

# Green's Functions for Fractional Difference Equations with Dirichlet Boundary Conditions

Alberto Cabada<sup>a</sup>, Nikolay D. Dimitrov<sup>b</sup> and Jagan Mohan Jonnalagadda<sup>c</sup>

<sup>a</sup>Departamento de Estatística, Análise Matemática e Optimización,  
Instituto de Matemáticas, Facultade de Matemáticas,  
Universidade de Santiago de Compostela,

Santiago de Compostela, Galicia, Spain; alberto.cabada@usc.es

<sup>b</sup>Department of Mathematics, University of Ruse, 7017 Ruse,  
Bulgaria; ndimitrov@uni-ruse.bg

<sup>3</sup> Department of Mathematics, Birla Institute of Technology  
and Science Pilani, Hyderabad - 500078, Telangana, India;  
j.jaganmohan@hotmail.com

## Abstract

This article is devoted to deduce the expression and the main properties of the Green's function related to a general constant coefficients fractional difference equation coupled to Dirichlet conditions. In particular, it is proved that such function has constant sign on their set of definition and, moreover, it satisfies some additional strong sign conditions that are fundamental to define suitable cones, where to ensure the existence of solutions of nonlinear problems.

**Key Words:** Fractional difference, Dirichlet conditions, Green's function, existence of solutions.

**AMS Classification:** 39A12, 39A70.

## 1 Introduction and Preliminaries

In recent years the study of fractional calculus has grown, this is due to the fact that it considers integrals and derivatives of arbitrary orders, which allows modeling many real phenomena in which the value that the solution takes at a given instant depends on the value of the solution in all the previous moments of the process. So, this calculus is useful when several fields as viscoelasticity, neurology or control theory, are considered [13, 14, 15, 17, 18, 19].

In [16] it is developed an analogous theory for fractional difference equations. One can see at the monograph [12] the main tasks of this theory. As usual, we will denote  $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$  for  $a \in \mathbb{R}$ , and  $[c, c+n_0]_{\mathbb{N}_c} = [c, c+n_0] \cap \mathbb{N}_c$ , for  $c \in \mathbb{R}$  and  $n_0 \in \mathbb{N}_1$ .

In [1, 2] the general expression of several linear  $n$ th order initial value problems is obtained. They use  $R_0(f(t))(s)$  the Laplace transform on the time scale of integers [7, 10], which is defined by the following expression:

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} f(t).$$

In [3] it is proved that for all  $(t, s) \in [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times [0, b+1]_{\mathbb{N}_0}$ , the following function

$$G_0(t, s) = \frac{1}{\Gamma(v)} \begin{cases} \frac{t^{(v-1)}(v+b-s)^{(v-1)}}{(v+b+1)^{(v-1)}} - (t-s-1)^{(v-1)}, & s < t-v+1, \\ \frac{t^{(v-1)}(v+b-s)^{(v-1)}}{(v+b+1)^{(v-1)}}, & t-v+1 \leq s, \end{cases}$$

is the related Green function to the Dirichlet problem

$$\begin{aligned} -\Delta^v y(t) &= h(t+v-1), \quad t \in [0, b+1]_{\mathbb{N}_0}, \\ y(v-2) &= y(v+b+1) = 0, \end{aligned}$$

with  $v \in \mathbb{R}$ ,  $1 < v < 2$  and  $b \in \mathbb{N}$ .

Moreover, they prove that  $G_0(t, s) > 0$  for all  $(t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [0, b+1]_{\mathbb{N}_0}$ .

Using previous expression, in [9] is ensured, for a suitable range of values of the nonconstant function  $a(t)$ , the positiveness of the Green's function related to the following Dirichlet problem

$$\begin{aligned} -\Delta^v y(t) + a(t+v-1)y(t+v-1) &= h(t+v-1), \\ y(v-2) &= y(v+b+1) = 0, \end{aligned}$$

for  $t \in [0, b+1]_{\mathbb{N}_0}$ , where  $v \in \mathbb{R}$  with  $1 < v < 2$  and  $b \in \mathbb{N}$ ,  $b \geq 5$ .

Continuing in the direction of the two previous papers, in this work we consider the problem

$$-\Delta^v y(t) + \alpha \Delta^\mu y(t+v-\mu-1) = h(t+v-1), \quad (1)$$

$$y(v-2) = y(v+b+1) = 0, \quad (2)$$

for  $t \in I \equiv [0, b+1]_{\mathbb{N}_0}$ .

Here  $\mu, v \in \mathbb{R}$  such that  $0 < \mu < 1$  and  $1 < v < 2$ ;  $\Delta^v$  and  $\Delta^\mu$  are the standard  $v$ -th and  $\mu$ -th order Riemann–Liouville fractional difference operators, respectively;  $\alpha$  is a real constant and  $h : I \rightarrow \mathbb{R}$ .

On next section we will use the Laplace transform  $R_0(f(t))(s)$  to obtain the general expression of equation (1) and, by applying such expression at the Dirichlet conditions, we deduce the explicit values of the Green's function related to Problem (1)–(2). As we will see in section 3, such Green's function has some symmetric properties and is positive on  $[v-1, v+b]_{\mathbb{N}_{v-1}} \times [0, b+1]_{\mathbb{N}_0}$  for all  $\alpha \geq 0$  and  $v-\mu-1 > 0$ , which improves the results given in [3].

Moreover, on section 4 we will deduce some strong positiveness conditions on the Green's function that allow to construct suitable cones where to deduce the existence of solutions of related nonlinear problems. We finalize the paper, in section 5, where we show the applicability of such properties to some nonlinear problems.

Before constructing the Green's function, we first recall some basic definitions and lemmas, which will be used till the end of this work.

**Definition 1.1.** We define  $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$ , for any  $t$  and  $v$  for which the right-hand side is well defined. We also appeal to the convention that if  $t+1-v$  is a pole of the Gamma function and  $t+1$  is not a pole, then  $t^{(v)} = 0$ .

**Definition 1.2.** The  $v$ -th fractional sum of a function  $f$ , for  $v > 0$  and  $t \in \mathbb{N}_{a+v}$ , is defined as

$$\Delta^{-v} f(t) = \Delta^{-v} f(t; a) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} f(s).$$

We also define the  $v$ -th fractional difference for  $v > 0$  by  $\Delta^v f(t) := \Delta^N \Delta^{v-N} f(t)$ , where  $t \in \mathbb{N}_{a+v}$  and  $N \in \mathbb{N}$  is chosen so that  $0 \leq N-1 < v \leq N$ .

**Lemma 1.3.** ([11, Lemma 2.3]) Let  $t$  and  $v$  be any numbers for which  $t^{(v)}$  and  $t^{(v-1)}$  be defined. Then  $\Delta t^{(v)} = vt^{(v-1)}$ .

## 2 Construction of the Green's Function

In this section, we will characterize in Theorem 2.1 the uniqueness of solution of Problem (1)–(2) and construct, in such a case, its related Green's function.

