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Extremal solutions of nonlinear functional discontinuous fractional equations

Alberto Cabada[1] and Wanassi Om Kalthoum[2]
[1]Departamento de Estatística, Análise Matemática e Optimización
Instituto de Matemáticas, Facultade de Matemáticas,
Universidade de Santiago de Compostela, Spain.
alberto.cabada@usc.gal
[2]Department of Mathematics
University of Monastir, Tunisia.
kalthoum.wannassi@gmail.com

Abstract

This paper is devoted to prove the existence of extremal solutions of Fractional equation with Riemann-Liouville derivative. The existence follows from the method of lower and upper solutions. Some jumps in the derivative of these functions are allowed. It is important to point out that a discontinuous and functional dependence on the nonlinear part of the equation with respect to the solution is allowed. The construction of the Green's function related to the linear part of the equation coupled to spectral theory is fundamental to deduce the results.

Key Words: Lower and Upper Solutions, Green's Functions, Discontinuous Equations, Functional Equations, Comparison Principles

AMS Subject Classifications: 34K37, 34A08, 26A33, 34B27

1 Introduction

The concept of Fractional Calculus has been developed up to now as an extension and a generalization of the classical calculus with non-integer orders. In recent years there has been an important growth on research activities on the application of fractional calculus to several scientific fields, from various physical phenomena to control systems in economics and finance. So, applications related to fractional calculus have appeared in many fields of science and engineering, such as a HIV [27, 28] and a fluid model [29].

The main advantage of fractional derivatives is that mathematical models involving fractional derivatives provide a very well description of the memory and hereditary properties of various materials. In fact, we have that the fractional derivative of a function f defined on a given interval evaluated at a particular point depends on all the values of the function on the given interval till the evaluated point.

On the other hand, the method of lower and upper solutions is a very well known tool used to deduce the existence of solutions of many kind of both

Ordinary and Partial Differential Equations coupled to many kind of boundary value problems. The reader can see the monographs [1, 17, 22] or the survey [2] and references therein. This method allows not only to ensure the existence of solutions of the considered problem but also to ensure the location of some of them, that lie between the given lower and upper solutions. In many cases, it is possible to deduce the existence of the greatest and the least of all the solutions on this set [16].

In the decade of 80s S. Heikkilä introduced the concept of Chain Iterative Monotones, see the monograph [21] and references therein. With this method it has been possible to study differential equations with discontinuities on the spatial variable. This method has been applied by many authors to different situations, see, for instance, [4, 5, 7, 8, 10, 13, 14, 15, 19, 25], and references therein.

In this paper, by considering $\alpha \in \mathbb{R}$, $1 < \alpha \leq 2$ be fixed and $I = [0, 1]$. For any $u : (0, 1] \to \mathbb{R}$, we define function $\overline{u} : I \to \mathbb{R}$ as follows:

$$\overline{u}(t) = \begin{cases} t^{2-\alpha} u(t) & \text{if} \quad t \in (0, 1], \\ \lim_{t \to 0^+} t^{2-\alpha} u(t) & \text{if} \quad t = 0, \end{cases} \tag{1}$$

provided that such limit exists.

First, we will study the existence of extremal solution of the following problem, in presence of a pair of lower and upper solutions,

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, \overline{u}(t), \overline{u}'(t)) = 0, \qquad t \in I, \\ \overline{u}(0) = \mu \int_0^1 u(s) ds, \quad u'(1) = \eta \int_0^1 u(s) ds. \end{cases}$$

Here μ and η are nonnegative real constants and f is a continuous function that satisfies the following condition:

$$(F) \qquad \forall\, C_1 > 0 \text{ such that } \|(x, y)\|_\infty \leq C_1 \ \exists\, C_2 > 0 \ |f(t, x, y)| \leq C_2\, t^{2-\alpha}.$$

To this end, we will prove in Section 3, some comparison results for the related linear equation, where some kind of jumps in the definition of the solutions are allowed. The idea can be found on, among others, [17] or [9]. As far as the authors know, this is the first time that such arguments are used in fractional equations. In this case, we must use some previous comparison results for solutions without jumps given in [6] for a Dirichlet problem, and in [11, 12] for mixed problems. The spectral results obtained in those references will be crucial in our analysis. Moreover the exact expression of the Green's function related to this equation, and some kind of Nagumo's condition concerning the growth with respect to the first derivative, will be also fundamental in our study. This study is done in Section 4, where the definition of lower and upper solutions with some kind of jumps of the first derivative is allowed.

In Section 5 we consider the functional and discontinuous problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, \overline{u}(t), \overline{u}'(t), \overline{u}) = 0, \qquad t \in I, \\ \overline{u}(0) = \mu \int_0^1 u(s) ds, \quad u'(1) = \eta \int_0^1 u(s) ds. \end{cases}$$

In this case, by means of the Heikkilä's theory coupled to the extremality results obtained in previous section, we deduce the existence of extremal solutions for such a problem.

An example, containing a particular case, is given at the end of each section.

2 Preliminary Results

In this section, we introduce some notations and definitions that will be used along the paper.

We introduce the following space for $\alpha \in \mathbb{R}$, $\alpha \in (1,2]$.

$$C^1_{2-\alpha}(I) := \{u : (0,1] \to \mathbb{R};\ \bar{u} \in C^1(I)\}.$$

It is not difficult to verify that $C^1_{2-\alpha}(I)$ is a Banach space endowed with the norm

$$\|u\|_1 = \max\{\max_{t\in I}\{|\bar{u}(t)|\}, \max_{t\in I}\{|\bar{u}'(t)|\}\}$$

We remark that $C^1_{2-\alpha}(I) \subset C^1((0,1])$ and point out that the right limit at $t = 0$ of the functions on this space may be $\pm\infty$.

Definition 1 *([23]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ t > 0,$$

where Γ is the Euler Gamma function, assuming that the right-hand side is point-wise defined on $(0,+\infty)$.

Definition 2 *([23]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds = \frac{d^n}{dt^n} I^{n-\alpha} f(t),$$

provided that the right-hand side is point-wise defined on the interval $(0,+\infty)$. Here $n = [\alpha]+1$, where $[\alpha]$ denotes the integer part of the real number α.

Definition 3 *[23, p. 42] A two parameter Mittag-Leffler function $E_{\alpha,\beta}(x)$ is defined by the series expansion*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta > 0,\ x \in \mathbb{R}.$$

For $\beta = 1$, $E_{\alpha,1}$ coincides with the usual Mittag–Leffler function E_α.

In [11] the following results are proved, provided $E_{\alpha,\alpha-1}(\lambda) \neq 0$,

$$v_1(t) = \Gamma(\alpha-1)\left(t^{\alpha-2}E_{\alpha,\alpha-1}(\lambda t^\alpha) - \frac{E_{\alpha,\alpha-2}(\lambda)}{E_{\alpha,\alpha-1}(\lambda)} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)\right) \tag{2}$$

is the unique solution in $C^1_{2-\alpha}(I)$ of problem

$$\begin{cases} D^\alpha v_1(t) - \lambda v_1(t) = 0, & t \in I, \\ \overline{v_1}(0) = 1, \quad v_1'(1) = 0, \end{cases} \tag{3}$$

and

$$v_2(t) = \frac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{E_{\alpha,\alpha-1}(\lambda)}, \tag{4}$$

the unique one in $C^1_{2-\alpha}(I)$ of

$$\begin{cases} D^\alpha v_2(t) - \lambda v_2(t) = 0, \quad t \in I, \\ \overline{v_2}(0) = 0, \quad v_2'(1) = 1. \end{cases} \tag{5}$$

Moreover, given $y \in L^\infty(I)$, the unique solution in $C^1_{2-\alpha}(I)$ of problem

$$\begin{cases} D^\alpha v(t) - \lambda v(t) + y(t) = 0, \quad t \in I, \\ \overline{v}(0) = v'(1) = 0, \end{cases} \tag{6}$$

follows the expression

$$v(t) = \int_0^1 G_1(t,s) y(s) ds,$$

with

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)}{(1-s)^{2-\alpha} E_{\alpha,\alpha-1}(\lambda)} - (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha), & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)}{(1-s)^{2-\alpha} E_{\alpha,\alpha-1}(\lambda)}, & 0 \le t < s < 1. \end{cases} \tag{7}$$

Finally, by denoting

$$\theta := \int_0^1 v_1(t) dt \quad \text{and} \quad \sigma := \int_0^1 v_2(t) dt,$$

we have the following result:

Theorem 4 *[12, Theorem 1] Let $y \in L^\infty(I)$, $1 < \alpha \le 2$, $\mu, \eta \ge 0$ and $\lambda \in \mathbb{R}$ be such that $E_{\alpha,\alpha-1}(\lambda) \neq 0$ and $1 - \mu\theta - \eta\sigma \neq 0$. Then the problem*

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) = 0, \quad t \in I, \\ \bar{u}(0) = \mu \int_0^1 u(s) ds, \quad u'(1) = \eta \int_0^1 u(s) ds, \end{cases} \tag{8}$$

has a unique solution u in $C^1_{2-\alpha}(I)$, which is given by the expression

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t,s) = G_1(t,s) + \frac{(\mu v_1(t) + \eta v_2(t))}{(1 - \mu\theta - \eta\sigma)} \left(\int_0^1 G_1(r,s) dr \right), \tag{9}$$

with v_1, v_2 and G_1 given in (2), (4) *and* (7) *respectively.*

Remark 5 *Notice that in [12, Theorem 1] it is assumed $y \in C((0,1]) \cap L^\infty(I)$. It is not difficult to verify that all the results are valid if $y \in L^\infty(I)$.*

3 Anti-Maximum Principle

In this section, we will prove an anti-maximum principle for the linear problem (8), where some constant sign jumps on the derivative of the solutions are allowed. Such result contains [11, Lemma 8] as a particular case (with no jumps).

First, we denote $\lambda_1^*(<0)$ as the greatest root of $E_{\alpha,\alpha-1}(\lambda)=0$, which is, as it is stated on [11, Theorem 6], the first eigenvalue of Problem (6). As it is pointed out on reference [20], it is clear that if $\alpha,\ \beta>0$ then $E_{\alpha,\beta}(\lambda)>0$ for all $\lambda>0$. Moreover, since $E_{\alpha,\beta}(0)=1/\Gamma(\beta)$, we have that the real zeros of this function, if they exists, must be negative. The existence of negative zeros can be consulted in [24, Theorem 2], where it is proved that $E_{\alpha,\beta}(\lambda)$ has real zeros for all $\beta\in(0,\alpha)$.

