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Research Article

Asymptotic Behavior of Solutions to Abstract Stochastic Fractional Partial Integrodifferential Equations

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The existence of asymptotically almost automorphic mild solutions to an abstract stochastic fractional partial integrodifferential equation is considered. The main tools are some suitable composition results for asymptotically almost automorphic processes, the theory of sectorial linear operators, and classical fixed point theorems. An example is also given to illustrate the main theorems.

1. Introduction

This paper is mainly concerned with the existence and uniqueness of square-mean asymptotically almost automorphic mild solutions to the following stochastic fractional partial integrodifferential equation in the form

$$\begin{aligned} d\left[x(t)-f(t,x(t))\right] &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} A\left[x(s)-f(s,x(s))\right] ds\, dt \\ &\quad + g(t,x(t))\, dW(t), \quad t \geq 0, \\ x(0) &= u_0, \end{aligned} \tag{1}$$

where $1<\alpha<2$, $A: D(A)\subset L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is a linear densely defined operator of sectorial type on a Hilbert space $L^2(\mathbb{P},\mathbb{H})$, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$, where $\mathcal{F}_t=\sigma\{W(u)-W(v); u,v\leq t\}$, and u_0 is an $\mathcal{F}_0$-adapted, $\mathbb{H}$-valued random variable independent of the Wiener process W. Here f and g are appropriate functions to be specified later. The convolution integral in (1) is understood in the Riemann-Liouville fractional integral (see, e.g., [1, 2]). We notice that fractional order can be complex in viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods to fractional equations, because they have recently proven to be valuable in various fields of science and engineering (see, e.g., [3–11] and references therein).

The concept of asymptotically almost automorphic functions was firstly introduced by N'Guérékata in [12]. Since then these functions have become of great interest to several mathematicians and gained lots of developments and applications, we refer the reader to [13–16] and the references listed therein.

Recently, the existence of almost automorphic and pseudo almost automorphic solutions to some stochastic differential equations has been considered in many publications such as [17–27] and the references therein. In a very recent paper [28], the authors introduced a new notation of square-mean asymptotically almost automorphic stochastic processes including a composition theorem. However, to the best of our knowledge, the existence of square-mean asymptotically almost automorphic mild solutions to the problem (1) is an untreated topic. Therefore, motivated by the works [16, 28], the main purpose of this paper is to investigate the existence and uniqueness of square-mean asymptotically almost automorphic mild solutions to the problem (1). Then, we present an example as an application of our main results.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and facts which will be used throughout this paper. In Section 3, we prove some existence results of square-mean asymptotically almost automorphic mild solutions to the problem (1). Finally, we give an example as an application of our abstract results.

2. Preliminaries

In this section, we introduce some basic definitions, notations, and preliminary facts which will be used in the sequel. For more details on this section, we refer the reader to [28–30].

Throughout the paper, $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ stands for a real separable Hilbert space. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space, and $L^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty. \tag{2}$$

Note that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm

$$\|x\|_2 := \left(\int_\Omega \|x\|^2 d\mathbb{P}\right)^{1/2}, \quad \text{for each } x \in L^2(\mathbb{P}, \mathbb{H}). \tag{3}$$

We denote by $C_0(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ the collection of all bounded continuous stochastic processes φ from $\mathbb{R}^+$ into $L^2(\mathbb{P}, \mathbb{H})$ such that $\lim_{t\to+\infty} E\|\varphi(t)\|^2 = 0$. It is then easy to check that $C_0(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ is a Banach space when it is endowed with the norm $\|\varphi\|_{C_0} := \sup_{t\in\mathbb{R}^+} \|\varphi(t)\|_2$. Similarly, $C_0(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$ stands for the space of the continuous stochastic processes $f : \mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{t\to+\infty} E\|f(t, x)\|^2 = 0 \tag{4}$$

uniformly for $x \in K$, where $K \subset L^2(\mathbb{P}, \mathbb{H})$ is any bounded subset. Additionally, $W(t)$ will be a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$.

2.1. Sectorial Linear Operators. A closed and linear operator A is said to be sectorial of type ϖ and angle θ if there exist $0 < \theta < \pi/2$, $M > 0$, and $\varpi \in \mathbb{R}$ such that its resolvent exists outside the sector $\varpi + S_\theta := \{\varpi + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \le M/|\lambda - \varpi|$, $\lambda \notin \varpi + S_\theta$.

