functions. However, as we will explain in [Section 5](#), the above problem does not exactly fit the framework of Logemann and Ryan [2] due to the constraint (2.18). For this reason, we have adapted said framework in [Section 6](#).

The following theorem summarizes the results concerning the reformulation of the coupled levitation problem as a causal differential equation, which we will obtain in [Section 7](#).

Theorem 2.4. The initial value problem (2.16)–(2.18) has a unique maximally defined solution, which we denote v , and the maximal interval of existence is of the form $[0, \tau_M)$, where $0 < \tau_M \leq T_f$. The function v is bounded on $[0, \tau_M)$. Therefore either $\tau_M = T_f$ or there exists a sequence $\{t_n\}$ in $[0, \tau_M)$ such that $\{t_n\} \rightarrow \tau_M$ and $\min(|(Pv)(t_n) - \underline{u}|, |(Pv)(t_n) - \bar{u}|) \rightarrow 0$.

Hence there is no blow of the velocity. Due to (4.2) and (4.3), the convergence $\min(|(Pv)(t_n) - \underline{u}|, |(Pv)(t_n) - \bar{u}|) \rightarrow 0$ physically means that the metallic body tends to touch the coil or the boundary of the computational domain. Therefore when this does not happen, the maximally defined solution exists over the entire interval $[0, T_f]$.

3. Obtention of the model

3.1. The electromagnetic model

In this subsection we will essentially follow [Section 2](#) of Bermúdez et al. [1] but we will make the appropriate modifications. The goal of the electromagnetic model is to determine the electromagnetic field produced by a coil on a cylindrical rigid metal body, assuming that the motion of the body is known. To ensure the cylindrical symmetry, we model the coil by several concentric rings with toroidal geometry, all carrying the same current intensity. On the other hand, to reduce the electromagnetic model to a bounded domain, we introduce a three dimensional cylinder $\tilde{\Omega}$ containing the coil and the metallic body with its boundary sufficiently far from them.

Because of the cylindrical symmetry, we are allowed to pose the problem in a meridian section of $\tilde{\Omega}$, which we denote by Ω . Since Ω is an open rectangle contained in the half-plane $\{(r, z) : r > 0, z \in \mathbb{R}\}$ with an edge lying on the axis $r = 0$, the domain Ω has the form $\Omega = (0, r_{\max}) \times (z_{\min}, z_{\max})$. We denote by Ω_t the meridian section of the metallic body at time t and $\Omega_S := \Omega_1 \cup \dots \cup \Omega_m$, where Ω_k ($k = 1, \dots, m$) are the meridian sections of the turns of the coil. We assume that Ω_t and the sets Ω_k are open and that $\Omega_t \cap \Omega_S = \emptyset$ for all t . Finally, we denote Γ_0 the intersection between $\partial\Omega$ and the axis $r = 0$, and $\Gamma_D := \partial\Omega \setminus \Gamma_0$. We will use standard notation in electromagnetism: E is the electric field, B is the magnetic induction, H is the magnetic field, J is the current density, μ is the magnetic permeability and σ is the electric conductivity.

The magnetic permeability μ is taken as a positive constant in the whole domain. The conductivity σ vanishes outside the metallic body. This body can be made of different materials, each with a different conductivity. We will make more precise assumptions on σ below (cf. (4.5) and (4.6)).

In this kind of problem, the electric displacement can be neglected in Ampère's law, leading to the so called eddy current model:

$$\text{curl } H = J, \quad (3.1)$$

$$\frac{\partial B}{\partial t} + \text{curl } E = 0, \quad (3.2)$$

$$\operatorname{div} \mathbf{B} = 0. \quad (3.3)$$

This system must be completed with the relations

$$\mathbf{B} = \mu \mathbf{H} \quad (3.4)$$

and

$$\mathbf{J} = \begin{cases} \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) & \text{in the metallic body,} \\ \mathbf{J}_S & \text{in the coil (data),} \\ \mathbf{0} & \text{in the air.} \end{cases}$$

Notice that since the source current density $\mathbf{J}_S$ is taken as a given data, the conductivity σ is taken as vanishing in the coil.

We assume that all the physical quantities are independent of the azimuthal coordinate θ and that the current density field has only azimuthal non-zero component, i.e.,

$$\mathbf{J}(t, r, \theta, z) = J(t, r, z) \mathbf{e}_\theta.$$

We also assume that the velocity has only a vertical component and that it depends only in time, namely, $\mathbf{v} = v(t) \mathbf{e}_z$. This corresponds to the motion of a rigid body under a vertical force, as it happens in levitation. We also assume that $\partial\Omega_0$ intersects Γ_0 in a set with a nonvanishing 1D measure. Proceeding as in Bermúdez et al. [6] and Bermúdez et al. [7], it can be shown that

$$\mathbf{H}(t, r, \theta, z) = H_r(t, r, z) \mathbf{e}_r + H_z(t, r, z) \mathbf{e}_z,$$

$$\mathbf{B}(t, r, \theta, z) = B_r(t, r, z) \mathbf{e}_r + B_z(t, r, z) \mathbf{e}_z,$$

$$\mathbf{E}(t, r, \theta, z) = E(t, r, z) \mathbf{e}_\theta \quad \text{in } \Omega_t$$

We introduce a magnetic vector potential $\mathbf{A}$ for $\mathbf{B}$, so that

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \quad (3.5)$$

Accordingly to Bermúdez et al. [7], the vector potential can be chosen of the form

$$\mathbf{A}(t, r, \theta, z) = A(t, r, z) \mathbf{e}_\theta \quad (3.6)$$

and such that (cf. (3.2))

$$-E = \frac{\partial A}{\partial t} \quad \text{in } \Omega_t$$

Using (3.1), (3.4) and (3.5), we have

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} \mathbf{A} \right) = \mathbf{J},$$

where

$$\mathbf{J} = J \mathbf{e}_\theta = \begin{cases} -\sigma \frac{\partial A}{\partial t} \mathbf{e}_\theta + \sigma \mathbf{v} \times \operatorname{curl}(A \mathbf{e}_\theta) & \text{in } \Omega_t, \\ J_S \mathbf{e}_\theta & \text{in } \Omega_S \text{ (data),} \\ \mathbf{0} & \text{in } \Omega \setminus (\overline{\Omega_S} \cup \overline{\Omega_t}). \end{cases} \quad (3.7)$$

By extending J_S by zero outside the coil and by understanding $\sigma \frac{\partial A}{\partial t}$ as zero outside the metallic body (even if $\frac{\partial A}{\partial t}$ is not defined pointwisely there), the relation above can be written in a single equation as follows:

$$\mathbf{J} = -\sigma \frac{\partial A}{\partial t} \mathbf{e}_\theta + \sigma \mathbf{v} \times \operatorname{curl}(A \mathbf{e}_\theta) + J_S \mathbf{e}_\theta.$$

Thus, we are led to the following parabolic-elliptic problem:

$$\sigma \frac{\partial A}{\partial t} \mathbf{e}_\theta + \operatorname{curl} \left[\frac{1}{\mu} \operatorname{curl} (A \mathbf{e}_\theta) \right] - \sigma \mathbf{v} \times \operatorname{curl}(A \mathbf{e}_\theta) = J_S \mathbf{e}_\theta \quad \text{in } (0, T) \times \Omega, \quad (3.8)$$

which in cylindrical coordinates is written as the PDE (2.1).

We complement this equation with the homogeneous Dirichlet boundary condition (2.2), which makes sense provided the boundary of $\tilde{\Omega}$ is sufficiently far from the coil and the workpiece at all time $t \in [0, T]$. Moreover, as will be seen in the following section, a homogeneous Dirichlet boundary condition will be automatically imposed on Γ_0 from the fact that $\mathbf{A} = A \mathbf{e}_\theta \in H(\operatorname{curl}, \Omega) := \{Z \in L^2(\tilde{\Omega})^3 : \operatorname{curl} Z \in L^2(\tilde{\Omega})^3\}$.

3.2. The mechanical model

To determine the position of the metallic body, we assume that it moves as a rigid body under the action of the 3D resultant of the Lorentz force and also the gravitational force. We denote by $\mathbf{f}$ the Lorentz force density, which is given by

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} \text{ in } \Omega_t.$$

From (3.5), (3.6) and (3.7), we obtain

$$\mathbf{B} = -\frac{\partial A}{\partial z} \mathbf{e}_r + \frac{1}{r} \frac{\partial(rA)}{\partial r} \mathbf{e}_z \quad (3.9)$$

and

$$\mathbf{J} = J \mathbf{e}_\theta = -\sigma \left(\frac{\partial A}{\partial t} + v \frac{\partial A}{\partial z} \right) \mathbf{e}_\theta \text{ in } \Omega_t, \quad (3.10)$$

hence

$$\mathbf{f}(t, r, z) = f_r(t, r, z) \mathbf{e}_r + f_z(t, r, z) \mathbf{e}_z,$$

where

$$f_r(t, r, z) = J \frac{1}{r} \frac{\partial(rA)}{\partial r}$$

and

$$f_z(t, r, z) = J \frac{\partial A}{\partial z} = -\sigma \left(\frac{\partial A}{\partial t} + v \frac{\partial A}{\partial z} \right) \frac{\partial A}{\partial z}$$

The 3D-resultant of the Lorentz force is given by

$$\mathbf{F} = \int_{\tilde{\Omega}_t} \mathbf{f} dV,$$

where $\tilde{\Omega}_t$ is the revolution volume around z axis whose meridian section is Ω_t . Because of the axisymmetric assumptions, the 3D-resultant of the radial component vanishes. In fact, an easy calculation gives

$$\mathbf{F} = f(t) \mathbf{e}_z,$$

with $f(t)$ given by (2.6). The vertical displacement $u = u(t)$ of the metallic body is governed by the initial value problem (2.4)–(2.5) where m , g and v_0 denote respectively the mass of the body, the gravity acceleration and the initial axial velocity. Thus, our levitation model is constituted by the coupling of the electromagnetic model with the motion equation.

4. Well-posedness and regularity properties of the electromagnetic model

The aims of this section are: to obtain a weak formulation of problem (2.1), to prove that it has a unique solution, to obtain a priori estimates of the solution in terms on data and to establish some regularity results of the solution. With this end, first we introduce the functional framework we will use.

Let $L^2_r(\Omega)$ be the weighted Lebesgue space of all measurable functions Z defined in Ω such that

$$\|Z\|_{L^2_r(\Omega)}^2 := \int_{\Omega} |Z|^2 r dr dz < \infty.$$

The weighted Sobolev space $H_r^k(\Omega)$ consists of all functions in $L^2_r(\Omega)$ whose derivatives up to the order k are also in $L^2_r(\Omega)$. We define the norms and semi-norms of these spaces in the standard way. Let $L^2_{1/r}(\Omega)$ be the space of all measurable functions Z defined in Ω such that

$$\|Z\|_{L^2_{1/r}(\Omega)}^2 := \int_{\Omega} \frac{|Z|^2}{r} dr dz < \infty$$

and let

$$\tilde{H}_r^1(\Omega) := H_r^1(\Omega) \cap L^2_{1/r}(\Omega),$$

endowed with its natural norm $\|Z\|_{\tilde{H}_r^1(\Omega)}^2 := \|Z\|_{H_r^1(\Omega)}^2 + \|Z\|_{L^2_{1/r}(\Omega)}^2$. Since the curl of a vector field of the form $Z(r, z) \mathbf{e}_\theta$ reads

$$\mathbf{curl}(Z \mathbf{e}_\theta) = -\frac{\partial Z}{\partial z} \mathbf{e}_r + \frac{1}{r} \frac{\partial(rZ)}{\partial r} \mathbf{e}_z,$$

it is well known that $Z \mathbf{e}_\theta \in H(\mathbf{curl}, \tilde{\Omega})$ if and only if $Z \in \tilde{H}_r^1(\Omega)$ (cf. [8]). Let

$$\mathcal{V} := \left\{ Z \in \tilde{H}_r^1(\Omega) : Z = 0 \text{ on } \Gamma_D \right\}.$$

Let us emphasize that the functions in $\tilde{H}_r^1(\Omega)$ have vanishing traces on Γ_0 , so that those in $\mathcal{V}$ actually vanish on the whole boundary of Ω .

Finally, following [8], we denote

$$\|Z\|_{\tilde{H}_r^2(\Omega)}^2 := \left\| \frac{1}{r} \frac{\partial(rZ)}{\partial r} \right\|_{H_r^1(\Omega)}^2 + \left\| \frac{\partial Z}{\partial z} \right\|_{\tilde{H}_r^1(\Omega)}^2 + \|Z\|_{\tilde{H}_r^1(\Omega)}^2$$

and introduce the space $\tilde{H}_r^2(\Omega) := \{Z \in \tilde{H}_r^1(\Omega) : \|Z\|_{\tilde{H}_r^2(\Omega)} < \infty\}$.

Notation like $L_r^2(\cdot)$, $\tilde{H}_r^1(\cdot)$, etc. will also be used with analogous meaning for functions spaces on other subdomains.

In this section we consider the vertical velocity v of the metallic body as a given data and we assume $v \in C([0, T])$. We take $\hat{\Omega} = \Omega_0$ as reference domain. Note that $\hat{\Omega} \subset \Omega$. Since we assume $u(0) = 0$, the displacement of the metallic body at time t is $u(t) = (Pv)(t)$, where for any $g \in L^1(0, T)$, Pg is given by

$$(Pg)(t) := \int_0^t g(s) ds, \quad 0 \leq t \leq T. \quad (4.1)$$

Since $\hat{\Omega} = \Omega_0$, we have $\Omega_t = \hat{\Omega} + u(t)e_z$.

Following [1], we impose the following geometric constraints on $\hat{\Omega}$:

- (i) $\hat{\Omega}$ is a Lipschitz bounded domain;
- (ii) either $\hat{\Omega}$ does not intersect the axis $r = 0$ or there exists $a > 0$ such that the set $\{(r, z) \in \hat{\Omega} : 0 < r < a\}$ is a trapezoid with parallel sides aligned with the axis $r = 0$; ¹
- (iii) $\hat{\Omega} \cap \bar{\Gamma}_D = \emptyset$.

For further details and consequences of assumption (ii), see Mercier and Raugel [9].

Let be now $\underline{u}, \bar{u} \in \mathbb{R}$ with $\underline{u} < 0 < \bar{u}$ some constants such that

$$\forall \xi \in (\underline{u}, \bar{u}), \Omega_0 + \xi e_z \subset \Omega \text{ and } (\bar{\Omega}_0 + \xi e_z) \cap \bar{\Omega}_S = \emptyset. \quad (4.2)$$

The existence of these constants is granted by the following facts: $\hat{\Omega} = \Omega_0 \subset \Omega$, $\bar{\Omega} \cap \bar{\Gamma}_D = \emptyset$ and $\bar{\Omega}_0 \cap \bar{\Omega}_S = \emptyset$. In the sequel we assume that the displacement satisfies

$$\underline{u} < u(t) < \bar{u} \quad \forall t \in [0, T]. \quad (4.3)$$

This restriction guarantees that, along the motion, the metallic body remains in the computational domain Ω and does not touch the coil.

In order to use some results of Bermúdez et al. [1], we introduce the mapping $X : [0, T] \times \bar{\Omega} \rightarrow [0, \infty) \times \mathbb{R}$ defined by

$$X(t, \hat{x}, \hat{z}) = (\hat{r}, \hat{z} + u(t)) = (\hat{r}, \hat{z}) + u(t)e_z \quad (4.4)$$

Due to (4.2) and (4.3), we have $X([0, T] \times \bar{\Omega}) \subset \bar{\Omega}$. Since $v \in C([0, T])$ and $u = Pv$, the application X fulfills the properties labelled

(i)–(iv) in Bermúdez et al. [1, Section 3]. Besides, $X(0, \hat{x}) = \hat{x}$ for all $\hat{x} \in \bar{\Omega}$ because of $u(0) = 0$.

