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Innovative Applications of O.R.

On the centrality analysis of covert networks using games with externalities

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ABSTRACT

The identification of the most potentially hazardous agents in a terrorist organisation helps to prevent further attacks by effectively allocating surveillance resources and destabilising the covert network to which they belong. In this paper, several mechanisms for the overall ranking of covert networks members in a general framework are addressed based on their contribution to the overall relative effectiveness in the event of a merger. In addition, the possible organisation of agents outside of each possible merger naturally influences their relative effectiveness score, which motivates the innovative use of games in partition function form and specific ranking indices for individuals. Finally, we apply these methods to analyse the effectiveness of the hijackers of the covert network supporting the 9/11 attacks.

1. Introduction

In an increasingly globalised world, communication interconnections between its components are crucial for the analysis of expert or intelligent multi-agent interactive situations. Cooperation ensures that not only the decisions of a merger influence in a global framework, but also the information and the outcomes of other groups of agents around them gain interest in joint decision-making. In a collaborative scheme, considering coalitional externalities of agents in a game theoretical framework can be used to model such cooperation. This provides a considerably more realistic fitting than that given by the well-known transferable utility (TU) games. Under this scheme, we mention cases of resource allocation problems, in which factors such as environmental or climate change influence, the data envelopment analysis or the analysis of covert networks as representatives of the above.

The aim of covert network analysis is to identify the key members in an internal organisation, despite being unaware of the many links between members. For instance, counterterrorism measures taken from intelligence national agencies are increasingly based on sophisticated and realistic techniques that identify dormant groups capable of causing terror and destruction. Thus, scarce surveillance enables efficient counterterrorism activities. This task has received growing interest in recent decades because of the increasing number of massive attacks committed, such as the 9/11 attacks in 2001, the Bali bombing in 2002, and, more recently, the attacks in Paris in 2015 and Brussels in 2016 by the Zerkani network. These have prompted numerous studies being conducted to model such situations in practice and serve to endow many of these formulations with realism. Most proposals focus on ranking the members of covert networks based on the leadership and influence they exert. For instance, the degree centrality, the betweeness centrality and the closeness centrality of the covert network structure are examples of standard network measures that justify well-known techniques in social network analysis (Koschade, 2006). However, their practical use is limited, as the results obtained are not always as realistic as desired. Sparrow (1991), Klerks (2001), Farley (2003), Guzman, Deckro, Robbins, Morris, & Ballester (2014), and McGuire, Deckro, & Ahner (2015) use the standard social network centrality approach to identify key agents in this type of structures.

Although *covert network* is a term often associated with activities of an illicit nature, it has many other areas of application in social network analysis, such as in the emergence of epidemics or propagation of information. As Baker & Faulkner (1993) justifies in the energetic field, a covert network generally arises when studying the functioning of any multi-agent organisation in which

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balance is simultaneously sought between maintaining secrecy and ensuring the necessary coordination and control of its members. However, the notion of a covert network can be further generalised by applying it to any social movement network whose members collectively identify with a particular project of social/political change to achieve. By using this framework, Crossley, Edwards, Harries, & Stevenson (2012) study the covert network arising from the UK suffragette relationships during 1906–1914. Thus, covert terrorist networks can be considered to be an extreme case of this type of network in which members use violence. When these concepts are applied to the sports field, a team can also be considered to be a complex network, in which players interact with the aim of overcoming the opponent's network. For instance, Buldú, Busquets, Echegoyen, & Seirul.lo (2019) study F.C. Barcelona, coached by Guardiola, as a covert network.

In any of the aforementioned applications, the existence of links between agents justifies the formation of coalitions within a covert network. The fact of not considering valuable information on the possible existing relationships between the members of the covert network really limits the usage of more conventional techniques. Lindelauf, Hamers, & Husslage (2013) overcome these drawbacks, by incorporating the available information on the heterogeneity of links and nodes to assess the effectiveness of each coalition. Thus, the natural collaboration of agents favoured the use of specific tools extracted from cooperative game theory for its analysis. In particular, transferable utility games (or TU games) are typically used to model such collaborative situations. For instance, Husslage, Borm, Burg, Hamers, & Lindelauf (2015) specify a TU game involving the links between the members of a given coalition of agents in a covert network; van Campen, Hamers, Husslage, & Lindelauf (2018) analyse the computational problems of ranking them, and recently, Algaba, Prieto, Saavedra-Nieves, & Hamers (2022) assume the incorporation of a coalitional structure to model the possible affinities for the cooperation of agents. A common issue is the consideration of specific mechanisms for ranking the members of a covert network taken from cooperative game theory for allocating the worth of the cooperation.

In this paper, we analyse the influence of agents on the overall relative effectiveness in the event of a merger from a game theoretical approach. For this purpose, we consider the approach in Lindelauf et al. (2013) as a basis to measure the effectiveness of each possible merge of agents based on the weights of their members and their links. The main innovation lies in the fact that such relative effectiveness for each possible group of agents is obtained under any possible configuration of the remaining agents. The underlying idea is that the functioning of a covert network needs to be conceived of as a *whole*, where the profitable character of cooperation of a group of its members is relative to the possibilities of cooperation of the remainders. Therefore, whether a merger is the most effective of all possible mergers to be formed largely depends on how its outsiders organise themselves. However, these natural cooperation structures, if they exist, are rarely made explicit in covert networks. This perspective allows us to use the model of the *partition function form games* (Thrall & Lucas, 1963) to represent these situations. More specifically, we use partition function form simple games to identify the most effective coalitions as winning among those induced by every coalitional structure on the whole set of agents. Various papers in which real-world problems have been addressed by applying the approach of general partition function form games have been conducted. For instance, Pintassilgo & Lindroos (2008) and Liu, Lindroos, & Sandal (2016) model the management of fishery issues with externalities; Csercsik & Kóczy (2017) use games with externalities in an energy context; Yang, Sun, Hou, & Xu (2019) analyse sequencing situations also from this perspective; Grabisch & Funaki (2012) and Mahdiraji, Razghandi, & Hatami-Marbini (2021) analyse the coalition formation process, and Basso, Basso, Rönnqvist, & Weintraub (2021) use them in the production and transportation analysis under collaboration. As in the case of cooperative games with transferable utility, a central issue in partition function form games is to reasonably split the worth of the grand coalition. However, when considering simple games, solutions for TU games can be tipically interpreted as tools for measuring the ability of each agent to make coalitions winning. In this paper we will use partition form simple games and the values derived from classical solutions for TU games, as the Shapley value (Shapley, 1953) and related values, the Banzhaf value (Banzhaf, 1964) and the Deegan-Packel index (Deegan & Packel, 1978), to rank the members of covert networks under these criteria. Prior to the use of these values, following the approaches proposed by Albizuri, Arin, & Rubio (2005), de Clippel & Serrano (2008), McQuillin (2009), and Hu & Yang (2010), we consider TU games associated with the partition function form game proposed. Computational problems arise owing to the large number of partitions that need to be considered. These problems are solved using the ideas developed by Saavedra-Nieves & Fiestras-Janeiro (2022) based on sampling techniques. To the best of our knowledge, this is the first study in which games in partition function form are used to analyse covert networks.

The remainder of this paper is organised as follows. Section 2 introduces basic concepts of covert networks. Section 3 innovatively introduces the class of covert network games under the presence of externalities and presents a discussion on their properties. Section 4 provides several alternatives for ranking the members of a covert network based on values for games in partition function form. Section 5 describes the application of this methodology using data from the covert network of the 9/11 attack. Finally, Section 6 presents concluding remarks.

2. On covert networks

This section introduces preliminaries on graph theory required for covert network analysis. Note that the notion of a graph and that of a network can be used interchangeably. For a thorough revision of graphs, refer to Bollobás (1998).

First, we recall the notion of a covert network. Formally, this is represented by a graph, denoted by $G = (N, E)$. In the following, N denotes the finite set of vertices of the graph (nodes), which can be interpreted as the set of individuals belonging to the covert network, and E denotes the set of links or edges that describe all relationships between the members of the graph. The relationship between members i and j is denoted by ij, where $ij \in E$. Distances and communications between agents reflect this.

In a natural manner, existing relationships between each pair of individuals can be assumed to be bidirectional, and thus, the structure of a covert organisation is modelled as an undirected graph. The following parameters are associated with any covert network $G=(N, E)$:

- Individual strengths are represented by set $\mathcal{I}$ of weights on the player set N, i.e., $\mathcal{I} = \{w_i\}_{i \in N}$ with $w_i \geq 0$. For instance, special skills of the members of G can be quantified using these.
- The relational strength between the members of G is given by a set $\mathcal{R}$ of weights on the edges in E, i.e. $\mathcal{R} = \{k_{ij}\}_{ij \in E}$ with $k_{ij} \geq 0$. In this case, issues concerning the strength of the communication between the members in the covert network are representative.

To identify key players in a covert network, the formation of groups under cooperation is essential. These joint actions should be based on the principle of coordination, to maintain secrecy while ensuring success. Consequently, we are interested only in analysing connected graphs that provide an overview of the organisation of the members. Thus, if a certain coalition $S \subseteq N$ forms, subnetwork $G_S = (S, E_S)$ is naturally defined by the members of S and its links in E that connect them, i.e.,

$$E_S = \{ij \in E : i, j \in S\}.$$

Coalition $S \subseteq N$ is said to be a connected coalition in (N, E), if subnetwork G_S is connected; otherwise, coalition S is called disconnected.

Under this scheme, quantitative models can be considered to evaluate the characteristics of each possible merge. In general, all of these use the available information on the members (on individuals and their links) of a covert network in social network analysis. This allows the definition of a context-specific and tailor-made nonnegative function f, namely the *effectiveness function* depending on $\mathcal{I}$ and $\mathcal{R}$, to worth the effectiveness of each possible coalition, $S \subseteq N$, is in covert network G. In the remainder of this paper, we denote a *multi-agent covert network problem* by $(G, f) = (N, E, f)$,[1] where (N, E) denotes the graph underlying the covert network, and f is the effectiveness function considered.

As previously considered, only the effectiveness of connected coalitions may be of interest in covert network analysis. This allows the definition of specific models to be used for analysing cooperation in covert networks, for instance, establishing rankings for members of a covert network (see Husslage et al., 2015). Note that although disconnected coalitions are not formally covered by most of the functions f considered, their cooperative schemes describing these situations involve them. For example, the monotonic approach considers that the effectiveness of a non-connected coalition corresponds to the worth of its most effective component.

3. On cooperation in covert networks under externalities

In this section, we analyse the problem of cooperation in a covert network from an innovative perspective with respect to those considered by Lindelauf et al. (2013), Husslage et al. (2015), or even Algaba et al. (2022), that assume the existence of a specific coalitional structure on the set of agents. Naturally, the overall effectiveness of the merger of a group of members of the covert network is influenced by the organisation of the remaining members and their possible mergers. In this sense, Thrall & Lucas (1963) introduce the model of *games in partition function form* to describe such situations in which the worth of a coalition substantially depends on how the remaining agents are organised. In this framework the basic organisation of agents is called an *embedded coalition*, which is a pair whose first component is an element of a partition and whose second component contains the remaining elements of the partition (sometimes, the whole partition).