To this end we use the main properties of the Laplace transform showed in [4]. Thus, from Lemma 2.2 of [4], we have

$$R_0 [\Delta^v y(t)](s) = s^v R_{v-2} [y(t)](s) - sA - B, \quad (3)$$

where  $A = [\Delta^{v-2} y(t)]_{t=0}$  and  $B = [\Delta^{v-1} y(t)]_{t=0}$ . Observe that

$$\begin{aligned} A &= [\Delta^{v-2} y(t)]_{t=0} = [\Delta^{-(2-v)} y(t)]_{t=0} \\ &= \left[ \frac{1}{\Gamma(2-v)} \sum_{s=v-2}^{t-(2-v)} (t-s-1)^{(2-v-1)} y(s) \right]_{t=0} \\ &= \frac{1}{\Gamma(2-v)} (1-v)^{(1-v)} y(v-2) = y(v-2), \end{aligned} \quad (4)$$

and

$$\begin{aligned} B &= [\Delta^{v-1} y(t)]_{t=0} \\ &= \left[ \frac{1}{\Gamma(1-v)} \sum_{s=v-2}^{t+(v-1)} (t-s-1)^{(1-v-1)} y(s) \right]_{t=0} \\ &= \frac{1}{\Gamma(1-v)} \left[ (1-v)^{(-v)} y(v-2) + (-v)^{(-v)} y(v-1) \right] \\ &= (1-v)y(v-2) + y(v-1). \end{aligned}$$

Denote by  $Y_1(t) = y(t + v - \mu - 1)$ . Then,

$$\begin{aligned}
R_{\mu-1} [Y_1(t)](s) &= \sum_{t=\mu-1}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} Y_1(t) \\
&= \sum_{t=\mu-1}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} y(t + v - \mu - 1) \\
&= \sum_{t=v-2}^{\infty} \left( \frac{1}{s+1} \right)^{t-v+\mu+1+1} y(t) \\
&= (s+1)^{v-\mu-1} \sum_{t=v-2}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} y(t) \\
&= (s+1)^{v-\mu-1} R_{v-2} [y(t)](s).
\end{aligned} \tag{5}$$

From Lemma 2.2 of [4], we have

$$R_0 [\Delta^\mu y(t)](s) = s^\mu R_{\mu-1} [y(t)](s) - [\Delta^{\mu-1} y(t)]_{t=0}. \tag{6}$$

From (5) and (6), we obtain

$$\begin{aligned}
R_0 [\Delta^\mu Y_1(t)](s) &= s^\mu R_{\mu-1} [Y_1(t)](s) - [\Delta^{\mu-1} Y_1(t)]_{t=0} \\
&= s^\mu [(s+1)^{v-\mu-1} R_{v-2} [y(t)](s)] - [\Delta^{\mu-1} Y_1(t)]_{t=0}.
\end{aligned} \tag{7}$$

Now, consider

$$\begin{aligned}
[\Delta^{\mu-1} Y_1(t)]_{t=0} &= [\Delta^{-(1-\mu)} Y_1(t)]_{t=0} \\
&= \left[ \frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} Y_1(s) \right]_{t=0} \\
&= \left[ \frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} y(s+v-\mu-1) \right]_{t=0} \\
&= \frac{1}{\Gamma(1-\mu)} (-\mu)^{(-\mu)} y(v-2) \\
&= y(v-2) = A.
\end{aligned} \tag{8}$$

Using (4) and (8) in (7), we have

$$R_0 [\Delta^\mu Y_1(t)](s) = s^\mu (s+1)^{v-\mu-1} R_{v-2} [y(t)](s) - A. \tag{9}$$

Denote by  $H_1(t) = h(t + v - 1)$ . Then,

$$\begin{aligned}
R_0 [H_1(t)](s) &= \sum_{t=0}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} H_1(t) \\
&= \sum_{t=0}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} h(t+v-1) \\
&= \sum_{t=v-1}^{\infty} \left( \frac{1}{s+1} \right)^{t-v+1+1} h(t) \\
&= (s+1)^{v-1} \sum_{t=v-1}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} h(t) \\
&= (s+1)^{v-1} \sum_{t=v-2}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} h(t) - h(v-2) \\
&= (s+1)^{v-1} R_{v-2} [h(t)](s) - h(v-2). \tag{10}
\end{aligned}$$

Apply  $R_0$  to each side of (1) and employ (3), (9) and (10), we obtain

$$\begin{aligned}
- [s^v R_{v-2} [y(t)](s) - sA - B] + \alpha [s^\mu (s+1)^{v-\mu-1} R_{v-2} [y(t)](s) - A] \\
= (s+1)^{v-1} R_{v-2} [h(t)](s) - h(v-2).
\end{aligned}$$

Rearrange terms and obtain

$$\begin{aligned}
(s^v - \alpha s^\mu (s+1)^{v-\mu-1}) R_{v-2} [y(t)](s) \\
= (s-\alpha)A + B - (s+1)^{v-1} R_{v-2} [h(t)](s) + h(v-2).
\end{aligned}$$

This implies that

$$\begin{aligned}
R_{v-2} [y(t)](s) &= \frac{(s-\alpha)}{(s^v - \alpha s^\mu (s+1)^{v-\mu-1})} A \\
&+ \frac{1}{(s^v - \alpha s^\mu (s+1)^{v-\mu-1})} B \\
&- \frac{(s+1)^{v-1}}{(s^v - \alpha s^\mu (s+1)^{v-\mu-1})} R_{v-2} [h(t)](s) \\
&+ \frac{1}{(s^v - \alpha s^\mu (s+1)^{v-\mu-1})} h(v-2). \tag{11}
\end{aligned}$$

Denote by  $Z(t) = y(t + n(v - \mu - 1))$ . Then,

$$\begin{aligned}
R_{v-2+n} [Z(t)] (s) &= \sum_{t=v-2+n}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} Z(t) \\
&= \sum_{t=v-2+n}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} y(t + n(v - \mu - 1)) \\
&= \sum_{t=(n+1)v-n\mu-2}^{\infty} \left( \frac{1}{s+1} \right)^{t-nv+n\mu+n+1} y(t) \\
&= (s+1)^{nv-n\mu-n} \sum_{t=(n+1)v-n\mu-2}^{\infty} \left( \frac{1}{s+1} \right)^{t+1} y(t) \\
&= (s+1)^{nv-n\mu-n} R_{(n+1)v-n\mu-2} [y(t)] (s). \tag{12}
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{s}{s^v - \alpha s^\mu (s+1)^{v-\mu-1}} \\
&= \frac{1}{s^{v-1} - \alpha s^{\mu-1} (s+1)^{v-\mu-1}} \\
&= \frac{1}{s^{v-1}} \frac{1}{1 - \alpha s^{\mu-v} (s+1)^{v-\mu-1}} \\
&= \frac{1}{s^{v-1}} \frac{1}{\left[ 1 - \frac{\alpha}{s} \left( \frac{s+1}{s} \right)^{v-\mu-1} \right]} \\
&= \frac{1}{s^{v-1}} \sum_{k=0}^{\infty} \left( \frac{\alpha}{s} \right)^k \left( \frac{s+1}{s} \right)^{kv-k\mu-k} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu-k} \frac{1}{s^{(k+1)v-k\mu-1}} \\
&= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu-k} \frac{R_{(k+1)v-k\mu-2} [t^{((k+1)v-k\mu-2)}] (s)}{\Gamma((k+1)v - k\mu - 1)} \quad (\text{By Lemma 2.1 of [4]}) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-2+k} [(t + k(v - \mu - 1))^{((k+1)v-k\mu-2)}] (s)}{\Gamma((k+1)v - k\mu - 1)}. \quad (\text{By (12)}) \tag{13}
\end{aligned}$$

