Existence of solutions of integral equations defined in unbounded domains via spectral theory[☿]

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Abstract

In this work we study integral equations defined on the whole real line. Using a suitable Banach space, we look for solutions which satisfy some certain kind of asymptotic behavior. We will consider spectral theory in order to find fixed points of the integral operator.

MSC: 45P05; 34K08; 45J05; 34K25

Keywords: asymptotic behavior; integral operators; unbounded domain; spectral theory

1 Introduction

In this paper we will study the existence of fixed points of the following integral operator

$$Tu(t) := \int_{-\infty}^{\infty} k(t,s)\,\eta(s)\,f(s,u(s))\,\mathrm{d}s.$$

When working with integral problems defined in unbounded intervals, the main difficulty is the lack of compactness of the operator. In the recent literature (see [4, 6, 12–14]), most of the authors use the following relatively compactness criterion (see [3, 16]) to deal with this problem:

Theorem 1.1 ([16, Theorem 1]). *Let E be a Banach space and $\mathscr{C}(\mathbb{R},E)$ the space of all bounded continuous functions $x\colon \mathbb{R}\to E$. For a set $D\subset \mathscr{C}(\mathbb{R},E)$ to be relatively compact, it is necessary and sufficient that:*

[☿] The three authors were partially supported by Ministerio de Economía y Competitividad, Spain, and FEDER, project MTM2013-43014-P, and by the Agencia Estatal de Investigación (AEI) of Spain under grant MTM2016-75140-P, co-financed by the European Community fund FEDER.

[♀] Supported by FPU scholarship, Ministerio de Educación, Cultura y Deporte, Spain.

1. *$\{x(t),\ x \in D\}$ is relatively compact in E for any $t \in \mathbb{R}$;*
2. *for each $a > 0$, the family $D_a := \{x|_{[-a,a]},\ x \in D\}$ is equicontinuous;*
3. *D is stable at $\pm\infty$, that is, for any $\varepsilon > 0$, there exists $T > 0$ and $\delta > 0$ such that if $\|x(T) - y(T)\| \le \delta$, then $\|x(t) - y(t)\| \le \varepsilon$ for $t \ge T$ and if $\|x(-T) - y(-T)\| \le \delta$, then $\|x(t) - y(t)\| \le \varepsilon$ for $t \le -T$, where x and y are arbitrary functions in D.*

In a recent paper [2], the authors presented a novel way of dealing with the problem of the lack of compactness of the integral operator. They defined a new kind of Banach space: the *space of continuously n-differentiable φ-extensions to infinity*. Moreover, this Banach space makes it possible to study the asymptotic behavior of the solutions of the problem. In that work, the authors used fixed point index methods in order to obtain existence and multiplicity results for boundary value problems and Hammerstein-type equations of the kind

$$Lu(t) := p(t) + \int_{-\infty}^{\infty} k(t,s)\,\eta(s)\,f(s,u(s))\,\mathrm{d}s. \tag{1.1}$$

Furthermore, those results included the location of the solutions, since they were found in a cone defined in a general abstract way –cf. [7]. Depending on the region of the cone, the index was to be proven zero or nonzero, thus providing a solution to the problem of study.

In this paper we complement those findings by approaching the problem in a different way. If in the previous work we had fairly restrictive conditions on the nonlinearity f, here we relax in a significant way those restrictions by studying the eigenvalues of some related linear operators. This approach has been used successfully previously, as we can see in the works of Infante et al. [10], Webb and Lan [18] or even in the case of linearly bounded nonlinear operators as shown in [1]. We note that, for the sake of simplicity, we do not include in this paper the function p occurring in (1.1). However, it could be included with minor adaptations, following the hypotheses for p in [2].

There is of course a price to pay for the advantage regarding the nonlinearity, and is that the conditions on the kernel k occurring in (1.1) are more restrictive. One can check that this is the case of the kernel in the application studied in [2]. There, we have studied the equation that describes the movement of a self-propelled projectile launched vertically from the surface of a planet, that is,

$$u''(t) = -\frac{g\,R^2}{(u(t)+R)^2} + h(t,u(t)), \quad t \in [0,\infty);\ u(0) = 0,\ u'(0) = v_0, \tag{1.2}$$

where u represents the distance from the surface of the planet, R is the radius of the planet, g the surface gravity constant, v_0 the initial velocity and $h(t,y)$ the acceleration generated by the propulsion system of the rocket.

Rewriting previous problem as an integral one, we can see that solutions of (1.2) coincide with fixed points of operator

$$Lu(t) = v_0\,t + \int_0^{\infty} k(t,s)\,f(s,u(s))\,\mathrm{d}s,$$

where

$$k(t,s) = \begin{cases} t-s, & 0 \le s \le t, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f(t,y)=-\frac{gR^2}{(y+R)^2}+h(t,y).$$

We note that the results in the present work could not be applied to this problem (even with the modification to include the term p) as, for instance, condition (C_2) does not hold. At the same time, we will show in Section 5 an example which is solved with the method developed in this paper but does not satisfy the hypotheses required in [2].

Thus, our two methods are not comparable but complementary, making it possible to deal with different kinds of differential and integral problems defined on unbounded intervals, either with more restrictive conditions on the linear part or on the nonlinear one.

This paper is divided in the following way: in Section 2 we summarize some definitions and results of spectral theory. In Section 3 we compile the theory regarding the space of continuously n-differentiable φ-extensions to infinity, which has been developed in [2]. Section 4 includes our results of existence of solutions of integral problems. Finally, Section 5 shows an example to which the results in Section 4 are applied.

2 Preliminaries

Let $(N_1,\|\cdot\|_1)$ and $(N_2,\|\cdot\|_2)$ be two normed spaces. Let $\Gamma:N_1\to N_2$ be a bounded linear operator, that is, such that its norm $\|\Gamma\|=\sup_{\|u\|_2=1}\|\Gamma u\|_1/\|u\|_2$ is finite. We recall that λ is an *eigenvalue* of a linear operator between normed spaces $\Gamma:(N_1,\|\cdot\|_1)\to(N_2,\|\cdot\|_2)$ with corresponding eigenfunction ϕ if $\phi\neq 0$ and $\lambda\phi=\Gamma\phi$. The reciprocals of nonzero eigenvalues are called *characteristic values* of Γ. We will denote the *spectral radius* of Γ by $r(\Gamma):=\lim_{n\to\infty}\|\Gamma^n\|^{\frac{1}{n}}$ and its *principal characteristic value* by $\mu(\Gamma):=1/r(\Gamma)$.

We recall now some known definitions and results.

Definition 2.1. We say that K is a *total cone* if $\overline{K-K}=X$.

Theorem 2.2 (Krein-Rutman [5, Theorem 1.1]). *Let K be a total cone and $\mathscr{L}:X\to X$ a compact linear operator that maps K to K with positive spectral radius $r(\mathscr{L})$. Then $r(\mathscr{L})$ is an eigenvalue with an eigenvector $\phi\in K\backslash\{0\}$.*

Theorem 2.3 ([17, Theorem 2.7]). *Let $\mathscr{L}$ be a bounded linear operator in a Banach space X and let K be a cone in X such that $\mathscr{L}(K)\subset K$. If there exists $\lambda_0>0$ and $v\in K\setminus\{0\}$ such that $Lv\succeq_K\lambda_0 v$, then $r(\mathscr{L})\geq\lambda_0$.*

Theorem 2.4 ([21, Theorem 1]). *Let the positive, completely continuous, linear operator $\mathscr{L}$ satisfy the inequality*

$$\mathscr{L}v\leq\lambda_0 v,$$

where v is a quasi-interior element of the cone K. Then $r(\mathscr{L})\leq\lambda_0$.

Remark 2.5. If the cone K has non empty interior, then the interior and the quasi-interior of the cone coincide (see [9]).

3 The space of continuously n-differentiable φ-extensions to infinity

In this section, we review the concepts introduced in [2].

Consider the space $\overline{\mathbb{R}} := [-\infty, +\infty]$ with the compact topology, that is, the topology generated by the basis

$$\{B(a,r) \ : \ a \in \mathbb{R}, \ r \in \mathbb{R}^+\} \cup \{[-\infty, a) \ : \ a \in \mathbb{R}\} \cup \{(a, +\infty] \ : \ a \in \mathbb{R}\}.$$

With this topology, $\overline{\mathbb{R}}$ is homeomorphic to any compact interval of $\mathbb{R}$ with the relative topology inherited from the usual topology of $\mathbb{R}$.

It is easy to check that $\mathscr{C}(\overline{\mathbb{R}}, \mathbb{R})$ is a Banach space with the usual supremum norm. We define, in a similar way,

$$\mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R}) := \left\{ f : \overline{\mathbb{R}} \to \mathbb{R} \ : \ f|_{\mathbb{R}} \in \mathscr{C}^n(\mathbb{R}, \mathbb{R}), \ \exists \lim_{t \to \pm\infty} f^{(j)}(t) \in \mathbb{R}, \ j = 0, \dots, n \right\},$$

for $n \in \mathbb{N}$. $\mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R})$, $n \in \mathbb{N}$, is a Banach space with the norm

$$\|f\|_{(n)} := \sup\left\{ \left\| f^{(k)} \right\|_\infty \ : \ k = 0, \dots, n \right\}.$$

Take now $\varphi \in \mathscr{C}^n(\mathbb{R}, \mathbb{R}^+)$, where $\mathbb{R}^+ = (0, \infty)$, and define the *space of continuously n-differentiable φ-extensions to infinity*

$$\widetilde{\mathscr{C}}^n_\varphi \equiv \widetilde{\mathscr{C}}^n_\varphi(\mathbb{R}, \mathbb{R}) = \left\{ f \in \mathscr{C}^n(\mathbb{R}, \mathbb{R}) \ : \ \exists \widetilde{f} \in \mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R}), \ f = \varphi\left(\widetilde{f}|_{\mathbb{R}}\right) \right\}.$$

We define the norm

$$\|f\|_\varphi := \|\widetilde{f}\|_{(n)}, \ f \in \widetilde{\mathscr{C}}_\varphi.$$

$\|\cdot\|_\varphi$ is well defined, since the extension $\widetilde{f}$ is unique for every f; indeed, assume there are $\widetilde{f}_1$, $\widetilde{f}_2$ such that $\widetilde{f}_1 \varphi = \widetilde{f}_2 \varphi = f$ in $\mathbb{R}$. Since $\mathbb{R}$ is dense in $\overline{\mathbb{R}}$ and $\widetilde{f}_1$ and $\widetilde{f}_2$ are continuous, $\widetilde{f}_1 = \widetilde{f}_2$.