Let N be a set of agents and $\Pi(S)$ the set of partitions of S for every $S \subseteq N$. Formally, an *embedded coalition* is given by a pair $(S; P)$ with $S \subseteq N$ and $P \in \Pi(N \setminus S)$. Note that if $S = N$, then we obtain the embedded coalition $(N; \emptyset)$. Moreover, if $S = \emptyset$, we obtain embedded coalitions of the form $(\emptyset; P)$ for every $P \in \Pi(N)$. The family of embedded coalitions is denoted by

$$EC^N = \{(S; P) : \ P \in \Pi(N \setminus S) \text{ and } S \subseteq N\}. \tag{1}$$

A *partition function form game* or a *game with externalities* is formally defined by a function $v : EC^N \longrightarrow \mathbb{R}$ such that $v(\emptyset; P) = 0$, for every $P \in \Pi(N)$. We denote the family of all games with externalities with player set N by $PG(N)$.

Next, a game with externalities associated with any multi-agent covert network problem (G, f) is defined as follows.

[1] To simplify the notation, whenever possible we avoid explicit mention of sets $\mathcal{I}$ and $\mathcal{R}$ considered in the definition of f, to represent the multi-agent covert network problem, (G, f).

Definition 3.1. Let $(G, f) = (N, E, f)$ be a multi-agent covert network problem. The associated *partition function form game* $\boldsymbol{v}_{G,f}$ is defined as follows:

$$\boldsymbol{v}_{G,f}(S; P) = \begin{cases} 1, & \text{if } \max_{T \in \Sigma_S} f(T, \mathcal{I}, \mathcal{R}) \geq \max_{T \in \Sigma_{S'}} f(T, \mathcal{I}, \mathcal{R}), \\ & \text{for all } S' \in P, \\ & \text{with } \emptyset \neq S \subseteq N, \ P \in \Pi(N \setminus S), \\ 0, & \text{otherwise,} \end{cases}$$

where Σ_S is the set of components (maximal connected coalitions) in G_S, for all $S \subseteq N$, and f is the considered context specific and tailor-made nonnegative function depending on T, $\mathcal{I}$, and $\mathcal{R}$. Thus, $\boldsymbol{v}_{G,f}$ is called a *covert network game*.

Roughly speaking, $\mathbf{v}_{G,f}$ assigns a worth of 1 to each $(S; P) \in EC^N$ if S is the most effective coalition among all the elements of the coalitional structure determined by $(S; P)$. Otherwise, if the most effective coalition is formed by the outsiders of S (i.e., an element of P), with $P \in \Pi(N \setminus S)$, S obtains a worth of zero. Note that the definition of covert network games acceptably ensures that more than one disjoint coalition in a certain coalitional structure for N can be recognized as the most efficient coalition by $\mathbf{v}_{G,f}$ as long as its relative efficiency is the maximum and equal (in case of ties).

In particular, a covert network game $\mathbf{v}_{G,f} \in PG(N)$ is said to be a simple game with externalities if it satisfies (i) $\mathbf{v}_{G,f}(S; P) \in \{0, 1\}$, for all $(S; P) \in EC^N$; (ii) $\mathbf{v}_{G,f}(N; \emptyset) = 1$; and (iii) $\mathbf{v}_{G,f}$ is monotonic, that is, $\mathbf{v}_{G,f}(S; P) \leq \mathbf{v}_{G,f}(T; Q)$, for all $(S; P), \ (T; Q) \in EC^N$ such that $(S; P) \sqsubseteq (T; Q)$.[2] For a more in-depth analysis of simple games in partition function form, refer to Alonso-Meijide, Álvarez-Mozos, & Fiestras-Janeiro (2017). Evidently, (i) and (ii) hold; however, the fulfilment of (iii), and therefore that covert network games are simple, depends on function f being considered.

In the remainder of the paper, we will use the following notation. For a given coalition $S \subseteq N$, $\lfloor S \rfloor$ denotes set $\{\{i\} : \ i \in S\}$. It represents that all agents in S act individually. On the other hand, $\lceil S \rceil$ indicates that coalition S is a whole element in the sense that all its members cooperate to act as a whole block. That is, $\lceil S \rceil = \{\{S\}\}$.

Next, we check the fulfilment of the most basic properties of games with externalities that covert network games can satisfy. Among others, superadditivity, efficiency, or the sense of externalities of partition function form games are some of the most natural properties studied in this setting. For more details, refer to Hafalir (2007). Previously, we introduce notation for the order relations between partitions. Recall that $\Pi(N)$ can be considered an ordered set with the following order relation. Given partitions $P, Q \in \Pi(N)$, *P precedes Q* (or *P is finer than Q*), $P \preceq Q$, if for every $S \in P$, there exists $T \in Q$ such that $S \subseteq T$. That is, the elements in Q are obtained from unions of the elements in P.

A partition function form game $v \in PG(N)$ has *positive externalities* if for every $S \subseteq N$, $P, Q \in \Pi(N \setminus S)$ such that $P \preceq Q$, $v(S; P) \leq v(S; Q)$. This means that the earnings of coalition S do not decrease according to ordering $\preceq$ on $\Pi(N \setminus S)$. The following example illustrates that a covert network game generally has no positive externalities.

Example 3.2. Let $(G, f) = (N, E, f)$ be a multi-agent covert network situation such that N is the set of agents, with $N = \{1, 2, 3, 4, 5\}$. Fig. 1 displays its associated graph. Table 1 presents the set of weights of agents in N, and the weights of the links that indicate the relational strength between members of the covert network are 1.

To worth the effectiveness of any coalition, we consider function f, as proposed in Lindelauf et al. (2013). That is, map

[2] Recall that if $(S; P) \in EC^N$ and $(T; Q) \in EC^N$, $(S; P) \sqsubseteq (T; Q)$ if and only if $S \subseteq T$ and for all $T' \in Q$, there exists $S' \in P$ such that $T' \subseteq S'$.

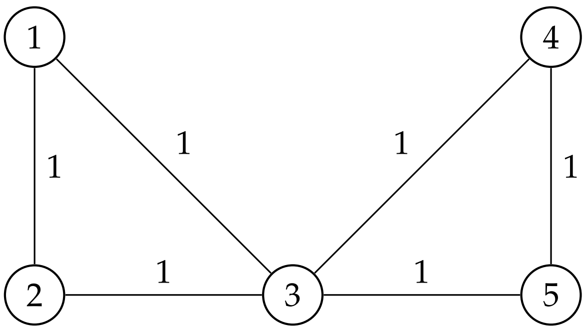

Fig. 1. Graph of covert network G.

Table 1
List of weights for nodes in G.

Agent i	w_i
1	2
2	3
3	3
4	1
5	4

$f : 2^N \longrightarrow \mathbb{R}$ is specifically defined for every connected coalition $S \subseteq N$ in $G = (N, E)$ as

$$f(S, \mathcal{I}, \mathcal{R}) = \begin{cases} (\sum_{i\in S} w_i) \cdot \max_{ij\in E_S} k_{ij}, & \text{if } |S| > 1, \\ w_S, & \text{if } |S| = 1. \end{cases} \quad (2)$$

Consider coalition $S = \{1, 2\}$ and the partitions for $N \setminus S$ given by $P = \{\{3\}, \{4\}, \{5\}\}$ and $Q = \{\{3, 5\}, \{4\}\}$ that satisfy $P \preceq Q$. In addition, consider f as the function in (2), which measures the effectiveness of each connected coalition $T \subseteq N$. Thus, we have $\mathbf{v}_{G,f}(S; P) = 1$ because

$$5 = f(\{1, 2\}, \mathcal{I}, \mathcal{R}) = \max_{T\in\Sigma_S} f(T, \mathcal{I}, \mathcal{R}) > \max_{T\in\Sigma_{S'}:\ S'\in P} f(T, \mathcal{I}, \mathcal{R}) = f(\{5\}, \mathcal{I}, \mathcal{R}) = 4.$$

However, $\mathbf{v}_{G,f}(S; Q) = 0$ because

$$5 = f(\{1, 2\}, \mathcal{I}, \mathcal{R}) = \max_{T\in\Sigma_S} f(T, \mathcal{I}, \mathcal{R}) < \max_{T\in\Sigma_{S'}:\ S'\in Q} f(T, \mathcal{I}, \mathcal{R}) = f(\{3, 5\}, \mathcal{I}, \mathcal{R}) = 7.$$

Then, we immediately obtain $\mathbf{v}_{G,f}(S; P) > \mathbf{v}_{G,f}(S; Q)$.

Reasonably, this example shows that the relative effectiveness of a coalition S with respect to a partition P of $N \setminus S$ may be reduced if the outsiders of S merge. Formally, a partition function form game $v \in PG(N)$ has *negative externalities* if for every $S \subseteq N$, $P, Q \in \Pi(N \setminus S)$ such that $P \preceq Q$, it satisfies $v(S; P) \geq v(S; Q)$. That is, the earnings of coalition S decrease according to ordering $\preceq$ on $\Pi(N \setminus S)$. Analogous to Example 3.2, determining counterexamples that illustrate that covert network games do not have negative externalities is generally straightforward.

In the following, we formally prove that covert network games, if the considered function f is monotonic, have negative or positive externalities depending on the sense of such monotonicity. Given a multi-agent covert network problem (G, f), the effectiveness function $f : 2^N \longrightarrow \mathbb{R}$ is monotonically increasing if it satisfies $f(S, \mathcal{I}, \mathcal{R}) \leq f(T, \mathcal{I}, \mathcal{R})$ for all $S \subseteq T \subseteq N$. Otherwise, f is said to be monotonically decreasing if it satisfies $f(S, \mathcal{I}, \mathcal{R}) \geq f(T, \mathcal{I}, \mathcal{R})$ for all $S \subseteq T \subseteq N$.

Proposition 3.3. *Let $(G, f) = (N, E, f)$ be a multi-agent covert network problem and $\mathbf{v}_{G,f}$ be the associated covert network game. Then, the following two statements hold.*

(a) *If effectiveness function f is monotonically increasing, $\mathbf{v}_{G,f}$ has negative externalities.*
(b) *If effectiveness function f is monotonically decreasing, $\mathbf{v}_{G,f}$ has positive externalities.*

Proof. Let $S \subseteq N$ and $P, Q \in \Pi(N \setminus S)$ such that $P \preceq Q$. We show that $\mathbf{v}_{G,f}(S; P) \geq \mathbf{v}_{G,f}(S; Q)$.