The comparison result is the following:

Theorem 6 *Let $y\in L^\infty(I)$, $y\geq 0$ a.e. on I, $1<\alpha\leq 2$ and $\lambda>\lambda_1^*$. Let $q\in\{0,1,\ldots\}$ and $0=t_0<t_1<\ldots<t_q<t_{q+1}=1$. Assume that $u:(0,1]\to\mathbb{R}$ is such that $\bar{u}\in C(I)\cap C^1(\cup_{l=0}^q(t_l,t_{l+1}))$ and satisfies the following equalities for some $A,\ B\geq 0$ and $k_j>0$, $j=1,\ldots,q$:*

$$\left\{\begin{array}{rcl} D^\alpha u(t)-\lambda u(t)+y(t) & = & 0,\quad t\in I\backslash\{t_1,\ldots,t_q\},\\ \bar{u}(0) & = & A,\\ u'(1) & = & B,\\ \bar{u}'(t_j^-)-\bar{u}'(t_j^+) & = & k_j,\quad j=1,\ldots,q. \end{array}\right.$$

Then, $\bar{u}\geq 0$ on I. Moreover, if y is not identically zero on I then $u>0$ on $(0,1]$.

Proof. In our case, by using the results obtained in [11], we have that function u is given by the following expression:

$$u(t)=\int_0^1 G_1(t,s)y(s)ds+Av_1(t)+Bv_2(t)+\sum_{l=1}^q k_l\,S_l(t), \tag{10}$$

with v_1, v_2 and G_1 given in (2), (4) and (7) respectively, and $S_l\in C(I)\cap C^1(I\backslash\{t_l\})$, the unique solution of problem

$$\left\{\begin{array}{rcl} D^\alpha S_l(t)-\lambda S_l(t) & = & 0,\quad t\in I\backslash\{t_l\},\\ \overline{S}(0) & = & 0,\\ S'(1) & = & 0,\\ \overline{S}_l'(t_j^-)-\overline{S}'(t_j^+) & = & \delta_{j\,l},\quad j=1,\ldots,q, \end{array}\right.$$

being $\delta_{j\,l}$ the classical Dirac delta function.

From [11, Lemmas 8 and 18], $G_1(t,s)>0$ for all $t,s\in(0,1)$ and v_1, v_2 are positive on $(0,1]$.
Now, let us see that $S_l>0$ on $(0,1]$ for all $l=1,\ldots,q$.

Assume that there exists $\bar{t}_l\in(0,t_k]$ for some $k=1,\cdots,p+1$, for which $\overline{S}_l(\bar{t}_l)=0$.

If $\bar{t}_l\leq t_l$, we have that $S_l\in C^1_{2-\alpha}[0,\bar{t}_l]$ is a solution of problem

$$D^\alpha S_l(t)-\lambda S_l(t)=0,\quad t\in[0,\bar{t}_l],\quad \overline{S}_l(0)=S_l(\bar{t}_l)=0. \tag{11}$$

As it has been shown in [6], the eigenvalues $\tilde{\lambda}_n$ of problem (11) are characterized by

$$E_{\alpha,\alpha}(\tilde{\lambda}_n\bar{t}_l^\alpha)=0,$$

and $\tilde{\lambda}_1:=\lambda_1/\bar{t}_l^\alpha$ is the first eigenvalue of (11). We remark, see [6, Lemma 3.4] and [11, Lemma 8], that $\lambda_1<0$ is the greatest root of $E_{\alpha,\alpha}(\lambda)=0$ and $\lambda_1^*>\lambda_1$.

Thus, if S_l is a nontrivial function then (λ, S_l) is an eigenvalue-eigenvector pair of problem (11) and $\lambda > \lambda_1^* > \lambda_1 > \tilde{\lambda}_1$, which contradicts the fact that $\tilde{\lambda}_1$ is the first eigenvalue of problem (11).

As a consequence, $\bar{S}_l(t) = 0$ for all $t \in [0, \bar{t}_l]$.

Now, when $\bar{t}_l < t_l$, it is fulfilled that $S_l \in C^1_{2-\alpha}[\bar{t}_l, t_l]$ is a solution of problem

$$D^\alpha S_l(t) - \lambda S_l(t) = 0, \quad t \in [\bar{t}_l, t_l], \quad \overline{S}_l(\bar{t}_l) = S_l'(\bar{t}_l) = 0.$$

Notice that, since $S_l(t) = 0$ for all $t \in [0, \bar{t}_l]$, we have that

$$I^\alpha S_l(t) = \frac{1}{\Gamma(\alpha)} \int_{\bar{t}_l}^t (t-s)^{\alpha-1} S_l(s) ds =: I^\alpha_{\bar{t}_l} S_l(t), \; t \in [\bar{t}_l, t_l],$$

and, as a consequence,

$$D^\alpha S_l(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{\bar{t}_l}^t (t-s)^{n-\alpha-1} S_l(s) ds =: D^\alpha_{\bar{t}_l} S_l(t).$$

Thus, $S_l \in C^1_{2-\alpha}[\bar{t}_l, t_l]$ satisfies the equation

$$D^\alpha_{\bar{t}_l} S_l(t) - \lambda S_l(t) = 0, \quad t \in [\bar{t}_l, t_l], \quad \overline{S}_l(\bar{t}_l) = S_l'(\bar{t}_l) = 0.$$

But, this problem has only the trivial solution.

As a consequence, it is fulfilled that $\bar{S}_l(t) = 0$ for all $t \in [0, t_l]$ whenever $\bar{t}_l \le t_l$.

Thus, since $\overline{S}_l'(t_l^-) - \overline{S}'(t_l^+) = 1$, we know that S_l is not identically zero at $[\bar{t}_l, 1]$. Therefore, we have that $S_l \in C^1_{2-\alpha}[t_l, 1]$ is a nontrivial solution of the problem

$$D^\alpha S_l(t) - \lambda S_l(t) = 0, \quad t \in [t_l, 1], \quad \overline{S}_l(t_l) = S_l'(1) = 0.$$

Arguing as before, we have that S_l is a solution of the following equation (with obvious notation):

$$D^\alpha_{t_l} S_l(t) - \lambda S_l(t) = 0, \quad t \in [t_l, 1], \quad \overline{S}_l(t_l) = S_l'(1) = 0.$$

It is immediate to verify that the eigenvalues of this problem, $\overline{\lambda}_n$, are given as the zeros of $E_{\alpha,\alpha-1}(\overline{\lambda}_n\,(1-t_l)^\alpha)$. Since they must be negative, it is obvious that all of then of are smaller than λ_1^*. As a direct consequence, we deduce that S_l cannot attain the value zero on $(0,1]$.

For the case $t_l < \bar{t}_l$, we attain a contradiction in an analogous way.

As a consequence, function S_l is a constant sign solution on the interval $(0,1]$. Let us see that it is positive for all $\lambda > \lambda_1^*$.

Notice that, if $\lambda = 0 (> \lambda_1^*)$, we have $S_l(1) = \frac{t_l^{3-\alpha}}{(\alpha-1)} > 0$ and so, $S_l > 0$ on $(0,1]$.

So, if there is $\lambda \neq 0$ for which $S_l < 0$ on $(0,1]$, by the continuity of function S_l with respect to λ, we have that there are $\tilde{\lambda}$ lying between λ and 0, and $\tilde{t} \in (0,1]$, for which $\bar{S}_l(\tilde{t}) = 0$.

If $\tilde{t} \in (0,1)$ we reach a contradiction with the previous cases.

If $\tilde{t} = 1$, we have that $\bar{S}_l \in C^1[\bar{t}_l, 1]$ is a solution of problem

$$D^\alpha S_l(t) - \lambda S_l(t) = 0, \quad t \in [t_l, 1], \quad S_l(1) = S_l'(1) = 0,$$

which has only the trivial solution.

So $S_l = 0$ on $[\bar{t}_l, 1]$ and, because of the jump condition, we arrive to a contradiction with the eigenvalue characterization in the interval $[0, t_l]$ as in the previous steps. ■

Arguing in an analogous way with the Dirichlet conditions, we can prove the following result

Theorem 7 *Let $y \in L^\infty(I)$, $y \geq 0$ a.e. on I, $1 < \alpha \leq 2$ and $\lambda > \lambda_1$. Let $q \in \{0, 1, \ldots\}$ and $0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = 1$. Assume that $u : (0, 1] \to \mathbb{R}$ is such that $\overline{u} \in C(I) \cap C^1(\cup_{l=0}^q (t_l, t_{l+1}))$ and satisfies the following equalities for some A, $B \geq 0$ and $k_j > 0$, $j = 1, \ldots, q$:*

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) &= 0, \quad t \in I \backslash \{t_1, \ldots, t_q\}, \\ \bar{u}(0) &= A, \\ u(1) &= B, \\ \bar{u}'(t_j^-) - \bar{u}'(t_j^+) &= k_j, \quad j = 1, \ldots, q. \end{cases}$$

Then, $\bar{u} \geq 0$ on I. Moreover, if y is not identically zero on I then $u > 0$ on $(0, 1)$.

In the sequel, we deduce a maximum principle for the case in which the mixed boundary conditions are attained on an sub interval $[a, b] \subset [0, 1]$.

Theorem 8 *Let $y \in L^\infty[0, 1]$, $y \geq 0$ on (a, b), $1 < \alpha \leq 2$ and $\lambda > \lambda_1^*/b^\alpha$. Let $q \in \{0, 1, \ldots\}$ and $0 \leq a = t_0 < t_1 < \ldots < t_q < t_{q+1} = b \leq 1$*

Assume that $u : (0, 1] \to \mathbb{R}$ is such that $\overline{u} \in C(I) \cap C^1(\cup_{l=0}^q (t_l, t_{l+1}))$ and satisfies the following equalities for some A, $B \geq 0$ and $k_j > 0$, $j = 1, \ldots, q$:

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) &= 0, \quad t \in [0, 1] \backslash \{t_1, \cdots, t_q\}, \\ \bar{u}(a) &= A, \\ u'(b) &= B, \\ \bar{u}'(t_j^-) - \bar{u}'(t_j^+) &= k_j, \quad j = 1, \ldots, q. \end{cases} \tag{12}$$

Then, $\bar{u} \geq 0$ on $[a, b]$. Moreover, if y is not identically zero on $(a, b]$ then $u > 0$ on $(a, b]$.