On the other hand, for every $\widetilde{f} \in \mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ there exists a unique $f \in \widetilde{\mathscr{C}}_\varphi$ such that $\widetilde{f}|_{\mathbb{R}} \varphi = f$ (just define $f := \widetilde{f} \varphi$ in $\mathbb{R}$).

This shows that there is an isometric isomorphism

$$\begin{aligned} \Phi : \mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R}) &\to \widetilde{\mathscr{C}}^n_\varphi \\ \tilde{f} &\mapsto \Phi(\tilde{f}) = \tilde{f}|_{\mathbb{R}} \varphi, \end{aligned}$$

of which the inverse isomorphism is

$$\begin{aligned} \Phi^{-1} : \widetilde{\mathscr{C}}^n_\varphi &\to \mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R}) \\ f &\mapsto \Phi^{-1}(f) = f / \varphi. \end{aligned}$$

Furthermore, Arcelà-Ascoli's Theorem applies to $\mathscr{C}^n(\overline{\mathbb{R}}, \mathbb{R})$ since $\overline{\mathbb{R}}$ is a Hausdorff compact topological space and $\mathbb{R}$ is a complete metric space. Using Φ we can apply the Theorem to $\widetilde{\mathscr{C}}^n_\varphi$. To be precise,

Theorem 3.1. *$F \subset \widetilde{\mathscr{C}}^n_\varphi$ has compact closure if and only if the two following conditions are satisfied:*

- *For each $t \in \mathbb{R}$, the set $\{\widetilde{f}(t),\ f \in F\}$ has compact closure or, which is the same (since $\widetilde{f}(t) \in \mathbb{R}$), $\{\widetilde{f}(t),\ f \in F\}$ is bounded, that is, for each $t \in \mathbb{R}$ there exists some constant $M > 0$ such that*
$$\left|\frac{\partial^j \widetilde{f}}{\partial t^j}(t)\right| = \left|\frac{\partial^j (f/\varphi)}{\partial t^j}(t)\right| \le M < \infty,$$
for all $j = 0, \dots, n$ and $f \in F$.

- *F is equicontinuous, that is, for all $\varepsilon \in \mathbb{R}^+$ there exists some $\delta \in \mathbb{R}^+$ such that*
$$\left|\frac{\partial^j \widetilde{f}}{\partial t^j}(r) - \frac{\partial^j \widetilde{f}}{\partial t^j}(s)\right| = \left|\frac{\partial^j (f/\varphi)}{\partial t^j}(r) - \frac{\partial^j (f/\varphi)}{\partial t^j}(s)\right| < \varepsilon,$$
for all $j = 0, \dots, n$, $f \in F$ and $r, s \in \mathbb{R}$ such that $|r - s| < \delta$.

More properties of these spaces can be found in [2].

4 Eigenvalue criteria

In this section we will study the existence of fixed points of an operator T on $\widetilde{\mathscr{C}}^n_\varphi$ given by equation (1.1). In particular, we will look for solutions of the previous integral equation in abstract cones, which will be defined following the line of [7]. In that work, the authors considered a real normed space $(N, \|\cdot\|)$ and a continuous functional $\alpha\colon N \to \mathbb{R}$. They proved that, when α satisfies the three following properties:

(P_1) $\alpha(u+v) \ge \alpha(u) + \alpha(v)$, for all $u, v \in N$;

(P_2) $\alpha(\lambda u) \ge \lambda\, \alpha(u)$, for all $u \in N$, $\lambda \ge 0$;

(P_3) $[\alpha(u) \ge 0,\ \alpha(-u) \ge 0] \Rightarrow u \equiv 0$;

then
$$K_\alpha = \{u \in N \ :\ \alpha(u) \ge 0\}$$
is a cone.

This way, we will consider the abstract cone
$$K_\alpha = \left\{u \in \widetilde{\mathscr{C}}^n_\varphi \ :\ \alpha(u) \ge 0\right\},$$
where $\alpha\colon \widetilde{\mathscr{C}}^n_\varphi \to \mathbb{R}$ is a functional satisfying (P_1)–(P_3).

Remark 4.1. If the cone K is defined by a continuous functional α (as it will occur with the cones considered in this paper), then v an element of the cone will belong to its interior if and only if $\alpha(v) > 0$.

In order to state our eigenvalue comparison results, we consider the following operator on $\widetilde{\mathscr{C}}^n_\varphi$:
$$L_1 u(t) := \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\, u(s)\, \mathrm{d}s, \quad t \in \mathbb{R}.$$

Consider P, the cone of nonnegative functions in $\widetilde{\mathscr{C}}^n_\varphi$, that is
$$P := \left\{u \in \widetilde{\mathscr{C}}^n_\varphi \ :\ u \ge 0 \text{ on } \mathbb{R}\right\}.$$

In this section we will assume the following hypotheses:

(C_1) The kernel $k:\mathbb{R}^2\to\mathbb{R}$, is such that $k(\cdot,s)\,\eta(s)\in\widetilde{\mathscr{C}}^n_\varphi$ for every $s\in\mathbb{R}$. Moreover,

- if $n=0$, then for every $\varepsilon>0$, there exist $\delta>0$ and a measurable function ω_0 such that if $|t_1-t_2|<\delta$ then

  (i)
  $$\left|\frac{k(t_1,s)\,\eta(s)}{\varphi(t_1)}-\frac{k(t_2,s)\,\eta(s)}{\varphi(t_2)}\right|<\varepsilon\,\omega_0(s)$$

  and

  (ii)
  $$\left|\frac{(k(t_1,s)\,\eta(s))^+}{\varphi(t_1)}-\frac{(k(t_2,s)\,\eta(s))^+}{\varphi(t_2)}\right|<\varepsilon\,\omega_0(s),$$

  for a. e. $s\in\mathbb{R}$. Here, as usual, $(k(t,s)\,\eta(s))^+=\max\{k(t,s)\,\eta(s),0\}$. Moreover, we note that (i) implies that

  $$\left|\frac{|k(t_1,s)\,\eta(s)|}{\varphi(t_1)}-\frac{|k(t_2,s)\,\eta(s)|}{\varphi(t_2)}\right|<\varepsilon\,\omega_0(s),$$

  for a. e. $s\in\mathbb{R}$.

- if $n>0$, $k(t,s)\,\eta(s)\geq 0$ and for every $\varepsilon>0$ and $j=0,\dots,n$, there exist $\delta>0$ and a measurable function ω_j such that if $|t_1-t_2|<\delta$ then

  $$\left|\frac{\partial^j(k/\varphi)}{\partial t^j}(t_1,s)\,\eta(s)-\frac{\partial^j(k/\varphi)}{\partial t^j}(t_2,s)\,\eta(s)\right|<\varepsilon\,\omega_j(s)$$

  for a. e. $s\in\mathbb{R}$.

(C_2) It holds that $\omega_j\,\varphi$, $\frac{\partial^j k}{\partial t^j}(t,\cdot)\,\eta(\cdot)\,\varphi(\cdot)\in\mathrm{L}^1(\mathbb{R})$ for every $t\in\mathbb{R}$, $j=0,\dots,n$; and

$$\frac{\partial^{j-l}}{\partial t^{j-l}}\frac{1}{\varphi}(t)\int_{-\infty}^{+\infty}\left|\frac{\partial^l k}{\partial t^l}(t,s)\,\eta(s)\right|\,\varphi(s)\,\mathrm{d}s\in\mathrm{L}^\infty(\mathbb{R}),$$

for all $j=0,\dots,n$; $l=0,\dots,j$.

Moreover, defining

$$z_{(\pm)}(s):=\lim_{t\to\pm\infty}\frac{|k(t,s)\,\eta(s)|}{\varphi(t)}$$

and

$$M_j(s):=\sup_{t\in\mathbb{R}}\left|\frac{\partial^j(k/\varphi)}{\partial t^j}(t,s)\,\eta(s)\right|,$$

it is satisfied that $z_{(\pm)}\,\varphi$, $M_j\,\varphi\in L^1(\mathbb{R})$, for $j=0,\dots,n$.

(C_3) $f:\mathbb{R}\times\mathbb{R}\to[0,+\infty)$ satisfies a sort of L^∞-Carathéodory conditions, that is, $f(\cdot,y)$ is measurable for each fixed $y\in\mathbb{R}$, $f(t,\cdot)$ is continuous for a. e. $t\in\mathbb{R}$, and, for each $r>0$, there exists $\phi_r\in\mathrm{L}^\infty(\mathbb{R})$ such that

$$\frac{f(t,x\varphi(t))}{\varphi(t)}\leq\phi_r(t),$$

for all $x\in[-r,r]$ and a. e. $t\in\mathbb{R}$.

(C_4) $\alpha(|k(\cdot,s)\,\eta(s)|)\geq 0$ for a. e. $s\in\mathbb{R}$.