We observe that

$$\left[ (t + k(v - \mu - 1))^{((k+1)v-k\mu-2)} \right]_{t=v-2+l} = 0, \quad l = 0, 1, \dots, (k-1).$$

Thus, from (13), we have

$$\begin{aligned}
\frac{s}{s^v - \alpha s^\mu (s+1)^{v-\mu-1}} &= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-2} [(t + k(v - \mu - 1))^{((k+1)v-k\mu-2)}] (s)}{\Gamma((k+1)v - k\mu - 1)} \\
&= R_{v-2} \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t + k(v - \mu - 1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v - k\mu - 1)} \right] (s) \\
&= R_{v-2} [F_1(t, \alpha, v, \mu)] (s). \tag{14}
\end{aligned}$$

Observe that  $F_1(t, \alpha, v, \mu)$  defines finite sum in the domain we choose to work here. Indeed,

$$F_1(t, \alpha, v, \mu) = \sum_{k=0}^n \alpha^k \frac{(t + k(v - \mu - 1))^{((k+1)v - k\mu - 2)}}{\Gamma((k+1)v - k\mu - 1)} \Big|_{t=v-2+n}, \quad n = 0, 1, 2, \dots.$$

Clearly,

$$F_1(v-2, \alpha, v, \mu) = \frac{(v-2)^{(v-2)}}{\Gamma(v-1)} = 1.$$

Similar to (12), we have

$$R_{v-1+n} [Z(t)](s) = (s+1)^{nv-n\mu-n} R_{(n+1)v-n\mu-1} [y(t)](s). \quad (15)$$

Consider

$$\begin{aligned} & \frac{1}{s^v - \alpha s^\mu (s+1)^{v-\mu-1}} \\ &= \frac{1}{s^v} \frac{1}{1 - \alpha s^{\mu-v} (s+1)^{v-\mu-1}} \\ &= \frac{1}{s^v} \frac{1}{\left[1 - \frac{\alpha}{s} \left(\frac{s+1}{s}\right)^{v-\mu-1}\right]} \\ &= \frac{1}{s^v} \sum_{k=0}^{\infty} \left(\frac{\alpha}{s}\right)^k \left(\frac{s+1}{s}\right)^{kv-k\mu-k} \\ &= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu-k} \frac{1}{s^{(k+1)v-k\mu}} \\ &= \sum_{k=0}^{\infty} \alpha^k (s+1)^{kv-k\mu-k} \frac{R_{(k+1)v-k\mu-1} [t^{((k+1)v-k\mu-1)}](s)}{\Gamma((k+1)v-k\mu)} \quad (\text{By Lemma 2.1 of [4]}) \\ &= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-1+k} [(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}](s)}{\Gamma((k+1)v-k\mu)}. \quad (\text{By (15)}) \quad (16) \end{aligned}$$

We observe that

$$\left[ (t+k(v-\mu-1))^{((k+1)v-k\mu-1)} \right]_{t=v-1+l} = 0, \quad l = 0, 1, \dots, (k-1).$$

Thus, from (16), we have

$$\begin{aligned}
& \frac{1}{s^v - \alpha s^\mu (s+1)^{v-\mu-1}} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{R_{v-1} [(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}] (s)}{\Gamma((k+1)v - k\mu)} \\
&= R_{v-1} \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v - k\mu)} \right] (s) \\
&= R_{v-2} \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v - k\mu)} \right] (s) \\
&\quad - (s+1)^{1-v} \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v - k\mu)} \right]_{t=v-2} \quad (\text{By (2.2) of [4]}) \\
&= R_{v-2} \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v - k\mu)} \right] (s) \\
&= R_{v-2} [F_2(t, \alpha, v, \mu)] (s). \tag{17}
\end{aligned}$$

Observe that  $F_2(t, \alpha, v, \mu)$  defines finite sum in the domain we choose to work here. Indeed,

$$F_2(t, \alpha, v, \mu) = \sum_{k=0}^n \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v - k\mu)} \Big|_{t=v-2+n}, \quad n = 0, 1, 2, \dots \tag{18}$$

Clearly,

$$F_2(v-2, \alpha, v, \mu) = \frac{(v-2)^{(v-1)}}{\Gamma(v)} = 0.$$

Using (14) and (17) in (11), we obtain

$$\begin{aligned}
R_{v-2} [y(t)] (s) &= AR_{v-2} [F_1(t, \alpha, v, \mu) - \alpha F_2(t, \alpha, v, \mu)] (s) \\
&\quad + BR_{v-2} [F_2(t, \alpha, v, \mu)] (s) \\
&\quad - (s+1)^{v-1} R_{v-2} [F_2(t, \alpha, v, \mu)] (s) R_{v-2} [h(t)] (s) \\
&\quad + R_{v-2} [F_2(t, \alpha, v, \mu)] (s) h(v-2).
\end{aligned}$$

That is,

$$\begin{aligned}
R_{v-2} [y(t)] (s) &= AR_{v-2} [F_1(t, \alpha, v, \mu) - \alpha F_2(t, \alpha, v, \mu)] (s) \\
&\quad + BR_{v-2} [F_2(t, \alpha, v, \mu)] (s) \\
&\quad - R_{v-2} [F_2(t, \alpha, v, \mu) *_{v-2} h] (s) \quad (\text{By Lemma 2.4 of [4]}) \\
&\quad + R_{v-2} [F_2(t, \alpha, v, \mu)] (s) h(v-2). \tag{19}
\end{aligned}$$

Apply to each side the inverse of  $R_{v-2}$ , we obtain

$$\begin{aligned}
y(t) &= A [F_1(t, \alpha, v, \mu) - \alpha F_2(t, \alpha, v, \mu)] + B [F_2(t, \alpha, v, \mu)] \\
&\quad - [F_2(t, \alpha, v, \mu) *_{v-2} h] + [F_2(t, \alpha, v, \mu)] h(v-2).
\end{aligned}$$

