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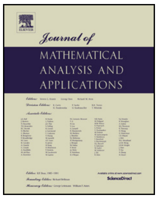

Regular Articles

On the connection between Stieltjes differential equations and ordinary differential equations

F. Adrián F. Tojo [a,b,*]

[a] *Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, Facultade de Matemáticas, Rúa de Lope Gómez de Marzoa, s/n, 15705, Santiago de Compostela, Spain*
[b] *CITMAga, Santiago de Compostela, Spain*

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ABSTRACT

In this work, we establish a link between Stieltjes differential equations and ordinary differential equations, allowing for the transformation of one into the other. As applications, we prove Binding's theorem for the equation $x'_g = f \circ x$, as well as a topological result used in the theory of differential equations.

1. Introduction

The current interest in Stieltjes differential equations can be traced back to [21], where they were presented as a useful unifying framework for studying differential and difference equations, as well as dynamic equations on time scales. The properties of the Stieltjes derivative are particularly suitable for researching models with periods of latency or sudden change, such as in population dynamics, whether it is insects [10,17], fish populations [19], single-celled organisms [18,20], or infectious diseases [1], in addition to problems of other nature [13].

It is the use of measure theory that makes the theory so powerful and versatile, but working with it also comes with some drawbacks. Proofs that are straightforward in the classical case require significant reworking or a lengthy prelude of technical lemmas in the Stieltjes case to arrive at the same (though much more general) conclusions, as can be seen in works such as [7,8,11].

It would be of great interest to find a way to connect Stieltjes differential equations to ordinary differential equations so that known results for ODEs can be transposed with less effort to Stieltjes differential equations.

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[*] Correspondence to: Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, Facultade de Matemáticas, Rúa de Lope Gómez de Marzoa, s/n, 15705, Santiago de Compostela, Spain.
E-mail address: fernandoadrian.fernandez@usc.es.

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This is the main objective of this work, and such a connection comes in the form of isomorphisms between the Stieltjes and classical function spaces related to these equations.

In particular, we will examine a Stieltjes differential problem of the form

$$u_g'(t) = f(t, u(t)),\ \mu_g\text{-a.e. } t \in [t_0, \infty), \quad u(t_0) = x_0,$$

where f is an L^1_g-Carathéodory function. This type of problem has been extensively studied, as seen in [13,24].

We will start by considering the equivalent integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s)) \,\mathrm{d}\,\mu_g(s).$$

Integral problems of this nature are the focus of quite a variety of works, many of them considering Lebesgue-Stieltjes measures or the Kurzweil-Stieltjes integral with respect to functions of bounded variation which generalize the case of our g—see, for instance, [25–27].

Our interest is to relate this equation to the following integral equation with respect to the usual Lebesgue measure:

$$v(x) = u_0 + \int_{x_0}^{x} f(g^\dagger(y), v(y)) \,\mathrm{d}\,y,$$

where $g^\dagger(y) := \inf\{t \in I : g(t) \geqslant y\}$, and the associated ordinary differential equation

$$\widetilde{v}'(x) = f(g^\dagger(x), \widetilde{v}(x))\chi_{\mathbb{R}\backslash D_g^*}(x),$$

where $\chi_{\mathbb{R}\backslash D_g^*}$ is the indicator function of the set $\mathbb{R}\backslash D_g^*$. We will show that, under certain conditions—see Theorem 5.4—there is a connection between the solutions to these problems.

The structure of the paper is as follows. In Section 2, we begin by reviewing the fundamentals of Stieltjes calculus necessary for the work ahead. Section 3 introduces the Sobolev spaces $W_g^{n,p}(I, \mathbb{F})$, which will be used in later results—e.g., Corollary 4.36—and proves some of their basic properties, such as their Banach space structure and their isomorphism with $\mathbb{F}^n \oplus L^1_g(I, \mathbb{F})$. Section 4 presents the main result of the paper: the aforementioned isomorphism—see Theorem 4.30—along with variants depending on the space in which we wish to work. However, before reaching this, we first develop a series of preliminary results in measure theory.

In Section 5, we present the applications of this new tool. First, we use it to establish a clear connection between Stieltjes differential equations and ordinary differential equations—see Theorem 5.4—and, as an application, we prove a version of Binding's famous existence theorem for the equation $v'(x) = f(v(x))$ [2] in the context of Stieltjes differential equations. Additionally, we explore applications to topological results relevant to the theory of differential equations, particularly in the context of topological methods. While we do not delve deeply into this aspect, we demonstrate, as an example, the compact embedding of $W_g^{1,1}([a, b), \mathbb{F})$ into $L^p_g([a, b), \mathbb{F})$.

Finally, in Section 6, we briefly discuss additional possible applications of the tools developed in this work, along with their limitations in certain cases.

2. The Stieltjes integral and derivative

In this section we will introduce the basics regarding the Stieltjes integral and derivative. The interested reader may check [9,11,13,17,21,23] and references therein for more details. For the rest of this work we will consider an interval $I \subset \mathbb{R}$ of the form $(-\infty, b)$ or $[a, b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$, that is, an interval on which the limit from the right can be considered at every point. We will also consider a left-continuous function $g : I \subset \mathbb{R} \to \mathbb{R}$ which we will call a *derivator*. We will denote by $\mathbb{F}$ the field $\mathbb{R}$ or $\mathbb{C}$. Id will denote the identity function or operator. Sometimes we will identify its domain X by writing Id_X. We shall write as μ_g the Lebesgue-Stieltjes measure associated to g given by

$$\mu_g([c, d)) = g(d) - g(c), \quad c, d \in \mathbb{R},\ c < d,$$

and denote by $\mathscr{L}$ and $\mathscr{B}$ the Lebesgue σ-algebra and the Borel σ-algebra (with respect to the usual topology) on $\mathbb{R}$ respectively, by μ the Lebesgue measure on $\mathbb{R}$, and by $\mathscr{M}_g$ the σ-algebra associated to μ_g. We will denote by μ_g^* and μ^* their respective associated exterior measures. It is important to remark that μ_g is a complete Borel measure (that is, $\mathscr{B} \subset \mathscr{M}_g$). We denote by $\mathscr{L}_g^1(X, \mathbb{F})$ the set of Lebesgue-Stieltjes μ_g-integrable functions on a μ_g-measurable set X with values in $\mathbb{F}$, whose integral we write as

$$\int\limits_X f(s)\, \mathrm{d}\, \mu_g(s), \quad f \in \mathscr{L}_g^1(X, \mathbb{F}).$$

$\mathscr{L}_g^1(X, \mathbb{F})$ is a complete pseudonormed space with the pseudonorm $\|f\|_{L^1} = \int_X |f|\, \mathrm{d}\, \mu_g$.

When integrating functions, we will use the following notation for convenience:

$$\int\limits_x^y f(s)\, \mathrm{d}\, \mu_g(s) = \begin{cases} \displaystyle\int\limits_{[x,y)} f(s)\, \mathrm{d}\, \mu_g(s), & y \geqslant x, \\ -\displaystyle\int\limits_{[y,x)} f(s)\, \mathrm{d}\, \mu_g(s), & y \leqslant x. \end{cases}$$

$L_g^1(X, \mathbb{F})$ will be the Banach space associated to $\mathscr{L}_g^1(X, \mathbb{F})$ by taking equivalence classes of functions that are equal μ_g-a.e. We will denote by $[f]$ the class of $f \in \mathscr{L}^1(I, \mathbb{F})$ in $L^1(I, \mathbb{F})$ when necessary (usually, we will abuse notation and write just f). The spaces $L_g^p(X, \mathbb{F})$ for $p \in [1, \infty]$ are defined as usual (they are L^p spaces with respect to the measure μ_g) and we will denote their respective norms by $\|\cdot\|_{L_g^p}$. Observe that

$$\|f\|_{L_g^\infty} := \operatorname{ess\,sup}_g |f| := \inf\{M \in [0, \infty] \ :\ |f(x)| \leqslant M \ \mu_g\text{-a.e.}\}.$$

We will also define

$$\operatorname{ess\,inf}_g f := \sup\{M \in [-\infty, \infty] \ : f(x) \geqslant M \ \mu_g\text{-a.e.}\}.$$

Define the sets

$$\begin{aligned} C_g &= \{t \in \mathbb{R} \ :\ g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}, \\ D_g &= \{t \in \mathbb{R} \ :\ \Delta g(t) > 0\}, \end{aligned}$$

where $\Delta g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+) := \lim_{s \to t^+} g(s)$ denotes the right hand side limit of g at t. We will use the notation $g^+(t) := g(t^+)$. The corresponding notation will be used for limits from the left. Observe that $C_g \cap D_g = \emptyset$. Furthermore, the set C_g is open in the usual topology of the real line, so it can be uniquely expressed as a countable union of open pairwise disjoint intervals, say

$$C_g = \bigcup_{n \in \Lambda} (a_n, b_n), \tag{2.1}$$

where $\Lambda \subset \mathbb{N}$.

Definition 2.1 *([9, Definition 3.7]).* We define the *Stieltjes derivative*, or *g-derivative*, of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R}$ as

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g \cup C_g, \\ \displaystyle\lim_{s \to t^+} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in D_g, \\ \displaystyle\lim_{s \to b_n^+} \frac{f(s) - f(b_n)}{g(s) - g(b_n)}, & t \in C_g,\ t \in (a_n, b_n), \end{cases}$$

where a_n, b_n are as in (2.1), provided the corresponding limits exist. In that case, we say that f is *g-differentiable at t*.

For more details on the definition and properties of the derivative, the reader may refer to [7,9].

Remark 2.2. Depending on the definition of g, it may not be possible to consider the g-derivative at some points. For instance, if g is constant on an interval of the form $[a, \infty)$, the g-derivative is not going to be well defined for any function on the points of $[a, \infty)$. From now on, whenever we consider a derivator $g : I \to \mathbb{R}$, we will assume it such that the derivative can be considered in all of I. Furthermore, as stated at the beginning, I will be open from the right. This condition is asked to avoid the possibility of g being defined on $\mathbb{R}$ and having a jump point at the right endpoint of I (say b). This would cause an ambiguity, as $g|_{\overline{I}}$ would not have a jump at b and, therefore, $f'_g(b) \neq f'_{g|_{\overline{I}}}(b)$ in general.

Definition 2.3 *([21, Definition 5.1]).* A function $F : I \to \mathbb{R}$ is *g-absolutely continuous* if for each $\varepsilon \in \mathbb{R}^+$ there is some $\delta \in \mathbb{R}^+$ such that for any family $\{(a_n, b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of I the inequality

$$\sum_{n=1}^{m} \left(g\left(b_n\right) - g\left(a_n\right)\right) < \delta$$

implies

$$\sum_{n=1}^{m} \left|F\left(b_n\right) - F\left(a_n\right)\right| < \varepsilon.$$

The following result will be a key component in the theory we want to develop.

Theorem 2.4 *(Fundamental Theorem of Calculus for the Lebesgue-Stieltjes Integral [21, Theorem 5.4]). A function $F : [a, b] \longrightarrow \mathbb{R}$ is g-absolutely continuous on $[a, b]$ if and only if the following three conditions are fulfilled:*

1. *There exists $F'_g(t)$ for g-almost all $t \in [a, b)$;*
2. *$F'_g \in \mathscr{L}^1_g([a, b))$; and*
3. *For each $t \in [a, b]$ we have*

$$F(t) = F(a) + \int\limits_{[a,t)} F'_g \, \mathrm{d}\, \mu_g.$$

2.1. The domain of the derivator

The interval on which the derivator is defined is not in general crucial when it comes to study Stieltjes function spaces (except for some minor circumstances such as those detailed in Remark 2.2). Nevertheless, it will be interesting to know when we can extend g to the whole real line and whether it has any implications regarding the basic spaces we will deal with.

Definition 2.5. Let $\overline{\mathbb{R}}\equiv[-\infty,\infty]$ with the topology that makes it homeomorphic to a compact interval of the real line. Any non-decreasing function $u:I\to\mathbb{R}$ can be extended to a non-decreasing function $\overline{u}:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ by defining

$$\overline{u}(t):=\begin{cases} u(t), & t\in I,\\ u(\inf I), & t<\inf I \text{ and } \inf I\in I,\\ \lim\limits_{y\to\inf I}u(y), & t\leqslant\inf I \text{ and } \inf I\notin I,\\ u(\sup I), & t>\sup I \text{ and } \sup I\in I,\\ \lim\limits_{y\to\sup I}u(y), & t\geqslant\sup I \text{ and } \sup I\notin I.\end{cases}$$

The fact that $\overline{g}$ takes values at ∞ and $-\infty$ does not affect the construction of the exterior measure $\mu_{\overline{g}}^*$ just by considering $\infty-\infty=0$, $(-\infty)-(-\infty)=0$ when studying the values of $\overline{g}(b)-\overline{g}(a)$ for $a,b\in\overline{\mathbb{R}}$, $b\geqslant a$. Furthermore, the associated σ-algebra, $\mathscr{M}_{\overline{g}}$, and measure, $\mu_{\overline{g}}$, relate to $\mathscr{M}_g$ and μ_g in a natural way, as Lemma 2.6 shows.

Lemma 2.6. *The following conditions hold:*

1. $\mathscr{M}_{\overline{g}}=\{A\sqcup B\ :\ A\in\mathscr{M}_g, B\subset\overline{\mathbb{R}}\backslash I\}$.
2. *If* $C=A\sqcup B$ *with* $A\in\mathscr{M}_g$ *and* $B\subset\overline{\mathbb{R}}\backslash I$, $\mu_{\overline{g}}(C)=\mu_g(A)$.
3. $L^p_{\overline{g}}(\mathbb{R},\mathbb{F})=\{f:\mathbb{R}\to\mathbb{R}\ :\ f|_I\in L^p_g(I,\mathbb{F})\}$.

Proof. 1. Since $\overline{g}$ is constant on $[-\infty,\inf I)$, $\mu_{\overline{g}}^*([-\infty,\inf I))=0$. Furthermore, $\mu_{\overline{g}}^*(\inf I)=0$ if $\inf I\notin I$. Analogously, $\mu_{\overline{g}}^*((\sup I,\infty])=0$ and $\mu_g^*(\sup I)=0$ if $\sup I\notin I$, so $A\in\mathscr{M}_{\overline{g}}$ for every $A\subset\overline{\mathbb{R}}\backslash I$, in particular, $\overline{\mathbb{R}}\backslash I\in\mathscr{M}_{\overline{g}}$ and $I\in\mathscr{M}_{\overline{g}}$. Since $\overline{g}|_I=g$, we have that $\mathscr{M}_g\subset\mathscr{M}_{\overline{g}}$, so any set of the form $A\sqcup B$ with $A\in\mathscr{M}_g$, $B\subset\mathbb{R}\backslash I$ belongs to $\mathscr{M}_{\overline{g}}$. On the other hand, if $C\in\mathscr{M}_{\overline{g}}$, $C\cap I\subset I$ and, since $\overline{g}|_I=g$, $C\in\mathscr{M}_g$.

2. It is a direct consequence of the fact that $\overline{g}|_I=g$ and $\mu_{\overline{g}}^*(\overline{\mathbb{R}}\backslash I)=0$.

3. $f\in L^p_{\overline{g}}(\overline{\mathbb{R}},\mathbb{F})$ if and only if $f|_I\in L^p_{\overline{g}}(I,\mathbb{F})$ and $f|_I\in L^p_{\overline{g}}(\overline{\mathbb{R}}\backslash I,\mathbb{F})$. $L^p_{\overline{g}}(I,\mathbb{F})=L^p_g(I,\mathbb{F})$ and any function $f:\overline{\mathbb{R}}\backslash I\to\mathbb{F}$ belongs to $L^p_{\overline{g}}(\overline{\mathbb{R}}\backslash I,\mathbb{F})$, so the result holds. ■

Given a derivator $g:I\to\mathbb{R}$, Lemma 2.6 lets us use Definition 2.5 to consider g as a map $g:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ whenever convenient.

3. $W_g^{n,p}(I,\mathbb{F})$ spaces

We now present the Stieltjes version of the classical Sobolev spaces.

Definition 3.1 *([11, Definition 5.1]).* Let $p\in[1,\infty]$. We define the *Stieltjes-Sobolev spaces*

$$W_g^{1,p}(I,\mathbb{F})=\left\{u\in L^p_g(I,\mathbb{F}):\ \exists\widetilde{u}\in L^p_g(I,\mathbb{F})\text{ such that }u(y)-u(x)=\int_x^y\widetilde{u}\,\mathrm{d}\mu_g,\ x,y\in\overline{I}\right\}.$$

We endow $W_g^{1,p}(I,\mathbb{F})$ with the norm

$$\|u\|_{W_g^{1,p}} := \|u\|_{L_g^p} + \|\widetilde{u}\|_{L_g^p}\,.$$

The spaces $W_g^{1,p}(I,\mathbb{F})$ are Banach spaces –see [11, Theorem 5.6]. We can further generalize this definition.