First, we prove (a). It readily follows that

$$\max_{W\in\Sigma_T:\ T\in P} f(W, \mathcal{I}, \mathcal{R}) \leq \max_{W\in\Sigma_{T'}:\ T'\in Q} f(W, \mathcal{I}, \mathcal{R}) \quad (3)$$

because of the increasing monotonicity of f, which ensures that $f(T, \mathcal{I}, \mathcal{R}) \leq f(T', \mathcal{I}, \mathcal{R})$ for all $T \subseteq T' \subseteq N$, and of the ordering relation $P \preceq Q$, that ensures that for every $T \in P$ there exists $T' \in Q$ with $T \subseteq T'$.

Now, consider covert network game $\mathbf{v}_{G,f}$. We distinguish two possibilities.

Case 1. First, we consider the case in which $\max_{W\in\Sigma_S} f(W, \mathcal{I}, \mathcal{R}) < \max_{W\in\Sigma_T:\ T\in P} f(W, \mathcal{I}, \mathcal{R})$. The inequality in (3) and Definition 3.1 ensure that $\mathbf{v}_{G,f}(S; P) = \mathbf{v}_{G,f}(S; Q) = 0$.

Case 2. Second, we consider the case in which

$$\max_{W\in\Sigma_S} f(W, \mathcal{I}, \mathcal{R}) \geq \max_{W\in\Sigma_T:\ T\in P} f(W, \mathcal{I}, \mathcal{R}).$$

Then, we distinguish two subcases.

Subcase 2.1. If

$$\max_{W\in\Sigma_S} f(W, \mathcal{I}, \mathcal{R}) < \max_{W\in\Sigma_{T'}:\ T'\in Q} f(W, \mathcal{I}, \mathcal{R})$$

is satisfied, we have $1 = \mathbf{v}_{G,f}(S; P) > \mathbf{v}_{G,f}(S; Q) = 0$.

Subcase 2.2. If

$$\max_{W\in\Sigma_S} f(W, \mathcal{I}, \mathcal{R}) \geq \max_{W\in\Sigma_{T'}:\ T'\in Q} f(W, \mathcal{I}, \mathcal{R})$$

holds, we have $\mathbf{v}_{G,f}(S; P) = \mathbf{v}_{G,f}(S; Q) = 1$.

Therefore, we can conclude that $\mathbf{v}_{G,f}$ has negative externalities.

The scheme of the proof for (b) is similar. Now, we must consider the monotonically decreasing character of f, which ensures that

$$\max_{W\in\Sigma_T:\ T\in P} f(W, \mathcal{I}, \mathcal{R}) \geq \max_{W\in\Sigma_{T'}:\ T'\in Q} f(W, \mathcal{I}, \mathcal{R}),$$

because of $f(T, \mathcal{I}, \mathcal{R}) \geq f(T', \mathcal{I}, \mathcal{R})$ for all $T \subseteq T' \subseteq N$ and the ordering relation $P \preceq Q$. □

Based on the definition of f given in (2) and the previous result, the following corollary immediately follows.

Corollary 3.4. *Let $(G, f) = (N, E, f)$ be a multi-agent covert network problem. If effectiveness function f is defined as in (2), the associated covert network game$\mathbf{v}_{G,f}$ has negative externalities.*

From the covert network perspective, the negative externalities ensure that, for any coalition $S \subseteq N$, its relative effectiveness with respect to a partition P of $N \setminus S$ decreases as the elements of such a partition progressively merge (i.e., the outsiders of S form larger blocks with elements of P).

In addition, we study the profitable character of cooperation in covert network games. To achieve this, we consider the following property. A partition function form game v is *superadditive* (Hafalir, 2007) if for every $S, T \subseteq N$ with $S \cap T = \emptyset$ and $P \in \Pi(N \setminus (S \cup T))$, the following holds:

$$v(S \cup T; P) \geq v(S; P \cup \lceil T \rceil) + v(T; P \cup \lceil S \rceil).$$

This property ensures that the joint relative effectiveness of merging increases with respect to the sum of the relative effectiveness of each coalition involved. Alternatively, a game with externalities

v is said to be *subadditive* if, for every $S, T \subseteq N$ with $S \cap T = \emptyset$ and $P \in \Pi(N \setminus (S \cup T))$, the following holds:

$$v(S \cup T; P) \leq v(S; P \cup \lceil T \rceil) + v(T; P \cup \lceil S \rceil).$$

This implies that the joint relative effectiveness of merging is less than the sum of the relative effectiveness of each coalition involved. Note that these notions of superadditivity and subadditivity are immediate extensions of the analogous notions in classical transferable utility cooperative games.

Example 3.2 shows that covert network games are generally neither superadditive nor subadditive.

Example 3.5. We revisit the multi-agent covert network situation, $(G, f) = (N, E, f)$, considered in Example 3.2. Let $S = \{3, 4\}$, $T = \{5\}$, and $P = \{\{1\}, \{2\}\}$ be partitions of $N \setminus \{3, 4, 5\}$ and $\boldsymbol{v}_{G,f}$ be the covert network game associated with (G, f). Thus, we have

$$\begin{aligned} 1 &= \boldsymbol{v}_{G,f}(\{3,4,5\}; \{\{1\},\{2\}\}) \\ &< \boldsymbol{v}_{G,f}(\{3,4\}; \{\{1\},\{2\},\{5\}\}) + \boldsymbol{v}_{G,f}(\{5\}; \{\{1\},\{2\},\{3,4\}\}) \\ &= 1 + 1. \end{aligned}$$

Hence, $\boldsymbol{v}_{G,f}$ is not superadditive. However, $\boldsymbol{v}_{G,f}$ is also not subadditive. Consider $S = \{2\}$, $T = \{3\}$, and $P = \{\{1\}, \{4, 5\}\}$. Thus, we have

$$\begin{aligned} 1 &= \boldsymbol{v}_{G,f}(\{2,3\}; \{\{1\},\{4,5\}\}) \\ &> \boldsymbol{v}_{G,f}(\{2\}; \{\{1\},\{3\},\{4,5\}\}) + \boldsymbol{v}_{G,f}(\{3\}; \{\{1\},\{2\},\{4,5\}\}) \\ &= 0 + 0. \end{aligned}$$

Hence, $\boldsymbol{v}_{G,f}$ is not subadditive.

Moreover, we can study if covert network games satisfy the property of efficiency or cohesiveness (cf. Hafalir, 2007). Formally, a partition function form game v is said to be *efficient (cohesive)* if for every $P \in \Pi(N)$,

$$v(N; \emptyset) \geq \sum_{S \in P} v(S; P_{-S}),$$

where P_{-S} denotes the partition induced by P on $N \setminus S$. That is, if $P \in \Pi(N)$, then a partition for $N \setminus S$ induced by P is given by $P_{-S} = \{T \setminus S : T \in P\}$ for every $S \subseteq N$. Let now $(G, f) = (N, E, f)$ be a multi-agent covert network problem and $\boldsymbol{v}_{G,f}$ be the associated covert network game. Again, Example 3.2 can be used as a counterexample to this property.

Example 3.6. Again, $(G, f) = (N, E, f)$ is the multi-agent covert network situation in Example 3.2 and $\boldsymbol{v}_{G,f}$ is the covert network game associated with (G, f). Consider the partition of N given by $P = \{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$. Thus, we have $\boldsymbol{v}_{G,f}(N; \emptyset) = 1$ and, by reusing the calculations from Example 3.5, it satisfies

$$\boldsymbol{v}_{G,f}(\{3,4\}; \{\{1\},\{2\},\{5\}\}) = 1 \text{ and } \boldsymbol{v}_{G,f}(\{5\}; \{\{1\},\{2\},\{3,4\}\}) = 1.$$

Hence,

$$\begin{aligned} 1 = \boldsymbol{v}_{G,f}(N; \emptyset) &< \boldsymbol{v}_{G,f}(\{1\}; \{\{2\},\{3,4\},\{5\}\}) \\ &+ \boldsymbol{v}_{G,f}(\{2\}; \{\{1\},\{3,4\},\{5\}\}) + \boldsymbol{v}_{G,f}(\{3,4\}; \{\{1\},\{2\},\{5\}\}) \\ &+ \boldsymbol{v}_{G,f}(\{5\}; \{\{1\},\{2\},\{3,4\}\}) = 2. \end{aligned}$$

Note that the efficiency property is unsatisfied, in general, as a consequence of the construction of $\mathbf{v}_{G,f}$. Given a partition in N, two disjoint coalitions can have the same associated global efficiency. Therefore, $\mathbf{v}_{G,f}$ can assign a value of 1 to both, which prevents the verification of the efficiency property.

4. On coalition formation, allocations and rankings

In the presence of externalities, the analysis of the formation of coalitions and the allocation of the joint benefits derived from their cooperation from a game theoretical approach is of interest. In this section, we consider allocations for games with externalities to rank the members of a covert network according to their contribution to the overall effectiveness.

For this purpose, we use some proposals in the literature that involve games with transferable utility. Before, we first recall the notion of a game with transferable utility. Formally, a *game with transferable utility or TU game* is a pair (N, w), where N is the finite set of agents and w is a map from 2^N to $\mathbb{R}$ satisfying $w(\emptyset) = 0$ (cf. González-Díaz, García-Jurado, & Fiestras-Janeiro, 2010). The class of TU games with a set of agents N is denoted by G^N. Given a game with externalities $v \in PG(N)$, the following TU games can be associated intuitively.

Formally, the TU games $(N, v^{\max})$ and $(N, v^{\min})$ in G^N associated with any partition function form game v are respectively given, for every $S \subseteq N$, by:

$$v^{\max}(S) = \max_{P \in \Pi(N \setminus S)} v(S; P) \quad \text{and} \quad v^{\min}(S) = \min_{P \in \Pi(N \setminus S)} v(S; P).$$

By definition, $v^{\max}$ and $v^{\min}$ assign the maximum (optimistic perspective) and minimum (pessimistic perspective) worth to each coalition $S \subseteq N$ that the members of S can obtain through their cooperation, among all possible structures of $N \setminus S$.

Although these games have been presented in the literature, they show a peculiarity in the partition function form games considered in this setup. For a covert network game $\mathbf{v}_{G,f}$, note that the expressions for optimistic and pessimistic TU games $\mathbf{v}_{G,f}^{\max}$ and $\mathbf{v}_{G,f}^{\min}$, respectively, can be explicitly provided for specific effectiveness functions. Moreover, the expressions obtained are computationally useful. From Proposition 3.3, which ensures the sense of the externalities of any covert network game $\mathbf{v}_{G,f}$ under the monotonicity of f, the following corollary can be readily established.

