A VARIATIONAL APPROACH TO PERTURBED IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

S. HEIDARKHANI, A. CABADA, G.A. AFROUZI, S. MORADI, AND G. CARISTI

Abstract. In this paper, perturbed systems of impulsive nonlinear fractional differential equations, including Lipschitz continuous nonlinear terms, are studied. The existence of at least three distinct weak solutions is obtained based on a recent three critical points theorem for differentiable functionals. In addition, examples are presented to illustrate the feasibility and effectiveness of the main results.

1. Introduction

In this paper we study the following perturbed impulsive fractional differential system

$(P_{\lambda,\mu}^{F,G})$

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0D_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + \mu G_{u_i}(t,u) + h_i(u_i), & t \in (0,T),\ t \neq t_j, \\ \Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), & j = 1,2,\dots,m, \\ u_i(0) = u_i(T) = 0 \end{cases}$$

for $1 \leq i \leq n$, where $u = (u_1, \dots, u_n)$, $0 < \alpha_i \leq 1$ for $1 \leq i \leq n$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $a_i \in L^\infty([0,T])$, $\bar{a}_i = ess\inf_{t\in[0,T]} a_i(t) > 0$ for $1 \leq i \leq n$, ${}_0D_t^\varsigma$ and ${}_tD_T^\varsigma$ denote the left and right Riemann-Liouville fractional derivatives of order ς, respectively, $n \geq 1$, $F, G : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are measurable with respect to t, for all $u \in \mathbb{R}^n$, continuously differentiable in u, for almost every $t \in [0,T]$, moreover $F(t,0,\dots,0) = G(t,0,\dots,0) = 0$ for every $t \in [0,T]$ and they satisfy the following standard summability condition:

$$\sup_{|\xi|\leq\varrho_1} (\max\{|F(.,\xi)|,\ |G(.,\xi)|,\ |F_{\xi_i}(.,\xi)|,\ |G_{\xi_i}(.,\xi)|,\ i = 1,\dots,n\}) \in L^1([0,T]) \tag{1.1}$$

for any $\varrho_1 > 0$ with $\xi = (\xi_1, \dots, \xi_n)$ and $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$, $h_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L_i > 0$, i.e.,

$$|h_i(\xi_1) - h_i(\xi_2)| \leq L_i|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}$, satisfying $h_i(0) = 0$ for $1 \leq i \leq n$, $I_{ij} \in C(\mathbb{R},\mathbb{R})$ for $i = 1,\dots,n$, $j = 1,\dots,m$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, the operator Δ is defined as $\Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u))(t_j) = {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^+) - {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^-)$ where ${}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^+) = \lim_{t\to t_j^+} {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t)$ and ${}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^-) = lim_{t\to t_j^-} {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t)$ and ${}_0^cD_t^{\alpha_i}$ is the left Caputo fractional derivatives of order

---

2000 *Mathematics Subject Classification.* 26A33; 34B15; 35A15; 34B15, 34K45, 58E05.

Key words and phrases. Three solutions; Fractional differential equation; Impulsive effect; Variational methods; Critical point theory.

α_i. Here, F_{u_i} and G_{u_i} denote respectively the partial derivatives of F and G with respect to u_i for $1 \leq i \leq n$.

Let a, b be real numbers and denote by $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$. For $0 < \alpha_i \leq 1$ for $1 \leq i \leq n$, $f \in AC([a,b])$ there are defined left and right Riemann-Liouville and Caputo fractional derivatives (see [3], pp. 69-93, [44]) by

$$ {}_aD_t^{\alpha_i} f(t) \equiv \frac{d}{dt} {}_aD_t^{\alpha_i-1} f(t) \equiv \frac{d}{dt} I_{a^+}^{1-\alpha_i} f(t) := \frac{1}{\Gamma(1-\alpha_i)} \frac{d}{dt} \Big(\int_a^t (t-s)^{-\alpha_i} f(s) ds \Big), $$

$$ {}_tD_b^{\alpha_i} f(t) \equiv -\frac{d}{dt} {}_tD_b^{\alpha_i-1} f(t) \equiv \frac{d}{dt} I_{b^-}^{1-\alpha_i} f(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \Big(\int_t^b (s-t)^{-\alpha_i} f(s) ds \Big), $$

$$ {}_a^cD_t^{\alpha_i} f(t) \equiv^c D_{a^+}^{\alpha_i} f(t) \equiv_a D_t^{\alpha_i-1} f'(t) := \frac{1}{\Gamma(1-\alpha_i)} \Big(\int_a^t (t-s)^{-\alpha_i} f'(s) ds \Big) $$

and

$$ {}_t^cD_b^{\alpha_i} f(t) \equiv^c D_{b^-}^{\alpha_i} f(t) \equiv -{}_tD_b^{\alpha_i-1} f'(t) := \frac{1}{\Gamma(1-\alpha_i)} \Big(\int_t^b (s-t)^{-\alpha_i} f'(s) ds \Big), $$

Note that when $\alpha_i = 1$, ${}_a^cD_t^1 f(t) = f'(t)$, ${}_t^cD_b^1 f(t) = -f'(t)$. The definition of the weak solutions of the system $(P_{\lambda,\mu}^{F,G})$ in given in Section 2.

Fractional differential equations (FDEs) have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, probability, electrical networks, etc, for details, see [1, 2, 3, 4, 5] and the references therein. Recently, the existence of solutions to boundary value problems for FDEs have been studied in many papers and we refer the reader to the papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the references therein. For example, in [8] using Krasnosel'skii fixed point theorem, the authors obtained the existence of positive solution for the FDEs. Kong in [13] under appropriate hypotheses based on variational methods, studied the existence of three solutions to the boundary value problem of the following form

$$ \begin{cases} \frac{d}{dt}\big(\frac{1}{2} {}_0D_t^{-\beta}(u'(t)) + \frac{1}{2} {}_tD_T^{-\beta}(u'(t))\big) + \lambda \nabla F(t, u(t)) = 0, & t \in [0,T], \\ u(0) = u(T) = 0 \end{cases} $$

where $T > 0$, $\lambda > 0$ is a parameter, $0 \leq \beta < 1$, and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function. Molica Bisci in [15] studied the existence of three weak solutions for a class of nonlocal fractional Laplacian problems depending on two real parameters via a recent abstract result by Ricceri. Galewski and Molica Bisci in [9] using variational methods, proved that a suitable class of one-dimensional fractional problems admits at least one non-trivial solution under an asymptotical behaviour of the nonlinear datum at zero, their problem was as the following

$$ \begin{cases} \frac{d}{dt}\big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\big) + f(t, u(t)) = 0, & t \in [0,T], \\ u(0) = u(T) = 0 \end{cases} $$

where $\alpha \in (\frac{1}{2}, 1]$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

The study of coupled systems involving fractional differential equations is quite important as such systems occur in various problems of applied nature, for instance, see [17, 18, 19, 20, 21, 22, 23, 24] and the references therein. For example, Bai and Fang in [18] using the Krasnoselskii's fixed-point theorem and the nonlinear alternative of Leray-Schauder theorem in a cone, studied the existence of a positive solution to singular coupled system of fractional order. Su in [19] based on Schauder fixed-point theorem, obtained an existence result for a boundary value problem for a coupled differential system of fractional order. Also, in [25] using variational methods and critical point theory the existence of one weak solution for the system $(P^{F,G}_{\lambda,\mu})$, in the case $\mu = 0$ and $I_{ij} \equiv 0$ for $i = 1, \dots, n,\ j = 1, \dots, m$ was studied.

On the other hand, impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The governing equations of such phenomena may be modeled as impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations as these equations provide a natural frame work for mathematical modelling of many real world phenomena, namely in the control theory, physics, chemistry, population dynamics, biotechnology, economics and medical fields. In the last few years, variational methods have been used to determine the existence of solutions for impulsive differential equations possessing variational structures under certain boundary conditions, see for instance [26, 27, 28] and the references therein. Due to the great development in the theory of fractional calculus and impulsive differential equations as well as having wide applications in several fields. Recently, the study of fractional differential equations with impulses has been studied by many authors using the variational methods, fixed-point theorems and critical point theory, see, for instance, [29, 30, 31, 32, 33, 35, 36, 37] and the references therein for detailed discussions. For example, Gao et al. in [31] using Schaefer fixed point theorem, established sufficient conditions for the existence and uniqueness of solutions for a class of impulsive integro-differential equations with nonlocal conditions involving the Caputo fractional derivative. In [36] based on fractional calculus and fixed point theorems the existence of PC-mild solutions for Cauchy problems and nonlocal problems for impulsive fractional evolution equations involving Caputo fractional derivative was discussed. By utilizing the theory of operators semigroup, probability density functions via impulsive conditions, a new concept on a PC-mild solution for the problem was introduced. In [32] existence results for the system $(P^{F,G}_{\lambda,\mu})$, in the case $\mu = 0$ under some algebraic conditions on the nonlinear part with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear and the impulsive terms were established. Moreover, by combining two algebraic conditions on the nonlinear term guaranteeing the existence of two weak solutions, applying the mountain pass theorem the existence of third weak solution for the system was ensured. In [33] using variational methods the existence of infinitely many solutions for the system $(P^{F,G}_{\lambda,\mu})$ was discussed. In [34] under suitable hypotheses and by using variational methods, some new criteria to guarantee that the system $(P^{F,G}_{\lambda,\mu})$, in the case $\mu = 0$ has at least two nontrivial and nonnegative solutions are obtained.

In this paper, a class of nonlinear impulsive fractional differential systems including Lipschitz continuous nonlinear terms is studied. Under suitable hypotheses and by using variational methods, some new criteria to guarantee that the fractional differential system has at least two nontrivial and nonnegative solutions are obtained. In addition, an example is presented to illustrate the applicability of the main results was discussed.

