Nonlinear Anal. TMA 240 (2024) 113470

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Research paper

Positive radial solutions for Dirichlet problems in the ball

Petru Jebelean [a], Radu Precup [b,c], Jorge Rodríguez-López [d,*]

[a] Institute for Advanced Environmental Research, West University of Timişoara, Blvd. V. Pârvan, No. 4, 300223 Timişoara, Romania
[b] Faculty of Mathematics and Computer Science and Institute of Advanced Studies in Science and Technology, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
[c] Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1, 400110 Cluj-Napoca, Romania
[d] CITMAga & Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782, Facultade de Matemáticas, Campus Vida, Santiago, Spain

ARTICLE INFO

Communicated by Enrico Valdinoci

MSC:
35J25
35J60
34B18
35J92
35J93

Keywords:
Dirichlet problem
Operator with Uhlenbeck structure
Positive radial solution
Fixed point index
Mean curvature operator
p-Laplacian

ABSTRACT

We are concerned with existence, localization and multiplicity of positive radial solutions to Dirichlet problems with ϕ-Laplacians in a ball, in both scalar and system cases. Our approach essentially relies on fixed point index computations and a main feature is that it avoids any Harnack type inequality. Applications to some problems involving operators with Uhlenbeck structure are discussed.

1. Introduction

In this paper, we first deal with existence and localization of positive solutions to the mixed boundary value problem

$$-\left(r^{n-1}\phi\left(u'\right)\right)' = r^{n-1}f(r,u), \qquad u'(0) = u(1) = 0, \tag{1.1}$$

where $\phi : (-a,a) \to (-b,b)$ $(0 < a, b \le +\infty)$ is an increasing odd homeomorphism and $f : [0,1]\times\mathbb{R}_+ \to [0,b)$ is continuous. According to an already usual terminology, ϕ is called *classical*, *singular* or *bounded* if $a = b = +\infty$, $a < +\infty$ and $b = +\infty$ or $a = +\infty$ and $b < +\infty$. To cover all possible cases, in the remaining situation $a < +\infty$ and $b < +\infty$, we will say that ϕ is *singular-bounded*. By a *solution* of (1.1) we mean a nonnegative function $u \in C^1[0,1]$ with $u'(0) = u(1) = 0$, $\left|u'(r)\right| < a$ for all $r \in [0,1]$, such that $r^{n-1}\phi(u') \in C^1[0,1]$ and (1.1) is satisfied. A solution is said to be *positive* if it is distinct from the identically zero function.

Our results here generalize those obtained in [18], where some monotonicity assumptions are required on f and also some additional restrictions are imposed on ϕ. Our approach is based on the fixed point index computation and it is worth to notice that no Harnack type inequality is needed.

* Corresponding author.
E-mail addresses: petru.jebelean@e-uvt.ro (P. Jebelean), r.precup@math.ubbcluj.ro (R. Precup), jorgerodriguez.lopez@usc.es (J. Rodríguez-López).

https://doi.org/10.1016/j.na.2023.113470
Received 4 September 2023; Accepted 14 December 2023
Available online 20 December 2023

A main motivation of this study concerns the class of Dirichlet problems in the unit open ball $\mathcal{B}$ in $\mathbb{R}^n$, involving operators with Uhlenbeck structure [20]. Such an operator has the form

$$\mathcal{U}_\theta v := \operatorname{div}(\theta(|\nabla v|)\nabla v),$$

where $\theta : (0,a) \to (0,+\infty)$ is a C^1 function. When $v(z) = u(|z|)$, setting $r = |z|$, it is straightforward to check that

$$\mathcal{U}_\theta v(z) = \frac{1}{r^{n-1}}\left(r^{n-1}\theta(|v'(r)|)v'(r)\right)'.$$

Assuming that

$$\lim_{x\to 0^+} x\,\theta(x) = 0 \tag{1.2}$$

together with the ellipticity condition

$$\theta(x) + x\,\theta'(x) > 0 \quad (x \in (0,a)) \tag{1.3}$$

and putting

$$\phi_\theta(x) := x\,\theta(|x|) \quad (x \in (-a,a)) \tag{1.4}$$

we have that $\phi_\theta : (-a,a) \to (-\phi_\theta(a), \phi_\theta(a))$ is an increasing odd homeomorphism. Thus, under assumptions (1.2) and (1.3), finding radially symmetric solutions (i.e., solutions of the form $v(x) = u(r)$ with $r = |x|$) to the Dirichlet problem

$$-\mathcal{U}_\theta v = f(|z|, v) \text{ in } \mathcal{B}, \quad v = 0 \text{ on } \partial\mathcal{B}$$

reduces to solving the mixed boundary value problem of type (1.1):

$$-\left(r^{n-1}\phi_\theta\left(u'\right)\right)' = r^{n-1} f(r,u), \qquad u'(0) = u(1) = 0, \tag{1.5}$$

with ϕ_θ in (1.4). Note that allowing a to be finite, unlike [20] and most of the subsequent works (see e.g. [3,5,7–10]), also includes singular and singular-bounded ϕ_θ homeomorphisms.

The following three particular cases are standard models in this context:

- $\theta : (0,+\infty) \to (0,+\infty)$, $\theta(x) = x^{p-2}$, where $p > 1$, when $\mathcal{U}_\theta$ becomes the *p-Laplacian operator*

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)$$

and the corresponding $\phi \equiv \phi_\theta : \mathbb{R} \to \mathbb{R}$, $\phi(x) = |x|^{p-2}x$ is classical;

- $\theta : (0,1) \to (0,+\infty)$, $\theta(x) = \left(1-x^2\right)^{-\frac{1}{2}}$, when $\mathcal{U}_\theta$ will be the *mean extrinsic curvature operator in Minkowski space*

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)$$

and the corresponding $\phi \equiv \phi_\theta : (-1,1) \to \mathbb{R}$, $\phi(x) = x\left(1-x^2\right)^{-\frac{1}{2}}$ is singular;

- $\theta : (0,+\infty) \to (0,+\infty)$, $\theta(x) = \left(1+x^2\right)^{-\frac{1}{2}}$, when $\mathcal{U}_\theta$ engenders the *mean curvature operator in Euclidean space*

$$\mathcal{E}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)$$

and the corresponding $\phi \equiv \phi_\theta : \mathbb{R} \to (-1,1)$, $\phi(x) = x\left(1+x^2\right)^{-\frac{1}{2}}$ is bounded.

To complete the table with the remaining case, let $\theta : (0,1) \to (0,+\infty)$ be given by $\theta(x) = \arcsin x$. Then the corresponding $\mathcal{U}_\theta$ becomes

$$\mathcal{A}v = \operatorname{div}(\nabla v \arcsin|\nabla v|)$$

with $\phi \equiv \phi_\theta : (-1,1) \to (-1,1)$, $\phi(x) = x\arcsin|x|$ which is singular-bounded.

Furthermore, we deal with existence and localization of positive solutions for the system

$$\begin{aligned} -\left(r^{n-1}\phi_1\left(u_1'\right)\right)' &= r^{n-1} f_1(r,u_1,u_2),\\ -\left(r^{n-1}\phi_2\left(u_2'\right)\right)' &= r^{n-1} f_2(r,u_1,u_2),\\ u_1'(0) &= u_1(1) = 0 = u_2'(0) = u_2(1), \end{aligned} \tag{1.6}$$

where for each $j = 1,2$, $\phi_j : \left(-a_j, a_j\right) \to \left(-b_j, b_j\right)$ $(0 < a_j, b_j \le +\infty)$ is an increasing odd homeomorphism and $f_j : [0,1]\times\mathbb{R}_+\times\mathbb{R}_+ \to [0,b_j)$ is continuous. Here, a notable feature is the fact that in the system the homeomorphisms ϕ_1 and ϕ_2 engendering the differential operators can be different. Among others, this allows the study of radial solutions in $\mathcal{B}$ for systems invoking, for example, any two of the operators Δ_p, $\mathcal{M}$, $\mathcal{E}$ or $\mathcal{A}$ or some combinations of them (see Example 3.5). We extend here existence and multiplicity results

obtained in [17], weakening the global monotonicity conditions on the nonlinearities f_1 and f_2 therein, and making use — as in the scalar case, of an approach that avoids any Harnack type inequality and relying exclusively on the fixed point index estimation.

The rest of the paper is organized as follows. In Section 2 we reformulate (1.1) as a fixed point problem and we obtain localization of the positive solutions under a homotopic boundary condition, as well as under conditions concerning the behavior of the nonlinearity f on some appropriate subintervals in connection with the range of ϕ^{-1}. Applications in terms of asymptotic conditions when the nonlinearity is with separated variables are provided. Section 3 is devoted to extensions of the results in Section 2 to system (1.6). Multiplicity of positive solutions of (1.6) is obtained in Section 4 by refining the estimations of the fixed point index under some additional suitable conditions on the nonlinearities f_1 and f_2.

2. Scalar case

The space $X := C[0,1]$ will be endowed with the usual sup-norm $|\cdot|_\infty$. We denote $B(d) := \{u \in X : |u|_\infty < d\}$ and let the cone $P := \{u \in X : u \geq 0 \text{ on } [0,1]\}$. It is not difficult to see that a nonnegative function u is a solution of problem (1.1) if and only if u is a fixed point of the operator $T : P \to P$ given by

$$T(u)(r) = \int_r^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1} f(s,u(s))\,ds\right)d\tau \quad (r \in [0,1]). \tag{2.1}$$

It is a standard matter that the operator T is completely continuous.