Corollary 4.1. *Let $(G, f) = (N, E, f)$ be a multi-agent covert network problem and let $\mathbf{v}_{G,f}$ be the associated covert network game. Then, the following two statements hold.*

(a) *For each $S \subseteq N$, if effectiveness function f is monotonically increasing, the following holds:*

$$\mathbf{v}_{G,f}^{\max}(S) = \mathbf{v}_{G,f}(S; \lfloor N \setminus S \rfloor) \quad \text{and} \quad \mathbf{v}_{G,f}^{\min}(S) = \mathbf{v}_{G,f}(S; \lceil N \setminus S \rceil). \tag{4}$$

(b) *For each $S \subseteq N$, if effectiveness function f is monotonically decreasing, it satisfies that*

$$\mathbf{v}_{G,f}^{\max}(S) = \mathbf{v}_{G,f}(S; \lceil N \setminus S \rceil) \quad \text{and} \quad \mathbf{v}_{G,f}^{\min}(S) = \mathbf{v}_{G,f}(S; \lfloor N \setminus S \rfloor).$$

Recall that for a given coalition $T \subseteq N$, $\lfloor T \rfloor$ denotes $\{\{j\} : j \in T\}$, and $\lceil T \rceil$ means that T is an element of the partition, that is, acting as a block. In particular, if effectiveness function f is monotonically increasing, the maximum worth of the cooperation of S is reached when the partition of $N \setminus S$ is formed by the individual agents, whereas the minimum worth is reached when the outsiders act as a whole block.

Alternatively, we can consider procedures based on averages that are particularly suitable in the presence of uncertainty. Let $v \in PG(N)$ be a game with externalities. Albizuri et al. (2005) proposed to assign the TU game $(N, \bar{v}) \in G^N$, given by

$$\bar{v}(S) = \frac{1}{|\Pi(N \setminus S)|} \sum_{P \in \Pi(N \setminus S)} v(S; P), \text{ for each coalition } S \subseteq N. \tag{5}$$

Here, coalition S obtains the expected value of the cooperation of their members in v, that is, the average over the whole set of the embedded coalitions $(S; P)$ with $P \in \Pi(N \setminus S)$, when they are equally likely.

Table 2
TU games $(N, \mathbf{v}^{min}_{G,f})$, $(N, \bar{\mathbf{v}}_{G,f})$, $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, and $(N, \mathbf{v}^{max}_{G,f})$.

S	$\mathbf{v}^{min}_{G,f}(S)$	$\bar{\mathbf{v}}_{G,f}(S)$	$\bar{\bar{\mathbf{v}}}_{G,f}(S)$	$\mathbf{v}^{max}_{G,f}(S)$
∅	0	0	0	0
{1}	0	0	0	0
{2}	0	0	0	0
{3}	0	0	0	0
{4}	0	0	0	0
{5}	0	0.26667	0.32692	1
{1, 2}	0	0.60000	0.71154	1
{1, 3}	1	1	1	1
{1, 4}	0	0	0	0
{1, 5}	0	0.60000	0.71154	1
{2, 3}	1	1	1	1
{2, 4}	0	0	0	0
{2, 5}	0	0.60000	0.71154	1
{3, 4}	0	0.60000	0.71154	1
{3, 5}	1	1	1	1
{4, 5}	0	0.60000	0.71154	1
{1, 2, 3}	1	1	1	1
{1, 2, 4}	0	0.50000	0.71154	1
{1, 2, 5}	1	1	1	1
{1, 3, 4}	1	1	1	1
{1, 3, 5}	1	1	1	1
{1, 4, 5}	0	0.50000	0.71154	1
{2, 3, 4}	1	1	1	1
{2, 3, 5}	1	1	1	1
{2, 4, 5}	1	1	1	1
{3, 4, 5}	1	1	1	1
{1, 2, 3, 4}	1	1	1	1
{1, 2, 3, 5}	1	1	1	1
{1, 2, 4, 5}	1	1	1	1
{1, 3, 4, 5}	1	1	1	1
{2, 3, 4, 5}	1	1	1	1
{1, 2, 3, 4, 5}	1	1	1	1

Finally, we mention the approach of Hu & Yang (2010), which considers the TU game $(N, \bar{\bar{v}}) \in G^N$ given by

$$\bar{\bar{v}}(S) = \frac{1}{|\Pi(N)|} \sum_{P \in \Pi(N)} v(S; P_{-S}), \text{ for each } S \subseteq N, \tag{6}$$

where P_{-S} denotes the partition induced by P on $N \setminus S$. Thus, $\bar{\bar{v}}(S)$ is, for each $S \subseteq N$, the expected worth of S in v over the set of embedded coalitions induced by any partition in $\Pi(N)$, assuming that all partitions $P \in \Pi(N)$ are equally likely.

The following example illustrates the obtaining of the TU games $(N, \mathbf{v}^{max}_{G,f})$, $(N, \mathbf{v}^{min}_{G,f})$, $(N, \bar{\mathbf{v}}_{G,f})$, and $(N, \bar{\bar{\mathbf{v}}}_{G,f})$ associated with a covert network game $\mathbf{v}_{G,f}$.

Example 4.2. Let $(G, f) = (N, E, f)$ be the multi-agent covert network situation considered in Example 3.2. Thus, from the associated covert network game $\boldsymbol{v}_{G,f}$, we obtain the TU games $(N, \boldsymbol{v}^{min}_{G,f})$, $(N, \bar{\mathbf{v}}_{G,f})$, $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, and $(N, \boldsymbol{v}^{max}_{G,f})$. Table 2 presents these results.

Note that although the analysis of the properties of these four TU games may be of interest, it is beyond the scope of this paper. Nonetheless, we note that, in this example, the games $\boldsymbol{v}^{min}_{G,f}$ and $\boldsymbol{v}^{max}_{G,f}$ are simple and that $\boldsymbol{v}^{min}_{G,f}$ is not a proper game because $\boldsymbol{v}^{min}_{G,f}(\{1,3\}) = \boldsymbol{v}^{min}_{G,f}(\{2,4,5\}) = 1$ and $\{1,3\}$ and $\{2,4,5\}$ are disjoint coalitions. The same occurs for $\boldsymbol{v}^{max}_{G,f}$. In addition, $\bar{\mathbf{v}}_{G,f}$ does not satisfy monotonicity as $\bar{\mathbf{v}}_{G,f}(\{1,5\}) > \bar{\mathbf{v}}_{G,f}(\{1,4,5\})$ and $\{1,5\} \subset \{1,4,5\}$.

Below, we address the main task of ranking agents in multi-agent covert network problems based on the concept of cooperation considered in this paper. For this purpose, tools from cooperative game theory, such as the values and solutions for TU games, are used. They assign a vector in $\mathbb{R}^N$ to every TU game $(N, w) \in G^N$ and are typically inspired by the concept of the *marginal contribution* of an agent. That is, with a fixed $i \in N$, a TU game $(N, w) \in G^N$ and a coalition S that does not contain it, the marginal contribution of agent i to $S \subseteq N \setminus \{i\}$ is specified as follows:

$$w(S \cup \{i\}) - w(S). \tag{7}$$

One of the most important values is the *Shapley value* (Shapley, 1953), which uses a marginalist approach and is defined, for every $i \in N$ and every TU game $(N, w) \in G^N$, by

$$Sh_i(N, w) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\ (|N| - |S| - 1)!}{|N|!} (w(S \cup \{i\}) - w(S)). \tag{8}$$

Under the presence of externalities, this value can be naturally extended to this new context. In analysing covert networks, the Shapley value can be interpreted as a measure of the ability of agents to modify the relative effectiveness of merging under cooperation.

Based on this solution, many extensions are justified according to different criteria. For instance, we mention the case of the generalization of the Shapley value in Choudhury, Borkotokey, Kumar, & Sarangi (2021), which seeks balance between the Shapley value and the Equal Division rule as representatives of marginalism and egalitarianism, respectively. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a collection of values in the interval $[0, 1]$ such that $0 = \alpha_0 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_n = 1$. The *Generalized Egalitarian Shapley value* (or α-GES value) for every $i \in N$ and every $(N, w) \in G^N$ is formally given by

$$Sh_i^{\alpha-GES}(N, w) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\ (|N| - |S| - 1)!}{|N|!} (\alpha_{|S \cup \{i\}|} w(S \cup \{i\}) - \alpha_{|S|} w(S)). \tag{9}$$

The α-GES value generalises other existing values studied in the literature, such as the *Solidarity value* for a TU game $(N, w) \in G^N$. This is first considered in Nowak & Radzik (1994) and is based on a marginalist approach. This corresponds to Expression 9, when $\alpha_{|S|} = \frac{1}{|S|+1}$ for every $S \subset N$ such that $1 < |S| < n$. We denote this as $Sh^s(N, w)$.

Handling simple games (with externalities) justifies the use of very popular solutions in these settings, such as the Banzhaf value and the Deegan-Packel index for a TU game $(N, w) \in G^N$, which are based on different criteria. The *Banzhaf value* for a TU game $(N, w) \in G^N$ (Banzhaf, 1964), denoted by $Bz(N, w)$, is defined under the marginalist approach as

$$Bz_i(N, w) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \setminus \{i\}} (w(S \cup \{i\}) - w(S)), \text{ for every } i \in N. \tag{10}$$

The *Deegan-Packel index* (Deegan & Packel, 1978) for a simple game $(N, w) \in G^N$ is based on the idea that only minimal winning coalitions matter when assessing the power of agents. In the following, for a simple game $(N, w) \in G^N$, $\mathcal{W}$ denotes the subset of coalitions of N that are winning, i.e., those coalitions $S \subseteq N$ satisfying $w(S) = 1$. A winning coalition is minimal if all its members are critical. The Deegan-Packel index of player i, with $i \in N$, for a simple game (N, w) is given by

$$DP_i(N, w) = \frac{1}{|\mathcal{M}|} \sum_{S \in \mathcal{M}_i} \frac{1}{|S|}, \tag{11}$$

where $\mathcal{M} = \{S \in \mathcal{W} : \forall T \subset S, T \notin \mathcal{W}\}$ is the set of minimal winning coalitions and $\mathcal{M}_i = \{S \in \mathcal{M} : \ i \in S\}$ denotes the subset of those that contain player i.

Let $v \in PG(N)$ be a game with externalities. In the following, we list a collection of those solutions originally proposed for games in partition function form, which we use to rank the members of a covert network according to the decreasing order of their components. First, we discuss those based on the Shapley value

Table 3
Ranking of agents based on the Shapley value (SH), the Generalized Egalitarian Shapley value (GES), the Solidarity value (S), the Banzhaf value (BZ) and the Deegan-Packel index (DP).