Definition 1 (see [2]). Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$. We call A the generator of a solution operator if there exist $\varpi \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \varpi\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt$, $Re(\lambda) > \varpi$, $x \in \mathbb{X}$. In this case, $S_\alpha(\cdot)$ is called the solution operator generated by A.

We note that if A is sectorial of type ϖ with $0 \le \theta < \pi(1 - \alpha/2)$, then A is the generator of a solution operator given by $S_\alpha(t) := (1/2\pi i)\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda$, $t \ge 0$, where γ is a suitable path lying outside the sector $\varpi + S_\theta$. Recently, Cuesta in [1] proved that if A is a sectorial operator of type $\varpi < 0$ for some $M > 0$ and $0 \le \theta < \pi(1 - \alpha/2)$, then there exists a constant $C > 0$ such that

$$\|S_\alpha(t)\| \le \frac{CM}{1 + |\varpi| t^\alpha}, \quad t \ge 0. \tag{5}$$

Remark 2. Note that $S_\alpha(t)$ is, in fact, integrable. For more details on the solution family $S_\alpha(t)$ and related issues, we refer the reader to [31–33].

2.2. Square-Mean Asymptotically Almost Automorphic Processes. We recall some basic facts for a symptotically almost automorphic processes which will be used in the sequel.

Definition 3 (see [22]). A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be stochastically continuous if

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0. \tag{6}$$

Definition 4 (see [17]). A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost automorphic if, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\begin{gathered}\lim_{n\to\infty} E\left\|x(t + s_n) - y(t)\right\|^2 = 0,\\ \lim_{n\to\infty} E\left\|y(t - s_n) - x(t)\right\|^2 = 0\end{gathered} \tag{7}$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Definition 5 (see [17]). A function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic if $f(t, x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $L^2(\mathbb{P}, \mathbb{H})$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\tilde{f} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\begin{gathered}\lim_{n\to\infty} E\left\|f(t + s_n, x) - \tilde{f}(t, x)\right\|^2 = 0,\\ \lim_{n\to\infty} E\left\|\tilde{f}(t - s_n, x) - f(t, x)\right\|^2 = 0\end{gathered} \tag{8}$$

for each $t \in \mathbb{R}$ and each $x \in \mathbb{K}$. Denote by $AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Lemma 6 (see [22]). *$(AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})), \|\cdot\|_\infty)$ is a Banach space equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} \left(E\|x(t)\|^2\right)^{1/2}, \tag{9}$$

for $x \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 7 (see [17]). *Let $f:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $(t,x)\to f(t,x)$ be square-mean almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each bounded subset $\mathbb{K}\subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}$; that is, for all $\varepsilon>0$, there exists $\delta>0$ such that $x,y\in\mathbb{K}$ and $E\|x-y\|^2<\delta$ imply that $E\|f(t,x)-f(t,y)\|^2<\varepsilon$ for all $t\in\mathbb{R}$. Then for any square-mean almost automorphic process $x:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$, the stochastic process $F:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ given by $F(\cdot):=f(\cdot,x(\cdot))$ is square-mean almost automorphic.*

Definition 8 (see [25]). A stochastically continuous process $f:\mathbb{R}^+\to L^2(\mathbb{P},\mathbb{H})$ is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f=g+h$, where $g\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$ and $h\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. Denote by $AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ the collection of all the square-mean asymptotically almost automorphic processes $f:\mathbb{R}^+\to L^2(\mathbb{P},\mathbb{H})$.

Definition 9 (see [28]). A function $f:\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $(t,x)\to f(t,x)$, which is jointly continuous, is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f=g+h$, where $g\in AA(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ and $h\in C_0(\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$. Denote by $AAA(\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ the set of all such functions.

Lemma 10 (see [28]). *If f, f_1, and f_2 are all square-mean asymptotically almost automorphic stochastic processes, then the following hold true:*

(I) *f_1+f_2 is square-mean asymptotically almost automorphic;*

(II) *λf is square-mean asymptotically almost automorphic for any scalar λ;*

(III) *there exists a constant $M>0$ such that $\sup_{t\in\mathbb{R}^+}E\|f(t)\|^2\le M$.*

Lemma 11 (see [28]). *Suppose that $f\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ admits a decomposition $f=g+h$, where $g\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$ and $h\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. Then $\{g(t):t\in\mathbb{R}\}\subset\overline{\{f(t):t\in\mathbb{R}^+\}}$.*

Corollary 12 (see [28]). *The decomposition of a square-mean asymptotically almost automorphic process is unique.*

Lemma 13 (see [28]). *$AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ is a Banach space when it is equipped with the norm*

$$\|f\|_{AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))}:=\sup_{t\in\mathbb{R}}\|g(t)\|_2+\sup_{t\in\mathbb{R}^+}\|h(t)\|_2, \tag{10}$$

where $f=g+h\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ with $g\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$, $h\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$.

