

## MULTIPLICITY RESULTS FOR FOURTH ORDER PROBLEMS RELATED TO THE THEORY OF DEFORMATIONS BEAMS

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**ABSTRACT.** The main purpose of this paper is to establish the existence and multiplicity of positive solutions for a fourth-order boundary value problem with integral condition. By using a new technique of construct a positive cone, we apply the Krasnoselskii compression/expansion and Leggett-Williams fixed point theorems in cones to show our multiplicity results. Finally, a particular case is studied, and the existence of multiple solutions is proved for two different particular functions.

**1. Introduction.** The study of fourth-order boundary value problems are useful for material mechanics because this kind of problems usually characterize the deformations of an elastic beam. They have been studied by many authors via various methods, such as Leray-Schauder continuation method, topological degree theory, shooting method, fixed point theorems on cones, critical point theory, the lower and upper solutions method or spectral theory, see for example [4, 6, 3, 13, 11, 16, 2] and references therein.

In [4], it is characterized the sign of the Green's function  $g_M$  related to the fourth order linear problem

$$L_M u(t) = u^{(4)}(t) + Mu(t) = \sigma(t), \quad t \in I := [0, 1] \quad (1)$$

$$u(0) = u'(0) = u''(0) = u(1) = 0. \quad (2)$$

There, by using the theory of disconjugation [10], the authors obtained the exact values on the real parameter  $M \in [-m_0^4, m_1^4]$ , for which the related Green's function  $g_M$  is strictly negative in  $(0, 1) \times (0, 1)$ . To be concise,  $m_0 \cong 4.73004$  is the first positive root of equation

$$\cos m \cosh m = 1,$$

and  $m_1 \cong 5.553$  is the first positive root of equation

$$\tan \frac{m}{\sqrt{2}} = \tanh \frac{m}{\sqrt{2}}.$$

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Such result has been extended in [8] (and further in [9]) for any  $n$ -th order differential operator, coupled to the so-called  $(k, n - k)$  boundary conditions, which are defined, for  $1 \leq k \leq n - 1$ , as follows:

$$u(0) = u'(0) = \dots = u^{(k-1)}(0) = u(1) = u'(1) = \dots = u^{(n-k-1)}(1) = 0.$$

There is proved that  $m_0^4$  is the least positive eigenvalue of operator  $u^{(4)}$  on the space

$$\{u \in C^4([0, 1]), \quad u(0) = u'(0) = u(1) = u'(1) = 0\}$$

and  $-m_1^4$  is the biggest negative eigenvalue of operator  $u^{(4)}$  in the space

$$\{u \in C^4([0, 1]), \quad u(0) = u'(0) = u''(0) = u(1) = 0\}.$$

Motivated by the above works, in this paper we study the existence and multiplicity of positive solutions for the fourth order equation:

$$u^{(4)}(t) + Mu(t) + f(t, u(t)) = 0, \quad t \in I, \quad (3)$$

subject to the perturbed functional boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \quad (4)$$

Here  $f$  is such that

$(H_0)$   $f : I \times [0, \infty) \rightarrow [0, \infty)$  is a continuous function,

$M \in [-m_0^4, m_1^4)$  and  $\lambda$  is a positive parameter bounded from above by a constant that will be introduced later.

The boundary conditions (4) model the deflection of beam fixed in 0 and has some mechanism in 1 which controls the displacement according to the feedback from devices measuring the displacements along parts of the beam.

In this paper, a function  $u \in C(I)$  is called nonnegative solution of Problem (3)-(4) if  $u$  is a solution of (3)-(4) and  $u(t) \geq 0$ , for all  $t \in I$ . A function  $u \in C(I)$  is called positive solution of Problem (3)-(4) if  $u$  is a nonnegative solution of (3)-(4) and  $u(t) > 0$ , for all  $t \in (0, 1)$ .

A standard approach to study positive solutions of a boundary value problem such as (3) – (4) consists on finding the corresponding Green's function  $G_M$  and seek solutions as fixed points of the following integral operator:

$$T(u)(t) = \int_0^1 G_M(t, s)f(t, u(t)) ds \quad (5)$$

in the cone  $P = \{u \in C(I), u \geq 0 \text{ on } I\}$  of non-negative functions in the space  $C(I)$  endowed with the usual supremum norm.

We will obtain the expression of the Green's function  $G_M$  related to the linear equation (1) coupled to the functional boundary conditions (4) as a combination of the expression of  $g_M$ , the Green's function related to Problem (1) – (2). In this case, we will give the exact values on the positive parameter  $\lambda$  for which  $G_M$  remains negative on  $(0, 1) \times (0, 1)$ , whenever  $M \in [-m_0^4, m_1^4)$ . Moreover there is no  $\lambda > 0$  for which  $G_M < 0$  when  $M > m_1^4$ . In the case of  $M < -m_0^4$  we have that it must exists a set of values of  $\lambda$  where such property is fulfilled, but this case has not been considered in this paper, and remains as an open problem.

Following the arguments of [20, 21], to ensure the positiveness of the solutions at some subinterval  $[a, b]$  of  $[0, 1]$ , it is convenient to work in a smaller cone than  $P$ ,

namely,

$$K = \{u \in P, \min_{t \in [a,b]} u(t) \geq c\|u\|\}$$

where  $c \in (0, 1)$  is a constant.

To construct such cone, it is usually convenient to establish the following type of inequality for some function  $\phi \in C(I)$ , such that  $\phi(s) > 0$  for all  $s \in (0, 1)$ .

$$(C) \quad c\phi(s) \leq G_M(t, s) \leq \phi(s), \quad t \in [a, b], s \in [0, 1].$$

It is possible to use this approach in our situation but, as we will see, the explicit form of Green’s function  $G_M$  is very complicated for  $M \neq 0$ , and so, condition (C) becomes hard to check.

So, we look for a condition on the line of the following one introduced in [7, Page 86]:

( $N_g$ ) There is a continuous function  $\phi(t) > 0$  for all  $t \in (0, 1)$  and  $k_1, k_2 \in L^1(I)$ , such that  $k_1(s) < k_2(s) < 0$  for a. e.  $s \in I$ , satisfying

$$\phi(t)k_1(s) \leq G_M(t, s) \leq \phi(t)k_2(s), \quad \text{for a. e. } (t, s) \in I \times I.$$

To this end, to avoid long computations concerning the expression of the Green’s function, we will study the limits at  $s = 0$  and  $s = 1$  of the quotient  $G_M(t, s)/G_M(1, s)$ .

The paper is organized as follows: in Section 2, we provide some necessary background material such as the compression/expansion and Leggett-Williams fixed point theorems in cones together with some properties of the Green’s function associated to the two-point homogeneous boundary value problem (1)–(2). In Section 3, the explicit expression of the Green’s function related equation (1) coupled to the integral boundary condition (4) and deduce some additional properties concerning its constant sign. Section 4 is devoted to prove the existence of countably many positive solutions for the nonlinear problem (3)-(4) under suitable conditions on  $f$ . In section 5, two examples are given to show the applicability of the obtained results.

**2. Preliminaries.** In this section we introduce some preliminary results which will be used along the paper. First, we provide some background definitions cited from cone theory in Banach spaces. After that, we introduce some definitions and properties of the Green’s function  $g_M$  related to problem (1) – (2).