Definition 3.2. Let $p \in [1,\infty]$. For $n \in \mathbb{N}$, $n > 1$, we define the *Stieltjes-Sobolev spaces*

$$W_g^{n,p}(I,\mathbb{F}) = \left\{ u \in L_g^p(I,\mathbb{F}) : \exists \widetilde{u} \in W_g^{n-1,p}(I,\mathbb{F}) \text{ such that } u(y) - u(x) = \int_x^y \widetilde{u}\,\mathrm{d}\,\mu_g,\ x,y \in \overline{I} \right\}.$$

We endow $W_g^{n,p}(I,\mathbb{F})$ with the norm

$$\|u\|_{W_g^{1,p}} := \|u\|_{L_g^p} + \|\widetilde{u}\|_{W_g^{n-1,p}}\,.$$

For convenience, we will write $W_g^{0,p}(I,\mathbb{F}) \equiv L_g^p(I,\mathbb{F})$. For $g = \mathrm{Id}$ we obtain the usual Sobolev spaces $W^{n,p}(I,\mathbb{F})$.

Remark 3.3. for $a, b \in \mathbb{R}$ $a < b$, every function $f \in W_g^{1,p}([a,b),\mathbb{F})$ for $n \in \mathbb{N}$ is g-absolutely continuous so, in particular, left continuous. Hence, we may as well consider the f to be defined on the interval $[a,b]$ instead of $[a,b)$, as the value at b is determined by the limit from the left at b. Clearly the norm in the space $W_g^{1,p}([a,b),\mathbb{F})$ does not change with this extension (it is assumed that $b \notin D_g$).

Let us show that these are Banach spaces (the proof is essentially that of [11, Theorem 5.6]).

Theorem 3.4. *Let $n \in \mathbb{N}$, $p \in [1,\infty]$. $W_g^{n,p}(I,\mathbb{F})$ is a Banach space.*

Proof. We know from [11, Lemma 5.5] that the embedding $W_g^{1,p}([a,b),\mathbb{F}) \subset \mathrm{BC}_g([a,b],\mathbb{F})$ is continuous for $a, b \in \mathbb{R}$, $a < b$. It is clear from the definition of the spaces $W_g^{1,p}$ and their norms that the embedding $W_g^{n,p}([a,b),\mathbb{F}) \subset W_g^{m,p}([a,b),\mathbb{F})$ is continuous for $n \geqslant m \geqslant 1$. Hence, the embedding $W_g^{n,p}([a,b),\mathbb{F}) \subset \mathrm{BC}_g([a,b],\mathbb{F})$ is continuous.

The spaces $W_g^{1,p}(I,\mathbb{F})$ are already known to be Banach spaces –see [11, Theorem 5.6]. We proceed now by induction. Assume $W_g^{n-1,p}(I,\mathbb{F})$ is a Banach space, $n \in \mathbb{N}$. Let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset W_g^{n,p}(I,\mathbb{F})$. We have, thanks to the definition of the space $W_g^{n,p}(I)$ and its norm, that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_g^p(I,\mathbb{F})$ and $(\widetilde{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_g^{n,p}(I,\mathbb{F})$. Moreover, due to the fact that the embedding $W_g^{n,p}([a,b),\mathbb{F}) \subset \mathrm{BC}_g([a,b],\mathbb{F})$ is continuous, $(u_n|_{[a,b]})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{BC}_g([a,b],\mathbb{F})$, for all a, $b \in I$, $a < b$.

Due to the completeness of $L_g^p(I,\mathbb{F})$ and $W_g^{n-1,p}(I,\mathbb{F})$, there exist elements $u \in L_g^p(I,\mathbb{F})$, $\widetilde{u} \in W_g^{n-1,p}(I,\mathbb{F})$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{L_g^p} = 0, \quad \lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{W_g^{n-1,p}} = 0.$$

This last equality implies that $\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{L_g^p} = 0$. Now, given x, $y \in \overline{I}$, $x < y$, and taking into account that $L_g^p([x,y),\mathbb{F}) \subset L_g^1([x,y),\mathbb{F})$ is a continuous embedding,

$$\left| \int_{[x,y)} (\widetilde{u}_n(s) - \widetilde{u}(s))\,\mathrm{d}\,\mu_g(s) \right| \leqslant \int_{[x,y)} |\widetilde{u}_n(s) - \widetilde{u}(s)|\,\mathrm{d}\,\mu_g(s) = \|\widetilde{u}_n - \widetilde{u}\|_{L_g^1} \leqslant C\, \|\widetilde{u}_n - \widetilde{u}\|_{L_g^p}\,, \tag{3.1}$$

for some constant $C \in \mathbb{R}^+$ which does not depend on $\widetilde{u}_n$ nor $\widetilde{u}$. Thus, because of inequality (3.1) and the fact that $\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{L^p_g} = 0$, we have that, for $x,\, y \in \overline{I}$, $x < y$,

$$\lim_{n\to\infty} \int\limits_{[x,y)} \widetilde{u}_n(s)\,\mathrm{d}\,\mu_g(s) = \int\limits_{[x,y)} \widetilde{u}(s)\;\mathrm{d}\,\mu_g(s). \tag{3.2}$$

On the other hand, let $a,\, b \in \mathbb{R}$, $a < b$. Since $\mathrm{BC}_g([a,b],\mathbb{F})$ is a Banach space –see [13, Theorem 3.4] and the embedding $W_g^{n,p}([a,b),\mathbb{F}) \subset \mathrm{BC}_g([a,b],\mathbb{F})$ is continuous, for all $p \in [1,\infty]$, we have that $u|_{[a,b]} \in \mathrm{BC}_g([a,b],\mathbb{F})$ and

$$\lim_{n\to\infty} u_n(t) = u(t),\ \forall t \in [a,b].$$

Since $[a,b]$ was fixed arbitrarily, $u \in \mathrm{BC}_g(I,\mathbb{F})$ and

$$\lim_{n\to\infty} u_n(t) = u(t),\ \forall t \in I. \tag{3.3}$$

Finally, thanks to (3.2) and (3.3), we can pass to the limit in the following expression

$$u_n(y) = u_n(x) + \int\limits_x^y \widetilde{u}_n(s)\;\mathrm{d}\,\mu_g(s),\ x,\, y \in I,\ n \in \mathbb{N},$$

and we obtain that

$$u(y) = u(x) + \int\limits_x^y \widetilde{u}(s)\;\mathrm{d}\,\mu_g(s),\ \forall x,\, y \in I.$$

Therefore, $(u_n)_{n\in\mathbb{N}}$ converges to u in $W_g^{n,p}(I)$. ■

We will now provide an equivalent norm in $W_g^{n,1}(I,\mathbb{F})$ through an embedding.

Theorem 3.5. *Let $x_0 \in I$, and $g : I \to \mathbb{R}$ be a non-constant derivator. Then, the map*

$$\begin{tikzcd} W_g^{n,1}(I,\mathbb{F}) \arrow[r, "\varphi_g"] & \mathbb{F}^n \oplus L^1_g(I,\mathbb{F}) \\ f \arrow[r, mapsto] & \left((f(x_0), f'_g(x_0), \dots, f_g^{(n-1)}(x_0)), f_g^{(n)}\right) \end{tikzcd}$$

is a linear embedding, where we consider in $\mathbb{F}^n \oplus L^1_g(I,\mathbb{F})$ the norm

$$\|((\alpha_0,\dots,\alpha_{n-1}),h)\| := |\alpha_0| + \cdots + |\alpha_{n-1}| + \|h\|_{L^1_g},$$

for $((\alpha_0,\dots,\alpha_{n-1}),h) \in \mathbb{F}^n \oplus L^1_g(I,\mathbb{F})$. Furthermore, if $\mu_g(I) < \infty$, it is an isomorphism.

Proof. φ_g is clearly linear. To see that it is continuous, we start by making some observations.

Given $f \in W^{n,1}(I,\mathbb{F})$, $f_g^{(k)}$ is g-absolutely continuous for $k = 0,\dots,n-1$. Furthermore, either $\mu_g(I) < \infty$ or $\operatorname{ess\,inf}_g |f^{(k)}| = 0$ for every $k = 0,\dots,n-1$, as $f_g^{(k)} \in L^1_g(I,\mathbb{F})$. We study these two cases:

1. **Case** $\mu_g(I) < \infty$**:** Fix $k \in \{0,\dots,n-1\}$ and let $\varepsilon \in \mathbb{R}^+$ and $c \in \mathbb{R}$ be such that $|f^{(k)}(c)| < \operatorname{ess\,inf}_g |f| + \varepsilon$. Then,

$$\|f_g^{(k)}\|_{L_g^1}=\int_{\mathbb{R}}|f_g^{(k)}|\,\mathrm{d}\,\mu_g\geqslant\int_{\mathbb{R}}(|f_g^{(k)}(c)|-\varepsilon)\,\mathrm{d}\,\mu_g=\mu_g(I)(|f_g^{(k)}(c)|-\varepsilon),$$

so

$$|f^{(k)}(c)|\leqslant\frac{1}{\mu_g(I)}\|f^{(k)}\|_{L_g^1}+\varepsilon.$$

On the other hand, for any $x\in I$,

$$\begin{aligned}|f_g^{(k)}(x_0)|-|f_g^{(k)}(x)|\leqslant&||f_g^{(k)}(x)|-|f_g^{(k)}(x_0)||\leqslant|f_g^{(k)}(x)-f_g^{(k)}(x_0)|\\&=\left|\int_{x_0}^{x}f_g^{(k+1)}\,\mathrm{d}\,\mu_g\right|\leqslant\int_I|f_g^{(k+1)}|\,\mathrm{d}\,\mu_g\leqslant\|f_g^{(k+1)}\|_{L_g^1},\end{aligned}$$

so

$$|f_g^{(k)}(x_0)|\leqslant|f_g^{(k)}(x)|+\|f_g^{(k+1)}\|_{L_g^1}.\tag{3.4}$$

Therefore, taking $x=c$,

$$|f_g^{(k)}(x_0)|\leqslant|f_g^{(k)}(c)|+\|f_g^{(k+1)}\|_{L_g^1}\leqslant\frac{1}{\mu_g(I)}\|f_g^{(k)}\|_{L_g^1}+\varepsilon+\|f_g^{(k+1)}\|_{L_g^1}.$$

Since $\varepsilon\in\mathbb{R}^+$ was fixed arbitrarily,

$$|f_g^{(k)}(x_0)|\leqslant\frac{1}{\mu_g(I)}\|f_g^{(k)}\|_{L_g^1}+\|f_g^{(k+1)}\|_{L_g^1}.$$

Thus,

$$\begin{aligned}\left\|\left((f(x_0),f_g'(x_0),\dots,f_g^{(n-1)}(x_0)),f^{n)}\right)\right\|\leqslant&\frac{1}{\mu_g(I)}\|f\|_{L_g^1}+\left(\frac{1}{\mu_g(I)}+1\right)\sum_{k=1}^{n-1}\|f_g^{(k)}\|_{L_g^1}+2\|f_g^{(n)}\|_{L_g^1}\\&\leqslant\left(\frac{1}{\mu_g(I)}+2\right)\sum_{k=0}^{n}\|f_g^{(k)}\|_{L_g^1}=\left(\frac{1}{\mu_g(I)}+2\right)\|f\|_{W_g^{n,1}},\end{aligned}$$

and we conclude that φ_g is continuous.

2. **Case** ess $\inf_g|f^{(k)}|=0$ **for every** $k=0,\dots,n-1$**:** Fix $k\in\{0,\dots,n-1\}$ and let $\varepsilon\in\mathbb{R}^+$ and $c\in I$ be such that $|f^{(k)}(c)|<\varepsilon$. Then, by the inequality (3.4),

$$|f_g^{(k)}(x_0)|\leqslant|f_g^{(k)}(c)|+\|f_g^{(k+1)}\|_{L_g^1}\leqslant\varepsilon+\|f_g^{(k+1)}\|_{L_g^1}.$$

Since $\varepsilon\in\mathbb{R}^+$ was fixed arbitrarily,

$$|f_g^{(k)}(x_0)|\leqslant\|f_g^{(k+1)}\|_{L_g^1}.$$

Thus,

$$\left\|\left((f(x_0),f_g'(x_0),\dots,f_g^{(n-1)}(x_0)),f^{n)}\right)\right\|\leqslant\sum_{k=1}^{n}\|f_g^{(k)}\|_{L_g^1}+\|f_g^{(n)}\|_{L_g^1}\leqslant2\sum_{k=1}^{n}\|f_g^{(k)}\|_{L_g^1}=2\|f\|_{W_g^{n,1}},$$

and we conclude that φ_g is continuous.

To see that φ_g is injective, suppose that $f \in W_g^{n,1}(I,\mathbb{R})$ is such that $f(x_0)$, $f_g'(x_0)$,…, $f_g^{(n-1)}(x_0)$, $f_g^{(n)} = 0$. Using Theorem 2.4 iteratively starting with $k = n-1$, we obtain that $f_g^{(k)} = 0$ for $k = n-1, n-2, \dots, 0$, so $f = 0$.

Now consider the case where $\mu_g(I) < \infty$ and let us define

$$\begin{tikzcd} \mathbb{F}^n \oplus L_g^1(I,\mathbb{F}) \arrow[r, "\psi_g"] & W_g^{n,1}(I,\mathbb{F}) \\ ((\alpha_0,\alpha_1,\dots,\alpha_{n-1}),h) \arrow[r, mapsto] & f(x) = \displaystyle\sum_{k=0}^{n-1}\frac{\alpha_k}{k!}g_n(x) + \int_{x_0}^{x}\int_{x_0}^{y_1}\cdots\int_{x_0}^{y_{n-1}} h(y_n)\,\mathrm{d}\,\mu_g(y_n)\cdots\mathrm{d}\,\mu_g(y_n) \end{tikzcd}$$

where $g_0 := 1$ and $g_n(x) := n\int_{x_0}^{x} g_{n-1}\,\mathrm{d}\,\mu_g$ for $n \in \mathbb{N}$. Since $\mu_g(I) < \infty$, ψ_g is well defined. Furthermore, if $f = \psi_g\left((\alpha_0,\alpha_1,\dots,\alpha_{n-1}),h\right)$, then $f_g^{(k)}(x_0) = \alpha_k$ for $k = 0,\dots,n$ and $f_g^{(n)} = h$, so $\varphi_g \circ \psi_g$ is the identity on $W_g^{n,1}(I,\mathbb{F})$, which implies that φ_g is surjective.

By the open mapping theorem, φ_g is an isomorphism. ■

4. An isomorphism between classical and Stieltjes spaces

The primary goal of this section is to demonstrate that, given a derivator g, an appropriate transformation of the domain allows us to understand g-measurability in terms of Lebesgue measurability, and vice versa. Consequently, g-integrals can also be interpreted in terms of Lebesgue integrals, and vice versa, which will lead to the isomorphism we want to define.

A naive approach to addressing this question might involve observing that, for intervals, $\mu_g([a,b)) = g(b) - g(a)$ and $\mu([a,b)) = b - a$, and thus, it would be reasonable to assume that for a Borel measurable set A, we have $\mu_g(A) = \mu(g(A))$. Moreover, if g is invertible, it follows that $\mu_g(g^{-1}(A)) = \mu(A)$.

However, this naive approach encounters two significant obstacles. First, in general, μ_g is not absolutely continuous with respect to μ, nor vice versa. As a result, the relationship between μ_g and μ is not straightforward. Second, g is not generally invertible. In fact, the most interesting applications involving g-derivatives and measures pertain to non-invertible derivators. To address this, we will define a pseudo-inverse $g^\dagger$ to serve in the role of an inverse. This function $g^\dagger$ is central to constructing the isomorphism.

We will formally define this pseudo-inverse and explore its properties in Theorem 4.9. Before doing so, however, we must establish several preliminary results that will help us relate the measures μ_g and μ in specific cases.

