NONTRIVIAL SOLUTIONS OF NON-AUTONOMOUS DIRICHLET FRACTIONAL DISCRETE PROBLEMS

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Abstract

In this paper, we introduce a two-point boundary value problem for a finite fractional difference equation with a perturbation term. By applying spectral theory, an associated Green's function is constructed as a series of functions and some of its properties are obtained. Under suitable conditions on the nonlinear part of the equation, some existence and uniqueness results are deduced.

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1. **Introduction**

In this paper we consider the following difference fractional Dirichlet boundary value problem (FBVP)

$$-\Delta^{\upsilon} y\,(t) + a\,(t+\upsilon-1)\, y\,(t+\upsilon-1) = w\,(t)\, f\,(t+\upsilon-1, y\,(t+\upsilon-1))\,,$$
$$y\,(\upsilon-2) = y\,(\upsilon+b+1) = 0, \tag{1.1}$$

for $t \in I \equiv [0, b+1]_{\mathbb{N}_0}$, where $\upsilon \in \mathbb{R}$ with $1 < \upsilon < 2$, $b \in \mathbb{N}$, $b \geq 5$, $\Delta^{\upsilon} y$ is the standard Riemann-Liouville fractional difference operator, $a(t),\ w(t) \in \mathbb{R}$ with $w \not\equiv 0$ on I, and $f : [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We use the following notation: $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for $a \in \mathbb{R}$, and $[c, c+n_0]_{\mathbb{N}_c} = [c, c+n_0] \cap \mathbb{N}_c$, for $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}_1$.

During the last decade, a lot of authors studied fractional difference equations. This is due to the fact that there has been a progress made in developing the basic theory in this field (see [5, 6, 9, 10, 11] and references

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therein). First, Díaz and Osler [9] define a fractional difference as an infinite series and a generalization of the binomial formula for the n-th order difference $\Delta^n f$ operator. Later, Miller and Ross [14] studied the linear υ-th order fractional differential equation as an analogue of the linear n-th order ordinary differential equation. In [1] and [2] Atici and Eloe have developed and applied a transform method for fractional finite differences and for fractional q-calculus problems, respectively. The same authors introduced in [3] and solved well-defined discrete fractional difference equations, while in [4] some properties of discrete fractional calculus in the sense of a backward difference, were introduced and developed as well as some properties of the Laplace transform for the nabla derivative on the time scale of integers.

Atici and Şengül introduced in [6] some analysis of discrete fractional variational problems. Their paper also provided some initial attempts at using the discrete fractional calculus to model biological processes as they showed a model of tumor growth. Similarly, Goodrich [10, 11] has established some important results on discrete fractional boundary value problems. However, due to the complexity of the fractional calculus, the Green's functions for fractional boundary value problems have not yet been well developed and only a few different discrete fractional problems have been studied.

In [5] Atici and Eloe showed that (see also Goodrich and Peterson [12]), for all $(t,s) \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I$

$$G_0(t,s) = \frac{1}{\Gamma(\upsilon)} \begin{cases} \frac{t^{(\upsilon-1)}(\upsilon+b-s)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}} - (t-s-1)^{(\upsilon-1)}, & s < t-\upsilon+1, \\ \frac{t^{(\upsilon-1)}(\upsilon+b-s)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}}, & t-\upsilon+1 \le s. \end{cases} \tag{1.2}$$

is the related Green's function to the Dirichlet problem

$$\begin{aligned} -\Delta^{\upsilon} y(t) &= h(t+\upsilon-1),\ t \in I, \\ y(\upsilon-2) &= y(\upsilon+b+1) = 0. \end{aligned}$$

This result was obtained by expressing the general solution of the first previous equation in terms of the υ-th fractional difference operator.

We point out that the method used in [5] fails to work if we study a more general problem such as

$$-\Delta^{\upsilon} y(t) + a(t+\upsilon-1)\, y(t+\upsilon-1) = 0,\ t \in I, \tag{1.3}$$

due to the complexity caused by the extra non-constant term $a(t+\upsilon-1)\, y(t+\upsilon-1)$.

This is the reason why we follow the approach given in [13], where Graef et al. studied the Dirichlet problem

$$\begin{aligned} -D_{0+}^{\alpha}u(t)+a\,(t)\,u(t) &= w\,(t)\,f\,(t,u(t))\,,\ 0<t<1, \\ u\,(0) &= u\,(1)=0, \end{aligned}$$

with D_{0+}^{α} the Riemann–Liouville fractional derivative for $1<\alpha<2$, $w\in C([0,1])$ such that $w\,(t)\geq 0$ on $[0,1]$ and $f\in C\,([0,1]\times\mathbb{R},\mathbb{R})$.

