Journal of Difference Equations and Applications
Vol. 00, No. 00, Month 200x, 1–18

RESEARCH ARTICLE

Existence of positive solutions for nth order periodic difference equations

A. Cabada[a*] and J.B. Ferreiro[b]

[a]*Dep. de Análise Matemática, Universidade de Santiago de Compostela, Facultade de Matemáticas, 15782 Santiago, Spain.*
[b]*Dep. de Matemática Aplicada, Universidade de Santiago de Compostela, Escola Politécnica Superior, 27002 Lugo, Spain.*
(*October 23, 2009*)

This paper is devoted to the study of difference equations coupled with periodic boundary value conditions. We deduce the existence of at least one positive solution provided that the nonlinear part of the equation satisfies some monotonicity assumptions and the existence of a positive upper solution. The result is obtained from a new fixed point theorem based on the classical Krasnoselskii's cone expansion/contraction theorem and the constant sign properties of the related Green's function.

Keywords: Periodic boundary value problem, nth order difference equations, non-zero fixed point, positive solutions.

1. Introduction

In the recent paper [7] the authors use the classical Krasnoselskii's fixed point theorem on cone expansions to prove a new fixed point theorem for nondecreasing operators on ordered Banach spaces. In contrast to the usual cases, in which some restrictions are imposed to the growth of the nonlinear term of the equation both at 0 and at $+\infty$ (see [9] for instance), this result can be used taking only into account the behaviour of the nonlinear part at $+\infty$.

This result has been applied in [7] to second order ordinary differential equations, and improves the result given by Persson in [8] for monotone operators defined in $\mathbb{R}^n_+$, the set of elements in $\mathbb{R}^n$ with all of their components nonnegative. In this paper we apply the new fixed point theorem given in [7] by combining two types of techniques: the lower and upper solution method ([2], [6]) with the classical cone contraction/expansion fixed point theorem [9]. So we deduce the existence of at least one positive solution of suitable nth order periodic boundary value problems.

This paper is organized as follows: in section 2 we present some definitions and some known results that will be necessary for the rest of the paper. Section 3 is devoted to prove the existence of at least one positive solution of a general nonlinear nth order periodic boundary value problem. We give in section 4 some suitable conditions to ensure the validity of some strong comparison results for linear operators. In section 5 we present some particular cases in which more concrete conditions for the existence of positive solutions are obtained. Finally, in section 6 we point out some conclusions for this paper and we suggest some new directions in the

*Corresponding author. Email: alberto.cabada@usc.es

ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

considered field.

2. Definitions and preliminaries

Let $(\mathcal{N}, \|\cdot\|)$ be a real Banach space, we say that a subset $\mathcal{K} \subset \mathcal{N}$ is a cone if it is closed and verifies that $\mathcal{K}+\mathcal{K} \subset \mathcal{K}$, $\lambda \mathcal{K} \subset \mathcal{K}$ for all $\lambda \geq 0$, and $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$.

A cone $\mathcal{K}$ defines the partial ordering $\preceq$ in $\mathcal{N}$ given by $x \preceq y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is said to be *normal* if there exists $c \in \mathbb{R}$, $c > 0$, such that $\|x\| \leq c\|y\|$ for all $x, y \in \mathcal{N}$ with $0 \preceq x \preceq y$. Whenever $int(\mathcal{K}) \neq \emptyset$, the relation $x \lll y$ means $y - x \in int(\mathcal{K})$.

As usual, we define a compact map $T : \mathcal{N} \to \mathcal{N}$ as a continuous map such that $\overline{T(\mathcal{N})}$ is a compact subset of $\mathcal{N}$. A map T is said to be completely continuous if it is continuous and $\overline{T(D)}$ is a compact subset of $\mathcal{N}$ for each bounded subset $D \subset \mathcal{N}$.

In the following we will use the usual component-wise order in $\mathbb{R}^{p+1}$, that is to say, given $x = \{x_0, \dots, x_p\}$ and $y = \{y_0, \dots, y_p\} \in \mathbb{R}^{p+1}$, we shall denote:

- $x \leq y$ if $x_k \leq y_k$ for all $k \in \{0, \dots, p\}$.
- $x < y$ if $x \leq y$ and there exists $k \in \{0, \dots, p\}$ such that $x_k < y_k$.
- $x \ll y$ if $x_k < y_k$ for all $k \in \{0, \dots, p\}$.

THEOREM 2.1 . [7, Theorem 2.1] *Let $\mathcal{N}$ be a real Banach space, $\mathcal{K}$ a normal cone with nonempty interior and $T : \mathcal{K} \to \mathcal{K}$ a nondecreasing and completely continuous operator. Define $S = \{x \in \mathcal{K} : Tx \preceq x\}$ and suppose that*

(1) There exists $\bar{x} \in S$ such that $\theta \lll \bar{x}$.
(2) S is bounded.

Then there exists $x \in \mathcal{K}$, $x \neq \theta$, such that $x = Tx$.

Taking $\mathcal{N} = \mathbb{R}^m$ and the cone $\mathcal{K} = \mathbb{R}^m_+$, the partial ordering defined by $\mathcal{K}$ is the component-wise order in $\mathbb{R}^m$, and a particular case of the previous theorem is the following:

COROLLARY 2.2. *(*[7, Corollary 2.1] and [8, Theorem 5]*) Assume that $f : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is continuous and nondecreasing. Let $S = \{x \in \mathbb{R}^m_+ : f(x) \leq x\}$. If S is bounded, and if there is an $x' \gg \theta$, $x' \in S$, then there is $x > \theta$ such that $x = f(x)$.*

Throughout the paper, given $n, N \in \mathbb{N}$ with $n \leq N$, we will denote $I := \{0, 1, \dots, N-1\}$, $J := \{0, 1, \dots, N+n-1\}$ and

$$\Omega^n_N = \{x \in \mathbb{R}^{N+n};\, x(i) = x(N+i),\, i = 0, \dots, n-1\}.$$

We will consider the following linear operator $T_n[K_0, \dots, K_n] : \mathbb{R}^{N+n} \to \mathbb{R}^N$ defined by

$$T_n[K_0, \dots, K_n]\, x(k) \equiv x(k+n) + \sum_{i=0}^{n} K_i\, x(k+i), \quad \text{for all } k \in I.$$

Let us introduce now the concepts of inverse positive (inverse negative) and strongly inverse positive (strongly inverse negative) operators:

Definition 2.3. Let $K_0, \dots, K_n \in \mathbb{R}$ be fixed, such that $1 + K_n \neq 0$ and $1 + \sum_{i=0}^{n} K_i > 0$. We say that the operator $T_n[K_0, \dots, K_n]$ is strongly inverse positive (inverse positive) on Ω^n_N if $T_n[K_0, \dots, K_n]\, x > \theta$ $(T_n[K_0, \dots, K_n]\, x \geq \theta)$ implies that $x \gg \theta$ $(x \geq \theta)$ for all $x \in \Omega^n_N$.

Definition 2.4. Let $K_0, \ldots, K_n \in \mathbb{R}$ be fixed, such that $1 + K_n \neq 0$ and $1 + \sum_{i=0}^{n} K_i < 0$. We say that the operator $T_n[K_0, \ldots, K_n]$ is strongly inverse negative (inverse negative) on Ω_N^n if $T_n[K_0, \ldots, K_n]\, x > \theta$ ($T_n[K_0, \ldots, K_n]\, x \geq \theta$) implies that $x \ll \theta$ ($x \leq \theta$) for all $x \in \Omega_N^n$.