In this paper, we establish the existence of at least three weak solutions for the system $(P^{F,G}_{\lambda,\mu})$, in which two parameters are involved. Precise estimates of these two parameters λ and μ will be given. Moreover, in our new results, we do not need any asymptotic conditions of the nonlinear term at infinity. The proof is based on a three critical points theorem due to Bonanno and Candito proved in [38] which we recall in the next section (Theorem 2.1). Our main result is Theorem 3.1. Theorem 3.2 is deduced as a consequence of it. Example 3.1 illustrates Theorem 3.2. As special cases of Theorems 3.1 and 3.2, we obtain Theorems 3.3 and 3.4 for the scalar situation ($n = 1$). Example 3.2 illustrates Theorem 3.4. In Theorem 3.5, we obtain the existence of at least two positive solutions under suitable conditions on the nonlinear term at zero and at infinity, while finally in Theorem 3.6 we ensure the existence of at least four non-negative solutions.

For more studies on the subject we refer to [39, 40, 41].

The present paper is organized as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple weak solutions for the eigenvalue system $(P^{F,G}_{\lambda,\mu})$.

2. Preliminaries

Our main tool is a three critical point theorem due to Bonanno and Candito that we recall here.

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 2.1. [38, Theorem 3.3] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* where X^* is the dual space of X, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

- (a_1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
- (a_2) *for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1 - s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

- (a_3) $\varphi(r_1) < \beta(r_1, r_2)$;
- (a_4) $\varphi(r_2) < \beta(r_1, r_2)$;
- (a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1),\ u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [13, 42, 43] in which Theorem 2.1 has been successfully employed to the existence of at least three solutions for boundary value problems.

We now introduce some necessary definitions and properties of the fractional calculus which are used further in this paper.

Let $C_0^\infty([0,T],\mathbb{R}^N)$ be the set of all functions $x \in C^\infty([0,T],\mathbb{R}^N)$ with $x(0) = x(T) = 0$ and the norm

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Denote the norm of the space $L^p([0,T],\mathbb{R}^N)$ for $1 \le p < \infty$ by

$$\|x\|_{L^p} = \Big(\int_0^T |x(s)|^p ds\Big)^{\frac{1}{p}}.$$

The following lemma yields the boundedness of the Riemann-Liouville fractional integral operators from the space $L^p([0,T],\mathbb{R}^N)$ to the space $L^p([0,T],\mathbb{R}^N)$ where $1 \le p < \infty$.

Definition 2.1. [3] Let f be a function defined on $[0,T]$ and $\alpha_i > 0$ for $1 \le i \le n$. The left and right Riemann-Liouville fractional integrals of order α_i for the function f are defined by

$$\begin{aligned} {}_0D_t^{-\alpha_i} f(t) &= \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} f(s) ds, \quad t \in [0,T], \\ {}_tD_T^{-\alpha_i} f(t) &= \frac{1}{\Gamma(\alpha_i)} \int_t^T (s-t)^{\alpha_i - 1} f(s) ds, \quad t \in [0,T] \end{aligned}$$

for $1 \le i \le n$, provided the right-hand sides are pointwise defined on $[0,T]$, where $\Gamma(\alpha_i)$ is the standard gamma function given by

$$\Gamma(\alpha_i) = \int_0^{+\infty} z^{\alpha_i - 1} e^{-z} dz.$$

Lemma 2.2. [44] *Let* $0 < \alpha_i \le 1$ *for* $1 \le i \le n$, $1 \le p < \infty$ *and* $f \in L^p([0,T],\mathbb{R}^N)$. *Then*

$$\|{}_0D_\xi^{-\alpha_i} f\|_{L^p([0,t])} \le \frac{t^{\alpha_i}}{\Gamma(\alpha_i + 1)} \|f\|_{L^p([0,t])} \text{ for } \xi \in [0,t],\ t \in [0,T],$$

where ${}_0D_t^{-\alpha_i}$ *is left Riemann-Liouville fractional integral of order* α_i.

Proposition 2.3. [45] *We have the following property of fractional integration:*

$$\int_0^T [{}_0D_t^{-\zeta} u(t)] v(t) dt = \int_0^T [{}_tD_T^{-\zeta} v(t)] u(t) dt, \quad \zeta > 0,$$

provided that $u \in L^p([0,T],\mathbb{R})$, $v \in L^q([0,T],\mathbb{R})$ *and* $p \ge 1$, $q \ge 1$, $\frac{1}{p} + \frac{1}{q} \le 1 + \zeta$ *or* $p \ne 1$, $q \ne 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \zeta$.

Definition 2.2. Let $0 < \alpha_i \le 1$, $1 \le i \le n$. The fractional derivative space $E_0^{\alpha_i}$ is defined by the closure $C_0^\infty([0,T],\mathbb{R})$, that is

$$E_0^{\alpha_i} = \overline{C_0^\infty([0,T],\mathbb{R})}$$

with respect to the weighted norm

$$\|u_i\|_{a_i,\alpha_i} = \Big(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u_i(t)|^2 dt + \int_0^T |u_i(t)|^2 dt \Big)^{\frac{1}{2}} \tag{2.1}$$

for every $u_i \in E_0^{\alpha_i}$ and for $1 \leq i \leq n$.

Remark 2.1. [44, Remark 3.1] It is obvious that the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_i \in L^2([0,T],\mathbb{R})$ having an α_i-order Riemann-Loiuville fractional derivative ${}_0D_t^{\alpha_i}u_i \in L^2([0,T],\mathbb{R})$ and $u_i(0) = u_i(T) = 0$ for $1 \leq i \leq n$.

Proposition 2.4. [44, Propostion 3.1] *Let $0 < \alpha_i \leq 1$, $1 \leq i \leq n$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha_i}$ is a reflexive and separable Banach space.*

Lemma 2.5. [23] *Let $0 < \alpha_i \leq 1$, $1 \leq i \leq n$. For any $u_i \in E_0^{\alpha_i}$, we have*

$$\|u_i\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)\sqrt{\bar{a}_i}}\Big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{\frac{1}{2}}, \tag{2.2}$$

moreover, if $\alpha_i > \frac{1}{2}$, then

$$\|u_i\|_{\infty} \leq \frac{T^{\alpha_i-\frac{1}{2}}}{\Gamma(\alpha_i)\sqrt{\bar{a}_i(2\alpha_i-1)}}\Big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{\frac{1}{2}}. \tag{2.3}$$

By (2.2) we can consider $E_0^{\alpha_i}$ with respect to the norm

$$\|u_i\|_{\alpha_i} = \Big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{\frac{1}{2}} \quad \textit{for every } u_i \in E_0^{\alpha_i}, \quad 1 \leq i \leq n,$$

which is equivalent to (2.1).

Then we have

$$\sum_{i=1}^{n}\|u_i\|_{L^2}^2 \leq S\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 \tag{2.4}$$

and if $\alpha_i > \frac{1}{2}$, then

$$\sum_{i=1}^{n}\|u_i\|_{\infty}^2 \leq M\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 \tag{2.5}$$

with

$$S = \max\{\frac{T^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2\bar{a}_i}, 1 \leq i \leq n\}$$

and

$$M = \max\{\frac{T^{2\alpha_i-1}}{(\Gamma(\alpha_i))^2\bar{a}_i(2\alpha_i-1)}, 1 \leq i \leq n\}.$$

Now, let E be the Cartesian product of n Sobolev spaces $E_0^{\alpha_1}$, …, and $E_0^{\alpha_n}$, i.e., $E = E_0^{\alpha_1} \times \cdots \times E_0^{\alpha_n}$, which is a reflexive Banach space endowed with the norm

$$\|(u_1,\ldots,u_n)\| = \sum_{i=1}^{n}\|u_i\|_{\alpha_i}.$$

Obviously, E is compactly embedded in $(C^0([0,T]) \times \cdots \times C^0([0,T]))$.

Definition 2.3. We mean by a (weak) solution of the system $(P^{F,G}_{\lambda,\mu})$, any function $u=(u_1,\dots,u_n)\in E$ such that

$$\begin{aligned}&\sum_{i=1}^{n}\Big(\int_0^T a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\,{}_0D_t^{\alpha_i}v_i(t)dt\Big)-\sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)dt\\&+\sum_{j=1}^{m}\sum_{i=1}^{n}a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j)-\lambda\sum_{i=1}^{n}\int_0^T F_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)dt\\&-\mu\sum_{i=1}^{n}\int_0^T G_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)dt=0 \qquad (2.6)\end{aligned}$$

for every $v=(v_1,\dots,v_n)\in E$.

The derivation details of the weak solution for the system $(P^{F,G}_{\lambda,\mu})$ are the same with the scale case as given in [30].

Put

$$H_i(x)=\int_0^x h_i(\xi)d\xi \ \text{ for all } x\in\mathbb{R} \text{ and } \ 1\le i\le n.$$

In this paper we assume throughout and without further mention, that the following conditions hold:

(H_1) $\frac{1}{2}<\alpha_i\le 1$ for $1\le i\le n$;

(H_2) $I_{ij}(0)=0$ and there exists a constant $L_{ij}>0$ such that

$$|I_{ij}(s_1)-I_{ij}(s_2)|\le L_{ij}|s_1-s_2| \text{ for any } s_1,s_2\in\mathbb{R},\ 1\le i\le n,\ 1\le j\le m;$$

(H_3) $\sum_{i=1}^{n}\frac{L_iT^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2\bar{a}_i}+MCm\|\tilde{a}\|_\infty<1$ where $C=\max_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}\{L_{ij}\}$ and $\tilde{a}=\max\{a_i(t),t\in[0,T],1\le i\le n\}$.

Put

$$\sigma=\min\{1-\frac{L_iT^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2\bar{a}_i},1\le i\le n\},$$

$$\rho=\max\{1+\frac{L_iT^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2\bar{a}_i},1\le i\le n\},$$

$$\varrho_1=\sigma-MCm\|\tilde{a}\|_\infty \text{ and } \varrho_2=\rho+MCm\|\tilde{a}\|_\infty.$$

Proposition 2.6. *Let $J:E\to E^*$ be the operator defined for every $u=(u_1,\dots,u_n),v=(v_1,\dots,v_n)\in E$, as*

$$\begin{aligned}J(u)(v)=&\sum_{i=1}^{n}\Big(\int_0^T a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\,{}_0D_t^{\alpha_i}v_i(t)dt\Big)+\sum_{i=1}^{n}\sum_{j=1}^{m}a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j)\\&-\sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)dt.\end{aligned}$$

Then, J admits a continuous inverse on E^*.