Lemma 14 (see [28]). *$AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ is a Banach space with the norm*

$$\|f\|_\infty:=\sup_{t\in\mathbb{R}^+}\|f(t)\|_2=\sup_{t\in\mathbb{R}^+}\left(E\|f(t)\|^2\right)^{1/2}. \tag{11}$$

Remark 15 (see [28]). In view of the previous lemmas it is clear that the two norms are equivalent in $AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$.

Lemma 16 (see [28]). *Let $f\in AA(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ and let $f(t,x)$ be uniformly continuous in any bounded subset $\mathbb{K}\subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}^+$. Then $f(t,x)$ is uniformly continuous in any bounded subset $\mathbb{K}\subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}$.*

Lemma 17 (see [28]). *Let $f\in AAA(\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ and suppose that $f(t,x)$ is uniformly continuous in any bounded subset $\mathbb{K}\subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}^+$. If $u(t)\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$, then $f(\cdot,u(\cdot))\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$.*

We now give the following concept of mild solution of (1).

Definition 18. Let $S_\alpha(t)$ be an integrable solution operator on $L^2(\mathbb{P},\mathbb{H})$ with generator A. An $\mathcal{F}_t$-adapted stochastic process $x:[0,+\infty)\to L^2(\mathbb{P},\mathbb{H})$ is called a mild solution of the problem (1) if $x(0)=u_0$ is $\mathcal{F}_0$-measurable and $x(t)$ satisfies the corresponding stochastic integral equation:

$$\begin{aligned}x(t)={}&S_\alpha(t)\left[u_0-f\left(0,u_0\right)\right]+f(t,x(t))\\&+\int_0^t S_\alpha(t-s)\,g(s,x(s))\,dW(s),\quad t\ge 0.\end{aligned} \tag{12}$$

3. Main Results

In this section, we establish the existence of square-mean asymptotically almost automorphic mild solutions to the problem (1). For that, we need the following technical results.

First, we list the following basic assumptions.

(H1) The operator A is a sectorial operator of type $\varpi<0$ for some $M>0$ and $0\le\theta<\pi(1-\alpha/2)$, and then there exists $C>0$ such that

$$\left\|S_\alpha(t)\right\|\le\frac{CM}{1+|\varpi|\,t^\alpha},\quad t\ge 0, \tag{13}$$

where $S_\alpha(t)$ is the solution operator generated by A.

(H2) The function $f\in AAA(\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ and there exists a continuous and nondecreasing function $L_f:[0,+\infty)\to[0,+\infty)$ such that for each $r\ge 0$ and for all $E\|x\|^2\le r$, $E\|y\|^2\le r$,

$$E\left\|f(t,x)-f(t,y)\right\|^2\le L_f(r)\,E\left\|x-y\right\|^2 \tag{14}$$

for all $t\in\mathbb{R}^+$.

(H3) The function $g\in AAA(\mathbb{R}^+\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H}))$ and there exists a continuous and nondecreasing function $L_g:[0,+\infty)\to[0,+\infty)$ such that for each $r\ge 0$ and for all $E\|x\|^2\le r$, $E\|y\|^2\le r$,

$$E\left\|g(t,x)-g(t,y)\right\|^2\le L_g(r)\,E\left\|x-y\right\|^2 \tag{15}$$

for all $t\in\mathbb{R}^+$.

(H4) We have

$$\sup_{r>0}\left[\frac{r}{6(CM)^2}-\frac{L_f(r)\,r}{(CM)^2}-\frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\,\pi L_g(r)\,r}{\alpha\sin(\pi/\alpha)}-\lambda r\right]>\left(1+\frac{1}{(CM)^2}\right)M_f+\frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\,\pi}{\alpha\sin(\pi/\alpha)}M_g, \tag{16}$$

where $M_f = \sup_{t\in\mathbb{R}^+}E\|f(t,x(t))\|^2$ and $M_g = \sup_{t\in\mathbb{R}^+}E\|g(t,x(t))\|^2$.