Similarly to them, we want to obtain the Green's function related to (1.3) as a series of functions by using the spectral theory. Then, under suitable conditions, we are able to obtain existence and uniqueness of solutions of FBVP (1.1). Our work provides a new approach for constructing Green's functions for discrete fractional boundary value problems. This method can be further extended to problems with more general boundary conditions.

The paper is organized as follows: In next section we recall some definitions and preliminary results. In Section 3 we obtain some technical results and properties of the Green's function related to the linear problem. After that, in Section 4, under suitable conditions, we establish and prove our existence, non-existence and uniqueness results. In the last section we give some examples to illustrate the applications of these results.

2. **Preliminaries**

Let us first recall some basic definitions and lemmas, which will be used till the end of this work.

Definition 2.1. We define $t^{(\upsilon)}=\frac{\Gamma(t+1)}{\Gamma(t+1-\upsilon)}$, for any t and υ for which the right-hand side is well defined. We also appeal to the convention that if $t+1-\upsilon$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\upsilon)}=0$.

Definition 2.2. The υ-th fractional sum of a function f, for $\upsilon>0$ and $t\in\mathbb{N}_{a+\upsilon}$, is defined as

$$\Delta^{-\upsilon}f(t)=\Delta^{-\upsilon}f\,(t;a):=\frac{1}{\Gamma\,(\upsilon)}\sum_{s=a}^{t-\upsilon}(t-s-1)^{(\upsilon-1)}\,f(s).$$

We also define the υ-th fractional difference for $\upsilon>0$ by $\Delta^{\upsilon}f(t):=\Delta^{N}\Delta^{\upsilon-N}f(t)$, where $t\in\mathbb{N}_{a+\upsilon}$ and $N\in\mathbb{N}$ is chosen so that $0\leq N-1<\upsilon\leq N$.

LEMMA 2.1. [10, Lemma 2.3] *Let t and υ be any number for which $t^{(\upsilon)}$ and $t^{(\upsilon-1)}$ are defined. Then $\Delta t^{(\upsilon)} = \upsilon t^{(\upsilon-1)}$.*

LEMMA 2.2. [10, Lemma 2.4] *Let $0 \leq N-1 < \upsilon \leq N$. Then $\Delta^{-\upsilon}\Delta^{\upsilon}y(t) = y(t) + C_1 t^{(\upsilon-1)} + C_2 t^{(\upsilon-2)} + \ldots + C_N t^{(\upsilon-N)}$ for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.*

Next result imporoves [5, Theorem 3.2].

LEMMA 2.3. *The Green's function $G_0(t,s)$ given by (1.2) satisfies:*
(i) $G_0(t,s) > 0$ for each $(t,s) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times I$;
(ii) $\max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}}\{G_0(t,s)\} = G_0(s+\upsilon-1,s)$ for each $s \in I$;
(iii) there exists $\gamma \in (0,1)$ such that

$$\min_{t\in\left[\frac{\upsilon+b}{4},\frac{3(\upsilon+b)}{4}\right]}\{G_0(t,s)\} \geq \gamma \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}}\{G_0(t,s)\} = \gamma G_0(s+\upsilon-1,s) \text{ for } s \in I.$$

P r o o f. (i) Consider first the case $s=0$. In this situation we have

$$\begin{aligned}
G_0(t,0) &= \frac{1}{\Gamma(\upsilon)}\left(\frac{t^{(\upsilon-1)}(\upsilon+b)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}} - (t-1)^{(\upsilon-1)}\right)\\
&= \frac{1}{\Gamma(\upsilon)}\left(\frac{\Gamma(t+1)\Gamma(b+\upsilon+1)\Gamma(b+3)}{\Gamma(t+2-\upsilon)\Gamma(b+2)\Gamma(b+\upsilon+2)} - \frac{\Gamma(t)}{\Gamma(t+1-\upsilon)}\right)\\
&= \frac{1}{\Gamma(\upsilon)}\left(\frac{t\Gamma(t)(b+2)}{(t+1-\upsilon)\Gamma(t+1-\upsilon)(b+\upsilon+1)} - \frac{\Gamma(t)}{\Gamma(t+1-\upsilon)}\right)\\
&= \frac{\Gamma(t)}{\Gamma(\upsilon)\Gamma(t+1-\upsilon)}\left(\frac{t(b+2)}{(t+1-\upsilon)(b+\upsilon+1)} - 1\right)\\
&= \frac{\Gamma(t)}{(b+\upsilon+1)\Gamma(\upsilon)\Gamma(t+2-\upsilon)}\left(\upsilon b + \upsilon^2 + t - t\upsilon - b - 1\right)\\
&> 0
\end{aligned}$$

since $t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$ and $\upsilon b + \upsilon^2 + t - t\upsilon - b - 1 = \upsilon - 1 > 0$ for $t = b+\upsilon$.