Here we write $\hat{x}$ (resp. $\mathbf{x}$ as shorthand notation for $(\hat{r}, \hat{z})$ (resp. (r, z)).

The conductivity σ is taken such that

$$\sigma(t, \mathbf{x}) = \hat{\sigma}(\hat{\mathbf{x}}), \quad \text{with } \hat{\mathbf{x}} \in \bar{\Omega} : \mathbf{x} = X(t, \hat{\mathbf{x}}), \quad (4.5)$$

where $\hat{\sigma}$ is the conductivity in the reference domain $\hat{\Omega}$, which is a given measurable function satisfying

$$0 < \underline{\sigma} \leq \hat{\sigma}(\hat{\mathbf{x}}) \leq \bar{\sigma}, \quad \hat{\mathbf{x}} \in \bar{\Omega}. \quad (4.6)$$

From a physical point of view, this means that the conductivity of each material point remains constant along the process. ²

Next, we introduce the non-cylindrical open subset of $(0, T) \times \Omega$,

$$Q := \{(t, \mathbf{x}) : \mathbf{x} \in \Omega_t, t \in (0, T)\}$$

and the following Hilbert spaces of functions defined in Q :

$$L_r^2(Q) := \left\{ \varphi : Q \rightarrow \mathbb{R} \text{ measurable with } \int_0^T \int_{\Omega_t} |\varphi|^2 r dr dz dt < \infty \right\},$$

$$H_r^1(Q) := \left\{ \varphi \in L_r^2(Q) : \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial r}, \frac{\partial \varphi}{\partial z} \in L_r^2(Q) \right\},$$

respectively endowed with their natural norms:

$$\|\varphi\|_{L_r^2(Q)}^2 := \int_0^T \int_{\Omega_t} |\varphi|^2 r dr dz dt,$$

¹ Although our mathematical results include both cases, the first one is incompatible with the assumption that $\partial\Omega_0$ intersects Γ_0 in a set with a nonvanishing 1D measure, which is needed to obtain the electromagnetic model.

² Since $\hat{\Omega} = \Omega_0$ and X is defined by (4.4), Eq. (4.5) amounts to the first case in (2.7).

$$\|\varphi\|_{H_r^1(Q)}^2 := \|\varphi\|_{L_r^2(Q)}^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L_r^2(Q)}^2 + \left\| \frac{\partial \varphi}{\partial r} \right\|_{L_r^2(Q)}^2 + \left\| \frac{\partial \varphi}{\partial z} \right\|_{L_r^2(Q)}^2.$$

In order to write a weak formulation of (2.1), we multiply (2.1) by rZ , where $Z \in \mathcal{V}$ is a test function, we integrate in Ω , and use a Green's formula, to obtain

$$\int_{\Omega_t} \sigma(\partial_t A(t)) Z r dr dz + a(A(t), Z) + b(t, A(t), Z) = \int_{\Omega_S} J_S Z r dr dz \quad \forall Z \in \mathcal{V}, \quad (4.7)$$

where $a(\cdot, \cdot)$ is the following continuous bilinear form defined in $\tilde{H}_r^1(\Omega)$ by

$$\begin{aligned} a(Y, Z) &:= \int_{\Omega} \frac{1}{\mu} \left[\frac{1}{r} \frac{\partial(rY)}{\partial r} \frac{\partial(rZ)}{\partial r} + r \frac{\partial Y}{\partial z} \frac{\partial Z}{\partial z} \right] dr dz \\ &= \left(\int_{\Omega} \frac{1}{\mu} \operatorname{curl}(Y e_{\theta}) \cdot \operatorname{curl}(Z e_{\theta}) r dr dz \right), \end{aligned}$$

and $b(t, \cdot, \cdot)$ is the t -dependent continuous bilinear form defined by

$$b(t, Y, Z) := \int_{\Omega_t} \sigma v \frac{\partial Y}{\partial z} Z r dr dz = \left(- \int_{\Omega_t} \sigma v \times \operatorname{curl}(Y e_{\theta}) \cdot Z e_{\theta} r dr dz \right).$$

We denote by M a continuity constant of $a(\cdot, \cdot)$, that is,

$$|a(Y, Z)| \leq M \|Y\|_{\tilde{H}_r^1(\Omega)} \|Z\|_{\tilde{H}_r^1(\Omega)} \quad \forall Y, Z \in \tilde{H}_r^1(\Omega) \quad (4.8)$$

It is shown in Gopalakrishnan and Pasciak [8, Proposition 2.1 & 3.1] that $a(\cdot, \cdot)$ is a $\mathcal{V}$ -elliptic bilinear form; namely, there exists $\alpha > 0$ such that

$$a(Z, Z) \geq \alpha \|Z\|_{\tilde{H}_r^1(\Omega)}^2 \quad \forall Z \in \mathcal{V}. \quad (4.9)$$

The right-hand side of Eq. (4.7) makes sense when $J_S \in L^2(0, T; L_r^2(\Omega_S))$. In order to deal with a less regular source term, namely $J_S \in L^2(0, T; \mathcal{V}')$, where $\mathcal{V}'$ stands for the topological dual space of $\mathcal{V}$, we consider the Gelfand triple $\mathcal{V} \subset L_r^2(\Omega) = L_r^2(\Omega)' \subset \mathcal{V}'$, where $L_r^2(\Omega)'$ (the topological dual space of $L_r^2(\Omega)$) is identified with $L_r^2(\Omega)$ by using the standard scalar product of $L_r^2(\Omega)$. Hence, if $J_S \in L^2(0, T; L_r^2(\Omega))$, we have $\langle J_S(t), Z \rangle = \int_{\Omega} J_S(t) Z r dr dz$ for all $Z \in \mathcal{V}$. When function $J_S \in L^2(0, T; L_r^2(\Omega_S))$, we still denote by $J_S \in L^2(0, T; L_r^2(\Omega))$ its extension by zero outside $(0, T) \times \Omega_S$.

Thus we are led to the weak problem that we have labelled as [Problem 1](#).

Notice that [Problem 1](#) is degenerate parabolic, because the term including the time derivative of A is only defined in the (moving) subdomain Ω_t .

In order to deal with this moving subdomain, we will use following Reynolds transport formula, which holds for all $\varphi, \psi \in H_r^1(Q)$:

$$\frac{d}{dt} \int_{\Omega_t} \sigma \varphi \psi r dr dz = \int_{\Omega_t} \sigma \frac{\partial(\varphi \psi)}{\partial t} r dr dz + \int_{\Omega_t} \sigma v \frac{\partial(\varphi \psi)}{\partial z} r dr dz \quad (4.10)$$

in $D'((0, T))$ and a.e. in $[0, T]$. This formula arises from specializing Bermúdez et al. [1, Theorem 3.1], taking into account that $v = v(t)e_z$. Note that it makes use of the assumption $v \in C([0, T])$.

We also specialize the t -dependent bilinear form denoted by $c(t, \cdot, \cdot)$ in Bermúdez et al. [1] to our case. This gives

$$c(t, Y, Z) := - \int_{\Omega_t} \sigma v \frac{\partial(YZ)}{\partial z} r dr dz$$

with $Y, Z \in \tilde{H}_r^1(\Omega)$.

The following Gårding-like inequality for this form is essentially that proved in Bermúdez et al. [1, Lemma 4.1]

Lemma 4.1. *There exists $\lambda > 0$ such that*

$$a(Z, Z) \pm c(t, Z, Z) + \lambda \int_{\Omega_t} \sigma |Z|^2 r dr dz \geq \frac{\alpha}{2} \|Z\|_{\tilde{H}_r^1(\Omega)}^2,$$

with α as in (4.9), for all $Z \in \tilde{H}_r^1(\Omega)$ and all $t \in [0, T]$.

In fact, for $v = v(t)e_z$, we can take $\lambda = \frac{2\bar{\sigma}\|v\|_{C([0,T])}^2}{\alpha}$.

The following property, which is specific to the case $v = v(t)e_z$, is crucial for obtaining estimates that neither grow exponentially with respect to velocity nor time:

$$\frac{1}{2} c(t, Z, Z) + b(t, Z, Z) = 0 \quad \forall Z \in \tilde{H}_r^1(\Omega) \quad \forall t \in [0, T]. \quad (4.11)$$

We will also use the inequality

$$|b(t, Y, Z)| \leq \sqrt{\bar{\sigma}} \|v\|_{C([0,T])} \|Y\|_{\tilde{H}_r^1(\Omega)} \left(\int_{\Omega_t} \sigma |Z|^2 r dr dz \right)^{1/2} \quad (4.12)$$

$$\forall Y \in \tilde{H}_r^1(\Omega), \forall Z \in L_r^2(\Omega), \forall t \in [0, T],$$

which obviously implies

$$\begin{aligned} |b(t, Y, Z)| &\leq \bar{\sigma} \|v\|_{C([0, T])} \|Y\|_{\tilde{H}_r^1(\Omega)} \|Z\|_{L_r^2(\Omega)} \\ \forall Y \in \tilde{H}_r^1(\Omega), \forall Z \in L_r^2(\Omega), \forall t \in [0, T]. \end{aligned} \quad (4.13)$$

Proof of Theorem 2.1. The proof follows essentially the lines of that of Bermúdez et al. [1, Theorem 4.3]. We take $Z = A(t)$ in the first equation of Problem 1 and use Reynolds formula (4.10). Owing to (4.11), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \sigma |A(t)|^2 r dr dz + a(A(t), A(t)) = \langle J_S(t), A(t) \rangle.$$

Next, we use (4.9) and a Young's inequality to deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \sigma |A(t)|^2 r dr dz + \frac{\alpha}{2} \|A(t)\|_{\tilde{H}_r^1(\Omega)}^2 \leq \frac{1}{2\alpha} \|J_S(t)\|_{\mathcal{V}'}^2. \quad (4.14)$$

Integration in time of this inequality leads to (2.9). $\square$

Proof of Theorem 2.2. The proof follows the lines of that of Bermúdez et al. [1, Theorem 4.2], which uses Galerkin approximation and parabolic regularization. The main difference is that, in order to avoid that the additional term leads to a priori estimates with exponential growth with respect to $\|v\|_{C([0, T])}$ and T , we obtain previously for the solution of the approximate parabolic problem the a priori estimate (4.21) (analogous to (2.9)) instead of using Gronwall's lemma. Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a "basis" of $\mathcal{V}$ in the following sense: $\{\phi_n\}_{n \in \mathbb{N}}$ is a set of linear independent elements of the space $\mathcal{V}$ whose linear span is dense in $\mathcal{V}$. Consider the family of finite-dimensional subspaces $\mathcal{V}_N := \langle \phi_1, \dots, \phi_N \rangle$. The first step of the proof is to find a function of the form $A_N(t, r, z) = \sum_{j=1}^N A_{jN}(t) \phi_j(r, z)$ satisfying

$$\begin{aligned} \int_{\Omega_t} \sigma (\partial_t A_N) \phi_i r dr dz + a(A_N, \phi_i) + b(t, A_N, \phi_i) &= \langle J_S, \phi_i \rangle, \\ \text{a.e. } t \in [0, T], \quad 1 \leq i \leq N, \end{aligned} \quad (4.15)$$

$$A_N(0)|_{\Omega_0} = A_N^0|_{\Omega_0}. \quad (4.16)$$

We choose the initial condition $A_N^0 \in \mathcal{V}_N$ as in Bermúdez et al. [1]. We recall that

$$A_N^0|_{\Omega_0} \xrightarrow{N} A^0 \quad \text{in } L_r^2(\Omega_0), \quad (4.17)$$

and

$$\|A_N^0\|_{\tilde{H}_r^1(\Omega)} \leq C \|A^0\|_{\tilde{H}_r^1(\Omega_0)}. \quad (4.18)$$

where the constant C depends only on Ω_0 . For small $\varepsilon > 0$, we replace (4.15) and (4.16) by the regularized approximated problem:

$$\begin{aligned} \int_{\Omega_t} \sigma (\partial_t A_N^\varepsilon) \phi_i r dr dz + \varepsilon \int_{\Omega} (\partial_t A_N^\varepsilon) \phi_i r dr dz + a(A_N^\varepsilon, \phi_i) + b(t, A_N^\varepsilon, \phi_i) \\ = \langle J_S, \phi_i \rangle, \quad \text{a.e. } t \in [0, T], \quad i = 1, \dots, N, \end{aligned} \quad (4.19)$$

$$A_N^\varepsilon(0) = A_N^0 \quad \text{in } \Omega. \quad (4.20)$$

This problem has a unique solution in $W^{1,1}(0, T; \mathcal{V}_N)$. Furthermore, $A_N^\varepsilon \in H^1(0, T; \mathcal{V}_N)$. Let us write $A_N^\varepsilon(t) = \sum_{j=1}^N A_{jN}^\varepsilon(t) \phi_j$. We multiply (4.19) by A_{iN}^ε and sum up from $i = 1$ to N to obtain

$$\begin{aligned} \int_{\Omega_t} \sigma [\partial_t A_N^\varepsilon(t)] A_N^\varepsilon(t) r dr dz + \varepsilon \int_{\Omega} \sigma [\partial_t A_N^\varepsilon(t)] A_N^\varepsilon(t) r dr dz \\ + a(A_N^\varepsilon(t), A_N^\varepsilon(t)) + b(t, A_N^\varepsilon(t), A_N^\varepsilon(t)) = \langle J_S(t), A_N^\varepsilon(t) \rangle. \end{aligned}$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the estimate

$$\begin{aligned} \frac{1}{2} \int_{\Omega_t} \sigma |A_N^\varepsilon(\tau)|^2 r dr dz + \frac{\varepsilon}{2} \int_{\Omega} |A_N^\varepsilon(\tau)|^2 r dr dz + \frac{\alpha}{2} \int_0^\tau \|A_N^\varepsilon(t)\|_{\tilde{H}_r^1(\Omega)}^2 dt \leq \\ \leq \frac{1}{2} \int_{\Omega_0} \sigma |A_N^0|^2 r dr dz + \frac{\varepsilon}{2} \int_{\Omega} \sigma |A_N^0|^2 r dr dz + \frac{2}{\alpha} \int_0^\tau \|J_S(t)\|_{\mathcal{V}'}^2 dt \end{aligned} \quad (4.21)$$

$$\forall \tau \in [0, T].$$

Now we multiply (4.19) by $\partial_t A_{iN}^\varepsilon$ and sum up from $i = 1$ to N to obtain

$$\begin{aligned} \int_{\Omega_t} \sigma |\partial_t A_N^\varepsilon(t)|^2 r dr dz + \varepsilon \int_{\Omega} |\partial_t A_N^\varepsilon(t)|^2 r dr dz \\ + \frac{1}{2} \frac{d}{dt} a(A_N^\varepsilon(t), A_N^\varepsilon(t)) + b(t, A_N^\varepsilon(t), \partial_t A_N^\varepsilon(t)) = \langle J_S(t), \partial_t A_N^\varepsilon(t) \rangle. \end{aligned}$$

Integrating in time from 0 to τ ($0 < \tau \leq T$) and using an integration by parts formula on the right-hand side, we derive

$$\begin{aligned} & \int_0^\tau \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt + \varepsilon \int_0^\tau \int_{\Omega} \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt \\ & + \frac{1}{2} a(A_N^\varepsilon(\tau), A_N^\varepsilon(\tau)) = \frac{1}{2} a(A_N^0, A_N^0) + \langle J_S(\tau), A_N^\varepsilon(\tau) \rangle - \langle J_S(0), A_N^0 \rangle \\ & - \int_0^\tau \langle \partial_t J_S(t), A_N^\varepsilon(t) \rangle dt - \int_0^\tau b(t, A_N^\varepsilon(t), \partial_t A_N^\varepsilon(t)) dt. \end{aligned} \quad (4.22)$$