Let us now consider the closed, convex cone

$$K = \{u \in P : u \text{ is nonincreasing on } [0,1]\}. \tag{2.2}$$

Proposition 2.1. *The operator T maps K into itself.*

Proof. Indeed, take $u \in K$ and let us show that $v := Tu$ belongs to K. Since f is nonnegative, $v \geq 0$ and, moreover, from the sign of the homeomorphism ϕ, one deduces that v is nonincreasing on $[0,1]$. Therefore, $v \in K$, as claimed. □

For any real numbers $\alpha > \beta > 0$ and $\eta \in (0,1)$, consider the bounded sets

$$U_\alpha := K \cap B(\alpha) \text{ and } V_\beta := \left\{u \in \overline{U}_\alpha : u(\eta) < \beta\right\}.$$

Observe that if $u \in K$ then $u(\eta) = \min_{r\in[0,\eta]} u$ and $u(0) = |u|_\infty$. Since K is a closed set, the closure of U_α in both X and K is the same and it is easily seen that the sets U_α and V_β are open in $\overline{U}_\alpha$. Also, as $\overline{U}_\alpha$ is closed, the closure of any of its subsets is the same in both X and $\overline{U}_\alpha$. In the sequel, the boundary of the subsets of $\overline{U}_\alpha$ will be understood with respect to the topology induced on $\overline{U}_\alpha$, unless otherwise specified. It is a standard matter to check that

$$\overline{U}_\alpha = \left\{u \in K : |u|_\infty \leq \alpha\right\}, \quad \overline{V}_\beta = \left\{u \in K : |u|_\infty \leq \alpha,\ u(\eta) \leq \beta\right\},$$

$$\overline{\overline{U}_\alpha \setminus \overline{V}_\beta} = \left\{u \in K : |u|_\infty \leq \alpha,\ u(\eta) \geq \beta\right\} = \overline{U}_\alpha \setminus V_\beta,$$

$$\partial V_\beta = \left\{u \in K : |u|_\infty \leq \alpha,\ u(\eta) = \beta\right\} = \partial\left(\overline{U}_\alpha \setminus \overline{V}_\beta\right).$$

Next, it is clear that $\overline{U}_\alpha$ being closed and convex, it is a retract. Thus, if $\mathcal{O}$ is an open subset in $\overline{U}_\alpha$, $T(\overline{\mathcal{O}}) \subset \overline{U}_\alpha$ and T is fixed points free on the boundary of $\mathcal{O}$, then $i(T,\mathcal{O}) \equiv i(T,\mathcal{O},\overline{U}_\alpha)$ – the *fixed point index* of the completely continuous operator $T : \overline{\mathcal{O}} \to \overline{U}_\alpha$ on $\mathcal{O}$ with respect to $\overline{U}_\alpha$ is well defined [1,11,12].

The following result guarantees the existence of a non-trivial fixed point in K of the operator T.

Theorem 2.2. *Assume that there exist numbers $\alpha > \beta > 0$, $\eta \in (0,1)$ such that $T(\overline{U}_\alpha) \subset \overline{U}_\alpha$ and there exists a function $h \in K$ such that $|h|_\infty \leq \alpha$, $h(\eta) > \beta$ and*

$$(1-\lambda)T(u) + \lambda h \neq u \quad \text{for } u \in K \text{ with } |u|_\infty \leq \alpha,\ u(\eta) = \beta \text{ and } \lambda \in [0,1]. \tag{2.3}$$

Then $i\left(T,\overline{U}_\alpha \setminus \overline{V}_\beta\right) = 1$ and thus problem (1.1) has a solution $u \in K$, satisfying

$$\beta < u(\eta) \quad \text{and} \quad |u|_\infty \leq \alpha.$$

Proof. This is a modified computation of the fixed point index in [18]. Since $T\left(\overline{U}_\alpha\right) \subset \overline{U}_\alpha$ and $\partial\overline{U}_\alpha = \emptyset$, we have that the homotopy $H : [0,1] \times \overline{U}_\alpha \to \overline{U}_\alpha$, given by

$$H(\lambda,u) = \lambda T(u) \quad (\lambda \in [0,1],\ u \in \overline{U}_\alpha)$$

is *admissible* (i.e., $H(\lambda,\cdot)$ is fixed points free on $\partial\overline{U}_\alpha$ for all $\lambda \in [0,1]$). Then, by the invariance under homotopy of the fixed point index, we get

$$i\left(T,\overline{U}_\alpha\right) = i\left(H(1,\cdot),\overline{U}_\alpha\right) = i\left(H(0,\cdot),\overline{U}_\alpha\right) = i\left(0,\overline{U}_\alpha\right) = 1,$$

where the last equality is due to the fact that $0 \in \overline{U}_\alpha$.

Next, as h lies in the convex set $\overline{U}_\alpha$, we can define the homotopy $\widetilde{H} : [0,1] \times \overline{U}_\alpha \to \overline{U}_\alpha$ by

$$\widetilde{H}(\lambda, u) := (1-\lambda)\, T(u) + \lambda\, h \quad (\lambda \in [0,1],\ u \in \overline{U}_\alpha).$$

Condition (2.3) means

$$\widetilde{H}(\lambda, u) \neq u \quad \text{for all } u \in \partial V_\beta \text{ and } \lambda \in [0,1],$$

showing that $\widetilde{H} : [0,1] \times \overline{V}_\beta \to \overline{U}_\alpha$ is admissible. Thus, we have

$$i\left(T, V_\beta\right) = i\left(\widetilde{H}(0,\cdot), V_\beta\right) = i\left(\widetilde{H}(1,\cdot), V_\beta\right) = i\left(h, V_\beta\right) = 0,$$

since $h \in \overline{U}_\alpha \setminus \overline{V}_\beta$.

Then, as $\{\overline{U}_\alpha \setminus \overline{V}_\beta, V_\beta, \partial V_\beta\}$ is a partition of $\overline{U}_\alpha$, using the excision-additivity property of the fixed point index, we infer

$$i\left(T, \overline{U}_\alpha \setminus \overline{V}_\beta\right) = i\left(T, \overline{U}_\alpha\right) - i\left(T, V_\beta\right) = 1,$$

which ensures the existence of a fixed point of T in the set $\overline{U}_\alpha \setminus \overline{V}_\beta$, that is, a solution of (1.1) with $\beta < u(\eta)$ and $|u|_\infty \le \alpha$. □

Remark 2.1. From the first part of the above proof, we observe that the simple hypothesis that there exists a number $\alpha > 0$ such that $T(\overline{U}_\alpha) \subset \overline{U}_\alpha$ yields $i\left(T, \overline{U}_\alpha\right) = 1$. This ensures that T has a fixed point in $\overline{U}_\alpha$, hence (1.1) is solvable; notice, the same conclusion also follows from Schauder's fixed point theorem.

Now we give sufficient conditions in order to ensure that the assumptions of the previous result hold.

For any $\alpha > \beta > 0$ and $\eta \in (0,1)$ denote

$$m_{\alpha,\beta} := \min\{f(r,x) : r \in [0,\eta],\ x \in [\beta,\alpha]\},$$
$$M_\alpha := \max\{f(r,x) : r \in [0,1],\ x \in [0,\alpha]\}$$

and note that $0 \le m_{\alpha,\beta} \le M_\alpha < b$.

Theorem 2.3. *If there exist $\alpha > \beta > 0$ and $\eta \in (0,1)$ such that*

$$\phi^{-1}\left(M_\alpha\right) \le \alpha, \tag{2.4}$$

$$(1-\eta)\phi^{-1}\left(\eta^n m_{\alpha,\beta}/n\right) > \beta, \tag{2.5}$$

then problem (1.1) has at least one solution $u \in K$ with $\beta < u(\eta)$ and $|u|_\infty \le \alpha$.

Proof. We shall apply Theorem 2.2. First, we show that T maps $\overline{U}_\alpha$ into itself. Indeed, for $u \in K$ with $|u|_\infty \le \alpha$, we have that

$$0 \le f(s, u(s)) \le M_\alpha \quad (s \in [0,1])$$

and thus from (2.4),

$$\begin{aligned} |T(u)(r)| &\le \int_0^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1} f(s,u(s))\,ds\right)d\tau \\ &\le \int_0^1 \phi^{-1}\left(\int_0^\tau f(s,u(s))\,ds\right)d\tau \le \phi^{-1}(M_\alpha) \le \alpha \quad (r \in [0,1]). \end{aligned}$$

So, $|T(u)|_\infty \le \alpha$ for all $u \in K$ with $|u|_\infty \le \alpha$, hence T maps $\overline{U}_\alpha$ into itself.

Next, we prove that (2.3) is fulfilled with $h \equiv \alpha$, which clearly satisfies $|h|_\infty = \alpha$ and $h(\eta) > \beta$. Assume that (2.3) does not hold. Then there exist $u \in K$ with $|u|_\infty \le \alpha$, $u(\eta) = \beta$ and $\lambda \in [0,1]$ such that

$$(1-\lambda)\, T(u) + \lambda\, h = u.$$

In particular, one has

$$\begin{aligned} \beta = u(\eta) &= (1-\lambda)T(u)(\eta) + \lambda\,\alpha \\ &= (1-\lambda)\int_\eta^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1} f(s,u(s))\,ds\right)d\tau + \lambda\,\alpha \\ &\ge (1-\lambda)\int_\eta^1 \phi^{-1}\left(\tau^{1-n}\int_0^\eta s^{n-1} f(s,u(s))\,ds\right)d\tau + \lambda\,\beta. \end{aligned}$$

Since $\beta \le u(s) \le \alpha$ for all $s \in [0,\eta]$, it follows

$$\beta \geq (1-\lambda)\int_{\eta}^{1}\phi^{-1}\left(\tau^{1-n}\int_{0}^{\eta}s^{n-1}m_{\alpha,\beta}\,ds\right)d\tau+\lambda\beta$$
$$\geq (1-\lambda)(1-\eta)\phi^{-1}\left(\frac{\eta^n}{n}m_{\alpha,\beta}\right)+\lambda\beta,$$

that is,

$$(1-\lambda)\beta \geq (1-\lambda)(1-\eta)\phi^{-1}\left(\eta^n m_{\alpha,\beta}/n\right),$$

which contradicts (2.5) for any $\lambda \in [0,1)$. In the case $\lambda = 1$ one has the contradiction

$$\beta = u(\eta) = \alpha > \beta.$$

Finally, the conclusion follows from Theorem 2.2. □

Remark 2.2. Conditions (2.4)–(2.5) in Theorem 2.3 are of compression type. If a Harnack inequality holds for all nonnegative supersolutions u of (1.1), that is

$$\min_{r\in[0,\eta]} u \geq c\,|u|_\infty\,,$$

for some $c \in (0,1)$, then one may consider the cone

$$\widehat{K} = \{u \in P : \min_{r\in[0,\eta]} u \geq c\,|u|_\infty\}.$$

In that case, the set $\widehat{V}_\beta = \{u \in \widehat{K} : \min_{r\in[0,\eta]} u < \beta\}$ is bounded (indeed, $\widehat{V}_\beta \subset \widehat{U}_{\beta/c} = \{u \in \widehat{K} : \ |u|_\infty < \beta/c\}$) and so expansion conditions are possible by means of standard fixed point index arguments. In addition, in that case, condition (2.5) can be weakened by replacing $m_{\alpha,\beta}$ with

$$\widetilde{m}_\beta := \min\{f(r,s) : r \in [0,\eta],\ s \in [\beta,\beta/c]\},$$

and the proof follows in a similar way.