(C_5) $\alpha(|k(\cdot,s)\,\eta(s)|)\,\varphi(s)\in \mathrm{L}^1(\mathbb{R})$ and

$$\alpha(L_1 u)\geq \int_{-\infty}^{\infty} \alpha(|k(\cdot,s)\,\eta(s)|)\,u(s)\,\mathrm{d}s \text{ for all } u\in P.$$

(C_6) There exists $A\subset\mathbb{R}$ such that A is a finite union of compact intervals and $k(t,s)\,\eta(s)\geq 0$, $k(\cdot,s)\,\eta(s)\not\equiv 0$ for every $t\in A$ and a. e. $s\in\mathbb{R}$. Moreover, it holds that

$$\frac{1}{\widetilde{M}(A)}=\frac{1}{\widetilde{M}}:=\inf_{t\in A}\int_A k(t,s)\eta(s)\,\mathrm{d}s>0.$$

We will also define the following auxiliary operator on $\widetilde{\mathscr{C}}^n_\varphi$:

$$L_2\,u(t):=\int_A (k(t,s)\,\eta(s))^+\,u(s)\,\mathrm{d}s,\quad t\in\mathbb{R}.$$

With regard to operator L_2, we will consider the following assumptions:

(C_7) $\alpha((k(\cdot,s)\eta(s))^+)\geq 0$ for a. e. $s\in\mathbb{R}$.

(C_8) $\alpha\,((k(\cdot,s)\,\eta(s))^+)\,\varphi(s)\in \mathrm{L}^1(A)$ and

$$\alpha\,(L_2 u)\geq \int_A \alpha\left((k(\cdot,s)\,\eta(s))^+\right)\,u(s)\,\mathrm{d}s \text{ for all } u\in P.$$

Finally, to ensure that operator T maps the cone K_α into itself, we need to ask for the following conditions:

(C_9) $\alpha(k(\cdot,s)\eta(s))\geq 0$ for a. e. $s\in\mathbb{R}$.

(C_{10}) $\alpha\,(k(\cdot,s)\,\eta(s))\,\varphi(s)\in \mathrm{L}^1(\mathbb{R})$ for a. e. $s\in\mathbb{R}$ and

$$\alpha\,(Tu)\geq \int_{-\infty}^{\infty} \alpha\,(k(\cdot,s)\,\eta(s))\,f(s,u(s))\,\mathrm{d}s \text{ for all } u\in K_\alpha.$$

Theorem 4.2. *If* (C_1), (C_2), (C_4) *and* (C_5) *hold, then operator* L_1 *is continuous, compact and maps* P *into* $P\cap K_\alpha$.

Proof. We will distinguish two different cases:

CASE I: $n=0$:

L_1 maps $(\widetilde{\mathscr{C}}_\varphi,\|\cdot\|_\varphi)$ *to* $(\widetilde{\mathscr{C}}_\varphi,\|\cdot\|_\varphi)$: Let $u\in\widetilde{\mathscr{C}}_\varphi$. From (C_1), (i), given $\varepsilon\in\mathbb{R}^+$, there exists some $\delta\in\mathbb{R}^+$ such that for t_1, $t_2\in\mathbb{R}$, $|t_1-t_2|<\delta$ it is satisfied that

$$\begin{aligned}\left|\widetilde{L_1 u}(t_1)-\widetilde{L_1 u}(t_2)\right| &\leq \int_{-\infty}^{+\infty}\left|\frac{|k(t_1,s)\,\eta(s)|}{\varphi(t_1)}-\frac{|k(t_2,s)\,\eta(s)|}{\varphi(t_2)}\right|\,|u(s)|\,\mathrm{d}s\\ &\leq \varepsilon\int_{-\infty}^{\infty}\omega_0(s)\,|u(s)|\,\mathrm{d}s\leq \varepsilon\int_{-\infty}^{\infty}\omega_0(s)\,\frac{|u(s)|}{\varphi(s)}\,\varphi(s)\,\mathrm{d}s \qquad (4.1)\\ &\leq \varepsilon\,\|u\|_\varphi\int_{-\infty}^{\infty}\omega_0(s)\,\varphi(s)\,\mathrm{d}s\end{aligned}$$

and since, by (C_2), $\omega_0\,\varphi \in \mathrm{L}^1(\mathbb{R})$, the previous expression is bounded from above by $\varepsilon\,\|u\|_\varphi\,c$ for some positive constant c. Hence, $\widetilde{L_1 u}$ is continuous in $\mathbb{R}$. Now we will prove that there exists

$$\lim_{t\to\pm\infty}\widetilde{L_1 u}(t)=\lim_{t\to\pm\infty}\frac{L_1u(t)}{\varphi(t)}=\lim_{t\to\pm\infty}\frac{1}{\varphi(t)}\int_{-\infty}^{\infty}|k(t,s)\,\eta(s)|\,u(s)\,\mathrm{d}s\in\mathbb{R}.$$

Since $k(\cdot,s)\,\eta(s)\in\widetilde{\mathscr{C}}_\varphi$, then, for all $s\in\mathbb{R}$, there exists

$$\lim_{t\to\pm\infty}\frac{|k(t,s)\,\eta(s)|}{\varphi(t)}=:z_{(\pm)}(s)\in\mathbb{R}.$$

On the other hand, for all $t\in\mathbb{R}$ and a. e. $s\in\mathbb{R}$:

$$\left|\frac{|k(t,s)\,\eta(s)|}{\varphi(t)}u(s)\right|\le M_0(s)\,|u(s)|=M_0(s)\,\frac{|u(s)|}{\varphi(s)}\varphi(s)\le\|u\|_\varphi\,M_0(s)\,\varphi(s)$$

and, from (C_2), $M_0\,\varphi\in\mathrm{L}^1(\mathbb{R})$. Thus, from Lebesgue's Dominated Convergence Theorem,

$$\lim_{t\to\pm\infty}\frac{1}{\varphi(t)}\int_{-\infty}^{\infty}|k(t,s)\,\eta(s)|\,u(s)\,\mathrm{d}s=\int_{-\infty}^{\infty}\lim_{t\to\pm\infty}\frac{|k(t,s)\,\eta(s)|}{\varphi(t)}u(s)\,\mathrm{d}s=\int_{-\infty}^{\infty}z_{(\pm)}(s)\,u(s)\,\mathrm{d}s,$$

and, since,

$$\left|\int_{-\infty}^{\infty}z_{(\pm)}(s)\,u(s)\,\mathrm{d}s\right|\le\int_{-\infty}^{\infty}z_{(\pm)}(s)\,|u(s)|\,\mathrm{d}s\le\|u\|_\varphi\int_{-\infty}^{\infty}z_{(\pm)}(s)\,\varphi(s)\,\mathrm{d}s\in\mathbb{R},$$

we deduce that $z_{(\pm)}\,u\in\mathrm{L}^1(\mathbb{R})$. Therefore there exists $\lim\limits_{t\to\pm\infty}\frac{L_1u(t)}{\varphi(t)}$. Consequently, $L_1u\in\widetilde{\mathscr{C}}_\varphi$.

Continuity: It is obvious from the linearity and boundedness of operator L_1.

Compactness: Let $B\subset\widetilde{\mathscr{C}}_\varphi$ a bounded set, that is, there exists some $R>0$ such that $\|u\|_\varphi\le R$ for all $u\in B$. Then,

$$\begin{aligned}\|L_1u\|_\varphi=\left\|\widetilde{L_1u}\right\|_\infty=\left\|\frac{L_1u}{\varphi}\right\|_\infty&=\left\|\frac{1}{\varphi(\cdot)}\int_{-\infty}^{+\infty}|k(\cdot,s)\,\eta(s)|\,u(s)\,\mathrm{d}s\right\|_\infty\\&\le\|u\|_\varphi\left\|\frac{1}{\varphi(\cdot)}\int_{-\infty}^{+\infty}|k(\cdot,s)\,\eta(s)|\;\varphi(s)\,\mathrm{d}s\right\|_\infty\\&\le R\left\|\frac{1}{\varphi(\cdot)}\int_{-\infty}^{+\infty}|k(\cdot,s)\,\eta(s)|\;\varphi(s)\,\mathrm{d}s\right\|_\infty<+\infty,\end{aligned}\tag{4.2}$$

and we have obtained an upper bound which does not depend on u. Therefore it is clear that the set $L_1(B)$ is totally bounded.

On the other hand, taking into account the upper bound found in (4.1), we have that if $t_1,\ t_2\in\mathbb{R}$ are such that $|t_1-t_2|<\delta$ then

$$\left|\widetilde{L_1u}(t_1)-\widetilde{L_1u}(t_2)\right|\le\varepsilon\,\|u\|_\varphi\int_{-\infty}^{\infty}\omega_0(s)\,\varphi(s)\,\mathrm{d}s\le\varepsilon R\int_{-\infty}^{\infty}\omega_0(s)\,\varphi(s)\,\mathrm{d}s,$$

and, since $\omega_0\,\varphi\in\mathrm{L}^1(\mathbb{R})$, we conclude that $L_1(B)$ is equicontinuous.

In conclusion, we derive, by application of Ascoli-Arzela's Theorem, that $L_1(B)$ is relatively compact in $\widetilde{\mathscr{C}}_\varphi$ and therefore L_1 is a compact operator.

L_1 maps P to $P \cap K_\alpha$: Since L_1 has a positive integral kernel, it clearly maps P into P. Finally, it maps P into $P \cap K_\alpha$ as a direct consequence of hypothesis (C_4) and (C_5).

CASE II: $n \neq 0$:

We note that in this case we have the additional hypothesis that $k(\cdot,s)\,\eta(s)$ is nonnegative for all $s \in \mathbb{R}$. As a consequence, we will omit the absolute value in the definition of $L_1 u$.