**Definition 2.1.** Let  $E$  be a real Banach space. A nonempty convex closed set  $K \subset E$  is said to be a cone provided that

- (i)  $\alpha u \in K$  for all  $u \in K$  and all  $\alpha \geq 0$ ;
- (ii)  $u, -u \in K$  implies  $u = 0$ .

In the sequel, we enunciate the celebrated compression/expansion Krasnoselskii’s fixed point theorem:

**Theorem 2.2.** [12]. *Let  $K$  be a cone and  $T : K \rightarrow K$  a completely continuous operator and  $0 < r < R$ . Moreover, if one of the following conditions are fulfilled:*

- (i)  $\|Tu\| \leq \|u\|$  for any  $u \in K$  with  $\|u\| = r$  and  $\|Tu\| \geq \|u\|$  for any  $u \in K$  with  $\|u\| = R$ , or
- (ii)  $\|Tu\| \geq \|u\|$  for any  $u \in K$  with  $\|u\| = r$  and  $\|Tu\| \leq \|u\|$  for any  $u \in K$  with  $\|u\| = R$ ,

*then operator  $T$  has a fixed point in  $K$  such that  $r \leq \|x\| \leq R$ .*

In order to enunciate the Leggett-Williams fixed point theorem, we introduce the following concepts.

**Definition 2.3.** A map  $\beta$  is said to be a nonnegative, continuous, concave functional on a cone  $K$  of a real Banach space  $E$ , if  $\beta : K \rightarrow [0, \infty)$  is continuous, and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y) \text{ for all } x, y \in K \text{ and } t \in I.$$

**Definition 2.4.** Let  $P$  be a cone in a real Banach space  $E$ ,  $0 < a < b$  and let  $\beta$  be a nonnegative continuous concave functional on  $K$ . Define the convex sets  $P_r$  and  $P(\beta, a, b)$  by

$$P_r = \{x \in K \mid \|x\| \leq r\}$$

and

$$P(\beta, a, b) = \{x \in K \mid a \leq \beta(x), \|x\| \leq b\}.$$

**Theorem 2.5.** (*Leggett-Williams fixed point theorem*) (see [15]) Let  $A : \overline{P_c} \rightarrow \overline{P_c}$  be completely continuous operator and  $\beta$  be a nonnegative continuous concave functional on  $K$  such that  $\beta(x) \leq \|x\|$  for  $x \in \overline{P_c}$ . Suppose there exist  $0 < a < b < d \leq c$  such that

(A<sub>1</sub>)  $\{x \in P(\beta, b, d); \beta(x) > b\} \neq \emptyset$  and  $\beta(Ax) > b$  for  $x \in P(\beta, b, d)$

(A<sub>2</sub>)  $\|Ax\| < a$  for  $\|x\| \leq a$ ,

(A<sub>3</sub>)  $\beta(Ax) > b$  for  $x \in P(\beta, b, c)$  with  $\|Ax\| > d$ .

Then  $A$  has at least three fixed points  $x_1, x_2, x_3$  in  $\overline{P_c}$  such that  $\|x_1\| < a$ ,  $\beta(x_2) > b$  and  $\|x_3\| > a$  with  $\beta(x_3) < b$ .

By the other hand, we point out that problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = 0, & t \in I, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = 1, \end{cases} \quad (6)$$

has no solution if and only the following equality holds

$$M > 0 \quad \text{and} \quad \tan\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right) = \tanh\left(\frac{\sqrt[4]{M}}{\sqrt{2}}\right). \quad (7)$$

In any other case, it has a unique solution, denoted by  $w_M$ , which is given by the following expression:

$$w_M(t) = \begin{cases} \frac{\cosh\left(\frac{mt}{\sqrt{2}}\right) \sin\left(\frac{mt}{\sqrt{2}}\right) - \cos\left(\frac{mt}{\sqrt{2}}\right) \sinh\left(\frac{mt}{\sqrt{2}}\right)}{\cosh\left(\frac{m}{\sqrt{2}}\right) \sin\left(\frac{m}{\sqrt{2}}\right) - \cos\left(\frac{m}{\sqrt{2}}\right) \sinh\left(\frac{m}{\sqrt{2}}\right)} & \text{if } m > 0 \text{ and } M = m^4, \\ t^3 & \text{if } M = 0, \\ \frac{\sinh(mt) - \sin(mt)}{\sinh(m) - \sin(m)} & \text{if } m > 0 \text{ and } M = -m^4. \end{cases} \quad (8)$$

It is not difficult to verify that  $w_M(t) > 0$  for all  $t \in I$  if and only if  $M < m_1^4$ . Moreover, by denoting

$$C_M = \int_0^1 w_M(\tau) d\tau,$$

we have that it is given by the following expression:

$$C_M = \begin{cases} \frac{-2\sqrt{2} \cos\left(\frac{m}{\sqrt{2}}\right) \sinh\left(\frac{m}{\sqrt{2}}\right) + 2\sqrt{2} \sin\left(\frac{m}{\sqrt{2}}\right) \cosh\left(\frac{m}{\sqrt{2}}\right)}{m(-\cos(\sqrt{2}m) + \cosh(\sqrt{2}m) - \sin(\sqrt{2}m) \sinh(\sqrt{2}m))} \\ + \frac{2\sqrt{2} \cos^2\left(\frac{m}{\sqrt{2}}\right) \sinh\left(\frac{m}{\sqrt{2}}\right) \cosh\left(\frac{m}{\sqrt{2}}\right)}{m(-\cos(\sqrt{2}m) + \cosh(\sqrt{2}m) - \sin(\sqrt{2}m) \sinh(\sqrt{2}m))} \\ - \frac{2\sqrt{2} \cos^2\left(\frac{m}{\sqrt{2}}\right) \sinh\left(\frac{m}{\sqrt{2}}\right) \cosh\left(\frac{m}{\sqrt{2}}\right)}{m(-\cos(\sqrt{2}m) + \cosh(\sqrt{2}m) - \sin(\sqrt{2}m) \sinh(\sqrt{2}m))}, & \text{if } m > 0 \text{ and } M = m^4, \\ \frac{1}{4} & \text{if } M = 0, \\ \frac{\cos(m) + \cosh(m) - 2}{m \sinh(m) - m \sin(m)} & \text{if } m > 0 \text{ and } M = -m^4. \end{cases} \tag{9}$$

Moreover, we enunciate the following result concerning the expression of the Green's function  $g_M$ , related to the linear Problem (1) – (2). The proof can be found in [4]

**Lemma 2.6.** *Let  $\sigma \in C(I)$  and  $M \in \mathbb{R}$  be such that (7) does not hold. Then problem*

$$\begin{cases} u^{(4)}(t) + Mu(t) = \sigma(t), & t \in I \\ u(0) = u'(0) = u''(0) = u(1) = 0, \end{cases} \tag{10}$$

has a unique solution given by

$$u(t) = \int_0^1 g_M(t, s) \sigma(s) ds.$$

Here, for  $M = -m^4 < 0$ , we have

$$g_M(t, s) = \begin{cases} g_1(t, s, m) & \text{if } 0 \leq s \leq t \leq 1 \\ g_2(t, s, m) & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

with

$$g_1(t, s, m) = \frac{-e^{m(s-t)} + e^{m(t-s)} - 2 \sin(m(t-s))}{4m^3} + g_2(t, s, m),$$

and

$$g_2(t, s, m) = \frac{e^{-m(s+t-1)} (e^{m(2s-1)} + 2e^{ms} \sin(m-ms) - e^m) (e^{2mt} - 2e^{mt} \sin(mt) - 1)}{4m^3 (e^{2m} - 2e^m \sin(m) - 1)}.$$