The case $1 \leq s \leq b+1$ has been proved in [5, Theorem 3.2]. So, (i) is proved.

(ii) First, let $1 \leq s \leq b+1$, since $G(\upsilon-2,s) = G(\upsilon+b+1,s) = 0$, it is proved in [5, Theorem 3.2] that

$$\max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}}\{G_0(t,s)\} = \max_{t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}}\{G_0(t,s)\} = G_0(s+\upsilon-1,s).$$

Now, if $s=0$, we need to show that

$$\max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}}\{G_0(t,0)\}=G_0(\upsilon-1,0).$$

To this end, from the expression of $G_0(t,0)$, we have that $G_0(\upsilon-2,0)=G_0(\upsilon+b+1,0)=0$. Moreover, we have that $\frac{\Gamma(t)\left(\upsilon b+\upsilon^2+t-t\upsilon-b-1\right)}{\Gamma(t+2-\upsilon)}$ is decreasing on $t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. Thus, $G_0(t,0)$ is decreasing on $t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$ and

$$\max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}}\{G_0(t,0)\}=\max_{t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}}\{G_0(t,0)\}=G_0(\upsilon-1,0),$$

and the assertion is concluded.

(iii) In this case, from *(ii)* we have that $G_0(t,0)$ is decreasing on t. Then,

$$\min_{t\in\left[\frac{\upsilon+b}{4},\frac{3(\upsilon+b)}{4}\right]}\{G_0(t,0)\}=G_0\left(\frac{\upsilon+b}{4},0\right)$$

and we need to show that there exists $\gamma\in(0,1)$ such that $G_0\left(\frac{\upsilon+b}{4},0\right)\geq\gamma G_0(\upsilon-1,0)$, which is equivalent to $\gamma\leq\frac{G_0\left(\frac{\upsilon+b}{4},0\right)}{G_0(\upsilon-1,0)}<1$ since $G_0(t,0)$ is decreasing on t.

The case $1\leq s\leq b+1$ has been proved in [5, Theorem 3.2]. □

LEMMA 2.4. [15, Theorem 1.B] *Let X be a Banach space, $A:X\to X$ be a linear operator with the operator norm $\|A\|$. Then, if $\|A\|<1$, we have that $(Id-A)^{-1}$ exists and $(Id-A)^{-1}=\sum_{n=0}^{\infty}A^n$. Where Id is the identity operator.*

3. **Previous Inequalities and Hypotheses**

In the beginning of this section, we will show an important technical result, which extends the one given in [5, Theorem 3.2 (ii)].

THEOREM 3.1. *Let $G_0(t,s)$ be defined by (1.2). Then, for any $(t,s)\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}\times I$, the following inequalities are fulfilled:*

$$G_0(t,s)\leq G_0(s+\upsilon-1,s)<\frac{(b+2\upsilon-1)^2\,\Gamma(b+3)}{4^{\upsilon-1}\,(b+1)^{2(2-\upsilon)}\,\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)}.$$

P r o o f. First inequality has been proved in Lemma 2.3, *(ii)*, so, we only need to prove the second one.

If $s=0$, then $G_0(\upsilon+1,0)=\frac{b+2}{b+\upsilon+1}$ and we only need to show that

$$\frac{b+2}{b+\upsilon+1}<\frac{(b+2\upsilon-1)^2\,\Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)}\,\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)},$$

which is equivalent to

$$4^{\upsilon-1}(b+1)^{2(2-\upsilon)}\,\Gamma(\upsilon)\,\frac{\Gamma(b+\upsilon+1)}{\Gamma(b+2)}<(b+2\upsilon-1)^2 .$$