Remark 2.1. We notice (see [3] for details) that Definition 2.3 implies the existence of the inverse operator $T_n^{-1}[K_0, \ldots, K_n]$ and, in consequence, there is a related Green's function of the operator $T_n^{-1}[K_0, \ldots, K_n]$ on Ω_N^n. In an analogous way, we can see that Definition 2.4 implies the existence of the inverse operator too.

Remark 2.2. Obviously, a strongly inverse positive (strongly inverse negative) operator on Ω_N^n is also an inverse positive (inverse negative) operator on Ω_N^n.

Remark 2.3. It is clear that condition $1 + K_n \neq 0$ is a necessary condition to ensure that operator $T_n[K_0, \ldots, K_n]$ is a nth order operator. On the other hand, we can see in [3] that condition on the sign of $1 + \sum_{i=0}^{n} K_i$ is not restrictive.

Remark 2.4. The opposite operator $-T_n[K_0, \ldots, K_n]$ can be written as $T_n[-K_0, \ldots, -K_{n-1}, -K_n - 2]$, and it is easy to verify that the operator $T_n[K_0, \ldots, K_n]$ is (strongly) inverse positive on Ω_N^n if and only if the operator $-T_n[K_0, \ldots, K_n]$ is (strongly) inverse negative on Ω_N^n.

3. A general existence result for periodic boundary value problems

Consider the general nth order difference equation

$$T_n[K_0, \ldots, K_n]\, x(k) = f(k, x(k), x(k+1), \ldots, x(k+n)),\ k \in I, \tag{1}$$

coupled with the periodic boundary value conditions

$$x(i) = x(N+i),\ i = 0, \ldots, n-1. \tag{2}$$

Here $f : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function.
Moreover, we assume the following condition:

(H_0) The operator $T_n[K_0, \ldots, K_n]$ is strongly inverse positive on Ω_N^n.

If operator $T_n[K_0, \ldots, K_n]$ is invertible on Ω_N^n, it is clear that $x \in \mathbb{R}^{N+n}$ is a solution of problem (1) – (2) if and only if x is a fixed point of the operator $T : \mathbb{R}^{N+n} \to \mathbb{R}^{N+n}$ given by

$$Tx(k) = \sum_{j=0}^{N-1} G(k,j)\, f(j, x(j), \ldots, x(j+n)), \quad \text{for all } k \in J,$$

where $G(k,j)$ is the Green's function associated to the operator $T_n^{-1}[K_0, \ldots, K_n]$.
We will see now that such a Green's function is strictly positive.

Lemma 3.1. *The operator $T_n[K_0, \ldots, K_n]$ is strongly inverse positive on Ω_N^n if and only if the Green's function associated to the operator $T_n^{-1}[K_0, \ldots, K_n]$ is strictly positive on $J \times I$.*

Proof. As we can see in [3], the operator $T_n[K_0, \ldots, K_n]$ is inverse positive on Ω_N^n if and only if the Green's function $G : J \times I \to \mathbb{R}$ associated to the operator $T_n^{-1}[K_0, \ldots, K_n]$ is nonnegative on $J \times I$.

Suppose that the operator $T_n[K_0,\dots,K_n]$ is strongly inverse positive on Ω_N^n and there exists $(k_0,j_0)\in J\times I$ such that $G(k_0,j_0)=0$. Define $f:I\to\mathbb{R}$ as

$$f(k):=\begin{cases}0\,, & \text{if } k\neq j_0,\\ 1\,, & \text{if } k=j_0,\end{cases}$$

and let $u\equiv\{u(0),\dots,u(N+n-1)\}$ be the unique solution of the problem

$$T_n[K_0,\dots,K_n]\,u(k)=f(k),\ k\in I,\quad u(i)=u(N+i),\ i=0,\dots,n-1.$$

Clearly $u\in\Omega_N^n$ and $T_n[K_0,\dots,K_n]\,u=f>\theta$, but

$$u(k_0)=\sum_{j=0}^{N-1}G(k_0,j)f(j)=0.$$

Due to this, we deduce that $u\not\gg\theta$ and we reach a contradiction.

On the other hand, suppose that function G is strictly positive, and take $u\in\Omega_N^n$ that verifies $T_n[K_0,\cdots,K_n]\,u=v>\theta$. Then there exists $j_0\in I$ such that $v(j_0)>0$ and thus

$$u(k)=\sum_{j=0}^{N-1}G(k,j)v(j)\geq G(k,j_0)v(j_0)>0,\quad \forall k\in J.$$

So $u\gg\theta$ and we have finished the proof. □

Remark 3.1. In a similar way we can conclude that the operator $T_n[K_0,\dots,K_n]$ is strongly inverse negative on Ω_N^n if and only if the Green's function associated to the operator $T_n^{-1}[K_0,\dots,K_n]$ is strictly negative on $J\times I$.

Now, we are in a position to prove the main result of this paper, in which the existence of at least one positive solution of problem (1) – (2) is deduced.

THEOREM 3.2. *Suppose that (H_0) is fulfilled and the following conditions hold:*

- (H_1) *$f(k,x)\geq 0$ for all $k\in I$ and all $x\in\mathbb{R}_+^{n+1}$.*
- (H_2) *$f(k,x)\leq f(k,y)$ for all $k\in I$, and for all $x,y\in\mathbb{R}_+^{n+1}$ such that $x\leq y$.*
- (H_3) *There exists $\bar{x}\in\Omega_N^n$, with $\min_{k\in J}\bar{x}(k)>0$, and*

$$T_n[K_0,\dots,K_n]\,\bar{x}(k)\geq f(k,\bar{x}(k),\bar{x}(k+1),\dots,\bar{x}(k+n)),\ \textit{for all}\ k\in I.$$

- (H_4) *There exists $i\in\{0,\dots,n\}$ such that $\displaystyle\lim_{x_i\to\infty}\frac{f(k,x_0,\dots,x_n)}{x_i}=\infty$ for all $x_j>0, j\in\{0,\dots,n\}\backslash\{i\}$ and for all $k\in I$.*

Then problem (1) – (2) has a positive solution.

Proof. Denote

$$m:=\min_{(k,j)\in J\times I}G(k,j)\qquad\text{and}\qquad M:=\max_{(k,j)\in J\times I}G(k,j).$$

By using (H_0) and Lemma 3.1 we have that $0<m<M$. Moreover, condition (H_3) implies that $\min_{k\in J}\bar{x}(k)>0$. So, we can choose $0<\xi<m/M<1$ small enough such that

$$\min_{k \in J} \bar{x}(k) > \xi \|\bar{x}\|_\infty.$$

We define now

$$\mathcal{K} := \{x \in \mathbb{R}^{N+n} : \min_{k \in J} x(k) \geq \xi \|x\|_\infty\},$$

which is a normal cone, with $c = 1$, in $\mathbb{R}^{N+n}$ and has nonempty interior.

Let "$\preceq$" be the order induced in $\mathbb{R}^{N+n}$ by the cone $\mathcal{K}$, i.e.,

$$x \preceq y \Longleftrightarrow \min_{k \in J}(y(k) - x(k)) \geq \xi \|y - x\|_\infty.$$

We define also

$$S := \{x \in \mathcal{K} : Tx \preceq x\}.$$

To apply Theorem 2.1 we prove the following claims.

Claim 1. $T(\mathcal{K}) \subset \mathcal{K}$.