Lemma 4.1. *Given $A \subset I$, $\mu^*(g(A)) \leqslant \mu_g^*(A)$.*

Proof. Given $\varepsilon \in \mathbb{R}^+$, take a countable pairwise disjoint family of bounded intervals of the form $[c_n, d_n)$ such that $[c_n, d_n] \subset I$, $A \subset \bigcup_{n\in\mathbb{N}}[c_n, d_n)$ for $n \in \mathbb{N}$, and $\sum_{n\in\mathbb{N}}(g(d_n) - g(c_n)) < \mu_g^*(A) + \varepsilon$, see [24, Theorem 2.3]. Since $A \subset \bigcup_{n\in\mathbb{N}}[c_n, d_n)$ and g is nondecreasing, we have that $g(A) \subset \bigcup_{n\in\mathbb{N}}[g(c_n), g(d_n)]$, so

$$\mu^*(g(A)) \leqslant \mu^*\left(\bigcup_{n\in\mathbb{N}}[g(c_n), g(d_n)]\right) \leqslant \sum_{n\in\mathbb{N}}\mu^*([g(c_n), g(d_n)]) = \sum_{n\in\mathbb{N}}(g(d_n) - g(c_n)) < \mu_g^*(A) + \varepsilon.$$

Since ε was fixed arbitrarily, $\mu^*(g(A)) \leqslant \mu_g^*(A)$. ■

Example 4.2. To illustrate Lemma 4.1 we will consider the following derivator $g : \mathbb{R} \to \mathbb{R}$ —see Fig. 4.1, which appears in [7, Example 3.3],

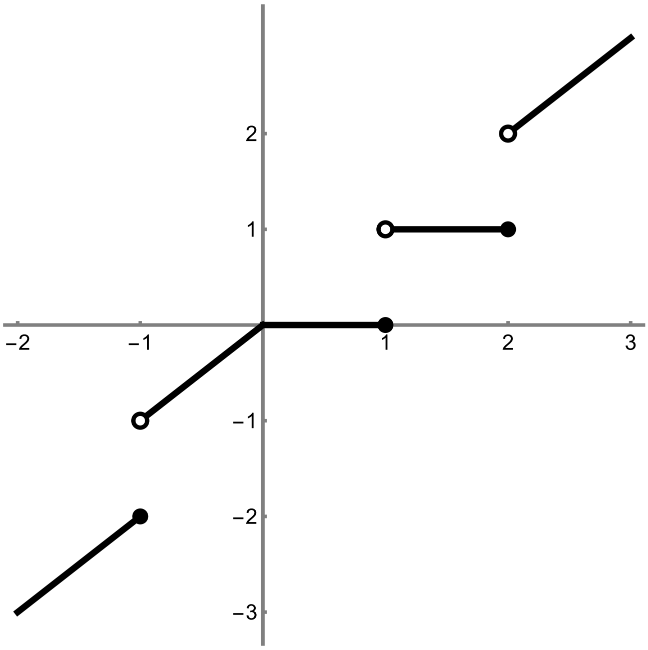

Fig. 4.1. The graph of function g in Example 4.2.

$$g(t)=\begin{cases} t-1, & t\le -1,\\ t, & -1<t\le 0,\\ 0, & 0<t\le 1,\\ 1, & 1<t\le 2,\\ t, & t>2.\end{cases}$$

Observe that, taking $A=\{1\}$, $\mu_g(A)=\Delta g(1)=1$, whereas $\mu(g(A))=\mu(\{0\})=0$, so $\mu(g(A))<\mu_g(A)$. This happens because $1\in D_g$. In Corollary 4.6 we show that if $A\cap D_g=\emptyset$, then $\mu_g^*(A)=\mu^*(g(A))$.

The next lemma shows that for intervals of the form $g^{-1}([g(t),g(s)))$, the g-measure is precisely $g(s)-g(t)$.

Lemma 4.3. *Given $t,s\in I$, $t\leqslant s$, the set*

$$A=\{r\in I\ :\ g(r)\in[g(t),g(s))\}=g^{-1}([g(t),g(s))),$$

is μ_g-measurable and $\mu_g(A)=g(s)-g(t)$.

Proof. If $g(t)=g(s)$ then $A=\emptyset$ and the result is obvious, so assume $g(t)<g(s)$. The measurability is straightforward as $[g(t),g(s))$ is an interval and, since g is monotone, $g^{-1}([g(t),g(s)))$ is an interval as well. Let $a=\inf A$, $b=\sup A$ (we can assume $a,b\in I$ as otherwise, we could just extend g to $\overline{\mathbb{R}}$ –see Definition 2.5– and go on with the proof in a similar fashion).

We have to take the following assertions into consideration:

- If $g(b)>g(s)$, then $b>s$ and, since g is left continuous at b, there must exist $d\in(s,b)$ such that $g(s)<g(d)\leqslant g(b)$ which implies that $d\notin A$ and, therefore, $(d,b]\cap A=\emptyset$, which contradicts that $b=\sup A$. We conclude that $g(b)\leqslant g(s)$.
- If $g(b)<g(s)$, then $b\in A$, $b<s$ and, for every $r\in(b,s]$, $g(r)=g(s)$ since, otherwise, we would have $r\in A$ which would contradict that $b=\sup A$. Hence, $g(s)=g(b^+)$.

- If $g(t) < g(a)$, then $t < a$, which is impossible. This is because $t \in A$, which implies $\inf A = a \leqslant t$. We conclude that $g(t) \leqslant g(a)$.
- If $g(a) < g(t)$, then $a < t$ and, for every $r \in (a,t]$, $g(r) = g(t)$ since, if there exists $r \in (a,t]$ with $g(r) < g(t)$, then $(a,r] \cap A = \emptyset$, which would contradict that $a = \inf A$. Hence, $g(t) = g(a^+)$.

We can now study four different cases:

- If $A = [a,b]$, then $\mu_g(A) = g(b^+) - g(a)$, $g(b) < g(s)$ and $g(a) \geqslant g(t)$, so $g(s) = g(b^+)$ and $g(t) = g(a)$. Therefore, $\mu_g(A) = g(s) - g(t)$.
- If $A = [a,b)$, then $\mu_g(A) = g(b) - g(a)$, $g(b) \geqslant g(s)$ and $g(a) \geqslant g(t)$, so $g(s) = g(b)$ and $g(t) = g(a)$. Therefore, $\mu_g(A) = g(s) - g(t)$.
- If $A = (a,b)$, then $\mu_g(A) = g(b) - g(a^+)$, $g(b) \geqslant g(s)$ and $g(a) < g(t)$, so $g(s) = g(b)$ and $g(t) = g(a^+)$. Therefore, $\mu_g(A) = g(s) - g(t)$.
- If $A = (a,b]$, then $\mu_g(A) = g(b^+) - g(a^+)$, $g(b) < g(s)$ and $g(a) < g(t)$, so $g(s) = g(b^+)$ and $g(t) = g(a^+)$. Therefore, $\mu_g(A) = g(s) - g(t)$. ■

The next result is proven in a similar way to Lemma 4.3. As expected, we only obtain an inequality in it due to the more general condition $g(A) \subset [c,d]$.

Lemma 4.4. *Let $A \subset I$. If $\sup A \notin A \cap D_g$ and $g(A) \subset [c,d]$ with $c,d \in \mathbb{R}$, then $\mu_g^*(A) \leqslant d - c$.*

Proof. Let $a = \inf A$, $b = \sup A$ (we can assume $a,b \in \mathbb{R}$ as otherwise, we could just extend g to $\overline{\mathbb{R}}$, see Definition 2.5). $g(t) \leqslant d$ for $t \in A$ and, since g is left-continuous,

$$g(b) = g(b^-) = \lim_{\substack{t \to b^- \\ t \in A}} g(t) \leqslant d.$$

Now, either $g(b^+) = g(b) \leqslant d$ or $b \in D_g$, in which case $b \notin A$.

Thus, we have two scenarios:

- If $A \subset [a,b)$, $\mu_g(A) \leqslant g(b) - g(a) \leqslant d - c$.
- If $A \subset [a,b]$ and $b \in A$, then $b \notin D_g$ and, therefore, $g(b) = g(b^+) \leqslant d$. Thus, $\mu_g(A) = g(b^+) - g(a) \leqslant d - c$. ■

Remark 4.5. The hypothesis $\sup A \notin A \cap D_g$ in Lemma 4.4 is necessary, as for any $t \in D_g$, $g(\{t\}) \subset [g(t), g(t) + \delta]$ for every $\delta \in (0, \Delta g(t))$, but $\mu_g(\{t\}) = \Delta g(t) > \delta = (g(t) + \delta) - g(t)$.

The following corollary shows that the thesis in Lemma 4.1 can be strenghen to an equality under certain conditions.

Corollary 4.6. *Let $A \subset I$. If $A \cap D_g = \emptyset$, then $\mu_g^*(A) = \mu^*(g(A))$.*

Proof. Given $\varepsilon \in \mathbb{R}^+$, take a countable pairwise disjoint family of bounded intervals of the form $[c_n, d_n)$, $n \in \mathbb{N}$, such that $g(A) \subset \bigcup_{n\in\mathbb{N}} [c_n, d_n)$ and $\sum_{n\in\mathbb{N}} (d_n - c_n) < \mu^*(g(A)) + \varepsilon$, see [24, Theorem 2.3]. Clearly, $A_n := A \cap g^{-1}([c_n, d_n))$ is such that $g(A_n) \subset [c_n, d_n]$ and $A_n \cap D_g = \emptyset$. Therefore, by Lemma 4.4, $\mu_g^*(A_n) \leqslant d_n - c_n$. Furthermore, $\bigcup_{n\in\mathbb{N}} A_n = A$. Thus,

$$\mu_g^*(A) \leqslant \sum_{n\in\mathbb{N}} \mu_g^*(A_n) \leqslant \sum_{n\in\mathbb{N}} (d_n - c_n) < \mu^*(g(A)) + \varepsilon.$$

Since ε was fixed arbitrarily, $\mu_g^*(A) \leqslant \mu^*(g(A))$. Lemma 4.1 provides the other inequality. ■

The next result will let us operate with the measure of intervals, making it easier to compute later on.

Lemma 4.7. *If $t, s \in \mathbb{R}$, then*

1. $g^{-1}((-\infty, g(t))) = I \cap (-\infty, t) \backslash g^{-1}(\{g(t)\}) \subset I \cap (-\infty, t)$.
2. $\mu_g(I \cap (-\infty, t)) \backslash g^{-1}((-\infty, g(t))) = 0$.
3. *Let* $[a, b] \subset I$. $\mu_g([a, b) \Delta g^{-1}([g(a), g(b)))) = 0$, *where* $A \Delta B := (A \backslash B) \cup (B \backslash A)$.

Proof. 1. If $s \in g^{-1}((-\infty, g(t)))$, then $g(s) < g(t)$, so $s \notin g^{-1}(g(\{t\}))$ and $s < t$. We deduce that $g^{-1}((-\infty, g(t))) \subset (-\infty, t) \backslash g^{-1}(\{g(t)\})$. On the other hand, if $s < t$ and $g(s) \neq g(t)$, then $g(s) < g(t)$ and, hence, $s \in g^{-1}((-\infty, g(t)))$.

2. Observe that

$$(-\infty, t) \backslash g^{-1}((-\infty, g(t))) = (-\infty, t) \cap g^{-1}(\{g(t)\}).$$

If $(-\infty, t) \cap g^{-1}(\{g(t)\}) = \emptyset$, the result is trivial. Otherwise, $(-\infty, t) \cap g^{-1}(\{g(t)\})$ is an interval of the form (a, t) or $[a, t)$ where g is constant. Therefore, $\mu_g((-\infty, t) \cap g^{-1}(\{g(t)\})) = 0$ and, hence, $\mu_g(g^{-1}((-\infty, g(t)) \backslash (-\infty, g(t)))) = 0$.

3. Let $A = I \cap (-\infty, a)$, $B = I \cap (-\infty, b)$, $C = g^{-1}((-\infty, g(a)))$ and $D = g^{-1}((-\infty, g(b)))$. We have that $[a, b) = B \backslash A$ and $g^{-1}([g(a), g(b))) = D \backslash C$. We also have that $A \subset B$, $C \subset D$ and, by point 1, $C \subset A$ and $D \subset B$. Thus,

$$(B \backslash A) \Delta (D \backslash C) = [(B \backslash A) \cap (B \backslash D)] \cup [(A \backslash C) \cap (D \backslash C)].$$

By point 2, $\mu_g(B \backslash D) = 0$ and $\mu_g(A \backslash C) = 0$, so

$$\mu_g((B \backslash A) \cap (B \backslash D)) = \mu_g((A \backslash C) \cap (D \backslash C)) = 0$$

and, therefore, $\mu_g((B \backslash A) \Delta (D \backslash C)) = 0$. ■

Example 4.8. We now illustrate the points of Lemma 4.7 by returning to the function g defined in Example 4.2. Observe that

$$g^{-1}((-\infty, g(t))) = (-\infty, t) \backslash g^{-1}(\{g(t)\}) = \begin{cases} (-\infty, t), & t \leq 0, \\ (-\infty, 0), & 0 < t \leq 1, \\ (-\infty, 1), & 1 < t \leq 2, \\ (-\infty, t), & t > 2. \end{cases}$$

This fact is clear from Fig. 4.1. Given that $C_g = (0, 1) \cup (1, 2)$, it is also evident that

$$\mu_g((0, 1)) = \mu_g((1, 2)) = 0,$$

so $\mu_g((-\infty, t)) \backslash g^{-1}((-\infty, g(t))) = 0$, as it is precisely at the points of C_g where there is a difference between $(-\infty, t)$ and $g^{-1}((-\infty, g(t)))$.

Finally, we already knew from Lemma 4.3 that $\mu_g([a, b)) = \mu_g(g^{-1}([g(a), g(b)))) = g(b) - g(a)$, but point 3 of Lemma 4.7 says more: that the set difference between $g^{-1}([g(a), g(b)))$ and $[a, b)$ is actually a g-null set. This is clear again for our particular choice of function g as the differences between the sets $(-\infty, t)$ and $g^{-1}((-\infty, g(t)))$ (which serve to construct $[a, b)$ and $g^{-1}([g(a), g(b)))$ respectively) only appear in the sets of g-measure zero $(0, 1)$ and $(1, 2)$.

For the next result, in which we define the function $g^\dagger$—used to construct the isomorphism and to generalize [16, Theorem 1.8]—and for the results that follow, we denote by $\operatorname{ce}(A)$ the *convex envelope* of $A \subseteq \mathbb{R}$, that is, the smallest interval containing A.

Recall that the interval $I \subseteq \mathbb{R}$ under consideration is always of the form $(-\infty, b)$ or $[a, b)$, where $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$. To simplify notation and avoid case distinctions, for a function $u : I \to \mathbb{R}$, we adopt the conventions $u(a^-) := u(a)$ and $u(b^+) := u(b)$ whenever $a, b \in \mathbb{R}$, $a = \inf I$, and $b = \sup I$.

Finally, we will use the notation $J = \operatorname{ce}(u(I))$.

Theorem 4.9. *Assume g is not constant on any interval of the form $(-\infty, x]$. Let $u : I \to \mathbb{R}$ be a non-decreasing function and let $u^\dagger : J \to \mathbb{R}$ be the function defined by*

$$u^\dagger(y) := \inf\{t \in I : u(t) \geqslant y\}, \quad y \in J.$$

Then,

1. *$u^\dagger$ is well defined and $u^\dagger(J) \subset I$,*
2. *$u^\dagger$ is non-decreasing and left continuous,*
3. *$u^\dagger$ jumps at some point $y_0 \in J \backslash \{\sup u\}$ if and only if $u(x) \equiv y_0$ for all x in some interval $(x_1, x_2) \subset I$, with $x_1 < x_2$,*
4. *$u^\dagger(u(t)) \leqslant t$ for every $t \in I$, with the strict inequality holding if and only if u is constant on some interval $[t_1, t] \subset I$, with $t_1 < t$,*
5. *$u^\dagger(y) = x_0$ for all y in some interval $(y_1, y_2) \subset I$, with $y_1 < y_2$, and for some $x_0 \in \mathbb{R}$ if and only if u jumps at x_0 and $(y_1, y_2) \subset (u(x_0^-), u(x_0^+))$,*
6. *$u(u^\dagger(y)^-) \leqslant y$, for every $y \in J$. Furthermore, if the strict inequality holds, there is a jump of u at $u^\dagger(y)$,*
7. *$u(u^\dagger(y)^+) \geqslant y$, for every $y \in J$. Furthermore, if u is continuous at $u^\dagger(y)$, then $u(u^\dagger(y)^+) = u(u^\dagger(y)) = y$,*
8. *$(u(t^-), u(t^+)] \subset (u^\dagger)^{-1}(\{t\}) \subset [u(t^-), u(t^+)]$ for $t \in I$.*

Proof. 1. To see that $u^\dagger$ is well defined, first observe that, given that g is nondecreasing, $\{x \in I : u(x) \geqslant y\} \neq \emptyset$ for $y \in J$, so the infimum can be considered. If I is of the form $[a, b)$, the infimum always belongs to I. If I is unbounded from below, since g is not constant on any interval of the form $(-\infty, x]$, the infimum also belongs to I. Thus, $u^\dagger$ is well defined and $u^\dagger(J) \subset I$.

2-5. Proven in [16, Theorem 1.8].

6. Suppose that $u(u^\dagger(y)^-) > y$. There exists $\delta \in \mathbb{R}^+$ such that $u(t) > y$ for $t \in (u^\dagger(y) - \delta, u^\dagger(y))$, but this contradicts the definition of $u^\dagger$, so $u(u^\dagger(y)^-) \leqslant y$.

If $u(u^\dagger(y)^-) < y$, taking into account that $u(t) \geqslant y$ for $t > u^\dagger(y)$, there is a jump at $u^\dagger(y)$.