Now, using Gautchi's inequality, we have that $\frac{\Gamma(x+r)}{\Gamma(x+1)}<x^{r-1}$ for a positive real number x and for $r\in(0,1)$. Thus

$$\frac{\Gamma(b+\upsilon+1)}{\Gamma(b+2)}=\frac{(b+\upsilon)\,\Gamma(b+\upsilon)}{\Gamma(b+2)}<(b+\upsilon)(b+1)^{\upsilon-2} .$$

Using the latter inequality and the fact that $\Gamma(\upsilon)<1$ for $\upsilon\in(1,2)$, it is enough to show that

$$4^{\upsilon-1}(b+1)^{2-\upsilon}(b+\upsilon)<(b+2\upsilon-1)^2 .$$

One can verify that $\frac{4^{\upsilon-1}(b+1)^{2-\upsilon}(b+\upsilon)}{(b+2\upsilon-1)^2}$ is decreasing on b, so we only need to check that the above inequality holds for $b=5$, i.e.,

$$4^{\upsilon-1}6^{2-\upsilon}(5+\upsilon)<(4+2\upsilon)^2 ,$$

which is true, since for $\upsilon\in(1,2)$

$$\frac{9(5+\upsilon)}{(4+2\upsilon)^2}<\frac{3}{2}<\left(\frac{3}{2}\right)^{\upsilon} .$$

Now, suppose that $1\le s\le b$. From (1.2) we have that

$$\begin{aligned} G_0(s+\upsilon-1,s) &= \frac{1}{\Gamma(\upsilon)}\frac{(s+\upsilon-1)^{(\upsilon-1)}(\upsilon+b-s)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}} \\ &= \frac{\Gamma(s+\upsilon)\,\Gamma(b-s+\upsilon+1)}{\Gamma(s+1)\,\Gamma(b-s+2)}\frac{\Gamma(b+3)}{\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)}. \end{aligned}$$

Using Gautchi's inequality again, we have

$$\frac{\Gamma(s+\upsilon)}{\Gamma(s+1)}=\frac{(s+\upsilon-1)\,\Gamma(s+\upsilon-1)}{\Gamma(s+1)}<(s+\upsilon-1)\,s^{\upsilon-2}$$

and

$$\frac{\Gamma(b-s+\upsilon+1)}{\Gamma(b-s+2)}=\frac{(b-s+\upsilon)\,\Gamma(b-s+\upsilon)}{\Gamma(b-s+2)}<(b-s+\upsilon)(b-s+1)^{\upsilon-2} .$$

Combining the last two inequalities we obtain that

$$\frac{\Gamma(s+\upsilon)\,\Gamma(b-s+\upsilon+1)}{\Gamma(s+1)\,\Gamma(b-s+2)}<(s+\upsilon-1)\,s^{\upsilon-2}(b-s+\upsilon)(b-s+1)^{\upsilon-2} .$$

Thus,

$$G_0(s+v-1,s) < (s+v-1)\,s^{v-2}\,(b-s+v)\,(b-s+1)^{v-2}\,\frac{\Gamma(b+3)}{\Gamma(v)\,\Gamma(b+v+2)}.$$

By denoting

$$g(s) := (s+v-1)\,s^{v-2}\,(b-s+v)\,(b-s+1)^{v-2} \text{ for } s\in[1,b]_{\mathbb{N}_1},$$

it is not difficult to verify that, for all $s\in[1,b]_{\mathbb{N}_1}$, $v\in(1,2)$ and $b\ge 5$, the following properties hold:

$$g(s) = g(b+1-s)$$

and

$$g''\left(\frac{b+1}{2}\right) = -2^{5-2v}(v-1)(b+1)^{2v-6}\left(b^2+b(4v-6)+4(v-2)v+1\right) < 0.$$

As consequence, since

$$\frac{g'(s)}{(v-1)\,s^{v-3}\,(b-s+1)^{v-3}} = (2s-b-1)\left(s^2-s\,(b+1)-(v-2)\,(b+v)\right),$$

it is clear that

$$\max_{1\le s\le b}\{g(s)\} = \max\left\{g(1),\, g\left(\frac{b+1}{2}\right)\right\}.$$

Let's see now that $g\left(\frac{b+1}{2}\right) > g(1)$, i.e.,

$$\left(\frac{b-1}{2}+v\right)^2\left(\frac{b+1}{2}\right)^{2(v-2)} > v\,(b+v-1)\,b^{v-2},$$

which is equivalent to

$$\frac{(b+2v-1)^2\,b^{2-v}}{(b+v-1)\,(b+1)^{4-2v}} > 4^{v-1}v.$$

Denote $p(b) := \frac{(b+2v-1)^2 b^{2-v}}{(b+v-1)(b+1)^{4-2v}}$. One may check that p is increasing for $b\ge 5$ and $v\in(1,2)$.