If $x \in \mathcal{K}$, then $x(k) \geq 0$ for all $k \in J$ and, using (H_1), we compute

$$\begin{aligned}
\min_{k \in J} Tx(k) &= \min_{k \in J} \sum_{j=0}^{N-1} G(k,j) f(j, x(j), \ldots, x(j+n)) \\
&\geq \sum_{j=0}^{N-1} m\, f(j, x(j), \ldots, x(j+n)) \\
&\geq \sum_{j=0}^{N-1} \xi\, M\, f(j, x(j), \ldots, x(j+n)) \\
&\geq \xi \max_{k \in J} \sum_{j=0}^{N-1} G(k,j) f(j, x(j), \ldots, x(j+n)) = \xi\, \|Tx\|_\infty,
\end{aligned}$$

so $Tx \in \mathcal{K}$, and the claim is proved.

Claim 2. $T : \mathcal{K} \to \mathcal{K}$ is monotone and nondecreasing.

Let $x,\ y \in \mathcal{K}$ be such that $x \preceq y$. It is clear that $x \leq y$ on J, so by using condition (H_2) and computing in analogous way to those of *Claim* 1 it follows that

$$\min_{k \in J} (Ty(k) - Tx(k)) \geq \xi \|Ty - Tx\|_\infty,$$

so $Tx \preceq Ty$.

Claim 3. $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.

It is obvious for this case.

Claim 4. $\bar{x} \in S$ and $\theta \lll \bar{x}$.

Due to the previous selection of ξ and the definition of $\mathcal{K}$, it is clear that $\bar{x} \in$ *int*$(\mathcal{K})$, that is to say, $\theta \lll \bar{x}$.

On the other hand, condition (H_3) ensures that there exists a nonnegative function $h : I \to \mathbb{R}$ such that

$$T_n[K_0, \ldots, K_n]\, \bar{x}(k) = f(k, \bar{x}(k), \bar{x}(k+1), \ldots, \bar{x}(k+n)) + h(k), \text{ for all } k \in I.$$

Due to the fact that operator $T_n[K_0, \ldots, K_n]$ is invertible on Ω_N^n, the previous equality is equivalent to the following expression

$$\bar{x}(k) - T\,\bar{x}(k) = T_n^{-1}[K_0, \ldots, K_n]\, h(k) \equiv \sum_{j=0}^{N-1} G(k,j)h(j) \qquad \text{for all } k \in J.$$

Now, by similar computations to those of *Claim* 1, we deduce that

$$\min_{k \in J} (\bar{x}(k) - T\bar{x}(k)) \geq \xi \|\bar{x} - T\bar{x}\|_\infty,$$

which implies $T\bar{x} \preceq \bar{x}$ and, therefore, $\bar{x} \in S$.

Claim 5. S is bounded.

By (H_4) there exist $i \in \{0, \ldots, n\}$ and $\delta > 0$ such that, if $x_i > \delta$ then

$$\frac{f(k, \frac{1}{\xi\, m\, N}, \ldots, \frac{1}{\xi\, m\, N}, x_i, \frac{1}{\xi\, m\, N}, \ldots, \frac{1}{\xi\, m\, N})}{x_i} > \frac{1}{\xi\, m\, N}$$

for all $k \in I$.

Take $\alpha = \max\{\frac{1}{\xi\, m\, N}, \delta\} > 0$. Thus, if $x \in \mathcal{K}$ is such that $\min_{j \in J} x(j) > \alpha$, as f is nondecreasing in virtue of (H_2), we have

$$\begin{aligned} \frac{f(j, x(j), \ldots, x(j+i), \ldots, x(j+n))}{x(j+i)} &\geq \frac{f(j, \frac{1}{\xi\, m\, N}, \ldots, \frac{1}{\xi\, m\, N}, x(j+i), \frac{1}{\xi\, m\, N}, \ldots, \frac{1}{\xi\, m\, N})}{x(j+i)} \\ &> \frac{1}{\xi\, m\, N} \end{aligned}$$

for all $j \in I$.

Because of this, if $x \in \mathcal{K}$ is such that $\min_{j \in J} x(j) > \alpha$ then

$$\begin{aligned} Tx(k) &= \sum_{j=0}^{N-1} G(k,j) f(j, x(j), \ldots, x(j+n)) \\ &> \sum_{j=0}^{N-1} G(k,j)\, \frac{x(j+i)}{\xi\, m\, N} \geq \sum_{j=0}^{N-1} m\, \frac{\xi \|x\|_\infty}{\xi\, m\, N} = \|x\|_\infty \geq x(k). \end{aligned}$$

So $Tx \not\preceq x$ and, in consequence, $x \notin S$. Therefore, whenever $x \in S$ we have $\min_{j \in J} x(j) \leq \alpha$, and from

$$\alpha \geq \min_{j \in J} x(j) \geq \xi \|x\|_\infty,$$

we deduce that $S \subset \overline{B(\theta, \alpha/\xi)}$ and the claim is proved.

Finally, from Theorem 2.1 and the above claims, we deduce the existence of a non trivial fixed point in $\mathcal{K}$ for the operator T, which is a positive solution of problem (1) – (2). □

Remark 3.2. It is clear that condition (H_0) implies that $\bar{y} \equiv 0$ is a lower solution for problem (1) – (2). As consequence, Theorem 2.1 in [3] ensures that such a problem has a solution u satisfying $0 \leq u(k) \leq \bar{x}(k)$ for all $k \in J$. However we cannot ensure that $u \not\equiv 0$ or, if it is the case, $u > 0$ on J. Moreover if there is a solution that vanish at some point of J, we can ensure, as a direct consequence of the previous result, the existence of two different solutions of problem (1) – (2).

Remark 3.3. In the proof of the previous theorem, using only conditions (H_0) and (H_3) we can choose an appropriate $0 < \xi < m/M < 1$ in order to construct a suitable cone $\mathcal{K}$ for our purposes. Because of this, it is possible to relax the hypothesis (H_4) as follows:

(H_4') There exists $i \in \{0, \ldots, n\}$ such that $\lim_{x_i \to \infty} \frac{f(k, x_0, \ldots, x_n)}{x_i} > \frac{1}{\xi\, m\, N}$ for all $x_j > \frac{1}{\xi\, m\, N}, j \in \{0, \ldots, n\} \setminus \{i\}$ and for all $k \in I$.

4. Positiveness criteria for the Green's function

As we have seen in Theorem 3.2, if, among other conditions, the Green's function related to operator $T_n[K_0, \ldots, K_n]$ is strictly positive on $J \times I$, we can deduce the existence of at least one positive solution of problem (1) – (2). The range of the parameters $K_0, \ldots, K_n$ for which this property holds has been studied for periodic (see [2–4]), Neumann [5] or, among others, two point higher order equations [1]. Despite this, the problem is far from being solved, only first and second order problems has been exhaustively studied.

We can see in [3, 4] some results about the inverse positive character of the operator $T_n[K_0, \ldots, K_n]$ depending on the values $K_0, \ldots, K_n$. Those results are obtained from the study of the sign of the Green's function associated to the operator $T_n^{-1}[K_0, \ldots, K_n]$, such expression is calculated by solving a $n \times n$ linear algebraic system. Here we introduce a particular case of [3, Theorem 5.1], given for the case $K_n = 0$, although the general case is derived directly from this one dividing by $1 + K_n$.