(H5) The operator A is a sectorial operator of type ϖ with $0\le\theta<\pi(1-\alpha/2)$, and there exists $\phi(\cdot)\in L^1(\mathbb{R}^+)$ such that

$$\|S_\alpha(t)\|^2\le\phi(t)\quad\forall t\ge 0,\ \lim_{t\to+\infty}\phi(t)=0, \tag{17}$$

where $S_\alpha(t)$ is the solution operator generated by A.

Lemma 19. *Suppose that assumption (H1) holds and let $f\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. If F is the function defined by*

$$F(t):=\int_0^t S_\alpha(t-s)\,f(s)\,dW(s),\quad t\ge 0, \tag{18}$$

then $F\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$.

Proof. Since $f\in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$, we have by definition that $f=g+h$, where $g\in AA(R;L^2(P,H))$ and $h\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. Then

$$\begin{aligned}F(t)&=\int_0^t S_\alpha(t-s)\,g(s)\,dW(s)+\int_0^t S_\alpha(t-s)\,h(s)\,dW(s)\\&=\int_{-\infty}^t S_\alpha(t-s)\,g(s)\,dW(s)-\int_{-\infty}^0 S_\alpha(t-s)\,g(s)\,dW(s)+\int_0^t S_\alpha(t-s)\,h(s)\,dW(s)\\&=G(t)+H(t),\end{aligned} \tag{19}$$

where $G(t)=\int_{-\infty}^t S_\alpha(t-s)g(s)dW(s)$ and $H(t)=-\int_{-\infty}^0 S_\alpha(t-s)g(s)dW(s)+\int_0^t S_\alpha(t-s)h(s)dW(s)$.

First we prove that $G(t)\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$. Let $\{s_n'\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $g\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s_n'\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{g}$

$$\begin{aligned}&\lim_{n\to\infty}E\|g(t+s_n)-\widetilde{g}(t)\|^2=0,\\&\lim_{n\to\infty}E\|\widetilde{g}(t-s_n)-g(t)\|^2=0\end{aligned} \tag{20}$$

hold for each $t\in\mathbb{R}$. Now, let $\widetilde{W}(\sigma):=W(\sigma+s_n)-W(s_n)$ for each $\sigma\in\mathbb{R}$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $\widetilde{G}(t)=\int_{-\infty}^t S_\alpha(t-s)\widetilde{g}(s)dW(s)$, then by making a change of variables $\sigma=s-s_n$ to get (see the equation (10.6.6) in [34])

$$\begin{aligned}E\|G(t+s_n)-\widetilde{G}(t)\|^2&=E\left\|\int_{-\infty}^{t+s_n}S_\alpha(t+s_n-s)\,g(s)\,dW(s)-\int_{-\infty}^t S_\alpha(t-s)\,\widetilde{g}(s)\,dW(s)\right\|^2\\&=E\left\|\int_{-\infty}^t S_\alpha(t-\sigma)\left[g(\sigma+s_n)-\widetilde{g}(\sigma)\right]d\widetilde{W}(\sigma)\right\|^2\end{aligned} \tag{21}$$

and, hence, using the Ito's isometry property of stochastic integral, we have the following estimations

$$\begin{aligned}E\|G(t+s_n)-\widetilde{G}(t)\|^2&\le E\left(\int_{-\infty}^t\|S_\alpha(t-\sigma)\left[g(\sigma+s_n)-\widetilde{g}(\sigma)\right]\|^2d\sigma\right)\\&\le\int_{-\infty}^t\|S_\alpha(t-\sigma)\|^2E\|g(\sigma+s_n)-\widetilde{g}(\sigma)\|^2d\sigma\\&\le\sup_{t\in\mathbb{R}}E\|g(t+s_n)-\widetilde{g}(t)\|^2\int_{-\infty}^t\left(\frac{CM}{1+|\varpi|(t-\sigma)^\alpha}\right)^2d\sigma\\&\le(CM)^2\sup_{t\in\mathbb{R}}E\|g(t+s_n)-\widetilde{g}(t)\|^2\int_0^\infty\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\&\le\frac{(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\,\pi}{\alpha\sin(\pi/\alpha)}\sup_{t\in\mathbb{R}}E\|g(t+s_n)-\widetilde{g}(t)\|^2.\end{aligned} \tag{22}$$

Then by (20), we obtain that $\lim_{n\to\infty}E\|G(t+s_n)-\widetilde{G}(t)\|^2=0$ for each $t\in\mathbb{R}$. In a similar way, we can show that $\lim_{n\to\infty}E\|\widetilde{G}(t-s_n)-G(t)\|^2=0$ for each $t\in\mathbb{R}$. Thus we conclude that $G(\cdot)\in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$.