If  $M = 0$ , it is given by

$$g_0(t, s) = \frac{1}{6} \begin{cases} (s-1)^3 t^3 + (t-s)^3 & \text{if } 0 \leq s \leq t \leq 1, \\ (s-1)^3 t^3 & \text{if } 0 < t \leq s \leq 1. \end{cases}$$

Moreover, when  $M = m^4 > 0$  it follows the expression

$$g_M(t, s) = \begin{cases} g_3(t, s, m) & \text{if } 0 \leq s \leq t \leq 1 \\ g_4(t, s, m) & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

$$g_3(t, s, m) = \frac{e^{-\sqrt{2}m(t-1)}}{2\sqrt{2}m^3 \left( (e^{\sqrt{2}m} - 1) \cos\left(\frac{m}{\sqrt{2}}\right) - (e^{\sqrt{2}m} + 1) \sin\left(\frac{m}{\sqrt{2}}\right) \right)} h_1(t, s, m),$$

being

$$\begin{aligned}
& h_1(t, s, m) \\
= & e^{\frac{m(t-s)}{\sqrt{2}}} \left( -e^{\sqrt{2}m(s+t-1)} + e^{\sqrt{2}ms} + e^{\sqrt{2}m(t-1)} - 1 \right) \cos \left( \frac{m(s-t-1)}{\sqrt{2}} \right) \\
& - e^{-\frac{m(s-3t)}{\sqrt{2}}} \left( e^{\sqrt{2}m(s-t-1)} + 1 \right) \cos \left( \frac{m(s-t+1)}{\sqrt{2}} \right) \\
& + e^{-\frac{m(s-3t)}{\sqrt{2}}} \cos \left( \frac{m(s+t-1)}{\sqrt{2}} \right) + \\
& + e^{\frac{m(s+t-2)}{\sqrt{2}}} \cos \left( \frac{m(s+t-1)}{\sqrt{2}} \right) + e^{-\frac{m(s-3t+2)}{\sqrt{2}}} \sin \left( \frac{m(s-t+1)}{\sqrt{2}} \right) - \\
& - e^{\frac{m(s+t)}{\sqrt{2}}} \sin \left( \frac{m(s-t+1)}{\sqrt{2}} \right) - e^{\frac{m(t-s)}{\sqrt{2}}} \sin \left( \frac{m(s+t-1)}{\sqrt{2}} \right) + \\
& + e^{\frac{m(s+3t-2)}{\sqrt{2}}} \sin \left( \frac{m(s+t-1)}{\sqrt{2}} \right).
\end{aligned}$$

Moreover

$$g_4(t, s, m) = \frac{-e^{-\frac{m(s+t-2)}{\sqrt{2}}}}{2\sqrt{2}m^3 \left( \left( e^{\sqrt{2}m} - 1 \right) \cos \left( \frac{m}{\sqrt{2}} \right) - \left( e^{\sqrt{2}m} + 1 \right) \sin \left( \frac{m}{\sqrt{2}} \right) \right)} h_2(t, s, m)$$

where

$$\begin{aligned}
& h_2(t, s, m) \\
= & \left( \left( e^{\sqrt{2}m(s-1)} - 1 \right) \cos \left( \frac{m(s-1)}{\sqrt{2}} \right) - \left( e^{\sqrt{2}m(s-1)} + 1 \right) \sin \left( \frac{m(s-1)}{\sqrt{2}} \right) \right) \times \\
& \times \left( e^{\sqrt{2}mt} - 1 \right) \cos \left( \frac{mt}{\sqrt{2}} \right) - \left( e^{\sqrt{2}mt} + 1 \right) \sin \left( \frac{mt}{\sqrt{2}} \right).
\end{aligned}$$

As a direct consequence of previous result, we deduce the following properties for function  $g_M$ :

**Corollary 1.** *Function  $g_M$  defined in Lemma 2.6 satisfies the following properties:*

1.  $g_M(0, s) = \frac{\partial g_M}{\partial t}(0, s) = \frac{\partial^2 g_M}{\partial t^2}(0, s) = g_M(1, s) = 0$ , for all  $s \in (0, 1)$ .
2.  $\frac{\partial^3 g_M}{\partial t^3}(0, s) < 0 < \frac{\partial g_M}{\partial t}(1, s)$ , for all  $s \in (0, 1)$ .
3.  $g_M(t, 1) = \frac{\partial g_M}{\partial s}(t, 1) = \frac{\partial^2 g_M}{\partial s^2}(t, 1) = g_M(t, 0) = 0$ , for all  $t \in (0, 1)$ .
4.  $\frac{\partial^3 g_M}{\partial s^3}(t, 1) > 0 > \frac{\partial g_M}{\partial s}(t, 0)$ , for all  $t \in (0, 1)$ .

On the other hand, if we consider the following boundary value problem:

$$v^{(4)}(t) + Mv(t) = \sigma(t) \quad t \in I, \quad v(0) = u(1) = v'(1) = v''(1) = 0, \quad (11)$$

in [7, Section 1.4] or [8, 9] one can see that Problem (11) is just the adjoint of Problem (10). So, the eigenvalues of both problems coincide and Green's function  $g_M^*$  related to this problem satisfies that  $g_M^*(t, s) = g_M(s, t)$  for all  $t, s \in I$ .

As a direct consequence, we have that

$$z_M(s) = \int_0^1 g_M^*(s, r) dr = \int_0^1 g_M(r, s) dr \quad (12)$$

is the unique solution of the following boundary value problem:

$$z^{(4)}(t) + Mz(t) = 1 \quad t \in I, \quad z(0) = z(1) = z'(1) = z''(1) = 0. \quad (13)$$

Moreover, we have that if  $M \in [-m_0^4, m_1^4)$  then  $z_M(s) < 0$  for all  $s \in (0, 1)$ , and  $z'_M(0) < 0 < z''_M(1)$ .

We point out that, by direct computations, it is possible to obtain the explicit expression of function  $z_M$ . However, it is, specially when  $M > 0$ , too long, which makes it very difficult to deal with.

**3. Expression of the Green's function.** In this section we will obtain the explicit expression of the Green's function related to the equation (1) coupled to boundary conditions (4). The result is the following.

**Lemma 3.1.** *Let  $\sigma \in L^1(I)$ ,  $\lambda > 0$  and  $M \in \mathbb{R}$  be such that (7) does not hold. Then problem*

$$\begin{cases} u^{(4)}(t) + Mu(t) + \sigma(t) = 0, & t \in I, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \int_0^1 u(s) ds, \end{cases} \tag{14}$$

has a unique solution if and only if

$$\lambda C_M \neq 1.$$

In such a case, it is given by the following expression

$$u_M(t) = \int_0^1 G_M(t, s)\sigma(s) ds \tag{15}$$

where

$$G_M(t, s) = -g_M(t, s) - \frac{\lambda w_M(t)}{1 - \lambda C_M} \int_0^1 g_M(\tau, s) d\tau \tag{16}$$

and  $w_M$  and  $C_M$  are defined in (8) and (9) respectively.

*Proof.* Let  $v_M$  and  $w_M$  be the unique solutions of Problems (10) and (6) respectively. Then, it is clear that  $u_M = v_M + \lambda w_M \int_0^1 u_M(s) ds$  is the unique solution of problem (14).