THEOREM 4.1. *Let $K_0, \ldots, K_{n-1} \in \mathbb{R}$ be fixed. Suppose that operator $T_n[K_0, \ldots, K_{n-1}, 0]$ is invertible on Ω_N^n. Then, for all $\sigma \in \mathbb{R}^N$ the unique solution of the problem*

$$T_n[K_0, \ldots, K_{n-1}, 0]\, u(k) = \sigma(k),\ k \in I, \quad u(i) = u(N+i),\ i = 0, \ldots, n-1 \quad (3)$$

is given by the expression

$$u(k) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} z(k-j-1)\,\sigma(j) + \sum_{j=k}^{N-1} z(N+k-j-1)\,\sigma(j) & \text{if } k \in I \\[2ex] \displaystyle\sum_{j=0}^{k-1-N} z(k-j-1-N)\,\sigma(j) + \sum_{j=k-N}^{N-1} z(k-j-1)\,\sigma(j) & \text{if } k \in J \setminus I, \end{cases}$$

where z is the unique solution of the problem

$$T_n[K_0,\dots,K_{n-1},0]\,z(k)=0\,,\qquad k\ge 0, \tag{4}$$

$$z(i)-z(N+i)=0\,,\qquad i=0,\dots,n-2, \tag{5}$$

$$z(n-1)-z(N+n-1)=1\,. \tag{6}$$

Moreover, taking into account the Lemma 3.1, it is not difficult to verify the following result

Corollary 4.2. *Suppose that* $1+K_n>0$ *and let* z *be the unique solution of* $T_n\left[\frac{K_0}{1+K_n},\dots,\frac{K_{n-1}}{1+K_n},0\right]z(k)=0$, $k\in I$ *satisfying the boundary conditions (5) – (6). Then, if* $1+\sum_{i=0}^n K_i>0$ *(*$1+\sum_{i=0}^n K_i<0$*) we have that operator* $T_n[K_0,\dots,K_n]$ *is strongly inverse positive (strongly inverse negative) on* Ω_N^n *if and only if* $z(k)>0$ *(*$z(k)<0$*) for all* $k\in I$.

Remark 4.1. For the case $1+K_n<0$, we only have to use the Remark 2.4 to conclude that, if $1+\sum_{i=0}^n K_i>0$ ($1+\sum_{i=0}^n K_i<0$), the operator $T_n[K_0,\dots,K_n]$ is strongly inverse positive (strongly inverse negative) on Ω_N^n if and only if $z(k)<0$ ($z(k)>0$) for all $k\in I$.

In [3, subsection 6.1] the authors give the expression of function z associated to operator $T_1[-\lambda,0]$ and deduce that function $z>0$ ($z<0$) on I if and only if $\lambda\in(0,1)$ ($\lambda>1$). We will make use of those calculations to deduce the next result:

Proposition 4.3. *Let* $K_0,\,K_1\in\mathbb{R}$ *be such that* $1+K_1>0$*, then the two following assertions hold:*

(1) Operator $T_1[K_0,K_1]$ *is strongly inverse positive on* Ω_N^1 *if and only if* $1+K_0+K_1>0$ *and* $K_0<0$.
(2) Operator $T_1[K_0,K_1]$ *is strongly inverse negative on* Ω_N^1 *if and only if* $1+K_0+K_1<0$.

Proof. As $1+K_1>0$, due to Corollary 4.2 operator $T_1[K_0,K_1]$ is strongly inverse positive (strongly inverse negative) on Ω_N^1 if and only if the function z associated to operator $T[\frac{K_0}{1+K_1},0]$ verifies $z(k)>0$ ($z(k)<0$) for all $k\in I$. As consequence we have that $z>0$ on I if and only if $0<-K_0/(1+K_1)<1$, which is equivalent to say that $K_0<0$ and $1+K_0+K_1>0$.

On the other hand, $z<0$ on I if and only if $-K_0/(1+K_1)>1$ or, what is the same, $1+K_0+K_1<0$. □

Remark 4.2. In case of $1+K_1<0$, we have, from remark 4.1, that operator $T_1[K_0,K_1]$ is strongly inverse positive on Ω_N^1 if and only if $1+K_0+K_1>0$, and is strongly inverse negative if and only if $1+K_0+K_1<0$ and $K_0>0$.

The expression of function z associated to operator $T_2[K_0,K_1,0]$ is deduced in [3, subsection 6.2]. Follow such expression, one can verify that if the characteristic polynomial of operator $T_2[K_0,K_1,0]$ has two real roots $\lambda_2\le\lambda_1$ then $z>0$ on I if and only if $0\le\lambda_2<\lambda_1<1$; $1<\lambda_2<\lambda_1$ or $0<\lambda_1=\lambda_2\ne 1$. On the other hand $z<0$ on I if and only if $0\le\lambda_2<1<\lambda_1$.

When the characteristic polynomial has complex roots $r\cos\theta\pm i\,r\sin\theta$, then function z is strictly positive on I if only if $\theta\in(0,\pi/N)$.

A careful analysis of these expressions allow us to obtain the following result:

Proposition 4.4. *Let* $K_0,\,K_1,\,K_2\in\mathbb{R}$ *be such that* $1+K_2>0$*. Moreover if*

$K_1^2 - 4\,K_0\,(1+K_2) < 0$ *we denote*

$$\theta = \arctan\left(\frac{-\sqrt{4K_0(1+K_2)-K_1^2}}{K_1}\right) \in (0,\pi).$$

The following assertions are fulfilled:

(1) If $1+K_0+K_1+K_2>0$ *then operator* $T_2[K_0,K_1,K_2]$ *is strongly inverse positive on* Ω_N^2 *if and only if one of the following conditions holds:*
- *(i)* $K_1^2-4K_0(1+K_2)<0$, *and* $\theta\in(0,\pi/N)$.
- *(ii)* $K_1^2-4K_0(1+K_2)=0$, $K_1<0$ *and* $K_1+2(1+K_2)\neq 0$.
- *(iii)* $K_1^2-4K_0(1+K_2)>0$, $K_1+2(1+K_2)>0$, $K_1<0$ *and* $K_0\geq 0$.
- *(iv)* $K_1^2-4K_0(1+K_2)>0$ *and* $K_1+2(1+K_2)<0$.

(2) If $1+K_0+K_1+K_2<0$ *then operator* $T_2[K_0,K_1,K_2]$ *is strongly inverse negative on* Ω_N^2 *if and only if* $K_1^2-4K_0(1+K_2)>0$, $K_1<0$ *and* $K_0\geq 0$.

Remark 4.3. In case of $1+K_2<0$ and $1+K_0+K_1+K_2>0$, we have that operator $T_2[K_0,K_1,K_2]$ is strongly inverse positive on Ω_N^2 if and only if $K_1^2-4\,K_0\,(1+K_2)>0$, $K_1>0$ and $K_0\leq 0$.

On the other hand, if $1+K_0+K_1+K_2<0$ operator $T_2[K_0,K_1,K_2]$ is strongly inverse negative on Ω_N^2 if and only if one of the following conditions holds:

- (i) $K_1^2-4K_0(1+K_2)<0$ and $\theta\in(0,\pi/N)$.
- (ii) $K_1^2-4K_0(1+K_2)=0$, $K_1>0$ and $K_1+2(1+K_2)\neq 0$.
- (iii) $K_1^2-4K_0(1+K_2)>0$, $K_1+2(1+K_2)<0$, $K_1>0$ and $K_0\leq 0$.
- (iv) $K_1^2-4K_0(1+K_2)>0$ and $K_1+2(1+K_2)>0$.