Next, let us show that $H(\cdot)\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. Since $h\in C_0(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ and $1/(1+|\varpi|s^\alpha)^2$ is integrable in $[0,+\infty)$, for any sufficiently small $\varepsilon>0$, there exists a constant $T>0$ such that $E\|h(s)\|^2\le\varepsilon$ and $\int_T^\infty(1/(1+|\varpi|s^\alpha)^2)ds\le\varepsilon$ for all $s\ge T$. Then, for all $t\ge 2T$, we obtain

$$E\|H(t)\|^2=E\left\|\int_0^{t/2}S_\alpha(t-s)\,h(s)\,dW(s)+\int_{t/2}^t S_\alpha(t-s)\,h(s)\,dW(s)-\int_{-\infty}^0 S_\alpha(t-s)\,g(s)\,dW(s)\right\|^2$$

$$\begin{aligned}
&\le 3E\left(\int_0^{t/2}\|S_\alpha(t-s)h(s)\|^2 ds\right)\\
&\quad + 3E\left(\int_{t/2}^{t}\|S_\alpha(t-s)h(s)\|^2 ds\right)\\
&\quad + 3E\left(\int_{-\infty}^{0}\|S_\alpha(t-s)g(s)\|^2 ds\right)\\
&\le 3\int_0^{t/2}\left(\frac{CM}{1+|\varpi|(t-s)^\alpha}\right)^2 E\|h(s)\|^2 ds\\
&\quad + 3\int_{t/2}^{t}\left(\frac{CM}{1+|\varpi|(t-s)^\alpha}\right)^2 E\|h(s)\|^2 ds\\
&\quad + 3\int_{-\infty}^{0}\left(\frac{CM}{1+|\varpi|(t-s)^\alpha}\right)^2 E\|g(s)\|^2 ds\\
&\le 3(CM)^2\sup_{t\in\mathbb{R}^+}E\|h(t)\|^2\int_{t/2}^{t}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\quad + 3(CM)^2\varepsilon\int_0^{t/2}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\quad + 3(CM)^2\sup_{t\in\mathbb{R}}E\|g(t)\|^2\int_t^{\infty}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\le 3(CM)^2\sup_{t\in\mathbb{R}^+}E\|h(t)\|^2\int_T^{\infty}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\quad + 3(CM)^2\varepsilon\int_0^{\infty}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\quad + 3(CM)^2\sup_{t\in\mathbb{R}}E\|g(t)\|^2\int_T^{\infty}\frac{1}{(1+|\varpi|s^\alpha)^2}ds\\
&\le 3(CM)^2\varepsilon\left(\sup_{t\in\mathbb{R}^+}E\|h(t)\|^2+\frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}\right.\\
&\qquad\left.+\sup_{t\in\mathbb{R}}E\|g(t)\|^2\right).
\end{aligned}\tag{23}$$

This inequality proves the assertion since ε is arbitrary. Recalling that $F(t) = G(t) + H(t)$ for all $t \ge 0$, we get $F(t) \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. The proof is completed. □

It is easy to see that, by arguments similar to those in the proof of Lemma 19, we have the following result.

Lemma 20. *Suppose that assumption (H5) holds and let $f \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. If F is the function defined by*

$$F(t) := \int_0^t S_\alpha(t-s)f(s)\,dW(s), \quad t \ge 0, \tag{24}$$

then $F \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$.

Now, we are ready to establish our main results.

Theorem 21. *Assume that (H1)–(H4) hold. Then there exists $\varepsilon > 0$ such that for each $u_0 \in B_\varepsilon(0, L^2(\mathbb{P},\mathbb{H}))$ there exists a unique square-mean asymptotically almost automorphic mild solution $x(\cdot, u_0)$ of the problem (1) on $[0,\infty)$ such that $x(0,u_0) = u_0$.*

Proof. We define a nonlinear operator Υ by

$$\begin{aligned}
(\Upsilon x)(t) &= S_\alpha(t)\left[u_0 - f(0,u_0)\right] + f(t,x(t))\\
&\quad + \int_0^t S_\alpha(t-s)g(s,x(s))\,dW(s) \quad t\ge 0.
\end{aligned}\tag{25}$$

First we prove that $\Upsilon(AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))) \subseteq AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. Given $x \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$, from the properties of $\{S_\alpha(t)\}_{t\ge0}$, f, and g, we infer that Υx is well defined and continuous. Since $x(t)$ is bounded, we can choose a bounded subset $\mathbb{K}$ of $L^2(\mathbb{P},\mathbb{H})$ such that $x(t) \in \mathbb{K}$ for all $t \in \mathbb{R}^+$. It follows from (H2) and (H3) that both $f(t,x)$ and $g(t,x)$ are uniformly continuous on the bounded subset $\mathbb{K}$ uniformly for $t \in \mathbb{R}^+$. Moreover, from Lemmas 17 and 19 and taking into account (H1), it follows that $\Upsilon x \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$.