7. Suppose $u(u^\dagger(y)^+) < y$. For $t \in (u^\dagger(y), u^\dagger(y) + \delta)$ and δ small enough, $u^\dagger(t) < y$, which contradicts the definition of $u^\dagger(y)$, so $u(u^\dagger(y)^+) \geqslant y$. In the case u is continuous at $u^\dagger$, then $u^\dagger(y)^- = u^\dagger(y) = u^\dagger(y)^+$ and we can apply point 6 to obtain the equality.

8. Let $\delta \in \mathbb{R}^+$ and y be such that $u^\dagger(y) = t$. By point 6,

$$u(t - \delta) = u(u^\dagger(y) - \delta) \leqslant u(u^\dagger(y)^-) \leqslant y.$$

Since δ was fixed arbitrarily, $u(t^-) \leqslant y$. Now, if $y < u(t)$, by point 7, $u(t^+) = u(u^\dagger(y)^+) \geqslant y$, so $y \in [u(t^-), u(t^+)]$, that is, $(u^\dagger)^{-1}(\{t\}) \subset [u(t^-), u(t^+)]$.

Finally, if $y \in (u(t^-), u(t^+)]$, we have that $u^\dagger(y) \leqslant t$ and, since $u(s) < y$ for every $s < t$, $u^\dagger(y) \geqslant s$ for every $s < t$, so we conclude that $u^\dagger(y) = t$, that is, $(u(t^-), u(t^+)] \subset (u^\dagger)^{-1}(\{t\})$. ■

Remark 4.10. We can extend Theorem 4.9 (even dropping the assumption that g is not constant on any interval of the form $(-\infty, x]$) to the case of an interval of the form $[-\infty, b)$ or $[a, b)$ with $a \in \overline{\mathbb{R}}$ and

$b \in \mathbb{R} \cup \{\infty\}$ and a function $u : I \to \overline{\mathbb{R}}$ (in this case $\operatorname{ce}(u(I))$ is the convex envelope in $\overline{\mathbb{R}}$), just by observing that $[-\infty, \infty]$ can be transformed into a compact interval of the real line through an order preserving homeomorphism and then applying Theorem 4.9. This allows to apply Theorem 4.9 to the extension considered in Definition 2.5.

We now define the auxiliary sets C_g^* and D_g^*.

Definition 4.11. We define

$$C_g^* := \{t \in I \ : \ g^\dagger(g(t)) \neq t\},$$
$$D_g^* := \{y \in J \ : \ g(g^\dagger(y)) \neq y\}.$$

Remark 4.12. Observe that, given a derivator g, from point 4 of Theorem 4.9, we deduce that $g^\dagger(g(t)) = t$ for $t \in \mathbb{R}\backslash g^{-1}(g(C_g))$, which implies $C_g^* \subset g^{-1}(g(C_g))$. Similarly, from point 6 of Theorem 4.9, we deduce that if $g(g^\dagger(y)) < y$, then $g^\dagger(y) \in D_g$, which implies $D_g^* \subset (g^\dagger)^{-1}(D_g)$. These results justify the choice of notation.

Intuitively, the set C_g^* consists of points where $g^\dagger$ fails to act as a left inverse of g, while D_g^* consists of points where $g^\dagger$ fails to act as a right inverse of g. Tracking these sets is crucial, as they identify those points where the intuitive relationship between the measures μ_g and μ may break down.

Example 4.13. For the function g in Example 4.1, we have that $C_g = (0, 1) \cup (1, 2)$, $g^{-1}(g(C_g)) = [0, 2]$, $C_g^* = (0, 2]$, $D_g = \{-1, 1, 2\}$, $(g^\dagger)^{-1}(D_g) = [-2, -1] \cup (0, 2]$ and $D_g^* = (-2, -1] \cup (0, 2]$.

In the following results, we will study the properties of the sets C_g^* and D_g^* and their relationships with the sets C_g and D_g.

Lemma 4.14. *$\mu_g(C_g^* \backslash D_g) = 0$.*

Proof. $C_g^*\backslash D_g \subset g^{-1}(g(C_g))\backslash D_g$, see Remark 4.12, so it is enough to see that $\mu_g(g^{-1}(g(C_g))\backslash D_g) = 0$. Let $y \in g(C_g)$, $C_y := g^{-1}(\{y\})\backslash D_g$. g is constant on C_y, so $\mu_g(C_y) = 0$.

Now,

$$g^{-1}(g(C_g))\backslash D_g = \bigsqcup_{y \in g(C_g)} C_y, \tag{4.1}$$

so, in order to finish the proof, it is enough to show that the union in equation (4.1) is actually a countable one. To see this, observe that C_g, being open in the usual topology, has a countable number of connected components and g is constant in each one, which implies that $g(C_g)$ is countable, so the union in equation (4.1) is a countable one and, therefore, $\mu_g(C_g^*\backslash D_g) = 0$. ■

Lemma 4.15. *$D_g^* \cap g(I) \subset g(C_g^*)$ and $C_g^* \cap g^\dagger(J) \subset g^\dagger(D_g^*)$.*

Proof. Let us show that $g(I\backslash C_g^*) \subset \mathbb{R}\backslash D_g^*$. If $t \notin C_g^*$, $g^\dagger(g(t)) = t$, so $g(g^\dagger(g(t))) = g(t)$ and, therefore, $g(t) \notin D_g^*$.

On the other hand, we will check that $g^\dagger(J\backslash D_g^*) \subset \mathbb{R}\backslash C_g^*$. If $y \notin D_g^*$, $g(g^\dagger(y)) = y$, so $g^\dagger(g(g^\dagger(y))) = g^\dagger(y)$ and, therefore, $g^\dagger(y) \notin C_g^*$. ■

Lemma 4.16. *$D_g^* = \bigcup_{t \in D_g} (g(t), g(t^+)]$.*

Proof. Let $x \in D_g^*$, that is, such that $g(g^\dagger(x)) \neq x$. By Theorem 4.9, point 6, $g(g^\dagger(x)) < x$ and there is a jump at $g^\dagger(x)$ and, by point 7, $g(g^\dagger(x)^+) \geqslant x$. Thus, $x \in (g(t), g(t^+)]$ for $t = g^\dagger(x) \in D_g$.

If $t \in D_g$ and $x \in (g(t), g(t^+)]$, then, since $g(t) < x < g(s)$ for every $s > t$, we have that $t \leqslant g^\dagger(x) \leqslant s$ for every $s > t$, so $g(t) \leqslant g(g^\dagger(x)) \leqslant g(t^+)$, but this is only possible if $g^\dagger(x) = t$, so $g(g^\dagger(x)) = g(t) < x$ and, thus, $t \in D_g^*$. ■

Remark 4.17. Observe that

$$\bigcup_{t\in D_g} (g(t), g(t^+)) \subset J\backslash g(I) \subset \bigcup_{t\in D_g} (g(t), g(t^+)] = D_g^*,$$

so $\mathbb{R}\backslash D_g^* \subset g(I)$ and the sets D_g^* and $J\backslash g(I)$ (or, equivalently, $J\backslash D_g^*$ and $g(I)$) are the same up to a countable set, that is, D_g^* is, essentially, what is missing of J in the image of g.

The following result will be useful in Corollary 4.37, where we study an embedding of $W_g^{n,1}(I,\mathbb{F})$ into $W^{n,1}(J,\mathbb{F})$.

Lemma 4.18. *Let $x_0, x \in J$, $x_0 < x$. Then,*

$$g^\dagger([x_0, x)) = \begin{cases} [g^\dagger(x_0), g^\dagger(x)) \cap g^\dagger(J), & x \in J\backslash D_g^*, \\ [g^\dagger(x_0), g^\dagger(x)] \cap g^\dagger(J), & x \in D_g^*. \end{cases}$$

Proof. Since $g^\dagger$ is non-decreasing, $g^\dagger([x_0, x))$ has to be an interval intersected with $g^\dagger(J)$ and the left endpoint of such interval has to be $g^\dagger(x_0)$. Furthermore,

$$g^\dagger([x_0, x)) = \bigcup_{y\in(x_0,x)} g^\dagger([x_0, y]) = g^\dagger(J) \cap \bigcup_{y\in(x_0,x)} [g^\dagger(x_0), g^\dagger(y)].$$

Therefore, taking into account that $g^\dagger$ is left continuous, $g^\dagger([x_0, x))$ has to be either $[g^\dagger(x_0), g^\dagger(x))$ or $[g^\dagger(x_0), g^\dagger(x)]$ intersected with $g^\dagger(J)$. We study two cases now:

If $x \in D_g^*$, then, by Lemma 4.16, $x \in (g(t), g(t^+)]$ for some $t \in D_g$ so, for $z \in (g(t), x)$, $g^\dagger(z) = g^\dagger(y)$ and, thus, $g^\dagger([x_0, x)) = [g^\dagger(x_0), g^\dagger(x)] \cap g^\dagger(J)$.

If $x \notin D_g^*$, then $g^\dagger(x) \notin D_g$ (see Remark 4.12) and, thus, $g^\dagger(y) < g^\dagger(x)$ for $y < x$ (see Theorem 4.9, point 5), so $g^\dagger(x) \notin g^\dagger([x_0, x))$ and, therefore, $g^\dagger([x_0, x)) = [g^\dagger(x_0), g^\dagger(x)) \cap g^\dagger(J)$. ■

Lemma 4.19. $(g^\dagger)^{-1}(A\backslash D_g) \subset g(A\backslash D_g)$ *and* $g(A\backslash C_g^*) \subset (g^\dagger)^{-1}(A\backslash C_g^*)$ *for every* $A \subset \mathbb{R}$.

Proof. Let $y \in (g^\dagger)^{-1}(A\backslash D_g)$, then $g^\dagger(y) \in A\backslash D_g$. By point 6 of Theorem 4.9, we have that $g(g^\dagger(y)) = y$. Thus, since $g^\dagger(y) \in A\backslash D_g$, we have that $y \in g(A\backslash D_g)$ and, therefore, $(g^\dagger)^{-1}(A\backslash D_g) \subset g(A\backslash D_g)$.

Let $y \in g(A\backslash C_g^*)$. Then there exists $x \in A\backslash C_g^*$ such that $g(x) = y$. By definition of C_g^*, $g^\dagger(g(x)) = x \in A\backslash C_g^*$, so $y = g(x) \in (g^\dagger)^{-1}(A\backslash C_g^*)$. ■

Now that we have examined the properties of C_g^* and D_g^* in detail, we are prepared to further explore the relationship between the measures μ and μ_g. To achieve this, we will utilize the change of variables formula, also known as the integration by substitution formula, for measure spaces.

Definition 4.20. Let (X, Σ_1, μ) be a measure space, and let $\varphi : (X, \Sigma_1) \to (Y, \Sigma_2)$ be measurable. We define the *pushforward* $\varphi_*\mu : \Sigma_2 \to [0, \infty]$ of μ by φ by the formula $\varphi_*\mu(E) := \mu\left(\varphi^{-1}(E)\right)$.

Theorem 4.21 *([29, Exercise 1.4.38]). Let (X,Σ_1,μ) be a measure space, and let $\varphi:(X,\Sigma_1)\to(Y,\Sigma_2)$ be measurable. Then,*

1. *$\varphi_*\mu$ is a measure on Σ_2, so that $(Y,\Sigma_2,\varphi_*\mu)$ is a measure space.*
2. *If $f:Y\to[0,+\infty]$ is measurable, then $\int_Y f\,\mathrm{d}\,\varphi_*\mu=\int_X f\circ\varphi\,\mathrm{d}\,\mu$.*

For the following result, recall that $\mathscr{M}_g$ denotes the σ-algebra associated with the measure μ_g.

Lemma 4.22. *$g^\dagger_*\mu(A)=\mu((g^\dagger)^{-1}(A))=\mu_g(A)$ for every $A\in\mathscr{M}_g$.*

Proof. Observe that $g^\dagger_*\mu$ is a measure on $\mathscr{M}_g$ by Theorem 4.21. Given $A\in\mathscr{M}_g$, we can write $A=A_1\sqcup A_2\sqcup A_3$ where $A_1=A\cap(C_g^*\backslash D_g)$, $A_2=A\cap D_g$, $A_3=A\backslash(C_g^*\cup D_g)$ and each of the sets is in $\mathscr{M}_g$.

On the one hand, by Lemma 4.14, $\mu_g(C_g^*\backslash D_g)=0$. Hence, for every set $E\subset C_g^*\backslash D_g$, by Lemma 4.19 and Corollary 4.6,

$$g^\dagger_*\mu(E)=\mu((g^\dagger)^{-1}(E))\leqslant\mu(g(E))=\mu_g(E)\leqslant\mu_g(C_g^*\backslash D_g)=0,$$

so $g^\dagger_*\mu(E)=\mu_g(E)=0$, in particular for $E=A_1$.

If $t\in D_g$, $\mu_g(\{t\})=\Delta g(t)$. On the other hand, by Theorem 4.9, point 8,

$$\mu((g^\dagger)^{-1}(\{t\}))=\mu([g(t),g(t)^+])=\Delta g(t)=\mu_g(\{t\}).$$

Taking into account that D_g is countable, we have proved that $\mu_g(E)=\mu((g^\dagger)^{-1}(E))$ for every $E\subset D_g$, in particular for $E=A_2$.

It is left to check the case where $E\in\mathscr{M}_g$, $E\cap(C_g^*\cup D_g)=\emptyset$. Using both statements in Lemma 4.19, $(g^\dagger)^{-1}(E)=g(E)$ and, by Corollary 4.6, $\mu_g(E)=\mu(g(E))$, so $\mu((g^\dagger)^{-1}(E))=\mu_g(E)$, in particular for $E=A_3$. ■

Corollary 4.23. *$\mu_g(A\cap g^\dagger(J))=\mu_g(A)$ for every $A\in\mathscr{M}_g$.*

Proof. $\mu_g(A\cap g^\dagger(J))=\mu((g^\dagger)^{-1}(A\cap g^\dagger(J)))=\mu((g^\dagger)^{-1}(A)\cap(g^\dagger)^{-1}(g^\dagger(J))))=\mu_g(A)$. ■

We will use the following result from measure theory to prove the measurability of g and $g^\dagger$.

Lemma 4.24 *([9, Lemma 2.2]). The following properties hold:*

1. *Given an element $E\in\mathscr{M}_g$ there exists $H\in G_\delta$ (that is, H is a countable intersection of open sets) and $N\in\mathscr{M}_g$ such that $E\subset H$, $N\subset H$, $\mu_g(N)=0$ and $E=H\setminus N$.*
2. *Given an element $E\in\mathscr{M}_g$ there exists $F\in F_\sigma$ (that is, F is a countable union of closed sets) and $N\in\mathscr{M}_g$ such that $\mu_g(N)=0$, $F\cap N=\emptyset$ and $E=F\cup N$.*

The next result is reminiscent of [9, Proposition 2.7 and Corollary 2.8], where the pseudo-inverse of the continuous part of g is used to relate the Lebesgue Stieltjes integral to the Lebesgue integral through a pullback.

Proposition 4.25. *The following functions are measurable morphisms:*

1. *$g:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{L})$.*
2. *$g^+:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{L})$ such that $g^+(t)=g(t^+)$ for every $t\in\mathbb{R}$.*

3. $g^\dagger:(\mathbb{R},\mathscr{L})\to(\overline{\mathbb{R}},\mathscr{M}_{\overline{g}})$.

Proof. 1. Let us consider $g:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{L})$ and $E\in\mathscr{L}$. Let $E_1=E\cap g(D_g)$ and $E_2=E\backslash g(D_g)$. Since D_g is countable, so is E_1. Therefore, $E_1,E_2\in\mathscr{L}$. Furthermore, g is nondecreasing, so the inverse image of any point of E_1 by g is an interval, from which we conclude that $g^{-1}(E_1)$ is the countable union of intervals, so $g^{-1}(E_1)\in\mathscr{B}\subset\mathscr{M}_g$.

On the other hand, since μ is a regular measure, we have that there exists $F\in F_\sigma$ and $N\in\mathscr{L}$ with $\mu(N)=0$ such that $F\cap N=\emptyset$ and $E_2=F\cup N$. It is clear that $g^{-1}(F)\in\mathscr{B}$ so, if we prove that $\mu_g^*(g^{-1}(N))=0$, we are done. $g^{-1}(N)\cap D_g=\emptyset$, therefore, by Corollary 4.6, $\mu_g^*(g^{-1}(N))=\mu^*(g(g^{-1}(N)))\leqslant\mu^*(N)=0$, that is, $\mu_g^*(g^{-1}(N))=0$.