For higher order equations only partial results has been given. In [4, Lemma 2.3, Lemma 2.4] the authors study an nth order operator by using the fact that the composition of inverse positive operators defined in suitable sets is an inverse positive operator too, and by explaining that operator as the composition of related first and second order inverse positive operators.

Following the ideas showed in [4, Lemma 2.1], we arrive at the following result

Lemma 4.5. *If operator* $T_n[K_0,\dots,K_n]$ *is strongly inverse positive on the space* Ω_N^n *and* $T_m[M_0,\dots,M_m]$ *is strongly inverse positive (strongly inverse negative) on* Ω_N^m *then the composition operator* $T_n[K_0,\dots,K_n]\circ T_m[M_0,\dots,M_m]$ *is strongly inverse positive (strongly inverse negative) on* Ω_N^{n+m}. *Moreover if both operators* $T_n[K_0,\dots,K_n]$ *and* $T_m[M_0,\dots,M_m]$ *are strongly inverse negatives on* Ω_N^n *and* Ω_N^m *respectively, then the composition operator* $T_n[K_0,\dots,K_n]\circ T_m[M_0,\dots,M_m]$ *is strongly inverse positive on* Ω_N^{n+m}.

Next, following the ideas in [4, Lemma 2.3, Lemma 2.4] we present a general result that allow us to ensure the positiveness of the Green's function depending on the roots of the characteristic polynomial.

Theorem 4.6. *Suppose that* $1+K_n>0$ *and* $1+\sum_{i=0}^n K_i>0$. *Now, let* $\lambda_s\in\mathbb{R}$, $s=1,\dots,k$, $0\leq k\leq n$ *and* $r_s\,(\cos\theta_s\pm i\,\sin\theta_s)$, $s=1,\dots,j$, $0\leq j\leq n/2$, *with* $k+j=n$, *be the roots of the polynomial*

$$p(\lambda)=(1+L_n)\,\lambda^n+L_{n-1}\,\lambda^{n-1}+\cdots+L_1\,\lambda+L_0.$$

Then, if $r_s>0$ *and* $\theta_s\in(0,\pi/N)$ *for all* $s=1,\dots,j$ *and there is an even number* $0\leq m\leq k$ *for which* $\lambda_s>1$ *for all* $s=1,\dots,m$ *and* $\lambda_s\in(0,1)$ *for all* $s=m+1,\dots,k$, *we have that operator* $T_n[K_0,\dots,K_n]$ *is strongly inverse positive on* Ω_N^n *provided that* $K_i\leq L_i$ *for all* $i=0,\dots,n$.

Proof. It is clear that operator $T_n[L_0,\dots,L_n]$ can be decomposed as follows:

$$T_n[L_0,\dots,L_n] = (1+L_n)\,T_1[-\lambda_1,0]\circ\cdots\circ T_1[-\lambda_m,0]\circ T_1[-\lambda_{m+1},0]\circ\cdots\circ T_1[-\lambda_k,0]\circ T_2[r_1^2,-2\,r_1\,\cos\theta_1,0]\circ\cdots\circ T_2[r_j^2,-2\,r_j\,\cos\theta_j,0].$$

From Proposition 4.3 we have that operators $T_1[-\lambda_s,0]$ are strongly inverse negative for $s=1,\dots,m$, and strongly inverse positive for $s=m+1,\dots,k$. Moreover, from Proposition 4.4 we have that operators $T_2[r_s^2,-2\,r_s\,\cos\theta_s,0]$ are strongly inverse positive for $s=1,\dots,j$. So, applying Lemma 4.5 and taking into account that $1+L_n\geq 1+K_n>0$, we deduce that operator $T_n[L_0,\dots,L_n]$ is strongly inverse positive on Ω_N^n.

Now, using [3, Theorem 4.1], since $1+\sum_{i=0}^n K_i>0$ and $1+K_n>0$, this property remains valid for operator $T_n[K_0,\dots,K_n]$. □

Remark 4.4. In case of $1+K_n<0$, if $1+L_n<0$ and we replace m even by m odd, then operator $T_n[K_0,\dots,K_n]$ remains strongly inverse positive on Ω_N^n.

Remark 4.5. If in the statement of previous theorem we replace m even by m odd, $1+\sum_{i=0}^n K_i>0$ by $1+\sum_{i=0}^n K_i<0$ and $K_i\leq L_i$ by $K_i\geq L_i$, then operator $T_n[K_0,\dots,K_n]$ is strongly inverse negative on Ω_N^n.

Example 4.7 The sixth order operator

$T_6[L_0,\dots,L_6]\,u(k)\equiv 6\,u(k+6)-(53+6\sqrt{3})\,u(k+5)+(137+53\sqrt{3})\,u(k+4)-(136+131\sqrt{3})\,u(k+3)+(146+83\sqrt{3})\,u(k+2)-(83+15\sqrt{3})\,u(k+1)+15\,u(k)$ is strongly inverse positive on Ω_{11}^6.

The proof follows directly from the fact that $1+\sum_{i=1}^6 L_i=32-16\sqrt{3}>0$, $1+L_6=6>0$, and the related characteristic polynomial is given by the expression

$$p(\lambda)=6\,(\lambda-1/2)(\lambda-1/3)(\lambda-3)(\lambda-5)((\lambda-\cos\pi/12)^2+\sin^2\pi/12).$$

Moreover, every sixth order operator $T_6[K_0,\dots,K_6]$ that satisfies $1+K_6>0$, $1+\sum_{i=1}^6 K_i>0$ and $K_0\leq 15$, $K_1\leq -83-15\sqrt{3}$, $K_2\leq 146+83\sqrt{3}$, $K_3\leq -136-131\sqrt{3}$, $K_4\leq 137+53\sqrt{3}$, $K_5\leq -53-6\sqrt{3}$ and $K_6\leq 5$, is strongly inverse positive on Ω_{11}^6.

Remark 4.6. Notice that, see [3, section 6.2], the previous operator is inverse positive on Ω_{12}^6. However, Theorem 4.6 does not allow us to ensure that it is strongly inverse positive on Ω_{12}^6. Since such theorem only provides sufficient conditions, this last assertion could be true or not. To answer the question we must obtain the expression of the Green's function and study its sign.

5. Particular cases

In this section, in order to obtain results about existence of positive solutions for some first, second and nth order periodic boundary value problems, we will take advantage of the previous calculations and of the arguments used in [4] for some nth order periodic operators.

In the following, we will denote $\Delta x(k)=x(k+1)-x(k)$ for $k\in I$, and $\Delta^l x(k)=\Delta(\Delta^{l-1}x(k))$ for $k\in I$ and $l\in\{2,\dots,n\}$.

Moreover, we rewrite condition (H_4) for a continuous function $f:I\times\mathbb{R}\to\mathbb{R}$ as follows:

(F_0) $\displaystyle\lim_{x\to\infty}\frac{f(k,x)}{x}=\infty$ for all $k\in I$.

5.1 First order Problems

At a first moment we consider the first order explicit problem, which appears in the Euler's discretization method of a first order differential equation:

$$\Delta x(k) = f(k, x(k)),\ k \in I; \qquad x(0) = x(N). \tag{7}$$

From Proposition 4.3, we know that operator $T_1[M-1, 0]\, x(k) \equiv \Delta\, x(k) + M\, x(k)$ is strongly inverse positive on Ω_N^1 if and only if $M \in (0, 1)$.