Now, by (H4), there exists a constant $r > 0$ such that

$$\begin{aligned}
&\frac{r}{6(CM)^2} - \frac{L_f(r)\,r}{(CM)^2} - \frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\pi L_g(r)\,r}{\alpha\sin(\pi/\alpha)} - \lambda r\\
&\qquad > \left(1+\frac{1}{(CM)^2}\right)M_f + \frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}M_g.
\end{aligned}\tag{26}$$

Let $0 < \lambda < 1$. We affirm that the assertion holds for $\varepsilon = \lambda r$. In fact, let $u_0 \in B_\varepsilon(0, L^2(\mathbb{P},\mathbb{H}))$. Define the space $\mathbb{D} = \{x \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H})) : x(0) = u_0, E\parallel x(t)\|^2 \le r, t \ge 0\}$. Then $\mathbb{D}$ is a closed subspace of $AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. We claim that $\Upsilon\mathbb{D} \subseteq \mathbb{D}$. If $x \in \mathbb{D}$ and $t \in \mathbb{R}^+$, we get

$$\begin{aligned}
&E\|(\Upsilon x)(t)\|^2\\
&\quad\le 3E\left\|S_\alpha(t)\left[u_0 - f(0,u_0)\right]\right\|^2 + 3E\left\|f(t,x(t))\right\|^2\\
&\qquad + 3E\left\|\int_0^t S_\alpha(t-s)g(s,x(s))\,dW(s)\right\|^2\\
&\quad\le 6(CM)^2\left[E\|u_0\|^2 + E\left\|f(0,u_0)\right\|^2\right]\\
&\qquad + 3E\left\|f(t,x(t)) - f(t,0) + f(t,0)\right\|^2\\
&\qquad + 3E\left(\int_0^t\left\|S_\alpha(t-s)g(s,x(s))\right\|^2 ds\right)\\
&\quad\le 6(CM)^2\lambda r + 6(CM)^2E\left\|f(0,u_0)\right\|^2\\
&\qquad + 6L_f(r)\,r + 6\sup_{t\in\mathbb{R}^+}E\left\|f(t,0)\right\|^2\\
&\qquad + 6(CM)^2\left[\int_0^t\left(\frac{1}{1+|\varpi|(t-s)^\alpha}\right)^2 L_g(r)\,r\,ds\right.\\
&\qquad\qquad\left.+\int_0^t\left(\frac{1}{1+|\varpi|(t-s)^\alpha}\right)^2 E\left\|g(s,0)\right\|^2 ds\right]
\end{aligned}$$

$$\leq 6(CM)^2\lambda r + 6(CM)^2 M_f + 6L_f(r)r + 6M_f + \frac{6(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}\left[L_g(r)r + M_g\right], \tag{27}$$

which from (26) implies that $E\|(\Upsilon x)(t)\|^2 \leq r$ for all $t \geq 0$, and so that $\Upsilon\mathbb{D} \subseteq \mathbb{D}$.

Next, to complete the proof, we need to show that $\Upsilon(\cdot)$ is a contraction from $\mathbb{D}$ into $\mathbb{D}$. By (26), we know that

$$\frac{r}{6(CM)^2} - \frac{L_f(r)r}{(CM)^2} - \frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\pi L_g(r)r}{\alpha\sin(\pi/\alpha)} - \lambda r > 0. \tag{28}$$

That is,

$$\frac{r}{6(CM)^2} > \frac{L_f(r)r}{(CM)^2} + \frac{|\varpi|^{-1/\alpha}(1-1/\alpha)\pi L_g(r)r}{\alpha\sin(\pi/\alpha)} + \lambda r. \tag{29}$$

Then, one has

$$6L_f(r) + \frac{6(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}L_g(r) + 6\lambda(CM)^2 < 1. \tag{30}$$