As a consequence, for all  $t \in I$ , the following equalities are fulfilled:

$$u_M(t) = - \int_0^1 g_M(t, s)\sigma(s) ds + \lambda w_M(t) \int_0^1 u_M(s) ds. \tag{17}$$

Let  $A_M = \int_0^1 u_M(\tau) d\tau$ , then, from the previous equality, we deduce that

$$A_M = - \int_0^1 \int_0^1 g_M(\tau, s)\sigma(s) ds d\tau + \lambda A_M \int_0^1 w_M(\tau) d\tau$$

or, which is the same,

$$A_M = - \frac{\int_0^1 \sigma(s) \int_0^1 g_M(\tau, s) d\tau ds}{1 - \lambda \int_0^1 w_M(\tau) d\tau}.$$

Replacing this value in (17), we arrive at the following expression for function  $u_M$ :

$$u_M(t) = - \int_0^1 g_M(t, s)\sigma(s) ds - \lambda w_M(t) \frac{\int_0^1 \sigma(s) \int_0^1 g_M(\tau, s) d\tau ds}{1 - \lambda \int_0^1 w_M(\tau) d\tau},$$

and the proof is concluded. □

A careful analysis of the Green’s function  $G_M$  allows us to deduce the following result:

**Theorem 3.2.** *Let  $G_M(t, s)$  be Green’s function related to problem (3)-(4) given by expression (16). Then if  $M \in [-m_0^4, m_1^4)$  and  $\lambda \in (0, 1/C_M)$  we have that  $G_M(t, s) > 0$  for all  $(t, s) \in (0, 1) \times (0, 1)$ . Moreover there exist  $R > 0$  and  $h \in C(I)$ , such that  $h(0) = 0$  and  $h > 0$  on  $(0, 1]$ , for which the following inequalities are fulfilled:*

$$h(t) \frac{\lambda}{\lambda C_M - 1} z_M(s) \leq G_M(t, s) \leq R \frac{\lambda}{\lambda C_M - 1} z_M(s), \text{ for all } (t, s) \in I \times I. \quad (18)$$

*Proof.* Since  $M \in [-m_0^4, m_1^4)$  we have that  $g_M < 0$  and, as a direct consequence of  $\lambda \in (0, 1/C_M)$  and the fact that  $w_M > 0$  on  $I$  for all  $M < m_1^4$ , we conclude, from (16), that  $G_M(t, s) > 0$  for all  $(t, s) \in (0, 1) \times (0, 1)$ .

Now, we denote by

$$\varphi(t, s) = \frac{G_M(t, s)}{G_M(1, s)} = \frac{1 - \lambda C_M}{\lambda} \frac{g_M(t, s)}{\int_0^1 g_M(r, s) dr} + w_M(t). \quad (19)$$

It is clear that function  $\varphi(t, s)$  is continuous on  $[0, 1] \times (0, 1)$ .

Using the properties of  $g_M$  showed in Lemma 2.6 and those of  $z_M$  explained in previous section, by means of L’Hôpital rule, we deduce, for all  $t \in (0, 1)$ :

$$\lim_{s \rightarrow 0^+} \frac{g_M(t, s)}{\int_0^1 g_M(r, s) dr} = \lim_{s \rightarrow 0^+} \frac{g_M(t, s)}{z_M(s)} = \lim_{s \rightarrow 0^+} \frac{\frac{\partial g_M}{\partial s}(t, s)}{z'_M(s)} = \frac{\frac{\partial g_M}{\partial s}(t, 0)}{z'_M(0)} > 0.$$

Thus,

$$\lim_{s \rightarrow 0^+} \varphi(t, s) = \frac{1 - \lambda C_M}{\lambda} \left( \frac{\frac{\partial g_M}{\partial s}(t, 0)}{z'_M(0)} \right) + w_M(t) := l_1(t) > 0 \text{ for all } t \in (0, 1].$$

Analogously, if  $t \in (0, 1)$ , we have

$$\lim_{s \rightarrow 1^-} \frac{g_M(t, s)}{\int_0^1 g_M(r, s) dr} = \lim_{s \rightarrow 1^-} \frac{g_M(t, s)}{z_M(s)} = \lim_{s \rightarrow 1^-} \frac{\frac{\partial^3 g_M}{\partial s^3}(t, s)}{z'''(s)} = \frac{\frac{\partial^3 g_M}{\partial s^3}(t, 1)}{z'''(1)} > 0$$

and

$$\lim_{s \rightarrow 1^-} \varphi(t, s) = \frac{1 - \lambda C_M}{\lambda} \left( \frac{\frac{\partial^3 g_M}{\partial s^3}(t, 1)}{z'''(1)} \right) + w_M(t) := l_2(t) > 0 \text{ for all } t \in (0, 1].$$

The limits  $l_1(t)$  and  $l_2(t)$  exist and are finite, so  $\varphi$  has removable discontinuities at  $s = 0, 1$ , and we can extend it to a function  $\tilde{\varphi} \in C(I \times I)$ .

Therefore  $h(t) = \min_{s \in [0, 1]} \tilde{\varphi}(t, s)$  is a continuous function such that

$$h(0) = 0 \text{ and } 0 < h(t) \leq \tilde{\varphi}(t, s) \leq R \text{ for all } (t, s) \in (0, 1] \times [0, 1],$$

where  $R = \max_{(t, s) \in I \times I} \tilde{\varphi}(t, s)$ .

The result follows from the expression of  $G_M(1, s)$ . □

**Corollary 2.** *Let  $G_M(t, s)$  be Green's function related to problem (3)-(4) given by expression (16). Then if  $M \in [-m_0^4, m_1^4]$  and  $\lambda \in (0, 1/C_M)$  we have that for all positive constant  $\delta \in (0, 1)$  there exists  $\gamma(\delta) \in (0, 1)$  for which the following inequality is fulfilled:*

$$\gamma \frac{\lambda}{\lambda C_M - 1} z_M(s) \leq G_M(t, s), \text{ for all } (t, s) \in [\delta, 1] \times I. \tag{20}$$

*Proof.* The result follows from the fact that function  $h$  is continuous on  $I$  and strictly positive on  $(0, 1]$ . □

**Remark 1.** If, instead of problem (1), (4), we consider the linear equation (1) coupled to the adjoint integral boundary conditions:

$$u(1) = u'(1) = u''(1) = 0, \quad u(0) = \lambda \int_0^1 u(s) ds. \tag{21}$$

It is immediate to verify that  $u$  is a solution of problem (1), (21) if and only if  $v(t) := u(1 - t)$  is a solution of (1), (4).

As a consequence, both problems has a unique solution if and only if  $\lambda C_M \neq 1$  and, after a suitable change of variables, we have that the Green's function  $\tilde{G}_M$  of problem (1), (21) is given by the following expression

$$\tilde{G}_M(t, s) = G_M(1 - t, 1 - s), \quad \text{for all } t, s \in I.$$

So, previous results can be directly adapted to this problem under a simple change of variables.

The same comment is valid for the results concerning nonlinear problems proved in next section.

**4. Nonlinear problems.** This section is devoted to prove existence and multiplicity of solutions of the nonlinear problem (3) – (4). To this, we will work on the Banach space  $C(I)$  endowed with the supremum norm  $\|u\| = \max_{t \in I} |u(t)|$ .

The following result is a direct consequence of the results showed in previous sections.