Note that if $0 < a < +\infty$, then condition (2.4) in Theorem 2.3 is trivially satisfied for α large enough (for instance, $\alpha = a$). Thus, in this case, it suffices to ensure the existence of a positive number β satisfying condition (2.5). Observe that, for β small enough, (2.5) can be rewritten as

$$\frac{m_{\alpha,\beta}}{\phi(\beta/(1-\eta))} > \frac{n}{\eta^n}. \tag{2.6}$$

Hence, it is natural to look for asymptotic conditions on the quotient f/ϕ at 0 which guarantee that the above inequality holds. Then a difficulty arises: the number $m_{\alpha,\beta}$ depends not only of the behavior of $f(r,\cdot)$ at β, but on the whole interval $[\beta,\alpha]$. That is why it is convenient to assume the following monotonicity assumption on f:

($\mathbf{H}_f$) the function $f(r,\cdot)$ is nondecreasing in $(0,a)$ for every $r \in [0,1]$.

Under assumption ($\mathbf{H}_f$), we have that

$$m_{\alpha,\beta} = \min\{f(r,\beta) : r \in [0,\eta]\} := m_\beta,$$

for any $\alpha > \beta$, so (2.6) reads as

$$\frac{m_\beta}{\phi(\beta/(1-\eta))} > \frac{n}{\eta^n}. \tag{2.7}$$

Below, we need to invoke the condition

$$\limsup_{x\to 0^+}\frac{\phi(\tau x)}{\phi(x)} < +\infty \quad \text{for all } \tau > 1, \tag{2.8}$$

that is employed in [7] in connection with a classical homeomorphism ϕ and in [2] relative to a singular ϕ.

Theorem 2.4. *Assume that $0 < a < +\infty$ and ($\mathbf{H}_f$), (2.8) are fulfilled. If there is some $\eta \in (0,1)$ such that*

$$\lim_{x\to 0^+}\frac{f(r,x)}{\phi(x)} = +\infty \quad \textit{uniformly with } r \in [0,\eta], \tag{2.9}$$

then problem (1.1) has at least one positive nonincreasing solution u with $u(\eta) > 0$.

Proof. This resembles the proof of [18, Theorem 3.6]. From (2.8) with $\tau = 1/(1-\eta)$, there exist numbers $L > 0$ and $\rho_1 \in (0, a(1-\eta))$ such that

$$\phi(x/(1-\eta)) \leq L\,\phi(x) \quad \text{for all } x \in (0,\rho_1). \tag{2.10}$$

Then, by (2.9), we can find $\rho_2 \in (0,\rho_1)$ so that

$$\frac{f(r,x)}{\phi(x)} > L\frac{n}{\eta^n} \quad \text{for all } (r,x) \in [0,\eta]\times(0,\rho_2).$$

Choosing $\beta \in (0, \rho_2)$, we deduce

$$\frac{m_\beta}{\phi(\beta)} > L\frac{n}{\eta^n},$$

which, together with (2.10), gives (2.7) and the conclusion follows by Theorem 2.3. □

In the case $a = +\infty$, the existence of a positive number α satisfying (2.4) is not trivial and it can be derived from a suitable asymptotic behavior of the ratio f/ϕ at infinity.

Theorem 2.5. *Assume that $a = +\infty$ and $(\boldsymbol{H}_f)$, (2.8) are fulfilled. If, in addition,*

$$\lim_{x\to+\infty} \frac{f(r,x)}{\phi(x)} < 1 \quad \text{uniformly with } r \in [0,1] \tag{2.11}$$

and there exists $\eta \in (0,1)$ such that (2.9) is satisfied, then problem (1.1) has at least one positive nonincreasing solution u with $u(\eta) > 0$.

Proof. Using $(\mathbf{H}_f)$, from (2.11) we easily get that there exists $\alpha > 0$ such that (2.4) holds true. Then, the arguments in the proof of Theorem 2.4 yield the conclusion. □

Next, we consider the case in which the function f is with separated variables i.e., $f(r,x) = \mu(r)g(x)$, and where μ and g satisfy the following hypotheses:

$(\mathbf{H}_\mu)$ $\mu : [0,1] \to (0,+\infty)$ is continuous;

$(\mathbf{H}_g)$ $g : \mathbb{R}_+ \to [0, b/\overline{\mu})$ is a continuous function such that $g(0) = 0$ and $g(x) > 0$ for all $x > 0$, where $\overline{\mu} := \max\{\mu(r) : r \in [0,1]\}$.

Notice that no monotonicity assumptions on f are required.

Theorem 2.6. *Assume that (2.8), $(\boldsymbol{H}_\mu)$ and $(\boldsymbol{H}_g)$ are fulfilled. If, in addition, one of the following conditions holds*

(i) $a < +\infty$ and

$$\lim_{x\to 0^+} \frac{g(x)}{\phi(x)} = +\infty; \tag{2.12}$$

(ii) $a = +\infty$ and

$$\lim_{x\to 0^+} \frac{g(x)}{\phi(x)} = +\infty, \qquad \lim_{x\to+\infty} \frac{g(x)}{\phi(x)} < \frac{1}{\overline{\mu}}; \tag{2.13}$$

then problem

$$-\left(r^{n-1}\phi\left(u'\right)\right)' = r^{n-1}\mu(r)g(u), \qquad u'(0) = u(1) = 0, \tag{2.14}$$

has at least one strictly decreasing solution $u > 0$ on $[0,1)$.

Proof. Let us show that hypotheses in Theorem 2.3 hold, that is, there exist $\alpha > \beta > 0$ such that inequalities (2.4) and (2.5) are satisfied.

First, suppose that condition (i) holds. Since $a < +\infty$, condition (2.4) is satisfied for any $\alpha \geq a$. Fix such a positive α and let us prove the existence of a number $\beta \in (0,\alpha)$ as in (2.5). Taking any $\eta \in (0,1)$, by (2.8) with $\tau = 1/(1-\eta)$, there exist $L > 0$ and $\rho_1 \in (0, a(1-\eta))$ such that (2.10) holds true. The asymptotic behavior of the quotient g/ϕ at zero in (2.12) implies that there exists $\rho_2 > 0$ (we may assume $\rho_2 < \rho_1 < \alpha$) such that

$$g(x) > \frac{L}{\underline{\mu}}\frac{n}{\eta^n}\phi(x) \quad \text{for all } x \in (0,\rho_2),$$

where $\underline{\mu}$ stands for $\min\{\mu(r) : r \in [0,\eta]\} > 0$. This and (2.10) give

$$\underline{\mu}\, g(x) > \frac{n}{\eta^n}\phi\left(\frac{x}{1-\eta}\right) \quad \text{for all } x \in (0,\rho_2). \tag{2.15}$$

Since g is continuous, vanishes at zero and it is positive on $(0,+\infty)$, there exists $\overline{x} \in (0,\rho_2)$ such that

$$g(\overline{x}) < \min\{g(x) : x \in [\rho_2,\alpha]\}.$$

We choose a number $\beta \in [\overline{x},\alpha]$ such that

$$g(\beta) = \min\{g(x) : x \in [\overline{x},\alpha]\}.$$

Clearly, $g(\beta) \leq g(\overline{x}) < \min\{g(x) : x \in [\rho_2,\alpha]\}$ implies that $\beta < \rho_2$. Moreover, it is obvious that

$$g(\beta) = \min\{g(x) : x \in [\beta,\alpha]\}.$$

Therefore, we obtain from (2.15) that

$$m_{\alpha,\beta} = \underline{\mu}\, g(\beta) > \frac{n}{\eta^n}\phi\left(\frac{\beta}{1-\eta}\right),$$

as wished.

Next, let us suppose that condition (ii) holds. If $b = +\infty$, the asymptotic behavior of g/ϕ at infinity in (2.13) guarantees that there exists $k > 0$ and $l \in (0,1)$ such that

$$\overline{\mu}\, g(x) \le k + l\,\phi(x) \quad \text{for all } x \ge 0.$$

Since ϕ is unbounded, there exists $\alpha > 0$ large enough, such that $k \le (1-l)\phi(\alpha)$. Hence, being ϕ an increasing function, we have

$$\overline{\mu}\, g(x) \le \phi(\alpha) \quad \text{for all } x \in [0,\alpha],$$

which implies that $M_\alpha \le \phi(\alpha)$. On the other hand, if $b < +\infty$, then condition (ii) implies that $\lim_{x\to+\infty} g(x) < b/\overline{\mu}$ and so there exists $l > 0$ and $x_0 > 0$ such that $\overline{\mu}\, g(x) \le l < b$ for all $x \ge x_0$. It suffices to choose $\alpha \ge x_0$ such that $\phi(\alpha) \ge \max\{l, \overline{\mu} \max_{x\in[0,x_0]} g(x)\}$ in order to ensure that $M_\alpha \le \phi(\alpha)$. Finally, the existence of a positive number $\beta < \alpha$ satisfying inequality (2.5) can be deduced exactly as in the previous case.

Now, Theorem 2.3 ensures that problem (2.14) has a positive solution u which, by virtue of [2, Lemma 1] is strictly decreasing. □

Let us emphasize the previous result in the particular cases of radial solutions to the Dirichlet problem in the unit ball $\mathcal{B}$ of $\mathbb{R}^n$ involving the operators Δ_p, $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{A}$ defined in Section 1.

Corollary 2.7. *Assume that conditions $(\boldsymbol{H}_\mu)$ and $(\boldsymbol{H}_g)$ with $b = +\infty$ hold. If*

$$\lim_{x\to 0^+} \frac{g(x)}{x^{p-1}} = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} \frac{g(x)}{x^{p-1}} < \frac{1}{\overline{\mu}},$$

then problem

$$-\Delta_p v = \mu(|z|)g(v) \text{ in } \mathcal{B}, \quad v = 0 \text{ on } \partial\mathcal{B}$$

has at least one radial solution $v(z) = u(|z|)$, with $u > 0$ on $[0,1)$ and strictly decreasing.

Corollary 2.8. *Assume that conditions $(\boldsymbol{H}_\mu)$ and $(\boldsymbol{H}_g)$ with $b = +\infty$ hold. If*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = +\infty,$$

then problem

$$-\mathcal{M}v = \mu(|z|)g(v) \text{ in } \mathcal{B}, \quad v = 0 \text{ on } \partial\mathcal{B}$$

has at least one radial solution $v(z) = u(|z|)$, with $u > 0$ on $[0,1)$ and strictly decreasing.

Corollary 2.9. *Assume that conditions $(\boldsymbol{H}_\mu)$ and $(\boldsymbol{H}_g)$ with $b = 1$ hold. If*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} g(x) < \frac{1}{\overline{\mu}},$$

then problem

$$-\mathcal{E}v = \mu(|z|)g(v) \text{ in } \mathcal{B}, \quad v = 0 \text{ on } \partial\mathcal{B}$$

has at least one radial solution $v(z) = u(|z|)$, with $u > 0$ on $[0,1)$ and strictly decreasing.