Proof. Since $h_i(0) = 0$, one has $|h_i(x)| \leq L_i|x|$ for $i = 1, \dots, n$. So, using Lemma 2.5 we have

$$\sum_{i=1}^{n} \Big(\int_0^T a_i(t)|_0D_t^{\alpha_i} u_i(t)|^2 dt \Big) - \sum_{i=1}^{n} \int_0^T h_i(u_i(t))u_i(t)dt$$

$$\geq \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^{n} \frac{L_i T^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2 \bar{a}_i} \|u_i\|_{\alpha_i}^2 \geq \sigma \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2.$$

Moreover, from the condition (H_2) we obtain

$$-MCm\|\tilde{a}\|_\infty \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 \leq -M \sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}\|a_i\|_\infty \|u_i\|_{\alpha_i}^2$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j) I_{ij}(u_i(t_j))u_i(t_j) \leq M \sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}\|a_i\|_\infty \|u_i\|_{\alpha_i}^2$$

$$\leq MCm\|\tilde{a}\|_\infty \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2.$$

Therefore, we have

$$\begin{aligned} J(u)(u) =& \sum_{i=1}^{n} \Big(\int_0^T a_i(t)|_0D_t^{\alpha_i} u_i(t)|^2 dt \Big) + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j) I_{ij}(u_i(t_j))u_i(t_j) \\ & - \sum_{i=1}^{n} \int_0^T h_i(u_i(t))u_i(t)dt \geq \varrho_1 \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2, \end{aligned}$$

and taking (H_3) into account, it follows that J is coercive. By using (H_2) again, we have

$$\begin{aligned}
<J(u)-J(v),u-v> \quad &= \quad \sum_{i=1}^{n}\Big(\int_0^T a_i(t){}_0D_t^{\alpha_i}u_i(t){}_0D_t^{\alpha_i}(u_i(t)-v_i(t))dt\Big)\\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)I_{ij}(u_i(t_j))(u_i(t_j)-v_i(t_j))\\
&- \quad \sum_{i=1}^{n}\int_0^T h_i(u_i(t))(u_i(t)-v_i(t))dt\\
&- \quad \Bigg(\sum_{i=1}^{n}\Big(\int_0^T a_i(t){}_0D_t^{\alpha_i}v_i(t){}_0D_t^{\alpha_i}(u_i(t)-v_i(t))dt\Big)\\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)I_{ij}(v_i(t_j))(u_i(t_j)-v_i(t_j))\\
&- \quad \sum_{i=1}^{n}\int_0^T h_i(v_i(t))(u_i(t)-v_i(t))dt\Bigg)\\
&= \quad \sum_{i=1}^{n}\Big(\int_0^T a_i(t){}_0D_t^{\alpha_i}(u_i(t)-v_i(t)){}_0D_t^{\alpha_i}(u_i(t)-v_i(t))dt\Big)\\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\big(I_{ij}(u_i(t_j))-I_{ij}(v_i(t_j))\big)(u_i(t_j)-v_i(t_j))\\
&- \quad \sum_{i=1}^{n}\int_0^T (h_i(u_i(t))-h_i(v_i(t)))(u_i(t)-v_i(t))dt\\
&\geq \quad \varrho_1\sum_{i=1}^{n}\|u_i-v_i\|_{\alpha_i}^2.
\end{aligned}$$

Thus, J is uniformly monotone. Hence, by [46, Theorem 26.A (d)], $J^{-1}: E^* \to E$ exists and is continuous. ☐

3. MAIN RESULTS

In this section, we formulate our main results on the existence of at least three weak solutions for the system $(P_{\lambda,\mu}^{F,G})$.

For any $\varsigma > 0$ we denote by $Q(\varsigma)$ the set $\big\{(t_1,\dots,t_n)\in\mathbb{R}^n : \sum_{i=1}^{n}|t_i|^2 \leq \varsigma\big\}$.

For positive constants θ and η set

$$G^{\theta} := \int_0^T \max_{(x_1,\dots,x_n)\in Q(\theta)} G(t,x_1,\dots,x_n)dt$$

and

$$G_{\eta} := \inf_{[0,T]\times[0,\Gamma(2-\alpha_1)\eta]\times\cdots\times[0,\Gamma(2-\alpha_n)\eta]} G(t,x_1,\dots,x_n).$$

In the remainder of this article, for positive constant θ and η, let Θ and $\bar{\eta}$ be the vectors in $\mathbb{R}^N$ defined by

$$\Theta = (\sqrt{\theta},\dots,\sqrt{\theta}) \text{ and } \bar{\eta} = (\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta),$$

respectively.

Set

$$\begin{aligned}P_i(\alpha_i,\gamma) =&\frac{1}{2\gamma^2T^2}\Bigg\{\int_0^T a_i(t)t^{2(1-\alpha_i)}dt+\int_{\gamma T}^T a_i(t)(t-\gamma T)^{2(1-\alpha_i)}dt\\ &+\int_{(1-\gamma)T}^T a_i(t)(t-(1-\gamma)T)^{2(1-\alpha_i)}dt-2\int_{\gamma T}^T a_i(t)(t^2-\gamma Tt)^{1-\alpha_i}dt\\ &-2\int_{(1-\gamma)T}^T a_i(t)(t^2-(1-\gamma)Tt)^{1-\alpha_i}dt\\ &+2\int_{(1-\gamma)T}^T a_i(t)(t^2-\gamma Tt+\gamma(1-\gamma)T^2)^{1-\alpha_i}dt\Bigg\}\end{aligned}$$

for $0<\gamma<\frac{1}{2}$,

$$K_1=\max\{P_i(\alpha_i,\gamma), 1\le i\le n\}$$

and

$$K_2=\min\{P_i(\alpha_i,\gamma), 1\le i\le n\}.$$

Fixing four positive constants θ_1, θ_2, θ_3 and η, put

(3.1)

$$\delta_{\lambda,G}:=\min\Bigg\{\frac{1}{2M}\min\Big\{\frac{\varrho_1\theta_1-2M\lambda\displaystyle\int_0^T F(t,\Theta_1)dt}{G^{\theta_1}},\frac{\varrho_1\theta_2-2M\lambda\displaystyle\int_0^T F(t,\Theta_2)dt}{G^{\theta_2}},$$

$$\frac{\varrho_1(\theta_3-\theta_2)-2M\lambda\displaystyle\int_0^T F(t,\Theta_3)dt}{G^{\theta_3}}\Big\},$$

$$\frac{K_1n\varrho_2\eta^2-\lambda\big(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt-\int_0^T F(t,\Theta_1)dt\big)}{TG_\eta-G^{\theta_1}}\Bigg\}$$

for $0<\gamma<\frac{1}{2}$.

Theorem 3.1. *Let $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be non-negative. Assume that there exist positive constants $\gamma<\frac{1}{2}$, θ_1,θ_2, θ_3 and η with $\theta_1<2MK_2n\eta^2$ and $\frac{2MK_1n\varrho_2}{\varrho_1}\eta^2<\theta_2<\theta_3$ such that*

(A$_1$)

$$\max\Big\{\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1},\frac{\int_0^T F(t,\Theta_2)dt}{\theta_2},\frac{\int_0^T F(t,\Theta_3)dt}{\theta_3-\theta_2}\Big\}$$
$$<\frac{\varrho_1}{2M}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt-\int_0^T F(t,\Theta_1)dt}{K_1n\varrho_2\eta^2}.$$

Then, for every

$$\lambda\in\Lambda:=\Big]\frac{K_1n\varrho_2\eta^2}{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt-\int_0^T F(t,\Theta_1)dt},$$

$$\frac{\varrho_1}{2M}\min\Big\{\frac{\theta_1}{\int_0^T F(t,\Theta_1)dt},\frac{\theta_2}{\int_0^T F(t,\Theta_2)dt},\frac{\theta_3-\theta_2}{\int_0^T F(t,\Theta_3)dt}\Big\}\Big[$$

and every non-negative function $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfying* $G^\theta \geq 0$, *there exists* $\delta_{\lambda,G} > 0$ *given by* (3.1) *such that, for each* $\mu \in [0, \delta_{\lambda,G}[$, *the system* $(P^{F,G}_{\lambda,\mu})$ *has at least three solutions* u_1, u_2, *and* u_3 *such that* $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, *and* $\max_{t\in[0,T]} |u_3(t)| < \theta_3$.

Proof. Our aim is to apply Theorem 2.1 to the system $(P^{F,G}_{\lambda,\mu})$. We take $X = E$ and introduce the functionals Φ, Ψ for $u = (u_1, ..., u_n) \in X$, as follows

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n} \|u_i\|^2_{\alpha_i} - \sum_{i=1}^{n}\int_0^T H_i(u_i(t))dt + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds \tag{3.2}$$

and

$$\Psi(u) = \int_0^T F(t, u(t))dt + \frac{\mu}{\lambda}\int_0^T G(t, u(t))dt, \tag{3.3}$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for $u \in X$. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. We easily observe that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Since X is compactly embedded in $(C^0([0, T]), \mathbb{R})^n$, it is well known that Ψ is a differentiable functional whose differential at the point $u = (u_1, \ldots, u_n) \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^{n} F_{u_i}(t, u(t))v_i(t)dt + \frac{\mu}{\lambda}\int_0^T \sum_{i=1}^{n} G_{u_i}(t, u(t))v_i(t)dt$$

for every $v = (v_1, \ldots, v_n) \in X$, and Ψ is sequentially weakly upper semicontinuous. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \sum_{i=1}^{n}\Big(\int_0^T a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\,{}_0D_t^{\alpha_i}v_i(t)dt\Big) + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j)$$

$$-\sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)dt$$

for every $v = (v_1, \ldots, v_n) \in X$. Furthermore, we can show that Φ defined by (3.2) is sequentially weakly lower semicontinuous. Indeed, taking the sequentially weakly lower semicontinuity property of the norm into account and since H_i is continuous for $i = 1, \ldots, n$, it is enough to prove that $\sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds$ is weakly continuous in X. In fact, for $\{u_k = (u_{1k}, \ldots, u_{nk})\} \subset X$, if $\{u_k\}$ converges to u in X, then, there exists $S_1 > 0$ such that $\|u_k\|_\infty \leq S_1$. Therefore, we have

$$\Big|\sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_{ik}(t_j)} I_{ij}(s)ds - \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds\Big|$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\Big|a_i(t_j)\int_{u_i(t_j)}^{u_{ik}(t_j)} I_{ij}(s)ds\Big| \leq S_2 mn\|\tilde{a}\|_\infty\|u_k - u\|_\infty \to 0$$

where $S_2 = \max_{i\in\{1,\ldots,n\}, j\in\{1,\ldots,m\}, |s|\leq S_1} |I_{ij}(s)|$. So, we have $|\Phi(u_k) - \Phi(u)| \to 0$, thus Φ is weakly continuous. Hence, Φ is sequentially weakly lower semicontinuous in X. Since $h_i(0) = 0$ one has $|h_i(x)| \leq L_i|x|$ for $i = 1, \dots, n$. So, from (3.2) and the condition (H_2) we have