Using the ellipticity of $a(\cdot, \cdot)$, (4.18), and a Young's inequality, we obtain

$$\begin{aligned} & \int_0^\tau \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt + \varepsilon \int_0^\tau \int_{\Omega} \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt + \frac{\alpha}{4} \|A_N^\varepsilon(\tau)\|_{\tilde{H}_r^1(\Omega)}^2 \\ & \leq C(M, \alpha, \Omega_0) \left[\|A^0\|_{\tilde{H}_r^1(\Omega_0)}^2 + \|J_S\|_{C([0,T];\mathcal{V}')}^2 \right] + \frac{1}{2} \int_0^\tau \|\partial_t J_S(t)\|_{\mathcal{V}'}^2 dt \\ & + \frac{1}{2} \int_0^\tau \|A_N^\varepsilon(t)\|_{\tilde{H}_r^1(\Omega)}^2 dt - \int_0^\tau b(t, A_N^\varepsilon(t), \partial_t A_N^\varepsilon(t)) dt. \end{aligned} \quad (4.23)$$

Now, using (4.12), (4.21) and (4.18), we deduce that

$$\begin{aligned} & \frac{1}{2} \int_0^\tau \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt + \varepsilon \int_0^\tau \int_{\Omega} \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt \\ & + \frac{\alpha}{4} \|A_N^\varepsilon(\tau)\|_{\tilde{H}_r^1(\Omega)}^2 \leq C(\bar{\sigma}, M, \alpha, \Omega_0) (1 + \|v\|_{C([0,T])})^2 \\ & \times \left[\|A^0\|_{\tilde{H}_r^1(\Omega_0)}^2 + \|J_S\|_{C([0,T];\mathcal{V}')}^2 + \|J_S\|_{L^2(0,T;\mathcal{V}')}^2 + \int_0^\tau \|\partial_t J_S(t)\|_{\mathcal{V}'}^2 dt \right] \end{aligned} \quad (4.24)$$

We omit the rest of the proof because it mimics that of Bermúdez et al. [1, Theorem 4.2]. $\square$

Remark 4.1. As a by-product of the proof of previous theorem we have the following facts:

- $\partial_t A_N^\varepsilon$ is bounded in $L_r^2(Q)$
- A_N^ε is bounded in $L^\infty(0, T; \mathcal{V})$
- for fixed N , there exists $A_N \in L^\infty(0, T; \mathcal{V})$ with $\partial_t A_N \in L_r^2(Q)$ and a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ converging to 0 such that:
 - (i) $\partial_t A_N^{\varepsilon_n} \rightharpoonup \partial_t A_N$ weakly in $L_r^2(Q)$.
 - (ii) $A_N^{\varepsilon_n} \rightharpoonup A_N$ weakly-star in $L^\infty(0, T; \mathcal{V})$.
 - (iii) A_N is a solution of the approximated problem (4.15)–(4.16).³

Concerning the obtention of time regularity, the adaptation of the proof of Bermúdez et al. [1, Theorem 5.1] is not immediate due to the presence of the term $b(t, A(t), Z)$. When we derive with respect to time the regularized approximated problem (4.19)–(4.20), we need to apply Reynolds' theorem to this new term. This makes appear spatial second-order spatial derivatives, which must be previously estimated. Hence, in order to obtain estimates of A_N^ε that lead to the time regularity of A , we need to establish the analogue of Remark 2.2 for A_N^ε . To do this it is necessary to implement Galerkin method with the spectral basis, that is, the basis formed by the eigenfunctions of the problem

Problem 2. Find $\lambda \in \mathbb{R}$ and $Y \in \mathcal{V}$, $Y \neq 0$, such that

$$a(Y, Z) = \lambda \int_{\Omega} Y Z r dr dz \quad \forall Z \in \mathcal{V} \quad (4.25)$$

Since $\mathcal{V}$ is compactly and densely embedded in $L_r^2(\Omega)$ (Mercier and Raugel [9, Lemma 4.2]), a classical result of abstract spectral theory (see Raviart and Thomas [10, Theorem 6.2-1] for instance) ensures that the eigenvalues λ_n , $n \in \mathbb{N}$, of Problem 2 form an increasing sequence of strictly positive real numbers which tends to $+\infty$ and there exists a Hilbert basis $\{\phi_n\}_{n \in \mathbb{N}}$ of the space $L_r^2(\Omega)$ such that for all $n \in \mathbb{N}$, ϕ_n is an eigenfunction associated to eigenvalue λ_n . Note that $\phi_n \in \tilde{H}_r^2(\Omega) \cap \mathcal{V}$ because of Gopalakrishnan and Pasciak [8, Theorem 4.1].

Lemma 4.2. Assume $J_S \in H^1(0, T; \mathcal{V}') \cap L^2(0, T; L_r^2(\Omega))$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$. Let A_N^ε be the solution of the regularized approximated problem (4.19)–(4.20) implemented with the spectral basis $\{\phi_n\}_{n \in \mathbb{N}}$ of $\mathcal{V}$. Then the following a priori estimate holds:

$$\begin{aligned} \|A_N^\varepsilon\|_{L^2(0,T;\tilde{H}_r^2(\Omega))} & \leq C_{reg} \mu \left[\|J_S\|_{L^2(0,T;L_r^2(\Omega))} + \|\sigma \partial_t A_N^\varepsilon\|_{L_r^2(Q)} + \right. \\ & \left. \varepsilon \|\partial_t A_N^\varepsilon\|_{L^2(0,T;L_r^2(\Omega))} + \bar{\sigma} \|v\|_{C([0,T])} \|A_N^\varepsilon\|_{L^2(0,T;\mathcal{V})} \right]. \end{aligned} \quad (4.26)$$

³ In fact, by the same arguments of Theorem 2.1, the approximated problem (4.15)–(4.16) has at most one solution $A_N \in L^2(0, T; \mathcal{V}_N)$ with $\partial_t A_N \in L_r^2(Q)$.

Proof. We have $A_N^\varepsilon \in C([0, T]; \tilde{H}_r^2(\Omega) \cap \mathcal{V})$ because $A_{jN}^\varepsilon \in H^1(0, T)$ and $\phi_n \in \tilde{H}_r^2(\Omega) \cap \mathcal{V}$. Let $\mathcal{L} : \tilde{H}_r^2(\Omega) \cap \mathcal{V} \mapsto L_r^2(\Omega)$ be the operator defined by

$$\mathcal{L}Y = -\frac{1}{\mu} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(rY)}{\partial r} \right) + \frac{\partial^2 Y}{\partial z^2} \right] \quad \forall Y \in \tilde{H}_r^2(\Omega) \cap \mathcal{V}. \quad (4.27)$$

We have

$$a(Y, Z) = \int_{\Omega} (\mathcal{L}Y) Z r dr dz \quad \forall Y \in \tilde{H}_r^2(\Omega) \cap \mathcal{V}, \forall Z \in \mathcal{V}$$

(see, Bermúdez et al. [1, Remark 3.1]). Hence ϕ_n satisfies

$$\mathcal{L}\phi_n = \lambda_n \phi_n.$$

Therefore $\mathcal{L}A_N^\varepsilon(t) \in \mathcal{V}_N$ for all $t \in [0, T]$. By rewriting (4.19) as

$$\begin{aligned} \int_{\Omega} (\mathcal{L}A_N^\varepsilon(t)) \phi_i r dr dz = \\ \int_{\Omega} [J_S(t) - \chi_Q(t) \sigma(t) \partial_t A_N^\varepsilon(t) - \varepsilon \partial_t A_N^\varepsilon(t) - \chi_Q(t) \sigma(t) v(t) \partial_z A_N^\varepsilon(t)] \phi_i r dr dz \\ \text{a.e. } t \in [0, T], \quad i = 1, \dots, N, \end{aligned}$$

where χ_Q stands for the characteristic function of Q , we realize that

$$\mathcal{L}A_N^\varepsilon(t) = P_N (J_S(t) - \chi_Q(t) \sigma(t) \partial_t A_N^\varepsilon(t) - \varepsilon \partial_t A_N^\varepsilon(t) - \chi_Q(t) \sigma(t) v(t) \partial_z A_N^\varepsilon(t))$$

where P_N stands for the orthogonal projection operator from $L_r^2(\Omega)$ onto $\mathcal{V}_N$ with respect to the standard scalar product of $L_r^2(\Omega)$. The result follows by applying now Gopalakrishnan and Pasciak [8, Theorem 4.1]. $\square$

The estimate (4.26) combined with (4.21), (4.18) and (4.24) yields to

$$\begin{aligned} \|A_N^\varepsilon\|_{L^2(0,T;\tilde{H}_r^2(\Omega))} &\leq C_{reg} \mu C_4(\bar{\sigma}, M, \alpha, \Omega_0) (1 + \|v\|_{C(0,T)}) \\ &\times \left[\|A^0\|_{\tilde{H}_r^1(\Omega)} + \|J_S\|_{C([0,T];\mathcal{V}')} + \|J_S\|_{L^2(0,T;L_r^2(\Omega))} + \|\partial_t J_S\|_{L^2(0,T;\mathcal{V}')} \right]. \end{aligned} \quad (4.28)$$

Proof of Theorem 2.3. We use again the spectral basis to implement the regularized approximated problem. Writing (4.19) in matrix form and taking into account the regularity of the data, it can be deduced that $A_{jN}^\varepsilon \in W^{2,1}(0, T) \quad j = 1, \dots, N$. Thus we are allowed to differentiate (4.19) with respect to time. For this purpose, we recall that

$$b(t, A_N^\varepsilon, \phi_i) = v(t) \int_{\Omega_i} \sigma(\partial_z A_N^\varepsilon) \phi_i r dr dz.$$

Using Reynolds formula (4.10), we obtain

$$\begin{aligned} \int_{\Omega_i} \sigma(\partial_{tt} A_N^\varepsilon) \phi_i r dr dz + \varepsilon \int_{\Omega} (\partial_{tt} A_N^\varepsilon) \phi_i r dr dz + a(\partial_t A_N^\varepsilon, \phi_i) \\ + v(t) \int_{\Omega_i} \sigma \partial_z ((\partial_t A_N^\varepsilon) \phi_i) r dr dz + v(t) \int_{\Omega_i} \sigma(\partial_{tz} A_N^\varepsilon) \phi_i r dr dz \\ + (v(t))^2 \int_{\Omega_i} \sigma \partial_z (\partial_z A_N^\varepsilon \phi_i) r dr dz + v'(t) \int_{\Omega_i} \sigma(\partial_z A_N^\varepsilon) \phi_i r dr dz = \langle \partial_t J_S(t), \phi_i \rangle \end{aligned}$$

Next we multiply the above equation by $t \partial_t A_{iN}^\varepsilon(t)$ and sum up from $i = 1$ to N . Thus, we obtain

$$\begin{aligned} t \int_{\Omega_i} \sigma(\partial_{tt} A_N^\varepsilon) (\partial_t A_N^\varepsilon) r dr dz + \varepsilon t \int_{\Omega} (\partial_{tt} A_N^\varepsilon) (\partial_t A_N^\varepsilon) r dr dz + t a(\partial_t A_N^\varepsilon, \partial_t A_N^\varepsilon) \\ + t v(t) \int_{\Omega_i} \sigma \partial_z (|\partial_t A_N^\varepsilon|^2) r dr dz + t v(t) \int_{\Omega_i} \sigma(\partial_{tz} A_N^\varepsilon) (\partial_t A_N^\varepsilon) r dr dz \\ + t (v(t))^2 \int_{\Omega_i} \sigma \partial_z (\partial_z A_N^\varepsilon \partial_t A_N^\varepsilon) r dr dz + t v'(t) \int_{\Omega_i} \sigma(\partial_z A_N^\varepsilon) (\partial_t A_N^\varepsilon) r dr dz \\ = t \langle \partial_t J_S(t), \partial_t A_N^\varepsilon(t) \rangle \end{aligned}$$

This equation together Reynolds formula (4.10) and (4.11) gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega_i} t \sigma |\partial_t A_N^\varepsilon|^2 r dr dz + \frac{\varepsilon}{2} \frac{d}{dt} \int_{\Omega} t |\partial_t A_N^\varepsilon|^2 r dr dz \\ + t (a(\partial_t A_N^\varepsilon, \partial_t A_N^\varepsilon) - c(t, \partial_t A_N^\varepsilon, \partial_t A_N^\varepsilon)) = \frac{1}{2} \int_{\Omega_i} \sigma |\partial_t A_N^\varepsilon|^2 r dr dz \\ + \frac{\varepsilon}{2} \int_{\Omega} |\partial_t A_N^\varepsilon|^2 r dr dz - t S_1 - t S_2 - t S_3 + t \langle \partial_t J_S(t), \partial_t A_N^\varepsilon(t) \rangle \end{aligned} \quad (4.29)$$

where

$$\begin{aligned} S_1 &= v'(t) \int_{\Omega_t} \sigma(\partial_z A_N^\varepsilon)(\partial_t A_N^\varepsilon) r dr dz, \\ S_2 &= (v(t))^2 \int_{\Omega_t} \sigma(\partial_{zt} A_N^\varepsilon)(\partial_z A_N^\varepsilon) r dr dz, \\ S_3 &= (v(t))^2 \int_{\Omega_t} \sigma(\partial_{zz} A_N^\varepsilon)(\partial_t A_N^\varepsilon) r dr dz. \end{aligned}$$

Using the bounds

$$\begin{aligned} |S_1| &\leq \left[\frac{1}{2} \bar{\sigma} \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{1}{2} \|A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 \right] |v'(t)| \\ |S_2| &\leq \delta_1 \|\partial_t A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 + \frac{\bar{\sigma}^2}{4\delta_1} \|v\|_{C([0,T])}^4 \|A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 \quad \forall \delta_1 > 0, \\ |S_3| &\leq \frac{\bar{\sigma}}{2} \|v\|_{C([0,T])}^2 \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{1}{2} \|v\|_{C([0,T])}^2 \|A_N^\varepsilon\|_{\tilde{H}_r^2(\Omega)}^2 \end{aligned}$$

in (4.29), we deduce the inequality

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} t \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{\varepsilon}{2} \frac{d}{dt} \int_{\Omega} t \left| \partial_t A_N^\varepsilon \right|^2 r dr dz \\ &\quad + t(a(\partial_t A_N^\varepsilon, \partial_t A_N^\varepsilon) - c(t, \partial_t A_N^\varepsilon, \partial_t A_N^\varepsilon)) \leq \frac{1}{2} \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz \\ &\quad + \frac{\varepsilon}{2} \int_{\Omega} \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + t |v'(t)| \left[\frac{1}{2} \bar{\sigma} \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{1}{2} \|A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 \right] \\ &\quad + \delta_1 t \|\partial_t A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 + \frac{\bar{\sigma}^2}{4\delta_1} t \|v\|_{C([0,T])}^4 \|A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 + \frac{\bar{\sigma}}{2} t \|v\|_{C([0,T])}^2 \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz \\ &\quad + \frac{1}{2} t \|v\|_{C([0,T])}^2 \|A_N^\varepsilon\|_{\tilde{H}_r^2(\Omega)}^2 + \delta_2 t \|\partial_t A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 + \frac{1}{4\delta_2} t \|\partial_t J_S\|_{\mathcal{V}}^2, \quad \forall \delta_1 > 0, \forall \delta_2 > 0, \end{aligned}$$