Corollary 2.10. *Assume that conditions $(\boldsymbol{H}_\mu)$ and $(\boldsymbol{H}_g)$ with $b = 1$ hold. If*

$$\lim_{x\to 0^+} \frac{g(x)}{x \arcsin x} = +\infty,$$

then problem

$$-\mathcal{A}v = \mu(|z|)g(v) \text{ in } \mathcal{B}, \quad v = 0 \text{ on } \partial\mathcal{B}$$

has at least one radial solution $v(z) = u(|z|)$, with $u > 0$ on $[0,1)$ and strictly decreasing.

3. Systems — existence of positive solutions

Let ϕ_j and f_j $(j = 1,2)$ be as in Section 1. We consider the cone $P \times P$ in the product Banach space $X \times X$ and the completely continuous operator $T = (T_1, T_2) : P \times P \to P \times P$, where, for $(u_1, u_2) \in P \times P$,

$$T_j(u_1,u_2)(r) = \int_r^1 \phi_j^{-1}\left(\frac{1}{\tau^{n-1}}\int_0^\tau s^{n-1} f_j\left(s, u_1(s), u_2(s)\right) ds\right) d\tau \quad (r \in [0,1],\ j = 1,2).$$

Clearly, $u=(u_1,u_2)$ is a solution of (1.6) (that is, u solves (1.6) and both u_1 and u_2 are nonnegative functions on $[0,1]$) iff u is a fixed point of T. In the case of (1.6), by a *positive solution*, we mean a solution $u=(u_1,u_2)$ such that both u_1 and u_2 are distinct from the identically zero function. Also, arguing as in the proof of Proposition 2.1 we have that T maps the cone $K\times K$ into itself.

In accordance with the previous section, for any real numbers $\eta\in(0,1)$, $\alpha_j>\beta_j>0$, $j=1,2$, we define the sets

$$U_{\alpha_j}:=K\cap B(\alpha_j)\ \text{ and }\ V_{\beta_j}:=\left\{u\in\overline{U}_{\alpha_j}\ :\ u(\eta)<\beta_j\right\}.$$

This time, the retract will be the bounded closed convex set $\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$. If $\mathcal{O}$ is an open subset of $\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$ and $A:\overline{\mathcal{O}}\to\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$ is a completely continuous operator which is fixed points free on the boundary of $\mathcal{O}$, we use the notation $i(A,\mathcal{O})$ for the fixed point index of A on $\mathcal{O}$ with respect to $\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$.

We can state the following *vector version* of Theorem 2.2, cf. [16,17,19].

Theorem 3.1. *Assume that there are numbers $\alpha_j>0$, $j=1,2$, such that $T(\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2})\subset\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$. In addition, suppose that there exist $\eta\in(0,1)$, $\beta_j\in(0,\alpha_j)$ and functions $h_j\in\overline{U}_{\alpha_j}\setminus\overline{V}_{\beta_j}$, $j=1,2$, such that*

$$(1-\lambda)T_j(u)+\lambda h_j\neq u_j \tag{3.1}$$

for all $u=(u_1,u_2)\in(\overline{U}_{\alpha_1}\setminus V_{\beta_1})\times(\overline{U}_{\alpha_2}\setminus V_{\beta_2})$ with $u_j(\eta)=\beta_j$ and $\lambda\in[0,1]$ $(j=1,2)$.

Then $i\left(T,\left(\overline{U}_{\alpha_1}\setminus\overline{V}_{\beta_1}\right)\times\left(\overline{U}_{\alpha_2}\setminus\overline{V}_{\beta_2}\right)\right)=1$ and thus problem (1.6) has a solution $u=(u_1,u_2)\in K\times K$, with

$$\beta_j<u_j(\eta)\quad\text{and}\quad\left|u_j\right|_\infty\le\alpha_j\quad(j=1,2).$$

Proof. For the sake of simplicity of the writing, set $Q:=\overline{U}_{\alpha_1}\times\overline{U}_{\alpha_2}$. Since $\overline{U}_{\alpha_i}\setminus V_{\beta_i}$ are closed and convex, there exist retractions $\rho_i:X\to\overline{U}_{\alpha_i}\setminus V_{\beta_i}$ $(i=1,2)$. Then, the map $\rho:Q\to\left(\overline{U}_{\alpha_1}\setminus V_{\beta_1}\right)\times\left(\overline{U}_{\alpha_2}\setminus V_{\beta_2}\right)$, defined by

$$\rho(u)=\left(\rho_1(u_1),\rho_2(u_2)\right)\quad(u=(u_1,u_2)\in Q)$$

is a retraction, too.

We introduce the operator $N=(N_1,N_2):Q\to Q$ defined by $N:=T\circ\rho$. Explicitly, this means

$$N_j(u)=T_j\left(\rho_1(u_1),\rho_2(u_2)\right)\quad(u=(u_1,u_2)\in Q,\ j=1,2)$$

and it is easily seen that N is completely continuous. From (3.1) we deduce that, for each $j\in\{1,2\}$, it holds

$$(1-\lambda)N_j(u)+\lambda h_j\neq u_j \tag{3.2}$$

for all $u=(u_1,u_2)\in Q$ with $u_j(\eta)=\beta_j$ and $\lambda\in[0,1]$.

We first consider the homotopy $H:[0,1]\times Q\to Q$, given by

$$H(\lambda,u)=\lambda N(u)\quad(\lambda\in[0,1],\ u\in Q),$$

which is admissible and so, we infer

$$i(N,Q)=i(H(1,\cdot),Q)=i(H(0,\cdot),Q)=i(0,Q)=1. \tag{3.3}$$

Next, denote

$$\begin{aligned}
\Omega_{11}&:=V_{\beta_1}\times V_{\beta_2}=\left\{(u_1,u_2)\in Q\ :\ u_1(\eta)<\beta_1,\ u_2(\eta)<\beta_2\right\},\\
\Omega_{12}&:=V_{\beta_1}\times\left(\overline{U}_{\alpha_2}\setminus\overline{V}_{\beta_2}\right)=\left\{(u_1,u_2)\in Q\ :\ u_1(\eta)<\beta_1,\ u_2(\eta)>\beta_2\right\},\\
\Omega_{21}&:=\left(\overline{U}_{\alpha_1}\setminus\overline{V}_{\beta_1}\right)\times V_{\beta_2}=\left\{(u_1,u_2)\in Q\ :\ u_1(\eta)>\beta_1,\ u_2(\eta)<\beta_2\right\},\\
\Omega_{22}&:=\left(\overline{U}_{\alpha_1}\setminus\overline{V}_{\beta_1}\right)\times\left(\overline{U}_{\alpha_2}\setminus\overline{V}_{\beta_2}\right)=\left\{(u_1,u_2)\in Q\ :\ u_1(\eta)>\beta_1,\ u_2(\eta)>\beta_2\right\},\\
\Gamma&:=\left\{(u_1,u_2)\in Q\ :\ u_1(\eta)=\beta_1\ \text{or}\ u_2(\eta)=\beta_2\right\}
\end{aligned}$$

and notice that $\left\{\Omega_{11},\Omega_{12},\Omega_{21},\Omega_{22},\Gamma\right\}$ is a partition of Q. Clearly, Γ is closed and Ω_{jk} are open $(j,k=1,2)$. Also, since, on account of (3.2) with $\lambda=0$, we have that $N(u)\neq u$ for $u\in\Gamma$, the excision-additivity property of the fixed point index and (3.3) yields

$$i(N,\Omega_{22})=1-i(N,\Omega_{11})-i(N,\Omega_{12})-i(N,\Omega_{21}). \tag{3.4}$$

To estimate $i(N,\Omega)$ with $\Omega\in\{\Omega_{11},\Omega_{12},\Omega_{21}\}$, first observe that $h\in Q\setminus\overline{\Omega}$. Then, let $\widetilde{H}:[0,1]\times\overline{\Omega}\to Q$ be the homotopy

$$\widetilde{H}(\lambda,u)=(1-\lambda)N(u)+\lambda h\quad(u=(u_1,u_2)\in\overline{\Omega},\ \lambda\in[0,1]).$$

From (3.2) we have that $\widetilde{H}(\lambda,u)\neq u$ for $u\in\partial\Omega\ (\subset\Gamma)$ and $\lambda\in[0,1]$, meaning that $\widetilde{H}$ is admissible. Thus, we get

$$i(N,\Omega)=i(\widetilde{H}(0,\cdot))=i(\widetilde{H}(1,\cdot))=i(h,\Omega)=0.$$

Therefore, as $N=T$ on $\overline{\Omega}_{22}=\left(\overline{U}_{\alpha_1}\setminus V_{\beta_1}\right)\times\left(\overline{U}_{\alpha_2}\setminus V_{\beta_2}\right)$, the conclusion follows from (3.4). □

Remark 3.1. Similarly to Remark 2.1, solely assuming that there exist numbers $\alpha_j > 0$, $j = 1, 2$, such that $T(\overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2}) \subset \overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2}$, we have that $i\left(T, \overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2}\right) = 1$. This immediately follows by the invariance under homotopy of the fixed point index, as in the beginning of the proof of Theorem 2.2

For any numbers $\eta \in (0, 1)$, $0 < \beta_j < \alpha_j$, $j = 1, 2$, we use the following notations

$$m^j_{\alpha,\beta} := \min\{f_j(r, x, y) : r \in [0, \eta],\ x \in [\beta_1, \alpha_1],\ y \in [\beta_2, \alpha_2]\},$$
$$M^j_{\alpha_1,\alpha_2} := \max\{f_j(r, x, y) : r \in [0, 1],\ x \in [0, \alpha_1],\ y \in [0, \alpha_2]\}.$$

Note that one has $0 \le m^j_{\alpha,\beta} \le M^j_{\alpha_1,\alpha_2} < b_j$ $(j = 1, 2)$. Also, in the writing of $m^j_{\alpha,\beta}$, actually, we mean $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. We have the following existence result for the system (1.6).