**Lemma 4.1.** *Assume that  $f$  satisfies condition  $(H_0)$ , then,  $u \in C(I)$  is a solution of (3)-(4) if and only if  $u$  is a fixed point of operator  $T$  defined on (5).*

Now, by considering function  $h$  and constant  $M$ , obtained in Theorem 3.2, we look for the fixed points of operator  $T$  at the following cone,

$$K = \{u \in C(I) \text{ and } u(t) \geq \frac{h(t)}{R} \|u\| \text{ for all } t \in I\}. \tag{22}$$

**Lemma 4.2.** *If condition  $(H_0)$  is fulfilled, then operator  $T : K \rightarrow K$ , defined in (5), is completely continuous.*

*Proof.* From the non-negativeness of functions  $f$  and  $G_M$  we deduce that  $T(u)(t) \geq 0$  for all  $t \in I$  and  $u \in K$ . The regularity of both functions allow us to deduce the completely continuous character of operator  $T$  as a direct application of Arzela-Ascoli Theorem [12].

Let  $u \in K$ , by (18), we have, for all  $t \in I$ , that the following inequalities are fulfilled for all  $t \in I$

$$T(u)(t) = \int_0^1 G_M(t, s) f(s, u(s)) ds$$

$$\begin{aligned}
 &\geq h(t) \frac{\lambda}{\lambda C_M - 1} \int_0^1 z_M(s) f(s, u(s)) ds \\
 &\geq \frac{h(t)}{R} \int_0^1 \max_{t \in I} \{G_M(t, s)\} f(s, u(s)) ds \\
 &\geq \frac{h(t)}{R} \max_{t \in I} \int_0^1 G_M(t, s) f(s, u(s)) ds \\
 &= \frac{h(t)}{R} \|T(u)\|.
 \end{aligned}$$

So,  $T(u) \in K$  for all  $u \in K$  and the proof is complete. □

In the sequel, for any pair  $\delta, \gamma$  satisfying (20) we introduce the following cone as follows:

$$K_\gamma^\delta = \{u \in K \text{ and } \min_{t \in [\delta, 1]} u(t) \geq \frac{\gamma}{R} \|u\|\}. \tag{23}$$

As in the proof of Lemma 4.2, one can verify the following result.

**Lemma 4.3.** *Assuming condition  $(H_0)$ , we have that  $T(K_\gamma^\delta) \subset K_\gamma^\delta$ .*

In order to deduce the existence and multiplicity of solutions of the nonlinear problem (3) – (4), we introduce the following constants for any  $M \in [-m_0^4, m_1^4]$  (in case of  $M = 0$  it must be considered the limit when  $M$  goes to zero):

$$\Lambda_1 = \frac{1}{\max_{t \in I} \int_\delta^1 G_M(t, s) ds}, \quad \Lambda_2 = \frac{1}{\|\frac{\lambda}{\lambda C_M - 1} z_M\|_1} = \frac{(1 - \lambda C_M)M}{\lambda(z_M'''(1) - z_M'''(0) - 1)}. \tag{24}$$

**Remark 2.** We note that, for  $M \neq 0$ , the identity

$$\|z_M\|_1 = \frac{z_M'''(1) - z_M'''(0) - 1}{M}$$

follows directly from the negative sign of  $z_M$  on  $(0, 1]$  for all  $M \in [-m_0^4, m_1^4]$ , and equality (13).

One can show the exact expression by using a suitable symbolic language package. On Figure 1 it is showed its graph on  $[-m_0^4, m_1^4]$ .

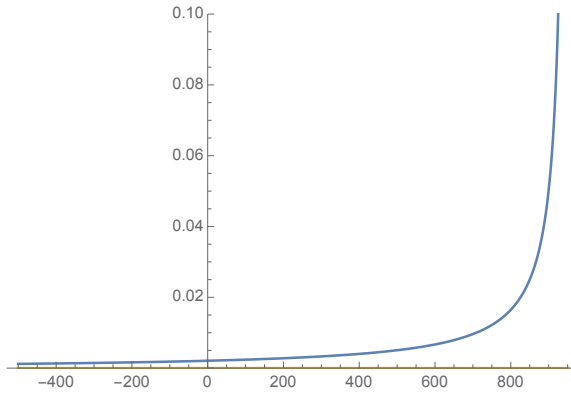


FIGURE 1. Graph of  $\frac{z_M'''(1) - z_M'''(0) - 1}{M}$  on  $[-m_0^4, m_1^4]$

**Theorem 4.4.** *Let  $m \in \mathbb{N} \cup +\infty$  and  $\{r_k\}_{k=1}^m$  and  $\{R_k\}_{k=1}^m$  be such that*

$$r_{k+1} < R_{k+1} < r_k < R_k, \quad k = 1, 2, 3, \dots, m - 1.$$

Where  $A \in (0, \Lambda_2)$  and  $B \in (\Lambda_1, +\infty)$ . Furthermore for each natural number  $k$  we assume that  $f$  satisfies:

(C<sub>1</sub>)  $f(t, u) \geq Br_k$ , for all  $\gamma r_k/R \leq u \leq r_k$ , and  $t \in [\delta, 1]$ .

(C<sub>2</sub>)  $f(t, u) \leq AR_k$ , for all  $0 \leq u \leq R_k$  and  $t \in [0, 1]$ .

Then, if function  $f$  satisfies condition (H<sub>0</sub>), the boundary value problem (3)-(4) has  $2m - 1$  positive solutions  $\{u_k\}_{k=1}^m$  such that

$$r_k < \|u_k\| < R_k, \quad k = 1, 2, 3, \dots, m,$$

and  $\{v_k\}_{k=1}^{m-1}$  such that

$$R_{k+1} < \|v_k\| < r_k, \quad k = 1, 2, 3, \dots, m - 1.$$

*Proof.* Notice that if  $u \in K$ , then  $u > 0$  on  $(0, 1]$ .

Consider the sequence  $\{\Omega_{1,k}\}_{k=1}^m$  and  $\{\Omega_{2,k}\}_{k=1}^m$  of open subsets of  $E$  defined by

$$\Omega_{1,k} = \{u \in K_\gamma^\delta : \|u\| < r_k\}, \quad k = 1, 2, \dots, m,$$

and

$$\Omega_{2,k} = \{u \in K_\gamma^\delta : \|u\| < R_k\}, \quad k = 1, 2, \dots, m.$$

For a fixed  $k$  and  $u \in K_\gamma^\delta \cap \partial\Omega_{1,k}$ , we have that  $\gamma r_k/R = \gamma \|u\|/R \leq \min_{t \in [\delta, 1]} |u(t)| \leq u(s) \leq \|u\| = r_k$ , for all  $s \in [\delta, 1]$ . By condition (C<sub>1</sub>), we get

$$\begin{aligned} \|T(u)\| &= \max_{t \in I} \int_0^1 G_M(t, s) f(s, u(s)) ds \geq \max_{t \in I} \int_\delta^1 G_M(t, s) f(s, u(s)) ds \\ &\geq B r_k \max_{t \in I} \int_\delta^1 G_M(t, s) ds > r_k = \|u\|. \end{aligned}$$

On the other hand, let  $u \in K_\gamma^\delta \cap \partial\Omega_{2,k}$ . Obviously,  $u(s) \leq \|u\| = R_k$  for all  $s \in I$ . By condition (C<sub>2</sub>) and equality (18) we get

$$\|T(u)\| = \max_{t \in I} \int_0^1 G_M(t, s) f(s, u(s)) ds \leq \frac{\lambda R A R_k}{\lambda C_M - 1} \int_0^1 z_M(s) ds < R_k = \|u\|.$$

Theorem 2.2, (ii), implies that  $T$  has a fixed point such that  $r_k \leq \|u_k\| \leq R_k$ .