Now, taking $\delta_1 = \delta_2 = \alpha/8$ and using Lemma 4.1, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} t \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{\varepsilon}{2} \frac{d}{dt} \int_{\Omega} t \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{\alpha}{4} t \|\partial_t A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 \\ &\leq \frac{\varepsilon}{2} \int_{\Omega} \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \left(\frac{1}{2} + \bar{\sigma} \left(\frac{2}{\alpha} + \frac{1}{2} \right) T \|v\|_{C([0,T])}^2 \right) \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz \\ &\quad + \frac{2\bar{\sigma}^2}{\alpha} T \|v\|_{C([0,T])}^4 \|A_N^\varepsilon\|_{L^\infty(0,T;\mathcal{V})}^2 + \frac{1}{2} T \|v\|_{C([0,T])}^2 \|A_N^\varepsilon\|_{\tilde{H}_r^2(\Omega)}^2 + \frac{2}{\alpha} T \|\partial_t J_S\|_{\mathcal{V}}^2 \\ &\quad + t |v'(t)| \left[\frac{1}{2} \bar{\sigma} \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon \right|^2 r dr dz + \frac{1}{2} \|A_N^\varepsilon\|_{\tilde{H}_r^1(\Omega)}^2 \right] \end{aligned}$$

and integrating in time from 0 to τ , we deduce that

$$\begin{aligned} &\frac{\tau}{2} \int_{\Omega_\tau} \sigma \left| \partial_t A_N^\varepsilon(\tau) \right|^2 r dr dz + \frac{\varepsilon}{2} \tau \int_{\Omega} \left| \partial_t A_N^\varepsilon(\tau) \right|^2 r dr dz \\ &\quad + \frac{\alpha}{4} \int_0^\tau t \|\partial_t A_N^\varepsilon(t)\|_{\tilde{H}_r^1(\Omega)}^2 dt \leq \frac{\varepsilon}{2} \int_0^\tau \int_{\Omega} \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt \\ &\quad + \left(\frac{1}{2} + \bar{\sigma} \left(\frac{2}{\alpha} + \frac{1}{2} \right) T \|v\|_{C([0,T])}^2 \right) \int_0^\tau \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt \\ &\quad + \frac{2\bar{\sigma}^2}{\alpha} T^2 \|v\|_{C([0,T])}^4 \|A_N^\varepsilon\|_{L^\infty(0,T;\mathcal{V})}^2 + \frac{1}{2} T \|v\|_{C([0,T])}^2 \int_0^\tau \|A_N^\varepsilon(t)\|_{\tilde{H}_r^2(\Omega)}^2 dt + \\ &\quad \frac{2}{\alpha} T \int_0^\tau \|\partial_t J_S\|_{\mathcal{V}}^2 dt + \frac{1}{2} T \|A_N^\varepsilon\|_{L^\infty(0,T;\mathcal{V})}^2 \int_0^\tau |v'(t)| dt \\ &\quad + \frac{1}{2} \bar{\sigma} \int_0^\tau t |v'(t)| \int_{\Omega_t} \sigma \left| \partial_t A_N^\varepsilon(t) \right|^2 r dr dz dt. \end{aligned}$$

By combining this inequality with (4.24) and (4.28), and setting

$$y(\tau) := \frac{\tau}{2} \int_{\Omega_\tau} \sigma \left| \partial_t A_N^\varepsilon(\tau) \right|^2 r dr dz + \frac{\alpha}{4} \int_0^\tau t \|\partial_t A_N^\varepsilon(t)\|_{\tilde{H}_r^1(\Omega)}^2 dt$$

we obtain an estimate of the form

$$y(\tau) \leq \bar{\sigma} \int_0^\tau |v'(t)| y(t) dt + \hat{P}$$

where

$$\begin{aligned} \hat{P} &= \hat{C}(\bar{\sigma}, M, \alpha, C_{reg}, \mu, \Omega_0) \\ &\times \left[(1 + T^2)(1 + \|v\|_{C([0,T])}^6) + T(1 + \|v\|_{C([0,T])}^2) \int_0^T |v'(t)| dt \right] \\ &\times \left[\|A^0\|_{\tilde{H}_r^1(\Omega_0)} + \|J_S\|_{C([0,T]; \mathcal{V}')} + \|\partial_t J_S\|_{L^2(0,T; \mathcal{V}')} + \|J_S\|_{L^2(0,T; L_r^2(\Omega))} \right]^2 \end{aligned} \quad (4.30)$$

Finally, by applying Gronwall's lemma, we obtain estimate for A_N^ε which is similar to (2.12). We conclude the proof by using the same arguments as in Bermúdez et al. [1, Theorem 5.1] $\square$

We recall the following notations introduced in Bermúdez et al. [1].

Let $\mathcal{V}^e := \{Z|_{\Omega_e} : Z \in \mathcal{V}\}$ and $\mathcal{V}_0^e := \{Z \in \mathcal{V}^e : Z|_{\Gamma_e} = 0\}$, where $\Omega_e = \Omega \setminus \bar{\Omega}_0$ and $\Gamma_e := \partial\Omega_e \cap \partial\Omega_0$. Let A^e be the solution of the following well-posed elliptic problem with non-homogeneous Dirichlet boundary data: Find $A^e \in \mathcal{V}^e$ such that $A^e|_{\Gamma_e} = A^0|_{\Gamma_e}$ and

$$\int_{\Omega_e} \frac{1}{\mu} \operatorname{curl}(A^e e_\theta) \cdot \operatorname{curl}(Z e_\theta) r dr dz = \int_{\Omega_S} J_S(0) Z r dr dz \quad \forall Z \in \mathcal{V}_0^e. \quad (4.31)$$

Finally, let $\bar{A}^0 \in \mathcal{V}$ be the following extension of A^0 to the whole Ω :

$$\bar{A}^0 := \begin{cases} A^0 & \text{in } \Omega_0, \\ A^e & \text{in } \Omega_e. \end{cases}$$

By the same arguments as in Bermúdez et al. [1, Theorem 5.3], we have the following result.

Theorem 4.1. Under the assumptions of Theorem 2.3 and $J_S(0) \in L_r^2(\Omega)$, if A is the solution to Problem 1, then $A(t) \rightharpoonup \bar{A}^0$ weakly in $\mathcal{V}$ as $t \rightarrow 0+$.

For the sake of completeness, we now address two additional spatial regularity results, although we will not need them in the study of levitation model. We have the following adaptation of Theorems 5.4 and 5.5 of Bermúdez et al. [1].

Theorem 4.2. Under the assumptions of Theorem 4.1, if the solution to Problem 1 satisfies $A \in H^1(0, T; \mathcal{V}) \cap L^\infty(0, T; \tilde{H}_r^2(\Omega))$, then $\bar{A}^0 \in \tilde{H}_r^2(\Omega)$.

Theorem 4.3. Under the assumptions of Theorem 4.1, if the initial data A^0 and $J_S(0)$ are such that $\bar{A}^0 \in \tilde{H}_r^2(\Omega)$, then the solution to Problem 1 satisfies $A \in H^1(0, T; \mathcal{V})$ and $\chi_Q \partial_t A \in L^\infty(0, T; L_r^2(\Omega))$. If we assume further $J_S \in L^\infty(0, T; L_r^2(\Omega))$, then $A \in L^\infty(0, T; \tilde{H}_r^2(\Omega))$.

The proof of the Theorem 4.2 is the same as that of Bermúdez et al. [1, Theorem 5.4]. The proof of Theorem 4.3 follows essentially the lines of that of Bermúdez et al. [1, Theorem 5.5] but with two changes. First, now we need to use the spectral basis (as in the proof of Theorem 2.3). Hence we cannot include $\bar{A}^0$ in the “basis” of $\mathcal{V}$ and take $A_N^0 = \bar{A}^0$ as it was done in Bermúdez et al. [1]. This was used to obtain a priori estimates of

$$\frac{1}{2} \int_{\Omega_0} \sigma |\partial_t A_N^\varepsilon(0)|^2 r dr dz + \frac{\varepsilon}{2} \int_{\Omega} \sigma |\partial_t A_N^\varepsilon(0)|^2 r dr dz.$$

Instead, we take $A_N^0 = P_N \bar{A}^0$, so we have $\mathcal{L} A_N^0 = \mathcal{L} P_N \bar{A}^0 = P_N \mathcal{L} \bar{A}^0$. This enables us obtain the required a priori estimates of the two integrals. Second, we need to use Gronwall's lemma to handle a term involving $|v'(t)|$ (as in the proof of Theorem 2.3).

The following estimate will be applied in the study of the levitation problem.

Theorem 4.4. Assume $v \in W^{1,1}(0, T)$, $J_S \in H^1(0, T; L_r^2(\Omega_S))$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$. We have an inequality of the form

$$\begin{aligned} &\int_0^\tau \int_{\Omega_0} \sigma |\partial_t A(t)|^2 r dr dz dt + \frac{\alpha}{4} \|A(\tau)\|_{\tilde{H}_r^1(\Omega)}^2 \\ &\leq C(\bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S) - \int_0^\tau v(t) \int_{\Omega_0} \sigma \partial_z A(t) \partial_t A(t) r dr dz dt \quad \forall \tau \in (0, T] \end{aligned} \quad (4.32)$$

In order to prove previous theorem, we need the following lemma.

Lemma 4.3. Assume $v \in W^{1,1}(0, T)$, $J_S \in H^1(0, T; L_r^2(\Omega_S))$ and $A^0 \in \tilde{H}_r^1(\Omega_0)$. Let A_N^ε be the solution of the regularized approximated problem (4.19)–(4.20) implemented with the spectral basis $\{\phi_n\}_{n \in \mathbb{N}}$ of $\mathcal{V}$ and let A_N be the solution introduced in Remark 4.1 of the approximated problem (4.15)–(4.16). Then $\partial_z A_N^\varepsilon \rightarrow \partial_z A_N$ strongly in $L^2(0, T; L_r^2(\Omega))$ as $\varepsilon_n \rightarrow 0+$. Analogously $\partial_z A_N \rightarrow \partial_z A$ strongly in $L^2(0, T; L_r^2(\Omega))$ as $N \rightarrow \infty$.

Proof. We have the bounds and convergences stated in Remark 4.1. Besides, from Lemma 4.2, we have that A_N^ε is bounded in $L^2(0, T; \tilde{H}_r^2(\Omega))$ and from the proof of Theorem 2.3, we also have that $\sqrt{t}\partial_t A_N^\varepsilon$ is bounded in $L^2(0, T; \mathcal{V})$. For all $\delta > 0$ small, we introduce the space $H_r^1((\delta, T) \times \Omega) = L^2(\delta, T; H_r^1(\Omega)) \cap H^1((\delta, T); L_r^2(\Omega))$. This space is compactly embedded into $L^2(\delta, T; L_r^2(\Omega))$ owing to Lions–Aubin lemma (see, for instance, Showalter [11, Proposition III.1.3]) and the compact injection $H_r^1(\Omega) \hookrightarrow L_r^2(\Omega)$ (Mercier and Raugel [9, Lemma 4.2]). From the mentioned boundedness properties, we have that for all $\delta > 0$ small, $\partial_z A_N^\varepsilon$ is bounded in $H_r^1((\delta, T) \times \Omega)$. Therefore, for fixed N , there exists a subsequence of $\partial_z A_N^\varepsilon$ which converges strongly in $L^2(\delta, T; L_r^2(\Omega))$. Clearly its limit is $\partial_z A_N$. Hence, a standard argument implies that the whole sequence $\partial_z A_N^\varepsilon \rightarrow \partial_z A_N$ strongly in $L^2(\delta, T; L_r^2(\Omega))$. This holds for every $\delta > 0$ small. Now, by writing

$$\begin{aligned} \int_0^T \|\partial_z A_N^\varepsilon(t) - \partial_z A_N(t)\|_{L_r^2(\Omega)}^2 dt &= \\ \int_0^\delta \|\partial_z A_N^\varepsilon(t) - \partial_z A_N(t)\|_{L_r^2(\Omega)}^2 dt &+ \int_\delta^T \|\partial_z A_N^\varepsilon(t) - \partial_z A_N(t)\|_{L_r^2(\Omega)}^2 dt \end{aligned}$$

and using the last convergence together with the boundedness of A_N^ε in $L^\infty(0, T; \mathcal{V})$, it is easy to deduce that $\partial_z A_N^\varepsilon \rightarrow \partial_z A_N$ strongly in $L^2(0, T; L_r^2(\Omega))$. The proof of the strong convergence $\partial_z A_N \rightarrow \partial_z A$ in $L^2(0, T; L_r^2(\Omega))$ uses analogous arguments. $\square$

Proof of Theorem 4.4. By combining inequalities (4.23), (4.21) and (4.18), we deduce an estimate of the form

$$\begin{aligned} \int_0^\tau \int_{\Omega_t} \sigma |\partial_t A_N^\varepsilon(t)|^2 r dr dz dt + \frac{\alpha}{4} \|A_N^\varepsilon(\tau)\|_{\tilde{H}_r^1(\Omega)}^2 \\ \leq C(\bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S) - \int_0^\tau v(t) \int_{\Omega_t} \sigma \partial_z A_N^\varepsilon(t) \partial_t A_N^\varepsilon(t) r dr dz dt \quad \forall \tau \in [0, T]. \end{aligned} \quad (4.33)$$

We take the limit inferior as $\varepsilon_n \rightarrow 0^+$. Note that we can pass to the limit in the integral of the right hand side because $\partial_z A_N^\varepsilon \rightarrow \partial_z A_N$ strongly in $L^2(0, T; L_r^2(\Omega))$ and $\partial_t A_N^\varepsilon \rightharpoonup \partial_t A_N$ weakly in $L_r^2(Q)$. Thus we obtain for a.e. $\tau \in (0, T]$ an analogous inequality with A_N^ε replaced by A_N . Next, by passing to the limit inferior as $N \rightarrow \infty$ using the same arguments, we obtain inequality (4.32), in principle for a.e. $\tau \in (0, T]$. Since all the terms involved in it are continuous functions of τ in $(0, T]$, inequality (4.32) holds in fact for all $\tau \in (0, T]$, as desired. $\square$

5. The need to adapt the framework of Logemann and Ryan [2] to analyze the formulation of the coupled levitation problem as a causal differential equation

The problem (2.16)–(2.18) does not exactly fit the framework of Logemann and Ryan [2] due to the constraint (2.18). This is because the constraint considered in Logemann and Ryan [2] for a case without delay is of the form $(t, \bar{u}(t)) \in \Delta$, where $\bar{u}$ is the unknown function (which takes values in $\mathbb{R}^N$) and Δ is a subset of $[0, \infty) \times \mathbb{R}^N$, open for the relative topology. At first glance, it might seem that a form to fit the coupled problem into the framework of Logemann and Ryan [2] is to reformulate the differential Eq. (2.4) as a system by proceeding in the usual way (recalling (2.14)):

$$\frac{du}{dt} = v, \quad (5.1)$$

$$\frac{dv}{dt} = (F(v))(t), \quad (5.2)$$

and taking into account the constraint $\underline{u} < u(t) < \bar{u}$. However, this requires the second member to be defined for “decoupled” functions u and v , that is, that do not satisfy $\frac{du}{dt} = v$, hence u does not coincide with the displacement associated with the velocity v . Therefore said constraint will not be the physical constraint required by the electromagnetic problem.

It is also not possible to replace (5.2) by $\frac{dv}{dt} = (F(u'))(t)$ because for $u \in W^{1,1}(0, T)$, u' does not have enough regularity to be able to define $S_T(u')$ and therefore $F(u')$. In summary, it is necessary to adapt the framework of Logemann and Ryan [2] to the case of constraint (2.18).

6. Adaptation of the framework of Logemann and Ryan [2]

In this section we follow closely Section 3 of Logemann and Ryan [2], from which we take some definitions and results. For an arbitrary interval (not necessarily compact) $I \subset \mathbb{R}$ and $p \in [1, +\infty]$, we define

$$L_{loc}^p(I) := \{u : I \rightarrow \mathbb{R}^N; u|_{[a,b]} \in L^p[a,b] \forall a, b \in I, a < b\},$$

$$W_{loc}^{1,1}(I) := \{u : I \rightarrow \mathbb{R}^N; u|_{[a,b]} \in W^{1,1}[a,b] \forall a, b \in I, a < b\}.$$

Obviously, if I is compact $L_{loc}^p(I) = L^p(I)$ and $W_{loc}^{1,1}(I) = W^{1,1}(I)$.