		Numerical results					Positions				
		Ag. 1	Ag. 2	Ag. 3	Ag. 4	Ag. 5	Ag. 1	Ag. 2	Ag. 3	Ag. 4	Ag. 5
(SH)	$(N, \mathbf{v}^{min}_{G,f})$	0.1167	0.2000	0.4500	0.0333	0.2000	4	2-3	1	5	2-3
	$(N, \bar{\mathbf{v}}_{G,f})$	0.1367	0.1783	0.3367	0.0533	0.2950	4	3	1	5	2
	$(N, \bar{\bar{\mathbf{v}}}_{G,f})$	0.1478	0.1718	0.3032	0.0644	0.3128	4	3	2	5	1
	$(N, \mathbf{v}^{max}_{G,f})$	0.1333	0.1333	0.2167	0.0500	0.4667	3–4	3–4	2	5	1
(GES)	$(N, \mathbf{v}^{min}_{G,f})$	0.1533	0.2033	0.3200	0.1200	0.2033	4	2-3	1	5	2-3
	$(N, \bar{\mathbf{v}}_{G,f})$	0.1707	0.1957	0.2673	0.1373	0.2290	4	3	1	5	2
	$(N, \bar{\bar{\mathbf{v}}}_{G,f})$	0.1785	0.1929	0.2503	0.1452	0.2330	4	3	1	5	2
	$(N, \mathbf{v}^{max}_{G,f})$	0.1833	0.1833	0.2167	0.1500	0.2667	3–4	3–4	2	5	1
(S)	$(N, \mathbf{v}^{min}_{G,f})$	0.1778	0.1986	0.2750	0.1500	0.1986	4	2-3	1	5	2-3
	$(N, \bar{\mathbf{v}}_{G,f})$	0.1728	0.1832	0.2325	0.1450	0.2665	4	3	2	5	1
	$(N, \bar{\bar{\mathbf{v}}}_{G,f})$	0.1733	0.1793	0.2211	0.1455	0.2808	4	3	2	5	1
	$(N, \mathbf{v}^{max}_{G,f})$	0.1444	0.1444	0.1722	0.1167	0.4222	3–4	3–4	2	5	1
(BZ)	$(N, \mathbf{v}^{min}_{G,f})$	0.1875	0.3125	0.6875	0.0625	0.3125	4	2-3	1	5	2-3
	$(N, \bar{\mathbf{v}}_{G,f})$	0.1958	0.2583	0.4958	0.0708	0.3667	4	3	1	5	2
	$(N, \bar{\bar{\mathbf{v}}}_{G,f})$	0.2115	0.2476	0.4447	0.0865	0.3774	4	3	1	5	2
	$(N, \mathbf{v}^{max}_{G,f})$	0.1875	0.1875	0.3125	0.0625	0.4375	3–4	3–4	2	5	1
(DP)	$(N, \mathbf{v}^{min}_{G,f})$	0.1667	0.2333	0.3000	0.0667	0.2333	4	2-3	1	5	2-3
	$(N, \mathbf{v}^{max}_{G,f})$	0.2000	0.2000	0.3000	0.1000	0.2000	2-3-4	2-3-4	1	5	2-3-4

of those TU games that can be associated with v. Albizuri et al. (2005) propose the Shapley value for the TU game $(N, \bar{v})$ as a solution for v, that is, $Sh(N, \bar{v})$. For v, Pham Do & Norde (2007) and de Clippel & Serrano (2008) propose the Shapley value for the TU game (N, v^{min}), i.e., $Sh(N, v^{min})$, the externality-free Shapley value. McQuillin (2009) considers the Shapley value for (N, v^{max}) as an allocation, that is, $Sh(N, v^{max})$. Finally, we discuss the proposal of Hu & Yang (2010), which is based on the Shapley value for $(N, \bar{\bar{v}})$.

However, the aforementioned solutions for TU games, based or not on the Shapley value, can be naturally applied in this setting. These solutions are listed as follows.

- The Generalized Egalitarian Shapley value, the α-GES value given in (9), can be considered for the TU games (N, v^{min}), $(N, \bar{v})$, $(N, \bar{\bar{v}})$, and $(N, v^{\max})$.
- Similarly, the Solidarity value for (N, v^{min}), $(N, \bar{v})$, $(N, \bar{\bar{v}})$, and $(N, v^{\max})$, given by the expression in (9) with $\alpha_{|S|} = \frac{1}{|S|+1}$ for every $S \subset N$ such that $1 < |S| < n$, can be considered.
- Alternatively, the Banzhaf value (10) is of interest, i.e., by ranking the members of the covert network according to $Bz(N, v^{min})$, $Bz(N, \bar{v})$, $Bz(N, \bar{\bar{v}})$, and $Bz(N, v^{\max})$.
- Finally, the Deegan-Packel index can be considered for the TU games (N, v^{min}) and $(N, v^{\max})$ because they are simple games.

Next, Example 4.3 illustrates the performance of all possible rankings on covert network games.

Example 4.3. Consider the multi-agent covert network situation $(G, f) = (N, E, f)$ in Example 3.2. Example 4.2 presents the four considered TU games derived from $\mathbf{v}_{G,f}$.

Table 3 presents the rankings based on the Shapley value (SH), the Generalized Egalitarian Shapley value (GES), the Solidarity value (S), and the Banzhaf value (BZ) for $(N, \mathbf{v}^{min}_{G,f})$, $(N, \bar{\mathbf{v}}_{G,f})$, $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, and $(N, \mathbf{v}^{max}_{G,f})$ and the Deegan-Packel index (DP) for $(N, \mathbf{v}^{min}_{G,f})$ and $(N, \mathbf{v}^{max}_{G,f})$.

As indicated by (SH) (rows 1–4), Agent 3 is in position 1 when using $\mathbf{v}^{min}_{G,f}$ and $\bar{\mathbf{v}}_{G,f}$. Otherwise, Agent 5 rises to this position. As a common aspect of all four approaches, Agent 4 always occupies position 5 and Agent 1 is usually at position 4.

Similar conclusions can be extracted using (GES) (rows 5–8) with $\alpha_{|S|} = \frac{|S|}{|N|}$ for all $S \subseteq N$. The clearest difference is that under approach $\bar{\bar{\mathbf{v}}}_{G,f}$, Agent 3 remains in position 1. Proposals based on (BZ) (rows 13–16) provide the same rankings.

As indicated by (S) (rows 9–12), Agent 3 occupies only position 1 when using the TU game $\mathbf{v}^{min}_{G,f}$. For the other ranking options, Agent 3 is second, and Agent 5 becomes first.

Besides, (DP) only for $(N, \mathbf{v}^{min}_{G,f})$ and $(N, \mathbf{v}^{max}_{G,f})$ (rows 17–18) assigns positions 1 and 5 to Agents 3 and 4 in both rankings. Moreover, Agent 1 occupies position 4 when using $\mathbf{v}^{min}_{G,f}$. In the remaining cases, the positions are indistinguishable (ties occur).

To conclude, we highlight that Agent 3 is typically in position 1 because of the critical position of its associated node in the graph in Fig. 1, which ensures the connectivity of many coalitions. In addition, Agent 4 has the lowest weight within the covert network.

To conclude this section, we focus on determining the best player of a team in a football match using the ranking proposals presented in this paper. In this sense, the term “covert” here refers to those pre-rehearsed tactics, moves and passes that should be unknown to the opposing team. The aim of this example is to essentially generalise the use of these techniques to all types of situations that can be modelled in terms of a covert network and those in which ordering its members according to their relevance in its operations is of interest.

Example 4.4. Consider now the multi-agent covert network situation $(G, f) = (N, E, f)$ from the organisation and performance of football teams. A team is considered as a complex network whose nodes (players) interact with the aim of overcoming the opponent network. Besides, the associated graph is derived from the networks of football passes, denoting the edges of those interactions between nodes.

We consider the football match between Portugal and Spain at the 2018 World Cup in Russia, which ended in a three-goal draw. Fig. 2 shows the passing pattern among the 11 players of the Spanish national football team. On the right, we list the players involved, as well as the number of passes in which a player is involved, and the number of goals he scored in the match (in brackets). The set of individual weights for each player in N is determined by the average of the proportion of passes that he is

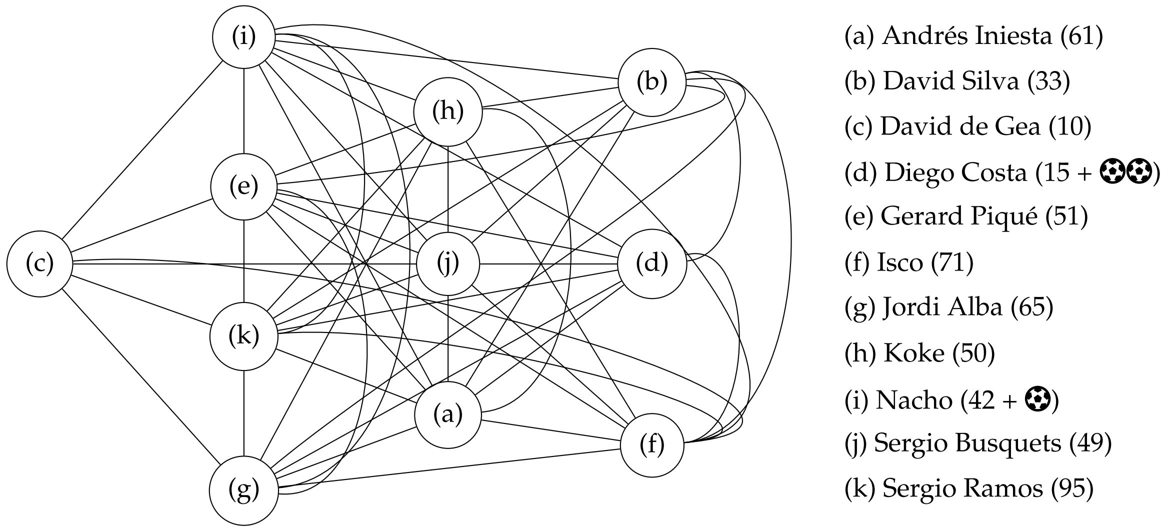

Fig. 2. Graph of covert network G associated with the Spanish national team in the Portugal-Spain match at the 2018 World Cup in Russia.

Table 4
Positions of players using the Shapley value (SH), the Generalized Egalitarian Shapley value (GES), the Solidarity value (S), the Banzhaf value (BZ), and the Deegan-Packel index (DP).