Theorem 3.2. *If there exist $\eta \in (0, 1)$, $\alpha_j > \beta_j > 0$, such that*

$$\phi_j^{-1}\left(M^j_{\alpha_1,\alpha_2}\right) \le \alpha_j \text{ and } (1-\eta)\phi_j^{-1}\left(\eta^n m^j_{\alpha,\beta}/n\right) > \beta_j \quad (j = 1, 2),$$

then problem (1.6) has at least one solution $u = (u_1, u_2) \in K \times K$ such that

$$\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \le \alpha_j, \quad j = 1, 2.$$

Proof. This can be easily adapted from that of Theorem 2.3 using Theorem 3.1 instead of Theorem 2.2. □

Now we consider the following system

$$\begin{aligned} -\left(r^{n-1}\phi_1\left(u_1'\right)\right)' &= r^{n-1}\mu_1(r)g_1(u_1, u_2), \\ -\left(r^{n-1}\phi_2\left(u_2'\right)\right)' &= r^{n-1}\mu_2(r)g_2(u_1, u_2), \\ u_1'(0) &= u_1(1) = 0 = u_2'(0) = u_2(1), \end{aligned} \tag{3.5}$$

under the hypotheses:

$(\mathbf{H}^s_\mu)$ $\mu_1, \mu_2 : [0, 1] \to (0, +\infty)$ are continuous;

$(\mathbf{H}^s_g)$ $g_j : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, b_j/\overline{\mu}_j)$ is a continuous function, where $\overline{\mu}_j := \max\{\mu_j(r) : r \in [0, 1]\}$ $(j = 1, 2)$. Also, $g_1(0, 0) = 0 = g_2(0, 0)$ and $g_1(\xi, 0) > 0 < g_2(0, \xi)$ for all $\xi > 0$.

We say that a function $g = g(x, y)$ is *nondecreasing* with respect to y (resp. x) if for fixed x (resp. y) one has

$$g(x, y_1) \le g(x, y_2) \text{ as } y_1 \le y_2 \quad (\text{resp. } g(x_1, y) \le g(x_2, y) \text{ as } x_1 \le x_2).$$

This property is also known as *quasi-monotonicity*, see for instance [4,13–15,21].

Proposition 3.3. *Assume $(\mathbf{H}^s_\mu)$ and $(\mathbf{H}^s_g)$ and that $g_1(x_1, x_2)$, $g_2(x_1, x_2)$ are nondecreasing with respect to x_2 and x_1, respectively. If $u = (u_1, u_2)$ is a positive solution of problem (3.5) then both of the components of u are > 0 on $[0, 1)$ and strictly decreasing.*

Proof. From

$$r^{n-1}\phi_1\left(u_1{}'\right) = -\int_0^r t^{n-1}\mu_1(t)g_1(u_1, u_2)\,dt \le -\int_0^r t^{n-1}\mu_1(t)g_1(u_1, 0)\,dt \tag{3.6}$$

we have $u_1' \le 0$, that is u_1 is decreasing. Since u_1 is positive (not identically zero), it follows that $u_1(0) > 0$. Then, using $(\mathbf{H}^s_g)$ and (3.6), we get $u_1' < 0$, meaning that u_1 is strictly decreasing and > 0 on $[0, 1)$. Similar reasoning for u_2. □

Now, as a consequence of Theorem 3.2, we prove the existence of positive solutions for problem (3.5) under a sublinear growth condition on g_1 and g_2. In the particular case of systems involving the mean curvature operator in Minkowski space, we obtain the following result in the line of [13, Theorem 3.1]. To state it, for $\delta > 0$, we introduce the notation

$$\Pi_j(\delta) := \begin{cases} [0, \delta] \times \mathbb{R}_+ & \text{if } j = 1, \\ \mathbb{R}_+ \times [0, \delta] & \text{if } j = 2. \end{cases}$$

Theorem 3.4. *Let conditions $(\mathbf{H}^s_\mu)$ and $(\mathbf{H}^s_g)$ hold and let condition (2.8) be satisfied with ϕ replaced by ϕ_j $(j = 1, 2)$. Assume that $g_1(x_1, x_2)$, $g_2(x_1, x_2)$ are nondecreasing with respect to x_2 and x_1, respectively, and*

$$\lim_{x_1 \to 0^+} \frac{g_1(x_1, 0)}{\phi_1(x_1)} = +\infty, \tag{3.7}$$

$$\lim_{x_2 \to 0^+} \frac{g_2(0, x_2)}{\phi_2(x_2)} = +\infty. \tag{3.8}$$

In case that for some $j \in \{1, 2\}$ one has $a_j = +\infty$, assume in addition that one of the following conditions is satisfied:

(A) *(i) if $b_j = +\infty$, then for any $\delta > 0$, g_j is bounded on $\Pi_j(\delta)$ and*

$$\lim_{x_j \to +\infty} \frac{g_j(x_1, x_2)}{\phi_j(x_j)} < \frac{1}{\overline{\mu}_j}, \quad \text{uniformly with } x_k \in \mathbb{R}_+ \ (k \neq j), \tag{3.9}$$

(ii) if $b_j < +\infty$ then

$$\sup_{x_1, x_2 \in \mathbb{R}_+} g_j(x_1, x_2) < \frac{b_j}{\overline{\mu}_j}; \tag{3.10}$$

(B) *$g_j(x_1, x_2)$ is also nondecreasing in x_j and*

$$\lim_{\tau \to +\infty} \frac{g_j(\tau, \tau)}{\phi_j(\tau)} < \frac{1}{\overline{\mu}_j}. \tag{3.11}$$

Then problem (3.5) has at least one solution $u = (u_1, u_2)$ with both of the components > 0 on $[0, 1)$ and strictly decreasing.

Proof. We first show that Theorem 3.2 applies.

Step 1: Finding α_1, α_2. First, we claim that there exists $\tau_1 > 0$ large enough, such that

$$\phi_1^{-1}\left(M^1_{\tau,\tau}\right) < \tau \quad \text{for all } \tau \geq \tau_1. \tag{3.12}$$

Indeed, if $a_1 < +\infty$, we pick $\tau_1 \geq a_1$ and obviously (3.12) holds true. Otherwise, meaning $a_1 = +\infty$, the existence of such a number τ_1 can be obtained under each of the conditions (A) and (B) (with $j = 1$), as follows.

Case A: Assume that condition (A) holds. So, if $b_1 = +\infty$, (3.9) implies the existence of $c_1 \geq 0$ and $l_1 \in (0, 1)$ satisfying

$$\overline{\mu}_1\, g_1(x_1, x_2) \leq c_1 + l_1\, \phi_1(x_1) \quad \text{for all } x_1, x_2 \geq 0.$$

Since ϕ_1 is unbounded, we can find $\tau_1 > 0$ such that $c_1 \leq (1 - l_1)\phi_1(\tau_1)$. For $\tau \geq \tau_1$ and $x_1, x_2 \in [0, \tau]$, we can estimate

$$\overline{\mu}_1\, g_1(x_1, x_2) \leq (1 - l_1)\phi_1(\tau) + l_1\, \phi_1(x_1) \leq \phi_1(\tau)$$

which gives (3.12). In case $b_1 < +\infty$, (3.10) implies that there is some $l_1 > 0$ such that

$$\overline{\mu}_1\, g_1(x_1, x_2) \leq l_1 < b_1 \quad \text{for all } x_1, x_2 \geq 0.$$

We choose $\tau_1 > 0$ with $\phi_1(\tau_1) > l_1$ (such an τ_1 exists by virtue of $\lim_{x \to +\infty} \phi_1(x) = b_1$). Then for $\tau \geq \tau_1$, it follows

$$\phi_1(\tau) \geq \phi_1(\tau_1) > M^1_{\tau,\tau},$$

which clearly implies (3.12).

Case B: Assume now that condition (B) holds. The asymptotic condition (3.11) guarantees the existence of l_1 and $\tau_1 \geq 0$ such that

$$\frac{g_1(\tau, \tau)}{\phi_1(\tau)} \leq l_1 < \frac{1}{\overline{\mu}_1} \quad \text{for all } \tau \geq \tau_1.$$

From this, using that $\overline{\mu}_1 l_1 < 1$, one has

$$\overline{\mu}_1\, g_1(\tau, \tau) \leq \overline{\mu}_1 l_1 \phi_1(\tau) < \phi_1(\tau) \quad \text{for all } \tau \geq \tau_1$$

and, by the monotonicity of g_1 in both variables, we get (3.12), as claimed.

As above, also there exists $\tau_2 > 0$ such that

$$\phi_2^{-1}\left(M^2_{\tau,\tau}\right) < \tau \quad \text{for all } \tau \geq \tau_2.$$

Thus, taking $\alpha_1 = \alpha_2 = \alpha_0 := \max\{\tau_1, \tau_2\}$, it follows

$$\phi_j^{-1}\left(M^j_{\alpha_1,\alpha_2}\right) < \alpha_j \quad (j = 1, 2). \tag{3.13}$$

Step 2: We prove the existence of numbers $\beta_1, \beta_2 \in (0, \alpha_0)$ as required in Theorem 3.2.

Fixed $\eta \in (0, 1)$, by (2.8) and (3.7), there exists $\rho_1 = \rho_1(\eta) \in (0, \alpha_0)$ such that

$$\underline{\mu}_1\, g_1(x, 0) > \frac{n}{\eta^n} \phi_1\left(\frac{x}{1 - \eta}\right) \quad \text{for all } x \in (0, \rho_1), \tag{3.14}$$

where $\underline{\mu}_1 := \min\{\mu_1(r) : r \in [0, \eta]\}$. Since g_1 is continuous and

$$g_1(0, 0) = 0 < \min\{g_1(x, 0) : x \in [\rho_1, \alpha_0]\},$$

there exists $x_1 \in (0, \rho_1)$ such that

$$0 < g_1(x_1, 0) < \min\{g_1(x, 0) : x \in [\rho_1, \alpha_0]\}.$$

Choose $\beta_1 \in [x_1, \rho_1)$ such that $g_1(\beta_1, 0) = \min\{g_1(x, 0) : x \in [x_1, \alpha_0]\}$. Since $g_1(x, y)$ is nondecreasing with respect to y, we have

$$g_1(\beta_1, 0) = \min\{g_1(x, y) : x \in [\beta_1, \alpha_0],\ y \in [0, \alpha_0]\},$$

and thus (3.14) implies that

$$\underline{\mu}_1 \min\{g_1(x, y) : x \in [\beta_1, \alpha_0],\ y \in [0, \alpha_0]\} > \frac{n}{\eta^n}\phi_1\left(\frac{\beta_1}{1-\eta}\right).$$

Clearly, the inequality $(1-\eta)\phi_1^{-1}\left(\eta^n m^1_{\alpha,\beta}/n\right) > \beta_1$ holds for any $\beta = (\beta_1, \sigma)$ with $\sigma \in (0, \alpha_0)$. The number β_2 can be fixed similarly using condition (3.8) instead of (3.7) and obtaining that $(1-\eta)\phi_2^{-1}\left(\eta^n m^2_{\alpha,\beta}/n\right) > \beta_2$ for all $\beta = (\sigma, \beta_2)$ with $\sigma \in (0, \alpha_0)$. We take the pair $\beta = (\beta_1, \beta_2)$.