2. We have, for $E\in\mathscr{L}$, that $(g^+)^{-1}(E)=[(g^+)^{-1}(E)\backslash D_g]\cup[(g^+)^{-1}(E)\cap D_g]$. g^+ differs from g only at the points of D_g, which is a countable set. Every countable set is in $\mathscr{B}$, so $(g^+)^{-1}(E)\cap D_g\in\mathscr{B}\subset\mathscr{M}_g$. On the other hand, $(g^+)^{-1}(E)\backslash D_g=g^{-1}(E)\backslash D_g$. Since $g:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{L})$ is measurable, as proven in point 1, and $g^{-1}(E)\in\mathscr{M}_g$, we have that $g^{-1}(E)\backslash D_g\in\mathscr{M}_g$ and, thus, $g^+:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{L})$ is measurable.

3. Let us consider $g^\dagger:(\mathbb{R},\mathscr{L})\to(\overline{\mathbb{R}},\mathscr{M}_{\overline{g}})$ and $E\in\mathscr{M}_{\overline{g}}$. By Lemma 2.6, $E=A\sqcup B$ where $A\in\mathscr{M}_g$ and $B\subset\overline{\mathbb{R}}\backslash I$. $B\cap g^\dagger(\mathbb{R})\subset\{-\infty,\inf I,\sup I\}$, so $(g^\dagger)^{-1}(B)$ is the inverse image of a finite number of points and, since $g^\dagger$ is nondecreasing, this implies that it is the union of a finite number of intervals. Thus, $(g^\dagger)^{-1}(B)\in\mathscr{B}\subset\mathscr{L}$. We just have to prove that $(g^\dagger)^{-1}(A)\in\mathscr{L}$.

We have that $A\subset\mathbb{R}$ and, thanks to Lemma 4.24, that there exists $F\in F_\sigma$ and $N\in\mathscr{M}_g$ such that $\mu_g(N)=0$, $F\cap N=\emptyset$ and $A=F\cup N$. Thus, we conclude that $(g^\dagger)^{-1}(A)=(g^\dagger)^{-1}(F)\cup(g^\dagger)^{-1}(N)$. Now, since $g^\dagger$ is increasing, it is a Borel map, so we have that $(g^\dagger)^{-1}(F)\in\mathscr{B}\subset\mathscr{L}$. Hence, if we prove that $\mu^*((g^\dagger)^{-1}(N))=0$, we are done.

Since $\mu_g(N)=0$, given $\varepsilon>0$, there exists a countable disjoint family $\{[c_n,d_n)\}_{n\in\mathbb{N}}$ such that $N\subset\bigcup_{n\in\mathbb{N}}[c_n,d_n)$ and $\sum_{n\in\mathbb{N}}(g(d_n)-g(c_n))<\varepsilon$. Now, since $g(N)\subset\bigcup_{n\in\mathbb{N}}[g(c_n),g(d_n)]$,

$$\mu^*(g(N))\leqslant\mu^*\left(\bigcup_{n\in\mathbb{N}}[g(c_n),g(d_n)]\right)\leqslant\sum_{n\in\mathbb{N}}\mu^*([g(c_n),(d_n)])\leqslant\sum_{n\in\mathbb{N}}(g(d_n)-g(c_n))<\varepsilon,$$

from which we conclude that $\mu^*(g(N))=0$. Now it is enough to show that $(g^\dagger)^{-1}(N)\subset g(N)$. First observe that, since $\mu_g(N)=0$, $N\cap D_g=\emptyset$. Thus, $N=N\backslash D_g$ and, by Lemma 4.19, $(g^\dagger)^{-1}(N)\subset g(N)$. ■

Remark 4.26. If g is not constant on any interval of the form $(-\infty,x]$ then $g^\dagger$ takes values on $\mathbb{R}$, $g^\dagger:(\mathbb{R},\mathscr{L})\to(\mathbb{R},\mathscr{M}_g)$ is a measurable morphism and the proof of Proposition 4.25, point 3, can be simplified.

We are now ready to show that $g^\dagger$ is the left inverse of g^+ μ_g-almost everywhere.

Lemma 4.27. *$(g^\dagger\circ g^+)(t)=t$ for $t\in\mathbb{R}\backslash(C_g^*\backslash D_g)$ (that is, for μ_g-a.e. $t\in\mathbb{R}$).*

Proof. By definition of C_g^*, $g^\dagger\circ g=\mathrm{Id}$ on $\mathbb{R}\backslash C_g^*$. Since $g^+=g$ on $\mathbb{R}\backslash D_g$, $g^\dagger\circ g^+=\mathrm{Id}$ on $\mathbb{R}\backslash(C_g^*\cup D_g)$. If $t\in D_g$, $g^\dagger(g(t^+))=\inf\{s\in I: g(s)\geqslant g(t^+)\}=t$, and we conclude that $g^\dagger\circ g^+(t)=t$ on $\mathbb{R}\backslash(C_g^*\backslash D_g)$. Since, by Lemma 4.14, $\mu_g(C_g^*\backslash D_g)=0$, the equality holds for μ_g-a.e. $t\in\mathbb{R}$. ■

We recall the definition of pullback by a function.

Definition 4.28. Given a function $\varphi:A\to B$, the *pullback by* φ is the operator φ^* such that $\varphi^*f:=f\circ\varphi$ for every function $f:B\to C$.

For the next results we will denote by $M_g(I,\mathbb{F})$ the set of g-measurable functions from A to $\mathbb{F}$ and by $M((A,\mu_A),\mathbb{F})$ the set of μ_A-measurable functions from I to $\mathbb{F}$ for a given measure space (A,μ_A).

It is clear that if we have three measure spaces (A, μ_A), (B, μ_B) and (C, μ_C) and a measurable function $\zeta : (A, \mu_A) \to (B, \mu_B)$ then, for every measurable function $f : (B, \mu_B) \to (C, \mu_C)$, $\zeta^* f : (A, \mu_A) \to (C, \mu_C)$ is measurable, so the pullback function $\zeta^* : M((B, \mu_B), \mathbb{F}) \to M((A, \mu_A), \mathbb{F})$ given in such a way is well defined. It might be the case that the pullback $\zeta^* : \mathscr{L}^1((B, \mu_B), \mathbb{F}) \to \mathscr{L}^1((A, \mu_A), \mathbb{F})$ is also well defined but, in general, this does not mean that $\zeta^* : L^1((B, \mu_B), \mathbb{F}) \to L^1((A, \mu_A), \mathbb{F})$ is well defined as well –see [28, pg. 18]. The next result provides a necessary and sufficient condition.

Lemma 4.29. *Let (A, μ_A), (B, μ_B) be measure spaces, $\zeta : (A, \mu_A) \to (B, \mu_B)$ measurable and assume that $\zeta^* : \mathscr{L}^1((B, \mu_B), \mathbb{F}) \to \mathscr{L}^1((A, \mu_A), \mathbb{F})$ is well defined. Then $\zeta^* : L^1((B, \mu_B), \mathbb{F}) \to L^1((A, \mu_A), \mathbb{F})$ is well defined if and only if $\zeta_* \mu_A$ is μ_B absolutely continuous.*

Proof. Assume $\zeta^* : L^1((B, \mu_B), \mathbb{F}) \to L^1((A, \mu_A), \mathbb{F})$ and take E μ_B-measurable with $\mu_B(E) = 0$. Consider $\chi_E \in \mathscr{L}^1((B, \mu_B), \mathbb{F})$. Since $[\chi_E] = [0]$ and ζ^* is well defined, $[\chi_E \circ \zeta] = [0 \circ \zeta] = 0$. On the other hand, $\chi_E \circ \zeta = \chi_{\zeta^{-1}(E)}$, so we conclude that $\zeta_* \mu_A(E) = \mu_A(\zeta^{-1}(E)) = 0$ and, therefore, $\zeta_* \mu_A$ is μ_B absolutely continuous.

Correspondingly, if $\zeta_* \mu_A$ is μ_B absolutely continuous, there exists the Radon-Nykodim derivative of $\zeta_* \mu_A$ with respect to μ_B, call it h. Now, given $[f] \in L^1((B, \mu_B), \mathbb{F})$ and $\widetilde{f} \in [f]$, then $f - \widetilde{f} = 0$ μ_B-a.e. Therefore, by Theorem 4.21,

$$\|f \circ \zeta - \widetilde{f} \circ \zeta\|_{L^1} = \int\limits_A |f - \widetilde{f}| \circ \zeta \, \mathrm{d}\, \mu_A = \int\limits_B |f - \widetilde{f}| \, \mathrm{d}\, \zeta_* \mu_A = \int\limits_B |f - \widetilde{f}| h \, \mathrm{d}\, \mu_B = 0,$$

proving that $[f \circ \zeta] = [\widetilde{f} \circ \zeta]$ and that $\zeta^* : L^1((B, \mu_B), \mathbb{F}) \to L^1((A, \mu_A), \mathbb{F})$ is well defined. ■

We are finally ready to present the main result of this work: an isometry that connects $L^1_g(I, \mathbb{F})$ and $L^1(J, \mathbb{F})$.

Theorem 4.30. *Let $g : I \to \mathbb{R}$ be a derivator, $J = \operatorname{ce}(g(I))$ and assume $g^\dagger(J) \subset I$.*[1] *The following properties hold:*

1. *If $f \in \mathscr{L}^1_g(I, \mathbb{F})$ then $f \circ g^\dagger \in \mathscr{L}^1(J, \mathbb{F})$ and $\int f \circ g^\dagger \, \mathrm{d}\, \mu = \int f \, \mathrm{d}\, \mu_g$. As a consequence, the pullback $(g^\dagger)^* : L^1_g(I, \mathbb{F}) \to L^1(J, \mathbb{F})$ is well defined and a linear isometry.*
2. *The pullback $(g^+)^* : \mathscr{L}^1(\mathbb{R}, \mathbb{F}) \to M_g(I, \mathbb{F})$ is well defined and $(g^\dagger \circ g^+)^*$ is the identity on $L^1_g(I, \mathbb{F})$. Furthermore, $(g^+)^*|_{(g^\dagger)^*(\mathscr{L}^1_g(I, \mathbb{F}))} : (g^\dagger)^*(\mathscr{L}^1_g(I, \mathbb{F})) \to \mathscr{L}^1_g(I, \mathbb{F})$ is an isometric isomorphism.*
3. *Let X be the space of functions $f \in L^1(J, \mathbb{F})$ such that for every $t \in D_g$, there exists $c \in \mathbb{F}$ satisfying $f(x) = c$ for a.e. $x \in (g(t), g(t^+)]$. Then $X = (g^\dagger)^*(L^1_g(I, \mathbb{F}))$, X is a closed subspace of $L^1(J, \mathbb{F})$ and $(g^\dagger)^* : L^1_g(I, \mathbb{F}) \to X$ is an isometric isomorphism with inverse $\xi : X \to L^1_g(I, \mathbb{F})$ given by $(\xi f)(t) = c$ where $f(x) = c$ for a.e. $x \in (g(t), g(t^+)]$ if $t \in D_g$ and $(\xi f)(t) = f \circ g^+(t) = f \circ g(t)$ if $t \in I \backslash D_g$.*

Proof. 1. If $f \in \mathscr{L}^1_g(I, \mathbb{F})$, then $f : (I, \mathscr{M}_g) \to (\mathbb{R}, \mathscr{B})$ is measurable and, since Proposition 4.25 guarantees that $g^\dagger : (J, \mathscr{L}) \to (I, \mathscr{M}_g)$ is measurable, so is $f \circ g^\dagger : (J, \mathscr{L}) \to (\mathbb{R}, \mathscr{B})$. Now, by Theorem 4.21 and Lemma 4.22,

$$\int\limits_J |f| \circ g^\dagger \, \mathrm{d}\, \mu = \int\limits_I |f| \, \mathrm{d}\, g^\dagger_* \mu = \int\limits_I |f| \, \mathrm{d}\, \mu_g < \infty.$$

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[1] If g were constant on some interval of the form $(-\infty, b]$, then $g^\dagger(g(b)) = -\infty$. We are avoiding cases such as this.

The same holds true if we substitute $|f|$ by f, so $f\circ g^\dagger \in \mathscr{L}^1(J,\mathbb{F})$ and $\int f\circ g^\dagger \,\mathrm{d}\,\mu = \int f \,\mathrm{d}\,\mu_g$, which proofs that $(g^\dagger)^*: \mathscr{L}^1_g(I,\mathbb{F}) \to \mathscr{L}^1(J,\mathbb{F})$ is an isometry.

Finally, by Lemma 4.22, $g^\dagger_*\mu = \mu_g$, so $g^\dagger_*\mu$ is trivially μ_g-absolutely continuous and, therefore, by Lemma 4.29, $(g^\dagger)^*: L^1_g(I,\mathbb{F}) \to L^1(J,\mathbb{F})$ is well defined.

2. If $f \in \mathscr{L}^1(\mathbb{R},\mathbb{F})$, then $f:(\mathbb{R},\mathscr{L}) \to (\mathbb{R},\mathscr{B})$ is measurable and, since Proposition 4.25 guarantees that $g^+:(I,\mathscr{M}_g) \to (\mathbb{R},\mathscr{L})$ is measurable, so is $f\circ g^+:(I,\mathscr{M}_g)\to(\mathbb{R},\mathscr{B})$. By Lemma 4.27, $(g^\dagger\circ g^+)(t)=t$ for μ_g-a.e. $t\in I$, so $(f\circ g^\dagger\circ g^+)(t) = f(t)$ for μ_g-a.e. $t \in I$. Clearly, the same happens for every $\widetilde{f}\in[f]$, so $(g^+)^*\circ(g^\dagger)^*$ is well defined and the identity on $L^1_g(J,\mathbb{F})$. Given $f\in\mathscr{L}^1(\mathbb{R},\mathbb{F})$,

$$\|(g^+)^*(g^\dagger)^*f\|_{L^1} = \|f\|_{L^1} = \|(g^\dagger)^*f\|_{L^1_g},$$

because $(g^\dagger)^*$ is a linear isometry, so $(g^+)^*|_{(g^\dagger)^*(\mathscr{L}^1_g(I,\mathbb{F}))}:(g^\dagger)^*(\mathscr{L}^1_g(I,\mathbb{F}))\to\mathscr{L}^1_g(I,\mathbb{F})$ is a linear isometry. Since it is surjective because $(g^+)^*\circ(g^\dagger)^*:\mathscr{L}^1_g(I,\mathbb{F})\to\mathscr{L}^1_g(I,\mathbb{F})$ is the identity, it is an isometric isomorphism.

3. Let $[f]\in L^1_g(I,\mathbb{F})$ (we consider f to be fixed element in the equivalence class), $t\in D_g$. We know by point 1 that $[f\circ g^\dagger]\in L^1(J,\mathbb{F})$ and, by Theorem 4.9, point 8, $(g(t),g(t^+)]\subset(g^\dagger)^{-1}(\{t\})$, so $f\circ g^\dagger$ is constant on $(g(t),g(t^+)]$. Thus, $f\circ g^\dagger\in X$.

On the other hand, if $[f]\in X$, consider $h = f\circ g^+ \in \mathscr{L}^1_g(I,\mathbb{F})$, by point 2. Since $g^+ = g$ on $I\backslash D_g$ and $g\circ g^\dagger = \mathrm{Id}$ on $(g^\dagger)^{-1}(I\backslash D_g)\subset g(I\backslash D_g)$ (see Theorem 4.9, point 6, and Lemma 4.19) we have that $h\circ g^\dagger = f\circ g^+\circ g^\dagger = f\circ g\circ g^\dagger = f$ on $g(I\backslash D_g)$. It is left to see what happens on $J\backslash g(I\backslash D_g)$. The connected components in $J\backslash g(I\backslash D_g)$ are intervals of endpoints $g(t)$ and $g(t^+)$ with $t\in D_g$. Both f and $h\circ g^\dagger$ are constant on $(g(t),g(t^+)]$. Furthermore, for $y\in(g(t),g(t^+))$, we have that $(h\circ g^\dagger)(y) = h(t) = f(g(t^+)) = f(y)$, so $h\circ g^\dagger = f$ on $(g(t),g(t^+)]$. We conclude that $[f]\in(g^\dagger)^*(L^1_g(I,\mathbb{F}))$.

To see that X is closed, take $(f_n)_{n\in\mathbb{N}}\in X$, $f_n\to f$ in $L^1(J,\mathbb{F})$. Let $t\in D_g$. For every $n\in\mathbb{N}$ there exists $c_n\in\mathbb{R}$ such that $f_n(x) = c_n$ for a.e. $x\in(g(t),g(t^+)]$. Since $(f_n)_{n\in\mathbb{N}}\to f$ in $L^1(J,\mathbb{F})$, there exists a subsequence, call it $(h_n)_{n\in\mathbb{N}}$, that converges pointwisely a.e. This implies that c_n has to converge to some value c, and, furthermore, that $f(x)=c$ for a.e. $x\in(g(t),g(t^+)]$, so we conclude that $f\in X$.