Players		(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
(SH)	$(N, \mathbf{v}_{G,f}^{min})$	6	10	11	1	8	4	3	7	5	9	2
	$(N, \bar{\mathbf{v}}_{G,f})$	6	10	11	4	8	3	2	7	5	9	1
	$(N, \mathbf{v}_{G,f}^{max})$	4	3	11	1	9	4	10	4	4	4	2
(GES)	$(N, \mathbf{v}_{G,f}^{min})$	6	10	11	1	8	4	3	7	5	9	2
	$(N, \bar{\mathbf{v}}_{G,f})$	6	10	11	2	8	4	3	7	5	9	1
	$(N, \mathbf{v}_{G,f}^{max})$	4	3	11	1	9	4	10	4	4	4	2
(S)	$(N, \mathbf{v}_{G,f}^{min})$	6	10	11	1	8	4	3	7	5	9	2
	$(N, \bar{\mathbf{v}}_{G,f})$	6	10	11	4	8	2	3	7	5	9	1
	$(N, \mathbf{v}_{G,f}^{max})$	4	3	11	1	9	4	10	4	4	4	2
(BZ)	$(N, \mathbf{v}_{G,f}^{min})$	6	10	11	1	8	4	3	7	4	9	2
	$(N, \bar{\mathbf{v}}_{G,f})$	6	10	11	2	8	4	3	7	5	9	1
	$(N, \mathbf{v}_{G,f}^{max})$	4	3	11	1	9	4	10	4	4	4	2
(DP)	$(N, \mathbf{v}_{G,f}^{min})$	3	10	11	1	9	4	6	7	5	8	2
	$(N, \mathbf{v}_{G,f}^{max})$	7	2	10	11	6	3	9	3	7	3	1

involved in relative to the total passes and proportion of goals he scores. In addition, the weights on the links that indicate the relational strength between members of the network account for the number of passes between each pair of connected players. Table A.1, in Appendix A in the Online Resource Section, contains all the information regarding the passes.

Using the effectiveness function of Lindelauf et al. in (2), we take $\boldsymbol{v}_{G,f}$ a covert network game. Table 4 lists the positions in the rankings of players based on the Shapley value (SH), the Generalized Egalitarian Shapley value (GES), the Solidarity value (S), and the Banzhaf value (BZ) for $(N, \mathbf{v}_{G,f}^{min})$, $(N, \bar{\mathbf{v}}_{G,f})$, and $(N, \boldsymbol{v}_{G,f}^{max})$, and the Deegan-Packel index (DP) for $(N, \boldsymbol{v}_{G,f}^{min})$ and $(N, \boldsymbol{v}_{G,f}^{max})$. Table A.2 in Appendix A in the Online Resource Section presents the numerical results. From Corollary 4.1, $(N, \mathbf{v}_{G,f}^{min})$ and $(N, \boldsymbol{v}_{G,f}^{max})$ are the TU games in (4). We highlight the case of the TU game $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, which cannot be obtained in feasible time because, with 11 players, we must evaluate 678,570 partitions per coalition.

Based on the results, Diego Costa (d) is typically in position 1 (he is responsible for two of the goals). Positions 2 and 3 are typically occupied by Sergio Ramos (k) and Jordi Alba (g), except for $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, where David Silva (b) moves to position 3 (in general, in position 10). Isco (f) is typically in position 4, except under (SH) and (S) for $(N, \bar{\bar{\mathbf{v}}}_{G,f})$, where Andrés Iniesta (a) moves up to this position. This indicates the key roles of defenders and strikers in the bid to win the match. The other players generally remain in the same positions (with differences of one or two). Notably, David de Gea (c) is placed at the bottom of the ranking, which may because of his responsibility to safeguard the goal posts, even under $DP(N, \mathbf{v}_{G,f}^{max})$, which yields different results.

As can be gathered from the previous example, computational problems pose when determining some of the TU games associated with any games with externalities v. Evidently, obtaining $\bar{v}$ and $\bar{\bar{v}}$ requires a large computational effort because the number of partitions to be evaluated increases dramatically with the number of players involved. To address this drawback, Saavedra-Nieves & Fiestras-Janeiro (2022) provide as solution two specific procedures based on sampling techniques (cf. Cochran, 2007) to approximate both TU games for any game with externalities v involving a large amount of agents. These problems are in addition to those known in the exact computation of, for example, the Shapley value or the Banzhaf value of a general TU game. Such drawbacks prompted the works of Fernández-García and Puerto-Albandoz (2006), Castro, Gómez, & Tejada (2009) and Maleki (2015) to estimate the Shapley value, and Bachrach et al. (2010) for the Banzhaf value approximation. However, Saavedra-Nieves & Fiestras-Janeiro (2022) also address the specific case of estimating the Shapley value of the TU games of Albizuri et al. (2005) and Hu & Yang (2010).

5. An application: the 9/11 attacks

In this section, we apply the proposed ranking methodologies to analyse the covert network of hijackers that supported the 9/11 attacks.

The September 11, attacks, more commonly referred to as 9/11, are considered among the most devastating in recent history. The number of deaths they have caused was dramatic, as well as the economic and social impact they had on the world. This chain of attacks was executed by 19 members of the Islamist terrorist group Al Qaeda. On the morning of Tuesday 11 September 2001, four commercial airliners travelling from the north-eastern parts of the United States to California were hijacked mid-flight by 19 terrorists. Two of these planes were flown into the Twin Towers of the World Trade Center in New York. A third plane was flown into the west side of the Pentagon, and a fourth plane crashed in Pennsylvania. A detailed description of these events was provided by Kean, Hamilton, & Ben-Veniste (2002). The attack on the United States was not a new idea, as several years earlier, Osama Bin Laden had declared a fight against Americans and their collaborators' interests because of what they considered intrusions and humiliations to the Islamic community. Bin Laden declared jihad against the United States in 1996. The base of this chain of attacks follows the existing proposal from other Islamic leaders and involved training pilots to fly into buildings and eventually motivated the 9/11 attack. Bin Laden, with his collaborators, selected hijackers who attacked the United States, trained them, and managed the logistics of the attack in terms of accommodation and funding.

The data considered for the analysis from a game theoretical approach were extracted from the studies conducted by Kean et al. (2002) and Krebs (2002). First, we consider the resulting covert network that can be built from the available information in Fig. 3.

Thus, $N = \{1, \ldots, 19\}$ denotes the set of nodes representing each hijacker involved. In this sense, the game theoretical analysis of covert networks adds additional information. For this purpose, according to the report of the 9/11 commissions in Kean et al. (2002), hijackers can be characterised according to their affiliation and degree of radicalisation, as accomplished in Lindelauf et al. (2013).

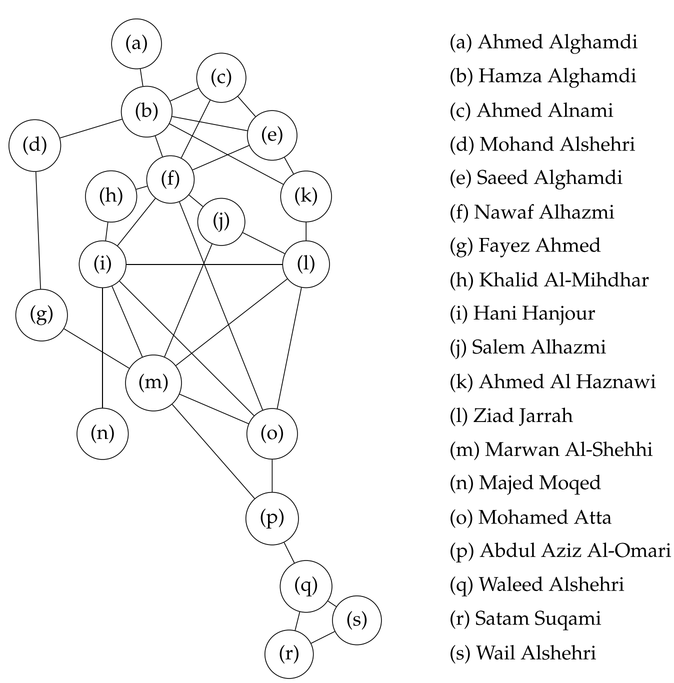

Fig. 3. Graph of covert network G responsible for the 9/11 attacks.

Table 5 summarises this information by assigning an initial weight of 1; however, this is increased according to the most relevant associated indicators. We highlight the case of Mohamed Atta (with a weight of 4), which is typically considered the leader of the attacks. However, no information on relational strength is available; therefore, we take $k_{ij} = 1$ for all $ij \in E$, as in Husslage et al. (2015).

In the following, we analyse the influence of the hijackers involved in achieving their objective of perpetrating the attacks. For this purpose, we consider the cooperative approach described in this study. The formation of coalitions under cooperation in this setting naturally increases the chances of a "successful" operation, with success being understood as an attack on certain previously designated objectives. However, the effectiveness of a given merger is clearly influenced by the organisation of its outsiders, as another merger may exist that is much more effective and that relegates it to the background. Based on its ability to increase the overall effectiveness in the event of a merger, we rank the 19 hijackers belonging to the covert network associated with the 9/11 attacks using covert network games. Therefore, this tool can be considered useful for surveillance services, as its effectiveness under cooperation is equivalent to a real threat in terrorism. Once a potential danger is identified, its neutralisation would be easier for intelligence services.

Let $\mathbf{v}_{G,f}$ be the covert network game associated with this situation, with function f of Lindelauf et al. (2013) given in (2). In particular, we obtain the rankings of the hijackers based on the Shapley value (Shapley, 1953), on the Generalized Egalitarian Shapley value (Choudhury et al., 2021), with $\alpha_{|S|} = \frac{|S|}{|N|}$ for all $S \subseteq N$, on the Solidarity value (Nowak & Radzik, 1994), and also on the Banzhaf value (Banzhaf, 1964) of such TU game. They will be denoted again, for the sake of simplicity, by (SH), (GES), (S) and (BZ), respectively. Because the TU games $(N, \mathbf{v}^{\min}_{G,f})$ and $(N, \mathbf{v}^{\max}_{G,f})$ are simple games, using the Deegan-Packel index (11) is justified and it will be denoted by (DP). Tables B.3 and B.4 in Appendix B in the Online Resource Section present all the numerical results.

First, we comment on the associated TU games that can be exactly obtained within a feasible time. Corollary 4.1 ensures that $(N, \mathbf{v}^{min}_{G,f})$ and $(N, \mathbf{v}^{max}_{G,f})$ are, also in this case, the TU games in (4). The total number of coalitions required to compute both characteristic functions is relatively small; thus, they can be obtained exactly, thereby avoiding the use of sampling. Table 6 provides the rankings obtained when considering the pessimistic TU game $(N, \mathbf{v}^{\min}_{G,f})$. Salem Alhazmi occupies position 1 in the four rankings, Khalid Al-Mihdhar occupies position 2, and Ziad Jarrah, occupies position 3. Mohamed Atta, which is typically considered the leader of the group, occupies position 4, and Hani Hanjour ranks position 5. Ahmed Al-Haznawi and Majed Moqed occupy positions 6 and 7,

Table 5
List of individual weights for the members of the covert network responsible for the 9/11 attacks.

Hijacker	w_i	Hijacker	w_i
Ahmed Alghamdi	1	Nawaf Alhazmi	2
Hamza Alghamdi	1	Khalid Al-Mihdhar	3
Mohand Alshehri	1	Hani Hanjour	1
Fayez Ahmed	1	Majed Moqed	1
Marwan Al-Shehhi	3	Mohamed Atta	4
Ahmed Alnami	1	Abdul Aziz Al-Omari	1
Saeed Alghamdi	1	Waleed Alshehri	1
Ahmed Al-Haznawi	1	Satam Suqami	1
Ziad Jarrah	4	Wail Alshehri	1
Salem Alhazmi	1		

Table 6
Rankings based on (SH), (GES), (S), (BZ), and (DP) for $(N, \mathbf{v}^{\min}_{G,f})$.