For any $x, y \in \mathbb{D}$ and $t \geq 0$, we have

$$\begin{aligned} &E\|(\Upsilon x)(t) - (\Upsilon y)(t)\|^2 \\ &\leq 2E\|f(t,x(t)) - f(t,y(t))\|^2 \\ &\quad + 2E\left\|\int_0^t S_\alpha(t-s)\left[g(s,x(s)) - g(s,y(s))\right]dW(s)\right\|^2 \\ &\leq 2L_f(r)\sup_{t\in\mathbb{R}^+}E\|x(t)-y(t)\|^2 \\ &\quad + 2(CM)^2\int_0^t\left(\frac{1}{1+|\varpi|(t-s)^\alpha}\right)^2 \\ &\qquad \times E\|g(s,x(s)) - g(s,y(s))\|^2 ds \\ &\leq 2L_f(r)\sup_{t\in\mathbb{R}^+}E\|x(t)-y(t)\|^2 \\ &\quad + \frac{2(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}L_g(r) \\ &\quad \times \sup_{t\in\mathbb{R}^+}E\|x(t)-y(t)\|^2 \\ &\leq \left[2L_f(r) + \frac{2(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}L_g(r)\right] \\ &\quad \times \sup_{t\in\mathbb{R}^+}E\|x(t)-y(t)\|^2. \end{aligned} \tag{31}$$

Thus, we get

$$\begin{aligned} &\|\Upsilon x - \Upsilon y\|_\infty \\ &= \sup_{t\in\mathbb{R}^+}\left(E\|(\Upsilon x)(t) - (\Upsilon y)(t)\|^2\right)^{1/2} \\ &\leq \sqrt{2L_f(r) + \frac{2(CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi}{\alpha\sin(\pi/\alpha)}L_g(r)}\|x-y\|_\infty. \end{aligned} \tag{32}$$

It follows from (30) that Υ is a contraction mapping on $\mathbb{D}$. So by the Banach contraction mapping principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Υ in $\mathbb{D}$. It is clear that x is a square-mean asymptotically almost automorphic mild solution of (1). The proof is complete. □

The next result is proved using the similar steps as in the proof of the previous result, so we omit the details.

Theorem 22. *Assume that (H1)–(H3) hold. If $L_f(r) \equiv L_f$ and $L_g(r) \equiv L_g$ for all $r \geq 0$ and $L_f + (CM)^2|\varpi|^{-1/\alpha}(1-1/\alpha)\pi L_g/\alpha\sin(\pi/\alpha) < 1/2$, then for every $u_0 \in L^2(\mathbb{P},\mathbb{H})$, there exists a unique square-mean asymptotically almost automorphic mild solution $x(\cdot,u_0)$ of the problem* (1) *on* $[0,\infty)$ *such that* $x(0,u_0) = u_0$.

Theorem 23. *Suppose that assumptions (H2), (H3), and (H5) hold. If $L_f(r) \equiv L_f$ and $L_g(r) \equiv L_g$ for all $r \geq 0$ and $L_f + L_g\|\phi\|_1 < 1/2$, then for every $u_0 \in L^2(\mathbb{P},\mathbb{H})$, there exists a unique square-mean asymptotically almost automorphic mild solution $x(\cdot,u_0)$ of the problem* (1) *on* $[0,\infty)$ *such that* $x(0,u_0) = u_0$.

Proof. Consider the nonlinear operator Υ given by

$$\begin{aligned} (\Upsilon x)(t) &= S_\alpha(t)\left[u_0 - f(0,u_0)\right] + f(t,x(t)), \\ &\quad + \int_0^t S_\alpha(t-s)g(s,x(s))dW(s), \quad t \geq 0. \end{aligned} \tag{33}$$

First we prove that Υ maps $AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ into itself. Given $x \in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$, from the properties of $\{S_\alpha(t)\}_{t\geq 0}$, f and g, we infer that Υx is well defined and continuous. Since $x(t)$ is bounded, we can choose a bounded subset $\mathbb{K}$ of $L^2(\mathbb{P},\mathbb{H})$ such that $x(t) \in \mathbb{K}$ for all $t \in \mathbb{R}^+$. It follows from conditions (H2) and (H3) that both $f(t,x)$ and $g(t,x)$ are uniformly continuous on the bounded subset $\mathbb{K}$ uniformly for $t \in \mathbb{R}^+$. Moreover, from Lemmas 17 and 20 and taking into account (H5), it follows that $\Upsilon x \in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$.