(3.4)

$$\begin{aligned}
&\frac{\varrho_1}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 \leq \frac{\sigma}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 - \frac{M}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}\|a_i\|_\infty\|u_i\|_{\alpha_i}^2\\
&\leq \frac{1}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^{n}\frac{L_iT^{2\alpha_i}}{2(\Gamma(\alpha_i+1))^2\bar{a}_i}\|u_i\|_{\alpha_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds\\
&\leq \Phi(u) \leq \frac{1}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 + \sum_{i=1}^{n}\frac{L_iT^{2\alpha_i}}{2(\Gamma(\alpha_i+1))^2\bar{a}_i}\|u_i\|_{\alpha_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds\\
&\leq \frac{\rho}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 + \frac{M}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}\|a_i\|_\infty\|u_i\|_{\alpha_i}^2 \leq \frac{\varrho_2}{2}\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2
\end{aligned}$$

and bearing the condition (H_3) in mind, it follows $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, namely Φ is coercive. Moreover, by Proposition 2.6, $(\Phi')^{-1} : X^* \to X$ exists and is continuous. For $0 < \gamma < \frac{1}{2}$, define $\omega = (\omega_1, \dots, \omega_n)$ by setting

$$\omega_i(t) = \begin{cases} \frac{\Gamma(2-\alpha_i)\eta}{\gamma T}t, & t \in [0, \gamma T[,\\ \Gamma(2-\alpha_i)\eta, & t \in [\gamma T, (1-\gamma)T],\\ \frac{\Gamma(2-\alpha_i)\eta}{\gamma T}(T-t), & t \in](1-\gamma)T, T] \end{cases}$$

for $1 \leq i \leq n$. Clearly $\omega_i(0) = \omega_i(T) = 0$ and $\omega_i \in L^2[0,T]$ for $1 \leq i \leq n$. A direct calculation shows that

$${}_0D_t^{\alpha_i}\omega_i(t) = \begin{cases} \frac{\eta}{\gamma T}t^{1-\alpha_i}, & t \in [0, \gamma T[,\\ \frac{\eta}{\gamma T}(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}), & t \in [\gamma T, (1-\gamma)T],\\ \frac{\eta}{\gamma T}(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i} - (t-(1-\gamma)T)^{1-\alpha_i}), & t \in](1-\gamma)T, T] \end{cases}$$

for $1 \leq i \leq n$. Furthermore,

$$\begin{aligned}
&\int_0^T a_i(t)|{}_0D_t^{\alpha_i}\omega_i(t)|^2dt = \int_0^{\gamma T} a_i(t)|{}_0D_t^{\alpha_i}\omega_i(t)|^2dt + \int_{\gamma T}^{(1-\gamma)T} a_i(t)|{}_0D_t^{\alpha_i}\omega_i(t)|^2dt\\
&+\int_{(1-\gamma)T}^{T} a_i(t)|{}_0D_t^{\alpha_i}\omega_i(t)|^2dt = \frac{\eta^2}{\gamma^2T^2}\Big\{\int_0^T a_i(t)t^{2(1-\alpha_i)}dt\\
&+\int_{\gamma T}^{T} a_i(t)(t-\gamma T)^{2(1-\alpha_i)}dt + \int_{(1-\gamma)T}^{T} a_i(t)(t-(1-\gamma)T)^{2(1-\alpha_i)}dt\\
&-2\int_{\gamma T}^{T} a_i(t)(t^2-\gamma Tt)^{1-\alpha_i}dt - 2\int_{(1-\gamma)T}^{T} a_i(t)(t^2-(1-\gamma)Tt)^{1-\alpha_i}dt\\
&+2\int_{(1-\gamma)T}^{T} a_i(t)(t^2-\gamma Tt+\gamma(1-\gamma)T^2)^{1-\alpha_i}dt\Big\} = 2P_i(\alpha_i,\gamma)\eta^2
\end{aligned}$$

for $1 \leq i \leq n$. Thus, $\omega \in X$ and

$$\|\omega_i\|_{\alpha_i}^2 = \int_0^T a_i(t)|{}_0D_t^{\alpha_i}\omega_i(t)|^2dt = 2P(\alpha_i,\gamma)\eta^2 \tag{3.5}$$

for $1 \leq i \leq n$. By using (3.4) and (3.5) we have

$$K_2n\varrho_1\eta^2 \leq \Phi(\omega) \leq K_1n\varrho_2\eta^2.$$

Choose $r_1 = \frac{\varrho_1}{2M}\theta_1$, $r_2 = \frac{\varrho_1}{2M}\theta_2$ and $r_3 = \frac{\varrho_1}{2M}(\theta_3 - \theta_2)$. From the conditions $\theta_3 > \theta_2$, $\theta_1 < 2MK_2n\eta^2$ and $\frac{2MK_1n\varrho_2}{\varrho_1}\eta^2 < \theta_2$ we achieve $r_3 > 0$ and $r_1 < \Phi(\omega) < r_2$. From the definition of Φ and considering equations (2.3), (2.5) and (3.4) one has

$$\begin{aligned}
\Phi^{-1}(-\infty, r_1) &= \{u \in X; \Phi(u) < r_1\} \subseteq \left\{u \in X; \sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 \leq \frac{2r_1}{\varrho_1}\right\} \\
&\subseteq \left\{u \in X; \frac{1}{M}\sum_{i=1}^{n} \|u_i\|_{\infty}^2 \leq \frac{2r_1}{\varrho_1}\right\} \\
&= \left\{u \in X; \sum_{i=1}^{n} \|u_i\|_{\infty}^2 \leq \frac{2Mr_1}{\varrho_1}\right\} \\
&= \left\{u \in X; \sum_{i=1}^{n} \|u_i\|_{\infty}^2 \leq \theta_1\right\}.
\end{aligned}$$

Hence, since F is non-negative, one has

$$\begin{aligned}
\sup_{u \in \Phi^{-1}(-\infty, r_1)} \int_0^T F(t, u(t))dt &\leq \int_0^T \max_{(x_1,\ldots,x_n) \in Q(\theta_1)} F(t, x_1, \ldots, x_n)dt \\
&\leq \int_0^T F(t, \Theta_1)dt.
\end{aligned}$$

In a similar way, we have

$$\sup_{u \in \Phi^{-1}(-\infty, r_2)} \int_0^T F(t, u(t))dt \leq \int_0^T F(t, \Theta_2)dt$$

and

$$\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \int_0^T F(t, u(t))dt \leq \int_0^T F(t, \Theta_3)dt.$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r_1) &= \inf_{u \in \Phi^{-1}(-\infty, r_1)} \frac{(\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)) - \Psi(u)}{r_1 - \Phi(u)} \\
&\leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} \\
&= \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \int_0^T [F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t))]dt}{r_1} \\
&\leq \frac{2M}{\varrho_1} \frac{\int_0^T F(t, \Theta_1)dt + \frac{\mu}{\lambda} G^{\theta_1}}{\theta_1}, \qquad (3.6)
\end{aligned}$$

$$\varphi(r_2) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u)}{r_2} = \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_0^T [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt}{r_2}$$

$$\le \frac{2M}{\varrho_1}\frac{\int_0^T F(t,\Theta_2)dt + \frac{\mu}{\lambda}G^{\theta_2}}{\theta_2} \tag{3.7}$$

and

$$\gamma(r_2,r_3) = \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3}$$

$$= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \int_0^T [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt}{r_3}$$

$$\le \frac{2M}{\varrho_1}\frac{\int_0^T F(t,\Theta_3)dt + \frac{\mu}{\lambda}G^{\theta_3}}{\theta_3-\theta_2}. \tag{3.8}$$

On the other hand, for each $u \in \Phi^{-1}(-\infty, r_1)$ one has

$$\beta(r_1,r_2) \ge \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt - \int_0^T F(t,\Theta_1)dt + \frac{\mu}{\lambda}(TG_\eta - G^{\theta_1})}{\Phi(w)-\Phi(u)}$$

$$\ge \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt - \int_0^T F(t,\Theta_1)dt + \frac{\mu}{\lambda}(TG_\eta - G^{\theta_1})}{K_1 n\varrho_2\eta^2}. \tag{3.9}$$

Since $\mu < \delta_{\lambda,G}$, one has

$$\mu < \frac{1}{2M}\frac{\varrho_1\theta_1 - 2M\lambda\int_0^T F(t,\Theta_1)dt}{G^{\theta_1}},$$

this means

$$\frac{\frac{\mu}{\lambda}G^{\theta_1} + \int_0^T F(t,\Theta_1)dt}{\frac{\varrho_1}{2M}\theta_1} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{K_1 n\varrho_2\eta^2 - \lambda\big(\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt - \int_0^T F(t,\Theta_1)dt\big)}{TG_\eta - G^{\theta_1}},$$

this means

$$\frac{\frac{\mu}{\lambda}(TG_\eta - G^{\theta_1}) + \big(\int_{\gamma T}^{(1-\gamma)T} F(t,\bar\eta)dt - \int_0^T F(t,\Theta_1)dt\big)}{K_1 n\varrho_2\eta^2} > \frac{1}{\lambda}.$$

Then,

$$\begin{aligned}\frac{\frac{\mu}{\lambda}G^{\theta_1}+\displaystyle\int_0^T F(t,\Theta_1)dt}{\frac{\varrho_1}{2M}\theta_1} &< \frac{1}{\lambda}\\ &< \frac{\frac{\mu}{\lambda}(TG_\eta - G^{\theta_1}) + \big(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\big)}{K_1 n\varrho_2\eta^2}.\end{aligned}\tag{3.10}$$