Indeed, we have that

$$p'(b) = \frac{(b-1)(v-1)b^{1-v}(b+1)^{2v-5}(b+2v-1)\left(3bv+(b-5)b+2v^2-5v+2\right)}{(b+v-1)^2}$$

It is immediate to verify that $w(b) := 3bv+(b-5)b+2v^2-5v+2 > 0$ for all $b\ge 5$. So, p is also increasing for $b\ge 5$.

Hence, we only need to show that $p(5) > 4^{v-1}v$, which is equivalent to

$$h(v) := \frac{(4+2v)^2}{v\,(4+v)} > 4\left(\frac{9}{5}\right)^{2-v} =: k(v). \tag{3.1}$$

Since for all $\upsilon \in (1,2)$ we have that

$$h^{(4)}(\upsilon) = 96\left(\frac{1}{\upsilon^5} - \frac{1}{(\upsilon+4)^5}\right) > 0$$

and

$$k^{(4)}(\upsilon) = 4\left(\frac{9}{5}\right)^{2-\upsilon} \log^4\left(\frac{9}{5}\right) > 0.$$

So, h''' and k''' are strictly increasing on $(1,2)$. Since

$$h'''(2) \approx -1.48148 < -1.46215 \approx k'''(1)$$

we deduce that $(h-k)''' < 0$ on $(1,2)$ and, as a direct consequence, $(h-k)'$ is concave on $(1,2)$.

Since $(h-k)(1) \approx 0.392064$ and $(h-k)(2) \approx 1.46226$ we conclude that $(h-k)' > 0$ on $(1,2)$, which implies that $h-k$ is strictly increasing in $(1,2)$. Since $(h-k)(1) = 0$, inequality (3.1) is fulfilled on $(1,2)$.

Thus,

$$g(s) \le g\left(\frac{b+1}{2}\right) = \frac{(b+2\upsilon-1)^2}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)}} \text{ for } s \in [1,b]_{\mathbb{N}_1}.$$

Now, if $s = b+1$, we have that

$$G_0(b+\upsilon, b+1) = \frac{\Gamma(b+\upsilon+1)\,\Gamma(b+3)}{\Gamma(b+2)\,\Gamma(b+\upsilon+2)}.$$

We will show that

$$G_0(b+\upsilon, b+1) < \frac{g(1)\,\Gamma(b+3)}{\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)}.$$

This is equivalent to

$$\frac{\Gamma(b+\upsilon+1)\,\Gamma(\upsilon)}{\Gamma(b+2)} < \upsilon\,(b+\upsilon-1)\,b^{\upsilon-2}.$$

Again, using Gautchi's inequality we have that

$$\frac{\Gamma(b+\upsilon+1)}{\Gamma(b+2)} = \frac{(b+\upsilon)\,\Gamma(b+\upsilon)}{\Gamma(b+2)} < (b+\upsilon)\,(b+1)^{\upsilon-2}.$$

Thus, it is enough to show that

$$\Gamma(\upsilon)\,(b+\upsilon)\,(b+1)^{\upsilon-2} < \upsilon\,(b+\upsilon-1)\,b^{\upsilon-2},$$

but, since $\Gamma(\upsilon) < 1$ for all $\upsilon \in (1,2)$, we only need to prove that

$$(b+\upsilon)\,b^{2-\upsilon} < \upsilon\,(b+\upsilon-1)\,(b+1)^{2-\upsilon}.$$

Denote

$$q(b) := \frac{(b+\upsilon-1)\,(b+1)^{2-\upsilon}}{(b+\upsilon)\,b^{2-\upsilon}}.$$

Let us see that q is increasing for $b \geq 5$ and $\upsilon \in (1,2)$. Indeed, we have that

$$q'(b) = (b+1)\left(\frac{b}{b+1}\right)^{\upsilon} \frac{(\upsilon-1)\left(b^2+(2\upsilon-3)b+\upsilon(\upsilon-2)\right)}{(b+\upsilon)^2 b^3}.$$

Moreover, it is not difficult to verify that $b^2+(2\upsilon-3)b+\upsilon(\upsilon-2) > 0$ for $b \geq 5$.