Let $T_f > 0$ (eventually $T_f = +\infty$). We restrict ourselves to the case where there is no delay (that is, $h = 0$ with the notation of Logemann and Ryan [2]) because it is enough to tackle the levitation problem. We consider the initial value problem (IVP)

$$\frac{du}{dt}(t) = (F(u))(t), \quad 0 \leq t < T_f \quad (6.1)$$

$$u(0) = \varphi \in \mathbb{R}^N, \quad (6.2)$$

where F is an operator acting in suitable function spaces u defined on intervals of the form $I = [0, \eta]$ (where $0 < \eta < T_f$) or $I = [0, \eta]$ (where $0 < \eta \leq T_f$) and such that $gr(Pu) \subset \Delta$, where

$$Pu(t) = \int_0^t u(\tau) d\tau, t \in I,$$

$$gr(Pu) = \{(t, (Pu)(t)); t \in I\}.$$

Here and hereafter, $\Delta = [0, T_f] \times G$, where G is an open subset of $\mathbb{R}^N$. (It is enough for us to restrict ourselves to this case, simpler than the more general one considered in Logemann and Ryan [2]). We assume that $0 \in G$ (hence $(0, (Pu)(0)) = (0, 0) \in \Delta$).

The basic difference with respect to Logemann and Ryan [2] is that we have replaced the condition $gr u \subset \Delta$ by $gr(Pu) \subset \Delta$ (where $gr u = \{(t, u(t)); t \in I\}$.) We need to analyze what modifications need to be made in the results of Section 3 of Logemann and Ryan [2]. The assumption $\Delta_0 := \{x \in \mathbb{R}^N; (0, x) \in \Delta\} \neq \emptyset$ made in Logemann and Ryan [2] is now not needed. The hypothesis that plays an analogous role is: $0 \in G$.

We define now

$$W_\Delta(I) := \{u \in W_{loc}^{1,1}(I), gr(Pu) \subset \Delta\},$$

$W_\Delta(\{0\}) = \mathbb{R}^N$ (since any $\varphi \in \mathbb{R}^N$ is an admissible initial condition.) Note that $W_{loc}^{1,1}(I) \subset C(I)$.

Let $\mathcal{J} := \{I \text{ interval } I = [0, \eta] \text{ (where } 0 < \eta < T_f) \text{ or } I = [0, \eta] \text{ (where } 0 < \eta \leq T_f); W_\Delta(I) \neq \emptyset\}$. Since $0 \in G$ and G is open in $\mathbb{R}^N$, we have that for $\alpha > 0$ small enough $[0, \alpha] \in \mathcal{J}$. This is equivalent to

$$T := \sup\{\alpha \in (0, T_f); W_\Delta[0, \alpha] \neq \emptyset\} > 0.$$

We assume that for all $I \in \mathcal{J}$, the operator F maps $W_\Delta(I)$ into $L_{loc}^1(I)$. We say that $u : I \rightarrow \mathbb{R}^N$ is a solution of (6.1)–(6.2) on the interval I if: $I \in \mathcal{J}$, $u \in W_\Delta(I)$, $u(0) = \varphi$ and u satisfies the differential Eq. (6.1) for almost every $t \in I$. Given $0 \leq \alpha < T$, $\beta > \alpha$, $\gamma > 0$ and $w \in W_\Delta[0, \alpha]$, we introduce when $\alpha > 0$

$$W(w; \alpha, \beta, \gamma) := \{u \in C[0, \beta]; u|_{[0, \alpha]} = w, u|_{[\alpha, \beta]} \in W^{1,1}[\alpha, \beta], \int_\alpha^\beta \|u'(s)\| ds \leq \gamma\}$$

(If $\alpha = 0$ and $w \in W_\Delta(\{0\}) = \mathbb{R}^N$,

$$W(w; 0, \beta, \gamma) := \{u \in W^{1,1}[0, \beta]; u(0) = w, \int_0^\beta \|u'(s)\| ds \leq \gamma\}.$$

Obviously $W(w; \alpha, \beta, \gamma) \subset W^{1,1}[0, \beta]$. We endow $W(w; \alpha, \beta, \gamma)$ with the distance μ defined by

$$\mu(u, v) = \int_\alpha^\beta \|u'(s) - v'(s)\| ds.$$

Lemma 3.1 of Logemann and Ryan [2] is still true (with the same proof).

Given $\alpha \in [0, T)$ and $w \in W_\Delta[0, \alpha]$, we define ⁴

$$Adm(w, \alpha) := \{(\beta, \gamma) \in (\alpha, T) \times (0, +\infty); W(w; \alpha, \beta, \gamma) \subset W_\Delta[0, \beta]\}.$$

Now we have that

$$(\beta, \gamma) \in (\alpha, T) \times (0, +\infty) \text{ is in } Adm(w, \alpha) \text{ iff } gr(Pu) \subset \Delta \quad \forall u \in W(w; \alpha, \beta, \gamma).$$

Lemma 3.2 of Logemann and Ryan [2] is still true but its proof needs to be slightly modified.

Lemma 6.1. *If $\alpha \in [0, T)$ and $w \in W_\Delta[0, \alpha]$, then:*

- (i) *The set $Adm(w, \alpha)$ is nonempty.*
- (ii) *If $(\beta, \gamma) \in Adm(w, \alpha)$, then $(b, c) \in Adm(w, \alpha)$ for all $b \in (\alpha, \beta]$ and all $c \in (0, \gamma]$.*

Proof. We have that $(\alpha, (Pw)(\alpha)) \in \Delta = [0, T_f] \times G$. Since G is open in $\mathbb{R}^N$, there exists $d > 0$ such that $\overline{B((Pw)(\alpha), d)} \subset G$. Hence

$$\forall \beta \in (\alpha, T_f) \quad [\alpha, \beta] \times \overline{B((Pw)(\alpha), d)} \subset [0, T_f] \times G = \Delta.$$

For every $\beta \in (\alpha, T_f)$, every $u \in W(w; \alpha, \beta, \gamma)$ and every $t \in [\alpha, \beta]$ we have:

$$\|u(t) - w(\alpha)\| = \|u(t) - u(\alpha)\| \leq \int_\alpha^t \|u'(s)\| ds \leq \int_\alpha^\beta \|u'(s)\| ds \leq \gamma$$

⁴ In [2], the notation $A(w; \alpha)$ is used instead of $Adm(w, \alpha)$. Here we use $Adm(w, \alpha)$ in order to avoid confusion with A , the solution of the electromagnetic problem.

and then

$$\|(Pu)(t) - (Pw)(\alpha)\| = \|(Pu)(t) - (Pu)(\alpha)\| \leq \int_{\alpha}^t \|u(s)\| ds \leq (\beta - \alpha)(\|w(\alpha)\| + \gamma).$$

Consequently, by taking $\beta \in (\alpha, T_f)$ and $\gamma > 0$ such that

$$(\beta - \alpha)(\|w(\alpha)\| + \gamma) \leq d,$$

we have $W(w; \alpha, \beta, \gamma) \subset W_{\Delta}[0, \beta]$. Therefore $(\beta, \gamma) \in \text{Adm}(w, \alpha)$. This proves (i). The proof of (ii) follows the same lines as the second statement of Lemma 3.2 of Logemann and Ryan [2] with minor modifications (gru and $gr\tilde{u}$ must be replaced by $gr(Pu)$ and $gr(P\tilde{u})$ respectively, where notation $\tilde{u}$ has the same meaning as in Logemann and Ryan [2].) $\square$

If $\alpha \in [0, T]$, $w \in W_{\Delta}[0, \alpha]$, $(\beta, \gamma) \in \text{Adm}(w, \alpha)$, $b \in (\alpha, \beta]$ and $c \in (0, \gamma]$, in virtue of previous lemma we have that $W(w; \alpha, b, c) \subset W_{\Delta}[0, b]$ and therefore $F(u)$ is well defined for all $u \in W(w; \alpha, b, c)$.

Hypotheses (H1) and (H2) are as in Logemann and Ryan [2]:

(H1) (Causality): if $I, J \in \mathcal{J}$, then, for all $\tau \in (I \cap J) \setminus \{0\}$, all $u \in W_{\Delta}(I)$ and all $v \in W_{\Delta}(J)$,

$$u|_{[0, \tau]} = v|_{[0, \tau]} \Rightarrow F(u)|_{[0, \tau]} = F(v)|_{[0, \tau]}$$

(H2) (Local Lipschitz-type condition): for every $\alpha \in [0, T]$ and every function $w \in W_{\Delta}[0, \alpha]$, there exists $\lambda \in (0, 1)$ and $(\beta, \gamma) \in \text{Adm}(w; \alpha)$ such that

$$\|F(u) - F(v)\|_{L^1[\alpha, \beta]} \leq \lambda \mu(u, v) \quad \forall u, v \in W(w; \alpha, \beta, \gamma).$$

Some result related to the maximally defined solution will also require the hypothesis (H3) (which resembles but differs from hypothesis (H3) of Logemann and Ryan [2]):

(H3) Integrability condition: For every $I \in \mathcal{J}$ and every function $u \in W_{\Delta}(I) \cap L^{\infty}(I)$ such that $\overline{gr(Pu)}$ is compact and contained in Δ , the function $F(u) \in L^1(I)$.

(As in Logemann and Ryan [2], the essence of (H3) is that it encompasses non-compact intervals $I \in \mathcal{J}$.) We need (as in Logemann and Ryan [2]) to consider the IVP

$$\frac{du}{dt}(t) = (F(u))(t), \quad \alpha \leq t < T_f \tag{6.3}$$

$$u|_{[0, \alpha]} = \psi \in W_{\Delta}[0, \alpha]. \tag{6.4}$$

For $\alpha = 0$, this IVP reduces to (6.1)–(6.2). We say that $u : I \mapsto \mathbb{R}^N$ is a solution of (6.3)–(6.4) on the interval I if $I \in \mathcal{J}$ with $\sup I > \alpha$, $u \in W_{\Delta}(I)$, $u|_{[0, \alpha]} = \psi$ and u satisfies the differential Eq. (6.3) for almost every $t \in I \cap [\alpha, T_f)$.

Lemma 3.3, Proposition 3.5 and Theorem 3.6 of Logemann and Ryan [2] are still true. For the sake of completeness, we recall the last one.

Theorem 6.1. Assume that F satisfies (H1) and (H2). Then:

1. The initial value problem (6.1)–(6.2) has a solution $u : [0, \eta] \mapsto \mathbb{R}^N$ for sufficiently small $\eta > 0$.
2. Given an interval $I \in \mathcal{J}$, there exists at most one solution $u : I \mapsto \mathbb{R}^N$ of the IVP (6.1)–(6.2).

Let $I \in \mathcal{J}$ and suppose that $u \in W_{\Delta}(I)$ is a solution of the IVP (6.1)–(6.2). We say that the interval I is a **maximal interval of existence** and u is a **maximally defined solution** if u does not have a proper extension which is also a solution of (6.1)–(6.2), that is, there does not exist $\tilde{I} \in \mathcal{J}$ and a solution $\tilde{u} \in W_{\Delta}(\tilde{I})$ of (6.1)–(6.2) such that $I \subset \tilde{I}$, $I \neq \tilde{I}$ and $\tilde{u}(t) = u(t) \quad \forall t \in I$.

The following result, which is the analogue of the first part of the statement of Logemann and Ryan [2, Theorem 3.7], is still true, with the same proof.

Theorem 6.2. Assume that F satisfies (H1) and (H2). Then there exists a unique maximally defined solution u of (6.1)–(6.2). The associated maximal interval of existence is of the form $[0, \tau_M)$, where $0 < \tau_M \leq T_f$.

An easy modification of the proof of the remaining part of Logemann and Ryan [2, Theorem 3.7] (which again amounts to replace gru and $gr\tilde{u}$ by $gr(Pu)$ and $gr(P\tilde{u})$ respectively) yields to the following result:

Theorem 6.3. Assume that F satisfies (H1), (H2) and (H3). Let $[0, \tau_M)$ be the maximal interval of existence and let $u : [0, \tau_M) \mapsto \mathbb{R}^N$ be the maximally defined solution of (6.1)–(6.2). If $\tau_M < +\infty$ and u is bounded, then, for every compact set $K \subset \Delta$ and every $\sigma \in (0, \tau_M)$, there exists $t \in (\sigma, \tau_M)$ such that $(t, (Pu)(t)) \notin K$.

Remark 6.1. Let us make the assumptions of previous theorem and also, that u is bounded in $[0, \tau_M)$ with $\tau_M < +\infty$. Since Pu is also bounded in $[0, \tau_M)$, we infer that there exists a sequence $\{t_n\}$ in $[0, \tau_M)$ such that $\{t_n\} \rightarrow \tau_M$ and $\text{dist}(t_n, (Pu)(t_n), \partial\Delta) \rightarrow 0$. As we also have $\Delta = [0, T_f) \times G$ with G open in $\mathbb{R}^N$, we conclude that:

1. If $T_f < +\infty$, then either $\tau_M = T_f$ or $\text{dist}((Pu)(t_n), \partial G) \rightarrow 0$.
2. If $T_f = \infty$, then $\text{dist}((Pu)(t_n), \partial G) \rightarrow 0$.

7. Application of the theory of causal differential equations to levitation model

From now on, T_f denotes the final time up to which the source term J_S is defined and we assume $T_f < +\infty$, $A^0 \in \tilde{H}_r^1(\Omega_0)$ and $J_S \in H^1(0, T_f; L_r^2(\Omega_S))$ (as we have done in Section 2 when presenting our approach to the study of the coupled problem.)

We include the reformulation (2.16)–(2.18) of the levitation problem in the framework of Section 6 by setting $G = (\underline{u}, \bar{u})$ and $\Delta = [0, T_f] \times G$, so if $I \in \mathcal{I}$ and $v \in C(I)$,

$$gr(Pv) \subset \Delta \Leftrightarrow \underline{u} < (Pv)(t) < \bar{u} \quad \forall t \in I.$$

We need to check that the operator F given in (2.14) acts from $W_\Delta(I)$ into $L_{loc}^1(I)$ for every interval $I \in \mathcal{I}$. In the case where $I = [0, \eta]$, $\eta > 0$, this is clear: it suffices to replace in the definition of the operator S_T (see (2.13)) the time T by η and notice that, since $W_\Delta[0, \eta] \subset \text{dom}(S_\eta)$, $S_\eta(v) = A$ is well defined, $A \in L^\infty(0, \eta; \mathcal{V})$ and $\partial_t A \in L_r^2(Q^\eta)$, where $Q^\eta := \{(t, r, z); t \in (0, \eta), (r, z) \in \Omega_r\}$. Hence $f = F(v)$ belongs to $L^2[0, \eta]$. In the case where $I = [0, \eta]$, given $v \in W_\Delta[0, \eta]$, we consider for every $\eta_1 \in (0, \eta)$, $v|_{[0, \eta_1]} \in W_\Delta[0, \eta_1]$, so there exists a unique solution of the electromagnetic problem on the interval $[0, \eta_1]$, $A_{\eta_1} \in L^\infty(0, \eta_1; \mathcal{V})$ with $\partial_t A \in L_r^2(Q^{\eta_1})$. We define A on $[0, \eta]$ by $A|_{[0, \eta_1]} = A_{\eta_1}$ for all $\eta_1 \in (0, \eta)$; therefore $F(v) \in L_{loc}^2[0, \eta]$ (it may happen that $A \notin L^\infty(0, \eta; \mathcal{V})$, $\partial_t A \notin L_r^2(Q^\eta)$ and $F(v) \notin L^2[0, \eta]$).

It is clear that the operator F is causal, that is, it satisfies (H1). Our goal is to prove that F also satisfies (H2). To do this, as a previous step we are going to establish a local Lipschitz-type condition for the solution operator S_T .

From here until the end of this section, the notation T keeps the same meaning as in Section 6. We consider $t_0 \in [0, T]$ (t_0 here has the meaning corresponding to notation α in Section 6.)