In a similar way, we have

$$\begin{aligned}\frac{\frac{\mu}{\lambda}G^{\theta_2}+\displaystyle\int_0^T F(t,\Theta_2)dt}{\frac{\varrho_1}{2M}\theta_2} &< \frac{1}{\lambda}\\ &< \frac{\frac{\mu}{\lambda}(TG_\eta - G^{\theta_1}) + \big(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\big)}{K_1 n\varrho_2\eta^2}\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\frac{\frac{\mu}{\lambda}G^{\theta_3}+\displaystyle\int_0^T F(t,\Theta_3)dt}{\frac{\varrho_1}{2M}(\theta_3-\theta_2)} &< \frac{1}{\lambda}\\ &< \frac{\frac{\mu}{\lambda}(TG_\eta - G^{\theta_1}) + \big(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\big)}{K_1 n\varrho_2\eta^2}.\end{aligned}\tag{3.12}$$

Hence from (3.6)-(3.12), we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let $u^* = (u_1^*,\ldots,u_n^*)$ and $u^{**} = (u_1^{**},\ldots,u_n^{**})$ be two local minima for I_λ. Then u^* and u^{**} are critical points for I_λ, and so, they are weak solutions for the system $(P_{\lambda,\mu}^{F,G})$. Since we assumed F is non-negative and since G is non-negative, for fixed $\lambda > 0$ and $\mu \geq 0$ we have $F(t, su^* + (1-s)u^{**}) + \frac{\mu}{\lambda}G(t, su^* + (1-s)u^{**}) \geq 0$, and consequently, $\Psi(su^* + (1-s)u^{**}) \geq 0$ for all $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \Bigg]\frac{K_1 n\varrho_2\eta^2}{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt},$$

$$\frac{\varrho_1}{2M}\min\Bigg\{\frac{\theta_1}{\int_0^T F(t,\Theta_1)dt}, \frac{\theta_2}{\int_0^T F(t,\Theta_2)dt}, \frac{\theta_3-\theta_2}{\int_0^T F(t,\Theta_3)dt}\Bigg\}\Bigg[$$

and $\mu \in [0, \delta_{\lambda,G}[$, the functional I_λ has three critical points u_i, $i = 1,2,3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is, $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \theta_2$, and $\max_{t\in[0,T]}|u_3(t)| < \theta_3$. Then, taking into account the fact that the weak solutions of the system $(P_{\lambda,\mu}^{F,G})$ are exactly critical points of the functional I_λ we have the desired conclusion. $\square$

For positive constants θ_1, θ_4 and η, set

$$\begin{aligned}\delta'_{\lambda,G} := \min\Bigg\{ & \frac{1}{2M}\min\Big\{\frac{\varrho_1\theta_1 - 2M\lambda\displaystyle\int_0^T F(t,\Theta_1)dt}{G^{\theta_1}}, \\ & \frac{\varrho_1\theta_4 - 4M\lambda\displaystyle\int_0^T F(t,\sqrt{\frac{\theta_4}{2}},\dots,\sqrt{\frac{\theta_4}{2}})dt}{2G^{\frac{\theta_4}{2}}}, \frac{\varrho_1\theta_4 - 4M\lambda\displaystyle\int_0^T F(t,\Theta_4)dt}{2G^{\theta_4}}\Big\}, \\ & \frac{K_1 n\varrho_2\eta^2 - \lambda\big(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\big)}{TG_\eta - G^{\theta_1}}\Bigg\} \end{aligned} \tag{3.13}$$

where $0 < \gamma < \frac{1}{2}$.

Theorem 3.2. *Let* $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ *satisfy the condition* $F(t,x_1,\dots,x_n) \geq 0$ *for all* $(t,x_1,\dots,x_n) \in [0,T]\times\mathbb{R}^n$. *Assume that there exist positive constants* $\gamma < \frac{1}{2}$, θ_1, θ_4 *and* η *with* $\theta_1 < \min\{\eta^2, 2MK_2n\eta^2\}$ *and* $\frac{4MK_1n\varrho_2}{\sigma - MCm\|\bar{a}\|_\infty}\eta^2 < \theta_4$ *such that*

(A$_2$)

$$\max\Big\{\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1}, \frac{2\int_0^T F(t,\Theta_4)dt}{\theta_4}\Big\} < \frac{\varrho_1}{\varrho_1 + 2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt}{\eta^2}.$$

Then, for every

$$\lambda \in \Lambda' := \Big]\frac{(\varrho_1 + 2MK_1n\varrho_2)\eta^2}{2M\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt}, \frac{\varrho_1}{2M}\min\Big\{\frac{\theta_1}{\int_0^T F(t,\Theta_1)dt}, \frac{\theta_4}{2\int_0^T F(t,\Theta_4)dt}\Big\}\Big[$$

and every non-negative function $G : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ *satisfying* $G^\theta \geq 0$, *there exists* $\delta'_{\lambda,G} > 0$ *given by* (3.13) *such that, for each* $\mu \in [0, \delta'_{\lambda,G}[$, *the system* $(P^{F,G}_{\lambda,\mu})$ *has at least three solutions* u_1, u_2, *and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \frac{\theta_4}{2}$, *and* $\max_{t\in[0,T]}|u_3(t)| < \theta_4$.

Proof. Choose $\theta_2 = \frac{1}{2}\theta_4$ and $\theta_3 = \theta_4$. So, by using (A_2) one has

$$\begin{aligned}\frac{\int_0^T F(t,\Theta_2)dt}{\theta_2} &= \frac{2\int_0^T F(t,\sqrt{\frac{\theta_4}{2}},\dots,\sqrt{\frac{\theta_4}{2}})dt}{\theta_4} \leq \frac{2\int_0^T F(t,\Theta_4)dt}{\theta_4} \\ &< \frac{\varrho_1}{\varrho_1 + 2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt}{\eta^2}\end{aligned} \tag{3.14}$$

and

$$\frac{\int_0^T F(t,\Theta_3)dt}{\theta_3 - \theta_2} = \frac{2\int_0^T F(t,\Theta_4)dt}{\theta_4} < \frac{\varrho_1}{\varrho_1 + 2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt}{\eta^2}. \tag{3.15}$$

Moreover, taking into account that $\theta_1 < \eta^2$, by using (A_2) we have

$$\frac{\varrho_1}{2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\bar{\eta})dt-\int_0^T F(t,\Theta_1)dt}{\eta^2} > \frac{\varrho_1}{2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\bar{\eta})dt}{\eta^2}$$

$$-\frac{\varrho_1}{2MK_1n\varrho_2}\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1} > \frac{\varrho_1}{2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\bar{\eta})dt}{\eta^2}$$

$$-\frac{\varrho_1^2}{2MK_1n\varrho_2(\varrho_1+2MK_1n\varrho_2)}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\bar{\eta})dt}{\eta^2} = \frac{\varrho_1}{\varrho_1+2MK_1n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\bar{\eta})dt}{\eta^2}.$$

Hence, from (A_2), (3.14) and (3.15), it is easy to see that the assumption (A_1) of Theorem 3.1 is satisfied, and since the critical points of the functional $\Phi - \lambda\Psi$ are the weak solutions of the system $(P_{\lambda,\mu}^{F,G})$ we have the conclusion. $\square$

Now, we present the following example in which the hypotheses of Theorem 3.2 are satisfied.

Example 3.1. Consider the following system

(3.16)

$$\begin{cases} {}_tD_1^{0.75}({}_0D_t^{0.75}u_1(t)) = \lambda F_{u_1}(u_1,u_2)+\mu G_{u_1}(u_1,u_2)+h_1(u_1), & t\in(0,1), t\neq t_1\\ {}_tD_1^{0.8}({}_0D_t^{0.8}u_2(t)) = \lambda F_{u_2}(u_1,u_2)+\mu G_{u_2}(u_1,u_2)+h_2(u_2), & t\in(0,1), t\neq t_2\\ \Delta({}_tD_1^{-0.25}({}_0^cD_t^{0.75}u_1))(t_j) = I_{1j}(u_1(t_j)), & j=1,2,\\ \Delta({}_tD_1^{-0.2}({}_0^cD_t^{0.8}u_2))(t_j) = I_{2j}(u_2(t_j)), & j=1,2,\\ u_1(0)=u_1(1)=0,\ u_2(0)=u_2(1)=0 & \end{cases}$$

where

$$F(x_1,x_2)=\begin{cases} e^{-\frac{1}{|x_1|}}+e^{-\frac{1}{|x_2|}} & if\ x_1x_2\neq 0,\\ e^{-\frac{1}{|x_2|}}, & if\ x_1=0, x_2\neq 0,\\ e^{-\frac{1}{|x_1|}}, & if\ x_1\neq 0, x_2=0,\\ 0, & if\ x_1=0, x_2=0,\end{cases}$$

$h_1(x_1)=\frac{1}{10^6}(1-\cos(x_1))$ and $h_2(x_2)=\frac{1}{10^6}(\ln(1+x_2^2))$ for every $x_1,x_2\in\mathbb{R}$, $t_1=\frac{1}{3}$, $t_2=\frac{1}{2}$, $I_{ij}(\xi)=\frac{1}{10^6}\xi$ for every $\xi\in\mathbb{R}$ and for $i,j=1,2$. By expressions of h_1 and h_2 we have $H_1(x_1)=\frac{1}{10^6}(x_1-\sin(x_1))$ and $H_2(x_2)=\frac{1}{10^6}(x_2\ln(1+x_2^2)-2x_2+2\arctan(x_2))$ for every $x_1,x_2\in\mathbb{R}$. Choosing $\gamma=\frac{1}{4}$, $\theta_1=10^{-4}$, $\theta_4=10^8$ and $\eta=1$, we clearly observe that all assumptions of Theorem 3.2 are satisfied. Hence, for every

$$\lambda\in\Bigg]\frac{0.5(\Gamma(0.75))^2\big(1-\frac{1}{10^6(\Gamma(1.75))^2}-\frac{2}{5\times10^5(\Gamma(0.75))^2}\big)}{e^{-\frac{1}{\Gamma(1.25)}}+e^{-\frac{1}{\Gamma(1.2)}}}$$

$$+\frac{31.8304\big(1+\frac{1}{10^6(\Gamma(1.75))^2}+\frac{2}{5\times10^5(\Gamma(0.75))^2}\big)}{e^{-\frac{1}{\Gamma(1.25)}}+e^{-\frac{1}{\Gamma(1.2)}}},$$

$$\frac{1-\frac{1}{10^6(\Gamma(1.75))^2}-\frac{2}{5\times10^5(\Gamma(0.75))^2}}{(\Gamma(0.75))^2}\frac{10^8}{4e^{-\frac{1}{10^4}}}\Bigg[$$

for every non-negative function $G:\mathbb{R}^n\to\mathbb{R}$ satisfying $G^\theta\geq 0$, there exists $\delta''_{\lambda,G}>0$ such that, for each $\mu\in[0,\delta''_{\lambda,G}[$, the system (3.16) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,1]}|u_1(t)|<10^{-4}$, $\max_{t\in[0,1]}|u_2(t)|<\frac{10^8}{2}$, and $\max_{t\in[0,1]}|u_3(t)|<10^8$.