L_1 maps $(\widetilde{\mathscr{C}}^n_\varphi, \|\cdot\|_\varphi)$ to $(\widetilde{\mathscr{C}}^n_\varphi, \|\cdot\|_\varphi)$: Let $u \in \widetilde{\mathscr{C}}^n_\varphi$. Since $\frac{k(\cdot,s)\,\eta(s)}{\varphi(\cdot)}$ is integrable for every $s \in \mathbb{R}$, we can use Leibniz's Integral Rule to get

$$\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t) = \frac{\partial^j (L_1 u/\varphi)}{\partial t^j}(t) = \int_{-\infty}^{+\infty} \frac{\partial^j (k/\varphi)}{\partial t^j}(t,s)\,\eta(s)\,u(s)\,\mathrm{d}s.$$

On the other hand, from (C_1), given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for $t_1,\ t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\begin{aligned}\left|\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t_1) - \frac{\partial^j \widetilde{L_1 u}}{\partial t^j}(t_2)\right| &\le \int_{-\infty}^{+\infty} \left|\frac{\partial^j (k/\varphi)}{\partial t^j}(t_1,s)\,\eta(s) - \frac{\partial^j (k/\varphi)}{\partial t^j}(t_2,s)\,\eta(s)\right|\,|u(s)|\,\mathrm{d}s \\ &\le \varepsilon \int_{-\infty}^{\infty} \omega_j(s)\,|u(s)|\,\mathrm{d}s \le \varepsilon\,\|u\|_\varphi \int_{-\infty}^{\infty} \omega_j(s)\,\varphi(s)\,\mathrm{d}s.\end{aligned} \tag{4.3}$$

Since $\omega_j\,\varphi \in \mathrm{L}^1(\mathbb{R})$, the previous expression is bounded from above by $\varepsilon\,\|u\|_\varphi\,c$ for some positive constant c. Hence, $\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}$ is continuous in $\mathbb{R}$ for $j = 0, \dots, n$, that is, $\widetilde{L_1 u} \in \mathscr{C}^n(\mathbb{R},\mathbb{R})$.

Analogously to Case I, it can be proved that there exists $\lim\limits_{t\to\pm\infty} \widetilde{L_1 u}(t)$ and, consequently, $L_1 u \in \widetilde{\mathscr{C}}^n_\varphi$.

Continuity: Again, it is obvious from the linearity and boundedness of operator L_1.

Compactness: Let $B \subset \widetilde{\mathscr{C}}^n_\varphi$ be a bounded set, that is, there exists $R > 0$ such that $\|u\|_\varphi \le R$ for all $u \in B$.

Using the General Leibniz's Rule (for differentiation), it is clear that

$$\frac{\partial^j \widetilde{L_1 u}}{\partial t^j} = \frac{\partial^j (L_1 u/\varphi)}{\partial t^j} = \sum_{l=0}^{j} \binom{j}{l} \frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}.$$

Moreover, from Leibniz's Integral Rule,

$$\frac{\partial^l L_1 u}{\partial t^l}(t) = \int_{-\infty}^{\infty} \frac{\partial^l k}{\partial t^l}(t,s)\,\eta(s)\,u(s)\,\mathrm{d}s.$$

Thus,

$$\begin{aligned}\left\|\frac{\partial^j \widetilde{L_1 u}}{\partial t^j}\right\|_\infty &= \left\|\sum_{l=0}^{j} \binom{j}{l} \frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}\right\|_\infty \le \sum_{l=0}^{j} \binom{j}{l} \left\|\frac{\partial^l L_1 u}{\partial t^l} \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}\right\|_\infty \\ &= \sum_{l=0}^{j} \binom{j}{l} \left\|\frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi(\cdot)} \int_{-\infty}^{+\infty} \frac{\partial^l k}{\partial t^l}(\cdot,s)\,\eta(s)\,u(s)\,\mathrm{d}s\right\|_\infty.\end{aligned}$$

It is satisfied that

$$\left|\frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \int_{-\infty}^{+\infty} \frac{\partial^l k}{\partial t^l}(t,s)\,\eta(s)\,u(s)\,\mathrm{d}s\right| \le \left|\frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t)\right| \int_{-\infty}^{+\infty} \left|\frac{\partial^l k}{\partial t^l}(t,s)\,\eta(s)\right|\,|u(s)|\,\mathrm{d}s$$

$$\leq R \left| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi}(t) \right| \int_{-\infty}^{+\infty} \left| \frac{\partial^l k}{\partial t^l}(t,s)\,\eta(s) \right| \varphi(s)\,\mathrm{d}s,$$

and so, from two previous inequalities and taking into account condition (C_2), we deduce that

$$\left\| \frac{\partial^j \widetilde{L_1 u}}{\partial t^j} \right\|_\infty \leq \|u\|_\varphi \sum_{l=0}^{j} \binom{j}{l} \left\| \frac{\partial^{j-l}}{\partial t^{j-l}} \frac{1}{\varphi(\cdot)} \int_{-\infty}^{+\infty} \left| \frac{\partial^l k}{\partial t^l}(\cdot,s)\,\eta(s) \right| \varphi(s)\,\mathrm{d}s \right\|_\infty < +\infty. \tag{4.4}$$

The rest of the proof is analogous to Case I but using equation (4.3) instead of (4.1).

L_1 maps P to $P \cap K_\alpha$: The proof is the same than in Case I. ■

Theorem 4.3. *If (C_1), (C_2) and (C_6)–(C_8) hold, then operator L_2 is continuous, compact and maps P into $P \cap K_\alpha$.*

Proof. We will distinguish two different cases:

CASE I: $n = 0$**:**

L_2 maps $(\widetilde{\mathscr{C}}_\varphi, \|\cdot\|_\varphi)$ to $(\widetilde{\mathscr{C}}_\varphi, \|\cdot\|_\varphi)$: Let $u \in \widetilde{\mathscr{C}}_\varphi$. Since $k(\cdot,s)\,\eta(s) \in \widetilde{\mathscr{C}}_\varphi$ for all $s \in \mathbb{R}$, it is clear that $\left(\frac{k(\cdot,s)\,\eta(s)}{\varphi(\cdot)}\right)^+ \equiv \frac{(k(\cdot,s)\,\eta(s))^+}{\varphi(\cdot)} \in \mathscr{C}(\mathbb{R})$ for all $s \in \mathbb{R}$.

Analogously to the proof for L_1, from (C_1), (ii), given $\varepsilon \in \mathbb{R}^+$, there exists some $\delta \in \mathbb{R}^+$ such that for t_1, $t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$ it is satisfied that

$$\left| \widetilde{L_2 u}(t_1) - \widetilde{L_2 u}(t_2) \right| \leq \varepsilon\, \|u\|_\varphi \int_A \omega_0(s)\,\varphi(s)\,\mathrm{d}s \tag{4.5}$$

and, since $\omega_0\,\varphi \in \mathrm{L}^1(\mathbb{R})$, it can be deduced that $\widetilde{L_2 u}$ is continuous in $\mathbb{R}$.

It is left to see that there exists

$$\lim_{t\to\pm\infty} \widetilde{L_2 u}(t) = \lim_{t\to\pm\infty} \frac{L_2 u(t)}{\varphi(t)} = \lim_{t\to\pm\infty} \frac{1}{\varphi(t)} \int_A (k(t,s)\,\eta(s))^+\, u(s)\,\mathrm{d}s \in \mathbb{R}.$$

Reasoning as before, since $k(\cdot,s)\,\eta(s) \in \widetilde{\mathscr{C}}_\varphi$, then for all $s \in \mathbb{R}$ it is ensured the existence of

$$0 \leq \lim_{t\to\pm\infty} \frac{(k(t,s)\,\eta(s))^+}{\varphi(t)} \leq \lim_{t\to\pm\infty} \frac{|k(t,s)\,\eta(s)|}{\varphi(t)} = z_{(\pm)}(s) \in \mathbb{R}.$$

On the other hand,

$$\left| \frac{(k(t,s)\,\eta(s))^+}{\varphi(t)} u(s) \right| \leq \left| \frac{|k(t,s)\,\eta(s)|}{\varphi(t)} |u(s)| \right| \leq M_0(s)\,|u(s)| = M_0(s) \frac{|u(s)|}{\varphi(s)} \varphi(s) \leq \|u\|_\varphi\, M_0(s)\,\varphi(s)$$

for all $t \in \mathbb{R}$. From (C_2), $M_0\,\varphi \in \mathrm{L}^1(\mathbb{R})$ and so $M_0\,\varphi \in \mathrm{L}^1(A)$. Thus, from Lebesgue's Dominated Convergence Theorem,

$$\lim_{t\to\pm\infty} \frac{1}{\varphi(t)} \int_A (k(t,s)\,\eta(s))^+\, u(s)\,\mathrm{d}s = \int_A \lim_{t\to\pm\infty} \frac{(k(t,s)\,\eta(s))^+}{\varphi(t)} u(s)\,\mathrm{d}s,$$

and since

$$\left| \int_A \lim_{t\to\pm\infty} \frac{(k(t,s)\,\eta(s))^+}{\varphi(t)} u(s)\,\mathrm{d}s \right| \leq \int_A z_{(\pm)}(s)\,|u(s)|\,\mathrm{d}s \leq \|u\|_\varphi \int_A z_{(\pm)}(s)\,\varphi(s)\,\mathrm{d}s \in \mathbb{R},$$

and $z_{(\pm)}\varphi \in \mathrm{L}^1(A)$, it can be concluded that there exists $\lim\limits_{t\to\pm\infty} \frac{L_2u(t)}{\varphi(t)}$ and consequently $L_2u \in \widetilde{\mathscr{C}}_\varphi$.

Continuity: It is obvious from the linearity and boundedness of operator L_2.

Compactness: The proof is analogous to the one for operator L_1 (Theorem 4.2) by considering inequalities

$$\left\|\widetilde{L_2u}\right\|_\infty = \left\|\frac{L_2u}{\varphi}\right\|_\infty = \left\|\frac{1}{\varphi(\cdot)}\int_A (k(\cdot,s)\,\eta(s))^+\,u(s)\,\mathrm{d}s\right\|_\infty \leq \|u\|_\varphi \left\|\frac{1}{\varphi(\cdot)}\int_A (k(\cdot,s)\,\eta(s))^+\,\varphi(s)\,\mathrm{d}s\right\|_\infty$$

$$\leq \|u\|_\varphi \left\|\frac{1}{\varphi(\cdot)}\int_A |k(\cdot,s)\,\eta(s)|\,\varphi(s)\,\mathrm{d}s\right\|_\infty \leq \|u\|_\varphi \left\|\frac{1}{\varphi(\cdot)}\int_{-\infty}^{\infty} |k(\cdot,s)\,\eta(s)|\,\varphi(s)\,\mathrm{d}s\right\|_\infty,$$

and (4.5) instead of (4.2) and (4.1), respectively.