That is,

$$\begin{aligned} y(t) &= A [F_1(t, \alpha, v, \mu) - \alpha F_2(t, \alpha, v, \mu)] + B [F_2(t, \alpha, v, \mu)] \\ &\quad - \sum_{s=v-2}^t F_2(t-s+v-2, \alpha, v, \mu) h(s) \quad (\text{By (2.4) of [4]}) \\ &\quad + [F_2(t, \alpha, v, \mu)] h(v-2). \end{aligned}$$

That is,

$$\begin{aligned} y(t) &= A [F_1(t, \alpha, v, \mu) - \alpha F_2(t, \alpha, v, \mu)] + B [F_2(t, \alpha, v, \mu)] \\ &\quad - \sum_{s=v-1}^t F_2(t-s+v-2, \alpha, v, \mu) h(s). \quad (20) \end{aligned}$$

Using  $y(v-2) = 0$  in (20), we have

$$\begin{aligned} 0 &= A [F_1(v-2, \alpha, v, \mu) - \alpha F_2(v-2, \alpha, v, \mu)] + B [F_2(v-2, \alpha, v, \mu)] \\ &\quad - \sum_{s=v-1}^{v-2} F_2(v-2-s+v-2, \alpha, v, \mu) h(s). \end{aligned}$$

That is,

$$0 = A. \quad (21)$$

Using  $y(v+b+1) = 0$  in (20) and taking  $A = 0$ , we have

$$0 = B [F_2(v+b+1, \alpha, v, \mu)] - \sum_{s=v-1}^{v+b+1} F_2(v+b+1-s+v-2, \alpha, v, \mu) h(s),$$

or

$$\begin{aligned} B &= \frac{1}{F_2(v+b+1, \alpha, v, \mu)} \sum_{s=v-1}^{v+b+1} F_2(v+b+1-s+v-2, \alpha, v, \mu) h(s) \\ &= \frac{1}{F_2(v+b+1, \alpha, v, \mu)} \sum_{s=v-1}^{v+b+1} F_2(2v+b-s-1, \alpha, v, \mu) h(s). \quad (22) \end{aligned}$$

Using (22) and  $A = 0$  in (20), we obtain

$$\begin{aligned} y(t) &= \left[ \frac{1}{F_2(v+b+1, \alpha, v, \mu)} \sum_{s=v-1}^{v+b+1} F_2(2v+b-s-1, \alpha, v, \mu) h(s) \right] [F_2(t, \alpha, v, \mu)] \\ &\quad - \sum_{s=v-1}^t F_2(t-s+v-2, \alpha, v, \mu) h(s). \end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned} y(t) &= \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} \sum_{s=v-1}^{v+b+1} F_2(2v+b-s-1, \alpha, v, \mu) h(s) \\ &\quad - \sum_{s=v-1}^t F_2(t-s+v-2, \alpha, v, \mu) h(s). \end{aligned}$$

That is,

$$\begin{aligned}
y(t) &= \sum_{s=v-1}^t \left[ \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) \right. \\
&\quad \left. - F_2(t-s+v-2, \alpha, v, \mu) \right] h(s) \\
&\quad + \sum_{s=t+1}^{v+b} \left[ \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) \right] h(s).
\end{aligned}$$

As a consequence of the previous calculations we arrive at the following result.

**Theorem 2.1.** *Problem (1) – (2) has a unique solution if and only if*

$$F_2(v+b+1, \alpha, v, \mu) \neq 0.$$

In such a case, denoting by

$$I_1 = \{(t, s) : v-1 \leq s \leq t \leq v+b+1\},$$

and

$$I_2 = \{(t, s) : v-1 \leq t+1 \leq s \leq v+b\},$$

we have that the expression of the related Green's function is given by the following one:

$$G_\alpha(t, s) = \begin{cases} \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) - F_2(t-s+v-2, \alpha, v, \mu) & \text{if } (t, s) \in I_1, \\ \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) & \text{if } (t, s) \in I_2. \end{cases}$$

### 3 Positiveness of the Green's function

In this section we study the sign of the Green's function for all  $\alpha \geq 0$ .

**Theorem 3.1.** *Let  $\alpha \geq 0$  and  $v - \mu - 1 > 0$ . Then Problem (1)–(2) has a unique solution and its related Green's function  $G_\alpha$  satisfies that*

$$G_\alpha(t, s) > 0, \quad (t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [v-1, v+b]_{\mathbb{N}_{v-1}}$$

*Proof.* First, notice that the following symmetry property holds:

$$G_\alpha(2v+b-1-t, 2v+b-1-s) = G_\alpha(t, s), \quad (t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [v-1, v+b]_{\mathbb{N}_{v-1}}. \quad (23)$$

Consider the expression of function  $F_2$  given by (18). Clearly,

$$F_2(v-2, \alpha, v, \mu) = 0.$$

Here  $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$ , that is  $t = v-2+n$ ,  $n = 1, 2, \dots, b+2$ . Then,

$$\begin{aligned}
F_2(t, \alpha, v, \mu) &= F_2(v-2+n, \alpha, v, \mu) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k)\Gamma(kv+v-k\mu)} \\
&= \sum_{k=0}^{n-1} \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k)\Gamma(kv+v-k\mu)} \\
&= \sum_{k=0}^{n-1} \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)}. \tag{24}
\end{aligned}$$

In particular,

$$\begin{aligned}
F_2(v-1, \alpha, v, \mu) &= F_2(v-2+1, \alpha, v, \mu) \\
&= \sum_{k=0}^0 \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \\
&= \frac{(v-1)^{(v-1)}}{\Gamma(v)} = 1. \tag{25}
\end{aligned}$$

From (24), we have

$$\begin{aligned}
F_2(t, \alpha, v, \mu) &= \frac{(v-2+n)^{(v-1)}}{\Gamma(v)} + \sum_{k=1}^{n-1} \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \\
&= \frac{t^{(v-1)}}{\Gamma(v)} + \sum_{k=1}^{t-v+1} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)}.
\end{aligned}$$

For each  $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$ ,

$$\frac{t^{(v-1)}}{\Gamma(v)} = \frac{\Gamma(t+1)}{\Gamma(t-v+2)\Gamma(v)} > 0.$$

For each  $1 \leq k \leq t-v+1$  and  $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$ , since  $\alpha \geq 0$  and  $v-\mu-1 > 0$ , we have that  $\alpha^k \geq 0$  and,

$$\frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} = \frac{\Gamma(t+k(v-\mu-1)+1)}{\Gamma(t-k-v+2)\Gamma((k+1)v-k\mu)} > 0.$$

Thus, we obtain

$$0 < \frac{t^{(v-1)}}{\Gamma(v)} \leq F_2(t, \alpha, v, \mu), \quad t \in [v-1, v+b]_{\mathbb{N}_{v-1}}. \tag{26}$$

For each  $\alpha \geq 0$  and  $(t, s) \in I_2$ , it follows from (26) that

$$G_\alpha(t, s) = \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) > 0.$$

Then, from (23), we obtain that  $G_\alpha(t, s) > 0$  for each  $\alpha \geq 0$  and  $(t, s) \in I_1$  and the proof is concluded.  $\square$

## 4 A priori bounds

In this section, once we have proved that the Green's function is positive for  $\alpha \geq 0$  and  $v - \mu - 1 > 0$ , we will obtain some strong properties on the sign of the Green's function. As we will see on next section, these strong properties will be useful to ensure the existence of solutions of nonlinear problems defined on suitable cones.