Remark 3.1. When F does not depend on t, in Theorem 3.1 the assumption (A_1) can be written as

$$\max\left\{\frac{F(\Theta_1)}{\theta_1},\ \frac{F(\Theta_2)}{\theta_2},\ \frac{F(\Theta_3)}{\theta_3-\theta_2}\right\} < \frac{\varrho_1}{2M}\frac{(1-2\gamma)F(\bar{\eta})-F(\Theta_1)}{K_1 n\varrho_2\eta^2},$$

as well as

$$\Lambda := \left]\frac{K_1 n\varrho_2\eta^2}{(1-2\gamma)TF(\bar{\eta})-TF(\Theta_1)},\ \frac{\varrho_1}{2TM}\min\left\{\frac{\theta_1}{F(\Theta_1)},\frac{\theta_2}{F(\Theta_2)},\frac{\theta_3-\theta_2}{F(\Theta_3)}\right\}\right[$$

and

$$\delta_{\lambda,G} := \min\left\{\frac{1}{2M}\min\left\{\frac{\varrho_1\theta_1-2M\lambda TF(\Theta_1)}{G^{\theta_1}},\frac{\varrho_1\theta_2-2M\lambda TF(\Theta_2)}{G^{\theta_2}},\right.\right.$$
$$\left.\left.\frac{\varrho_1(\theta_3-\theta_2)-2M\lambda TF(\Theta_3)}{G^{\theta_3}}\right\},\frac{K_1 n\varrho_2\eta^2-\lambda\big((1-2\gamma)TF(\bar{\eta})-TF(t,\Theta_1)\big)}{TG_\eta-G^{\theta_1}}\right\}.$$

In this case, in Theorem 3.2 the assumption (A_2) follows the form

$$\max\left\{\frac{F(\Theta_1)}{\theta_1},\ \frac{2F(\Theta_4)}{\theta_4}\right\} < \frac{\varrho_1}{\varrho_1+2MK_1 n\varrho_2}\frac{(1-2\gamma)F(\bar{\eta})}{\eta^2}.$$

as well as

$$\Lambda' := \left]\frac{(\varrho_1+2MK_1 n\varrho_2)\eta^2}{2M(1-2\gamma)TF(\bar{\eta})},\frac{\varrho_1}{2TM}\min\left\{\frac{\theta_1}{F(\Theta_1)},\frac{\theta_4}{2F(\Theta_4)}\right\}\right[$$

and

$$\delta'_{\lambda,G} := \min\left\{\frac{1}{2M}\min\left\{\frac{\varrho_1\theta_1-2M\lambda TF(\Theta_1)}{G^{\theta_1}},\frac{\varrho_1\theta_4-4M\lambda TF(\sqrt{\frac{\theta_4}{2}},\dots,\sqrt{\frac{\theta_4}{2}})}{2G^{\frac{\theta_4}{2}}},\right.\right.$$
$$\left.\left.\frac{\varrho_1\theta_4-4M\lambda TF(\Theta_4)}{2G^{\theta_4}}\right\},\frac{K_1 n\varrho_2\eta^2-\lambda\big((1-2\gamma)TF(\bar{\eta})-TF(\Theta_1)\big)}{TG_\eta-G^{\theta_1}}\right\}.$$

Remark 3.2. We observe that, in our results, no asymptotic conditions on F and G are needed and only algebraic conditions on F are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since the values of $F_{x_i}(t,0,\dots,0)$ and $G_{x_i}(t,0,\dots,0)$ for every $t\in[0,T]$, $1\le i\le n$, are not determined.

As an application of Theorem 3.1, we consider the following problem

$$\left\{\begin{array}{ll} {}_tD_T^\alpha(a(t)_0D_t^\alpha u(t)) = \lambda f(t,u)+\mu g(t,u)+h(u), & t\in(0,T), t\neq t_j\\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = I_j(u(t_j)), & j=1,2,\dots,m,\\ u(0)=u(T)=0 & \end{array}\right. \tag{3.17}$$

where $\frac{1}{2}<\alpha\le 1$, $\lambda>0$, $\mu\ge 0$, $T>0$, ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ denote the left and right Riemann-Liouville fractional derivatives of order α, respectively, $a_0 = ess\inf_{t\in[0,T]} a(t) > 0$, $f, g : [0,T]\times\mathbb{R}\to\mathbb{R}$ are L^1-Carathéodory functions, $h:\mathbb{R}\to[0,+\infty)$ is a Lipschitz continuous function with Lipschitz constant $L>0$, i.e.,

$$|h(\xi_1)-h(\xi_2)|\le L|\xi_1-\xi_2|$$

for every $\xi_1,\ \xi_2\in\mathbb{R}$, satisfying $h(0)=0$, $I_j\in C(\mathbb{R},\ \mathbb{R})$ for $j=1,2,\dots,m$, such that $I_j(0)=0$ and there exists a constant $L_j>0$ such that

$$|I_j(s_1)-I_j(s_2)|\le L_j|s_1-s_2| \text{ for any } s_1,s_2\in\mathbb{R},\quad \text{for } j=1,\dots,m,$$

$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$.

Put

$$F(t,x) = \int_0^x f(t,\xi)d\xi \quad \text{for every } (t,x) \in [0,T] \times \mathbb{R},$$
$$G(t,x) = \int_0^x g(t,\xi)d\xi \quad \text{for every } (t,x) \in [0,T] \times \mathbb{R}$$

and

$$H(x) = \int_0^x h(\xi)d\xi \quad \text{for every } x \in \mathbb{R}.$$

Set

$$\bar{\sigma} = 1 - \frac{LT^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0}, \ \bar{\rho} = 1 + \frac{LT^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0}, \ \bar{M} = \frac{T^{2\alpha-1}}{(\Gamma(\alpha))^2 a_0 (2\alpha-1)},$$
$$\bar{\varrho}_1 = \bar{\sigma} - \bar{M}\bar{C}m\|a\|_\infty, \ \bar{\varrho}_2 = \bar{\rho} + \bar{M}\bar{C}m\|a\|_\infty$$

and

$$\begin{aligned} P(\alpha,\gamma) =& \frac{1}{2\gamma^2 T^2}\Bigg\{ \int_0^T a(t)t^{2(1-\alpha)}dt + \int_{\gamma T}^T a(t)(t-\gamma T)^{2(1-\alpha)}dt \\ &+ \int_{(1-\gamma)T}^T a(t)(t-(1-\gamma)T)^{2(1-\alpha)}dt - 2\int_{\gamma T}^T a(t)(t^2-\gamma Tt)^{1-\alpha}dt \\ &-2\int_{(1-\gamma)T}^T a(t)(t^2-(1-\gamma)Tt)^{1-\alpha}dt \\ &+2\int_{(1-\gamma)T}^T a(t)(t^2-\gamma Tt+\gamma(1-\gamma)T^2)^{1-\alpha}dt \Bigg\} \end{aligned}$$

where $0 < \gamma < \frac{1}{2}$. We suppose that

$$\bar{K} = \frac{LT^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0} + \bar{M}\bar{C}m\|a\|_\infty < 1$$

where

$$\bar{C} = \max_{j \in \{1,\ldots,m\}} \{L_j\}.$$

For positive constants θ and η set

$$G^\theta := \int_0^T \max_{|x| \le \sqrt{\theta}} G(t,x)dt \text{ and } G_\eta := \inf_{[0,T]\times[0,\Gamma(2-\alpha)\eta]} G.$$

Obviously, if g changes sing on $[0,T]$ then clearly $G^\theta \ge 0$.

Now, we give the following straightforward consequences of Theorem 3.1 and Theorem 3.2, respectively.

Theorem 3.3. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a non-negative L^1-Carathéodory function. Assume that there exist positive constants $\gamma < \frac{1}{2}$, θ_1, θ_2, θ_3 and η with $\theta_3 > \theta_2$, $\theta_1 < 2\bar{M}P(\alpha,\gamma)\eta^2$ and $\frac{2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2}{\bar{\varrho}_1}\eta^2 < \theta_2$ such that*

$$\max\left\{ \frac{\int_0^T F(t,\sqrt{\theta_1})dt}{\theta_1}, \ \frac{\int_0^T F(t,\sqrt{\theta_2})dt}{\theta_2}, \ \frac{\int_0^T F(t,\sqrt{\theta_3})dt}{\theta_3-\theta_2} \right\}$$
$$< \frac{\bar{\varrho}_1}{2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2} \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt - \int_0^T F(t,\sqrt{\theta_1})dt}{\eta^2}.$$

Then, for every

$$\lambda \in \Lambda'' := \Big]P(\alpha,\gamma)\bar{\varrho}_2 \frac{\eta^2}{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt - \int_0^T F(t,\sqrt{\theta_1})dt},$$

$$\frac{\bar{\varrho}_1}{2\bar{M}} \min\left\{\frac{\theta_1}{\int_0^T F(t,\sqrt{\theta_1})dt}, \frac{\theta_2}{\int_0^T F(t,\sqrt{\theta_2})dt}, \frac{\theta_3-\theta_2}{\int_0^T F(t,\sqrt{\theta_3})dt}\right\}\Big[$$

and every non-negative L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^\star_{\lambda,g} > 0$ given by

$$\delta^\star_{\lambda,g} = \min\left\{\frac{1}{2\bar{M}} \min\left\{\frac{\bar{\varrho}_1\theta_1 - 2\bar{M}\lambda \displaystyle\int_0^T F(t,\sqrt{\theta_1})dt}{G^{\theta_1}}, \frac{\bar{\varrho}_1\theta_2 - 2\bar{M}\lambda \displaystyle\int_0^T F(t,\sqrt{\theta_2})dt}{G^{\theta_2}},\right.\right.$$

$$\left.\frac{\bar{\varrho}_1(\theta_3-\theta_2) - 2\bar{M}\lambda \displaystyle\int_0^T F(t,\sqrt{\theta_3})dt}{G^{\theta_3}}\right\},$$

$$\left.\frac{P(\alpha,\gamma)\bar{\varrho}_2\eta^2 - \lambda(\displaystyle\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha))dt - \int_0^T F(t,\sqrt{\theta_1})dt)}{TG_\eta - G^{\theta_1}}\right\},$$

such that, for each $\mu \in [0, \delta^\star_{\lambda,g}[$, the problem (3.17) *has at least three non-negative weak solutions u_1, u_2, and u_3 such that* $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, *and* $\max_{t\in[0,T]} |u_3(t)| < \theta_3$.