L_2 maps P to $P \cap K_\alpha$: Since L_2 has a positive integral kernel, it clearly maps P into P. Finally, it maps P into $P \cap K_\alpha$ as a direct consequence of hypothesis (C_7) and (C_8).

CASE II: $n \neq 0$**:** The proof is analogous to the one made for operator L_1, with some small changes in the line of those introduced in Case I. ■

Analogously to the two previous theorems, it can be proved that operator T satisfies the following properties.

Theorem 4.4. *If (C_1)–(C_3), (C_9) and (C_{10}) hold, the operator T is continuous, compact and maps K_α into K_α.*

Proof. The proof, except for the continuity, is analogous to previous theorems but using the following inequality

$$f(s,u(s)) = f\left(s, \frac{u(s)}{\varphi(s)}\varphi(s)\right) \leq \phi_{\|u\|_\varphi}(s)\,\varphi(s) \leq \left\|\phi_{\|u\|_\varphi}\right\|_\infty \varphi(s),$$

instead of $u(s) \leq \|u\|_\varphi\,\varphi(s)$.

Continuity: Since T is not a linear operator, continuity can not be deduced from boundedness, on the contrary to previous theorems. Therefore, we shall prove that operator T is continuous in a different way:

Let $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathscr{C}}_\varphi$ be a sequence which converges to u in $\widetilde{\mathscr{C}}_\varphi$. Then, there exists some $R \in \mathbb{R}$ such that $\|u_n\|_\varphi \leq R$ for all $n \in \mathbb{N}$ and so it holds that

$$f(s,u_n(s)) = f\left(s, \frac{u_n(s)}{\varphi(s)}\varphi(s)\right) \leq \phi_R(s)\,\varphi(s) \leq \|\phi_R\|_\infty\,\varphi(s),$$

where we have used condition (C_3).

Moreover, from (C_3), it holds that $f(s,u_n(s)) \to f(s,u(s))$ for a. e. $s \in \mathbb{R}$.

It is clear that, for all $t \in \mathbb{R}$ and $j \in \{0,\dots,n\}$,

$$\left|\frac{\partial^j \widetilde{Tu_n}}{\partial t^j}(t) - \frac{\partial^j \widetilde{Tu}}{\partial t^j}(t)\right| \leq \int_{-\infty}^{\infty} \left|\frac{\partial^j (k/\varphi)}{\partial t^j}(t,s)\,\eta(s)\right| |f(s,u_n(s)) - f(s,u(s))|\,\mathrm{d}s$$

and, using (C_2),

$$\left|\frac{\partial^j \widetilde{Tu_n}}{\partial t^j}(t) - \frac{\partial^j \widetilde{Tu}}{\partial t^j}(t)\right| \leq \int_{-\infty}^{\infty} M_j(s)\,|f(s,u_n(s)) - f(s,u(s))|\,\mathrm{d}s$$

$$\leq 2\,\|\phi_R\|_\infty \int_{-\infty}^{\infty} M_j(s)\,\varphi(s)\,\mathrm{d}s < \infty.$$

Now we deduce, by application of Lebesgue's Dominated Convergence Theorem, that

$$\lim_{n\to\infty}\left\|\frac{\partial^j\widetilde{Tu_n}}{\partial t^j}-\frac{\partial^j\widetilde{Tu}}{\partial t^j}\right\|_\infty \le \lim_{n\to\infty}\int_{-\infty}^{\infty} M_j(s)\,|f(s,u_n(s))-f(s,u(s))|\,\mathrm{d}s$$
$$=\int_{-\infty}^{\infty}\lim_{n\to\infty} M_j(s)\,|f(s,u_n(s))-f(s,u(s))|\,\mathrm{d}s=0.$$

Therefore, we deduce that $Tu_n \to Tu$ in $\widetilde{\mathscr{C}}^n_\varphi$ and so T is continuous.

■

The following theorem proves that the spectral radius of both operators L_1 and L_2 are positive and their eigenfunctions have constant sign. This result is analogous to [10, Theorem 4.5] and is proven using the facts that the considered operators leave P invariant and that P is a total cone, combined with Krein-Rutman Theorem.

Theorem 4.5. *Assume conditions* (C_1), (C_2) *and* (C_4)–(C_8) *hold. Then, it holds that* $r(L_1)>0$ *is an eigenvalue of* L_1 *with an eigenfunction in* $P\setminus\{0\}$.

Analogously, $r(L_2)>0$ *is an eigenvalue of* L_2 *with an eigenfunction in* $P\setminus\{0\}$.

Proof. We will prove the result for L_1. Consider $v\in P$ such that $v\equiv 1$ in A. Then, for $t\in A$,

$$L_1v(t)=\int_{-\infty}^{\infty}|k(t,s)\,\eta(s)|\,v(s)\,\mathrm{d}s\ge\int_A|k(t,s)\,\eta(s)|\,v(s)\,\mathrm{d}s=\int_A k(t,s)\,\eta(s)\,\mathrm{d}s\ge\frac{1}{\widetilde{M}},$$

with $\frac{1}{\widetilde{M}}$ given in (C_6).

Then, there exists some open and bounded set B, with $A\subset B$, such that when $t\in B$,

$$\int_A|k(t,s)\,\eta(s)|\,\mathrm{d}s\ge\frac{1}{2\widetilde{M}}.$$

Now, defining $u(t)=1$ for $t\in A$ and $u(t)=0$ when $t\notin B$, from Whitney's Extension Theorem [20, Theorem I], u can be extended to $\mathbb{R}$ (and this extension will be also denoted by u) as a function of class n. Moreover, from the proof of Whitney's Extension Theorem, it is possible to deduce that this extension will be nonnegative and upperly bounded by 1.

Finally, since $u(t)=0$ when $t\notin B$ and B is a bounded set, then it is clear that $\lim\limits_{t\to\pm\infty}u(t)=0$ and $u\in\widetilde{\mathscr{C}}^n_\varphi$, with independence of the choice of φ.

Therefore, for $t\in B$, it holds that

$$L_1u(t)=\int_{-\infty}^{\infty}|k(t,s)\,\eta(s)|\,u(s)\,\mathrm{d}s\ge\int_A|k(t,s)\,\eta(s)|\,u(s)\,\mathrm{d}s=\int_A|k(t,s)\,\eta(s)|\,\mathrm{d}s\ge\frac{1}{2\widetilde{M}}$$
$$\ge\frac{1}{2\widetilde{M}}\,u(t),$$

and for $t\notin B$,

$$L_1u(t)=\int_{-\infty}^{\infty}|k(t,s)\,\eta(s)|\,u(s)\,\mathrm{d}s\ge 0=\frac{1}{2\widetilde{M}}\,u(t).$$

Thus, as a consequence of Theorem 2.3, we conclude that $r(L_1)\ge\frac{1}{2\widetilde{M}}>0$.

Finally, since P is a total cone and L_1 maps P into P, Krein-Rutman Theorem assures that $r(L_1)$ is an eigenvalue with an eigenvector $\phi\in P\setminus\{0\}$. ■

Remark 4.6. As a consequence of Theorems 4.2 and 4.5, we know that the eigenfunctions mentioned above are in $P \cap K_\alpha$.

We will define now the following operator on $\mathscr{C}^n(A,\mathbb{R})$

$$\bar{L}u(t) := \int_A k(t,s)\,\eta(s)\,u(s)\,\mathrm{d}s, \quad t \in A,$$

and consider the cone P_A of positive functions in $\mathscr{C}^n(A,\mathbb{R})$.

As with previous operators, we will prove that $\bar{L}$ satisfies the following properties.

Theorem 4.7. *Assume that conditions* (C_1), (C_2) *and* (C_6)–(C_8) *hold. Then, operator* $\bar{L}$ *is compact and maps* P_A *into* P_A.

Proof. Let $f \in \mathscr{C}^n(A,\mathbb{R})$ and $B \subset \mathbb{R}$ an open and bounded set such that $A \subset B$. Define now $g(t) = f(t)$ for $t \in A$ and $g(t) = 0$ for $t \in \mathbb{R} \setminus B$. Then, from Whitney's Extension Theorem [20, Theorem I], g can be extended to $\mathbb{R}$ as a function of class n, that is, there exists an extension of f to $\mathbb{R}$ as a function of class n such that this extension vanishes for $t \in \mathbb{R} \setminus B$. Obviously, this extension of f belongs to $\mathscr{C}^n_\varphi(\mathbb{R})$.

Now, denote by i the function which maps a function in $\mathscr{C}^n(A,\mathbb{R})$ to the aforementioned extension in $\mathscr{C}^n_\varphi(\mathbb{R})$ and by π the map which takes every function in $\mathscr{C}^n_\varphi(\mathbb{R})$ to its restriction to the set A (which clearly belongs to $\mathscr{C}^n(A,\mathbb{R})$). We obtain the following diagram:

$$\begin{tikzcd}
\mathscr{C}^n_\varphi(\mathbb{R}) \arrow[r, "L_2"] & \mathscr{C}^n_\varphi(\mathbb{R}) \arrow[d, "\pi"] \\
\mathscr{C}^n(A,\mathbb{R}) \arrow[u, "i"] \arrow[r, "\bar{L}"] & \mathscr{C}^n(A,\mathbb{R})
\end{tikzcd}$$

Let us show now that it is commutative. Consider $f \in \mathscr{C}^n(A,\mathbb{R})$. It holds that

$$\begin{aligned}(\pi \circ L_2 \circ i)(f)(t) &= \pi\left(\int_A (k(t,s)\,\eta(s))^+\, i(f)(s)\,\mathrm{d}s\right) = \pi\left(\int_A (k(t,s)\,\eta(s))^+\, f(s)\,\mathrm{d}s\right) \\ &= \int_A k(t,s)\,\eta(s)\,f(s)\,\mathrm{d}s = \bar{L}(f)(t), \quad t \in A.\end{aligned}$$

Now, since L_2 is compact and both i and π are continuous, we deduce that $\bar{L}$ is a compact operator.