$(g^\dagger)^*:L^1_g(I,\mathbb{F})\to X$ is, therefore a surjective isometry between Banach spaces and, hence, an isometric isomorphism. To prove that ξ is, indeed, the inverse of $(g^\dagger)^*:L^1_g(I,\mathbb{F})\to X$, we have to check first that it is well defined. Let $[f]\in X$, $\widetilde{f}\in[f]$ and $E:=\{x\in J \ : \ f(x)\neq\widetilde{f}(x)\}$. By Corollary 4.6,

$$\mu_g(g^{-1}(E)\backslash D_g) = \mu(g(g^{-1}(E)\backslash D_g)) \subset \mu(g(g^{-1}(E))) \subset \mu(E) = 0.$$

Therefore, for $t\in I\backslash D_g$, $(\xi f)(t)$ is well defined. On the other hand, if $t\in I\cap D_g$, f and $\widetilde{f}$ are constant a.e. on $(g(t),g(t^+)]$ and, since $\mu((g(t),g(t^+)])>0$ and f and $\widetilde{f}$ are equal a.e., they have to take the same constant as value a.e. on $(g(t),g(t^+)]$. Hence, $(\xi f)(t)$ is well defined as well.

Furthermore, since $f\circ g^+|_{I\backslash D_g}\in L^1_g(I\backslash D_g,\mathbb{F})$ and $\xi f|_{D_g}\in L^1_g(D_g,\mathbb{F})$, we have that $\xi\in L^1_g(I,\mathbb{F})$. ■

Remark 4.31. X in Theorem 4.30 is a complemented subspace of $L^1(J,\mathbb{F})$. To see this, just define $P_X: L^1(J,\mathbb{F})\to X$ such that

$$P_X f(x) = \begin{cases} \dfrac{1}{\Delta g(t)}\displaystyle\int_{g(t)}^{g(t^+)} f, & x\in(g(t),g(t^+)] \text{ with } t\in D_g,\\[2ex] f(x), & x\in J\backslash D_g^*.\end{cases}$$

It is clear that P_X is well defined, linear and $P_Xf = f$ for $f\in X$, so P_X is the projection of $L^1(I,\mathbb{F})$ onto X. Furthermore, this projection is continuous since

$$\|P_X f\|_{L^1} = \|P_X f^+ - P_X f^-\|_{L^1} \leqslant \|P_X f^+\|_{L^1} + \|P_X f^-\|_{L^1} = \|f^+\|_{L^1} + \|f^-\|_{L^1} = \|f\|_{L^1},$$

where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ for any $f \in L^1(I, \mathbb{F})$, so we conclude that X is complemented and, thus, $L^1(I, \mathbb{F})$ can be written as $X \oplus Y$ where $Y = \ker P_X$ is a closed subspace of $L^1(I, \mathbb{F})$.

Theorem 4.30 and Remark 4.31 can be extended to the L^p spaces.

Corollary 4.32. *Let $p \in [1, \infty]$, $g : I \to \mathbb{R}$ be a derivator, $J = \operatorname{ce}(g(I))$ and assume $g^\dagger(J) \subset I$. The following properties hold:*

1. *If $f \in \mathscr{L}^p_g(I, \mathbb{F})$ then $f \circ g^\dagger \in \mathscr{L}^p(J, \mathbb{F})$ and $\int f \circ g^\dagger \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu_g$. As a consequence, the pullback $(g^\dagger)^* : L^p_g(I, \mathbb{F}) \to L^p(J, \mathbb{F})$ is well defined and a linear isometry.*
2. *The pullback $(g^+)^* : \mathscr{L}^p(\mathbb{R}, \mathbb{F}) \to M_g(I, \mathbb{F})$ is well defined and $(g^\dagger \circ g^+)^*$ is the identity on $L^p_g(I, \mathbb{F})$. Furthermore, $(g^+)^*|_{(g^\dagger)^*(\mathscr{L}^p_g(I,\mathbb{F}))} : (g^\dagger)^*(\mathscr{L}^p_g(I, \mathbb{F})) \to \mathscr{L}^p_g(I, \mathbb{F})$ is an isometric isomorphism.*
3. *Let X be the space of functions $f \in L^p(J, \mathbb{F})$ such that for every $t \in D_g$, there exists $c \in \mathbb{F}$ satisfying $f(x) = c$ for a.e. $x \in (g(t), g(t^+)]$. Then $X = (g^\dagger)^*(L^p_g(I, \mathbb{F}))$, X is a closed subspace of $L^p(J, \mathbb{F})$ and $(g^\dagger)^* : L^p_g(I, \mathbb{F}) \to X$ is an isometric isomorphism with inverse $\xi : X \to L^p_g(I, \mathbb{F})$ given by $(\xi f)(t) = c$ where $f(x) = c$ for a.e. $x \in (g(t), g(t^+)]$ if $t \in D_g$ and $(\xi f)(t) = f \circ g^+(t) = f \circ g(t)$ if $t \in I \backslash D_g$.*

Proof. 1. The case where $p < \infty$ is evident from Theorem 4.30 as $\|f\|^p_{L^p_g} = \|f^p\|_{L^1_g}$ for $f \in L^p_g(I, \mathbb{F})$ (in particular for $g = \mathrm{Id}$).

If $p = \infty$, let $f \in L^\infty_g(I, \mathbb{F})$. As before, $f : (I, \mathscr{M}_g) \to (\mathbb{R}, \mathscr{B})$ is measurable and, since Proposition 4.25 guarantees that $g^\dagger : (J, \mathscr{L}) \to (I, \mathscr{M}_g)$ is measurable, so is $f \circ g^\dagger : (J, \mathscr{L}) \to (\mathbb{R}, \mathscr{B})$. Let $N \in \mathscr{M}_g$ with $\mu_g(N) = 0$ such that $|f(x)| \leqslant \|f\|_{L^\infty_g}$ for $x \in I \backslash N$. Taking $x = g^\dagger(y)$, we have that $|f(g^\dagger(y))| \leqslant \|f\|_{L^\infty_g}$ for $y \in J \backslash (g^\dagger)^{-1}(N)$. By Lemma 4.22, $\mu((g^\dagger)^{-1}(N)) = \mu_g(N) = 0$, so $\|f \circ g^\dagger\|_{L^\infty} \leqslant \|f\|_{L^\infty_g}$ and $(g^\dagger)^*$ is continuous.

Now let $M \in \mathscr{L}$ with $\mu(M) = 0$ such that $|f \circ g^\dagger(y)| \leqslant \|f \circ g^\dagger\|_{L^\infty}$ for $y \in J \backslash M$. $f \circ g^\dagger$ is constant on $(g(t), g(t^+)]$ for every $t \in D_g$ and, given that $(g(t), g(t^+)]$ has positive Lebesgue measure, $M \cap (g(t), g(t^+)] = \emptyset$, so $(g^+)^{-1}(M) \cap D_g = \emptyset$. Since $g(t) = g^+(t)$ for $t \notin D_g$, $(g^+)^{-1}(M) = g^{-1}(M)$. Taking $y = g^+(x)$, we have that $|(f \circ g^\dagger \circ g^+)(x)| \leqslant \|f \circ g^\dagger\|_{L^\infty}$ for $x \in I \backslash (g^+)^{-1}(M)$. By Lemma 4.27, $g^\dagger \circ g^+(t) = t$ for μ_g-a.e. $t \in I$. Hence, there exists $T \in \mathscr{M}_g$ such that $\mu_g(T) = 0$ and $|(f \circ g^\dagger \circ g^+)(x)| = |f(x)|$ for $x \in I \backslash T$. Thus, $|f(x)| \leqslant \|f \circ g^\dagger\|_{L^\infty}$ for $x \in I \backslash ((g^+)^{-1}(M) \cup T)$.

We have that $\mu_g(T) = 0$ and, combining the fact that $(g^+)^{-1}(M) \cap D_g = \emptyset$ with Corollary 4.6, we obtain

$$\mu_g((g^+)^{-1}(M)) = \mu_g(g^{-1}(M)) = \mu(g(g^{-1}(M)) \leqslant \mu(M) = 0.$$

We conclude that $|f(x)| \leqslant \|f \circ g^\dagger\|_{L^\infty}$ μ_g-a.e. so $\|f\|_{L^\infty_g} \leqslant \|f \circ g^\dagger\|_{L^\infty}$.

Since $\|f\|_{L^\infty_g} = \|f \circ g^\dagger\|_{L^\infty}$, $(g^\dagger)^*$ is a linear isometry.

Points 2 and 3 are proven as in Theorem 4.30. ■

The question now is whether this good behavior extends to the spaces $W^{1,p}_g(I, \mathbb{F})$. In general this is not true. To prove this, we need the following result.

Theorem 4.33 *([11, Theorem 5.17]). Let $a < b$ and $1 \leqslant p \leqslant \infty$. Then there exists a continuous linear operator $E : W^{1,p}_g([a, b)) \to W^{1,p}_g(\mathbb{R})$, called the* extension operator*, such that*

1. *$Ef|_{[a,b]} = f$, $\forall f \in W^{1,p}_g([a, b))$,*
2. *$\|Ef\|_{L^p_g(\mathbb{R})} \leqslant \widetilde{C} \, \|f\|_{L^p_g([a,b))}$, $\forall f \in W^{1,p}_g([a, b))$,*

3. $\|Ef\|_{W_g^{1,p}(\mathbb{R})} \leqslant \widetilde{C}\,\|f\|_{W_g^{1,p}([a,b))}$, $\forall f \in W_g^{1,p}([a,b))$,
4. $\|Ef\|_0 \leqslant \widetilde{C}\,\|f\|_0$, $\forall f \in W_g^{1,p}([a,b))$,

where $\|\cdot\|_0$ *is the supremum norm and* $\widetilde{C} > 0$ *is a constant depending only on* p *and* $\mu_g([a,b))$.

The operator E used in the proof of Theorem 4.33 is defined as

$$Ef(x) = \begin{cases} f(a)\,\exp_g(\lambda_-, b)(t), & x \in (-\infty, a), \\ f(x), & x \in [a,b), \\ f(b)\,\exp_g(-\lambda_+, b)(t), & x \in [b, \infty), \end{cases}$$

where $\lambda_\pm$ are positive constants and the function $\exp_g(\lambda, x_0)$ satisfies $\exp_g(\lambda, x_0)(x_0) = 1$ and $\exp_g(\lambda, x_0)'_g = \lambda \exp_g(\lambda, x_0)$ on $[x_0, \infty)$—see [11, Lemma 5.13].

Theorem 4.34. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a derivator,* $n \in \mathbb{N}$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a function such that the pullback* $\varphi^* : W_g^{1,p}(\mathbb{R}, \mathbb{F}) \to W^{1,p}(\mathbb{R}, \mathbb{F})$ *is an isometry. Then,*

1. φ *is absolutely continuous.*
2. *Either* $D_g = \emptyset$ *or* D_g *has at most one point* t*, in which case, either* $\mu_g((-\infty, t)) = 0$ *or* $\mu_g((t, \infty)) = 0$.

Proof. 1. Let $f \in W_g^{1,p}(\mathbb{R}, \mathbb{F})$, $f \geqslant 0$, $x, y \in \mathbb{R}$, $x \leqslant y$. There exists $h \in L_g^p(\mathbb{R}, \mathbb{F})$ such that

$$f(y) - f(x) = \int_x^y h \,\mathrm{d}\mu_g, \; x, y \in \mathbb{R}.$$

Evaluating on $\varphi(x)$ and $\varphi(y)$, we obtain

$$\varphi^* f(y) - \varphi^* f(x) = \int_{\varphi(x)}^{\varphi(y)} h \,\mathrm{d}\mu_g.$$

On the other hand, $\varphi^* f \in W^{1,p}(\mathbb{R}, \mathbb{F})$ so there exists $\widetilde{h} \in L^p(\mathbb{R}, \mathbb{F})$ such that

$$\varphi^* f(y) - \varphi^* f(x) = \int_x^y \widetilde{h} \,\mathrm{d}\mu.$$

We conclude that

$$\int_{\varphi(x)}^{\varphi(y)} h \,\mathrm{d}\mu_g = \int_x^y \widetilde{h} \,\mathrm{d}\mu. \tag{4.2}$$

Now, since $g \in W_g^{1,p}([x,y), \mathbb{F})$, we use Theorem 4.33 to obtain $f \in W_g^{1,p}(\mathbb{R}, \mathbb{F})$ such that $f|_{[x,y)} = g$. With this choice of f, by Theorem 2.4, h in equation (4.2) satisfies $h = 1$ μ_g-a.e. on $[x,y)$ and, therefore,

$$\varphi(y) - \varphi(x) = \int_x^y \widetilde{h} \,\mathrm{d}\mu.$$

Since this happens for every $x, y \in \mathbb{R}$, $x < y$, we deduce that φ is absolutely continuous.

2. Take $z \in \mathbb{R}$ and $x, y \in \mathbb{R}$ such that $x < \varphi(z) < y$ and $f(t) = \int_{[x,t)} \chi_{\{\varphi(z)\}} \,\mathrm{d}\,\mu_g$ for $t \in [x, y)$. $f \in W_g^{1,p}([x, y), \mathbb{F})$, and we can extend it to $W_g^{1,p}(\mathbb{R}, \mathbb{F})$. Furthermore, for every $t \in (\varphi(z), y)$,

$$f(t) - f(\varphi(z)) = \Delta g(\varphi(z)) = \int\limits_{[\varphi(z),t)} \widetilde{h} \,\mathrm{d}\,\mu.$$

Taking the limit when $t \to \varphi(z)^+$,

$$f(\varphi(z)^+) - f(\varphi(z)) = \Delta g(\varphi(z)) = \int\limits_{\{\varphi(z)\}} \widetilde{h} \,\mathrm{d}\,\mu = 0.$$

This implies that $\varphi(z) \notin D_g$ and, therefore $\varphi(\mathbb{R}) \cap D_g = \emptyset$. Since φ is continuous, $\varphi(\mathbb{R})$ is contained in a connected component of $\mathbb{R} \backslash D_g$. Let $t \in D_g$ and assume $\mu_g((-\infty, t)), \mu_g((t, \infty)) \neq 0$. Then $\varphi(x) < t$ for every $x \in \mathbb{R}$ or $\varphi(x) > t$ for every $x \in \mathbb{R}$. In the first case, take

$$f_\lambda(x) := \begin{cases} 0, & x \in (-\infty, t), \\ \exp_g(-\lambda; b, t), & x \in [t, \infty), \end{cases}$$

for some $\lambda > 0$ for which the exponential is well defined on $[t, \infty)$—see [11, Lemma 5.13]. We have that $f_\lambda \in W_g^{1,p}(\mathbb{R}, \mathbb{F})$. Since $\mu_g((t, \infty)) > 0$, $\|f\|_{L_g^p} \neq 0$ and, therefore, $f \neq 0$, but $f \circ \varphi = 0$, which contradicts that φ^* is an isometry. The same reasoning works for the case $\varphi(x) > t$ for every $x \in \mathbb{R}$. We conclude that either $D_g = \emptyset$ or D_g has at most one point t, in which case either $\mu_g((-\infty, t)) = 0$ or $\mu_g((t, \infty)) = 0$. ■

The proof of Theorem 4.34, done for the case of the domain being $\mathbb{R}$, can be adapted to other intervals. Furthermore, Theorem 4.34 establishes strong limitations to those cases where a pullback isometry can be found, so one may look for other kinds of maps.

One may consider the following alternative operator T such that $Tf(x) := f(0) + \int_0^x f_g' \circ g^\dagger \,\mathrm{d}\,\mu$ for $x \in \mathbb{R}$ and $f \in W_g^{1,p}(\mathbb{R}, \mathbb{F})$. Clearly, $\|(T \circ f)'\|_{L^p} = \|f_g'\|_{L_g^p}$, but $\|Tf\|_{L^p} \neq \|f\|_{L_g^p}$ in general, so we will look for an isomorphism rather than an isometry.

The proof of the following corollary is straightforward from Theorem 4.30.

Corollary 4.35. *Let $p \in [1, \infty]$, $g : I \to \mathbb{R}$ be a derivator, $J = \operatorname{ce}(g(I))$ and assume $g^\dagger(J) \subset I$. The following properties hold:*

1. *The map $\mathrm{Id}_{\mathbb{F}^n} \times (g^\dagger)^* : \mathbb{F}^n \oplus L_g^1(I, \mathbb{F}) \to \mathbb{F}^n \oplus L^1(J, \mathbb{F})$ such that $\mathrm{Id}_{\mathbb{F}^n} \times (g^\dagger)^*(x, f) := (x, f \circ g^\dagger)$ is a linear isometry.*
2. *Let X be the space of pairs $(x, f) \in \mathbb{F}^n \oplus L^1(J, \mathbb{F})$ such that for every $t \in D_g$, there exists $c \in \mathbb{F}$ satisfying $f(x) = c$ for a.e. $x \in (g(t), g(t^+)]$. Then $X = \mathrm{Id}_{\mathbb{F}^n} \times (g^\dagger)^*(\mathbb{F}^n \oplus L_g^1(I, \mathbb{F}))$ is a closed subspace of $\mathbb{F}^n \oplus L^1(I, \mathbb{F})$ and $(g^\dagger)^* : \mathbb{F}^n \oplus L_g^1(I, \mathbb{F}) \to X$ is an isometric isomorphism.*

Now we make use of Theorem 3.5.