Proof. By a similar argument as given in the proof of Theorem 3.1 we ensure the existence of the weak solutions u_1, u_2, and u_3 with $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$. Now, we show that the weak solutions u_1, u_2, and u_3 are non-negative. To this end, let u_0 be a non-trivial weak solution of the problem (3.17). Arguing by a contradiction, assume that the set $\mathcal{A} = \{t \in]0,T] : u_0(t) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(t) = \min\{0, u_0(t)\}$ for all $t \in [0,T]$. Clearly, $\bar{v} \in E_\alpha$ and one has

$$\int_0^T a(t)\,{}_0D_t^\alpha u_0(t)\,{}_0D_t^\alpha \bar{v}(t)dt - \int_0^T h(u_0(t))\bar{v}(t)dt + \sum_{j=1}^m a(t_j)I_j(u_0(t_j))\bar{v}(t_j)$$

$$- \lambda \int_0^T f(t,u_0(t))\bar{v}(t)dt - \mu \int_0^T g(t,u_0(t))\bar{v}(t)dt = 0.$$

Thus, from our sign assumptions on the data we have

$$0 \le (1-\bar{K}) \int_{\mathcal{A}} a(t)|{}_0D_t^\alpha u_0(t)|^2 dt \le \int_{\mathcal{A}} a(t)|{}_0D_t^\alpha u_0(t)|^2 dt - \int_{\mathcal{A}} h(u_0(t))u_0(t)dt$$

$$+ \sum_{j=1}^m a(t_j)I_j(u_0(t_j))u_0(t_j) \le 0.$$

Hence, since $\bar{K} < 1$, $u_0 = 0$ in $\mathcal{A}$ and we arrive to a contradiction. □

Theorem 3.4. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a non-negative L^1-Carathéodory function. Assume that there exist positive constants $\gamma < \frac{1}{2}$, θ_1, θ_4 and η with $\theta_1 <$*

$\min\{\eta^2, 2\bar{M}P(\alpha,\gamma)\eta^2\}$ *and* $\frac{4\bar{M}P(\alpha,\gamma)\bar{\varrho}_2}{\bar{\varrho}_1}\eta^2 < \theta_4$ *such that*

$$\max\left\{\frac{\int_0^T F(t,\sqrt{\theta_1})dt}{\theta_1}, \frac{\int_0^T F(t,\sqrt{\theta_4})dt}{\theta_4}\right\}$$
$$< \frac{\bar{\varrho}_1}{\bar{\varrho}_1 + 2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2} \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt}{\eta^2}.$$

Then, for every

$$\lambda \in \left]\frac{(\bar{\varrho}_1 + 2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2)\eta^2}{2\bar{M}\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt}, \frac{\bar{\varrho}_1}{2\bar{M}} \min\left\{\frac{\theta_1}{\int_0^T F(t,\sqrt{\theta_1})dt}, \frac{\theta_4}{2\int_0^T F(t,\sqrt{\theta_4})dt}\right\}\right[$$

and for every non-negative L^1*-Carathéodory function* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$*, there exists* $\delta^{\star\star}_{\lambda,g} > 0$ *given by*

$$\delta^{\star\star}_{\lambda,g} = \min\left\{\frac{1}{2\bar{M}} \min\left\{\frac{\bar{\varrho}_1\theta_1 - 2\bar{M}\lambda\int_0^T F(t,\sqrt{\theta_1})dt}{G^{\theta_1}}, \frac{\bar{\varrho}_1\theta_4 - 4\bar{M}\lambda\int_0^T F(t,\sqrt{\frac{\theta_4}{2}})dt}{2G^{\frac{\theta_4}{2}}},\right.\right.$$
$$\left.\frac{\bar{\varrho}_1\theta_4 - 4\bar{M}\lambda\int_0^T F(t,\sqrt{\theta_4})dt}{2G^{\theta_4}}\right\},$$
$$\left.\frac{P(\alpha,\gamma)\bar{\varrho}_2\eta^2 - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha))dt - \int_0^T F(t,\sqrt{\theta_1})dt\right)}{TG_\eta - G^{\theta_1}}\right\},$$

such that, for each $\mu \in [0, \delta^{\star\star}_{\lambda,g}[$*, the problem* (3.17) *has at least three non-negative weak solutions* u_1*,* u_2*, and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < \theta_1$*,* $\max_{t\in[0,T]}|u_2(t)| < \frac{\theta_4}{2}$*, and* $\max_{t\in[0,T]}|u_3(t)| < \theta_4$*.*

Here, in order to illustrate Theorem 3.4, we present the following example

Example 3.2. Consider the following problem

$$\begin{cases} {}_tD_1^{0.8}({}_0D_t^{0.8}u(t)) = \lambda f(u) + \mu g(u) + h(u), & t \in (0,1), t \neq t_j \\ \Delta({}_tD_1^{-0.2}({}_0^cD_t^{0.8}u))(t_j) = I_j(u(t_j)), & j = 1,2, \\ u(0) = u(1) = 0 \end{cases} \tag{3.18}$$

where

$$f(x) = \begin{cases} 7x^6, & x \leq 1, \\ \frac{7}{x}, & x > 1, \end{cases}$$

$h(x) = \frac{1}{10^8}\ln(1+x^2)$ for every $x \in \mathbb{R}$, $t_1 = \frac{1}{6}$, $t_2 = \frac{1}{5}$ and $I_j(\xi) = \frac{1}{10^8}\xi$ for every $\xi \in \mathbb{R}$ and for $j = 1,2$. By expression of f we have

$$F(x) = \begin{cases} x^7, & x \leq 1, \\ 7\ln(x) + 1, & x > 1. \end{cases}$$

Taking $\gamma = \frac{1}{4}$, $\theta_1 = 10^{-4}$, $\theta_4 = 10^4$ and $\eta = 1$, we clearly observe that all assumptions of Theorem 3.4 are satisfied. Then, for each

$$\lambda \in \left]\frac{0.6(\Gamma(0.8))^2\left(1 - \frac{1}{10^8(\Gamma(1.8))^2} - \frac{1}{3\times 10^7(\Gamma(0.8))^2}\right)}{(\Gamma(1.2))^7}\right.$$

$$+\frac{8.9282\big(1+\frac{1}{10^8(\Gamma(1.8))^2}+\frac{1}{3\times 10^7(\Gamma(0.8))^2}\big)}{(\Gamma(1.2))^7},$$

$$\frac{1-\frac{1}{10^8(\Gamma(1.8))^2}-\frac{1}{3\times 10^7(\Gamma(0.8))^2}}{1.2(\Gamma(0.8))^2}\frac{10^4}{7\ln(10^2)+1}\Big[$$

and every non-negative continuous function $g:\mathbb{R}\to\mathbb{R}$, there exists $\bar{\delta}_{\lambda,g}>0$ such that, for each $\mu\in[0,\bar{\delta}_{\lambda,g}[$, the problem (3.18) has at least three non-negative weak solutions u_1, u_2 and u_3 such that $\max_{t\in[0,1]}|u_1(t)|<10^{-4}$, $\max_{t\in[0,1]}|u_2(t)|<\frac{10^4}{2}$ and $\max_{t\in[0,1]}|u_3(t)|<10^4$.

Now, we list some consequences of Theorem 3.4 as follows.

Theorem 3.5. *Let f be a non-negative continuous and nonzero function such that*

$$\lim_{x\to 0^+}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{f(x)}{x}=0 \tag{3.19}$$

for every $\lambda>\lambda^*$ where

$$\lambda^*=\inf\{\frac{(\bar{\varrho}_1+2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2)\eta^2}{2\bar{M}TF(\Gamma(2-\alpha)\eta)}:\eta>0,F(\Gamma(2-\alpha)\eta)>0\}.$$

Then there exists

$$\tilde{\mu}_{\lambda,g}=\min\Bigg\{\frac{1}{2\bar{M}}\min\Big\{\frac{\bar{\varrho}_1\theta_1-2\bar{M}\lambda TF(\sqrt{\theta_1})}{G^{\theta_1}}, \frac{\bar{\varrho}_1\theta_4-4\bar{M}\lambda TF(\sqrt{\frac{\theta_4}{2}})}{2G^{\frac{\theta_4}{2}}},\frac{\bar{\varrho}_1\theta_4-4\bar{M}\lambda TF(\sqrt{\theta_4})}{2G^{\theta_4}}\Big\}, \frac{P(\alpha,\gamma)\bar{\varrho}_2\eta^2-\lambda((1-2\gamma)TF(\Gamma(2-\alpha))-TF(\sqrt{\theta_1}))}{TG_\eta-G^{\theta_1}}\Bigg\} \tag{3.20}$$

where θ_1, θ_4 and γ are positive constants with $\gamma<\frac{1}{2}$, such that for each $\mu\in[0,\tilde{\mu}_{\lambda,g}[$, the problem

$$\begin{cases} {}_tD_T^{\alpha}(a(t)_0D_t^{\alpha}u(t))=\lambda f(u)+\mu g(u)+h(u), & t\in(0,T),\\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j)=I_j(u(t_j)), & j=1,2,\dots,m,\\ u(0)=u(T)=0 \end{cases} \tag{3.21}$$

where $g:\mathbb{R}\to\mathbb{R}$ is a non-negative continuous and nonzero function, has at least two distinct positive weak solutions.