Proof. It is clear that the unique solution of problem (16) is given by

$$u(t) = U(t) + Av_1(t) + Bv_2(t) + \sum_{l=1}^q k_l \, S_l(t), \tag{13}$$

where $U(t) = \int_a^b G_{1,a}(t, s) y(s) ds$ is the unique solution of

$$\begin{cases} D^\alpha U(t) - \lambda U(t) + \sigma(t) = 0, \quad t \in [0, 1], \\ U(a) = U'(b) = 0, \end{cases} \tag{14}$$

v_1 is a solution of problem

$$\begin{cases} D^{\alpha} v_1(t) - \lambda v_1(t) = 0, \quad t \in [0,1], \\ \bar{v}_1(a) = 1, \\ v_1'(b) = 0, \end{cases}$$

v_2 is a solution of problem

$$\begin{cases} D^{\alpha} v_2(t) - \lambda v_2(t) = 0, \quad t \in [0,1], \\ \bar{v}_2(a) = 0, \\ v_2'(b) = 1, \end{cases}$$

and S_l is a solution of problem

$$\left\{ \begin{array}{rcl} D^{\alpha} S_l(t) - \lambda S_l(t) & = & 0, \quad t \in [0,1] \backslash \{t_l\}, \\ \overline{S}(a) & = & 0, \\ S'(b) & = & 0, \\ \overline{S}_l'(t_j^-) - \overline{S}'(t_j^+) & = & \delta_{j\,l}, \quad j = 1, \ldots, q, \end{array} \right.$$

Now we will study the sign of the Green's function $G_{1,a} : I \times I \to \mathbb{R}$.

It is clear that the enunciate of the theorem is a direct consequence of the tw previous properties.

For any $s \in [0,1]$, $V_s(t) := G_{1,a}(t,s)$ satisfies

$$\begin{cases} D^{\alpha} V_s(t) - \lambda V_s(t) = 0, \quad t \in [0,1], \ t \neq s, \\ V_s(a) = 0 = V_s^{'}(b). \end{cases} \tag{15}$$

Now, consider the following eigenvalue problem

$$\begin{cases} D^{\alpha} w(t) - \lambda w(t) = 0, \quad t \in [0,1], \\ w(a) = 0 = w'(b). \end{cases}$$

So, by [23, Theorem 5.1, p. 284], its general solution is given by

$$w(t) = C_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) + C_2 t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^{\alpha}),$$

with $C_1, C_2 \in \mathbb{R}$.
Then $w(a) = 0 = w'(b)$ implies that

$$\frac{E_{\alpha,\alpha-2}(\lambda b^{\alpha})}{b E_{\alpha,\alpha-1}(\lambda b^{\alpha})} = \frac{E_{\alpha,\alpha-1}(\lambda a^{\alpha})}{a E_{\alpha,\alpha}(\lambda a^{\alpha})}.$$

Denote now

$$f(x,\lambda) = \frac{E_{\alpha,\alpha-1}(\lambda x^{\alpha})}{x E_{\alpha,\alpha}(\lambda x^{\alpha})} \quad \text{and} \quad g(x,\lambda) = \frac{E_{\alpha,\alpha-2}(\lambda x^{\alpha})}{x E_{\alpha,\alpha-1}(\lambda x^{\alpha})},$$

Functions f and g are defined for all $x \neq 0$ and $E_{\alpha,\alpha}(\lambda x^{\alpha}) \neq 0$ and $E_{\alpha,\alpha-1}(\lambda x^{\alpha}) \neq 0$ respectively.

As we know, λ_1, the first eigenvalue of the Dirichlet problem (11) on $[0,1]$, is given as the biggest solution of $E_{\alpha,\alpha}(\lambda) = 0$ and λ_1^*, the first eigenvalue of the Mixed problem (3), is given as the biggest solution of $E_{\alpha,\alpha-1}(\lambda)=0$. Then, one can show that if $\lambda > \frac{\lambda_1}{a^\alpha}$, $f(a,\lambda)>0$ and if $\lambda > \frac{\lambda_1^*}{b^\alpha}$, $g(b,\lambda)<0$. So, from the fact that

$$\frac{\lambda_1}{a^\alpha} < \frac{\lambda_1}{b^\alpha} < \frac{\lambda_1^*}{a^\alpha} < \frac{\lambda_1^*}{b^\alpha} < \lambda_1^* < 0,$$

we have that, if $\lambda > \frac{\lambda_1^*}{b^\alpha}$, then $g(b,\lambda) < 0 < f(a,\lambda)$, and, as a consequence, λ is not an eigenvalue of (14).

So, since problem (15) has a nontrivial solution if and only if λ is an eigenvalue of (14), we conclude that

$$G_{1,a}(t,s)=0 \quad \text{if either } 0 \le s \le \min\{a,t\} \text{ or } \max\{t,b\} \le s \le 1.$$

Now, if $G_{1,a}(t_0,s_0)=0$ for some $(t_0,s_0)\in(a,b)\times(a,b)$, we have that: If $a \le s_0 \le t_0$, V_{s_0} is a nontrivial solution of

$$\begin{cases} D^\alpha V_{s_0}(t) - \lambda V_{s_0}(t) = 0, \\ V_{s_0}(t_0) = 0 = V_{s_0}^{'}(b), \end{cases}$$

which implies that $\lambda < \frac{\lambda_1^*}{b^\alpha}$.
And, if $a \le t_0 \le s_0$, V_{s_0} is a nontrivial solution of

$$\begin{cases} D^\alpha V_{s_0}(t) - \lambda V_{s_0}(t) = 0, \\ V_{s_0}(a) = 0 = V_{s_0}(t_0), \end{cases}$$

which implies that $\lambda < \frac{\lambda_1}{b^\alpha}$.
Hence, for $\lambda > \frac{\lambda_1^*}{b^\alpha}$, $G_{1,a}(t,s) \neq 0$, $\forall\ (t,s)\in(a,b)\times(a,b)$.
Since we know that $G_{1,a}>0$ if $a=0,\ b=1$, we deduce that

$$G_{1,a}(t,s) > 0 \qquad \forall\ (t,s)\in(a,b)\times(a,b),$$

$$G_{1,a}(t,s) = 0 \qquad \forall\ t\in[a,b],\ \forall\ s\in[0,\min\{a,t\}]\cup[\max\{t,b\},1].$$

The same arguments are valid to deduce the positiveness of functions v_1 and v_2.

The comparison result holds immediately from the previous properties. ■

With similar arguments, one can proof the following result for the Dirichlet problem

Theorem 9 *Let* $y\in L^\infty[0,1]$, $y\ge 0$ *on* (a,b), $1<\alpha\le 2$ *and* $\lambda > \lambda_1/b^\alpha$. *Let* $q\in\{0,1,\ldots\}$ *and* $0 \le a = t_0 < t_1 < \ldots < t_q < t_{q+1} = b \le 1$

Assume that $u:(0,1]\to\mathbb{R}$ *is such that* $\bar u \in C(I)\cap C^1(\cup_{l=0}^q (t_l,t_{l+1}))$ *and satisfies the following equalities for some* A, $B \ge 0$ *and* $k_j>0$, $j=1,\ldots,q$:

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) &= 0, \quad t\in[0,1]\backslash\{t_1,\cdots,t_q\}, \\ \bar u(a) &= A, \\ u(b) &= B, \\ \bar u'(t_j^-) - \bar u'(t_j^+) &= k_j, \quad j=1,\ldots,q. \end{cases} \tag{16}$$

Then, $\bar u \ge 0$ *on* $[a,b]$. *Moreover, if* y *is not identically zero on* $(a,b]$ *then* $u>0$ *on* $(a,b]$.

4 Boundary value problem with derivative dependence

In this section some sufficient conditions of existence and location of solutions for the following problem are obtained:

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, \overline{u}(t), \overline{u}'(t)) = 0, \qquad t \in I, \\ \overline{u}(0) = \mu \int_0^1 u(s) ds, \quad u'(1) = \eta \int_0^1 u(s) ds, \end{cases} \tag{17}$$

with f a continuous function and $\lambda > \lambda_1^*$.

We are looking for solutions of Problem (17) on the space $C^1_{2-\alpha}(I)$.

In a first moment, we will ensure the existence of solutions by assuming the presence of a pair of well ordered lower and upper solutions. After this, we prove the existence of extremal solutions for (17) between such pair of functions.

First of all, we need to define the concept of lower and upper solutions for problem (17).

Definition 10 *Let $k \geq 0$ and $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$. A function $\gamma : (0, 1] \to \mathbb{R}$ satisfying $\overline{\gamma} \in C(I) \cap C^1(\cup_{l=0}^k (t_l, t_{l+1}))$ is a lower solution of problem (17) if $D^\alpha \gamma(t) - \lambda \gamma(t) + f(t, \overline{\gamma}(t), \overline{\gamma}'(t)) \in L^\infty(I)$ and the following inequalities are fulfilled:*

$$\begin{cases} \begin{array}{rcll} D^\alpha \gamma(t) - \lambda \gamma(t) + f(t, \overline{\gamma}(t), \overline{\gamma}'(t)) & \leq & 0, & t \in I \backslash \{t_1, \ldots, t_k\}, \\ \overline{\gamma}(0) - \mu \int_0^1 \gamma(s) ds & \leq & 0, & \\ \gamma'(1) - \eta \int_0^1 \gamma(s) ds & \leq & 0, & \\ \overline{\gamma}'(t_l^\pm) \in \mathbb{R} \quad \textit{and} \quad \overline{\gamma}'(t_l^-) - \overline{\gamma}'(t_l^+) & < & 0, & l = 1, \ldots, k. \end{array} \end{cases}$$

Definition 11 *Let $p \geq 0$ and $0 = s_0 < s_1 < \ldots < s_k < s_{p+1} = 1$. A function $\delta : (0, 1] \to \mathbb{R}$ satisfying $\overline{\delta} \in C(I) \cap C^1(\cup_{l=0}^p (s_l, s_{l+1}))$ is an upper solution of problem (17) if $D^\alpha \delta(t) - \lambda \delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t)) \in L^\infty(I)$ and the following inequalities are satisfied:*

$$\begin{cases} \begin{array}{rcll} D^\alpha \delta(t) - \lambda \delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t)) & \geq & 0, & t \in I \backslash \{s_1, \ldots, s_l\}, \\ \overline{\delta}(0) - \mu \int_0^1 \delta(s) ds & \geq & 0, & \\ \delta'(1) - \eta \int_0^1 \delta(s) ds & \geq & 0, & \\ \overline{\delta}'(s_l^\pm) \in \mathbb{R} \quad \textit{and} \quad \overline{\delta}'(s_l^-) - \overline{\delta}'(s_l^+) & > & 0, & l = 1, \ldots, p. \end{array} \end{cases}$$

In the sequel we will use the following notation: for a couple of functions $\overline{\gamma}, \overline{\delta} \in C(I)$ such that $\overline{\gamma} \leq \overline{\delta}$ on I, we define

$$[\overline{\gamma}, \overline{\delta}] := \{\overline{u} \in C(I) : \ \overline{\gamma}(t) \leq \overline{u}(t) \leq \overline{\delta}(t), \ \forall t \in I\}.$$

The following condition plays a fundamental role in the prior bounds of all the possible solutions of the considered problem. It is in the line of the very well known Nagumo's growth condition [1].