Next we prove that Υ is a contraction mapping from $AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ into itself. Note that we have already proved $\Upsilon(AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))) \subseteq AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$. Moreover, for any $x, y \in AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))$ and $t \geq 0$, we have

$$\begin{aligned} &E\|(\Upsilon x)(t) - (\Upsilon y)(t)\|^2 \\ &\leq 2E\|f(t,x(t)) - f(t,y(t))\|^2 \end{aligned}$$

$$+ 2E\left\| \int_0^t S_\alpha (t-s) \left[g(s, x(s)) - g(s, y(s)) \right] dW(s) \right\|^2$$
$$\leq 2L_f \sup_{t\in\mathbb{R}^+} E\|x(t) - y(t)\|^2$$
$$+ 2\int_0^t \phi(t-s) E\|g(s,x(s)) - g(s,y(s))\|^2 ds$$
$$\leq 2L_f \sup_{t\in\mathbb{R}^+} E\|x(t) - y(t)\|^2$$
$$+ 2L_g \sup_{t\in\mathbb{R}^+} E\|x(t) - y(t)\|^2 \int_0^\infty \phi(s)\, ds$$
$$\leq \left[2L_f + 2L_g\|\phi\|_1\right] \sup_{t\in\mathbb{R}^+} E\|x(t) - y(t)\|^2. \tag{34}$$

Therefore

$$\begin{aligned} \|\Upsilon x - \Upsilon y\|_\infty &= \sup_{t\in\mathbb{R}^+}\left(E\|(\Upsilon x)(t) - (\Upsilon y)(t)\|^2\right)^{1/2} \\ &\leq \sqrt{2L_f + 2L_g\|\phi\|_1}\,\|x-y\|_\infty. \end{aligned} \tag{35}$$

That is, Υ is a contraction mapping on $AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$. By the Banach contraction mapping principle, Υ has a unique fixed point $x(t) \in AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$. It is clear that x is a square-mean asymptotically almost automorphic mild solution of (1). The proof is then complete. □

4. An Example

In this section, we apply the results obtained previously to investigate the existence of square-mean asymptotically almost automorphic mild solutions for the following partial stochastic fractional differential system:

$$\begin{gathered} \frac{\partial}{\partial t}\left[u(t,\xi) - \int_{-\infty}^t a(t) a_1(t-s) u(s,\xi)\, ds\right] \\ = J_t^{\alpha-1}\left(\frac{\partial^2}{\partial \xi^2} - \nu\right)\left[u(t,\xi) - \int_{-\infty}^t a(t) a_1(t-s) u(s,\xi)\, ds\right] \\ + \int_{-\infty}^t a(t) a_2(t-s) u(s,\xi)\, ds dW(t), \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \\ u(0,\xi) = u_0(\xi), \quad \xi \in I = [0,\pi], \end{gathered} \tag{36}$$

where $a \in AAA(\mathbb{R}^+; \mathbb{R})$, $a_1, a_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, and $\nu > 0$ is a fixed constant.

Let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$ and $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ be the operator defined by $Ax = x'' - \nu x$ domain $D(A) = \{x \in \mathbb{H} : x'' \in \mathbb{H}, x(0) = x(\pi) = 0\}$. It is well known that $\Delta x = x''$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on $\mathbb{H}$. Furthermore, A is sectorial of type $\varpi = -\nu < 0$.

In the sequel, we assume

$$\begin{aligned} L_f &= \|a\|_\infty \left(\int_{-\infty}^0 |a_1(-s)|^2 ds\right)^{1/2} < \infty, \\ L_g &= \|a\|_\infty \left(\int_{-\infty}^0 |a_2(-s)|^2 ds\right)^{1/2} < \infty. \end{aligned} \tag{37}$$

Now, we can define the functions $f, g : \mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ by

$$\begin{aligned} f(t,x)(\xi) &= a(t) \int_{-\infty}^0 a_1(-s) x(s,\xi)\, ds, \\ g(t,x)(\xi) &= a(t) \int_{-\infty}^0 a_2(-s) x(s,\xi)\, ds, \\ J_t^{\alpha-1} h(t) &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\, ds, \end{aligned} \tag{38}$$

which permits to transform the system (36) into the abstract system (1). Moreover, it is not difficult to see that f, g are continuous and Lipschitz in the second variable with Lipschitz constants L_f and L_g, respectively.

The next result is a consequence of Theorem 22.

Theorem 24. *Under the previous assumptions,* (36) *has a unique mild solution $x \in AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ whenever L_f and L_g are small enough.*

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