Then, we only need to check that the inequality holds for $b = 5$, i.e., $5^{2-\upsilon}(\upsilon+5) < \upsilon(\upsilon+4)6^{2-\upsilon}$. It clearly holds, since

$$\frac{\upsilon+5}{\upsilon+4} = 1+\frac{1}{\upsilon+4} < \frac{6}{5} < \left(\frac{6}{5}\right)^{2-\upsilon}\upsilon.$$

As a result, we obtain that for all $s \in I$, the following inequality holds

$$G_0(t,s) \leq G_0(s+\upsilon-1,s) < \frac{(b+2\upsilon-1)^2\,\Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)}\,\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)}$$

and the result is proved. □

Let X represents all maps from $[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$ into $\mathbb{R}$, equipped with the standard maximum norm $\|\cdot\|$. Clearly, X is a Banach space.

Throughout the paper, let us assume the following condition

(A) There exists $\overline{a} > 0$ such that $|a(t)| \leq \overline{a} < \frac{4^{\upsilon-1}(b+1)^{3-2\upsilon}\Gamma(\upsilon)\Gamma(b+\upsilon+2)}{(b+2\upsilon-1)^2\Gamma(b+3)}$. for all $t \in [\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.

Define $G : [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I \to \mathbb{R}$ by

$$G(t,s) = \sum_{n=0}^{\infty}(-1)^n G_n(t,s), \tag{3.2}$$

where $G_0(t,s)$ is given by (1.2), and set

$$G_n(t,s) = \sum_{\tau=0}^{b+1} a(\tau+\upsilon-1)\,G_0(t,\tau)\,G_{n-1}(\tau+\upsilon-1,s) \text{ for } n \geq 1. \tag{3.3}$$

We have the following result

Theorem 3.2. *The function $G(t,s)$ defined by (3.2) as a series of functions is convergent for $(t,s) \in [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I$. Moreover, $G(t,s)$ is the Green's function for FBVP (1.3).*

P r o o f. For any $h \in X$ and $t \in I$, let us consider the following FBVP

$$\begin{aligned} -\Delta^{\upsilon} y(t) + a(t+\upsilon-1)\, y(t+\upsilon-1) &= h(t+\upsilon-1), \quad t \in I, \\ y(\upsilon-2) &= y(b+\upsilon+1) = 0. \end{aligned} \tag{3.4}$$

The solution y of this problem satisfies

$$y(t) = \sum_{s=0}^{b+1} G_0(t,s)\left(h(s+\upsilon-1) - a(s+\upsilon-1)\, y(s+\upsilon-1)\right),$$

where $G_0(t,s)$ is given by (1.2).

The last equality is equivalent to

$$y(t) + \sum_{s=0}^{b+1} a(s+\upsilon-1)\, G_0(t,s)\, y(s+\upsilon-1) = \sum_{s=0}^{b+1} G_0(t,s)\, h(s+\upsilon-1). \tag{3.5}$$

Now, define A and $B: X \to X$ by

$$(Ah)(t) := \sum_{s=0}^{b+1} G_0(t,s)\, h(s+\upsilon-1) \tag{3.6}$$

and

$$(By)(t) := \sum_{s=0}^{b+1} a(s+\upsilon-1)\, G_0(t,s)\, y(s+\upsilon-1). \tag{3.7}$$

Then (3.5) can be written as

$$(Id + B)\, y = Ah. \tag{3.8}$$

Using Theorem 3.1 and condition (A), it follows that $\|B\| = \max_{\|y\|=1} \|By\| < 1$. Then, by Lemma 2.4 we have that

$$y = \sum_{n=0}^{\infty} (-B)^n\, Ah. \tag{3.9}$$

First, we claim that for $n = 0, 1, 2, \ldots$

$$((-B)^n\, Ah)(t) = \sum_{s=0}^{b+1} (-1)^n\, G_n(t,s)\, h(s+\upsilon-1). \tag{3.10}$$

We will prove our claim by induction. Clearly, (3.10) holds for $n = 0$. Let us assume that (3.10) holds for some $n = k$. We will show that (3.10) holds for $n = k + 1$. Indeed, using (3.3), (3.6) and (3.7) we have

$$\begin{aligned}\left((-B)^{k+1} Ah\right)(t) &= \left(-B(-B)^k Ah\right)(t) \\ &= \sum_{\tau=0}^{b+1} -a(\tau+\upsilon-1)\, G_0(t,\tau) \sum_{s=0}^{b+1} (-1)^k G_k(\tau+\upsilon-1,s)\, h(s+\upsilon-1) \\ &= \sum_{\tau=0}^{b+1} (-1)^{k+1} \sum_{s=0}^{b+1} a(\tau+\upsilon-1)\, G_0(t,\tau)\, G_k(\tau+\upsilon-1,s)\, h(s+\upsilon-1) \\ &= \sum_{s=0}^{b+1} (-1)^{k+1} G_{k+1}(t,s)\, h(s+\upsilon-1),\end{aligned}$$

which proves our claim.