Definition 12 *Consider $\overline{\gamma}$ and $\overline{\delta} \in C(I)$ such that $\overline{\gamma} \leq \overline{\delta}$ on I. Define*

$$E := \{(t, \overline{u}, \overline{v}) \in I \times \mathbb{R}^2 \ \textit{such that} \ \overline{\gamma}(t) \leq \overline{u}(t) \leq \overline{\delta}(t)\}, \tag{18}$$

and suppose that $f : E \to \mathbb{R}$ is a continuous function that satisfies:

$$|f(t, u, v)| \leq h(|v|) \quad \forall (t, u, v) \in E, \tag{19}$$

where $h : [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function such that

$$\lim_{s\to+\infty} \frac{s}{h(s)} = +\infty. \tag{20}$$

Lemma 13 *Assume that there exists $\overline{\gamma}$ and $\overline{\delta}$ a pair of lower and upper solutions, satisfying the conditions imposed on definitions 10 and 11, such that $\overline{\gamma}(t) \le \overline{\delta}(t)$ for any $t \in I$ and that function f satisfies the conditions of Definition 12. Then, there exists a constant $C > 0$, depending on h, α, γ and δ, such that any solution $u \in [\overline{\gamma}, \overline{\delta}]$ of (17) satisfies $\|\overline{u}'\|_\infty \le C$.*

Proof. From Theorem 4, we know that the solutions of Problem (17) are characterized as the solutions of the following integral equation:

$$u(t) = \int_0^1 G(t,s) f(s, \overline{u}(s), \overline{u}'(s)) ds. \tag{21}$$

Thus, we denote

$$\overline{G}(t,s) = \left\{ \begin{array}{lll} t^{2-\alpha} G(t,s) & \text{if} & (t,s) \in (0,1] \times [0,1), \\ \lim_{t\to0^+} t^{2-\alpha} G(t,s) & \text{if} & (t,s) \in \{0\} \times [0,1), \end{array} \right. \tag{22}$$

which, as it has been proved in [12, lemma 2], is well defined.

Differentiating in (21), we obtain

$$\overline{u}'(t) = \int_0^1 \frac{\partial \overline{G}}{\partial t}(t,s) f(s, \overline{u}(s), \overline{u}'(s)) ds. \tag{23}$$

So, with obvious notation, we have that

$$\frac{\partial \overline{G}}{\partial t}(t,s) = \frac{\partial \overline{G_1}}{\partial t}(t,s) + (\mu\, \overline{v}_1'(t) + \eta\, \overline{v}_2'(t)) \frac{\int_0^1 G_1(r,s) dr}{1 - \mu\theta - \eta\sigma}. \tag{24}$$

For $t, s \in I \times [0,1)$, we have

$$\frac{\partial \overline{G_1}}{\partial t}(t,s) = H_1(t,s) + H_2(t,s),$$

where

$$H_1(t,s) \quad = \quad (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha) g_1(t),$$

being

$$g_1(t) := \frac{1}{E_{\alpha,\alpha-1}(\lambda)} \left[E_{\alpha,\alpha}(\lambda t^\alpha) - \lambda t^\alpha ((\alpha-1) E_{\alpha,2\alpha}(\lambda(t)^\alpha) - E_{\alpha,2\alpha-1}(\lambda t^\alpha)) \right],$$

and

$$H_2(t,s) = \left\{ \begin{array}{ll} 0, & \text{if} \quad t \le s, \\ t^{1-\alpha}(t-s)^{\alpha-2} g_2(t,s), & \text{if} \quad s \le t, \end{array} \right.$$

with

$$g_2(t,s) := -(t+s(\alpha-2)) E_{\alpha,\alpha}(\lambda(t-s)^\alpha) + t(t-s)^\alpha \lambda((\alpha-1) E_{\alpha,2\alpha}(\lambda(t-s)^\alpha) - E_{\alpha,2\alpha-1}(\lambda(t-s)^\alpha)).$$

For $t \in I$, we have

$$\begin{aligned}\overline{v}_1'(t) &= \Gamma(\alpha-1)\Big[\frac{E_{\alpha,\alpha-2}(\lambda)E_{\alpha,\alpha}(\lambda t^\alpha)}{E_{\alpha,\alpha-1}(\lambda)} - \frac{t^\alpha\lambda E_{\alpha,\alpha-2}(\lambda)(-(\alpha-1)E_{\alpha,2\alpha}(\lambda t^\alpha)+E_{\alpha,2\alpha-1}(\lambda t^\alpha))}{E_{\alpha,\alpha-1}(\lambda)} \\ &\quad + t^{\alpha-1}\lambda(E_{\alpha,2\alpha-2}(\lambda t^\alpha)-(\alpha-2)E_{\alpha,2\alpha-1}(\lambda t^\alpha))\Big]\end{aligned}$$

and

$$\overline{v}_2'(t) = \frac{E_{\alpha,\alpha}(\lambda t^\alpha) - t^\alpha\lambda((\alpha-1)E_{\alpha,2\alpha}(\lambda t^\alpha) - E_{\alpha,2\alpha-1}(\lambda t^\alpha)}{E_{\alpha,\alpha-1}(\lambda)}.$$

Thus, from the continuity of functions g_1 and g_2, we deduce that

$$\begin{aligned}\Big|\frac{\partial \overline{G_1}}{\partial t}(t,s)\Big| &\leq |H_1(t,s)| + |H_2(t,s)| \\ &\leq K_1(\alpha,\lambda)(1-s)^{\alpha-2} + K_2(\alpha,\lambda)t^{1-\alpha}\,q(t,s),\end{aligned}$$

for some suitable positive constants $K_1(\alpha,\lambda)$ and $K_2(\alpha,\lambda)$ and

$$q(t,s) = \begin{cases} (t-s)^{\alpha-2} & \text{if} \quad s \leq t \\ 0 & \text{if} \quad s \geq t \end{cases}$$

And so, since $\overline{v_1}$, $\overline{v_2} \in C^1(I)$ and $1-\mu\theta-\eta\sigma > 0$, we have that there are positive constants K_2, K_3 and M such that

$$\begin{aligned}\Big|\frac{\partial \overline{G}}{\partial t}(t,s)\Big| &\leq K_1(\alpha,\lambda)(1-s)^{\alpha-2} + K_2(\alpha,\lambda)t^{1-\alpha}(t-s)^{\alpha-2} \\ &\quad + \bigg(\frac{\mu K_3(\alpha,\lambda)+\eta K_4(\alpha,\lambda)}{(1-\mu\theta-\eta\sigma)\,\alpha}\bigg) M(\alpha,\lambda)(1-s)^{\alpha-2}. \qquad (25)\end{aligned}$$

Using (19), and the fact that h is nondecreasing, we get

$$\begin{aligned}|\overline{u}'(t)| &\leq \int_0^1 \Big|\frac{\partial \overline{G}}{\partial t}(t,s)\Big| |f(s,\overline{u}(s),\overline{u}'(s))|ds \\ &\leq \int_0^1 \Big(K_1(\alpha,\lambda)(1-s)^{\alpha-2} + K_2(\alpha,\lambda)t^{1-\alpha}\,q(t,s) \\ &\quad + \frac{\mu K_3(\alpha,\lambda)+\eta K_4(\alpha,\lambda)}{(1-\mu\theta-\eta\sigma)\,\alpha} M(\alpha,\lambda)(1-s)^{\alpha-2}\Big) h(|\overline{u}'(s)|)ds \\ &\leq h(\|\overline{u}'\|_\infty)\bigg(\frac{K_1(\alpha,\lambda)}{(\alpha-1)} + \frac{K_2(\alpha,\lambda)}{(\alpha-1)} + \frac{\mu K_3(\alpha,\lambda)+\eta K_4(\alpha,\lambda)}{(1-\mu\theta-\eta\sigma)(\alpha-1)\alpha} M(\alpha,\lambda)\bigg) \\ &:= K(\alpha,\lambda,\mu,\eta)h(\|\overline{u}'\|_\infty), \quad \text{for all } t \in I,\end{aligned}$$

which implies that

$$\frac{\|\overline{u}'\|_\infty}{h(\|\overline{u}'\|_\infty)} \leq K(\alpha,\lambda,\mu,\eta).$$

Then, from (20), we deduce that there exists $C(\alpha,\lambda,\mu,\eta) > 0$ such that $\|\overline{u}'\|_\infty < C$.

This completes the proof. ■

In the following result, we prove the existence of solutions of problem (17) by means of lower and upper solutions method.