Now, Theorem 3.2 ensures that system (3.5) has a positive solution $u = (u_1, u_2)$ and the conclusion follows by Proposition 3.3. □

Example 3.5. As emphasized in Section 1, the form of the system (3.5) allows to treat Dirichlet problems in the unit ball $\mathcal{B}$ with any two of the operators Δ_p, $\mathcal{M}$, $\mathcal{E}$ or $\mathcal{A}$, or combinations of them. To exemplify, we consider the autonomous system

$$\begin{aligned} -\Delta_p v_1 &= a\, v_1^\alpha + b\, v_2^r, \\ -\mathcal{M}v_2 - \sigma\,\mathcal{E}v_2 &= c\, v_1^s + d\, v_2^q, \\ v_1 &= v_2 = 0 \quad \text{on } \partial\mathcal{B} \end{aligned} \tag{3.15}$$

where $a, d > 0$, $\sigma, b, c, s \geq 0$, $\alpha \in [0, p-1)$, $q \in [0, 1)$ and $r \in [0, p-1]$. Then (3.15) has at least one radial solution $v = (v_1, v_2)$ with $v_1, v_2 > 0$ in $\mathcal{B}$ and radially strictly decreasing, provided that either $r < p-1$ or $r = p-1$ and $b < 1$. Indeed, finding radial solutions reduces to a problem of type (3.5) with

$$\begin{aligned} &\phi_1(x) = |x|^{p-2}x, \qquad \phi_2(x) = x\left(1-x^2\right)^{-\frac{1}{2}} + \sigma x\left(1+x^2\right)^{-\frac{1}{2}}, \\ &\mu_1 = \mu_2 \equiv 1, \qquad g_1(x_1, x_2) = a\, x_1^\alpha + b\, x_2^r, \qquad g_2(x_1, x_2) = c\, x_1^s + d\, x_2^q \end{aligned}$$

and the conclusion follows from Theorem 3.4 (condition (B)). Here, the nonlinearities g_1 and g_2 are of the type considered in [6,22], where they are superlinear and associated with the classical Laplacian operator. It is worth to observe that — as therein, the presence of the p-Laplacian operator Δ_p in the first equation induces restrictive conditions on both of the exponents α and r, while the singular operator $\mathcal{M}$ in the second equation allows only the sublinearity restriction on the exponent q – a situation that, having in view [13, Example 3.4 (ii)], is somehow to be expected. A situation when condition (A) (i) in Theorem 3.4 is fulfilled is, for instance, if in system (3.15) the first equation is replaced by

$$-\Delta_p v_1 = a\,(1 + \arctan v_2)v_1^\alpha + |\sin v_1|$$

and $a, \alpha, \sigma, c, s, d, q$ are kept as above.

4. Systems — multiplicity of positive solutions

We emphasize the fact that the results in this section have a simpler counterpart in the scalar case, and this is quite easy to formulate following the systems model. For the sake of brevity of the exposition, we will restrict ourselves only to the case of systems.

Clearly, Theorem 3.2 provides much more information about solutions of (1.6) than Theorem 3.4 since it gives a concrete localization of them. This is the key in order to derive multiplicity results from Theorem 3.2. Obviously, if there exist several pairs of numbers α and β satisfying the conditions of such existence result, we get multiplicity. In addition, the fixed point index computation obtained in Theorem 3.1 can be useful to this aim. In this line, we can prove the following three-solution theorem.

Theorem 4.1. *Under the hypotheses of Theorem 3.1, assume that there exist $\widetilde{\alpha}_1 \in (0, \alpha_1]$ and $\widetilde{\alpha}_2 \in (0, \alpha_2]$ such that either $\widetilde{\alpha}_1 < \beta_1$ or $\widetilde{\alpha}_2 < \beta_2$ and, moreover,*

$$T(u) \neq \lambda u \quad \text{for all } u \in \partial(U_{\widetilde{\alpha}_1} \times U_{\widetilde{\alpha}_2}) \text{ and all } \lambda \geq 1. \tag{4.1}$$

Then problem (1.6) has at least three solutions: $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in K \times K$ such that

$$\begin{aligned} &\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \leq \alpha_j \quad (j = 1, 2); \\ &\left|v_j\right|_\infty < \widetilde{\alpha}_j \quad (j = 1, 2); \\ &\left|w_j\right|_\infty \leq \alpha_j \quad (j = 1, 2),\ \widetilde{\alpha}_1 < \left|w_1\right|_\infty \text{ or } \widetilde{\alpha}_2 < \left|w_2\right|_\infty, \text{ and } w_1(\eta) < \beta_1 \text{ or } w_2(\eta) < \beta_2. \end{aligned}$$

Proof. To simplify the writing, as in the proof of Theorem 3.1, denote $\Omega_{22} := \left(\overline{U}_{\alpha_1} \setminus \overline{V}_{\beta_1}\right) \times \left(\overline{U}_{\alpha_2} \setminus \overline{V}_{\beta_2}\right)$, $Q := \overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2}$ and let $Q_\sharp := U_{\widetilde{\alpha}_1} \times U_{\widetilde{\alpha}_2}$. Since the open sets Ω_{22} and $Q_\sharp$ are disjoint, we can write

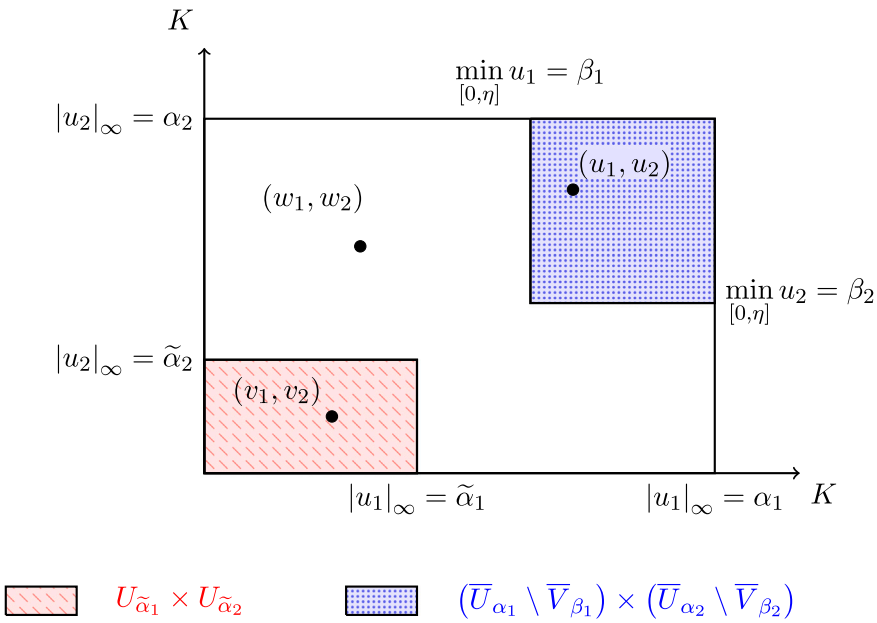

Fig. 1. Localization of the solutions provided by Theorem 4.1.

$$\begin{aligned} Q &= \overline{\Omega_{22} \cup Q_\sharp} \cup \left[Q \setminus \overline{\Omega_{22} \cup Q_\sharp}\right] \\ &= (\Omega_{22} \cup \partial\Omega_{22}) \cup (Q_\sharp \cup \partial Q_\sharp) \cup \left[Q \setminus \overline{\Omega_{22} \cup Q_\sharp}\right] \\ &= \Omega_{22} \cup Q_\sharp \cup \left[Q \setminus \overline{\Omega_{22} \cup Q_\sharp}\right] \cup \partial\left(\Omega_{22} \cup Q_\sharp\right) \end{aligned}$$

and hence, $\{\Omega_{22} \cup Q_\sharp,\ Q \setminus \overline{\Omega_{22} \cup Q_\sharp},\ \partial\left(\Omega_{22} \cup Q_\sharp\right)\}$ is a partition of Q. By Theorem 3.1 and Remark 3.1, we have that

$$i\left(T, \Omega_{22}\right) = 1 = i\left(T, Q\right). \tag{4.2}$$

From (4.1) we deduce that the homotopy $H_\sharp : [0,1] \times \overline{Q}_\sharp \to Q$ defined by

$$H_\sharp(\lambda, u) = \lambda T(u) \quad (u = (u_1, u_2) \in \overline{Q}_\sharp,\ \lambda \in [0,1])$$

is admissible and, by the invariance under homotopy of the fixed point index, it follows

$$i\left(T, Q_\sharp\right) = i\left(H_\sharp(1,\cdot), Q_\sharp\right) = i\left(H_\sharp(0,\cdot), Q_\sharp\right) = i\left(0, Q_\sharp\right) = 1. \tag{4.3}$$

Thus, using (4.2) and (4.3) and the excision-additivity property of the fixed point index, we infer

$$i\left(T, \overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2} \setminus \left[\left(\overline{U}_{\alpha_1} \setminus V_{\beta_1}\right) \times \left(\overline{U}_{\alpha_2} \setminus V_{\beta_2}\right) \cup \overline{U}_{\widetilde{\alpha}_1} \times \overline{U}_{\widetilde{\alpha}_2}\right]\right) = i(T, Q \setminus \overline{\Omega_{22} \cup Q_\sharp}) = -1.$$

Finally, the conclusion follows in a straightforward way from the existence property of the index. □

Remark 4.1. Obviously, under the assumptions of Theorem 4.1, both components of the solution (u_1, u_2) are positive, (v_1, v_2) may be the trivial solution while (w_1, w_2) may be semi-trivial, meaning that either w_1 or w_2 may be the identically zero function (see Fig. 1).

As a direct consequence of Theorem 4.1, we deduce the following multiplicity result.