Finally, from (C_6) it is clear that $\bar{L}$ maps P_A into P_A. ■

Remark 4.8. We point out that, in the previous proof, Whitney's extension theorem can be used as a consequence of the fact that A is a finite union of compact intervals.

Theorem 4.9. *It holds that* $r(\bar{L}) > 0$ *and it is an eigenvalue of* $\bar{L}$ *with an eigenfunction in* P_A.

Proof. Let ψ be the eigenfunction related to L_2 whose existence is proved in Theorem 4.5. Then, if we consider its restriction to A, $\psi|_A$, it is clear that for $t \in A$

$$\bar{L}\psi|_A(t) = L_2\psi(t) = r(L_2)\,\psi(t) = r(L_2)\,\psi|_A(t),$$

and so from Theorems 2.3 and 4.5, we deduce that $r(\bar{L}) \geq r(L_2) > 0$. ■

We define the following numbers in the extended real line:

$$f^0=\overline{\lim_{x\to 0}}\frac{\sup\limits_{t\in\mathbb{R}}\dfrac{f(t,x\varphi(t))}{\varphi(t)}}{|x|},\quad f_0=\underline{\lim_{x\to 0}}\frac{\inf\limits_{t\in A}\dfrac{f(t,x\varphi(t))}{\varphi(t)}}{|x|},$$

$$f^\infty=\overline{\lim_{|x|\to+\infty}}\frac{\sup\limits_{t\in\mathbb{R}}\dfrac{f(t,x\varphi(t))}{\varphi(t)}}{|x|},\quad f_\infty=\underline{\lim_{|x|\to+\infty}}\frac{\inf\limits_{t\in A}\dfrac{f(t,x\varphi(t))}{\varphi(t)}}{|x|}.$$

To prove that the index of some subsets of a cone is 1 or 0, we will use the following well-known sufficient conditions.

Let K be a cone in a Banach space X. If $\Omega\subset X$ is an open and bounded subset of K (in the relative topology), we denote by $\overline{\Omega}$ and $\partial\Omega$, respectively, its closure and its boundary relative to K. Moreover, we will note $\Omega_K=\Omega\cap K$, which is an open subset of K in the relative topology.

Lemma 4.10. *Let Ω be an open bounded set in K with $0\in\Omega_K$ and $\overline{\Omega_K}\neq K$. Assume that $F:\overline{\Omega_K}\to K$ is a continuous compact map such that $x\neq Fx$ for all $x\in\partial\Omega_K$. Then the fixed point index $i_K(F,\Omega_K)$ has the following properties.*

(1) If there exists $e\in K\backslash\{0\}$ such that $x\neq Fx+\lambda e$ for all $x\in\partial\Omega_K$ and all $\lambda\geq 0$, then $i_K(F,\Omega_K)=0$.

(2) If $\lambda x\neq Fx$ for all $x\in\partial\Omega_K$ and for every $\lambda\geq 1$, then $i_K(F,\Omega_K)=1$.

(3) If $i_K(F,\Omega_K)\neq 0$, then F has a fixed point in Ω_K.

(4) Let Ω^1 be open in X with $\overline{\Omega^1}\subset\Omega_K$. If $i_K(F,\Omega_K)=1$ and $i_K(F,\Omega^1_K)=0$, then F has a fixed point in $\Omega_K\backslash\overline{\Omega^1_K}$. The same result holds if $i_K(F,\Omega_K)=0$ and $i_K(F,\Omega^1_K)=1$.

Next, we will give a result in which we will prove that, under suitable conditions, the index of some subsets is 1 or 0. Before that, we shall give the following definition that will be implicitly used in Theorem 4.12.

Definition 4.11. Let, X,Y,Z be topological spaces, Y Hausdorff. Let $f:X\to Y$, $g:X\to Z$. Let $z_0\in g(X)'$. We say that L is the limit of f when $g(x)$ tends to z_0 if for every neighborhood N_Y of L there exists a neighborhood N_Z of z_0 such that $f\left(g^{-1}\left(N_Z\backslash\{z_0\}\right)\right)\subset N_Y$. We write

$$\lim_{g(x)\to z_0}f(x)=L.$$

A particular case of this definition would be the notion of limit in the case of the topology occurring when studying Stieltjes derivatives with respect to a function g (cf. [8, 15]).

In order to prove the following theorem, we adapt some of the proofs of [18, Theorems 3.2-3.5] to this new context.

Theorem 4.12. *Assume that (C_1)–(C_{10}) hold. Assume also that there exists $\beta:\mathcal{C}^n_\varphi\to[0,\infty)$ such that*

$$\lim_{\beta(u)\to 0}\|u\|_\varphi=0,\quad \lim_{\beta(u)\to+\infty}\|u\|_\varphi=+\infty,$$

and

$$\beta(u)\neq 0\Rightarrow u\neq 0.$$

Consider $K^{\beta,\rho}_\alpha:=\{u\in K_\alpha\ :\ \beta(u)<\rho\}$. We have the following:

(1) *If* $0 \le f^0 < \mu(L_1)$*, then there exists* $\rho_0 > 0$ *such that* $i_{K_\alpha}(T, K_\alpha^{\beta,\rho}) = 1$ *for each* $\rho \in (0, \rho_0]$*.*

(2) *If* $0 \le f^\infty < \mu(L_1)$*, then there exists* $R_0 > 0$ *such that* $i_{K_\alpha}(T, K_\alpha^{\beta,R}) = 1$ *for each* $R > R_0$*.*

(3) *If* $\mu(L_2) < f_0 \le \infty$*, then there exists* $\rho_0 > 0$ *such that* $i_{K_\alpha}(T, K_\alpha^{\beta,\rho}) = 0$ *for each* $\rho \in (0, \rho_0]$*.*

(4) *If* $\mu(L_2) < f_\infty \le \infty$*, then there exists* $R_1 > 0$ *such that* $i_{K_\alpha}(T, K_\alpha^{\beta,R}) = 0$ *for each* $R \ge R_1$*.*

Proof. (1) Let $\tau > 0$ be such that $f^0 < \mu(L_1) - \tau =: \xi$. Then there exists $\widetilde{\rho}_0 \in (0,1)$ such that, for all $x \in [-\widetilde{\rho}_0, \widetilde{\rho}_0]$ and almost every $t \in \mathbb{R}$, we have

$$f(t, x\varphi(t)) \le \xi |x| \varphi(t).$$

Also, since $\lim_{\beta(u)\to 0} \|u\|_\varphi = 0$, there is $\rho_0 < \widetilde{\rho}_0$ such that $\|u\|_\varphi < \widetilde{\rho}_0$ for $u \in \overline{K_\alpha^{\beta,\rho_0}}$.

Let $\rho \in (0, \rho_0]$. We prove that $Tu \ne \lambda u$ for $u \in \partial K_\alpha^{\beta,\rho}$ and $\lambda \ge 1$, which implies that $i_K(T, K_\alpha^{\beta,\rho}) = 1$. In fact, if we assume otherwise, then there exists $u \in \partial K_\alpha^{\beta,\rho}$, (that is, $\beta(u) = \rho$ and therefore, $u \not\equiv 0$) and $\lambda \ge 1$ such that $\lambda u = Tu$. Therefore, for $t \in \mathbb{R}$,

$$\begin{aligned} |u(t)| \le \lambda |u(t)| = |Tu(t)| &= \left| \int_{-\infty}^{\infty} k(t,s)\,\eta(s)\,f(s,u(s))\,\mathrm{d}s \right| \\ &\le \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\,f\left(s, \frac{u(s)}{\varphi(s)}\varphi(s)\right) \mathrm{d}s \le \xi \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)||u(s)|\,\mathrm{d}s \\ &= \xi\,(L_1|u|)(t). \end{aligned}$$

We conclude that $|u| \le \xi L_1 |u|$. Thus, as L_1 is a nondecreasing operator, iterating, we have that

$$|u| \le \xi\, L_1|u| \le \xi\, L_1\,(\xi\, L_1|u|) = \xi^2\, L_1^2|u| \le \cdots \le (\xi\, L_1)^n\,|u|.$$

That is,

$$\|u\|_\varphi \le \xi^n\, \|L_1^n|u|\,\|_\varphi$$

and, hence,

$$1 \le \xi^n\, \frac{\|L_1^n|u|\,\|_\varphi}{\|u\|_\varphi} \le \xi^n\, \|L_1^n\|_\varphi,$$

where $\|L_1^n\|_\varphi$ denotes the norm of the operator, namely

$$\|L_1^n\|_\varphi = \sup_{u \ne 0} \frac{\|L_1^n u\|_\varphi}{\|u\|_\varphi}.$$

Taking the n-th square root and the limit when $n \to \infty$,

$$1 \le \xi\, \left(\|L_1^n\|_\varphi\right)^{\frac{1}{n}} \to \xi\, r(L_1),$$

a contradiction.