Theorem 14 *Let $\overline{\gamma}, \overline{\delta}$ be a pair of lower and upper solutions of (17) such that $\overline{\gamma} \leq \overline{\delta}$ on I, satisfying the conditions imposed in definitions 10 and 11. Then problem (17) has at least one solution $\overline{u} \in [\overline{\gamma}, \overline{\delta}]$ which, moreover, satisfies that $\|\overline{u}'\|_\infty < C$, with C given in Lemma 13.*

Proof. First, we denote by $A_0 = \max\big(C, \|\overline{\gamma}'\|_\infty, \|\overline{\delta}'\|_\infty\big)$, and define the truncated functions

$$r(x) = \max\{-A_0, \min\{x, A_0\}\}$$

and

$$p(t,x) = \max\{\overline{\gamma}(t), \min\{x, \overline{\delta}(t)\}\}.$$

Consider now the following modified problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f\big(t, p(t, \overline{u}(t)), r\big(\frac{d}{dt}(p(t, \overline{u}(t)))\big)\big) = 0, \quad a.e.\ t \in I, \\ \overline{u}(0) = \mu \int_0^1 p(s, u(s))ds, \quad u'(1) = \eta \int_0^1 p(s, u(s))ds. \end{cases} \tag{26}$$

To ensure the regularity of previous problem, we cite the following result:

Lemma 15 *[26, Lemma 2] For each $u \in C^1(I)$ the following properties hold:*

- *(i)* $\dfrac{d}{dt}p(t, u(t))$ *exists a.e.* $t \in I$.
- *(ii)* *If $u, u_m \in C^1(I)$ and u_m converges in $C^1(I)$ to u, then*

$$\lim_{m\to\infty}\left\{\frac{d}{dt}p(t, u_m(t))\right\} = \frac{d}{dt}p(t, u(t)) \ a.e.\ t \in I.$$

So, we can ensure the existence of $\frac{d}{dt}p(t, \bar{u}(t))$ in the whole interval I except on, at most, a set of Lebesgue measure equals to zero. As a consequence, as it is pointed out in [3, Chapter 4], it is not difficult to deduce that $p(\cdot, \bar{u}(\cdot))$ is an absolutely continuous function on I. Thus, $r\left(\frac{d}{dt}p(\cdot, u(\cdot))\right)$ belongs to $L^\infty(I)$ and we have that, provided that f is a continuous function, the right hand side of the previous equation is in $L^1(I)$ for all $u \in C^1_{2-\alpha}(I)$.

We divide the existence result into several steps:

Step 1: If u is a solution of Problem (26) then $\overline{u} \in [\overline{\gamma}, \overline{\delta}]$.

Assume, on the contrary, that

$$\min_{t\in I}(\overline{u} - \overline{\gamma})(t) < 0.$$

If for some $\tilde{t} \in \{t_1, t_2, ..., t_k\} \subset (0,1)$, we get

$$\min_{t\in I}(\overline{u} - \overline{\gamma})(t) = (\overline{u} - \overline{\gamma})(\tilde{t}) < 0,$$

we have the necessary condition

$$(\overline{u} - \overline{\gamma})'(\tilde{t}^-) \leq (\overline{u} - \overline{\gamma})'(\tilde{t}^+).$$

But, since $\bar{u} \in C^1(I)$, from the fact that $\overline{\gamma}'(\tilde{t}^-) < \overline{\gamma}'(\tilde{t}^+)$, we arrive at a contradiction.

Hence, there is some $\tau_0 \in I \backslash \{t_1, t_2, ..., t_k\}$ for which

$$\min_{t\in I}(\overline{u} - \overline{\gamma})(t) = (\overline{u} - \overline{\gamma})(\tau_0) < 0.$$

By the continuity of function $\bar{u}$ on I, we have that there is $[a,b] \subset I$ such that $\bar{u} \leq \bar{\gamma}$ on $[a,b]$.

From the definitions of the lower and upper solutions and functions p and r, we have that

$$f(t, p(t, \overline{u}(t)), r\big(\frac{d}{dt}(p(t, \overline{u}(t)))\big)\big) = f(t, \overline{\gamma}(t), \overline{\gamma}'(t)), \quad \text{for a. e.} \quad t \in [a,b],$$

and, as a consequence, there is $h_\gamma \in L^\infty(I)$, $h_\gamma \geq 0$ on $(0,1)$, such that

$$D^\alpha(u-\gamma) - \lambda(u-\gamma) + h_\gamma(t) = 0, \quad \text{a. e. } t \in [a,b] \backslash \{t_1, \cdots, t_k\}. \tag{27}$$

Using the linearity of the Riemann-Liouville derivative, we have that, if $a = 0$:

$$\overline{(u-\gamma)}(0) \geq \mu \int_0^1 p(s, u(s))ds - \mu \int_0^1 \gamma(s)ds \geq 0.$$

If $b = 1$, we deduce that

$$(u-\gamma)'(1) \geq \eta \int_0^1 p(s, u(s))ds - \eta \int_0^1 \gamma(s)ds \geq 0.$$

It is obvious that if $a \in (0,1)$ we have that $(u-\gamma)(a) = 0$ and if $b \in (0,1)$ it is showed that $(u-\gamma)(b) = 0$.

So, in all the possible combinations we are under the assumptions of Theorem 6 or Theorems 8 and 9. Thus, we deduce immediately that $u \geq \gamma$ on I, which is a contradiction with our assumption.

Using similar arguments, we deduce that $\overline{u} \leq \overline{\delta}$ on I.
Thus, we conclude that every solution of problem (26) is such that $\overline{\gamma} \leq \overline{u} \leq \overline{\delta}$ on I.

Step 2: Problem (26) has at least one solution u, with $\bar{u} \in C^1(I)$.

Let us consider the operator $\overline{T} : C^1(I) \to C^1(I)$ as follows

$$\overline{T}u(t) = \int_0^1 \overline{G}(t,s) f(s, p(s, \overline{u}(s)), r\big(\frac{d}{ds}(p(s, \overline{u}(s)))\big)\big)ds, \quad t \in I. \tag{28}$$

Since there exists $M > 0$ such that

$$\left| f(s, p(s, \overline{u}(s)), r\big(\frac{d}{ds}(p(s, \overline{u}(s)))\big)\big) \right| \leq M\, s^{2-\alpha} \qquad \text{a.e. } s \in I,$$

arguing as in the proof of Lemma 13, we have that there is $K > 0$ such that $\max\{\|\bar{T}(u)\|_\infty, \|(\bar{T}(u))'\|_\infty\} \leq K$.

In consequence, by defining $\Omega = \{u \in C^1(I), ; \max\{\|\bar{u}\|_\infty, \|\bar{u}'\|_\infty\} \leq K\}$, we have that $\bar{T} : \Omega \to \Omega$.

To verify that $\bar{T}$ is completely continuous on Ω, it must be proved that the set $\bar{T}(\Omega)$ is equicontinuous on $C^1(I)$.

The equicontinuity in $C(I)$ is analogous to the one done in [11, Lemma 4.2].

To prove the equicontinuity in $C^1(I)$ we use the expressions obtained in Lemma 13 for $\partial\bar{G}/\partial t$.

From expression (22), since $\bar{v}_1$ and $\bar{v}_2$ are in $C^1(I)$, we only need to verify the equicontinuity for the first part of the sum on such inequality $\frac{\partial G_1}{\partial t}$. So, we must work with operator

$$\overline{T_1}u(t)=\int_0^1 \overline{G_1}(t,s)f(s,p(s,\overline{u}(s)),r\big(\frac{d}{ds}(p(s,\overline{u}(s))))\big)ds,\quad t\in I. \tag{29}$$

Thus, for any $t_1,t_2\in I$ such that $0<t_1<t_2<1$ and $u\in\Omega$, we have

$$\begin{aligned}\Big|\left(\overline{T_1}u\right)'(t_2)-\left(\overline{T_1}u\right)'(t_1)\Big| &\le \int_0^1\Big|\frac{\partial\overline{G_1}}{\partial t}(t_2,s)-\frac{\partial\overline{G_1}}{\partial t}(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\big(\frac{d}{ds}(p(s,\overline{u}(s))))\big)\Big|ds\\ &\le \int_0^1\Big|H_1(t_2,s)-H_1(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\big(\frac{d}{ds}(p(s,\overline{u}(s))))\big)\Big|ds\\ &\quad+\int_0^1\Big|H_2(t_2,s)-H_2(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\big(\frac{d}{ds}(p(s,\overline{u}(s))))\big)\Big|ds.\end{aligned}$$

Note that the function $H_1(t,s)=(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})g_1(t)$, and $g_1(t)$ is continuous on I. Thus, g_1 is uniformly continuous on I and, as a consequence, for every $\varepsilon>0$ there exists $\delta>0$ such that if $|t_2-t_1|<\delta$, we obtain

$$|g_1(t_2)-g_1(t_1)|<\varepsilon,$$

and so, the first integral is bounded by

$$M\varepsilon\int_0^1(1-s)^{\alpha-2}ds\le\frac{M}{\alpha-1}\varepsilon.$$

Moreover,

$$\begin{aligned}\textstyle\int_0^1\Big|H_2(t_2,s)-H_2(t_1,s)\Big||f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s))))\Big)\Big)\Big|ds &\le \textstyle\int_0^{t_1}\Big|H_2(t_2,s)-H_2(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s)))\Big)\Big)\Big|ds\\ &\quad\textstyle+\int_{t_1}^{t_2}\Big|H_2(t_2,s)-H_2(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s)))\Big)\Big)\Big|ds\\ &\quad\textstyle+\int_{t_2}^{1}\Big|H_2(t_2,s)-H_2(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s)))\Big)\Big)\Big|ds\\ &= \textstyle\int_0^{t_1}\Big|H_2(t_2,s)-H_2(t_1,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s)))\Big)\Big)\Big|ds\\ &\quad\textstyle+\int_{t_1}^{t_2}\Big|H_2(t_2,s)\Big|\Big|f(s,p(s,\overline{u}(s)),r\Big(\frac{d}{ds}(p(s,\overline{u}(s)))\Big)\Big)\Big|ds\\ &\equiv I_1+I_2.\end{aligned}$$

For $0<t_1<s<t_2<1$, we have that $H_2(t_2,s)=t_2^{1-\alpha}(t_2-s)^{\alpha-2}g_2(t_2,s)$.

Since function $g_2(t,s)$ is continuous on the triangle $\{(t,s)\in I\times I,\ 0\le s\le t\le 1\}$, there exists a constant $M>0$ such that

$$|g_2(t,s)|<K_2\quad\text{for all } 0\le s\le t\le 1.$$

Thus,

$$\begin{aligned}I_2 &\le K_2M\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-2}\,s^{2-\alpha}ds\\ &= K_2Mt_2^{2-\alpha}\left(B_1(3-\alpha,\alpha-1)-B_{\frac{t_1}{t_2}}(3-\alpha,\alpha-1)\right),\end{aligned}$$

where by $B_a(p,q)$ it is denoted the classical Beta function.