Proof. Fix $\lambda>\lambda^*$, put $F(x)=\int_0^x f(\xi)d\xi$ for all $x\in\mathbb{R}$ and let $\eta>0$ such that $F(\Gamma(2-\alpha)\eta)>0$ and

$$\lambda>\frac{(\bar{\varrho}_1+2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2)\eta^2}{2\bar{M}TF(\Gamma(2-\alpha)\eta)}.$$

From (3.19) there is $\theta_1>0$ such that $\theta_1<\min\{\eta^2,2\bar{M}P(\alpha,\gamma)\eta^2\}$ and $\frac{F(\sqrt{\theta_1})}{\theta_1}<\frac{\bar{\varrho}_1}{2\bar{M}T\lambda}$, and $\theta_4>0$ such that $\frac{4\bar{M}P(\alpha,\gamma)\bar{\varrho}_2}{\bar{\varrho}_1}\eta^2<\theta_4$ and $\frac{F(\sqrt{\theta_4})}{\theta_4}<\frac{\bar{\varrho}_1}{4\bar{M}T\lambda}$. Therefore, Theorem 3.2 ensures the conclusion. □

Finally, by the way of example, we point out the following simple consequence of Theorem 3.5 when $\mu=0$.

Theorem 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $xf(x) > 0$ for all $x \neq 0$ and*

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{|x|\to +\infty} \frac{f(x)}{x} = 0.$$

Then, for every $\lambda > \overline{\lambda}$ where

$$\overline{\lambda} = \frac{(\bar{\varrho}_1 + 2\bar{M}P(\alpha,\gamma)\bar{\varrho}_2)}{2\bar{M}T} \times \max\{\inf_{\eta>0} \frac{\eta^2}{F(\Gamma(2-\alpha)\eta)};\ \inf_{\eta<0} \frac{(-\eta)^2}{F(\Gamma(2-\alpha)\eta)}\},$$

the problem (3.21)*, in the case $\mu = 0$ has at least four distinct non-trivial weak solutions.*

Proof. Setting

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0, \\ f(x), & \text{if } x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if } x < 0, \\ -f(-x), & \text{if } x \geq 0, \end{cases}$$

and applying Theorem 3.5 to f_1 and f_2 we have the result. ☐

References

[1] K. Diethelm, *The Analysis of Fractional Differential Equation,* Spring, Heidelberg (2010).

[2] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, Singapore (2000).

[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier Science, Amsterdam (2006).

[4] I. Podlubny, *Fractional Differential Equations,* Mathematics in Science and Engineering, vol. 198. Academic Press, New York (1999).

[5] V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity,* Mathematics of the USSR-Izvestiya **29** (1987) 33-66.

[6] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions,* Acta Appl. Math. **109** (2010) 973-1033.

[7] C. Bai, *Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem,* Electron. J. Differ. Equ., Vol. **2013** (2013) No. 136, pp. 1-12.

[8] Z. Bai, Z. Qiu, *Existence of positive solution for singular fractional differential equation,* Appl. Math. Comput. **215** (2009) 2761-2767.

[9] M. Galewski, G. Molica Bisci, *Existence results for one-dimensional fractional equations,* Math. Meth. Appl. Sci. **39** (2016) 1480-1492.

[10] J.R. Graef, L. Kong, B. Yang, *Positive solutions for a semipositone fractional boundary value problem with a forcing term,* Fract. Calc. Appl. Anal. **15** (2012) 8-24.

[11] S. Heidarkhani, *Infinitely many solutions for nonlinear perturbed fractional boundary value problems,* Annals of the University of Craiova, Mathematics and Computer Science Series, **41** (2014) 88-103.

[12] S. Heidarkhani, *Multiple solutions for a nonlinear perturbed fractional boundary value problem,* Dynamic Sys. Appl. **23** (2014) 317-331.

[13] L. Kong, *Existence of solutions to boundary value problems arising from the fractional advection dispersion equation,* Electron. J. Diff. Equ., Vol. **2013** (2013) No. 106, pp. 1-15.

[14] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations,* Nonlinear Anal. TMA **69** (2008) 2677-2682.

[15] G. Molica Bisci, *Fractional equations with bounded primitive,* Appl. Math. Lett. **27** (2014) 53-58.

[16] W. Xie, J. Xiao, Z. Luo, *Existence of solutions for fractional boundary value problem with nonlinear derivative dependence,* Abstr. Appl. Anal. (2014) Art. ID 812910, 8 pp.

[17] B. Ahmad, J.R. Graef, *Coupled systems of nonlinear fractional differential equations with nonlocal boundary conditions,* PanAmer. Math. J. **19** (2009) 29-39.

[18] C. Bai, J. Fang, *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations,* Appl. Math. Comput. **150** (2004) 611-621.

[19] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations,* Appl. Math. Lett. **22** (2009) 64-69.

[20] S. Sun, Q. Li, Y. Li, *Existence and uniqueness of solutions for a coupled system of multi-term nonlinear fractional differential equations,* Comput. Math. Appl. **64** (2012) 3310-3320.

[21] Y. Zhang, Z. Bai, T. Feng, *Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance,* Comput. Math. Appl. **61** (2011) 1032-1047.

[22] Y. Zhao, H. Chen, B. Qin, *Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods,* Appl. Math. Comput. **257** (2015) 417-427.

[23] Y. Zhao, H. Chen, Q. Zhang, *Infinitely many solutions for fractional differential system via variational method,* J. Appl. Math. Comput. **50** (2016) 589-609.

[24] Y. Zhao, H. Chen, Q. Zhang, *Multiplicity of solutions for perturbed nonlinear fractional differential system via variational method,* preprint.

[25] S. Heidarkhani, Y. Zhou, G. Caristi, G.A. Afrouzi, S. Moradi, *Existence results for fractional differential systems through a local minimization principle,* Comput. Math. Appl., (2016), http://dx.doi.org/10.1016/j.camwa.2016.04.012.

[26] S. Heidarkhani, M. Ferrara, A. Salari, *Infinitely many periodic solutions for a class of perturbed second-order differential equations with impulses,* Acta. Appl. Math. **139** (2015) 81-94.

[27] J.J. Nieto, D. O'Regan, *Variational approach to impulsive differential equations,* Nonlinear Anal. RWA **10** (2009) 680-690.

[28] Y. Tian, W. Ge, *Applications of variational methods to boundary-value problem for impulsive differential equations,* Proc. Edinb. Math. Soc. **51** (2008) 509-527.

[29] C. Bai, *Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative,* J. Math. Anal. Appl. **384** (2011) 211-231.

[30] G. Bonanno, R. Rodríguez-López, S. Tersian, *Existence of solutions to boundary-value problem for impulsive fractional differential equations,* Fract. Calc. Appl. Anal. **17** (2014) 717-744.

[31] Z. Gao, L. Yang, G. Liu, *Existence and uniqueness of solutions to impulsive fractional integro-differential equations with nonlocal conditions,* Appl. Math. **4** (2013) 859-863.

[32] S. Heidarkhani, A. Salari, *Nontrivial Solutions for impulsive fractional differential systems through variational methods,* Comput. Math. Appl. (2016), http://dx.doi.org/10.1016/j.camwa.2016.04.016.

[33] S. Heidarkhani, Y. Zhao, G. Caristi, G.A. Afrouzi, S. Moradi, *Infinitely many solutions for perturbed impulsive fractional differential systems,* Appl. Anal. **96** (2017) 1401-1424.

[34] Y. Zhao, Y. Zhao, *Nontrivial solutions for a class of perturbed fractional differential systems with impulsive effects,* Bound. Value Probl. **2016** (2016):129.

[35] T.D. Ke, D. Lan, *Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces,* Fract. Calc. Appl. Anal. **17** (2014) 96-121.

[36] J. Wang, M. Fečkan, Y. Zhou, *On the new concept of solutions and existence results for impulsive fractional evolution equations,* Dyn. Partial Differ. Equ. **8** (2011) 345-361.

[37] J. Wang, X. Li, W. Wei, *On the natural solution of an impulsive fractional differential equation of order $q \in (1, 2)$,* Commun. Nonlinear Sci. Numer. Simul. **17** (2012) 4384-4394.

[38] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities,* J. Differ. Equ. **244** (2008) 3031-3059.

[39] A. Cabada, T. Kisela, *Existence of positive periodic solutions of some nonlinear fractional differential equations,* Commun. Nonlinear Sci. Numer. Simul. **50** (2017) 51-67.

[40] G. Wang, *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval,* Appl. Math. Lett. **47** (2015) 1-7.

[41] L. Zhang, B. Ahmad, G. Wang, *Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line,* Bull. Aust. Math. Soc. **91** (2015) 116-128.

[42] G. Bonanno, B. Di Bella, *A boundary value problem for fourth-order elastic beam equations,* J. Math. Anal. Appl. **343** (2008) 1166-1176.

[43] S. Heidarkhani, A.L.A. De Araujo, G.A. Afrouzi, S. Moradi, *Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions,* Math. Nachr., to appear.

[44] F. Jiao, Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory,* Comput. Math. Appl. **62** (2011) 1181-1199.
[45] F. Jiao, Y. Zhou, *Existence results for fractional boundary value problem via critical point theory,* Inter. J. Bifurcat. Chaos. **22** (2012) 1250086.
[46] E. Zeidler, *Nonlinear Functional Analysis and its Applications,* Vol. II, Springer, Berlin-Heidelberg- New York (1985).

Department of Mathematics, Faculty of sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: `s.heidarkhani@razi.ac.ir`

Departamento de Anàlise Matemàtica, Facultade de Matemàticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
E-mail address: `alberto.cabada@usc.es`

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: `afrouzi@umz.ac.ir`

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: `shahin.moradi86@yahoo.com`

Department of Economics, University of Messina, via dei Verdi, 75, Messina, Italy
E-mail address: `gcaristi@unime.it`