	(SH)	(GES)	(S)	(BZ)	(DP)
1	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi
2	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar
3	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah
4	Mohamed Atta	Mohamed Atta	Mohamed Atta	Mohamed Atta	Mohamed Atta
5	Hani Hanjour	Hani Hanjour	Hani Hanjour	Hani Hanjour	Hani Hanjour
6	Ahmed Al-Haznawi	Ahmed Al-Haznawi	Ahmed Al-Haznawi	Majed Moqed	Ahmed Al-Haznawi
7	Majed Moqed	Majed Moqed	Majed Moqed	Ahmed Al-Haznawi	Majed Moqed
8	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi
9	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Nawaf Alhazmi
10	Nawaf Alhazmi	Nawaf Alhazmi	Nawaf Alhazmi	Nawaf Alhazmi	Hamza Alghamdi
11	Saeed Alghamdi	Saeed Alghamdi	Saeed Alghamdi	Saeed Alghamdi	Saeed Alghamdi
12	Fayez Ahmed	Fayez Ahmed	Fayez Ahmed	Fayez Ahmed	Fayez Ahmed
13	Mohand Alshehri	Mohand Alshehri	Mohand Alshehri	Mohand Alshehri	Ahmed Alnami
14	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami	Mohand Alshehri
15	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari	Satam Suqami
16	Satam Suqami	Satam Suqami	Satam Suqami	Satam Suqami	Abdul Aziz Al-Omari
17	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi
18	Wail Alshehri	Wail Alshehri	Waleed Alshehri	Waleed Alshehri	Waleed Alshehri
19	Waleed Alshehri	Waleed Alshehri	Wail Alshehri	Wail Alshehri	Wail Alshehri

Table 7
Rankings based on (SH), (GES), (S), (BZ), and (DP) for $(N, \mathbf{v}^{\max}_{G,f})$.

	(SH)	(GES)	(S)	(BZ)	(DP)
1	Ziad Jarrah	Mohamed Atta	Ziad Jarrah	Ziad Jarrah	Hamza Alghamdi
2	Mohamed Atta	Ziad Jarrah	Mohamed Atta	Mohamed Atta	Salem Alhazmi
3	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Saeed Alghamdi
4	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Majed Moqed
5	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Ahmed Al-Haznawi
6	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Ahmed Alnami
7	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami	Mohand Alshehri
8	Majed Moqed	Majed Moqed	Majed Moqed	Majed Moqed	Hani Hanjour
9	Ahmed Al-Haznawi	Ahmed Al-Haznawi	Nawaf Alhazmi	Ahmed Al-Haznawi	Khalid Al-Mihdhar
10	Nawaf Alhazmi	Hani Hanjour	Ahmed Al-Haznawi	Saeed Alghamdi	Fayez Ahmed
11	Hani Hanjour	Nawaf Alhazmi	Hani Hanjour	Hani Hanjour	Ahmed Alghamdi
12	Saeed Alghamdi	Saeed Alghamdi	Satam Suqami	Mohand Alshehri	Marwan Al-Shehhi
13	Satam Suqami	Mohand Alshehri	Saeed Alghamdi	Fayez Ahmed	Nawaf Alhazmi
14	Mohand Alshehri	Fayez Ahmed	Mohand Alshehri	Nawaf Alhazmi	Satam Suqami
15	Fayez Ahmed	Satam Suqami	Fayez Ahmed	Ahmed Alghamdi	Ziad Jarrah
16	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi	Satam Suqami	Mohamed Atta
17	Wail Alshehri	Abdul Aziz Al-Omari	Waleed Alshehri	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari
18	Waleed Alshehri	Waleed Alshehri	Wail Alshehri	Waleed Alshehri	Waleed Alshehri
19	Abdul Aziz Al-Omari	Wail Alshehri	Abdul Aziz Al-Omari	Wail Alshehri	Wail Alshehri

respectively, except in terms of (BZ), for which they exchange positions. From position 8 onwards, the hijackers occupy the same positions.

Analogously, Table 7 provides the rankings, based on the aforementioned criteria, for the TU game $(N, \mathbf{v}^{\max}_{G,f})$. Recall that the Shapley value for this game is the solution proposed by McQuillin (2009) for $\mathbf{v}_{G,f}$. Regarding the rankings based on $\mathbf{v}^{\max}_{G,f}$, more differences are easily observed. However, from positions 1 to 8, the resulting rankings are identical. Ziad Jarrah and Mohamed Atta tie in position 1 (note that they have the highest individual weights). Khalid Al-Mihdhar occupies position 2; Marwan Al-Shehhi, position 4; Salem Alhazmi, position 5; Hamza Alghamdi and Ahmed Alnami, positions 6 and 7, respectively; and Majed Moqed, position 8. In addition, Wail Alshehri, Waleeed Alshehri and Abdul Aziz Al-Omari occupy the last three positions in the four rankings (tied). We now list the main changes in the rankings. Ahmed Al-Haznawi typically occupies position 9, except in terms of (S). Nawaf Alhazmi occupies position 10 using (SH), position 11 under (GES), position 9 under (S), and position 14 when (BZ). Hani Hanjour occupies position 11 using (SH), (S), and (BZ) and position 10 in terms of (GES). Saeed Alghamdi occupies position 12 under (SH) and (GES). However, he moves to position 13 with (S) and to position 10 with (BZ). Satam Suqami occupies position 13 for (SH); position 15 for (GES); position 12 for (S); and position 16 for (BZ). Using (SH) and (S), Mohand Alshehri occupies position 14, position 13 under (GES), and position 12 under (BZ). Similar conclusions are drawn for Fayez Ahmed. He occupies position 15 in terms of (SH) and (S). If (GES) is considered, he occupies position 14, and position 13 under (BZ). Finally, we mention the case of Ahmed Alghamdi, who typically occupies position 16, except for (BZ), for which he moves to position 15. In terms of (DP), we highlight the case of Mohamed Atta who, despite being considered the leader, the Deegan-Packel index relegates him to the last position (tied with other agents). Under $(N, \mathbf{v}^{\max}_{G,f})$, the first three positions are occupied by Hamza Alghamdi, Salem Alhazmi and Saeed Alghamdi. A more thorough discussion concerning the results can be conducted; however, the differences are now clearer than those with the other rankings considered.

Finally, we analyse the results using the TU games $(N, \bar{\mathbf{v}}_{G,f})$ and $(N, \bar{\bar{\mathbf{v}}}_{G,f})$. As the number of partitions involved is excessively large, sampling techniques are required to approximate the characteristic functions of both TU games. Take $v \in PG(N)$. Saavedra-Nieves & Fiestras-Janeiro (2022) propose that, for a fixed coalition $S \subseteq N$, the estimation of $\bar{v}(S)$ (Albizuri et al., 2005) corresponds to the sample mean of $v(S; P)$, with P an element of a sample of partitions in $\Pi(N \setminus S)$ obtained under simple random sampling

Table 8
Rankings based on (SH), (GES), (S), and (BZ) for the estimation of $(N, \bar{\mathbf{v}}_{G,f})$.

	(SH)	(GES)	(S)	(BZ)
1	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi
2	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar
3	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah
4	Mohamed Atta	Mohamed Atta	Mohamed Atta	Mohamed Atta
5	Hani Hanjour	Hani Hanjour	Hani Hanjour	Hani Hanjour
6	Ahmed Al-Haznawi	Ahmed Al-Haznawi	Ahmed Al-Haznawi	Majed Moqed
7	Majed Moqed	Majed Moqed	Majed Moqed	Ahmed Al-Haznawi
8	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi
9	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi
10	Nawaf Alhazmi	Nawaf Alhazmi	Nawaf Alhazmi	Nawaf Alhazmi
11	Saeed Alghamdi	Saeed Alghamdi	Saeed Alghamdi	Saeed Alghamdi
12	Fayez Ahmed	Fayez Ahmed	Fayez Ahmed	Fayez Ahmed
13	Mohand Alshehri	Mohand Alshehri	Mohand Alshehri	Mohand Alshehri
14	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami	Ahmed Alnami
15	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari
16	Satam Suqami	Satam Suqami	Satam Suqami	Satam Suqami
17	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi
18	Waleed Alshehri	Waleed Alshehri	Waleed Alshehri	Waleed Alshehri
19	Wail Alshehri	Wail Alshehri	Wail Alshehri	Wail Alshehri

with replacement. Alternatively, the estimation of $\bar{\bar{v}}(S)$ (Hu & Yang, 2010) is the average of the worth of the game in partition function form over partitions for $N \setminus S$ of the form P_{-S}, induced by a sample of partitions in $\Pi(N)$. To obtain these, we take samples of 50 partitions per coalition using simple random sampling with replacement whenever the size of the target partitions' population is larger. Otherwise, as for some components of the characteristic function of the TU game of Albizuri et al. (2005), we consider the entire sampling population.

In addition, the analysis of the estimation error is a task covered by Saavedra-Nieves & Fiestras-Janeiro (2022), which provide a probabilistic bound of the incurred error depending on the sample size, confidence level, maximum desired error, and range of values of v. In our setting, this range is 1 for all $S \subseteq N$. Thus, as we take samples of 50 partitions per coalition in both cases, we have ensured a maximum error of 0.192, with a probability of 95%, in the estimation of each of the components of both TU games. Note that we use sampling methods only when the number of partitions required to obtain the exact TU game of Albizuri et al. (2005) exceeds 50. Otherwise, the exact worth is obtained. Although the sample size may appear small, evaluating the effectiveness of all connected coalitions in a given partition is a nontrivial computational task.

Table 8 lists the rankings based on the estimation of the TU game $(N, \bar{\mathbf{v}}_{G,f})$. Verifying that (SH), (GES), (S) and (BZ) prescribe the same hijacker rankings under this approach is easy. Only numerical differences exist; however, they are based on the basis that justifies each of the solutions for the TU games considered here. Again, we mention the case of Mohamed Atta, who is typically considered the leader of the group and now ranks position 4. Salem Alhazmi occupies position 1, Khalid Al-Mihdhar, position 2, and Ziad Jarrah, position 3. Although comments can also be made on the hijackers placed in the intermediate zone of the ranking without changes, we focus on those considered less influential under this approach: Abdul Aziz Al-Omari, Satam Suqami, Ahmed Alghamdi, Waleed Alshehri and Wail Alshehri, who occupy the last five positions.