(2) Let $\tau \in \mathbb{R}^+$ such that $f^\infty < \mu(L_1) - \tau =: \xi$. Then there exists $R_1 > 0$ such that for every $|x| \ge R_1$ and almost every $t \in \mathbb{R}$

$$f(t, x\varphi(t)) \le \xi\, |x| \varphi(t).$$

Also, by (C_3) there exists $\phi_{R_1} \in L^\infty(\mathbb{R})$ such that

$$\frac{f(t, x\varphi(t))}{\varphi(t)} \le \phi_{R_1}(t),$$

for all $x \in [-R_1, R_1]$ and almost every $t \in \mathbb{R}$. Hence,

$$f(t, x\varphi(t)) \le \xi |x| \varphi(t) + \varphi(t)\phi_{R_1}(t) \text{ for all } x \in \mathbb{R} \text{ and a. e. } t \in \mathbb{R}. \tag{4.6}$$

Moreover, since $\xi < \frac{1}{r(L_1)}$ we deduce that

$$r(\xi L_1) = \xi\, r(L_1) < 1.$$

Thus, if we denote by Id the identity operator, since ξL_1 has spectral radius less than one, $\mathrm{Id} - \xi L_1$ is invertible. Furthermore, by the Neumann series expression,

$$(\mathrm{Id} - \xi L_1)^{-1} = \sum_{k=0}^{\infty} (\xi L_1)^k$$

and therefore, $(\mathrm{Id} - \xi L_1)^{-1}$ maps P into $P \cap K_\alpha$, since L_1 does.

Since $\phi_{R_1} \in L^\infty(\mathbb{R})$,

$$C(t) := \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\,\varphi(s)\,\phi_{R_1}(s)\,\mathrm{d}s \le \left\|\phi_{R_1}\right\|_\infty \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\,\varphi(s)\,\mathrm{d}s,$$

and so, from (C_2), it is clear that $C \in \widetilde{\mathscr{C}}_\varphi^n$. Furthermore, since $C(t) \ge 0$ for all $t \in \mathbb{R}$, $C \in P$. Therefore $(\mathrm{Id} - \xi L_1)^{-1} C \in P \cap K_\alpha$ and $R_0 := \|(\mathrm{Id} - \xi L_1)^{-1} C\|_\varphi < +\infty$.

Because $\lim_{\beta(u) \to +\infty} \|u\|_\varphi = +\infty$, there exists $R_2 > R_0$ such that $\|u\|_\varphi > R_1$ for every $u \in \partial K_\alpha^{\beta,R}$ for $R > R_2$. Now we prove that for each $R > R_2$, $Tu \ne \lambda u$ for all $u \in \partial K_\alpha^{\beta,R}$ and $\lambda \ge 1$, which implies $i_K(T, K_\alpha^{\beta,R}) = 1$. Assume, otherwise, that there exist $u \in \partial K_\alpha^{\beta,R}$ and $\lambda \ge 1$ such that $\lambda u = Tu$. Taking into account the inequality (4.6), we have, for $t \in \mathbb{R}$,

$$\begin{aligned}
|u(t)| &\le \lambda |u(t)| = |Tu(t)| = \left| \int_{-\infty}^{\infty} k(t,s)\,\eta(s)\, f(s, u(s))\,\mathrm{d}s \right| \\
&\le \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\, f\left(s, \frac{u(s)}{\varphi(s)}\varphi(s)\right) \mathrm{d}s \le \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)| \left[\xi \left| \frac{u(s)}{\varphi(s)} \right| \varphi(s) + \varphi(s)\phi_{R_1}(s) \right] \mathrm{d}s \\
&\le \xi \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\,|u(s)|\,\mathrm{d}s + C(t) = \xi\, L_1 |u|(t) + C(t),
\end{aligned}$$

which implies

$$(\mathrm{Id} - \xi L_1)|u|(t) \le C(t).$$

Since $(\mathrm{Id} - \xi L_1)^{-1}$ is non-negative, we have

$$|u(t)| \le (\mathrm{Id} - \xi L_1)^{-1} C(t)$$

and, consequently,

$$\|u\|_\varphi \le \|(\mathrm{Id} - \xi L_1)^{-1} C\|_\varphi = R_0.$$

Therefore, we have $\|u\|_\varphi \le R_0 < R$, a contradiction.

(3) There exists $\rho_0 > 0$ such that for all $x \in (0, \rho_0]$ and all $t \in A$ we have

$$f(t, x\varphi(t)) \ge \mu(L_2)\, x\varphi(t).$$

Since $\lim_{\beta(u) \to 0} \|u\|_\varphi = 0$, there exists $\rho_1 \in (0, \rho_0]$ such that $\|u\|_\varphi < \rho_0$ for every $u \in K_\alpha^{\beta,\rho}$, $\rho \in (0, \rho_1]$.

Let $\rho \in (0,\rho_1]$ be fixed. Let us prove that $u \neq Tu + \lambda\varphi_1$ for all u in $\partial K_\alpha^{\beta,\rho}$ and $\lambda \geq 0$, where $\varphi_1 \in K_\alpha \cap P$ is the eigenfunction of L_2 with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/\mu(L_2)$ of which the existence is proved in Theorem 4.5. This implies that $i_K(T,K_\rho) = 0$.

Assume, on the contrary, that there exist $u \in \partial K_\alpha^{\beta,\rho}$ and $\lambda \geq 0$ such that $u = Tu + \lambda\varphi_1$. We distinguish two cases.

Firstly, we discuss the case $\lambda > 0$. We have, for $t \in A$ in the conditions of (C_6),

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} k(t,s)\,\eta(s) f(s,u(s))\,\mathrm{d}s + \lambda\,\varphi_1(t) \geq \int_A k(t,s)\,\eta(s) f\left(s, \frac{u(s)}{\varphi(s)}\varphi(s)\right)\mathrm{d}s + \lambda\,\varphi_1(t) \\ &\geq \mu(L_2) \int_A k(t,s)\,\eta(s)\,u(s)\,\mathrm{d}s + \lambda\varphi_1(t) = \mu(L_2)\,L_2u(t) + \lambda\varphi_1(t). \end{aligned}$$

Note that the equality $u = Tu + \lambda\,\varphi_1$ implies that $u(t) \geq 0$ for $t \in A$. Therefore, $\mu(L_2)\,L_2u(t) \geq 0$ for $t \in A$ and we deduce, from previous inequalities, that

$$u(t) \geq \lambda\,\varphi_1(t) \quad \text{for } t \in A.$$

Hence, for $t \in A$,

$$(L_2u)(t) \geq \lambda(L_2\varphi_1)(t) = \frac{\lambda}{\mu(L_2)}\varphi_1(t),$$

in such a way that we obtain

$$u(t) \geq \mu(L_2)\,L_2u(t) + \lambda\varphi_1(t) \geq 2\lambda\varphi_1(t), \quad \text{for } t \in A.$$

By iteration, we deduce that, for $t \in A$, we get

$$u(t) \geq n\lambda\varphi_1(t) \text{ for every } n \in \mathbb{N},$$

a contradiction because $u(t)$ is finite and $\varphi_1|_A \not\equiv 0$.

Now we consider the case $\lambda = 0$. Let $\varepsilon > 0$ be such that for all such that for all $x \in (0,\rho_0]$ and and almost every $t \in A$ we have

$$f(t, x\varphi(t)) \geq (\mu(L_2) + \varepsilon)\,x\varphi(t).$$

We have, for $t \in A$,

$$u(t) = \int_{-\infty}^{\infty} k(t,s)\,\eta(s) f(s,u(s))\,\mathrm{d}s \geq \int_A (k(t,s)\,\eta(s))^+ f(s,u(s))\,\mathrm{d}s \geq (\mu(L_2) + \varepsilon)\,L_2u(t).$$

From previous expression together with (C_6), it is immediately deduced that $u(t) > 0$ for $t \in A$.

Since $L_2\varphi_1(t) = r(L_2)\,\varphi_1(t)$ for $t \in \mathbb{R}$, we have, for $t \in A$,

$$\bar{L}\varphi_1(t) = L_2\varphi_1(t) = r(L_2)\,\varphi_1(t),$$

and we obtain $r(\bar{L}) \geq r(L_2)$. On the other hand, we have, for $t \in A$,

$$u(t) \geq (\mu(L_2) + \varepsilon)\,L_2u(t) = (\mu(L_2) + \varepsilon)\,\bar{L}u(t),$$

where $u(t) > 0$. Thus, using Theorem 2.4, we have $r(\bar{L}) \leq \dfrac{1}{\mu(L_2) + \varepsilon}$ and therefore $r(L_2) \leq \dfrac{1}{\mu(L_2) + \varepsilon}$. This gives $\mu(L_2) + \varepsilon \leq \mu(L_2)$, a contradiction.

(4) Let $R_1 > 0$ be such that

$$f(t, x\varphi(t)) > \mu(L_2)\, x\varphi(t)$$

for all $x > R_1$ and all $t \in A$.

Moreover, since $\lim_{\beta(u)\to+\infty} \|u\|_\varphi = +\infty$, there exists R_2 such that $\|u\| > R_1$ for every $u \in \partial K_\alpha^{\beta,R}$ for $R > R_2$.

Let $R \geq R_2$. Now, proceeding as in the proof of the statement (3), it is easy to prove that $u \neq Tu + \lambda\varphi_1$ for all u in $\partial K_\alpha^{\beta,R}$ and $\lambda \geq 0$, which implies $i_K(T, K_\alpha^{\beta,R}) = 0$. ■

The following Theorem, in the line of [19], applies the index results in Theorem 4.12 in order to get some results on existence of nontrivial solutions for the equation (1.1).

Theorem 4.13. *Assume that conditions* $(C_1)-(C_{10})$ *hold. Suppose also that one of the following conditions is satisfied*

(T_1) $0 \leq f^0 < \mu(L_1)$ *and* $\mu(L_2) < f_\infty \leq \infty$.

(T_2) $0 \leq f^\infty < \mu(L_1)$ *and* $\mu(L_2) < f_0 \leq \infty$.

Then the integral equation (1.1) *has at least one non-trivial solution in* K_α.

Proof. We will prove (T_1), being (T_2) analogous.

Take $\beta(u) = \|u\|_\varphi$. Clearly β is in the conditions of Theorem 4.12. Then, the existence of $\rho_0 > 0$ and $R_1 > 0$ such that $i_{K_\alpha}(T, K_\alpha^{\beta,\rho}) = 1$ for each $\rho \in (0, \rho_0]$ and $i_{K_\alpha}(T, K_\alpha^{\beta,R}) = 0$ for each $R \geq R_1$ is ensured.