Corollary 4.36. *Let $t_0 \in I$ and $x_0 \in J$ such that $g(t_0) = x_0$ and $t_0 = g^\dagger(x_0)$, $\mu_g(I) < \infty$. Then the map*

$$\begin{array}{ccc} W_g^{n,1}(I,\mathbb{F}) & \xrightarrow{\qquad\qquad\varphi_n\qquad\qquad} & W^{n,1}(J,\mathbb{F}) \\ f & \longmapsto & h(x)=\displaystyle\sum_{k=0}^{n-1}\frac{f_g^{(k)}(t_0)}{k!}(x-x_0)^k+\frac{1}{(n-1)!}\int\limits_{x_0}^{x}(x-y)^{n-1}\,f_g^{(n)}(g^\dagger(y))\,\mathrm{d}\,y \end{array}$$

is a linear embedding (and thus $W_g^{n,1}(I,\mathbb{F})$ *is isomorphic to* $\varphi_n(W_g^{n,1}(I,\mathbb{F}))$*).*

Proof. It is enough to observe that $\varphi_n=\psi_{\mathrm{Id}}\circ(\mathrm{Id}_{\mathbb{F}^n}\times(g^\dagger)^*)\circ\varphi_g$, where φ_g and ψ_{Id} are the maps defined in Theorem 3.5. The map ψ_{Id} acts as

$$\psi_{\mathrm{Id}}((\alpha_0,\dots,\alpha_{n-1}),h)=\sum_{k=0}^{n-1}\frac{\alpha_k}{k!}(x-x_0)^k+\frac{1}{(n-1)!}\int\limits_{x_0}^{x}(x-t)^{n-1}\,h(t)\,\mathrm{d}\,t$$

by Cauchy's formula for repeated integration [12, p. 193]. ■

For the last result of this section, we will show the relation of the φ_1 to the pullback $g^\dagger$. The advantage of defining φ_1 as we did in Corollary 4.36 is that, from that definition, it is obvious that φ_n takes $W_g^{1,1}(I,\mathbb{F})$ to $W^{1,1}(J,\mathbb{F})$.

Corollary 4.37. *Let* $I=[a,b)$, $g:I\to\mathbb{R}$ *a derivator. Then* φ_1*, as defined in Corollary 4.36 for* $x_0=g(a)$*, satisfies*

$$\varphi_1 f(x)=\begin{cases} f\circ g^\dagger(x), & x\in J\backslash\{g(t^+)\ :\ t\in D_g\},\\ f\circ(g^\dagger)^+(x), & x\in\{g(t^+)\ :\in D_g\},\end{cases}\tag{4.3}$$

that is, $\varphi_1 f=f\circ g^\dagger$ *a.e.*

Proof. Using Theorem 4.21, Lemma 4.22, Theorem 2.4 and taking into account that $g^\dagger(g(a))=a$,

$$\varphi_1 f(x)=f(a)+\int\limits_{[g(a),x)} f_g'\circ g^\dagger\,\mathrm{d}\,\mu=f(a)+\int\limits_{g^\dagger([g(a),x))} f_g'\,\mathrm{d}\,g_*^\dagger\mu=f(a)+\int\limits_{g^\dagger([g(a),x))} f_g'\,\mathrm{d}\,\mu_g.$$

We study now two cases using Lemma 4.18 and Corollary 4.23. If $x\notin D_g^*$,

$$\varphi_1 f(x)=f(a)+\int\limits_{[a,g^\dagger(x))} f_g'\,\mathrm{d}\,\mu_g=f(g^\dagger(x)).$$

If $x\in D_g^*$, then $g^\dagger(x)\in D_g$ (see Remark 4.12) and

$$\varphi_1 f(x)=f(a)+\int\limits_{[a,g^\dagger(x)]} f_g'\,\mathrm{d}\,\mu_g=f(g^\dagger(x^+)).$$

Now, if $x\in D_g^*$, then there exists $t\in D_g$ such that $x\in(g(t),g(t^+)]$. $g^\dagger$ is constant on $(g(t),g(t^+))$, so, if $x\in(g(t),g(t^+))$, then $\varphi_1 f(x)=f(g^\dagger(x^+))=f(g^\dagger(x))$. We then obtain equation (4.3). Furthermore, since $\{g(t^+)\ :\in D_g\}$ is countable, $\mu(\{g(t^+)\ :\in D_g\})=0$, so $\varphi_1 f=f\circ g^\dagger$ a.e. ■

5. Applications

In this section we will summarize some applications of the isomorphisms presented above. We will not enter into details, as there is much to be said about each, and they may be explored in detail in the future.

5.1. Differential and integral equations

Consider $t_0, x_0 \in \mathbb{R}$ and a derivator $g : [t_0, \infty) \to \mathbb{R}$ such that $g(t_0) = x_0$. Observe that $t_0 = g^\dagger(x_0)$. If instead g were defined as $g : \mathbb{R} \to \mathbb{R}$, the condition $t_0 = g^\dagger(x_0)$ might no longer hold. However, we can always restrict such a function g to the interval $[t_0, \infty)$.

Consider as well a Stieltjes differential problem of the form

$$u_g'(t) = f(t, u(t)),\ \mu_g\text{-a.e. } t \in [t_0, \infty), \quad u(t_0) = x_0. \tag{5.1}$$

We will assume here, as it is frequently done, that f is an L^1_g-Carathéodory function.

Definition 5.1 *([13])*. Let $X \subset \mathbb{R}^n$, $X \neq \emptyset$. We say that $f : [a, b] \times X \to \mathbb{R}^n$ is L^1_g-*Carathéodory* if it satisfies the following conditions:

1. For every $x \in X, f(\cdot, x)$ is g-measurable.
2. For μ_g-a.e. $t \in [a, b], f(t, \cdot)$ is continuous on X.
3. For every $r \in \mathbb{R}^+$, there exists $h_r \in \mathscr{L}^1_g([a, b))$ such that $\|f(t, x)\| \leq h_r(t)$ for μ_g-a.e. $t \in [a, b)$, and for all $x \in X, \|x\| \leq r$.

We can turn problem (5.1) into an equivalent integral equation by the Fundamental Theorem of Calculus (Theorem 2.4):

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, \mathrm{d}\, \mu_g(s). \tag{5.2}$$

We can now try to use the theory developed above in order to transform this problem into another integral equation, this time with respect to the usual Lebesgue measure:

$$v(x) = u_0 + \int_{x_0}^{x} f(g^\dagger(y), v(y))\, \mathrm{d}\, y. \tag{5.3}$$

Equation (5.3) is classical integral equation to which a solution can be guaranteed using a variety of well known results. We will also be considering the equations

$$\widetilde{v}(x) = u_0 + \int_{x_0}^{g(g^\dagger(x))} f(g^\dagger(y), \widetilde{v}(y))\, \mathrm{d}\, y, \tag{5.4}$$

$$\widetilde{v}'(x) = f(g^\dagger(x), \widetilde{v}(x))\chi_{\mathbb{R}\backslash D_g^*}(x), \tag{5.5}$$

where $\chi_{\mathbb{R}\backslash D_g^*}$ is the indicator function of the set $\mathbb{R}\backslash D_g^*$, and the integral equation with deviated argument

$$v(x) = u_0 + \int_{x_0}^{x} f(g^\dagger(y), v(g(g^\dagger(y))))\, \mathrm{d}\, y. \tag{5.6}$$

We first need the following Lemma.

Lemma 5.2. *Let* t_0, x_0 *be such that* $g(t_0) = x_0$ *and* $t_0 = g^\dagger(x_0)$, $t > t_0$. *Then*

$$\mu((g^\dagger)^{-1}([g^\dagger(x_0), t))\Delta[x_0, g^\dagger(g(t)^+))) = 0.$$

In particular, if $t = g^\dagger(x)$, *then*

$$\mu((g^\dagger)^{-1}([g^\dagger(x_0), g^\dagger(x)))\Delta[x_0, g(g^\dagger(x)))) = 0.$$

Proof. We start by considering the case $t = g^\dagger(x)$ for $x = g(t)$. In this case, $t \notin C_g^*$ and, by Lemma 4.27, $g^\dagger(g(t)^+) = t$. Taking into account that $g(t_0) = x_0$ and $t_0 = g^\dagger(x_0)$ we deduce that $(g^\dagger)^{-1}(\{g^\dagger(x_0)\}) \subset [x_0, \infty)$. Since $g^\dagger : \mathbb{R} \to \mathbb{R}$ is a derivator, by Lemma 4.7, point 1, we have that

$$\begin{aligned}(g^\dagger)^{-1}([g^\dagger(x_0), g^\dagger(x))) =&[(-\infty, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\})]\backslash[(-\infty, x_0)\backslash(g^\dagger)^{-1}(\{g^\dagger(x_0)\})]\\ =&[x_0, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\}).\end{aligned}$$

Furthermore, $[x_0, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\}) \subset [x_0, g(g^\dagger(x))]$. Indeed, if $y \in [x_0, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\})$, since $y < x$, we have that $g^\dagger(y) \leqslant g^\dagger(x)$. But $g^\dagger(y) \neq g^\dagger(x)$, so $g^\dagger(y) < g^\dagger(x)$. Suppose that $y > g(g^\dagger(x))$. Then $g^\dagger(y) \geqslant g^\dagger(x)$, and we arrive at a contradiction. Also, if $y \in [x_0, g(\{g^\dagger(x)\}))$, then $g^\dagger(y) < g^\dagger(x)$ so $y \in [x_0, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\})$. We conclude that the difference between the sets $[x_0, x)\backslash(g^\dagger)^{-1}(\{g^\dagger(x)\})$ and $[x_0, g(g^\dagger(x)))$ is at most one point, so

$$\mu((g^\dagger)^{-1}([g^\dagger(x_0), g^\dagger(x)))\Delta[x_0, g(g^\dagger(x)))) = 0,$$

and we get the result.

Now, if $g^\dagger(g(t)) \neq t$, then $t \in C_g^*$ and g is constant on $(g^\dagger(g(t)), t]$, so $(g^\dagger)^{-1}([g^\dagger(x_0), t)) = [x_0, g^\dagger(g(t)^+)]$, and we conclude the same. ■

Remark 5.3. We already knew that $\mu((g^\dagger)^{-1}([t, s))) = \mu([g(t), g(s)))$. To see this it is just enough to observe that, by Lemma 4.22,

$$\mu((g^\dagger)^{-1}([t, s)) = g^\dagger_*\mu([t, s)) = \mu_g([t, s)) = g(s) - g(t) = \mu([g(t), g(s))).$$

Lemma 5.2 makes a stronger statement.

The following theorem establishes the relation between equations (5.2)–(5.6).

Theorem 5.4. *The following statements hold true:*

1. *If a function* $u : [t_0, b] \to \mathbb{R}$ *is a solution of equation* (5.2) *then* $\widetilde{v} := u \circ g^\dagger : (g^\dagger)^{-1}([t_0, b]) \to \mathbb{R}$ *satisfies equation* (5.4).
2. *If* $\widetilde{v} : [x_0, c] \to \mathbb{R}$ *satisfies equation* (5.4), *then it satisfies equation* (5.3) *for every* $x \in (g^\dagger)^{-1}([t_0, b])\backslash D_g^*$. *Furthermore,* $\widetilde{v}$ *is constant on* $(g(t), g(t^+)] \cap (g^\dagger)^{-1}([t_0, b])$ *for every* $t \in D_g$.
3. *If* $\mu(\partial D_g^*) = 0$ *and* $\widetilde{v} : [x_0, c] \to \mathbb{R}$ *satisfies equation* (5.4), *then* $\widetilde{v}$ *satisfies equation* (5.5) *a.e.*
4. *If* $v : [x_0, c] \to \mathbb{R}$ *is a solution of the equation* (5.6) *then* $u := v \circ g : g^{-1}([x_0, c]) \to \mathbb{R}$ *is a solution of equation* (5.2).
5. *If* $v : [x_0, c] \to \mathbb{R}$ *is a solution of equation* (5.3) *and* v *is constant on* $[g(t), g(t^+)]$ *for every* $t \in D_g$, *then* $u := v \circ g : g^{-1}([x_0, c]) \to \mathbb{R}$ *is a solution of equation* (5.6).

Proof. 1. Assume u is a solution of equation (5.2) and $t > t_0$. Using the map $g^\dagger$, Lemma 4.22, Corollary 4.23 and Theorem 4.21 for $\varphi = g^\dagger : (g^\dagger)^{-1}([t_0, t)) \to [t_0, t) \cap g^\dagger(\mathbb{R})$, we can rewrite the equation (5.2) as

$$u(t) = u_0 + \int\limits_{[t_0,t)\cap g^\dagger(\mathbb{R})} f(s, u(s))\,\mathrm{d}\, g^\dagger_* \mu(s) = u_0 + \int\limits_{(g^\dagger)^{-1}([t_0,t))} f(g^\dagger(y), u(g^\dagger(y)))\,\mathrm{d}\, y,$$

which, evaluated on $g^\dagger(x)$ instead of t, provides, by Lemma 5.2,

$$\begin{aligned}\widetilde{v}(x) :=& u(g^\dagger(x)) = u_0 + \int\limits_{(g^\dagger)^{-1}([g^\dagger(x_0), g^\dagger(x)))} f(g^\dagger(y), \widetilde{v}(y))\,\mathrm{d}\, y \\ =& u_0 + \int\limits_{[x_0, g(g^\dagger(x)))} f(g^\dagger(y), \widetilde{v}(y))\,\mathrm{d}\, y,\end{aligned}$$

and equation (5.4) holds.

2. It is clear that $\widetilde{v}$ satisfies equation (5.3) by the definition of D_g^*. Furthermore, by Corollary 4.9, point 8, $g^\dagger$ is constant on $(g(t), g(t^+)]$ for every $t \in D_g$, so $\widetilde{v}$ is constant on $(g(t), g(t^+)] \cap (g^\dagger)^{-1}([t_0, b])$ as well.

3. It is enough to differentiate $\widetilde{v}$ on $(x_0, c) \cap \mathring{D}_g^*$, where $\widetilde{v}' = 0$ and on $(x_0, c)\backslash\overline{D_g^*}$, where $\widetilde{v}(x) = f(g^\dagger(x), \widetilde{v}(x))$ due to point 1. It is important to remark that $\widetilde{v}$ is not necessarily continuous, and may have jumps in (x_0, c) where $g^\dagger$ does, that is, at the points a such that (a, b) is a connected component of C_g.

4. Assume now that we have a solution v of equation (5.6). Consider $u := v \circ g$, that is,

$$u(t) = u_0 + \int\limits_{[x_0, g(t))} f(g^\dagger(y), u(g^\dagger(y))\,\mathrm{d}\, y.$$

We will check now that u satisfies equation (5.2).

By Theorem 4.21 and Lemma 4.22,

$$\begin{aligned}\int\limits_{[x_0,g(t))} f(g^\dagger(y), u(g^\dagger(y)))\,\mathrm{d}\, y =& \int\limits_{g^\dagger([x_0,g(t)))} f(s, u(s))\,\mathrm{d}(g^\dagger)_*\mu(s) \\ =& \int\limits_{g^\dagger([x_0,g(t)))} f(s, u(s))\,\mathrm{d}\,\mu_g(s).\end{aligned}$$

Now we study the cases in Lemma 4.18 using Corollary 4.23. If $g(t) \in D_g^*$,

$$\int\limits_{g^\dagger([x_0,g(t)))} f(s, u(s))\,\mathrm{d}\,\mu_g(s) = \int\limits_{[t_0, g^\dagger(g(t))]\cap g^\dagger(\mathbb{R})} f(s, u(s))\,\mathrm{d}\,\mu_g(s) = \int\limits_{[t_0, g^\dagger(g(t))]} f(s, u(s))\,\mathrm{d}\,\mu_g(s).$$

By definition of $g^\dagger$, $g^\dagger(g(t)) \leqslant t$ and $g(s) = g(t)$ for $s \in (g^\dagger(g(t)), t]$, so $\mu_g((g^\dagger(g(t)), t)) = 0$ and

$$\int\limits_{[t_0, g^\dagger(g(t))]} f(s, u(s))\,\mathrm{d}\,\mu_g(s) = \int\limits_{[t_0,t)} f(s, u(s))\,\mathrm{d}\,\mu_g(s).$$

If $g(t) \notin D_g^*$,

$$\int_{g^\dagger([x_0,g(t)))} f(s,u(s))\,\mathrm{d}\,\mu_g(s) = \int_{[t_0,g^\dagger(g(t)))\cap g^\dagger(\mathbb{R})} f(s,u(s))\,\mathrm{d}\,\mu_g(s) = \int_{[t_0,g^\dagger(g(t)))} f(s,u(s))\,\mathrm{d}\,\mu_g(s).$$

Since $g(t) \notin D_g^*$, we have that $g(g^\dagger(g(t))) = g(t)$, so $\mu_g([g^\dagger(g(t)),t)) = 0$ and

$$\int_{[t_0,g^\dagger(g(t)))} f(s,u(s))\,\mathrm{d}\,\mu_g(s) = \int_{[t_0,t)} f(s,u(s))\,\mathrm{d}\,\mu_g(s).$$

We conclude that u is a solution of equation (5.2).