Since, as we have proved, when one of the previous equality holds, we have that  $\|T(u)\| \neq \|u\|$ , we deduce that the fixed points satisfies that  $r_k < \|u_k\| < R_k$  for all  $k = 1, 2, \dots, m$ .

On the other hand, since  $R_{k+1} < r_k$ ,  $k = 1, \dots, m - 1$ , by a direct application of Theorem 2.2, (i), we deduce the existence of  $m - 1$  fixed points of operator  $T$  such that  $R_{k+1} < \|v_k\| < r_k$ ,  $k = 1, 2, 3, \dots, m - 1$ , and the result is proved.  $\square$

In order to use Theorem 2.5, let  $\beta : K \rightarrow [0, +\infty)$  be a functional defined by:

$$\beta(u) = \min_{t \in [\delta, 1]} u(t).$$

Then, it is easy to see that  $\beta$  is a nonnegative continuous and concave functional on  $K$ , moreover, for each  $u \in K$ , one has

$$\beta(u) \leq \|u\|.$$

For convenience, we introduce the following notations:

$$\Lambda_3 = \left\| \frac{M \lambda}{\lambda C_M - 1} z_M \right\|_1 = \frac{\lambda(z_M'''(1) - z_M'''(0) - 1)}{(1 - \lambda C_M)M}, \Lambda_4 = \min_{t \in [\delta, 1]} \int_\delta^1 G_M(t, s) ds.$$

Our first existence result is the following:

**Theorem 4.5.** *Choose  $0 < \gamma < 1/R$  and let  $a, b, c$  in  $\mathbb{R}$  be such that  $0 < a < b < \frac{b}{\gamma R} \leq c$ . Assume that  $B \in (\Lambda_3, +\infty)$  and  $A \in (0, \Lambda_4)$ , the following properties hold:*

(H<sub>1</sub>) *For all  $u \in [0, c]$ , we have*

$$f(t, u) \leq \frac{c}{B}, \quad t \in [0, 1].$$

(H<sub>2</sub>) *For all  $u \in [0, a]$ , we have*

$$f(t, u) < \frac{a}{B}, \quad t \in [0, 1].$$

(H<sub>3</sub>) *For all  $u \in \left[b, \frac{b}{\gamma R}\right]$  we have*

$$f(t, u) \geq \frac{b}{A}, \quad t \in [\delta, 1].$$

Then, if condition (H<sub>0</sub>) holds, the boundary value problem (3)-(4) has at least two positive solutions  $u_1$  and  $u_2$  in  $\overline{P_c} = \{u \in K_\gamma^\delta, \|u\| \leq c\}$ , such that  $\min_{t \in [\delta, 1]} \{u_1(t)\} > b$  and  $\|u_2\| > a$  with  $\min_{t \in [\delta, 1]} \{u_2(t)\} < b$ .

*Proof.* First, let us prove that the operator  $T$  maps  $\overline{P_c}$  into itself. Indeed, if  $u \in \overline{P_c}$ , then  $\|u\| \leq c$ .

Thus, from hypothesis (H<sub>2</sub>), we have

$$\|T(u)\| = \max_{t \in I} \int_0^1 G_M(t, s) f(s, u(s)) ds \leq \max_{t \in I} \int_0^1 G_M(t, s) ds \frac{c}{B} = c.$$

Hence,  $\|T(u)\| \leq c$ , that is,  $T : \overline{P_c} \rightarrow \overline{P_c}$ .

In the same way, condition (H<sub>2</sub>) implies that condition (A<sub>2</sub>) of Theorem 2.5 holds.

Let us see now that condition (A<sub>1</sub>) of Theorem 2.5 is also fulfilled. Clearly, if  $u(t) = \frac{b}{\gamma R}$  then,  $\beta(u) > b$  and  $\|u\| \leq \frac{b}{\gamma R}$ , that is

$$\left\{ u \in P \left( \beta, b, \frac{b}{\gamma R} \right); \beta(u) > b \right\} \neq \emptyset.$$

Let  $u \in P \left( \beta, b, \frac{b}{\gamma R} \right)$  so, we have

$$b \leq u(t) \leq \frac{b}{\gamma R}, \quad t \in [\delta, 1].$$

Moreover, from (H<sub>3</sub>),

$$\beta(T(u)) = \min_{t \in [\delta, 1]} \int_0^1 G_M(t, s) f(s, u(s)) ds \geq \min_{t \in [\delta, 1]} \int_\delta^1 G_M(t, s) ds \frac{b}{A} > b,$$

and condition (A<sub>1</sub>) of Theorem 2.5 is satisfied for  $d = b/(\gamma R)$ .

Finally, if

$$u \in P(\beta, b, c) \quad \text{and} \quad \|T(u)\| > \frac{b}{\gamma R},$$

then

$$\beta(T(u)) = \min_{t \in [\delta, 1]} T(u)(t) \geq \frac{\gamma}{R} \|T(u)\| > b.$$

Therefore, condition (A<sub>3</sub>) in Theorem 2.5 is also satisfied.

By Theorem 2.5, there exist three nonnegative solutions  $u_1, u_2$  and  $u_3$  such that  $\|u_1\| < a$ ,  $\beta(u_2) > b$  and  $\|u_3\| > a$  with  $\beta(u_3) < b$ .

Since, we cannot ensure that  $u_1 \neq 0$  and  $u_2$  and  $u_3$  are positive on  $(0, 1]$ , the result is concluded.  $\square$

**Remark 3.** In case of  $f$  satisfies the following condition

$(H_4)$   $f(t, 0) \neq 0$  on  $I$ ,

Theorem 4.5 give us three positive solutions of problem (3)-(4).

From the proof of Theorem 4.5, it is easy to see that, if the conditions in the line of  $(H_1)$ - $(H_3)$  are appropriately combined, we can obtain an arbitrary number of positive solutions of problem (3)-(4). More precisely, let  $n$  be an arbitrary positive integer with  $n \geq 1$ . Assume that there exist numbers  $b_j$  ( $1 \leq j \leq n - 1$ ) and  $c_l$  ( $1 \leq l \leq n$ ) such that

$$0 < c_1 < b_1 < \frac{b_1}{\gamma R} \leq c_2 < b_2 < \frac{b_2}{\gamma R} \leq c_3 < \dots \leq c_{n-1} < b_{n-1} < \frac{b_{n-1}}{\gamma R} \leq c_n,$$

then, if we replace the hypothesis  $(H_1)$ - $(H_3)$  of Theorem 4.5 by the following ones:

$(H_{n,1})$  For all  $1 \leq l \leq n$  and  $u \in \mathbb{R}$  such that  $u \in [0, c_l]$ , we have

$$f(t, u) \leq \frac{c_l}{B}, \quad t \in [0, 1].$$

$(H_{n,2})$  For all  $1 \leq j \leq n - 1$  and  $u \in \mathbb{R}$  such that  $Bb_j < Ac_{j+1}$  and  $u \in [b_j, \frac{b_j}{\gamma R}]$ , we have

$$f(t, u) \geq \frac{b_j}{A}, \quad t \in [\delta, 1].$$

we obtain the following result:

**Theorem 4.6.** Under hypothesis  $(H_0)$ ,  $(H_4)$  and  $(H_{n,1}) - (H_{n,2})$ , problem (3)-(4) has at least  $2n - 1$  positive solutions in  $\overline{P_{c_n}}$ .