Next, we will show that for $n = 0, 1, 2, \dots$

$$|(-1)^n G_n(t,s)| \le \overline{a}^n \left(\frac{(b+2\upsilon-1)^2 \Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)} \Gamma(\upsilon)\Gamma(b+\upsilon+2)} \right)^{n+1}. \quad (3.11)$$

Clearly, for $n = 0$, (3.11) is Theorem 3.1. Assume that (3.11) holds for some $n = k$. Then

$$\begin{aligned}\left|(-1)^{k+1} G_{k+1}(t,s)\right| &\le \sum_{\tau=0}^{b+1} |a(\tau+\upsilon-1)|\, G_0(t,\tau)\, G_k(\tau+\upsilon-1,s) \\ &\le \sum_{\tau=0}^{b+1} \overline{a} \frac{(b+2\upsilon-1)^2 \Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)} \Gamma(\upsilon)\Gamma(b+\upsilon+2)} G_k(\tau+\upsilon-1,s) \\ &= \sum_{\tau=0}^{b+1} \overline{a}^{n+1} \left(\frac{(b+2\upsilon-1)^2 \Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)} \Gamma(\upsilon)\Gamma(b+\upsilon+2)} \right)^{n+2},\end{aligned}$$

i.e., (3.11) holds for $n = k + 1$.

By induction, (3.11) holds for any $n = 0, 1, 2, \dots$

Finally, using condition (A) and inequality (3.11), for all $(t,s) \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I$, we obtain that

$$|G(t,s)| = \left| \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \right| \le \sum_{n=0}^{\infty} \overline{a}^n \left(\frac{(b+2\upsilon-1)^2 \Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)} \Gamma(\upsilon)\Gamma(b+\upsilon+2)} \right)^{n+1} < \infty.$$

Hence, $G(t,s)$ is convergent on $[\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-1}} \times I$.

From (3.2), (3.9) and (3.10) it follows that

$$y(t) = \sum_{n=0}^{\infty} \sum_{s=0}^{b+1} (-1)^n G_n(t,s)\, h(s+\upsilon-1) = \sum_{s=0}^{b+1} G(t,s)\, h(s+\upsilon-1). \quad (3.12)$$

On the other hand, let y be defined by (3.12). By (3.2), (3.6) and (3.7), y satisfies (3.9). Thus, (3.8) holds. Again, by (3.6) and (3.7), y satisfies (3.5). Therefore, y is the unique solution of FBVP (3.4) and G is the Green's function related to FBVP (1.3). □

LEMMA 3.1. *Let G be defined by* (3.2) *and*

$$\overline{G}\left(s\right) := G_0(s+\upsilon-1,s)\left(\frac{4^{\upsilon-1}\left(b+1\right)^{2(2-\upsilon)}\Gamma\left(\upsilon\right)\Gamma\left(b+\upsilon+2\right)}{4^{\upsilon-1}\left(b+1\right)^{2(2-\upsilon)}\Gamma\left(\upsilon\right)\Gamma\left(b+\upsilon+2\right)-\overline{a}\left(b+2\upsilon-1\right)^2\Gamma\left(b+3\right)}\right). \tag{3.13}$$

Then, $|G\left(t,s\right)| \leq \overline{G}\left(s\right)$ for all $(t,s) \in [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I$.

P r o o f. We claim that for $n = 0,1,2,\ldots$

$$\left|(-1)^n G_n\left(t,s\right)\right| \leq G_0(s+\upsilon-1,s)\left(\frac{\overline{a}\left(b+2\upsilon-1\right)^2\Gamma\left(b+3\right)}{4^{\upsilon-1}\left(b+1\right)^{2(2-\upsilon)}\Gamma\left(\upsilon\right)\Gamma\left(b+\upsilon+2\right)}\right)^n. \tag{3.14}$$

Clearly, from Theorem 3.1 the inequality (3.14) is true for $n = 0$.