Now we arrive to the following existence result for problem (7).

THEOREM 5.1. *Suppose that condition (F_0) holds together with the following ones:*

(1) *There exists $M \in (0,1)$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.*
(2) *There exists $\bar{x} \in \Omega_N^1$, such that $\bar{x} \gg 0$ and $\Delta\bar{x}(k) \geq f(k, \bar{x}(k))$ for all $k \in I$.*

Then problem (7) has a positive solution.

Proof. Let $M \in (0,1)$ be given, we can rewrite problem (7) as

$$T_1[M-1, 0]\, x(k) = f(k, x(k)) + M\, x(k) \equiv g(k, x(k)),\ k \in I; \quad x(0) = x(N).$$

Since operator $T_1[M-1, 0]$ is strongly inverse positive on Ω_N^1, condition (H_0) is fulfilled. Clearly g satisfies (H_1), (H_2) and (H_4), and $\bar{x}$ is a positive upper solution. Due to Theorem 3.2 we can assure the existence of a positive solution for problem (7). □

Example 5.2 The following problem

$$\Delta x(k) = x^{k+2}(k) - \frac{k}{k+1}\sin\left(x(k)\right),\ k \in I = \{0,1,2\}; \quad x(0) = x(3),$$

has a positive solution.

Proof. This problem is a particular case of problem (7) taking $f(k,x) = x^{k+2} - k\,\sin(x)/(k+1)$ and $N = 3$.

So that $f(k,x)/x = x^{k+1} - k\sin(x)/((k+1)x)$, and it is clear that $\lim_{x\to\infty} f(k,x)/x = \infty$ for all $k \in I$.

Taking $M = 2/3$ we have that $f(k,x) + M\,x = x^{k+2} - k\sin(x)/(k+1) + 2x/3$, thus $f(k,x) + Mx$ is nonnegative and nondecreasing for all $k \in I$ and $x \in \mathbb{R}_+$.

Finally, we can verify that $\bar{x} = \{1/3, 4/9, 388/729 - \sin(4/9)/2, 1/3\}$ is a positive upper solution for this problem.

Theorem 5.1 guarantees the existence of a positive solution for this problem. □

Next we consider the implicit case,

$$\Delta x(k) = f(k, x(k+1)),\ k \in I; \qquad x(0) = x(N). \tag{8}$$

Proposition 4.3 provides us that operator $T_1[-1, M]\, x(k) \equiv \Delta\, x(k) + M\, x(k+1)$ is strongly inverse positive on Ω_N^1 if and only if $M > 0$.

As in the proof of the previous theorem, we can obtain the following existence result for problem (8).

THEOREM 5.3. *Suppose that condition (F_0) holds together with the following ones:*

(1) There exists $M > 0$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^1$, such that $\bar{x} \gg 0$ and $\Delta \bar{x}(k) \geq f(k, \bar{x}(k+1))$ for all $k \in I$.

Then problem (8) has a positive solution.

Example 5.4 The following problem

$$\Delta x(k) = x^2(k+1) - k\,x(k+1),\ k \in I = \{0,1,2\}; \quad x(0) = x(3),$$

has a positive solution.

Proof. This problem is a particular case of problem (8) taking $f(k,x) = x^2 - k\,x$ and $N = 3$.

So that $\lim_{x\to\infty} f(k,x)/x = \infty$ for all $k \in I$, and taking $M = 2$ we have that $f(k,x) + Mx$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.

We can check that $\bar{x} = \{1/4, 1/2, 1 - \sqrt{2}/2, 1/4\}$ is a positive upper solution for this problem.

The existence of a positive solution for this problem is a direct consequence of Theorem 5.3. □

Consider now the following problems:

$$-\Delta x(k) = f(k, x(k)),\ k \in I; \qquad x(0) = x(N) \tag{9}$$

and

$$-\Delta x(k) = f(k, x(k+1)),\ k \in I; \qquad x(0) = x(N). \tag{10}$$

By using Remark 4.2, we have that operator $T_1[M+1, -2] \equiv -\Delta x(k) + M\,x(k)$ is strongly inverse positive on Ω_N^1 if and only if $M > 0$, while, on the other hand, operator $T_1[1, M-2] \equiv -\Delta x(k) + M\,x(k+1)$ is strongly inverse positive on Ω_N^1 if and only if $M \in (0,1)$.

Therefore, as in the proof of Theorem 5.1, we can establish the following existence results.

Theorem 5.5. *Suppose that condition (F_0) holds together with the following ones:*

(1) There exists $M > 0$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^1$, such that $\bar{x} \gg 0$ and $-\Delta \bar{x}(k) \geq f(k, \bar{x}(k))$ for all $k \in I$.

Then problem (9) has a positive solution.

Theorem 5.6. *Suppose that condition (F_0) holds together with the following ones:*

(1) There exists $M \in (0,1)$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^1$, such that $\bar{x} \gg 0$ and $-\Delta \bar{x}(k) \geq f(k, \bar{x}(k+1))$ for all $k \in I$.

Then problem (10) has a positive solution.

5.2 Second order problems

We will pay attention now to the following second order problems:

$$\Delta^2 x(k) = f(k, x(k)),\ k \in I; \qquad x(i) = x(N+i),\, i \in \{0,1\}, \tag{11}$$

$$\Delta^2 x(k) = f(k, x(k+1)),\ k \in I; \qquad x(i) = x(N+i),\, i \in \{0,1\} \tag{12}$$

and

$$\Delta^2 x(k) = f(k, x(k+2)),\ k \in I; \qquad x(i) = x(N+i),\, i \in \{0,1\}. \tag{13}$$

Applying Proposition4.4, we know that the following properties hold:

(1) Operator $T_2[M_0+1,-2,0]\,x(k) \equiv \Delta^2\,x(k) + M_0\,x(k)$ is strongly inverse positive on Ω_N^2 if and only if $M_0 \in (0, \tan^2 \frac{\pi}{N})$.

(2) Operator $T_2[1, M_1-2, 0]\,x(k) \equiv \Delta^2\,x(k) + M_1\,x(k+1)$ is strongly inverse positive on Ω_N^2 if and only if $M_1 \in (0, 4\,\sin^2 \frac{\pi}{2\,N})$.

(3) Operator $T_2[1,-2,M_2]\,x(k) \equiv \Delta^2\,x(k) + M_2\,x(k+2)$ is strongly inverse positive on Ω_N^2 if and only if $M_2 \in (0, \tan^2 \frac{\pi}{N})$.

Thus, as in the proof of Theorem 5.1, we can obtain the following existence results.

THEOREM 5.7 . *Assume that condition (F_0) is fulfilled, and the following assertions are satisfied:*

(1) There exists $M \in (0, \tan^2 \frac{\pi}{N})$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $\Delta^2 \bar{x}(k) \geq f(k, \bar{x}(k))$ for all $k \in I$.

Then problem (11) has a positive solution.

THEOREM 5.8 . *Assume that condition (F_0) holds, and the following assertions are satisfied:*

(1) There exists $M \in (0, 4\,\sin^2 \frac{\pi}{2\,N})$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $\Delta^2 \bar{x}(k) \geq f(k, \bar{x}(k+1))$ for all $k \in I$.

Then problem (12) has a positive solution.

THEOREM 5.9 . *Suppose that condition (F_0) holds, and the following assertions are fulfilled:*

(1) There exists $M \in (0, \tan^2 \frac{\pi}{N})$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $\Delta^2 \bar{x}(k) \geq f(k, \bar{x}(k+2))$ for all $k \in I$.