Let $t_0 \in [0, T]$ and $w \in W_\Delta[0, t_0]$ (i. e., $w \in W^{1,1}[0, t_0]$ such that $\underline{u} < (Pw)(t) < \bar{u} \quad \forall t \in [0, t_0]$.) Let $(\beta, \gamma) \in \text{Adm}(w; t_0)$. (We recall that this means that $t_0 < \beta < T$, $\gamma > 0$ and $W(w; t_0, \beta, \gamma) \subset W_\Delta[0, \beta]$.) Since $W(w; t_0, \beta, \gamma) \subset W^{1,1}[0, \beta]$, we have that

$$\begin{aligned} W(w; t_0, \beta, \gamma) \subset W_\Delta[0, \beta] &\Leftrightarrow gr(Pv) \subset \Delta = [0, T_f] \times G \quad \forall v \in W(w; t_0, \beta, \gamma) \\ &\Leftrightarrow \underline{u} < (Pv)(t) < \bar{u} \quad \forall v \in W(w; t_0, \beta, \gamma). \end{aligned}$$

Let $v_1, v_2 \in W(w; t_0, \beta, \gamma)$, i.e.

$$v_i \in W^{1,1}[0, \beta], \quad v_i|_{[0, t_0]} = w \text{ and } \int_{t_0}^\beta |v_i'(s)| ds \leq \gamma, \quad i = 1, 2.$$

(If $t_0 = 0$, then $w = v_0 \in \mathbb{R}$ and $v_i(0) = v_0$, $i = 1, 2$.) Note that

$$\|v_i\|_{C[0, \beta]} \leq |w(t_0)| + \gamma, \quad i = 1, 2. \quad (7.1)$$

Let $A_i = S_\beta(v_i)$, $i = 1, 2$ be the respective azimuthal components of the magnetic potential: A_i is the unique solution of the following problem:

Problem 3. Find $A_i \in L^2(0, \beta; \mathcal{V})$ with $\partial_t A_i \in L_r^2(Q_i^\beta)$, such that

$$\int_{\Omega_r(u_i)} \sigma_i(t) \frac{\partial A_i}{\partial t}(t) Z r dr dz + a(A_i(t), Z) + v_i(t) \int_{\Omega_r(u_i)} \sigma_i(t) \frac{\partial A_i}{\partial z}(t) Z r dr dz \quad (7.2)$$

$$= \int_{\Omega_S} J_S(t) Z r dr dz \quad \forall Z \in \mathcal{V}, \quad \text{a.e. } t \in (0, \beta),$$

$$A_i(0) = A^0 \quad \text{in } \Omega_0, \quad (7.3)$$

where, for $i = 1, 2$:

$$u_i(t) = (Pv_i)(t) = \int_0^t v_i(s) ds, \quad 0 \leq t \leq \beta,$$

$$\Omega_r(u_i) := \Omega_0 + u_i(t) e_z$$

$$Q_i^\beta = \{(t, r, z); t \in (0, \beta), (r, z) \in \Omega_r(u_i)\}$$

$$\sigma_i(t)(r, z) = \begin{cases} \hat{\sigma}(r, z - u_i(t)) & \text{if } (r, z) \in \overline{\Omega_r(u_i)} \\ 0 & \text{otherwise.} \end{cases} \quad (7.4)$$

Note that the functions $\sigma_i(t) \in L^\infty(\Omega)$ for all $t \in [0, \beta]$.

In the case $t_0 > 0$ the following observation is crucial: $A_i|_{[t_0, \beta]}$ is determined by $J_S|_{[t_0, \beta]}$, $A_i(t_0) \in \mathcal{V}$, $v_i|_{[t_0, \beta]}$ and $u_i(t_0) = (Pv_i)(t_0) = (Pw)(t_0) = \int_0^{t_0} w(s) ds$. Hence we do not need to solve Problem 3 from $t = 0$; the “history” w only affects the state $A_1(t_0) = A_2(t_0)$, $v_1(t_0) = v_2(t_0) = w(t_0)$ and $u_1(t_0) = u_2(t_0) = (Pw)(t_0)$; this is not a general property of causal operators.

Hence, $A_i|_{[t_0, \beta]}$ is the unique solution of the following problem:

Problem 4. Find $A_i \in L^2(t_0, \beta; \mathcal{V})$ with $\partial_t A_i \in L_r^2(Q_i^\beta \cap \{t > t_0\})$, such that

$$\int_{\Omega_r(u_i)} \sigma_i(t) \frac{\partial A_i}{\partial t}(t) Z r dr dz + a(A_i(t), Z) + v_i(t) \int_{\Omega_r(u_i)} \sigma_i(t) \frac{\partial A_i}{\partial z}(t) Z r dr dz \quad (7.5)$$

$$\begin{aligned}
&= \int_{\Omega_S} J_S(t) Z r dr dz \quad \forall Z \in \mathcal{V}, \quad \text{a.e. } t \in (t_0, \beta), \\
A_i(t_0) &= A_i(t_0^-) \quad \text{in } \Omega_{t_0}(Pw).
\end{aligned} \tag{7.6}$$

where we have denoted by $A_i(t_0^-)$ the value provided by the integration of [Problem 3](#) from $t = 0$ to $t = t_0$. (For $t_0 = 0$, $A_i(0^-) := A^0$, $i = 1, 2$).

Since $v_1|_{[0, t_0]} = v_2|_{[0, t_0]} = w$, we have that $A_1^-(t_0) = A_2^-(t_0)$ (in fact $A_1(t) = A_2(t) \quad \forall t \in [0, t_0]$.) Besides

$$\begin{cases} v_1(t_0) = v_2(t_0) = w(t_0) \text{ if } t_0 > 0; v_1(0) = v_2(0) = v_0 \\ u_1(t_0) = u_2(t_0) = (Pw)(t_0) = \int_0^{t_0} w(s) ds \text{ if } t_0 > 0; u_1(0) = u_2(0) = 0 \\ u_i(t) = (Pw)(t_0) + \int_{t_0}^t v_i(s) ds, \quad t_0 \leq t \leq \beta \quad i = 1, 2 \end{cases} \tag{7.7}$$

The difficulty of the study lies in the fact that $\Omega_t(u_i)$ and Q_i^β depend on the velocity v_i . The worst case is $t_0 = 0$ because for $t_0 > 0$ we can take advantage of the regularity $A_i \in L^\infty(t_0, \beta; \tilde{H}_r^2(\Omega)) \cap H^1(t_0, \beta; \mathcal{V})$, due to [Theorem 2.3](#) (Here we use the fact that functions v_i are not merely continuous but they belong to $W^{1,1}[0, \beta]$.) In order to unify the writing of the two cases, we consider [Problem 3](#) as a particular case of [Problem 4](#); to do this it is enough to take into account that $A_i(0^-) := A^0$, $i = 1, 2$.

From [Eq. \(7.5\)](#) we deduce that

$$\begin{aligned}
&\int_{\Omega_t(u_1)} \sigma_1(t) \left(\frac{\partial A_1}{\partial t}(t) - \frac{\partial A_2}{\partial t}(t) \right) Z r dr dz + a(A_1(t) - A_2(t), Z) \\
&\quad + v_1(t) \int_{\Omega_t(u_1)} \sigma_1(t) \left(\frac{\partial A_1}{\partial z}(t) - \frac{\partial A_2}{\partial z}(t) \right) Z r dr dz \\
&= \int_{\Omega_t(u_2)} \sigma_2(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz - \int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz + \\
&\quad v_2(t) \int_{\Omega_t(u_2)} \sigma_2(t) \frac{\partial A_2}{\partial z}(t) Z r dr dz - v_1(t) \int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial A_2}{\partial z}(t) Z r dr dz \\
&\quad \forall Z \in \mathcal{V}, \quad \text{a.e. } t \in (t_0, \beta),
\end{aligned} \tag{7.8}$$

Let $g_1 = \sigma_2 \chi_{Q_2^\beta} - \sigma_1 \chi_{Q_1^\beta}$ and $g_2 = v_2 \sigma_2 \chi_{Q_2^\beta} - v_1 \sigma_1 \chi_{Q_1^\beta}$, that is,

$$g_1(t, r, z) = \sigma_2(t, r, z) \chi_{\Omega_t(u_2)}(r, z) - \sigma_1(t, r, z) \chi_{\Omega_t(u_1)}(r, z) \tag{7.9}$$

$$g_2(t, r, z) = v_2(t) \sigma_2(t, r, z) \chi_{\Omega_t(u_2)}(r, z) - v_1(t) \sigma_1(t, r, z) \chi_{\Omega_t(u_1)}(r, z) \tag{7.10}$$

We rewrite [\(7.8\)](#) with the aid of g_1 and g_2 as follows

$$\begin{aligned}
&\int_{\Omega_t(u_1)} \sigma_1(t) \left(\frac{\partial A_1}{\partial t}(t) - \frac{\partial A_2}{\partial t}(t) \right) Z r dr dz + a(A_1(t) - A_2(t), Z) \\
&\quad + v_1(t) \int_{\Omega_t(u_1)} \sigma_1(t) \left(\frac{\partial A_1}{\partial z}(t) - \frac{\partial A_2}{\partial z}(t) \right) Z r dr dz \\
&= \langle g_1(t) \frac{\partial A_2}{\partial t}(t), Z \rangle + \langle g_2(t) \frac{\partial A_2}{\partial z}(t), Z \rangle \quad \forall Z \in \mathcal{V}, \quad \text{a.e. } t \in (t_0, \beta),
\end{aligned} \tag{7.11}$$

(Note that $\langle g_1(t) \frac{\partial A_2}{\partial t}(t), Z \rangle = \int_{\Omega} g_1(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz$ and $\langle g_2(t) \frac{\partial A_2}{\partial z}(t), Z \rangle = \int_{\Omega} g_2(t) \frac{\partial A_2}{\partial z}(t) Z r dr dz$ because $\frac{\partial A_2}{\partial t}(t)$ and $\frac{\partial A_2}{\partial z}(t)$ are in $L_r^2(\Omega)$ for a.e. $t \in (0, \beta)$ thanks to [Theorem 2.3](#).)

Let $\varepsilon > 0$ small enough (such that $t_0 + \varepsilon < \beta$). We have

$\delta A := A_1 - A_2 \in H^1(t_0 + \varepsilon, \beta; \mathcal{V}) \cap L^\infty(t_0 + \varepsilon, \beta; \tilde{H}_r^2(\Omega))$ (even in the case $t_0 = 0$; if $t_0 > 0$ we can take in fact $\varepsilon = 0$ thanks again to [Theorem 2.3](#).) Function δA is the unique solution of the following problem:

Problem 5. Find $\delta A \in L^2(t_0 + \varepsilon, \beta; \mathcal{V})$ with $\partial_t(\delta A) \in L_r^2(Q_1^\beta \cap \{t > t_0 + \varepsilon\})$, such that

$$\int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial(\delta A)}{\partial t}(t) Z r dr dz + a((\delta A)(t), Z) + v_1(t) \int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial(\delta A)}{\partial z}(t) Z r dr dz \tag{7.12}$$

$$= \langle g_1(t) \frac{\partial A_2}{\partial t}(t), Z \rangle + \langle g_2(t) \frac{\partial A_2}{\partial z}(t), Z \rangle \quad \forall Z \in \mathcal{V}, \quad \text{a.e. } t \in (t_0, \beta),$$

$$(\delta A)(t_0 + \varepsilon) = (A_1 - A_2)(t_0 + \varepsilon) \quad \text{in } \Omega_{t_0 + \varepsilon}(u_1). \tag{7.13}$$

7.1. Estimation of $\|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})}$

By applying estimate [\(2.9\)](#) to [Problem 5](#), we deduce that

$$\sup_{t_0 + \varepsilon \leq t \leq \beta} \left[\int_{\Omega_t(u_1)} \sigma_1(t) |A_1(t) - A_2(t)|^2 r dr dz \right]^{1/2} + \|A_1 - A_2\|_{L^2(t_0 + \varepsilon, \beta; \mathcal{V})} \tag{7.14}$$

$$\leq C_1(\alpha) \left[\left(\int_{\Omega_{t_0+\epsilon}(u_1)} \sigma(t_0+\epsilon) |A_1 - A_2(t_0+\epsilon)|^2 r dr dz \right)^{1/2} + \left\| g_1 \frac{\partial A_2}{\partial t} + g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0+\epsilon, \beta; \mathcal{V}')} \right].$$

If $t_0 > 0$ we can take $\epsilon = 0$ and use that $A_1(t_0) = A_2(t_0)$. In the case $t_0 = 0$, we take $\epsilon > 0$ and we pass to the limit as $\epsilon \rightarrow 0^+$ in (7.14) by using that $A_i(\epsilon) \rightharpoonup \tilde{A}^0$ weakly in $\mathcal{V}$ (by Theorem 4.1) and the compactness of the injection $\tilde{H}_r^1(\Omega) \subset L_r^2(\Omega)$ Mercier and Raugel [9, Lemma 4.1]. In both cases, we have the estimate

$$\sup_{t_0 \leq t \leq \beta} \left[\int_{\Omega_t(u_1)} \sigma_1(t) |A_1(t) - A_2(t)|^2 r dr dz \right]^{1/2} + \|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \leq C_1(\alpha) \left\| g_1 \frac{\partial A_2}{\partial t} + g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \quad (7.15)$$

It is clear that $\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(t_0, \beta; \mathcal{V}')} + \left\| g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')} < +\infty$ when $t_0 > 0$ due to regularizing effect (Theorem 2.3). We will see that these norms are also finite when $t_0 = 0$.

7.1.1. Bounds for $\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(t_0, \beta; \mathcal{V}')}$ and $\left\| g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')}$

Due to the definition of g_1 (cf. (7.9)), we have for a.e. $t \in [t_0, \beta]$

$$\begin{aligned} &< g_1(t) \frac{\partial A_2}{\partial t}(t), Z > \\ &= \int_{\Omega_t(u_2)} \sigma_2(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz - \int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz \quad \forall Z \in \mathcal{V} \end{aligned} \quad (7.16)$$

Let us denote $\eta := u_2 - u_1$ and let us consider an arbitrary $t \in [t_0, \beta]$. Note that

$$\Omega_t(u_2) = \Omega_t(u_1) + \eta(t) e_z. \quad (7.17)$$

We introduce the fictitious motion $X^f : [0, 1] \times \overline{\Omega_t(u_1)} \rightarrow \overline{\Omega}$ defined by

$$X^f(\tau, r, z) := (r, z + \tau \eta(t)) \quad (7.18)$$

where the variable τ plays the role of a fictitious time. The mapping X^f satisfies the properties labelled (i)–(iv) in Bermúdez et al. [1, Section 3] (with $\Omega_t(u_1)$ as reference domain) and its associated velocity is the constant vector $\eta(t) e_z$. Note that $X^f(\tau, \cdot, \cdot)(\overline{\Omega_t(u_1)}) = \overline{\Omega_t(u_1) + \tau \eta(t) e_z}$. Analogous to (4.5), we introduce the fictitious conductivity

$$\sigma^f(\tau)(r, z) = \begin{cases} \sigma_1(t)(r, z - \tau \eta(t)) & \text{if } (r, z) \in \overline{\Omega_t(u_1)} + \tau \eta(t) e_z \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sigma^f(0) = \sigma_1(t)$ and, using (7.4) and (7.17), it is easy to check that $\sigma^f(1) = \sigma_2(t)$. Now we are in a position to apply Reynolds theorem to fictitious motion X^f . This gives for a.e. $t \in (t_0, \beta)$

$$\begin{aligned} &\int_{\Omega_t(u_2)} \sigma_2(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz - \int_{\Omega_t(u_1)} \sigma_1(t) \frac{\partial A_2}{\partial t}(t) Z r dr dz \\ &= \int_0^1 \frac{d}{d\tau} \int_{\Omega_t(u_1) + \tau \eta(t) e_z} \sigma^f(\tau) \frac{\partial A_2}{\partial t}(t) Z r dr dz \\ &= \eta(t) \int_0^1 \int_{\Omega_t(u_1) + \tau \eta(t) e_z} \sigma^f(\tau) \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial t}(t) Z \right) r dr dz. \end{aligned}$$

This, together with (7.16) and (4.6), implies for a.e. $t \in (t_0, \beta)$

$$| < g_1(t) \frac{\partial A_2}{\partial t}(t), Z > | \leq \bar{\sigma} |\eta(t)| \left\| \frac{\partial A_2}{\partial t}(t) \right\|_{H_r^1(\Omega)} \|Z\|_{H_r^1(\Omega)} \quad \forall Z \in \mathcal{V},$$

that is,

$$\left\| g_1(t) \frac{\partial A_2}{\partial t}(t) \right\|_{\mathcal{V}'} \leq \bar{\sigma} |\eta(t)| \left\| \frac{\partial A_2}{\partial t}(t) \right\|_{H_r^1(\Omega)} \quad \text{a.e. } t \in (t_0, \beta). \quad (7.19)$$

Therefore

$$\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \leq \bar{\sigma} \left(\int_{t_0}^{\beta} |\eta(t)|^2 \left\| \frac{\partial A_2}{\partial t}(t) \right\|_{H_r^1(\Omega)}^2 dt \right)^{1/2} \quad (7.20)$$

In the case $t_0 > 0$, the regularizing effect (Theorem 2.3) guarantees that $\frac{\partial A_2}{\partial t} \in L^2(t_0, \beta; \tilde{H}_r^1(\Omega))$. More precisely, by using estimate (2.12) and noting that $\eta = P(v_2 - v_1)$, we obtain the following lemma.