The obtained result is the following

**Theorem 4.1.** *Assume that  $v - \mu - 1 > 0$ . Let  $G_\alpha$  be the related Green's function to Problem (1)–(2). Then, there exists  $\alpha^* > 0$  such that the following inequalities hold for any  $\alpha \in (0, \alpha^*)$ :*

$$(G) \quad l_1(t)\Phi(s) \leq G_\alpha(t, s) \leq l_2(t)\Phi(s), \quad \text{for all } (t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [v-1, v+b]_{\mathbb{N}_{v-1}},$$

where

$$l_1(t) = \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} > 0, \quad t \in [v-1, v+b]_{\mathbb{N}_{v-1}},$$

$$l_2(t) = F_2(t, \alpha, v, \mu) > 0, \quad t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$$

and

$$\Phi(s) = \frac{F_2(2v+b-s-1, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} > 0, \quad s \in [v-1, v+b]_{\mathbb{N}_{v-1}}.$$

*Proof.* First, we consider the following value:

$$\frac{G_\alpha(t, s)}{G_\alpha(s, s)} = \begin{cases} \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} - \frac{F_2(v+b+1, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} \frac{F_2(t-s+v-2, \alpha, v, \mu)}{F_2(2v+b-s-1, \alpha, v, \mu)} & \text{if } (t, s) \in I_1, \\ \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} & \text{if } (t, s) \in I_2. \end{cases}$$

In view of (26), for  $(t, s) \in I_1$ , we have that  $F_2(v+b+1, \alpha, v, \mu) > 0$ ,  $F_2(s, \alpha, v, \mu) > 0$ ,  $F_2(2v+b-s-1, \alpha, v, \mu) > 0$ , and  $F_2(t-s+v-2, \alpha, v, \mu) \geq 0$ .

Consequently, we have

$$\frac{G_\alpha(t, s)}{G_\alpha(s, s)} \begin{cases} \leq \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} & \text{if } (t, s) \in I_1, \\ = \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} & \text{if } (t, s) \in I_2. \end{cases}$$

That is,

$$\frac{G_\alpha(t, s)}{G_\alpha(s, s)} \leq \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)}, \quad (t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [v-1, v+b]_{\mathbb{N}_{v-1}}. \quad (27)$$

For any  $t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}$  and  $\alpha \geq 0$ , we denote by

$$g(t, \alpha) = (v-2) \frac{t^{(v-3)}}{\Gamma(v-1)} + \sum_{k=1}^{t-v+3} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)}. \quad (28)$$

Clearly,

$$g(t, 0) = (v-2) \frac{t^{(v-3)}}{\Gamma(v-1)} = (v-2) \frac{\Gamma(t+1)}{\Gamma(t-v+4)\Gamma(v-1)} < 0,$$

for each  $t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}$ . Consider

$$\begin{aligned} \frac{\partial g}{\partial \alpha}(t, \alpha) &= \sum_{k=1}^{t-v+3} k\alpha^{k-1} \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)} \\ &= \frac{(t+(v-\mu-1))^{(2v-\mu-3)}}{\Gamma(2v-\mu-2)} + \sum_{k=2}^{t-v+3} k\alpha^{k-1} \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)}. \end{aligned}$$

Clearly, for each  $t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}$ ,

$$\frac{(t+(v-\mu-1))^{(2v-\mu-3)}}{\Gamma(2v-\mu-2)} = \frac{\Gamma(t+v-\mu)}{\Gamma(t-v+3)\Gamma(2v-\mu-2)} > 0,$$

and for each  $k \geq 2$ ,  $k\alpha^{k-1} \geq 0$  and,

$$\frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)} = \frac{\Gamma(t+k(v-\mu-1)+1)}{\Gamma(t-k-v+4)\Gamma(k(v-\mu-1)+k+v-2)} > 0.$$

Thus, from the above discussion, we have

$$0 < \frac{\partial g}{\partial \alpha}(t, \alpha), \quad \alpha \geq 0, \quad t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}},$$

and

$$\lim_{\alpha \rightarrow \infty} g(t, \alpha) = \infty, \quad t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}.$$

Therefore, there exists a unique  $\bar{\alpha}(t) > 0$  such that

$$g(t, \bar{\alpha}(t)) = 0, \quad t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}, \quad \text{and } g(t, \alpha) < 0 \text{ for all } \alpha \in [0, \bar{\alpha}(t)).$$

Now, we define

$$\alpha^* := \min\{\bar{\alpha}(t); \quad t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}\} > 0. \quad (29)$$

Throughout, we assume that the parameter  $\alpha$  satisfies

$$\alpha \in (0, \alpha^*).$$

Consider

$$\begin{aligned} \Delta_t F_2(t, \alpha, v, \mu) &= \Delta_t \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right] \\ &= \sum_{k=0}^{\infty} \alpha^k \frac{\Delta_t [(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}]}{\Gamma((k+1)v-k\mu)} \\ &= \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)}. \end{aligned} \quad (30)$$

Here  $t \in [v-1, v+b-1]_{\mathbb{N}_{v-1}}$ , that is  $t = v-2+n$ ,  $n = 1, 2, \dots, b+1$ . Then, from (30), we have

$$\begin{aligned}
\Delta_t F_2(t, \alpha, v, \mu) &= \Delta_t F_2(v-2+n, \alpha, v, \mu) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k+1)\Gamma(kv+v-k\mu-1)} \\
&= \sum_{k=0}^n \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k+1)\Gamma(kv+v-k\mu-1)} \\
&= \sum_{k=0}^n \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)}.
\end{aligned}$$

Again, consider

$$\begin{aligned}
\Delta_t F_2(t, \alpha, v, \mu) &= \sum_{k=0}^{t-v+2} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)} \\
&= \frac{t^{(v-2)}}{\Gamma(v-1)} + \sum_{k=1}^{t-v+2} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)}.
\end{aligned}$$