Then problem (13) has a positive solution.

Example 5.10 The following problem

$$\Delta^2 x(k) = x(k+1)\log\left(x(k+1) + \frac{k+1}{k+2}\right), \; k \in \{0,1,2\}; \; x(0) = x(3), x(1) = x(4),$$

has a positive solution.

Proof. This problem is a particular case of problem (12) taking $N = 3$ and $f(k,x) = x\log(x + (k+1)/(k+2))$.

It is immediate to verify that condition (F_0) holds.

Taking $M = \log(2) < 1 = 4\sin^2\frac{\pi}{6}$, we have that $f(k,x) + Mx$ is nonnegative and nondecreasing for all $k \in I$ and $x \in \mathbb{R}_+$.

Moreover, we can verify that $\bar{x} = \{1/4, \ldots, 1/4\}$ is a positive upper solution for this problem.

Then, Theorem 5.8 ensures that this problem has a positive solution. □

When we consider the problems

$$-\Delta^2 x(k) = f(k, x(k)), \; k \in I; \qquad x(i) = x(N+i), \, i \in \{0,1\}, \tag{14}$$

$$-\Delta^2 x(k) = f(k, x(k+1)), \; k \in I; \qquad x(i) = x(N+i), \, i \in \{0,1\} \tag{15}$$

and

$$-\Delta^2 x(k) = f(k, x(k+2)), \; k \in I; \qquad x(i) = x(N+i), \, i \in \{0,1\}, \tag{16}$$

from Remark 4.3 we know that operators $-\Delta^2 x(k) + M\,x(k)$, $-\Delta^2 x(k) + M\,x(k+1)$ and $-\Delta^2 x(k) + M\,x(k+2)$ are strongly inverse positive on Ω_N^2 if and only if $M \in (0,1]$, $M > 0$ and $M \in (0,1)$ respectively. Due to this, we arrive at the following results

THEOREM 5.11. *Assume that condition (F_0) is fulfilled, and the following assertions are satisfied:*

(1) There exists $M \in (0,1]$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $-\Delta^2\bar{x}(k) \geq f(k,\bar{x}(k))$ for all $k \in I$.

Then problem (14) has a positive solution.

THEOREM 5.12. *Assume that condition (F_0) holds, and the following assertions are satisfied:*

(1) There exists $M > 0$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.
(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $-\Delta^2\bar{x}(k) \geq f(k,\bar{x}(k+1))$ for all $k \in I$.

Then problem (15) has a positive solution.

THEOREM 5.13. *Suppose that condition (F_0) holds, and the following assertions are fulfilled:*

(1) There exists $M \in (0,1)$ such that $f(k,x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$ and $x \geq 0$.

(2) There exists $\bar{x} \in \Omega_N^2$, such that $\bar{x} \gg 0$ and $-\Delta^2 \bar{x}(k) \geq f(k, \bar{x}(k+2))$ for all $k \in I$.

Then problem (16) has a positive solution.

5.3 nth order problems.

In this subsection, we will pay attention first to the following nth order problem

$$\Delta^n x(k) = f(k, x(k)),\ k \in I;\ x(i) = x(N+i),\ i \in \{0, \dots, n-1\}. \tag{17}$$

We can use the arguments given in [4, Lemma 2.3] and apply Theorem 4.6 in order to obtain the result below for operator $\Delta^n\, x(k) + M\, x(k)$.

LEMMA 5.14. *Let be $M > 0$, then operator $\Delta^n\, x(k) + M\, x(k)$ is strongly inverse positive on Ω_N^n provided that one of the following properties is fulfilled:*

(1) $n = 4p$, $p \in \{1, 2, \dots\}$ and $M < \left[\frac{\tan\frac{\pi}{N}}{\left(1+\tan\frac{\pi}{N}\right)\cos\frac{\pi}{n}}\right]^n$,
(2) $n = 2 + 4\,p$, $p \in \{0, 1, \dots\}$ and $M < \left[\frac{\tan\frac{\pi}{N}}{1+\tan\frac{\pi}{N}\cos\frac{\pi}{n}}\right]^n$,
(3) n odd and $M < \left[\frac{\tan\frac{\pi}{N}}{\tan\frac{\pi}{N}\cos\frac{2\pi}{n}+\cos\frac{\pi}{2\,n}}\right]^n$.

Thus, as in the proof of Theorem 5.1, by using Lemma 5.14, we can obtain the following existence result for problem (17).

THEOREM 5.15. *Let f be satisfying condition (F_0). If there exists $\bar{x} \in \Omega_N^n$, such that $\bar{x} \gg 0$ and $\Delta^n \bar{x}(k) \geq f(k, \bar{x}(k))$ for all $k \in I$, and one of the following properties is fulfilled:*

(1) $n = 4\,p$, $p \in \{1, 2, \dots\}$, $f(k, x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$, $x \geq 0$ and some $0 < M < \left[\frac{\tan\frac{\pi}{N}}{\left(1+\tan\frac{\pi}{N}\right)\cos\frac{\pi}{n}}\right]^n$.
(2) $n = 2 + 4\,p$, $p \in \{0, 1, \dots\}$, $f(k, x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$, $x \geq 0$ and some $0 < M < \left[\frac{\tan\frac{\pi}{N}}{1+\tan\frac{\pi}{N}\cos\frac{\pi}{n}}\right]^n$.
(3) n is odd, $f(k, x) + M\,x$ is nonnegative and nondecreasing for all $k \in I$, $x \geq 0$ and some $0 < M < \left[\frac{\tan\frac{\pi}{N}}{\tan\frac{\pi}{N}\cos\frac{2\pi}{n}+\cos\frac{\pi}{2\,n}}\right]^n$.

Then problem (17) has a positive solution.

Example 5.16 Given $\lambda > 0$, the following problem

$$\Delta^4 x(k) = \lambda\left(x^{k+3}(k) - \frac{x^{k+2}(k)}{k+2}\right),\ k \in I = \{0, \dots, 4\};\ x(i) = x(5+i), i \in \{0, \dots, 3\},$$

has a positive solution for all $0 < \lambda < 12\left[\frac{\tan\frac{\pi}{5}}{\left(1+\tan\frac{\pi}{5}\right)\cos\frac{\pi}{4}}\right]^4 \approx 1,505136$.

Proof. This problem is a particular case of problem (17) taking $n = 4$, $N = 5$ and $f(k, x) = \lambda\,(x^{k+3} - x^{k+2}/(k+2))$.

Obviously $\lim_{x\to\infty} f(k, x)/x = \infty$ for all $k \in I$.

On the other hand, it is clear that if

$$M \geq h(x) \equiv \lambda\,(-x^{k+2} + x^{k+1}/(k+2))$$

and

$$M \geq g(x) \equiv \lambda\,(-(k+3)x^{k+2} + x^{k+1}),$$

function $f(k,x) + Mx$ is nonnegative and nondecreasing in $\mathbb{R}_+$ for all $k \in I$.

We can check that, for all $k \in I$, function h reaches the maximum value in $\mathbb{R}_+$ when $x = (k+1)/(k+2)^2$, while function g reaches the maximum value in $\mathbb{R}_+$ when $x = (k+1)/((k+2)(k+3))$. So that, taking $M = \lambda/12$, we have that the two previous inequalities hold. Moreover, from our hypotheses, we have that

$$M = \frac{\lambda}{12} < \left[\frac{\tan\frac{\pi}{5}}{(1+\tan\frac{\pi}{5})\,\cos\frac{\pi}{4}}\right]^4 \approx 0.125428.$$

Finally, we can verify that $\bar{x} = \{1/6, \ldots, 1/6\}$ is a positive upper solution for this problem.