*Proof.* In order to prove Theorem 4.6, observe that for  $n = 1$ , we know from  $(H_2)$  that  $T : \overline{P_{c_1}} \rightarrow P_{c_1}$ . Then it follows from Schauder fixed point theorem that (3)-(4) has at least one positive solution in  $\overline{P_{c_1}}$ . Moreover, for  $n = 2$ , it is clear that Theorem 4.5 holds (with  $a = c_1$ ,  $b = b_1$  and  $c = c_2$ ). Then, we can obtain three positive solutions  $x_2$ ,  $x_3$ , and  $x_4$ .

Along this way, we can finish the proof by the induction method. To this aim, we suppose that there exist numbers  $b_j$  ( $1 \leq j \leq n$ ) and  $c_l$  ( $1 \leq l \leq n + 1$ ) such that

$$0 < c_1 < b_1 < \frac{b_1}{\gamma R} \leq c_2 < b_2 < \frac{b_2}{\gamma R} \leq \dots \leq c_n < b_n < \frac{b_n}{\gamma R} \leq c_{n+1},$$

and  $(H_{n+1,1})$  and  $(H_{n+1,2})$  hold true. We know by the inductive hypothesis that (3)-(4) has at least  $2n - 1$  positive solutions  $u_i$  ( $i = 1, 2, \dots, 2n - 1$ ) in  $\overline{P_{c_n}}$ . At the same time, it follows from Theorem 4.5,  $(H_{n+1,1})$  and  $(H_{n+1,2})$  that (3)-(4) has at least three positive solutions  $u$ ,  $v$  and  $w$  in  $P_{c_{n+1}}$  such that  $\|u\| < c_n$ ,  $\beta(v) > b_n$  and  $\|w\| > c_n$  with  $\beta(w) < b_n$ . Obviously,  $v$  and  $w$  are not in  $\overline{P_{c_n}}$ . Therefore, (3)-(4) has at least  $2n + 1$  nonnegative solutions in  $P_{c_{n+1}}$ . By using condition  $(H_4)$ , Problem (3)-(4) has at least  $2n + 1$  positive solutions.  $\square$

5. **Examples.** In the sequel, we will obtain the different bounds and results for the particular case when  $M = 0$ . That is, we want to prove the existence of multiple positive solutions of the problem:

$$L_0 u(t) = u^{(4)}(t) = -f(t, u(t)), \quad t \in [0, 1] \quad (25)$$

subject to the boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \quad (26)$$

with  $0 < \lambda < 4$ .

Now, let us obtain the correspondent  $\delta$ ,  $\gamma$  and  $R$ . We have to calculate the related Green's function. By means of the Mathematica program developed in Cabada et al [5] we obtain

$$G_0(t, s) = \begin{cases} \frac{1}{6} (-(1+s)^3 t^3 - (-s+t)^3) + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} & \text{if } 0 < s < t < 1 \\ -\frac{1}{6} (1+s)^3 t^3 + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} & \text{if } 0 \leq t \leq s \leq 1 \end{cases}$$

Thus, clearly, for this case

$$\tilde{\varphi}(t, s) = \begin{cases} \frac{6(-4+\lambda) \left( \frac{1}{6} (-(1+s)^3 t^3 - (-s+t)^3) + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1+s)^3 s \lambda} & \text{if } 0 < s < t < 1 \\ \frac{t^2 (-3(-4+\lambda) + 4t(-3+\lambda))}{\lambda} & \text{if } s = 0 \\ \frac{4t^3}{\lambda} & \text{if } s = 1 \\ \frac{6(-4+\lambda) \left( -\frac{1}{6} (1+s)^3 t^3 + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1+s)^3 s \lambda} & \text{if } 0 < t \leq s < 1. \end{cases}$$

So we have

$$\frac{\partial \tilde{\varphi}}{\partial s}(t, s) = \begin{cases} \phi_1(t, s) & \text{if } 0 < s < t < 1 \\ 0 & \text{if } s = 0 \text{ or } s = 1 \\ \phi_2(t, s) & \text{if } 0 < t \leq s < 1 \end{cases}$$

where

$$\begin{aligned} \phi_1(t, s) &= \frac{6(-4+\lambda) \left( \frac{1}{6} (-3(-1+s)^2 t^3 + 3(-s+t)^2) + \frac{(1-s)^3 t^3 \lambda}{6(4-\lambda)} - \frac{(1-s)^2 s t^3 \lambda}{2(4-\lambda)} \right)}{(-1+s)^3 s \lambda} \\ &- \frac{6(-4+\lambda) \left( \frac{1}{6} (-(1+s)^3 t^3 - (-s+t)^3) + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1+s)^3 s^2 \lambda} \\ &- \frac{18(-4+\lambda) \left( \frac{1}{6} (-(1+s)^3 t^3 - (-s+t)^3) + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1+s)^4 s \lambda} \end{aligned}$$

and

$$\begin{aligned} \phi_2(t, s) &= \frac{6(-4+\lambda) \left( -\frac{1}{2} (1+s)^2 t^3 + \frac{(1-s)^3 t^3 \lambda}{6(4-\lambda)} - \frac{(1-s)^2 s t^3 \lambda}{2(4-\lambda)} \right)}{(-1+s)^3 s \lambda} \\ &- \frac{6(-4+\lambda) \left( -\frac{1}{6} (1+s)^3 t^3 + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1+s)^3 s^2 \lambda} \end{aligned}$$

$$- \frac{18(-4 + \lambda) \left( -\frac{1}{6}(-1 + s)^3 t^3 + \frac{(1-s)^3 s t^3 \lambda}{6(4-\lambda)} \right)}{(-1 + s)^4 s \lambda}$$

By denoting

$$\alpha(t) := \frac{-1 + 2t + 2t^2 - \sqrt{1 - t - 3t^2 + 5t^3 - 2t^4}}{1 + t + t^2}, \tag{27}$$

we have

$$\begin{cases} \alpha(t) \leq 0 & \text{if } t \in [0, \frac{1}{2}] \\ 0 \leq \alpha(t) \leq t & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \tag{28}$$

and for all  $t \in [\frac{1}{2}, 1]$ ,

$$\frac{\partial \tilde{\varphi}(t, \alpha(t))}{\partial s} = 0. \tag{29}$$

- If  $t \in [0, \frac{1}{2}]$ , we get  $\frac{\partial \tilde{\varphi}(t,s)}{\partial s} \leq 0$  for all  $s \in I$ . Then  $\max_{s \in I} \tilde{\varphi}(t, s) = \tilde{\varphi}(t, 0) = \frac{t^2(-3(-4+\lambda)+4t(-3+\lambda))}{\lambda}$  and  $\min_{s \in I} \tilde{\varphi}(t, s) = \tilde{\varphi}(t, 1) = \frac{4t^3}{\lambda}$ .
- If  $t \in [\frac{1}{2}, 1]$ , we get

$$\begin{cases} \frac{\partial \tilde{\varphi}(t,s)}{\partial s} \geq 0 & \text{if } s \in [0, \alpha(t)] \\ \frac{\partial \tilde{\varphi}(t,s)}{\partial s} \leq 0 & \text{if } s \in [\alpha(t), 1]. \end{cases} \tag{30}$$