Clearly,

$$\frac{t^{(v-2)}}{\Gamma(v-1)} = \frac{\Gamma(t+1)}{\Gamma(t-v+3)\Gamma(v-1)} > 0, \quad t \in [v-1, v+b-1]_{\mathbb{N}_{v-1}},$$

and for each  $k \geq 1$ ,  $\alpha^k \geq 0$  and,

$$\frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-2)}}{\Gamma((k+1)v-k\mu-1)} = \frac{\Gamma(t+k(v-\mu-1)+1)}{\Gamma(t-k-v+3)\Gamma((k+1)v-k\mu-1)} > 0,$$

for  $t \in [v-1, v+b-1]_{\mathbb{N}_{v-1}}$ . Thus, from the above discussion, we have

$$0 < \Delta_t F_2(t, \alpha, v, \mu), \quad t \in [v-1, v+b-1]_{\mathbb{N}_{v-1}}. \quad (31)$$

Consider

$$\begin{aligned}
\Delta_t^2 F_2(t, \alpha, v, \mu) &= \Delta_t^2 \left[ \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}}{\Gamma((k+1)v-k\mu)} \right] \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{\Delta_t^2 [(t+k(v-\mu-1))^{((k+1)v-k\mu-1)}]}{\Gamma((k+1)v-k\mu)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)}. \quad (32)
\end{aligned}$$

Here  $t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}$ , that is  $t = v-2+n$ ,  $n = 1, 2, \dots, b$ . Then, from

(32), we have

$$\begin{aligned}
\Delta_t^2 F_2(t, \alpha, v, \mu) &= \Delta_t^2 F_2(v-2+n, \alpha, v, \mu) \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)} \\
&= \sum_{k=0}^{\infty} \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k+2)\Gamma(kv+v-k\mu-2)} \\
&= \sum_{k=0}^{n+1} \alpha^k \frac{\Gamma(v-1+n+kv-k\mu-k)}{\Gamma(n-k+2)\Gamma(kv+v-k\mu-1)} \\
&= \sum_{k=0}^{n+1} \alpha^k \frac{(v-2+n+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)}. \tag{33}
\end{aligned}$$

Again, from (28) and (33), for each  $t \in [v-1, v+b-2]_{\mathbb{N}_{v-1}}$ , we have

$$\begin{aligned}
\Delta_t^2 F_2(t, \alpha, v, \mu) &= \sum_{k=0}^{t-v+3} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)} \\
&= \frac{t^{(v-3)}}{\Gamma(v-2)} + \sum_{k=1}^{t-v+3} \alpha^k \frac{(t+k(v-\mu-1))^{((k+1)v-k\mu-3)}}{\Gamma((k+1)v-k\mu-2)} \\
&= g(t, \alpha) < g(t, \alpha^*) = 0. \tag{34}
\end{aligned}$$

From (31) and (34), it follows that  $F_2(t, \alpha, v, \mu)$  is strictly increasing on  $[v-1, v+b-1]_{\mathbb{N}_{v-1}}$ , and  $\Delta_t F_2(t, \alpha, v, \mu)$  is strictly decreasing on  $[v-1, v+b-2]_{\mathbb{N}_{v-1}}$ .

Now, for  $(t, s) \in I_1$ , we observe that  $t < v+b+1$  and  $t-s+v-2 < 2v+b-s-1$ .

Then, we have

$$F_2(t, \alpha, v, \mu) < F_2(v+b+1, \alpha, v, \mu),$$

and

$$\Delta_s F_2(2v+b-s-1, \alpha, v, \mu) < \Delta_s F_2(t-s+v-2, \alpha, v, \mu).$$

Consider

$$\begin{aligned}
\Delta_s G_\alpha(t, s) &= \Delta_s \left[ \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} F_2(2v+b-s-1, \alpha, v, \mu) - F_2(t-s+v-2, \alpha, v, \mu) \right] \\
&= \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} \Delta_s F_2(2v+b-s-1, \alpha, v, \mu) - \Delta_s F_2(t-s+v-2, \alpha, v, \mu) \\
&< \Delta_s F_2(2v+b-s-1, \alpha, v, \mu) - \Delta_s F_2(t-s+v-2, \alpha, v, \mu) \\
&< 0,
\end{aligned}$$

implying that  $G_\alpha(t, s)$  decreases in  $s$  for each  $(t, s) \in I_1$ . Thus, we have

$$G_\alpha(t, t) \leq G_\alpha(t, s) \leq G_\alpha(t, v-1), \quad (t, s) \in I_1.$$

Then, for  $(t, s) \in I_1$ , we have

$$\frac{G_\alpha(t, s)}{G_\alpha(s, s)} \geq \frac{G_\alpha(t, t)}{G_\alpha(s, s)} = \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} \frac{F_2(2v+b-t-1, \alpha, v, \mu)}{F_2(2v+b-s-1, \alpha, v, \mu)}. \tag{35}$$

Also, for  $(t, s) \in I_2$ , we have

$$\frac{G_\alpha(t, s)}{G_\alpha(s, s)} = \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)}. \quad (36)$$

For  $(t, s) \in I_1$ , we observe that  $2v + b - t - 1 \leq 2v + b - s - 1$ . Then, we have

$$F_2(2v + b - t - 1, \alpha, v, \mu) \leq F_2(2v + b - s - 1, \alpha, v, \mu),$$

implying that

$$\frac{F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)} \leq 1,$$

implying that

$$\frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} \frac{F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)} \leq \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)}.$$

Thus, in view of (27), (35) and (36), we have

$$\frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)} \frac{F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)} \leq \frac{G_\alpha(t, s)}{G_\alpha(s, s)} \leq \frac{F_2(t, \alpha, v, \mu)}{F_2(s, \alpha, v, \mu)}.$$

That is,

$$\frac{F_2(t, \alpha, v, \mu) F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(v + b + 1, \alpha, v, \mu)} \leq G_\alpha(t, s) \leq \frac{F_2(t, \alpha, v, \mu) F_2(2v + b - s - 1, \alpha, v, \mu)}{F_2(v + b + 1, \alpha, v, \mu)}.$$

That is,

$$\begin{aligned} \frac{F_2(t, \alpha, v, \mu) F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)} \left[ \frac{F_2(2v + b - s - 1, \alpha, v, \mu)}{F_2(v + b + 1, \alpha, v, \mu)} \right] &\leq G_\alpha(t, s) \\ &\leq F_2(t, \alpha, v, \mu) \left[ \frac{F_2(2v + b - s - 1, \alpha, v, \mu)}{F_2(v + b + 1, \alpha, v, \mu)} \right]. \end{aligned} \quad (37)$$

We know that  $v - 1 \leq 2v + b - s - 1 \leq v + b$ . Then, we have

$$F_2(v - 1, \alpha, v, \mu) \leq F_2(2v + b - s - 1, \alpha, v, \mu) \leq F_2(v + b, \alpha, v, \mu),$$

implying that

$$\frac{1}{F_2(v + b, \alpha, v, \mu)} \leq \frac{1}{F_2(2v + b - s - 1, \alpha, v, \mu)} \leq \frac{1}{F_2(v - 1, \alpha, v, \mu)}.$$

Similarly, we have  $v - 1 \leq 2v + b - t - 1 \leq v + b$ , implying that

$$F_2(v - 1, \alpha, v, \mu) \leq F_2(2v + b - t - 1, \alpha, v, \mu) \leq F_2(v + b, \alpha, v, \mu).$$