Theorem 4.2. *Under the hypotheses of Theorem 3.2, assume that there exist $\widetilde{\alpha}_1 \in (0, \alpha_1]$ and $\widetilde{\alpha}_2 \in (0, \alpha_2]$ such that either $\widetilde{\alpha}_1 < \beta_1$ or $\widetilde{\alpha}_2 < \beta_2$ and, moreover,*

$$\phi_j^{-1}\left(M^j_{\widetilde{\alpha}_1, \widetilde{\alpha}_2}\right) < \widetilde{\alpha}_j \quad (j = 1, 2). \tag{4.4}$$

Then problem (1.6) has at least three solutions: $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in K \times K$ such that

$$\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \le \alpha_j \quad (j = 1, 2);$$

$$\left|v_j\right|_\infty < \widetilde{\alpha}_j \quad (j = 1, 2);$$

$$\left|w_j\right|_\infty \le \alpha_j \quad (j = 1, 2),\ \widetilde{\alpha}_1 < |w_1|_\infty \text{ or } \widetilde{\alpha}_2 < |w_2|_\infty,\ \text{and } w_1(\eta) < \beta_1 \text{ or } w_2(\eta) < \beta_2.$$

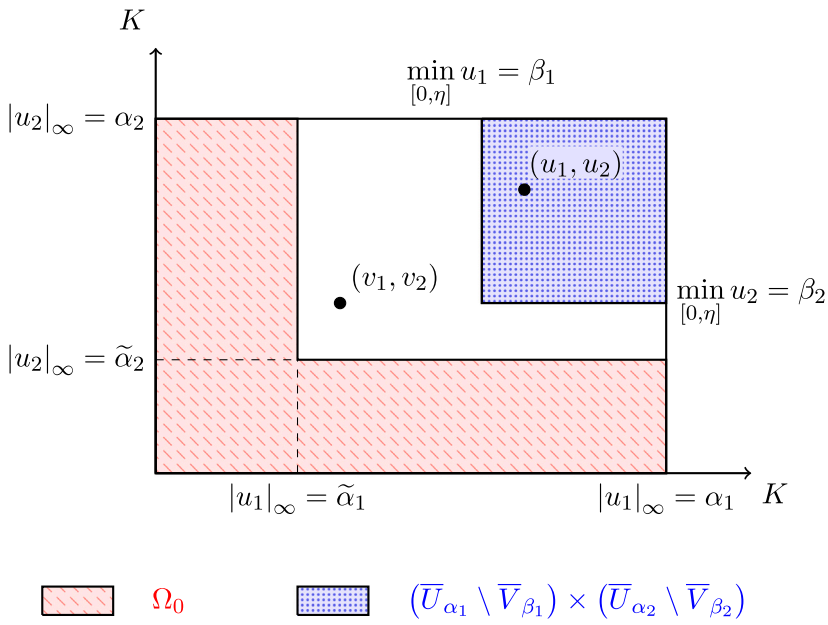

Fig. 2. Localization of the solutions provided by Theorem 4.4.

Proof. Hypotheses of Theorem 3.2 guarantee that conditions of Theorem 3.1 are satisfied. So, to apply Theorem 4.1, we only have to verify (4.1). But, (4.4) ensures that

$$T\left(\overline{U}_{\widetilde{\alpha}_1} \times \overline{U}_{\widetilde{\alpha}_2}\right) \subset U_{\widetilde{\alpha}_1} \times U_{\widetilde{\alpha}_2}$$

(see the proof of Theorem 2.3), which implies (4.1). □

In the particular case when $\widetilde{\alpha}_1 = \alpha_1$, we obtain the following corollary and a similar result also holds true when $\widetilde{\alpha}_2 = \alpha_2$.

Corollary 4.3. *Assume that there are numbers $\eta \in (0,1)$, $\alpha_1 > \beta_1 > 0$, $\alpha_2 > \beta_2 > \widetilde{\alpha}_2 > 0$, such that*

$$\phi_1^{-1}\left(M^1_{\alpha_1,\alpha_2}\right) < \alpha_1, \quad \phi_2^{-1}\left(M^2_{\alpha_1,\alpha_2}\right) \le \alpha_2, \quad \phi_2^{-1}\left(M^2_{\alpha_1,\widetilde{\alpha}_2}\right) < \widetilde{\alpha}_2$$

and

$$(1-\eta)\phi_j^{-1}\left(\eta^n m^j_{\alpha,\beta}/n\right) > \beta_j \quad (j = 1,2).$$

Then problem (1.6) has at least three solutions: $(u_1,u_2),(v_1,v_2),(w_1,w_2) \in K \times K$ such that

$$\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \le \alpha_j \quad (j = 1,2);$$
$$|v_1|_\infty < \alpha_1, \quad |v_2|_\infty < \widetilde{\alpha}_2;$$
$$\left|w_j\right|_\infty \le \alpha_j \quad (j = 1,2), \ \widetilde{\alpha}_2 < |w_2|_\infty, \text{ and } w_1(\eta) < \beta_1 \text{ or } w_2(\eta) < \beta_2.$$

Existence of at least two positive solutions can be obtained if we strengthen the previous assumptions. See their localization in Fig. 2.

Theorem 4.4. *Under the hypotheses of Theorem 3.1, assume that there exist $\widetilde{\alpha}_1 \in (0,\beta_1)$ and $\widetilde{\alpha}_2 \in (0,\beta_2)$ such that*

$$T_1\left(\overline{U}_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) \subset U_{\widetilde{\alpha}_1}, \tag{4.5}$$
$$T_2\left(\overline{U}_{\alpha_1} \times \overline{U}_{\widetilde{\alpha}_2}\right) \subset U_{\widetilde{\alpha}_2}. \tag{4.6}$$

Then problem (1.6) has at least two positive solutions: $(u_1,u_2),(v_1,v_2) \in K \times K$ such that

$$\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \le \alpha_j \quad (j = 1,2);$$
$$\widetilde{\alpha}_j < \left|v_j\right|_\infty \le \alpha_j \quad (j = 1,2), \text{ and } v_1(\eta) < \beta_1 \text{ or } v_2(\eta) < \beta_2.$$

Proof. Let Q, $Q_\sharp$ and Ω_{22} be as in the proof of Theorem 4.1. First, observe that conditions (4.5) and (4.6) imply $T\left(\overline{Q}_\sharp\right) \subset Q_\sharp$. Then, using

$$\begin{aligned}
\partial\left(Q_\sharp\right) &= (\partial U_{\widetilde{\alpha}_1} \times \overline{U}_{\widetilde{\alpha}_2}) \cup (\overline{U}_{\widetilde{\alpha}_1} \times \partial U_{\widetilde{\alpha}_2}) \\
&= \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty = \widetilde{\alpha}_1 \, , \, |u_2|_\infty \le \widetilde{\alpha}_2 \text{ or } |u_1|_\infty \le \widetilde{\alpha}_1 \, , \, |u_2|_\infty = \widetilde{\alpha}_2\right\}, \\
\partial\left(U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) &= \partial U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2} \\
&= \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty = \widetilde{\alpha}_1 \, , \, |u_2|_\infty \le \alpha_2\right\}, \\
\partial\left(\overline{U}_{\alpha_1} \times U_{\widetilde{\alpha}_2}\right) &= \overline{U}_{\alpha_1} \times \partial U_{\widetilde{\alpha}_2} \\
&= \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty \le \alpha_1 \, , \, |u_2|_\infty = \widetilde{\alpha}_2\right\}
\end{aligned}$$

and the invariance under homotopy of the fixed point index (as in the proof of Theorem 4.1), we deduce that

$$i\left(T, Q_\sharp\right) = i\left(T, U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) = i\left(T, \overline{U}_{\alpha_1} \times U_{\widetilde{\alpha}_2}\right) = 1.$$

Let

$$\begin{aligned}
\gamma_1 &:= \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty \le \widetilde{\alpha}_1, \, |u_2|_\infty = \widetilde{\alpha}_2 \text{ or } |u_1|_\infty = \widetilde{\alpha}_1, \, |u_2|_\infty \le \alpha_2\right\}, \\
\gamma_2 &:= \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty = \widetilde{\alpha}_1, \, |u_2|_\infty \le \widetilde{\alpha}_2 \text{ or } |u_1|_\infty \le \alpha_1, \, |u_2|_\infty = \widetilde{\alpha}_2\right\}
\end{aligned}$$

and notice that γ_1, γ_2 are closed, $\gamma_1 \subset \partial Q_\sharp \cup \partial\left(U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right)$, $\gamma_2 \subset \partial Q_\sharp \cup \partial\left(\overline{U}_{\alpha_1} \times U_{\widetilde{\alpha}_2}\right)$ and T is fixed points free on both γ_1 and γ_2. Also, we have that $\{Q_\sharp, U_{\widetilde{\alpha}_1} \times \left(\overline{U}_{\alpha_2} \setminus \overline{U}_{\widetilde{\alpha}_2}\right), \gamma_1\}$ and $\{Q_\sharp, \left(\overline{U}_{\alpha_1} \setminus \overline{U}_{\widetilde{\alpha}_1}\right) \times U_{\widetilde{\alpha}_2}, \gamma_2\}$ are partitions of $\overline{U}_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}$ and $\overline{U}_{\alpha_1} \times \overline{U}_{\widetilde{\alpha}_2}$, respectively. Then, excision-additivity property of the index yields

$$\begin{aligned}
i\left(T, U_{\widetilde{\alpha}_1} \times \left(\overline{U}_{\alpha_2} \setminus \overline{U}_{\widetilde{\alpha}_2}\right)\right) &= i\left(T, U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) - i\left(T, Q_\sharp\right) = 0, \\
i\left(T, \left(\overline{U}_{\alpha_1} \setminus \overline{U}_{\widetilde{\alpha}_1}\right) \times U_{\widetilde{\alpha}_2}\right) &= i\left(T, U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) - i\left(T, Q_\sharp\right) = 0.
\end{aligned}$$

Consider the relatively open set

$$\Omega_0 := \left(U_{\widetilde{\alpha}_1} \times \overline{U}_{\alpha_2}\right) \cup \left(\overline{U}_{\alpha_1} \times U_{\widetilde{\alpha}_2}\right)$$

and observe that setting

$$\gamma := \gamma_1 \cup \gamma_2 = \left\{(u_1,u_2) \in Q \ : \ |u_1|_\infty = \widetilde{\alpha}_1, \, |u_2|_\infty \le \alpha_2 \text{ or } |u_1|_\infty \le \alpha_1, \, |u_2|_\infty = \widetilde{\alpha}_2\right\}.$$

one has that $\{Q_\sharp, U_{\widetilde{\alpha}_1} \times \left(\overline{U}_{\alpha_2} \setminus \overline{U}_{\widetilde{\alpha}_2}\right), \left(\overline{U}_{\alpha_1} \setminus \overline{U}_{\widetilde{\alpha}_1}\right) \times U_{\widetilde{\alpha}_2}, \gamma\}$ is a partition of $\overline{\Omega}_0$. Thus, we obtain

$$i\left(T, \Omega_0\right) = i\left(T, Q_\sharp\right) + i\left(T, U_{\widetilde{\alpha}_1} \times \left(\overline{U}_{\alpha_2} \setminus \overline{U}_{\widetilde{\alpha}_2}\right)\right) + i\left(T, \left(\overline{U}_{\alpha_1} \setminus \overline{U}_{\widetilde{\alpha}_1}\right) \times U_{\widetilde{\alpha}_2}\right) = 1. \tag{4.7}$$

Finally, consider the relative complement of $\overline{\Omega}_0 \cup \overline{\Omega}_{22}$ with respect to the set Q, that is,

$$\Omega := \overline{U}_{\alpha_1} \times \overline{U}_{\alpha_2} \setminus \left[\overline{\Omega}_0 \cup \left(\overline{U}_{\alpha_1} \setminus V_{\beta_1}\right) \times \left(\overline{U}_{\alpha_2} \setminus V_{\beta_2}\right)\right].$$

Using (4.7), Theorem 3.1 and Remark 3.1, we deduce

$$i\left(T, \Omega\right) = i\left(T, Q\right) - i\left(T, \Omega_0\right) - i\left(T, \Omega_{22}\right) = -1. \tag{4.8}$$

In conclusion, the existence property of the index together with (4.8) and Theorem 3.1 ensure that T has at least two non-trivial positive fixed points located in the disjoint open sets Ω and $\left(\overline{U}_{\alpha_1} \setminus \overline{V}_{\beta_1}\right) \times \left(\overline{U}_{\alpha_2} \setminus \overline{V}_{\beta_2}\right)$. □

Remark 4.2. In the proof of Theorem 4.4 we obtain that $i(T, \Omega_0) = 1$, which ensures that T has a fixed point in Ω_0. Hence, problem (1.6) has a third solution, but it may be the trivial one.