Therefore, if we choose $\rho \leq \rho_0$ and $R \geq R_1$ such that $\rho < R$, $\overline{K_\alpha^{\beta,\rho}} \subset K_\alpha^{\beta,R}$ and from (4) in Lemma 4.10 we deduce that T has a fixed point in $K_\alpha^{\beta,R} \setminus \overline{K_\alpha^{\beta,\rho}}$. ■

The following Lemma establishes some relations between the characteristic values of some of the considered operators.

Lemma 4.14. *It holds that* $\widetilde{M}(A) \geq \mu(L_2) \geq \mu(L_1)$.

Proof. First, we prove that $\mu(L_2) \geq \mu(L_1)$. Let ϕ be an eigenfunction of L_1 related to the eigenvalue $r(L_1)$. We have that

$$\begin{aligned} r(L_1)\,\phi(t) = L_1\phi(t) &= \int_{-\infty}^{\infty} |k(t,s)\,\eta(s)|\,\phi(s)\,\mathrm{d}s \geq \int_A |k(t,s)\,\eta(s)|\,\phi(s)\,\mathrm{d}s \\ &\geq \int_A (k(t,s)\,\eta(s))^+\,\phi(s)\,\mathrm{d}s = L_2\phi(t). \end{aligned}$$

Therefore, Theorem 2.4 yields that $r(L_2) \leq r(L_1)$ or, equivalently, $\mu(L_2) \geq \mu(L_1)$.

Now we prove $\widetilde{M}(A) \geq \mu(L_2)$. Let $\phi \in P \cap K_\alpha$ be a corresponding eigenfunction of norm 1 of $1/\mu(L_2)$ for the operator L_2, that is $\phi = \mu(L_2)\, L_2(\phi)$ and $\|\phi\| = 1$. Then, for $t \in A$, we have

$$\phi(t) = \mu(L_2) \int_a^b k(t,s)\,\eta(s)\,\phi(s)\,\mathrm{d}s \geq \mu(L_2) \min_{t\in A} \phi(t) \int_a^b k(t,s)\,\eta(s)\,\mathrm{d}s.$$

Taking the infimum over A, we obtain

$$\min_{t\in A}\phi(t)\geq\mu(L_2)\min_{t\in A}\phi(t)/\widetilde{M}(A),$$

that is, $\widetilde{M}(A)\geq\mu(L_2)$. ■

Remark 4.15. We note that the previous results could also be formulated for $\widetilde{\mathscr{C}}_\varphi([a,+\infty))$ or $\widetilde{\mathscr{C}}_\varphi((-\infty,a])$ (with obvious notation) for any $a\in\mathbb{R}$ (see [2]).

5 An example

We will consider now the problem

$$Tu(t)=\int_{-\infty}^{\infty}e^{-\frac{|s|}{2}}\sin t\,\sqrt{|u(s)|}\,\sin^2 s\,\mathrm{d}s,$$

that is, $k(t,s)=e^{-\frac{|s|}{2}}\sin t$, $\eta(s)=1$ and $f(s,y)=\sqrt{|y|}\,\sin^2 s$.

We will take

$$\varphi(t)=|t|,$$

and

$$\alpha(u)=\min_{t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]}u(t)-\frac{\sqrt{2}}{2}\|u\|_\infty.$$

We will verify that conditions (C_1)–(C_{10}) are satisfied for the case $n=0$:

(C_1) First of all, since $k(\cdot,s)\in\mathscr{C}(\mathbb{R})$ and there exist

$$\lim_{t\to\pm\infty}\frac{k(t,s)}{\varphi(t)}=\lim_{t\to\pm\infty}\frac{e^{-\frac{|s|}{2}}\sin t}{|t|}=0,$$

it is clear that $k(\cdot,s)\in\widetilde{\mathscr{C}}_\varphi$ for all $s\in\mathbb{R}$.

Moreover, for every $\varepsilon>0$ there exists $\delta>0$ such that when $|t_1-t_2|<\delta$,

(i)

$$\left|\frac{k(t_1,s)}{\varphi(t_1)}-\frac{k(t_2,s)}{\varphi(t_2)}\right|=\left|\frac{e^{-\frac{|s|}{2}}\sin t_1}{|t_1|}-\frac{e^{-\frac{|s|}{2}}\sin t_2}{|t_2|}\right|\leq\varepsilon\,e^{-\frac{|s|}{2}},$$

and

(ii)

$$\left|\frac{(k(t_1,s)^+}{\varphi(t_1)}-\frac{(k(t_2,s))^+}{\varphi(t_2)}\right|=\left|\frac{e^{-\frac{|s|}{2}}(\sin t_1)^+}{|t_1|}-\frac{e^{-\frac{|s|}{2}}(\sin t_2)^+}{|t_2|}\right|\leq\varepsilon\,e^{-\frac{|s|}{2}},$$

so we will take $\omega_0(s)=e^{-\frac{|s|}{2}}$.

(C_2) Clearly, it holds that $\omega_0\, \varphi \in \mathrm{L}^1(\mathbb{R})$. Also,

$$\frac{1}{\varphi(t)} \int_{-\infty}^{\infty} |k(t,s)|\, \varphi(s)\, \mathrm{d}s = \frac{|\sin t|}{|t|} \int_{-\infty}^{\infty} e^{-\frac{|s|}{2}}\, |s|\, \mathrm{d}s = 8\, \frac{|\sin t|}{|t|} \in \mathrm{L}^{\infty}(\mathbb{R}).$$

Moreover, in this case

$$z_{(\pm)}(s) = \lim_{t\to\pm\infty} \frac{e^{-\frac{|s|}{2}}\, |\sin t|}{|t|} = 0,$$

$$M_0(s) = \sup_{t\in\mathbb{R}} \frac{e^{-\frac{|s|}{2}}\, |\sin t|}{|t|} = e^{-\frac{|s|}{2}},$$

and it holds that $z_{(\pm)}\, \varphi,\ M_0\, \varphi \in \mathrm{L}^1(\mathbb{R})$.

(C_3) It is clear that $f(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}$ and $f(t,\cdot)$ is continuous for a. e. $t \in \mathbb{R}$. Finally, for each $r > 0$, there exists $\phi_r(t) = \frac{\sqrt{r}\,\sin^2 t}{\sqrt{|t|}} \in \mathrm{L}^{\infty}(\mathbb{R})$ such that

$$\frac{f(t, x\varphi(t))}{\varphi(t)} = \frac{\sqrt{|x\, t|}\, \sin^2 t}{|t|} \le \phi_r(t),$$

for all $x \in [-r, r]$ and a. e. $t \in \mathbb{R}$.

(C_4) In this case,

$$\alpha(|k(\cdot,s)|) = \min_{t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]} |k(t,s)| - \frac{\sqrt{2}}{2}\, \|k(\cdot,s)\|_{\infty} = e^{-\frac{|s|}{2}} \min_{t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]} |\sin t| - \frac{\sqrt{2}}{2}\, e^{-\frac{|s|}{2}} = 0.$$

(C_5) It is clear that $\alpha(|k(t,s)|)\, \varphi(s) \in \mathrm{L}^1(\mathbb{R})$. Moreover, for all $u \in P$, it holds that

$$\begin{aligned}
\alpha(L_1 u) &= \min_{t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]} \int_{-\infty}^{\infty} |k(t,s)|\, u(s)\, \mathrm{d}s - \frac{\sqrt{2}}{2} \left\| \int_{-\infty}^{\infty} |k(t,s)|\, u(s)\, \mathrm{d}s \right\|_{\infty} \\
&\ge \int_{-\infty}^{\infty} \min_{t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]} |k(t,s)|\, u(s)\, \mathrm{d}s - \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} \|k(t,s)\|_{\infty}\, u(s)\, \mathrm{d}s \\
&= \int_{-\infty}^{\infty} \alpha(|k(\cdot,s)|)\, u(s)\, \mathrm{d}s.
\end{aligned}$$

(C_6) We can take $A = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. For such A, we obtain

$$\frac{1}{\widetilde{M}(A)} = \inf_{t\in A} \int_A e^{-\frac{|s|}{2}}\, \sin t\, \mathrm{d}s = \inf_{t\in A} \left\{ 2\, e^{\frac{-3\pi}{8}} \left(-1 + e^{\frac{\pi}{4}}\right) \sin t \right\} = \sqrt{2}\, e^{\frac{-3\pi}{8}} \left(-1 + e^{\frac{\pi}{4}}\right) > 0.$$

(C_7) It is analogous to (C_4). The same occurs to (C_9).

(C_8) It is analogous to (C_5). The same occurs to (C_{10}).

Finally, we obtain the following values for the limits f^{∞} and f_0:

$$f^{\infty} = \overline{\lim_{|x|\to+\infty}} \frac{\sup_{t\in\mathbb{R}} \dfrac{\sqrt{|x|}\, \sin^2 t}{\sqrt{|t|}}}{|x|} \le \overline{\lim_{|x|\to+\infty}} \frac{\sqrt{|x|}}{|x|} = 0,$$

and so $f^{\infty}=0$. Analogously,

$$f_0=\lim_{|x|\to 0}\frac{\inf\limits_{t\in A}\dfrac{\sqrt{|x|}\,\sin^2 t}{\sqrt{|t|}}}{|x|}=\lim_{|x|\to 0}\frac{\sqrt{|x|}}{\sqrt{3\,\pi}\,|x|}=+\infty.$$

On the other hand, since both $r(L_1)$ and $r(L_2)$ are positive (as it has been proved in Theorem 4.5), it holds that $\mu(L_1)>0$ and $\mu(L_2)<+\infty$.

Thus, from (T_2) in Theorem 4.13, we deduce that our problem has at least a non-trivial solution in $K_\alpha\subset\mathscr{C}_\varphi$.

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