Then

$$\begin{aligned} \max_{s \in I} \tilde{\varphi}(t, s) &= \tilde{\varphi}(t, \alpha(t)) \\ &= - \frac{15t^2(-4 + \lambda) + 2 \left( 1 + \sqrt{-(-1 + t)^3(1 + 2t)} \right) (-4 + \lambda)}{27\lambda} \\ &\quad + \frac{t \left( 3 + 4\sqrt{-(-1 + t)^3(1 + 2t)} \right) (-4 + \lambda) - 4t^3(7 + 5\lambda)}{27\lambda} \end{aligned}$$

and

$$\begin{aligned} h(t) &= \min_{s \in I} \tilde{\varphi}(t, s) \\ &= \min \left\{ \frac{4t^3}{\lambda}, \frac{t^2(-3(-4 + \lambda) + 4t(-3 + \lambda))}{\lambda} \right\} \\ &= \begin{cases} \frac{4t^3}{\lambda} & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{t^2(-3(-4 + \lambda) + 4t(-3 + \lambda))}{\lambda} & \text{if } t \in [\frac{3}{4}, 1]. \end{cases} \end{aligned}$$

So we obtain

$$\max_{s \in I} \tilde{\varphi}(t, s) = \begin{cases} \frac{t^2(-3(-4 + \lambda) + 4t(-3 + \lambda))}{\lambda} & \text{if } t \in [0, \frac{1}{2}] \\ - \frac{15t^2(-4 + \lambda) + 2 \left( 1 + \sqrt{-(-1 + t)^3(1 + 2t)} \right) (-4 + \lambda)}{27\lambda} \\ \quad - \frac{t \left( 3 + 4\sqrt{-(-1 + t)^3(1 + 2t)} \right) (-4 + \lambda) - 4t^3(7 + 5\lambda)}{27\lambda} & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$h(t) = \begin{cases} \frac{4t^3}{\lambda} & \text{if } t \in [0, \frac{3}{4}] \\ \frac{t^2(-3(-4 + \lambda) + 4t(-3 + \lambda))}{\lambda} & \text{if } t \in [\frac{3}{4}, 1]. \end{cases}$$

We choose  $\lambda = 2.5$ ,  $\delta = 0.25$ ,  $R = \max_{(t,s) \in I \times I} \tilde{\varphi}(t, s) = \max_{s \in I} \tilde{\varphi}(1, s) = 1.6$ ,  $\gamma = \min_{t \in [\delta, 1]} h(t) = h(\delta) = 0.025$ ,  $\frac{1}{R} = 0.625$ ,  $\frac{\gamma}{R} = 0.015625$ ,  $\Lambda_1 = 113.778$ ,  $\Lambda_2 = 71.9999$ ,  $\Lambda_3 = 0.0138889$  and  $\Lambda_4 = 0.000343323$ .

**Example 1.** Choosing  $m \in \mathbb{N}$ ,  $\alpha \in \mathbb{R}$ ,  $A \in (0, \Lambda_2)$  and  $B \in (\Lambda_1, +\infty)$  such that  $\alpha > \max\left(10, \sqrt{\frac{B}{A}}\right)$ ,  $R_k = \alpha^{-4k}$  and  $r_k = \alpha^{-(4k+2)}$  for  $k \in \{1, 2, \dots, m-1\}$ .

We have that  $r_{k+1} < R_{k+1} < r_k < R_k$ ,  $[10^{-2}r_k, \frac{\gamma}{R}r_k] \subset [R_{k+1}, r_k]$ ,  $R_{k+1} < r_k < Br_k$  and  $Br_k < AR_k$ .

Let  $f$  a function defined as follows:

$$f(t, u) = \begin{cases} R_{k+1} & \text{if } u \in [0, R_{k+1}] \\ \frac{Br_k - R_{k+1}}{10^{-2}r_k - R_{k+1}}(u - R_{k+1}) + R_{k+1} & \text{if } u \in [R_{k+1}, 10^{-2}r_k] \\ Br_k & \text{if } u \in [10^{-2}r_k, r_k] \\ \frac{Br_k - AR_k}{r_k - R_k}(u - R_k) + AR_k & \text{if } u \in [r_k, R_k] \\ AR_k & \text{if } u \in [R_k, +\infty) \end{cases}$$

Moreover  $f$  satisfies conditions  $(H_0)$ ,  $(C_1)$  and  $(C_2)$ .

Then all the conditions of Theorem 4.4 are satisfied. Therefore, by Theorem 4.4 we know that boundary value problem (25)-(26) has  $2m - 1$  positive solutions  $\{u_k\}_{k=1}^m$  such that  $\alpha^{-(4k+2)} \leq \|u_k\| \leq \alpha^{-4k}$  for each  $k = 1, 2, \dots, m$  and  $\{v_k\}_{k=1}^m$  such that  $\alpha^{-(4(k+1))} \leq \|v_k\| \leq \alpha^{-(4k+2)}$  for each  $k = 1, 2, \dots, m - 1$ .

**Example 2.** As an application of Theorem 4.6, we consider the boundary value problem (25) with

$$f(t, u) = \begin{cases} \frac{\lfloor (t - \frac{1}{2})^2 + 1 \rfloor}{10} + 3334u^3 & \text{if } 0 \leq u \leq 1 \\ 3333 + \frac{\lfloor (t - \frac{1}{2})^2 + 1 \rfloor}{10} + u & \text{if } u > 1. \end{cases}$$

Choosing  $n = 2$ ,  $c_1 = \frac{1}{30}$ ,  $b_1 = 1$ ,  $\frac{b_1}{\gamma R} = 25$ ,  $c_2 = 70$ ,  $B = 0.014$  and  $A = 0.0003$ , it holds that  $Bb_1 < Ac_2$ ,

$$f(t, u) = \frac{\lfloor (t - \frac{1}{2})^2 + 1 \rfloor}{10} + 3334u^3 \leq 0.2234 \leq \frac{c_1}{B} \approx 2.38095, \text{ for } (t, u) \in I \times [0, \frac{1}{30}],$$

$$f(t, u) = \frac{\lfloor (t - \frac{1}{2})^2 + 1 \rfloor}{10} + 3333 + u \leq 3410.1 \leq \frac{c_2}{B} \approx 5000, \text{ for } (t, u) \in I \times [0, 70]$$

and

$$f(t, u) = \frac{\lfloor (t - \frac{1}{2})^2 + 1 \rfloor}{10} + 3333 + u \geq 3334.025 \geq \frac{b_1}{A} \approx 3333.3333, \text{ for } (t, u) \in [0.25, 1] \times [1, 25].$$

From Theorem 4.6, the boundary value problem (25)-(26) has at least three positive solutions  $u_1, u_2, u_3$  satisfying  $\|u_1\| < \frac{1}{30}$ ,  $\beta(u_2) > 1$  and  $\|u_3\| > \frac{1}{30}$  with  $\beta(u_3) < 1$ .

It is clear that  $f(t, u) = f(1 - t, u)$ , then  $v_1 : t \mapsto u_1(1 - t)$ ,  $v_2 : t \mapsto u_2(1 - t)$  and  $v_3 : t \mapsto u_3(1 - t)$  are three positive solutions of (25) with boundary conditions  $u(1) = u'(1) = u''(1) = 0$  and  $u(0) = \lambda \int_0^1 u(s) ds$ .

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