Consequently, we have

$$\frac{F_2(t, \alpha, v, \mu) F_2(v - 1, \alpha, v, \mu)}{F_2(v + b, \alpha, v, \mu)} \leq \frac{F_2(t, \alpha, v, \mu) F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)},$$

that is,

$$\frac{F_2(t, \alpha, v, \mu)}{F_2(v + b, \alpha, v, \mu)} \leq \frac{F_2(t, \alpha, v, \mu) F_2(2v + b - t - 1, \alpha, v, \mu)}{F_2(2v + b - s - 1, \alpha, v, \mu)}, \quad (38)$$

where we have used the fact that  $F_2(v-1, \alpha, v, \mu) = 1$ , from (25). Combining (37) and (38), we obtain that

$$\begin{aligned} \left[ \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} \right] \left[ \frac{F_2(2v+b-s-1, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} \right] &\leq G_\alpha(t, s) \\ &\leq [F_2(t, \alpha, v, \mu)] \left[ \frac{F_2(2v+b-s-1, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} \right]. \end{aligned}$$

The above inequality is of the form

$$l_1(t)\Phi(s) \leq G_\alpha(t, s) \leq l_2(t)\Phi(s),$$

for all  $(t, s) \in [v-1, v+b]_{\mathbb{N}_{v-1}} \times [v-1, v+b]_{\mathbb{N}_{v-1}}$ , where, from (26),

$$l_1(t) = \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} > 0,$$

$$l_2(t) = F_2(t, \alpha, v, \mu) > 0,$$

and

$$\Phi(s) = \frac{F_2(2v+b-s-1, \alpha, v, \mu)}{F_2(v+b+1, \alpha, v, \mu)} > 0.$$

We know that  $v-1 \leq t \leq v+b$ . Then,

$$F_2(v-1, \alpha, v, \mu) \leq F_2(t, \alpha, v, \mu) \leq F_2(v+b, \alpha, v, \mu).$$

From (25), we obtain that

$$1 \leq F_2(t, \alpha, v, \mu) \leq F_2(v+b, \alpha, v, \mu),$$

implying that

$$0 < \frac{F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} \leq F_2(t, \alpha, v, \mu).$$

That is,  $0 < l_1(t) \leq l_2(t)$  and the proof is concluded.  $\square$

## 5 Existence of multiple solutions

This section is devoted to show the applicability of previous results, in order to deduce the existence of multiple solutions of the nonlinear problem

$$\begin{cases} -\Delta^v y(t) + \alpha \Delta^\mu y(t+v-\mu-1) &= f(t+v-1, y(t+v-1)), t \in I, \\ y(v-2) &= y(v+b+1) = 0. \end{cases} \quad (39)$$

with  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  a continuous function.

As we have seen on Theorem 4.1, we have that for all  $\alpha \in (0, \alpha^*)$  and  $v-\mu-1 > 0$ , condition (G) is fulfilled.

Moreover, since  $\Delta_t F_2(t, \alpha, v, \mu) > 0$  for all  $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$ , we have that

$$L_1 := \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} l_1(t) = \frac{\max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} = \frac{F_2(v+b, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} = 1,$$

$$\begin{aligned}
m &:= \min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} l_1(t) = \frac{\min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} F_2(t, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} \\
&= \frac{F_2(v-1, \alpha, v, \mu)}{F_2(v+b, \alpha, v, \mu)} = \frac{1}{F_2(v+b, \alpha, v, \mu)}
\end{aligned}$$

and

$$L_2 := \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} l_2(t) = \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} F_2(t, \alpha, v, \mu) = F_2(v+b, \alpha, v, \mu).$$

Define the Banach space  $X$  of real valued functions on  $[v-2, v+b+1]_{\mathbb{N}_{v-2}}$  with the standard maximum norm.

We have the following uniqueness result

**Theorem 5.1.** *Assume that  $\alpha \in (0, \alpha^*)$  and  $f : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times \mathbb{R} \rightarrow \mathbb{R}$  is uniformly Lipschitz in its second variable with constant  $0 < k < \frac{1}{L_2 \sum_{s=v-1}^{v+b} \Phi(s)}$ , then FBVP (39) has an unique solution.*

*Proof.* We will show that  $T$  is a contraction map. For all  $t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}$ , we have

$$\begin{aligned}
\|Ty_1(t) - Ty_2(t)\| &= \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \left| \sum_{s=v-1}^{v+b} G(t, s) [f(s+v-1, y_1(s+v-1)) \right. \\
&\quad \left. - f(s+v-1, y_2(s+v-1))] \right| \\
&\leq k \|y_1 - y_2\| \max_{t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}} \sum_{s=v-1}^{v+b} G(t, s) \\
&\leq kL_2 \sum_{s=v-1}^{v+b} \Phi(s) \|y_1 - y_2\| \leq \|y_1 - y_2\|.
\end{aligned}$$

Then, the Contraction mapping theorem gives us that  $T$  is a contraction on  $X$ . Thus,  $T$  has an unique fixed point in  $X$ .  $\square$

Now, consider the cone

$$P = \left\{ y : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \rightarrow [0, \infty), y(t) \geq \frac{l_1(t)}{L_2} \|y\|, t \in I \right\},$$

where

$$\|y\| := \max_{t \in I} |y(t)|.$$

It is well known that the solutions of (39) are given as the fixed points of the difference operator

$$(Ty)(t) = \sum_{s=v-1}^{v+b} G_\alpha(t, s) f(s+v-1, y(s+v-1)), \quad t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}},$$

where  $G_\alpha(t, s)$  is its related Green's function.

Thus, by assuming that  $f : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times [0, \infty) \rightarrow [0, \infty)$  is a continuous function, and  $\alpha \in (0, \alpha^*)$ , with  $\alpha^*$  given by (29). If we take

$y \in P$  be arbitrarily chosen, from condition (G), we have that  $Ty \geq 0$  on  $[v-2, v+b+1]_{\mathbb{N}_{v-2}}$  and, moreover, we deduce that the following inequalities are fulfilled for all  $t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}$  :

$$\begin{aligned}
(Ty)(t) &= \sum_{s=v-1}^{v+b} G_{\alpha}(t, s) f(s+v-1, y(s+v-1)) \\
&\geq l_1(t) \sum_{s=v-1}^{v+b} \Phi(s) f(s+v-1, y(s+v-1)) \\
&= \frac{l_1(t)}{L_2} \sum_{s=v-1}^{v+b} \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} l_2(t) \Phi(s) f(s+v-1, y(s+v-1)) \\
&\geq \frac{l_1(t)}{L_2} \sum_{s=v-1}^{v+b} \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \{G_{\alpha}(t, s)\} f(s+v-1, y(s+v-1)) \\
&= \frac{l_1(t)}{L_2} \sum_{s=v-1}^{v+b} \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \{G_{\alpha}(t, s) f(s+v-1, y(s+v-1))\} \\
&\geq \frac{l_1(t)}{L_2} \|Ty\|.
\end{aligned}$$

In other words,  $T : P \rightarrow P$ .