Since

$$\lim_{t_1\to t_2^-}\left\{B_{\frac{t_1}{t_2}}(3-\alpha,\alpha-1)\right\}=B_1(3-\alpha,\alpha-1),$$

we have that for every $\varepsilon>0$ there exists $\delta>0$ such that if $|t_2-t_1|<\delta$, we obtain

$$I_2\leq K_2M\varepsilon.$$

To get estimates for I_1, we separate the expression $H_2(t_2,s)-H_2(t_1,s)$, on the integral I_1, as

$$t_2^{1-\alpha}(t_2-s)^{-2+\alpha}\big(g_2(t_2,s)-g_2(t_1,s)\big)+\big(t_2^{1-\alpha}(t_2-s)^{-2+\alpha}-t_1^{1-\alpha}(t_1-s)^{-2+\alpha}\big)g_2(t_1,s).$$

Since g_2 is continuous when $0\leq s\leq t\leq 1$. Then, for $\varepsilon>0$ there exists $\delta>0$ such that if $|t_2-t_1|<\delta$, we obtain

$$|g_2(t_2,s)-g_2(t_1,s)|<\varepsilon.$$

Hence, the first part of I_1 is bounded by

$$\varepsilon K_2\int_0^{t_1}t_2^{1-\alpha}(t_2-s)^{-2+\alpha}ds=\varepsilon K_2\frac{1-(\frac{t_2}{t_2-t_1})^{1-\alpha}}{\alpha-1}.$$

Since $t_2>t_1$, we have $\frac{t_2}{t_2-t_1}>1$, so $0<(\frac{t_2}{t_2-t_1})^{1-\alpha}<1$ and $1-(\frac{t_2}{t_2-t_1})^{1-\alpha}<1$ and $\frac{1-(\frac{t_2}{t_2-t_1})^{1-\alpha}}{\alpha-1}<\frac{1}{\alpha-1}$.
Thus, the previous integral is bounded by

$$\frac{\varepsilon\,K_2}{\alpha-1}.$$

Now, as we mention above, we have that $|g_2(t,s)|\leq K_2$ for $0\leq s\leq t\leq 1$. So, the second part of I_1 is bounded by

$$MK_2\int_0^{t_1}\big(t_2^{1-\alpha}(t_2-s)^{-2+\alpha}-t_1^{1-\alpha}(t_1-s)^{-2+\alpha}\big)s^{2-\alpha}ds=$$

$$MK_2\left(t_2^{2-\alpha}B_{\frac{t_1}{t_2}}(3-\alpha,\alpha-1)-t_1^{2-\alpha}B_1(3-\alpha,\alpha-1)\right).$$

Using now that Since

$$\lim_{t_2\to t_1^+}\left\{t_2^{1-\alpha}B_{\frac{t_1}{t_2}}(3-\alpha,\alpha-1)\right\}=t_1^{1-\alpha}B_1(3-\alpha,\alpha-1),$$

we have that for every $\varepsilon>0$ there exists $\delta>0$ such that if $|t_2-t_1|<\delta$, we obtain that the previous integral is bounded from above by

$$M\,K_2\,\varepsilon.$$

So, by using all the previous inequalities, we deduce that for all $\varepsilon>0$, there exists $\delta>0$ such that if $|t_2-t_1|<\delta$ we arrive at the following inequalities

$$\left|\left(\overline{T_1}u\right)'(t_2)-\left(\overline{T_1}u\right)'(t_1)\right|\leq\left(\frac{M+K_2}{\alpha-1}+2\,M\,K_2\right)\varepsilon.$$

So, we conclude that operator $\bar{T}$ is a completely continuous operator and it satisfies the assumptions of Schauder's fixed point theorem [30], which ensures us the existence of a fixed point of $\bar{T}$ in Ω. Therefore, this fixed point is a solution of problem (26).

Step 3: Every solution of problem (26) is a solution of problem (17).

From Step 1, we have that $p(t, \overline{u}(t)) = \overline{u}(t)$ and

$$f\big(t, p(t, \overline{u}(t)), r\big(\frac{d}{dt}(p(t, \overline{u}(t)))\big)\big) = f(t, \overline{u}(t), r(\overline{u}'(t))).$$

Notice that f satisfies (19) with respect to $\overline{\gamma}$ and $\overline{\delta}$ and h is monotone nondecreasing. Thus, arguing as in the proof of Lemma 13, we deduce that $\|\overline{u}'\|_\infty < C$. And so $r(\overline{u}'(t)) = \overline{u}'(t)$ and the claim is concluded. ∎

Now, we are able to ensure the existence of extremal solutions for problem (17).
First, we introduce the notion of minimal and maximal solutions for (17).

Definition 16 *Given $\overline{\gamma}$, $\overline{\delta} \in C^1(I)$, $\overline{\gamma} \leq \overline{\delta}$ on I. A solution $u_{min} \in E$ (E defined in (18)) (resp. $u_{max} \in E$) of (17) is said to be a minimal solution (resp. maximal solution) in $[\overline{\gamma}, \overline{\delta}]$ if any other solution $\overline{u}$ of (17), with $\overline{\gamma} \leq \overline{u} \leq \overline{\delta}$, is such that $u_{min} \leq \overline{u}$ (resp. $\overline{u} \leq u_{max}$).*

Theorem 17 *Assume hypotheses of Theorem 14 hold. Then problem* (17) *has extremal solutions between $\overline{\gamma}$ and $\overline{\delta}$.*

Proof. The first step of the proof of this theorem follows as [18, Theorem 5].
Set

$$S := \{\overline{u} \in E \mid \overline{u} \textit{ is a solution of (17)}\}$$

and

$$\overline{u}_{min}(t) := \inf\{\overline{u}(t) \mid \overline{u} \in S\}.$$

Step 1: Let us prove that for any $\varepsilon > 0$, there exists an upper solution $\widehat{\overline{\delta}}$ such that

$$\overline{u}_{min}(t) \leq \widehat{\overline{\delta}} \leq \overline{u}_{min}(t) + 3\varepsilon. \tag{30}$$

It is clear that the function $\overline{u}_{min}$ is continuous. Hence, for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_i \in [0, 1]$ and all $|t - t_i| \leq \delta$,

$$\overline{u}_{min}(t_i) - \varepsilon \leq \overline{u}_{min}(t) \leq \overline{u}_{min}(t_i) + \varepsilon.$$

Also, any function $\overline{u}_i \in S$ is continuous. Then for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_i \in [0, 1]$ and all $|t - t_i| \leq \delta$,

$$\overline{u}_i(t_i) - \varepsilon \leq \overline{u}_i(t) \leq \overline{u}_i(t_i) + \varepsilon.$$

Now, consider the points $t_i = i\delta$ and the functions $\overline{u}_i \in S$ such that for $i \geq 1$,

$$\overline{u}_{min}(t_i) \leq \overline{u}_i(t_i) \leq \overline{u}_{min}(t_i) + \varepsilon$$

and for $i \geq 2$,

$$\overline{u}_i(t_i) \leq \overline{u}_{i-1}(t_i).$$

Then, we obtain

$$\begin{array}{rcl}\overline{u}_{min}(t) \leq \overline{u}_i(t) & \leq & \overline{u}_i(t_i) + \varepsilon \leq \overline{u}_{min}(t_i) + \varepsilon + \varepsilon \\ & \leq & \overline{u}_{min}(t) + 2\varepsilon + \varepsilon = \overline{u}_{min}(t) + 3\varepsilon.\end{array}$$

Next, we shall use the same construction of step 5 in the proof of [18, Theorem 5] to prove the inequality (30).
So, we consider the interval $[0, t_3]$ and we define

$$s_1 := \inf\{s \in [0, t_2],\ \forall t \in [s, t_2],\ \overline{u}_1(t) \geq \overline{u}_2(t)\}$$

Thus, we can set

$$\widehat{\overline{\delta_1}}(t) = \begin{cases} \overline{u}_1(t), & if\ t \in [0, s_1], \\ \\ \overline{u}_2(t), & if\ t \in]s_1, 1], \end{cases} \tag{31}$$

which allow us to introduce $\widehat{\overline{\delta_1}}$ as an upper solution satisfying, for all $t \in [0, t_3]$,

$$\overline{u}_{min}(t) \leq \widehat{\overline{\delta_1}} \leq \overline{u}_{min}(t) + 3\varepsilon.$$

We repeat the same argument to construct upper solutions $\widehat{\overline{\delta_k}}$. And so, for all $t \in [0, t_{k+2}]$, $k \in \mathbb{N}^*$, we get

$$\overline{u}_{min}(t) \leq \widehat{\overline{\delta_k}} \leq \overline{u}_{min}(t) + 3\varepsilon.$$

After a finite number of steps we obtain the result.

Step 2:The function $\overline{u}_{min}$ is a minimal solution.
From the previous step, we get for any $\varepsilon > 0$, there exists a solution $\overline{u}$ of (17) satisfying

$$\overline{u}_{min} \leq \overline{u} \leq \overline{u}_{min} + 3\varepsilon.$$

Taking $\varepsilon = 1/n$ we have that there exists $(\overline{u}_n) \subset S$ a sequence of solutions of (17) such that $\overline{u}_n \longrightarrow \overline{u}_{min}$ in $C(I)$. From Lemma 13, $\|\overline{u}_n'\|_\infty \leq C$.

Now, arguing as in Step 2 of Theorem 14, we have that for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $\delta > 0$ (independent of n) such that it $|t_1 - t_2| < \delta$ then

$$|\bar{u}_n'(t_2) - \bar{u}_n'(t_1)| \leq \int_0^1 \left| \frac{\partial}{\partial t}\bar{G}(t_2, s) - \frac{\partial}{\partial t}\bar{G}(t_1, s) \right| |f(s, \bar{u}_n(s), \bar{u}_n'(s))|\, ds < \epsilon.$$

As a consequence of the Ascoli-Arzela theorem we know that there is a subsequence $(\bar{u}_m) \to \bar{u}_{min}$ in $C^1(I)$.