Lemma 7.1. If $t_0 > 0$, the estimation

$$\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \leq \frac{C_5}{\sqrt{t_0}} (\beta - t_0) \|v_1 - v_2\|_{C([t_0, \beta])}, \quad (7.21)$$

holds, where the constant $C_5 = C_5(\underline{\sigma}, \bar{\sigma}, M, \alpha, C_{reg}, \mu, \Omega_0, A^0, J_S, \|w\|_{W^{1,1}_{[0,t_0]}}, \gamma, \beta)$ neither depends on v_1 nor v_2 , and it is increasing with respect to γ and β .

(We recall that $\beta \leq T_f$).

In the case $t_0 = 0$, in general $\frac{\partial A_2}{\partial t} \notin L^2(0, \beta; \tilde{H}_r^1(\Omega))$. Nevertheless, by using again estimate (2.12), the following lemma can be easily proved.

Lemma 7.2.

$$\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(0, \beta; \mathcal{V}')} \leq C_5 \sqrt{\beta} \|v_1 - v_2\|_{C([0, \beta])} \quad (7.22)$$

Now we address the obtention of a bound for $\left\| g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')}.$ We do not need to distinguish between the case $t_0 > 0$ and the case $t_0 = 0$.

First we write

$$g_2 = v_2 g_1 + (v_2 - v_1) \sigma_1 \chi_{Q_1^\beta} \quad (7.23)$$

We have an estimate similar to (7.19) but with $\frac{\partial A_2}{\partial t}(t)$ replaced by $\frac{\partial A_2}{\partial z}(t)$. Hence

$$\left\| g_1 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \leq \bar{\sigma} \|\eta\|_{C([t_0, \beta])} \left(\int_{t_0}^{\beta} \left\| \frac{\partial A_2}{\partial z}(t) \right\|_{H_r^1(\Omega)}^2 dt \right)^{1/2} \quad (7.24)$$

Using the spatial regularity estimate (2.11) together with estimates (2.10) and (2.9), we deduce the bound

$$\|A_2\|_{L^2(0, \beta; \tilde{H}_r^2(\Omega))} \leq C_6, \quad (7.25)$$

where $C_6 = C_6(\underline{\sigma}, \bar{\sigma}, M, \alpha, C_{reg}, \mu, \Omega_0, A^0, J_S, \|w\|_{C([0, t_0])}, \gamma)$. Using again that $\eta = P(v_2 - v_1)$, the following inequality is immediate:

$$\left\| g_1 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \leq C_6 \bar{\sigma} (\beta - t_0) \|v_1 - v_2\|_{C([t_0, \beta])}. \quad (7.26)$$

On the other hand

$$\left\| (v_1 - v_2) \chi_{Q_1^\beta} \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; L_r^2(\Omega))} \leq \|v_1 - v_2\|_{L^2(t_0, \beta)} \left\| \chi_{Q_1^\beta} \frac{\partial A_2}{\partial z} \right\|_{L^\infty(t_0, \beta; L_r^2(\Omega))} \quad (7.27)$$

Using (2.10), it turns out that

$$\left\| \chi_{Q_1^\beta} \frac{\partial A_2}{\partial z} \right\|_{L^\infty(t_0, \beta; L_r^2(\Omega))} \leq C_7, \quad (7.28)$$

where $C_7 = C_7(\underline{\sigma}, \bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S, \|w\|_{C([0, t_0])}, \gamma)$.

Thus, we have proven the following lemma:

Lemma 7.3.

$$\left\| g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')} \leq \left[(C_6 \bar{\sigma} (|w(t_0)| + \gamma) (\beta - t_0) + C_7 \bar{\sigma} \sqrt{\beta - t_0}) \|v_1 - v_2\|_{C([t_0, \beta])} \right] \quad (7.29)$$

By combining (7.29) with (7.21) (in the case $t_0 > 0$) or (7.22) (in the case $t_0 = 0$), we obtain a bound for $\left\| g_1 \frac{\partial A_2}{\partial t} \right\|_{L^2(t_0, \beta; \mathcal{V}')} + \left\| g_2 \frac{\partial A_2}{\partial z} \right\|_{L^2(t_0, \beta; \mathcal{V}')}$, which depends on the case. By using this bound in (7.15), we obtain the following result, which can be interpreted as a local Lipschitz-type condition for the operator S_β :

Theorem 7.1. We have the following inequalities:

i) If $t_0 > 0$,

$$\begin{aligned} & \sup_{t_0 \leq t \leq \beta} \left[\int_{\Omega_t(u_1)} \sigma_1(t) |A_1(t) - A_2(t)|^2 r dr dz \right]^{1/2} + \|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \\ & \leq C_1(\alpha) \left\{ \left[\frac{C_5}{\sqrt{t_0}} + C_6 \bar{\sigma} (|w(t_0)| + \gamma) \right] (\beta - t_0) + C_7 \bar{\sigma} \sqrt{\beta - t_0} \right\} \|v_1 - v_2\|_{C([t_0, \beta])} \end{aligned} \quad (7.30)$$

ii) If $t_0 = 0$,

$$\begin{aligned} & \sup_{0 \leq t \leq \beta} \left[\int_{\Omega_t(u_1)} \sigma_1(t) |A_1(t) - A_2(t)|^2 r dr dz \right]^{1/2} + \|A_1 - A_2\|_{L^2(0, \beta; \mathcal{V})} \\ & \leq C_1(\alpha) \left\{ C_6 \bar{\sigma} (|v_0| + \gamma) \beta + (C_5 + C_7 \bar{\sigma}) \sqrt{\beta} \right\} \|v_1 - v_2\|_{C([t_0, \beta])} \end{aligned} \quad (7.31)$$

Note that in both cases we have an estimate

$$\|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \leq C_8(\beta - t_0)^{1/2} \|v_1 - v_2\|_{C([t_0, \beta])}, \quad (7.32)$$

where C_8 is increasing with respect to γ and β , and neither depends on v_1 nor v_2 . It can be assumed that C_8 is also independent of β since $\beta \leq T_f < +\infty$.

7.2. Local Lipschitz-type condition for $\mathcal{F}$

In order to establish the Lipschitz-type condition (H2) for operator $\mathcal{F}$, it seems natural to try to prove an analogous condition for the $L^2(t_0, \beta; L_r^2(\Omega))$ -norm of $\chi_{Q_1^\beta} \left(\frac{\partial A_1}{\partial t} - \frac{\partial A_2}{\partial t} \right)$ with respect to $\|v_1 - v_2\|_{W^{1,1}([t_0, \beta])}$. Unfortunately, even if we neglect the term $\sigma v \frac{\partial A}{\partial z}$ in Eq. (2.1), namely we neglect the term $\sigma v \times \text{curl}(Ae_\theta)$ in Eq. (3.8) (which amounts to the electromagnetic model presented in Bermúdez et al. [1] with $v = v(t)e_z$) and we consider $t_0 > 0$ (the most favorable case), we have only been able to obtain a Hölder-type condition of exponent $1/2$. There seems to be an essential difference here with respect to parabolic problems. Therefore, we explore an alternative route.

We recall that operator $\mathcal{F}$ is defined by (2.14) and (2.15), and that we need to prove condition (H2), that is, that for every $t_0 \in [0, T)$ and every $w \in W_\Delta[0, t_0]$, there exists $(\beta, \gamma) \in \text{Adm}(w; t_0)$ and $\lambda \in (0, 1)$ such that

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^1[t_0, \beta]} \leq \lambda \|v_1' - v_2'\|_{L^1[t_0, \beta]} (= \lambda \mu(v_1, v_2)) \quad \forall v_1, v_2 \in W(w; t_0, \beta, \gamma). \quad (7.33)$$

Let $t_0 \in [0, T)$, $w \in W_\Delta[0, t_0]$, $v_1, v_2 \in W(w; t_0, \beta, \gamma)$, $u_i = P(v_i)$ and $A_i = S_\beta(v_i)$, $i = 1, 2$. We recall that $A_i \in L^\infty(0, \beta; \mathcal{V})$, $\frac{\partial A_i}{\partial t} \in L_r^2(Q_i^\beta)$ and $A_i \in L^2(0, \beta; \tilde{H}_r^2(\Omega))$. Hence

$$\begin{aligned} & \mathcal{F}(v_1)(t) - \mathcal{F}(v_2)(t) \\ &= -\frac{2\pi}{m} \left[\int_{\Omega_i(u_1)} \sigma_1(t) \left(\frac{\partial A_1}{\partial t}(t) + v_1(t) \frac{\partial A_1}{\partial z}(t) \right) \frac{\partial A_1}{\partial z}(t) r dr dz - \int_{\Omega_i(u_2)} \sigma_2(t) \left(\frac{\partial A_2}{\partial t}(t) + v_2(t) \frac{\partial A_2}{\partial z}(t) \right) \frac{\partial A_2}{\partial z}(t) r dr dz \right]. \end{aligned} \quad (7.34)$$

We can write

$$\mathcal{F}(v_1)(t) - \mathcal{F}(v_2)(t) = -\frac{2\pi}{m} (\varphi_1(t) + \varphi_2(t)), \quad (7.35)$$

where

$$\varphi_1(t) = \int_{\Omega} \left[\sigma_1(t) \chi_{Q_1^\beta}(t) \left(\frac{\partial A_1}{\partial t}(t) + v_1(t) \frac{\partial A_1}{\partial z}(t) \right) - \sigma_2(t) \chi_{Q_2^\beta}(t) \left(\frac{\partial A_2}{\partial t}(t) + v_2(t) \frac{\partial A_2}{\partial z}(t) \right) \right] \frac{\partial A_1}{\partial z}(t) r dr dz \quad (7.36)$$

and

$$\varphi_2(t) = \int_{\Omega} \sigma_2(t) \chi_{Q_2^\beta}(t) \left(\frac{\partial A_2}{\partial t}(t) + v_2(t) \frac{\partial A_2}{\partial z}(t) \right) \left(\frac{\partial A_1}{\partial z}(t) - \frac{\partial A_2}{\partial z}(t) \right) r dr dz. \quad (7.37)$$

For almost every $t \in (t_0, \beta)$,

$$|\varphi_2(t)| \leq \bar{\sigma} \left[\left\| \chi_{Q_2^\beta}(t) \frac{\partial A_2}{\partial t}(t) \right\|_{L_r^2(\Omega)} + \|v_2\|_{C([t_0, \beta])} \left\| \frac{\partial A_2}{\partial z}(t) \right\|_{L_r^2(\Omega)} \right] \|A_1(t) - A_2(t)\|_{H_r^1(\Omega)}. \quad (7.38)$$

Integrating on $[t_0, \beta]$ yields

$$\begin{aligned} \int_{t_0}^{\beta} |\varphi_2(t)| dt &\leq \bar{\sigma} \left[\left\| \frac{\partial A_2}{\partial t} \right\|_{L_r^2(Q_2^\beta)} + \|v_2\|_{C([t_0, \beta])} \|A_2\|_{L^2(0, \beta; \mathcal{V})} \right] \|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \\ &\leq C_9(\underline{\sigma}, \bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S, \|w\|_{W^{1,1}([0, t_0])}, \gamma) \|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \\ &\leq C_8 C_9(\underline{\sigma}, \bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S, \|w\|_{W^{1,1}([0, t_0])}, \gamma) (\beta - t_0)^{1/2} \|v_1 - v_2\|_{C([t_0, \beta])}. \end{aligned} \quad (7.39)$$

where we have used (2.10), (2.9) and (7.1) in the penultimate inequality, and (7.32) in the last one.

We now need to estimate $\int_{t_0}^{\beta} |\varphi_1(t)| dt$ but without bounding the $L^2(t_0, \beta; L_r^2(\Omega))$ -norm of the function defined by the expression that is between brackets in (7.36). The key idea is essentially to replace the Cauchy-Schwarz inequality in $L_r^2(\Omega)$ by the duality $\langle \cdot, \cdot \rangle$ between $\mathcal{V}'$ and $\mathcal{V}$. Note that $\frac{\partial A_1}{\partial z}(t) \notin \mathcal{V}$ but $\frac{\partial A_1}{\partial z}(t) \in \tilde{H}_r^1(\Omega)$ for a.e. $t \in (0, \beta)$ (because $A_1 \in L^2(0, \beta; \tilde{H}_r^2(\Omega))$). Since $\varphi_1(t)$ reduces to an integral over the set $\Omega_i(u_1) \cup \Omega_i(u_2)$, which is at positive distance from $\Gamma_D (= ([0, r_{\max}] \times \{z_{\min}\}) \cup (\{r_{\max}\} \times [z_{\min}, z_{\max}]) \cup ([0, r_{\max}] \times \{z_{\max}\}))$, we can construct a cut-off function θ independent of v_1 and v_2 such that $\theta(t) \frac{\partial A_1}{\partial z}(t) \in \mathcal{V}$ and $\theta(t) = 1$ in $\Omega_i(u_1) \cup \Omega_i(u_2)$.