Finally, Table 9 presents the rankings of the hijackers supporting 9/11, based on the solutions considered for the estimated $\bar{\bar{\mathbf{v}}}_{G,f}$. In brief, we draw some conclusions. Ziad Jarrah and Mohamed Atta rank positions 1 and 2, except under (BZ) for which they exchange positions. Khalid Al-Mihdhar always ranks position 3. Marwan Al-Shehhi and Salem Alhazmi occupy positions 4 and 5 under (SH), (GES), and (S). However, they exchange positions when (BZ) is used. Hamza Alghamdi always ranks position 6. In addition, Hani Hanjour occupies position 7 under (SH), (GES), and (S), and occupies position 8 under (BZ). Majed Moqed ranks position 8 under (SH) and (S), position 9 under (GES), and position 10 under (BZ). Ahmed Al-Haznawi typically occupies position 9, except under (GES), which moves him to position 10. Ahmed Alnami ranks position 10 under (SH), position 8 under (GES), position 11 under (S), and position 7 under (BZ). Nawaf Alhazmi moves to position 11 under (SH) and (GES), to position 10 under (S), and to position 12 under (BZ). Similarly, Saeed Alghamdi occupies position 12 under (SH) and (GES), position 13 under (S), and position 11 under (BZ). Satam Suqami ranks position 13 under (SH), position 15 under (GES), position 12 under (S), and position 17 under (BZ). Mohand Alshehri and Fayez Ahmed occupy positions 14 and 16 under (SH) and (S), respectively; however, they occupy positions 13 and 14 under (GES) and (BZ), respectively. Abdul Aziz Al-Omari typically occupies position 15, except under (GES), which moves him to position 16. Finally, Ahmed Alghamdi typically occupies position 17, except under (BZ), and Waleed and Wail Alshehri are always in the final two positions.

5.1. A brief comparison with other standard centrality measures

The results presented there considerably differ those obtained studies using standard tools or other game theoretical methods, underlining the relevance of the use of games in the form of partition function. In this section, we revisit some well-known standard centrality measures to compare the performance of our ranking proposal.

Given a network $G = (N, E)$, we briefly list some classical indices, used in standard network analysis, that also provide rankings according to their decreasing order.

- If $d(i)$ is the number of the direct connections of i, the *normalized degree centrality* (Proctor & Loomis, 1951) of agent i is given by

$$C_{deg}(i) = \frac{d(i)}{|N| - 1}. \tag{12}$$

- The *normalized betweenness centrality* of agent i is defined as

$$C_{bet}(i) = \frac{2}{(|N| - 1)(|N| - 2)} \sum_{k,j \in N \setminus \{i\}: k<j} \frac{s_{kij}}{s_{kj}}, \tag{13}$$

  where s_{kj} is the total number of shortest paths between agents k and j, and s_{kij} the number of shortest paths between k and j containing agent i.

Table 9
Rankings based on (SH), (GES), (S), and (BZ) for the estimation of $(N, \bar{\bar{\mathbf{v}}}_{G,f})$.

	(SH)	(GES)	(S)	(BZ)
1	Ziad Jarrah	Ziad Jarrah	Ziad Jarrah	Mohamed Atta
2	Mohamed Atta	Mohamed Atta	Mohamed Atta	Ziad Jarrah
3	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar	Khalid Al-Mihdhar
4	Marwan Al-Shehhi	Marwan Al-Shehhi	Marwan Al-Shehhi	Salem Alhazmi
5	Salem Alhazmi	Salem Alhazmi	Salem Alhazmi	Marwan Al-Shehhi
6	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi	Hamza Alghamdi
7	Hani Hanjour	Hani Hanjour	Hani Hanjour	Ahmed Alnami
8	Majed Moqed	Ahmed Alnami	Majed Moqed	Hani Hanjour
9	Ahmed Al-Haznawi	Majed Moqed	Ahmed Al-Haznawi	Ahmed Al-Haznawi
10	Ahmed Alnami	Ahmed Al-Haznawi	Nawaf Alhazmi	Majed Moqed
11	Nawaf Alhazmi	Nawaf Alhazmi	Ahmed Alnami	Saeed Alghamdi
12	Saeed Alghamdi	Saeed Alghamdi	Satam Suqami	Nawaf Alhazmi
13	Satam Suqami	Mohand Alshehri	Saeed Alghamdi	Mohand Alshehri
14	Mohand Alshehri	Fayez Ahmed	Mohand Alshehri	Fayez Ahmed
15	Abdul Aziz Al-Omari	Satam Suqami	Abdul Aziz Al-Omari	Abdul Aziz Al-Omari
16	Fayez Ahmed	Abdul Aziz Al-Omari	Fayez Ahmed	Ahmed Alghamdi
17	Ahmed Alghamdi	Ahmed Alghamdi	Ahmed Alghamdi	Satam Suqami
18	Waleed Alshehri	Waleed Alshehri	Waleed Alshehri	Wail Alshehri
19	Wail Alshehri	Wail Alshehri	Wail Alshehri	Waleed Alshehri

Table 10
Spearman correlations between the rankings.

	$\mathbf{v}^{min}_{G,f}$				$\mathbf{v}^{max}_{G,f}$				$\mathbf{v}^{min}_{G,f}$	$\mathbf{v}^{max}_{G,f}$
	(SH)	(GES)	(S)	(BZ)	(SH)	(GES)	(S)	(BZ)	(DP)	(DP)
C_{deg}	0.328	0.328	0.342	0.326	0.344	0.381	0.358	0.316	0.335	-0.019
C_{bet}	0.137	0.137	0.158	0.144	0.089	0.163	0.123	0.089	0.135	-0.293
C_{close}	**0.616**	**0.616**	**0.619**	**0.611**	**0.549**	**0.607**	**0.551**	**0.542**	**0.612**	0.107
C_L	0.163	0.163	0.188	0.174	0.086	0.170	0.114	0.121	0.142	-0.212

	$\bar{\mathbf{v}}_{G,f}$				$\bar{\bar{\mathbf{v}}}_{G,f}$			
	(SH)	(GES)	(S)	(BZ)	(SH)	(GES)	(S)	(BZ)
C_{deg}	0.342	0.342	0.342	0.326	0.439	0.447	0.442	0.421
C_{bet}	0.158	0.158	0.158	0.144	0.209	0.223	0.226	0.198
C_{close}	**0.619**	**0.619**	**0.619**	**0.611**	**0.642**	**0.661**	**0.644**	**0.660**
C_L	0.188	0.188	0.188	0.174	0.254	0.254	0.265	0.225

- The *normalized closeness centrality* of agent i is defined by

$$C_{close}(i) = \frac{|N|-1}{\sum_{j\in N} l_{ij}}, \quad (14)$$

where l_{ij} is the shortest distance between agents i and j.

These three measures have already been computed by Lindelauf et al. (2013) for the covert network depicted in Fig. 3. Table 6 in that work provides the associated rankings and the ranking induced by their game theoretical proposal (denoted by C_L). For each, Spearman correlations are obtained among the rankings to compare the degree of similarity. Table 10 lists these values.

These results reveal that the greatest similarities originate from the rankings obtained under the normalised closeness centrality (in bold). However, note that in the ranking given by the Deegan-Packel index, correlations are typically negative. This indicates that the positions in this ranking are generally opposite to those compared. Because the correlations are not considerably high in the remaining cases, the results shown are not invalidated but emphasize the effort required by investigation teams when determining the criteria to be used in each case. The chosen criteria largely determine the resulting rankings. As observed, this decision is key when deciding where to centralise terrorist surveillance resources to control the most potentially active and dangerous terrorists.

Our results contribute to a more comprehensive analysis of the covert network responsible for the 9/11 attacks that can be compared with the results previously obtained by Lindelauf et al. (2013) and Husslage et al. (2015), among others. Note that one of the most relevant differences is that, here, Salem Alhazmi was determined to rank in the first five positions with all games and values considered. Although this agent has a weight of 1, its position in the associated graph favours Nawaf Alhazmi (weight of 2), Ziad Jarrah (4), and Marwan Al-Shehhi (3), who are three of the most relevant agents, to be part of the same connected coalition. In addition, Waleed Alshehri and Wail Alshehri are always ranked in the final five positions. This result considerably differs from that obtained in the literature when network centrality was studied using standard tools or other game theoretical methods, which underlines the relevance of using games in the form of partition function.

6. Concluding remarks

Recently, covert network analysis has been attracting increasing attention because of its usefulness in handling global problems such as social movements, terrorism, and more day-to-day events such as the analysis of sports teams. Classic techniques in social network analysis have been used to identify the most relevant members of covert networks without considering the possible cooperation of their members. This problem is partially solved by using an approach based on cooperative game theory that considers possible links between them. In this paper, we have analysed the impact of the cooperation between members of a covert network on their overall effectiveness. In this framework, such a value can be influenced by the organisation of the outsiders to the covert network of any formed coalition, allowing us to consider partition function games (Thrall & Lucas, 1963) to model these situations.

Starting from a covert network and a system of weights defined on the nodes and arcs that integrate it, we introduced a game in partition function form to model the covert network in the presence of externalities. Besides, we formally study the game theoretical properties that satisfy this new class of games. Based on a thorough review of the literature, we use some values defined for games in partition function form to rank the members of a covert network according to their contribution to the overall effectiveness. In particular, some of these proposals are based on the Shapley value (Shapley, 1953); however, we also use others well-known solutions reported in the literature, such as the Banzhaf value (Banzhaf, 1964) and the Deegan-Packel index (Deegan & Packel, 1978), along with some associated TU games for any game in partition function form, by applying the approaches proposed in the literature (Albizuri et al., 2005; de Clippel & Serrano, 2008; Hu & Yang, 2010; McQuillin, 2009). These TU games, given a merger of agents, assume different structures for the remaining agents: their union, the formation of all individual coalitions, or an equiprobable distribution of the different possible partitions, particularly suitable in situations of increased uncertainty.

The complexity of the calculations performed should be noted. In this framework, the role of the sampling techniques used to approximately obtain some rankings is crucial, being all computation times achieved considerably diminished by using a supercomputing centre. Comparative studies on the performance of other effectiveness functions for any merge and those on new formulations of partition function form games to describe the reality of each covert network more reliably should be conducted in the future. In addition, the application of these techniques to different covert networks, which were not considered here, may be valuable.

As an application, we use these methods to analyse the covert network supporting the 9/11 attacks to determine, under the proposed approach, the degree of involvement of each member in achieving their objectives. Naturally, different rankings are obtained depending on the aforementioned assumptions inherent in the models used. The most appropriate results can be selected according to the information available and depending on the counterterrorism agents and potential users of the tool created. All rankings can be used simultaneously, allowing for the consideration of different scenarios.

Although we demonstrated the performance of our proposal using the effectiveness function proposed by Lindelauf et al. (2013), the method is easily extensible to any general framework with a multiagent covert network scheme, in which establishing rankings of its members is of interest. The usage of such a function is justified as it is a function that encompasses more information about the members of each coalition, such as their connectivity and hierarchy and communication between them. Naturally, a general family of covert network games arises if any effectiveness function f is considered.

Declaration of Competing Interest

The authors declare that there is no conflict of interest.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2023.02.023.

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