The fact that u_{min} is a solution of of (17) follows from the expression

$$\bar{u}_m(t) = \int_0^1 \bar{G}(t, s)\, f(s, \bar{u}_m(s), \bar{u}_m'(s))\, ds$$

and the dominated convergence Lebesgue Theorem. ■

Example 18 *Let us consider the Problem (17) with* $\lambda = 0$*,* $\alpha = 1.6$*,* $\mu = 1/30$ *and* $\eta = 1/25$*.*

In this case we have that $\theta \approx 2.08333$ *and* $\sigma \approx 1.04167$*, So* $1 - \mu\theta - \eta\sigma \approx 0.888889 > 0$

Define the function

$$f(t, x, y) = t^{\alpha-2}\left(1 - t^2 + \cos(x) + \arctan(y)\right),$$

which satisfies inequality (19) *and property* (20) *for function* $h(y) = \arctan(y) + 2$*.*

In this case, we define the following functions

$$\gamma(t) = \begin{cases} -\pi t^{\alpha-2}, & if \quad t \in (0, 1/2] \\ \\ -7\pi(t-1)(t-1/2) - \pi(\frac{1}{2})^{\alpha-2} & if \quad t \in [1/2, 1] \end{cases}$$

and

$$\delta(t) = \begin{cases} t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^{\alpha}), & if \quad t \in (0, 1/2] \\ \\ 2(t-1)(t-1/2) + 2^{2-\alpha}\, E_{\alpha,\alpha-1}(2^{-\alpha}\lambda) & if \quad t \in [1/2, 1]. \end{cases}$$

It is not difficult to verify that

$$D^{\alpha}\gamma(t) - \lambda\gamma(t) + f(t, \overline{\gamma}(t), \overline{\gamma}'(t)) < 0, \quad \textit{for all } t \in I \backslash \{1/2\}.$$

$$\bar{\gamma}(0) = -\pi \leq \mu \int_0^1 \gamma(s)ds \approx -0.168966$$

$$\gamma'(1) = \frac{-7\pi}{2} \leq \eta \int_0^1 \gamma(s)ds \approx -0.20276$$

$$\bar{\gamma}'(1/2^-) = 0 < \bar{\gamma}'(1/2^+) \approx 5.81981$$

and

$$D^{\alpha}\delta(t) - \lambda\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t)) > 0, \quad \textit{for all } t \in I \backslash \{1/2\}.$$

$$\bar{\delta}(0) = \frac{1}{\Gamma(\alpha-1)} \approx 0.671505 \geq \mu \int_0^1 \delta(s)ds = \mu\left(\frac{\alpha\, 2^{1-\alpha}}{\Gamma(\alpha)} - \frac{1}{24}\right) \approx \mu 1.13974$$

$$\delta'(1) = \frac{1}{2} \geq \eta \int_0^1 \delta(s)ds = \eta\left(\frac{\alpha\, 2^{1-\alpha}}{\Gamma(\alpha)} - \frac{1}{24}\right) \approx \eta 1.13974$$

$$\bar{\delta}'(1/2^-) = 0 > \bar{\delta}'(1/2^+) \approx -0.220654.$$

As a consequence of Theorems 14 and 17, we deduce the existence of extremal solutions lying between $\overline{\gamma}$ *and* $\overline{\delta}$*.*

5 Existence results for discontinuous functional fractional differential equations

The aim of this section is to extend Theorem 17 to cover a class of functional fractional equations. The nonlinear term is allowed to be discontinuous in the spacial variable u.

To this end, we consider the following problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, \overline{u}(t), \overline{u}'(t), \overline{u}) = 0, & t \in I, \\ \overline{u}(0) = \mu \int_0^1 u(s)ds, \quad u'(1) = \eta \int_0^1 u(s)ds. \end{cases} \tag{32}$$

We introduce the following space

$$C_{2-\alpha}(I) := \{u : (0,1] \to \mathbb{R};\ \bar{u} \in C(I)\}.$$

It is not difficult to verify that $C_{2-\alpha}(I)$ is a Banach space endowed with the norm

$$\|u\| = \max_{t \in I} \{|\bar{u}(t)|\}$$

In this case, it is clear that $C_{2-\alpha}(I) \subset C((0,1])$ and the functions on this space may be discontinuous at $t = 0$.

We assume the existence of a pair of lower and upper solutions for this problem, the definitions are the natural extension of previous section to this situation:

Definition 19 *Let $k \in \{0, 1, \ldots\}$ and $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$. A function $\gamma : (0,1] \to \mathbb{R}$ satisfying $\overline{\gamma} \in C(I) \cap C^1(\cup_{l=0}^k (t_l, t_{l+1}))$ is a lower solution of problem (32) if it satisfies the following inequalities:*

$$\left\{ \begin{array}{rcll} D^\alpha \gamma(t) - \lambda \gamma(t) + f(t, \overline{\gamma}(t), \overline{\gamma}'(t), \overline{\gamma}) & \leq & 0, & t \in I \backslash \{t_1, \ldots, t_k\}, \\ \overline{\gamma}(0) - \mu \int_0^1 \gamma(s)ds & \leq & 0, & \\ \gamma'(1) - \eta \int_0^1 \gamma(s)ds & \leq & 0, & \\ \overline{\gamma}'(t_l^\pm) \quad \text{and} \quad \overline{\gamma}'(t_l^-) - \overline{\gamma}'(t_l^+) & < & 0, & l = 1, \ldots, k. \end{array} \right.$$

Definition 20 *Let $p \in \{0, 1, \ldots\}$ and $0 = s_0 < s_1 < \ldots < s_k < s_{p+1} = 1$. A function $\delta : (0,1] \to \mathbb{R}$ satisfying $\overline{\delta} \in C(I) \cap C^1(\cup_{l=0}^p (s_l, s_{l+1}))$ is an upper solution of problem (32) if it satisfies the following inequalities:*

$$\left\{ \begin{array}{rcll} D^\alpha \delta(t) - \lambda \delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \overline{\delta}) & \geq & 0, & t \in I \backslash \{s_1, \ldots, s_l\}, \\ \overline{\delta}(0) - \mu \int_0^1 \delta(s)ds & \geq & 0, & \\ \delta'(1) - \eta \int_0^1 \delta(s)ds & \geq & 0, & \\ \overline{\delta}'(s_l^\pm) \quad \text{and} \quad \overline{\delta}'(s_l^-) - \overline{\delta}'(s_l^+) & > & 0, & l = 1, \ldots, p. \end{array} \right.$$

Moreover, we denote

$$[\overline{\gamma}, \overline{\delta}]_1 := \{\overline{u} \in AC(I) :\ \overline{\gamma}(t) \leq \overline{u}(t) \leq \overline{\delta}(t),\ t \in I\}, \tag{33}$$

where $AC(I)$ refers to the set of the absolutely continuous functions on I.

In the development of this section, we shall assume the following hypotheses:

(H$_1$) For $z \in \mathbb{R}$, $f(\cdot, \cdot, \cdot, z) : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function that satisfies condition (F).

(H$_2$) For $t \in I$, $x \in [\overline{\gamma}(t), \overline{\delta}(t)]$ and $y \in \mathbb{R}$, we have that function $f(t,x,y,\cdot) : C_{2-\alpha}(I) \to \mathbb{R}$ is monotone decreasing.

(H$_3$) For $\overline{\gamma} \leq z \leq \overline{\delta}$, $x \in [\overline{\gamma}(t), \overline{\delta}(t)]$ and $C > 0$ such that $|y| \leq C$, there exists $h_C : (0,1] \to [0,\infty)$ such that $\overline{h}_C \in L^\infty(I)$ and

$$|f(t,x,y,z)| \leq h_C(t) \quad \text{for all } t \in I.$$

Next, the following result is also needed in the proof of our main result, it is a particular case of [21, Theorem 1.4.7, p. 58],

Theorem 21 *Given a nonempty order interval $[\gamma,\delta] \subset AC(I)$ and a monotone nondecreasing mapping $G : [\gamma,\delta] \longrightarrow [\gamma,\delta]$ such that there is $y \in AC(J)$ such that*

$$|Gx(s) - Gx(t)| \leq |y(s) - y(t)|, \quad \textit{for } x \in [\gamma,\delta], \quad s,t \in I. \tag{34}$$

Then

$$x_* = \min\{x \in [\gamma,\delta] / \ Gx \leq x\}, \quad x^* = \max\{x \in [\gamma,\delta] / \ x \leq Gx\}. \tag{35}$$

are, respectively, the smallest and the greatest fixed points of operator G in $[\gamma,\delta]$.

Moreover we must assume the analogous to the Nagumo's condition introduced on Definition 12

Definition 22 *Consider $\overline{\gamma}$ and $\overline{\delta} \in C(I)$ be such that $\overline{\gamma} \leq \overline{\delta}$ on I, and suppose that $f : E \times [\overline{\gamma}, \overline{\delta}] \to \mathbb{R}$ is a continuous function that satisfies:*

$$|f(t,u,v,\overline{\gamma})| \leq h_\gamma(|v|) \quad \textit{and} \quad |f(t,u,v,\overline{\delta})| \leq h_\delta(|v|) \quad \forall (t,u,v) \in E, \tag{36}$$

where h_γ, $h_\delta : [0,+\infty) \to [0,+\infty)$ are continuous functions such that $h(v) := \mathrm{m\bar{a}x}\{h_\gamma(v),\, h_\delta(v)\}$ is a nondecreasing function such that

$$\lim_{s \to +\infty} \frac{s}{\overline{h}(s)} = +\infty. \tag{37}$$

Now we are in a position to prove our fundamental result.

Theorem 23 *Assume the existence of $\gamma \leq \delta$ a pair of well ordered lower and upper solutions of Problem* (32) *and that hypotheses (H_1),(H_2) and (H_3) hold together Definition 22. Then Problem (32) has extremal solutions in $[\overline{\gamma}, \overline{\delta}]$.*

Proof. For $\bar{v} \in [\overline{\gamma}, \overline{\delta}]$, consider the following problem

$$(P_v) \begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, \overline{u}(t), \overline{u}'(t), \overline{v}) = 0, & t \in I, \\ \overline{u}(0) = \mu \int_0^1 u(s)ds, \quad u'(1) = \eta \int_0^1 u(s)ds. \end{cases}$$

Because of the monotonicity of function f on the last variable (hypothesis (H_2)) is it not difficult to verify that γ and δ are a pair of well ordered lower and upper solutions for problem (P$_v$) for any $v \in [\gamma,\delta]$. As a consequence, from the regularity assumptions on f (hypothesis (H_1)) we deduce, from Theorems 14 and 17, that Problem (P_v) has extremal solutions in $C^1_{2-\alpha}(I)$ lying between γ and δ.