We need to detail the construction of θ in order to bound some of its norms. To do this, we begin by using some ideas from the proof of the Lemma 6.1. Given $t_0 \in [0, T)$ and $w \in W_\Delta[0, t_0]$, in order to obtain that $\text{Adm}(w; t_0) \neq \emptyset$, it was proved there that there exists $\beta \in (t_0, T_f)$ and $\gamma > 0$ such that

$$(\beta - t_0)(|w(t_0)| + \gamma) \leq d, \quad (7.40)$$

where $d > 0$ is small enough so that

$$[(Pw)(t_0) - d, (Pw)(t_0) + d] \subset G = (\underline{u}, \bar{u}). \quad (7.41)$$

Condition (7.40) ensures that for all $v \in W(w; t_0, \beta, \gamma)$ we have

$$|(Pv)(t) - (Pw)(t_0)| \leq d \quad \forall t \in [t_0, \beta].$$

Let $z_b := \min_{(r,z) \in \bar{\Omega}_0} z$, $z_h := \min_{(r,z) \in \bar{\Omega}_0} z$ and $r_d := \max_{(r,z) \in \bar{\Omega}_0} r$. We have that

$$\bigcup_{\substack{t \in [t_0, \beta] \\ v \in W(w; t_0, \beta, \gamma)}} \bar{\Omega}_t(Pv) \subset \bar{\Omega}_0 + [(Pw)(t_0) - d, (Pw)(t_0) + d]e_z \subset K,$$

where $K := [0, r_d] \times [z_b + (Pw)(t_0) - d, z_h + (Pw)(t_0) + d]$. Assumption (4.2) implies the inequalities $z_{\min} \leq z_b + \underline{u}$ and $z_h + \bar{u} \leq z_{\max}$, which in turn, together with (7.41) imply $[z_b + (Pw)(t_0) - d, z_h + (Pw)(t_0) + d] \subset (z_{\min}, z_{\max})$, hence

$H := \min(z_{\max} - (z_h + (Pw)(t_0) + d), z_b + (Pw)(t_0) - d - z_{\min}) > 0$. Besides, the fact that $\Omega_0 = \bar{\Omega}$ and the assumption $\bar{\Omega} \cap \bar{\Gamma}_D = \emptyset$ imply that $r_d < r_{\max}$. Let $\delta > 0$ be such that $\delta < \min(H/3, (r_{\max} - r_d)/3)$ and let

$K_\delta := [0, r_d + \delta] \times [z_b + (Pw)(t_0) - d - \delta, z_h + (Pw)(t_0) + d + \delta]$. The compact set $K_\delta \subset [0, r_{\max}] \times (z_{\min}, z_{\max})$. We now introduce two cut-off functions such that:

$$\begin{aligned} \theta_1 &\in C^\infty([0, r_{\max}]), 0 \leq \theta_1 \leq 1, \theta_1 = 1 \text{ in } [0, r_d + \delta], \theta_1 = 0 \text{ in } [r_{\max} - \delta, r_{\max}], \\ |\theta_1'| &\leq C_{10}\delta^{-1}, \\ \theta_2 &\in C^\infty([z_{\min}, z_{\max}]), 0 \leq \theta_2 \leq 1, \\ \theta_2 &= 1 \text{ en } [z_b + (Pw)(t_0) - d - \delta, z_h + (Pw)(t_0) + d + \delta], \\ \theta_2 &= 0 \text{ in } [z_{\min}, z_{\min} + \delta] \cup [z_{\max} - \delta, z_{\max}], |\theta_2'| \leq C_{10}\delta^{-1} \end{aligned}$$

where C_{10} is a constant. Let $\theta(r, z) := \theta_1(r)\theta_2(z)$. We have: $\theta \in C^\infty([0, r_{\max}] \times [z_{\min}, z_{\max}])$, $\theta = 1$ in $K_\delta \supset K$, $\theta = 0$ in a neighborhood of Γ_D (with respect to the relative topology in $\bar{\Omega}$) and $\|\theta\|_{C^1(\bar{\Omega})} \leq C_{10}\delta^{-1}$.

Since $\Omega_t(u_1) \cup \Omega_t(u_2) = \Omega_t(Pv_1) \cup \Omega_t(Pv_2) \subset K$, we can write

$$\varphi_1(t) = \int_{\Omega} \left[\sigma_1(t) \chi_{Q_1^\beta}(t) \left(\frac{\partial A_1}{\partial t}(t) + v_1(t) \frac{\partial A_1}{\partial z}(t) \right) - \sigma_2(t) \chi_{Q_2^\beta}(t) \left(\frac{\partial A_2}{\partial t}(t) + v_2(t) \frac{\partial A_2}{\partial z}(t) \right) \right] \theta \frac{\partial A_1}{\partial z}(t) r dr dz \quad (7.42)$$

and, since function $\theta \frac{\partial A_1}{\partial z}(t) \in \mathcal{V}$, using the Eq. (7.5) (that is, the weak formulation of the PDE (2.1) with velocity v_i on the time interval $[t_0, \beta]$) it turns out that

$$\varphi_1(t) = -a(A_1(t) - A_2(t), \theta \frac{\partial A_1}{\partial z}(t)). \quad (7.43)$$

Hence, recalling (4.8), we have

$$|\varphi_1(t)| \leq M \|A_1(t) - A_2(t)\|_{\mathcal{V}} \|\theta \frac{\partial A_1}{\partial z}(t)\|_{\mathcal{V}} \leq C_{10}\delta^{-1} M \|A_1(t) - A_2(t)\|_{\mathcal{V}} \|A_1(t)\|_{\tilde{H}_r^2(\Omega)}, \quad (7.44)$$

and then, in virtue of (7.25) and (7.32),

$$\begin{aligned} \int_{t_0}^{\beta} |\varphi_1(t)| dt &\leq C_{10}\delta^{-1} M \|A_1 - A_2\|_{L^2(t_0, \beta; \mathcal{V})} \|A_1\|_{L^2(t_0, \beta; \tilde{H}_r^2(\Omega))} \\ &\leq C_{10}\delta^{-1} M C_8 C_6 (\beta - t_0)^{1/2} \|v_1 - v_2\|_{C([t_0, \beta])} \end{aligned} \quad (7.45)$$

which, together with (7.35) and (7.39) implies that

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^1[t_0, \beta]} \leq \frac{2\pi}{m} (C_8 C_9 + C_{10}\delta^{-1} M C_8 C_6) (\beta - t_0)^{1/2} \|v_1 - v_2\|_{C([t_0, \beta])} \quad (7.46)$$

We recall that $v_1(t_0) = v_2(t_0)$, so $\|v_1 - v_2\|_{C([t_0, \beta])} \leq \|v_1' - v_2'\|_{L^1[t_0, \beta]}$. Hence, it is enough to take $\beta \in (t_0, T_f)$ with $\beta - t_0$ sufficiently small so that both (7.40) and $\lambda = \frac{2\pi}{m} (C_8 C_9 + C_{10}\delta^{-1} M C_8 C_6) (\beta - t_0)^{1/2} < 1$ hold. Thus we have proved the following result.

Theorem 7.2. The operator $\mathcal{F}$ fulfills the Lipschitz-type condition (H2).

Applying now Theorems 6.1 and 6.2, we obtain the following conclusions for the initial value problem (2.16)–(2.18), that is, for the reformulation of the coupled levitation problem as a causal differential equation:

- (i) There is a local solution.
- (ii) There is uniqueness of solution.
- (iii) There is a unique maximally defined solution and the maximal interval of existence is of the form $[0, \tau_M)$, where $0 < \tau_M \leq T_f$.

Thus, we have proven the first statement of Theorem 2.4. Next subsection is devoted to the proof of the second one.

7.3. The maximally defined solution

The goal of this subsection is to establish that velocity does not present blow-up, so the maximally defined solution exists as long as the metallic body neither touches the coil nor the boundary of the "computational domain" Ω . This will result from Theorem 6.3 and Remark 6.1 once we will have checked the fulfillment of condition (H3) and we will have proved that the velocity is bounded.

7.3.1. Fulfillment of condition (H3)

We must prove that for every $I \in \mathcal{J}$ and every function $v \in W_\Delta(I) \cap L^\infty(I)$ such that $\overline{gr(Pv)}$ is compact and is contained in $\Delta = [0, T_f) \times (\underline{u}, \bar{u})$, the function $F(v) \in L^1(I)$.

At the beginning of this section, we have seen that operator F acts from $W_\Delta(I)$ into $L^1_{loc}(I)$ for every interval $I \in \mathcal{J}$. We recall that if $I = [0, \eta]$ with $0 < \eta < T_f$ and $v \in W_\Delta[0, \eta]$, we have $A \in L^\infty(0, \eta; \mathcal{V})$ and $\partial_t A \in L^2_r(Q^\eta)$, where $Q^\eta := \{(t, r, z); t \in (0, \eta), (r, z) \in \Omega_t\}$, and then $F(v) = f$ belongs to $L^2[0, \eta]$.

If $I = [0, \eta]$ with $0 < \eta \leq T_f$ and $v \in W_\Delta[0, \eta]$ (that is, $v \in W^{1,1}_{loc}[0, \eta]$ and $\underline{u} < (Pv)(t) < \bar{u}$ for every $t \in [0, \eta]$), we have that for all $\eta_1 \in (0, \eta)$, $v|_{[0, \eta_1]} \in W_\Delta[0, \eta_1]$, $A \in L^\infty(0, \eta_1; \mathcal{V})$ with $\partial_t A \in L^2_r(Q^{\eta_1})$ and, by virtue of (2.10), we also have the estimate

$$\|\partial_t A\|_{L^2_r(Q^{\eta_1})} + \|A\|_{L^\infty(0, \eta_1; \mathcal{V})} \leq C(1 + \|v\|_{C([0, \eta_1])}) \leq C(1 + \|v\|_{L^\infty(0, \eta)}) \quad (7.47)$$

where $C = C(\hat{\sigma}, M, \alpha, \Omega_0, A^0, J_S)$. Therefore $A \in L^\infty(0, \eta; \mathcal{V})$, $\partial_t A \in L^2_r(Q^\eta)$ (due to Lebesgue's monotone convergence theorem), and then $F(v) \in L^2[0, \eta]$.

7.3.2. An a priori bound for velocity

Theorem 7.3. Let $v : [0, \tau_M] \mapsto \mathbb{R}$ be the maximally defined solution of problem (2.16)–(2.18). The function v is bounded on $[0, \tau_M]$

Proof. For all $\tau \in (0, \tau_M)$, we have $v \in W^{1,1}[0, \tau]$. Let $A = S_\tau(v)$. By substituting (2.14) and (2.15) in (2.16), we obtain

$$\frac{dv}{dt}(t) = -g - \frac{2\pi}{m} \int_{\Omega_t} \sigma(t) \left(\frac{\partial A}{\partial t}(t) + v(t) \frac{\partial A}{\partial z}(t) \right) \frac{\partial A}{\partial z}(t) r dr dz. \quad (7.48)$$

In this proof we write Ω_t in lieu of $\Omega_t(Pv)$. Multiplying the above equation by $v(t)$, we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} v^2(t) \right) + \frac{2\pi}{m} v^2(t) \int_{\Omega_t} \sigma(t) \left(\frac{\partial A}{\partial z}(t) \right)^2 r dr dz \\ = -g v(t) - \frac{2\pi}{m} v(t) \int_{\Omega_t} \sigma(t) \frac{\partial A}{\partial t}(t) \frac{\partial A}{\partial z}(t) r dr dz. \end{aligned} \quad (7.49)$$

Noticing that the second term of the left-hand side is ≥ 0 and integrating in $[0, \tau]$, we deduce that

$$\begin{aligned} \frac{1}{2} v^2(\tau) \leq \frac{1}{2} v_0^2 + \int_0^\tau |v(t)| \left(\frac{2\pi}{m} \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \frac{\partial A}{\partial z}(t) \right| r dr dz + g \right) dt, \\ 0 \leq \tau < \tau_M. \end{aligned} \quad (7.50)$$

By applying Brézis [12, Lemma A.5], we obtain

$$|v(\tau)| \leq |v_0| + g\tau + \frac{2\pi}{m} \int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \frac{\partial A}{\partial z}(t) \right| r dr dz dt, \quad 0 \leq \tau < \tau_M, \quad (7.51)$$

hence

$$\begin{aligned} |v(\tau)|^2 \leq 2(|v_0| + gT_f)^2 + \frac{8\pi^2}{m^2} \left(\int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \frac{\partial A}{\partial z}(t) \right| r dr dz dt \right)^2, \\ 0 \leq \tau < \tau_M. \end{aligned} \quad (7.52)$$

By using Cauchy-Schwarz inequality and (2.9), we have

$$\begin{aligned} \int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \frac{\partial A}{\partial z}(t) \right| r dr dz dt \\ \leq \sqrt{\bar{\sigma}} \left(\int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \right|^2 r dr dz dt \right)^{1/2} \left(\int_0^\tau \int_{\Omega_t} \left| \frac{\partial A}{\partial z}(t) \right|^2 r dr dz dt \right)^{1/2} \\ \leq C_{11} \left(\int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \right|^2 r dr dz dt \right)^{1/2} \end{aligned} \quad (7.53)$$

with $C_{11} = C_{11}(\alpha, \bar{\sigma}, \Omega_0, A^0, J_S)$. Inserting this bound in (7.52), we obtain

$$|v(\tau)|^2 \leq 2(|v_0| + gT_f)^2 + \frac{8\pi^2 C_{11}^2}{m^2} \int_0^\tau \int_{\Omega_t} \sigma(t) \left| \frac{\partial A}{\partial t}(t) \right|^2 r dr dz dt, \quad 0 \leq \tau < \tau_M. \quad (7.54)$$

On the other hand, by applying Theorem 4.4, we have for all $\tau \in [0, \tau_M]$,

$$\begin{aligned} \int_0^\tau \int_{\Omega_t} \sigma(t) |\partial_t A(t)|^2 r dr dz dt \leq C_{12} - \int_0^\tau v(t) \int_{\Omega_t} \sigma(t) \partial_z A(t) \partial_t A(t) r dr dz dt \\ \leq C_{12} + \int_0^\tau |v(t)| \int_{\Omega_t} \sigma(t) |\partial_z A(t) \partial_t A(t)| r dr dz dt \end{aligned} \quad (7.55)$$

$$\begin{aligned} &\leq C_{12} + \sqrt{\bar{\sigma}} \int_0^\tau |v(t)| \left(\int_{\Omega_t} |\partial_z A(t)|^2 r dr dz \right)^{1/2} \left(\int_{\Omega_t} \sigma(t) |\partial_t A(t)|^2 r dr dz \right)^{1/2} dt \\ &\leq C_{12} + \frac{1}{2} \int_0^\tau \int_{\Omega_t} \sigma(t) |\partial_t A(t)|^2 r dr dz dt + \frac{1}{2} \bar{\sigma} \int_0^\tau |v(t)|^2 \int_{\Omega_t} |\partial_z A(t)|^2 r dr dz dt, \end{aligned}$$

where $C_{12} = C_{12}(\bar{\sigma}, M, \alpha, \Omega_0, A^0, J_S)$. Hence

$$\int_0^\tau \int_{\Omega_t} \sigma(t) |\partial_t A(t)|^2 r dr dz dt \leq 2C_{12} + \bar{\sigma} \int_0^\tau |v(t)|^2 \int_{\Omega_t} |\partial_z A(t)|^2 r dr dz dt, \quad (7.56)$$

Inserting this bound in (7.54), it turns out that

$$|v(\tau)|^2 \leq 2(|v_0| + gT_f)^2 + \frac{16C_{12}\pi^2 C_{11}^2}{m^2} + \frac{8\pi^2 C_{11}^2}{m^2} \bar{\sigma} \int_0^\tau |v(t)|^2 \int_{\Omega_t} |\partial_z A(t)|^2 r dr dz dt, \quad 0 \leq \tau < \tau_M. \quad (7.57)$$

Finally, as by virtue of the estimation (2.9), we have that

$$\begin{aligned} &\int_0^\tau \int_{\Omega_t} |\partial_z A(t)|^2 r dr dz dt \leq C(\alpha) \left[\bar{\sigma} \int_{\Omega_0} |A^0|^2 r dr dz + \int_0^\tau \|J_S(t)\|_{V'}^2 dt \right] \\ &\leq C(\alpha) \left[\bar{\sigma} \int_{\Omega_0} |A^0|^2 r dr dz + \int_0^{T_f} \|J_S(t)\|_{V'}^2 dt \right] =: C_{13} < +\infty \quad 0 \leq \tau < \tau_M, \end{aligned} \quad (7.58)$$

we can apply Gronwall's lemma to the inequality (7.57). This yields

$$\begin{aligned} &|v(\tau)|^2 \\ &\leq \left(2(|v_0| + gT_f)^2 + \frac{16C_{12}\pi^2 C_{11}^2}{m^2} \right) \exp \left(\frac{8\pi^2 C_{11}^2}{m^2} \bar{\sigma} \int_0^\tau \int_{\Omega_t} |\partial_z A(t)|^2 r dr dz dt \right) \\ &\leq \left(2(|v_0| + gT_f)^2 + \frac{16C_{12}\pi^2 C_{11}^2}{m^2} \right) \exp \left(\frac{8\pi^2 C_{11}^2}{m^2} \bar{\sigma} C_{13} \right), \quad \forall \tau \in [0, \tau_M). \end{aligned} \quad (7.59)$$

Therefore the result is proved. $\square$

As a consequence of previous theorem, we can apply Theorem 6.3 and Remark 6.1. This proves the second statement of Theorem 2.4. So the levitation problem does not present a velocity blow-up, it can happen that $\tau_M = T_f$ or that at an instant $\tau_M < T_f$ the metallic body tends to touch the coil or the boundary of the computational domain.

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