Then Theorem 5.15 guarantees that this problem has a positive solution for this range of λ. □

We will consider now the following problem

$$-\Delta^n x(k) = f(k, x(k)),\ k \in I;\ x(i) = x(N+i),\ i \in \{0, \ldots, n-1\}. \tag{18}$$

We know that the operator $-\Delta^n\, x(k) + M\, x(k)$ is strongly inverse positive if and only if the opposite operator $\Delta^n\, x(k) - M\, x(k)$ is strongly inverse negative. So that, we can use analogous arguments to the used in the proof of [4, Lemma 2.4] in order to obtain the following result as a consequence of Theorem 4.6

LEMMA 5.17. *Let be $M < 0$, then operator $-\Delta^n\, x(k) + M\, x(k)$ is strongly inverse positive on Ω_N^n provided that one of the following properties is fulfilled:*

(1) $n = 4p$, $p \in \{1, 2, \ldots\}$ *and* $M < \left[\frac{\tan\frac{\pi}{N}}{1+\tan\frac{\pi}{N}\,\cos\frac{2\pi}{n}}\right]^n$,

(2) $n = 2 + 4\,p$, $p \in \{0, 1, \ldots\}$ *and* $M < \left[\frac{\tan\frac{\pi}{N}}{\cos\frac{\pi}{n} + \tan\frac{\pi}{N}\,\cos\frac{2\pi}{n}}\right]^n$,

(3) n *odd and* $M < \left[\frac{\tan\frac{\pi}{N}}{\tan\frac{\pi}{N}\,\cos\frac{\pi}{n} + \cos\frac{\pi}{2\,n}}\right]^n$.

Hence, as in the proof of Theorem 5.1, by using Lemma 5.17, we can get the following existence result for problem (18).

THEOREM 5.18. *Let f be satisfying condition (F_0). If there exists $\bar{x} \in \Omega_N^n$, such that $\bar{x} \gg 0$ and $-\Delta^n \bar{x}(k) \geq f(k, \bar{x}(k))$ for all $k \in I$, and one of the following properties is fulfilled:*

(1) $n = 4\,p$, $p \in \{1, 2, \ldots\}$, $f(k,x) + M\,x$ *is nonnegative and nondecreasing for all* $k \in I$, $x \geq 0$ *and some* $0 < M < \left[\frac{\tan\frac{\pi}{N}}{1+\tan\frac{\pi}{N}\,\cos\frac{2\pi}{n}}\right]^n$.

(2) $n = 2 + 4\,p$, $p \in \{0, 1, \ldots\}$, $f(k,x) + M\,x$ *is nonnegative and nondecreasing for all* $k \in I$, $x \geq 0$ *and some* $0 < M < \left[\frac{\tan\frac{\pi}{N}}{\cos\frac{\pi}{n} + \tan\frac{\pi}{N}\,\cos\frac{2\pi}{n}}\right]^n$.

(3) n *is odd,* $f(k,x) + M\,x$ *is nonnegative and nondecreasing for all* $k \in I$, $x \geq 0$ *and some* $0 < M < \left[\frac{\tan\frac{\pi}{N}}{\tan\frac{\pi}{N}\,\cos\frac{\pi}{n} + \cos\frac{\pi}{2\,n}}\right]^n$.

Then problem (18) has a positive solution.

Example 5.19 Given $\lambda > 0$, the following problem

$$-\Delta^4 x(k) = \lambda \left(x^{k+3}(k) - \frac{x^{k+2}(k)}{k+2} \right), \; k \in I = \{0, \ldots, 4\}; \; x(i) = x(5{+}i), i \in \{0, \ldots, 3\},$$

has a positive solution for all $0 < \lambda < 12 \, \tan^4 \frac{\pi}{5} \approx 3.34369$.

Proof. This problem is a particular case of problem (18) taking $n = 4$, $N = 5$ and $f(k, x) = \lambda \, (x^{k+3} - x^{k+2}/(k+2))$. We are considering the same values for n, N and the same function as in the Example 5.16, so f satisfies (F_0) and we only have to check that

$$M = \frac{\lambda}{12} < \left[\frac{\tan \frac{\pi}{5}}{1 + \tan \frac{\pi}{5} \, \cos \frac{\pi}{2}} \right]^4 = \tan^4 \frac{\pi}{5} \approx 0.27864,$$

which is a direct consequence of our assumption.

Clearly $\bar{x} = \{1/6, \ldots, 1/6\}$ is also an upper solution for this problem thus, as a direct consequence of Theorem 5.18, this problem has a positive solution for such λ. □

6. Conclusions and Future directions

In this paper we have obtained some existence results for nth order difference equations. In this case we combine two types of techniques: the lower and upper solution method with the classical cone contraction/expansion fixed point theorem. In this situation we deduce the existence of a solution by assuming that there is at least one upper solution and by imposing some growth condition on the nonlinear part of the equation at infinity. However we are not able to give a more precise location of the solution as it is usual when a lower solution is obtained or some growth condition at 0 is imposed. The location of the solution is an interesting matter that can be studied in further developments of this theory.

As we have seen along the paper, the sign of the related Green's function is fundamental in our arguments. The study of the expression of the Green's function will allow to obtain better estimates on the values of the parameters $K_0, \ldots, K_n$ for which the sign is constant and, as consequence, to deduce more general existence results for the considered nonlinear problems.

Acknowledgements

This work has been partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2007-61724, and by Xunta de Galicia, Spain, project PGIDIT06PXIB207023PR.

References

[1] R. P. Agarwal, *Difference equations and inequalities. Theory, methods, and applications.* Second edition. Monographs and Textbooks in Pure and Applied Mathematics, 228. Marcel Dekker, Inc., New York, 2000.

[2] F.M. Atici and G.Sh. Guseinov, *Positive periodic solutions for nonlinear difference equations with periodic coefficients*, J. Math. Anal. Appl. 232 (1999), pp. 166–182.

[3] A. Cabada and V. Otero-Espinar, *Optimal existence results for nth order periodic boundary value difference problems*, J. Math. Anal. Appl. 247 (2000), pp. 67–86.

[4] A. Cabada and V. Otero-Espinar, *Comparison Results for n-th Order Periodic Difference Equations*, Nonlinear Anal. 47 (2001), pp. 2395–2406.

[5] A. Cabada and V. Otero-Espinar, *Fixed sign solutions of second-order difference equations with Neumann boundary conditions*, Advances in difference equations, IV. Comput. Math. Appl. 45 (2003), 6-9, 1125–1136.

[6] A. Cabada, *Extremal solutions for the difference ϕ-Laplacian problem with nonlinear functional boundary conditions*, Comput. Math. Appl. 42 (2001), pp. 593–601.

[7] A. Cabada and J.A. Cid, *Existence of a non-zero fixed point for nondecreasing operators via Krasnoselskii's fixed point theorem*, Nonlinear Anal. 71 (2009), pp. 2114–2118.

[8] H. Persson, *A fixed point theorem for monotone functions*, Appl. Math. Lett. 19 (2006), no. 11, pp. 1207–1209.

[9] P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via Krasnoselskii fixed point theorem*, J. Differ. Equations, 190 (2003), pp. 643–662.