Moreover, from Hypothesis (H_2), we have that, if $\bar{\gamma} \leq \bar{v} \leq \bar{\delta}$, then, for all $(t,x,u) \in E$, the following inequalities hold

$$f(t,x,u,\bar{\delta}) \leq f(t,x,u,\bar{v}) \leq f(t,x,u,\bar{\gamma})$$

and, as a consequence, from Definition 22 it is not difficult to verify that there is $K > 0$ for which any solution u_v of problem (P_v) satisfies that $\|u_v\| \leq K$. Such bound K is valid for all $v \in [\gamma, \delta]$.

As a consequence, we define the operator $T : [\gamma,\delta] \longrightarrow [\gamma,\delta]$ as follows

$$Tv := \text{maximal solutions of } (P_v) \text{ in } [\gamma,\delta]. \tag{38}$$

From the construction given in previous section, we know that Tv satisfies the following equality

$$Tv(t) = \int_0^1 G(t,s) f(s, \overline{T}v(s), \overline{T}v'(s), \bar{v}) ds, \quad t \in I.$$

Thus, to deduce the existence of extremal solutions of Problem (32) is equivalent to ensure the existence of extremal fixed points of operator T. To this end, we divide the proof in several steps.

Step 1: T is monotone nondecreasing on $[\gamma,\delta]$.

Let $v_1, v_2 \in [\gamma,\delta]$ be such that $v_1 \leq v_2$ on I. From (H_2), we have for all $t \in I$ that

$$\begin{array}{rcl} 0 & = & D^{\alpha}(Tv_1)(t) - \lambda(Tv_1)(t) + f(t, \overline{T}v_1(t), \overline{T}v_1'(t), \bar{v}_1) \\ & \geq & D^{\alpha}(Tv_1)(t) - \lambda(Tv_1)(t) + f(t, \overline{T}v_1(t), \overline{T}v_1'(t), \bar{v}_2), \end{array}$$

subject to boundary condition

$$\overline{T}v_1(0) = \mu \int_0^1 Tv_1(s) ds, \quad (Tv_1)'(1) = \eta \int_0^1 Tv_1(s) ds.$$

Which implies, since $Tv_1 \in C^1_{2-\alpha}(I)$, that Tv_1 is a lower solution of problem (P_{v_2}).
Now, using again (H_2), we obtain for all $t \in I \backslash \{s_1, \ldots, s_l\}$ that

$$\begin{array}{rcl} 0 & \leq & D^{\alpha}\delta(t) - \lambda\,\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \bar{\delta}) \\ & \leq & D^{\alpha}\delta(t) - \lambda\,\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \bar{v}_2), \end{array}$$

which means that δ is an upper solution of problem (P_{v_2}).

Thus, from Theorem 17, Problem (P_{v_2}) has extremal solutions on $[Tv_1, \delta]$. And so, Tv_2 is the maximal solution of (P_{v_2}) on $[Tv_1,\delta]$, and we deduce that $Tv_1(t) \leq Tv_2(t)$ for all $t \in I$.

Step 2: $T([\gamma,\delta]) \subset [\gamma,\delta]$.
Let $T\gamma$ be the maximal solution of problem (P_{γ})

$$\begin{cases} D^{\alpha}\gamma(t) - \lambda\gamma(t) + f(t, \overline{\gamma}(t), \overline{\gamma}'(t), \overline{\gamma}) = 0, & t \in I, \\ \overline{\gamma}(0) = \mu \int_0^1 \gamma(s) ds, \quad \gamma'(1) = \eta \int_0^1 \gamma(s) ds. \end{cases}$$

By definition, γ is a lower solution of problem (P_γ) and, from (H_2), we have for all $t \in I\backslash\{s_1, \ldots, s_l\}$ that

$$\begin{aligned} 0 &\leq D^\alpha\delta(t) - \lambda\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \overline{\delta}) \\ &\leq D^\alpha\delta(t) - \lambda\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \overline{\gamma}). \end{aligned}$$

Thus, δ is an upper solution of problem (P_γ) and so,

$$\gamma \leq T\gamma \leq \delta.$$

In an analogous way, we prove that

$$\gamma \leq T\delta \leq \delta.$$

Since T is nondecreasing on $[\gamma, \delta]$, we obtain $\gamma \leq T\gamma \leq T\delta \leq \delta$, and it is immediate to deduce that $T([\gamma, \delta]) \subset [\gamma, \delta]$.

Step 3: For $t_1, t_2 \in I$ it is satisfied that there is $y \in AC(I)$ such that

$$|\overline{T}v(t_2) - \overline{T}v(t_1)| \leq |y(t_2) - y(t_1)|. \tag{39}$$

Let $t \in I$, we get

$$(\overline{T}v)'(t) = \int_0^1 \frac{\partial}{\partial t}(\overline{G}(t,s))\, f(s, \overline{T}v(s), (\overline{T}v)'(s), \bar{v})ds$$

Using (25) and (H_3), we obtain that

$$\begin{aligned} |(\overline{T}v)'(t)| \;\leq\; & K_1(\alpha,\lambda) \int_0^1 (1-s)^{\alpha-2}\; h_K(s)ds + t^{1-\alpha}\, K_2(\alpha,\lambda) \int_0^1 (1-s)^{\alpha-2} h_K(s)ds \\ & + \left(\frac{\mu K_3(\alpha,\lambda) + \eta K_4(\alpha,\lambda)}{(1-\mu\theta-\eta\sigma)(\alpha-1)} \right) M \int_0^1 s(1-s)^{\alpha-2} h_K(s)ds. \end{aligned}$$

So, by defining

$$y(t) := \left[K_1(\alpha,\lambda)\, t + \frac{t^{2-\alpha}}{2-\alpha}\, K_2(\alpha,\lambda) + \left(\frac{\mu K_3(\alpha,\lambda) + \eta K_4(\alpha,\lambda)}{(1-\mu\theta-\eta\sigma)(\alpha-1)} \right) M\, t \right] \|\bar{h}_K\|_{L^\infty(I)},$$

we deduce that, for $t_1, t_2 \in I$,

$$|\overline{T}v(t_2) - \overline{T}v(t_1)| \leq \int_{t_2}^{t_1} |(\overline{T}v)'(s)|ds \leq |y(t_2) - y(t_1)|.$$

Therefore, from Theorem 21, operator $\overline{T}$ has extremal fixed points $\overline{x}^*$ and $\overline{x}_*$ in $[\overline{\gamma}, \overline{\delta}]$. By construction, it is immediate to verify that x^* is the greatest fixed point in $C_{2-\alpha}(I)$ of operator T and it coincides with the greatest solution in $[\gamma, \delta]$ of problem (32).

By considering the operator $\widehat{T} : [\overline{\gamma}, \overline{\delta}] \longrightarrow [\overline{\gamma}, \overline{\delta}]$ as

$$\widehat{T}v := \text{minimal solutions in } [\gamma, \delta] \text{ of } (P_v),$$

by dual arguments, we can ensure the existence of the minimal solution of problem (32). ■

Example 24 *Consider, the Problem (32) with* $\lambda_1^* < \lambda = -0.1$*,* $\alpha = 1.4$*,* $\mu = \frac{1}{25}$ *and* $\eta = \frac{1}{12}$*. We have, for* $\theta \approx 4.10999$ *and* $\sigma \approx 2.25619$*, that* $1 - \mu\theta - \eta\sigma \approx 0.647585 > 0$*.*

Let

$$f(t,x,y,z) = t^{2-\alpha}\left(-[z+1] + (1-t^2)\sqrt{y} + \sin(x)\right),$$

In this case, the following functions

$$\delta(t) = \begin{cases} t^{\alpha-2}e^t, & if \quad t \in (0, 1/3] \\ \\ (\frac{1}{3})^{\alpha-2}e^{\frac{1}{3}} + 4(t-1)(t-\frac{1}{3}) & if \quad t \in [1/3, 1]. \end{cases}$$

and $\gamma(t) = 0$ *are, respectively, lower and upper solutions to this problem. It is clear that* $\gamma \leq \delta$*.*
For $1 < \alpha \leq 2$*, we have*

$$D^\alpha \delta(t) = D^\alpha(t^{\alpha-2}e^t) = \sum_{m=0}^{\infty} \binom{\alpha}{m} D^{\alpha-m}(t^{\alpha-2}) D^m(e^t).$$

Moreover, by the definition of Riemann-Liouville fractional derivative, we obtain, for m=n+1,

$$\begin{aligned} D^m(e^t) &= \frac{d^n}{dt^n}\left(I^{n-m}(e^t)\right) \\ &= \frac{d^{m+1}}{dt^{m+1}}\left(I^1(e^t)\right) \\ &= \frac{d^{m+1}}{dt^{m+1}}\left(\int_0^t e^z dz\right) = \frac{d^m}{dt^m}e^t = e^t, \end{aligned}$$

and $D^{\alpha-m}(t^{\alpha-2}) = 0$*. Thus* $D^\alpha\delta(t) = 0$*.*
Next, it is not difficult to verify that

$$D^\alpha\delta(t) - \lambda\delta(t) + f(t, \overline{\delta}(t), \overline{\delta}'(t), \overline{\delta}) > 0, \quad for\ all\ t \in I\backslash\{1/3\}.$$

$$\bar{\delta}(0) = 1 \geq \mu \int_0^1 \delta(s)ds \approx 0.135268$$

$$\delta'(1) = 2.66667 \geq \eta \int_0^1 \delta(s)ds \approx 0.281808$$

$$\bar{\delta}'(1/3^-) \approx 1.39561 > \bar{\delta}'(1/3^+) \approx 1.13268.$$

As the function f *satisfies (*H_1*)-(*H_3*) in*

$$E^* = \{(t,x,y,z) \in I \times \mathbb{R}^3 : 0 \leq x \leq e^t\},$$

then, by Theorem 23 the problem (32) has extremal solutions in $[0, e^t]$*.*

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