Remark 4.3. When we speak about two or three solutions, it is understood that they are supposed to be distinct. In case of systems (of two equations), where by a solution we mean a pair (u_1, u_2), two solutions are distinct if they differ at least on one of their components, not necessarily on both. Thus regarding Theorem 4.4, there is the possibility to have $v_2 = u_2$ in case where $v_1(\eta) < \beta_1$, or $v_1 = u_1$ in case where $v_2(\eta) < \beta_2$.

Theorem 4.5. *Under the hypotheses of Theorem 3.2, assume that there exist $\widetilde{\alpha}_1 \in (0, \beta_1)$ and $\widetilde{\alpha}_2 \in (0, \beta_2)$ such that*

$$\phi_1^{-1}\left(M^1_{\widetilde{\alpha}_1, \alpha_2}\right) < \widetilde{\alpha}_1, \tag{4.9}$$

$$\phi_2^{-1}\left(M^2_{\alpha_1, \widetilde{\alpha}_2}\right) < \widetilde{\alpha}_2. \tag{4.10}$$

Then problem (1.6) has at least two positive solutions: $(u_1, u_2), (v_1, v_2) \in K \times K$ *such that*

$$\beta_j < u_j(\eta) \quad \text{and} \quad \left|u_j\right|_\infty \le \alpha_j \quad (j = 1, 2);$$
$$\widetilde{\alpha}_j < \left|v_j\right|_\infty \le \alpha_j \quad (j = 1, 2), \text{ and } v_1(\eta) < \beta_1 \text{ or } v_2(\eta) < \beta_2.$$

Proof. Hypotheses of Theorem 3.2 ensure that conditions of Theorem 3.1 are fulfilled. So, to apply Theorem 4.4, we only have to verify (4.5) and (4.6). But, this is straightforward from (4.9) and (4.10), respectively (see the proof of Theorem 2.3). □

Example 4.6. For any $\gamma > \sqrt{3}\, n2^n/3$, the system

$$\begin{aligned} -\left(r^{n-1}\left(u_1'/\sqrt{1-(u_1')^2}\right)\right)' &= r^{n-1}u_1^2\left[16\gamma + (u_2 - 1/4)^2\right], \\ -\left(r^{n-1}\left(u_2'/\sqrt{1-(u_2')^2}\right)\right)' &= r^{n-1}u_2^2\left[16\gamma + (u_1 - 1/4)^2\right], \\ u_1'(0) &= u_1(1) = 0 = u_2'(0) = u_2(1), \end{aligned} \tag{4.11}$$

has at least two positive solutions: $(u_1, u_2), (v_1, v_2) \in K \times K$ such that

$$\frac{1}{4} < u_j\left(\frac{1}{2}\right) \quad \text{and} \quad \left|u_j\right|_\infty \le 1 \quad (j = 1, 2);$$
$$\widetilde{\alpha} < \left|v_j\right|_\infty \le 1 \quad (j = 1, 2), \text{ and } v_1\left(\frac{1}{2}\right) < \frac{1}{4} \text{ or } v_2\left(\frac{1}{2}\right) < \frac{1}{4}$$

where $\widetilde{\alpha} \in (0, 1/4)$ is such that

$$\widetilde{\alpha}\left(16\gamma + \frac{9}{16}\right)\left[1 + \widetilde{\alpha}^4\left(16\gamma + \frac{9}{16}\right)^2\right]^{-\frac{1}{2}} < 1. \tag{4.12}$$

To see this, first observe that the choice of γ ensures that

$$\frac{\gamma}{n2^n}\left[1 + \left(\frac{\gamma}{n2^n}\right)^2\right]^{-\frac{1}{2}} > \frac{1}{2}, \tag{4.13}$$

then Theorem 4.5 applies with the following choices:

$$\phi_j(x) = x(1 - x^2)^{-\frac{1}{2}}, \quad f_j(r, x_1, x_2) = x_j^2\left[16\gamma + (x_k - 1/4)^2\right] \quad (j, k \in \{1, 2\} \text{ and } k \ne j)$$

$$\eta = 1/2, \quad \alpha_1 = \alpha_2 = 1, \quad \beta_1 = \beta_2 = 1/4, \quad \widetilde{\alpha}_1 = \widetilde{\alpha}_2 = \widetilde{\alpha} \text{ (see (4.12))}.$$

Indeed, using that

$$\phi_j^{-1}(x) = x(1 + x^2)^{-\frac{1}{2}}, \quad m^j_{\alpha,\beta} \equiv m^j_{(1,\ 1),(1/4,\ 1/4)} = \gamma \quad (j \in \{1, 2\}),$$

$$M^1_{\widetilde{\alpha}_1,\alpha_2} = M^2_{\alpha_1,\widetilde{\alpha}_2} = \widetilde{\alpha}^2\left(16\gamma + \frac{9}{16}\right),$$

together with (4.12) and (4.13), hypotheses of Theorem 4.5 are easily checked by direct computations. Notice that none of the functions $f_j(r, x_1, x_2)$ is nondecreasing with respect to x_k ($j, k \in \{1, 2\}$ and $k \ne j$). As a consequence of the above, we have that, for any $\gamma > \sqrt{3}\, n2^n/3$, the Dirichlet system in the unit ball $\mathcal{B} \subset \mathbb{R}^n$,

$$\begin{aligned} -\mathcal{M}v_1 &= v_1^2\left[16\gamma + (v_2 - 1/4)^2\right], \\ -\mathcal{M}v_2 &= v_2^2\left[16\gamma + (v_1 - 1/4)^2\right], \\ v_1 &= v_2 = 0 \quad \text{on } \partial\mathcal{B} \end{aligned}$$

has at least two radial solutions, each of them having nonnegative and nontrivial components.

Acknowledgments

J. Rodríguez–López has been partially supported by Ministerio de Ciencia y Tecnología (Spain), AIE and Feder, grant PID2020-113275GB-I00.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.
[2] C. Bereanu, P. Jebelean, P.J. Torres, Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space, J. Funct. Anal. 264 (2013) 270–287.
[3] P. Candito, U. Guarnotta, R. Livrea, Existence of two solutions for singular Φ-Laplacian problems, Adv. Nonlinear Stud. 22 (2022) 659–683.
[4] X. Cheng, H. Lü, Multiplicity of positive solutions for a $(p1, p2)$– Laplacian system and its applications, Nonlinear Anal. RWA 13 (2012) 2375–2390.
[5] F.J.S.A. Corrêa, M.L. Carvalho, J.V.A. Gonçalves, K.O. Silva, Positive solutions of strongly nonlinear elliptic problems, Asymptot. Anal. 93 (2015) 1–20.
[6] R. Dalmasso, Existence of positive radial solutions for a semilinear elliptic system without variational structure, Nonlinear Anal. 67 (2007) 1255–1259.

[7] P. Drábek, M. García-Huidobro, R. Manásevich, Positive solutions for a class of equations with a p-Laplace like operator and weights, Nonlinear Anal. 71 (2009) 1281–1300.
[8] M. García-Huidobro, R. Manásevich, K. Schmitt, Positive radial solutions of quasilinear elliptic partial differential equations on a ball, Nonlinear Anal. 35 (1999) 175–190.
[9] M. García-Huidobro, R. Manásevich, P. Ubilla, Existence of positive solutions for some Dirichlet problems with an asymptotically homogeneous operator, Electron. J. Differential Equations 1995 (1995) 1–22.
[10] M. García-Huidobro, R. Manásevich, F. Zanolin, Infinitely many solutions for a Dirichlet problem with a nonhomogeneous p-Laplacian-like operator in a ball, Adv. Differential Equations 2 (1997) 203–230.
[11] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.
[12] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.
[13] D. Gurban, P. Jebelean, Positive radial solutions for systems with mean curvature operator in Minkowski space, Rend. Istit. Mat. Univ. Trieste 49 (2017) 245–264.
[14] D. Gurban, P. Jebelean, Positive radial solutions for multiparameter Dirichlet systems with mean curvature operator in Minkowski space and Lane–Emden type nonlinearities, J. Differential Equations 266 (2019) 5377–5396.
[15] Y.-H. Lee, Existence of multiple positive radial solutions for a semilinear elliptic system on an unbounded domain, Nonlinear Anal. 47 (2001) 3649–3660.
[16] R. Precup, A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions of nonlinear systems, J. Fixed Point Theory Appl. 2 (2007) 141–151.
[17] R. Precup, J. Rodríguez-López, Multiplicity results for operator systems via fixed point index, Results Math. 74 (2019) 1–14.
[18] R. Precup, J. Rodríguez-López, Positive radial solutions for Dirichlet problems via a Harnack-type inequality, Math. Methods Appl. Sci. (2022) 1–14, http://dx.doi.org/10.1002/mma.8682.
[19] J. Rodríguez-López, A fixed point index approach to Krasnosel'skiĭ–Precup fixed point theorem in cones and applications, Nonlinear Anal. 226 (113138) (2023) 19.
[20] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, Acta Math. 138 (1977) 219–240.
[21] W. Walter, Differential and Integral Inequalities, Springer-Verlag, New York, 1970.
[22] H. Zou, A priori estimates for a semilinear elliptic system without variational structure and their applications, Math. Ann. 323 (2002) 713–735.