5. Let $v : [x_0,b] \to \mathbb{R}$ be a solution of equation (5.3) such that v is constant on $[g(t),g(t^+)]$ for every $t \in D_g$ —cf. Theorem 4.30. Observe that, by Theorem 4.9 points 6 and 7, $g(g^\dagger(z)) \leqslant z \leqslant g^+(g^\dagger(z))$. Since v is constant on $[g(g^\dagger(z)),g^+(g^\dagger(z))]$, we deduce that $v(z) = v(g(g^\dagger(z)))$ for every z, so equations (5.3) and (5.6) are equivalent. ■

Remark 5.5. Observe that in Theorem 5.4, point 5, since v is constant on $[g(t),g(t^+)]$ for every $t \in D_g$ we have that $u = v \circ g = v \circ g^+$. This is consistent with Theorem 4.30.

Theorem 5.4 provides an straightforward link between Stieltjes differential equations and ordinary differential equations through their integral equation counterparts. This equivalence can be exploited not only to evaluate the existence or multiplicity of solutions, but also to obtain information concerning the stability of Stieltjes differential equations—cf. [5]—as well as other qualitative results.

As a direct application of Theorem 5.4, we will prove the Stieltjes counterpart to the following classical result due to Binding[2]:

Theorem 5.6 *([2]). The problem*

$$v'(x) = f(v(x)),\ \text{a.e. } x \geqslant x_0; \quad v(x_0) = 0, \tag{5.7}$$

has at least one absolutely continuous solution if, and only if, one of the following conditions holds:

1. $f(0) = 0$;
2. *There exists* $\alpha \in \mathbb{R}^+$ *such that* $f(y) > 0$ *for a.e.* $y \in [0,\alpha]$ *and* $1/f \in L^1([0,\alpha],\mathbb{R})$;
3. *There exists* $\beta \in \mathbb{R}^+$ *such that* $f(y) < 0$ *for a.e.* $y \in [-\beta,0]$ *and* $1/f \in L^1([-\beta,0],\mathbb{R})$.

Theorem 5.6 can be easily extended to non-homogeneous initial conditions in (5.7), as the following corollary shows.

Corollary 5.7. *The problem*

$$\widetilde{v}'(x) = \widetilde{f}(\widetilde{v}(x)),\ \text{a.e. } x \geqslant x_0; \quad \widetilde{v}(x_0) = v_0, \tag{5.8}$$

has at least one absolutely continuous solution if, and only if, one of the following conditions holds:

1. $\widetilde{f}(v_0) = 0$;
2. *There exists* $\alpha \in \mathbb{R}^+$ *such that* $\widetilde{f}(y) > 0$ *for a.e.* $y \in [v_0,v_0+\alpha]$ *and* $1/\widetilde{f} \in L^1([v_0,v_0+\alpha],\mathbb{R})$;

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[2] This extension of the result of Binding to the Stieltjes case was proposed as an open problem by Prof. Rodrigo López Pouso at the "Introduction to Stieltjes differential equations and their applications" Workshop held at the Faculty of Sciences, Mohammed V University in Rabat, from the 27 to 29 May 2024.

3. *There exists* $\beta\in\mathbb{R}^+$ *such that* $\widetilde{f}(y)<0$ *for a.e.* $y\in[v_0-\beta,v_0]$ *and* $1/\widetilde{f}\in L^1([v_0-\beta,v_0],\mathbb{R})$.

Proof. Just define $v(x)=\widetilde{v}(x)-v_0$ and $f(x):=\widetilde{f}(x+v_0)$. Then equation (5.7) holds if and only if (5.8) holds. Expressing conditions 1-3 in Theorem 5.6 in terms of $\widetilde{f}$, we get the result. ■

Theorem 5.8. *Let* $g:[t_0,\infty)\to\mathbb{R}$ *be a derivator,* $x_0=g(t_0)$, $t_0=g^\dagger(x_0)=g^{-1}(t_0)$, *and assume that there exists* $T>t_0$ *such that* $(g^\dagger)^{-1}([t_0,T])\backslash D_g^*$ *is a non-degenerate interval. Then the problem*

$$u_g'(t)=f(u(t)),\ \mu_g\text{-a.e. } t\geqslant t_0;\quad u(t_0)=u_0, \tag{5.9}$$

has at least one g-absolutely continuous solution if, and only if, one of the following conditions holds:

1. $f(u_0)=0$;
2. *There exists* $\alpha\in\mathbb{R}^+$ *such that* $f(y)>0$ *for a.e.* $y\in[u_0,u_0+\alpha]$ *and* $1/f\in L^1([u_0,u_0+\alpha],\mathbb{R})$;
3. *There exists* $\beta\in\mathbb{R}^+$ *such that* $f(y)<0$ *for a.e.* $y\in[u_0-\beta,u_0]$ *and* $1/f\in L^1([u_0-\beta,u_0],\mathbb{R})$.

Proof. Assume that a g-absolutely continuous solution $u:[t_0,T]\to\mathbb{R}$ of problem (5.9) exists, where T can be taken as close to t_0 as necessary as long as $(g^\dagger)^{-1}([t_0,T])\backslash D_g^*$ is a non-degenerate interval of the form $[x_0,c]$. Then, integrating on both sides of (5.9) between t_0 and $t\in[t_0,T]$,

$$u(t)=u_0+\int_{t_0}^{t} f(u(s))\,\mathrm{d}\,\mu_g(s),\quad t\in[t_0,T].$$

f is L^1-Carathéodory as it does not depend on the variable t. Therefore, by Theorem 5.4, points 1 and 2, $v:=u\circ g^\dagger$ satisfies

$$v(x)=u_0+\int_{x_0}^{x} f(v(y))\,\mathrm{d}\,y, \tag{5.10}$$

for every $x\in[x_0,c]:=(g^\dagger)^{-1}([t_0,T])\backslash D_g^*$, so we have that v satisfies (5.10) on $[x_0,c]$. Since $v(x_0)=u(g^\dagger(x_0))$, differentiating on both sides, we get the problem

$$v'(x)=f(v(x)),\ \text{a.e. } x\in[x_0,c];\quad v(x_0)=u(g^\dagger(x_0)).$$

Now, by Corollary 5.7, we conclude that one of conditions 1-3 holds.

On the other hand, if one of conditions 1-3 holds, by Corollary 5.7, the problem

$$v'(x)=f(v(x)),\quad v(x_0)=u_0,$$

has a solution v. Assume that this solution is defined on $[x_0,d]$. Integrating, v satisfies the equation

$$v(x)=u_0+\int_{x_0}^{x} f(v(y))\,\mathrm{d}\,y,\quad x\in[x_0,d].$$

By Theorem 5.4, point 4, $u:=v\circ g$ is a g-absolutely continuous solution of the equation

$$u(t)=u_0+\int_{g^\dagger(x_0)}^{t} f(s,u(s))\,\mathrm{d}\,\mu_g(s)=u_0+\int_{t_0}^{t} f(s,u(s))\,\mathrm{d}\,\mu_g(s),\quad t\in g^{-1}([x_0,d]).$$

$$\begin{tikzcd}
W^{1,1}(J,\mathbb{F}) \arrow[r, "\Gamma"] & L^p(J,\mathbb{F}) \\
W_g^{1,1}([a,b),\mathbb{F}) \arrow[u, "\varphi_1"'] \arrow[r, dashed, "\widetilde{\Gamma}"] & L_g^p([a,b),\mathbb{F}) \arrow[u, "(g^\dagger)^*"]
\end{tikzcd}$$

Fig. 5.1. The relation between the different embeddings.

Taking the g-derivative, we see that u is a solution of problem (5.9) for $t \in g^{-1}([x_0, d])$. Finally, we observe that, since g is monotone and $[x_0, d]$ is an interval, $g^{-1}([x_0, d])$ is an interval with left endpoint t_0. ■

Remark 5.9. The existence of a non degenerate interval contained in $[x_0,\infty)\backslash D_g^*$ is equivalent to the condition $\operatorname{Int}([x_0,\infty)\backslash D_g^*) \neq \emptyset$ and it is clearly satisfied if $g = \operatorname{Id}$ as, in this case, $D_g^* = \emptyset$, so Theorem 5.8 extends Theorem 5.6. Furthermore, this condition ensures that the topology of $g(\mathbb{R})$ (remember that $g(\mathbb{R})$ is, up to a countable set, $[x_0,\infty)\backslash D_g^*$ —see Remark 4.17) is not totally disconnected. There are situations where $g(\mathbb{R})$ is indeed totally disconnected, for instance in the case $g(t) := \inf\{k \in \mathbb{Z} \ : \ k \geqslant t\}$, which corresponds to difference equations, but in this case it is clear that a theorem such as Theorem 5.8 cannot hold, as we show now.

Example 5.10. Let $g(t) := \inf\{k \in \mathbb{Z} \ : \ k \geqslant t\}$ for $t \in [0,\infty)$. $\mu_g([0,\infty)) = 0$, so when studying equation (5.9) for $t_0 = x_0 = 0$, we just have to consider

$$u_g'(k) = u(k+1) - u(k) = f(u(k)), \quad k \in \mathbb{Z},\ k \geqslant 0; \quad u(0) = 0.$$

Clearly, this problem has a unique solution given recursively by $u(0) = 0$, $u(k+1) = f(u(k)) + u(k)$ for $k \in \mathbb{Z}$ and $u(t) = u(\lceil t \rceil)$ for every other $t \in [0,\infty)$ (to guarantee that u is g-absolutely continuous), regardless of any properties of f. This implies that, in general, Theorem 5.8 cannot hold without the hypothesis $\operatorname{Int}([x_0,\infty)\backslash D_g^*) \neq \emptyset$, although it might be possible to weaken it with sufficient care.

5.2. Topological results

Having embedded the Stieltjes-Sobolev spaces into the classical Sobolev spaces we deduce, immediately, that any weakly hereditary property of the latter holds for the former. An example of these kind of properties is separability (in the case of metric spaces), among other separability axioms.

Furthermore, any continuous embedding of the classical spaces translates to an embedding of the respective Stieltjes-Sobolev spaces just by composing the embeddings. This does not immediately translate to an analogous embedding for the case of Stieltjes-Sobolev spaces. We illustrate this point with the compact embedding of $W_g^{1,1}([a,b),\mathbb{F})$ into $L_g^p([a,b),\mathbb{F})$ [11] (we denote this embedding by $\widetilde{\Gamma}$). The result for the case $g = \operatorname{Id}$ is well known (we denote this embedding, which is the set inclusion, by Γ), so a compact embedding of $W_g^{1,1}([a,b),\mathbb{F})$ into $L^p([a,b),\mathbb{F})$ can be derived in a straightforward way by composing φ_1, defined as in Corollary 4.36, with Γ. If we want to derive the result in [11], we will show that $\widetilde{\Gamma}$ exists and makes the diagram in Fig. 5.1, where φ_1 is defined as in Corollary 4.36, commutative.

Theorem 5.11. *There exists a continuous map $\widetilde{\Gamma}$ such that the diagram in Fig. 5.1 is well defined and commutative. As a consequence, $\widetilde{\Gamma}$ is a compact embedding of $W_g^{1,1}([a,b),\mathbb{F})$ into $L_g^p([a,b),\mathbb{F})$.*

Proof. Let $I = [a,b)$, $J = [g(a), g(b))$. We will follow the diagram in Fig. 5.2. We start by considering $\varphi_1 : W_g^{1,1}(I,\mathbb{F}) \to W_g^{1,1}(I,\mathbb{F})$ in Corollary 4.37, which is a linear embedding, as the hypotheses of Corollary 4.36 hold. We take now the inclusion $\Gamma : W_g^{1,1}(J,\mathbb{F}) \to L_g^p(J,\mathbb{F})$ which is a compact embedding, we decompose $L^p(I,\mathbb{F})$ into $X_p \oplus Y$ as in Remark 4.31 (this decomposition is also valid for $L^p(I,\mathbb{F})$) and we project onto X_p using the continuous projection P_{X_p}. Finally, we use the isomorphism $\xi : X_p \to L_g^p(I,\mathbb{F})$.

$$\widetilde{\Gamma}: W_g^{1,1}(I,\mathbb{F}) \xrightarrow{\varphi_1} W^{1,1}(J,\mathbb{F}) \xrightarrow{\Gamma} L^p(J,\mathbb{F}) \xrightarrow{\sim} X \oplus Y \xrightarrow{P_{X_p}} X_p \xrightarrow{\xi} L_g^p(I,\mathbb{F}).$$

Fig. 5.2. Chain of maps that defines $\widetilde{\Gamma}$.

Let $\widetilde{\Gamma}$ be the composition of all of these maps—see Fig. 5.2. $\widetilde{\Gamma}$ is linear and bounded, as all of the maps are, and compact, because Γ is compact as well. It is left to see that $\widetilde{\Gamma}$ is injective. This is not trivial, as P_{X_p} is not injective. Suppose $\widetilde{\Gamma}f = 0$. Then, since ξ is an isomorphism, $h := (\Gamma \circ \varphi_1)f \in Y$, that is, $h \in \ker P_{X_p}$. Thus, $h \in L^p([a,b),\mathbb{F})$, $h(x) = 0$ if $x \notin D_g^*$ and $\int_{g(t)}^{g(t^+)} h = 0$ for $t \in D_g$. Γ is the natural inclusion, so $\varphi_1 f = h$. Now, by Corollary 4.37,

$$h(x) = \varphi_1 f(x) = \begin{cases} f \circ g^\dagger(x), & x \in J\backslash\{g(t^+) \ : \ t \in D_g\}, \\ f \circ (g^\dagger)^+(x), & x \in \{g(t^+) \ : \in D_g\}. \end{cases}$$

Since $g^\dagger$ is constant on $(g(t), g(t^+)]$ for any $t \in D_g$, $f \circ (g^\dagger)^+$ takes the value $f(t)$ on $(g(t), g(t^+))$ and

$$0 = \int\limits_{g(t)}^{g(t^+)} h = f(t)\Delta g(t),$$

so $f(t) = 0$. This implies that h, which is constant on $(g(t), g(t^+))$ and continuous, satisfies $h(x) = 0$ for $x \in [g(t), g(t^+)]$ for any $t \in D_g$ and, since $h(x) = 0$ if $x \notin D_g^*$, we deduce that $h = 0$. Given that $\varphi_1 f = h$ and φ_1 is injective, we conclude that $f = 0$, and we have proven that $\widetilde{\Gamma}$ is injective.

To check the commutativity of the diagram in Fig. 5.1, observe that $((g^\dagger)^* \circ \widetilde{\Gamma})f = ((g^\dagger)^* \circ \xi \circ \varphi_1)f = \varphi_1 f$. We have, by Corollary 4.3, that $\varphi_1 f = f \circ g^\dagger$ a.e., so the diagram in Fig. 5.1 commutes. ■

6. Other applications and conclusions

We have barely scratched the surface of what can be achieved with the tools presented, especially in the context of topological results. For instance, from Theorem 4.30 and Corollary 4.36, we could attempt to derive various characterizations of compact subsets, such as the Ascoli-Arzelà theorem [14, Theorem 4.14] or the Kolmogorov-Riesz theorems [11, Theorem 4.12]—among other results in [11], which have already been proven for the case of Stieltjes spaces. Furthermore, the isomorphisms introduced here could provide a way to approach density theorems, such as the Weierstrass approximation theorem or the density of g-smooth functions in the space of uniformly g-continuous functions.

Theorem 4.30 and its corollaries also provide a way to incorporate the theory of Stieltjes differential equations with notions of convolution or transforms, such as the Fourier or Laplace transforms. The theory of dynamic equations on time scales uses versions of these tools—see [3,4,6,15,22], so it would be interesting to determine whether we recover these definitions in the specific case of a g that defines a time scale. In fact, we should point out that the isomorphisms presented here work for time scales as well (as they are a particular instance of Stieltjes calculus), so further results of the theory might be amenable to the approach presented in this paper.

Finally, many qualitative properties of solutions of Stieltjes differential equations will be preserved through the isomorphisms, so the study of asymptotic behavior, oscillation, etc., could be advanced in this way.

Despite all these ideas, we do not want to give the impression that the isomorphisms defined here are a silver bullet that trivializes the theory of Stieltjes differential equations. As seen in Theorem 4.30 and Theorem 5.4, it is clear that we will not have a one-to-one correspondence between Stieltjes differential equations and ordinary differential equations. Some careful case-by-case treatment will be necessary, and, in some cases, the approach might not work at all due to the specific characteristics of the setting.

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