Moreover, due to the continuity of function  $f$ , it is clear that  $T$  is a completely continuous operator.

Now, by using fixed point theorems given in [5, 6], and arguing as in the proofs made in [8], we can deduce the following multiplicity results for the nonlinear Problem (39).

**Theorem 5.2.** *Let  $f : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times [0, \infty) \rightarrow [0, \infty)$  be a continuous function, and  $\alpha \in (0, \alpha^*)$ , with  $\alpha^*$  given by (29). Suppose that there exist positive integers  $p, q$  and  $r$  such that  $p < q < r$ , and assume that function  $f$  satisfies the following conditions:*

$$(i) \ f(t, y) \geq \frac{y}{\sum_{s=v-1}^{v+b} \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [r, \frac{L_2}{m} r], \text{ being}$$

the inequality strict at  $y = r$ ,

$$(ii) \ f(t, y) \leq \frac{q}{L_2 \sum_{s=v-1}^{v+b} \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [0, \frac{L_2}{m} q],$$

being the inequality strict at  $y = q$ ,

$$(iii) \ f(t, y) > \frac{y L_2}{L_1 \sum_{s=v-1}^{v+b} l_1(s) \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [\frac{m}{L_2} p, p].$$

Then the operator  $T$  has at least two fixed points,  $y_1$  and  $y_2$ , such that

$$p < \|y_1\|, \quad \max_{t \in I} y_1(t) < q < \max_{t \in I} y_2(t), \quad \min_{t \in I} y_2(t) < r.$$

**Remark 5.3.** *We point out that due to the properties that the fixed points  $y_1$  and  $y_2$  satisfy, both of them are not trivial.*

**Theorem 5.4.** *Let  $f : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \times [0, \infty) \rightarrow [0, \infty)$  be a continuous function, and  $\alpha \in (0, \alpha^*)$ , with  $\alpha^*$  given by (29). Let  $p, q$  and  $r$  be positive*

integers such that:

$$p < q < \frac{L_2}{m}q \leq r.$$

Suppose that the function  $f$  satisfies the assumptions below:

$$(a) f(t, y) \leq \frac{r}{L_2 \sum_{s=v-1}^{v+b} \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [0, r],$$

$$(b) f(t, y) < \frac{p}{L_2 \sum_{s=v-1}^{v+b} \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [0, p],$$

$$(c) f(t, y) \geq \frac{y}{m \sum_{s=v-1}^{v+b} \Phi(s)} \text{ for all } t \in [v-1, v+b]_{\mathbb{N}_{v-1}} \text{ and } y \in [q, \frac{L_2}{m}q], \text{ being}$$

the inequality strict for  $y = q$ .

Then, operator  $T$  has at least three fixed points  $y_1, y_2, y_3 \in \{y \in P \mid \|y\| \leq r\}$  such that  $\max_{t \in I} y_1(t) < p$ ,  $q < \min_{t \in I} y_2(t)$  and  $p < \max_{t \in I} y_3(t)$  with  $\max_{t \in I} y_3(t) < q$ .

**Remark 5.5.** We point out that the fixed points  $y_1$  and  $y_2$  obtained in previous result are not trivial. However, without additional assumptions on the data of operator  $T$ , we cannot ensure such property for  $y_3$ .

**Acknowledgements** First author is partially supported by Xunta de Galicia (Spain), project EM2014/032 and AIE, Spain and FEDER, grant PID2020-113275GB-I00 The second author is supported by the Bulgarian National Science Foundation under Project DN 12/4 "Advanced analytical and numerical methods for nonlinear differential equations with applications in finance and environmental pollution", 2017.

The authors thank the anonymous referee for her/his careful reading and interesting suggestions that have contributed to improving this paper

## References

- [1] Atici, F M, Eloe, P W, *A transform method in discrete fractional calculus.* Int. J. Difference Equ. 2 (2007), 2, 165-176.
- [2] Atici, F M, Eloe, P W, *Initial value problems in discrete fractional calculus.* Proc. Amer. Math. Soc. 137 (2009), no. 3, 981-989.
- [3] Atici, F M, Eloe, P W, *Two-point boundary value problems for finite fractional difference equations.* J. Difference Equ. Appl. 17 (2011), 4, 445-456.
- [4] Atici, F M, Eloe, P W, *Linear Forward Fractional Difference Equations,* Com. Appl. Anal. 19 (2015), 31-42.
- [5] Avery, R I, *A generalization of the Leggett-Williams fixed point theorem,* MSR Hot-Line 2 (1999) 9-14.
- [6] Avery, R I, Henderson, J, *Two positive fixed points of nonlinear operators on ordered Banach spaces,* Comm. Appl. Nonlinear Anal. 8 (2001) 27-36.
- [7] Bohner, M, Peterson, A, *Dynamic equations on time scales. An introduction with applications.,* Birkhäuser Boston Inc., Boston, MA, 2001.

- [8] Cabada, A, Dimitrov, N, *Existence of solution of nonlocal perturbation of Dirichlet discrete nonlinear problems*, Acta Mathematica Scientia, Vol.37, Ser.B, 4, (2017), 911–927.
- [9] Cabada, A, Dimitrov, N, *Nontrivial solutions of non-autonomous Dirichlet fractional discrete problems.*, Fract. Calc. Appl. Anal. 23 (2020), no. 4, 980-995.
- [10] Donahue, R, *The development of a transform method for use in solving difference equations*, Master's thesis, University of Dayton, 1987.
- [11] C.S. Goodrich, On positive solutions to nonlocal fractional and integer-order difference equations. *Appl. Anal. Discrete Math* 5 (2011), 122–132.
- [12] Goodrich, C, Peterson, A C, *Discrete fractional calculus*. Springer, Cham, 2015.
- [13] Heymans, N, Podlubny, I, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*. Rheol. Acta 45, 765-771 (2006)
- [14] Kilbas, AA, Srivastava, HH, Trujillo, J J, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006)
- [15] Magin, R L, *Fractional Calculus in Bioengineering*. Begell House Publisher, Connecticut (2006)
- [16] Miller, K S, Ross, B, *Fractional difference calculus*. In Univalent functions, fractional calculus, and their applications (Koriyama, 1988), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, (1989), 139-152.
- [17] Podlubny, I, *Fractional Differential Equations*. Academic Press, San Diego (1999)
- [18] Samko, S G, Kilbas, A A, Marichev, O I, *Fractional Integrals and Derivatives - Theory and Applications*. Gordon & Breach, Linghorne (1993)
- [19] West, B J, Bologna, M, Grigolini, P, *Physics of Fractal Operators*. Springer, New York (2003)