Assume that (3.14) holds for some $n = k$. Then, using Theorem 3.1 again, for all $(t,s) \in [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times I$, we have

$$\begin{aligned}
\left|(-1)^{k+1} G_{k+1}\left(t,s\right)\right| &\leq \sum_{\tau=0}^{b+1}\left|a\left(\tau+\upsilon-1\right)\right| G_0\left(t,\tau\right)\left|G_k\left(\tau+\upsilon-1,s\right)\right| \\
&\leq \sum_{\tau=0}^{b+1}\overline{a}G_0\left(\tau+\upsilon-1,\tau\right)G_0(s+\upsilon-1,s) \\
&\quad\times\left(\frac{\overline{a}\left(b+2\upsilon-1\right)^2\Gamma\left(b+3\right)}{4^{\upsilon-1}\left(b+1\right)^{2(2-\upsilon)}\Gamma\left(\upsilon\right)\Gamma\left(b+\upsilon+2\right)}\right)^k \\
&\leq G_0(s+\upsilon-1,s)\left(\frac{\overline{a}\left(b+2\upsilon-1\right)^2\Gamma\left(b+3\right)}{4^{\upsilon-1}\left(b+1\right)^{2(2-\upsilon)}\Gamma\left(\upsilon\right)\Gamma\left(b+\upsilon+2\right)}\right)^{k+1}.
\end{aligned}$$

Hence, (3.14) holds for $n = k+1$, which proves our claim by induction.

Combining (3.2), (3.13) and (3.14), we obtain that

$$\begin{aligned} |G(t,s)| &= \left| \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \right| \\ &\leq G_0(s+\upsilon-1,s) \sum_{n=0}^{\infty} \left(\frac{\overline{a}\,(b+2\upsilon-1)^2\,\Gamma(b+3)}{4^{\upsilon-1}(b+1)^{2(2-\upsilon)}\,\Gamma(\upsilon)\,\Gamma(b+\upsilon+2)} \right)^n \\ &= \overline{G}(s). \end{aligned}$$

□

As a direct consequence of previous result, we deduce the following one for a nonpositive potential $a(t)$.

Corollary 3.1. *Assume that condition* (A) *is fulfilled and* $-\bar{a} < a(t) \leq 0$ *on* $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. *Then* $G(t,s) > 0$ *for each* $(t,s) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times I$.

P r o o f. From Lemma 2.3, (i), we know, that $G_0 > 0$ on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times I$.

The result holds immediately from (3.2) and (3.3). □

Remark 3.1. We point out that previous property allows us to develop the method of lower and upper solution method coupled to monotone iterative technique (see [7, 8] and references therein) for this situation.

4. **Main Results**

In this section we prove the main results of this paper. We obtain two existence results for the nonlinear problem (1.1).

To this end, define the operator $T : X \to X$ by

$$(Ty)(t) = \sum_{s=0}^{b+1} G(t,s)\, w(s)\, f(s+\upsilon-1, y(s+\upsilon-1)), \ y \in X. \qquad (5.1)$$

Clearly, T is completely continuous and, as a direct consequence of the results proved in previous section, y is a solution of FBVP (1.1) if and only if y is a fixed point of operator T in X.

Now, we are in a position to establish our main results.

Theorem 5.1. *Assume that* f *satisfies the Lipschitz condition on its second variable, i.e.,*

$$|f(t,x_1) - f(t,x_2)| \leq K\,|x_1 - x_2| \ \text{ for } \ (t,x_1),(t,x_2) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R}$$

with $K \in \left(0, \left(\sum_{s=0}^{b+1} \overline{G}(s)\, w(s)\right)^{-1}\right)$. *Then the FBVP* (1.1) *has a unique solution. In addition, if* $f(t,0) \equiv 0$ *on* $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$, *then FBVP* (1.1) *has no nontrivial solution.*

P r o o f. For any $y_1, y_2 \in X$ and $t \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$ we have, from Lemma 3.1, that

$$\begin{aligned} |(Ty_1 - Ty_2)(t)| &= \sum_{s=0}^{b+1} G(t,s)\, w(s) \left(f(s+\upsilon-1, y_1(s+\upsilon-1)) - f(s+\upsilon-1, y_2(s+\upsilon-1))\right) \\ &\le \sum_{s=0}^{b+1} \overline{G}(s)\, w(s) \left|f(s+\upsilon-1, y_1(s+\upsilon-1)) - f(s+\upsilon-1, y_2(s+\upsilon-1))\right| \\ &\le \left(K \sum_{s=0}^{b+1} \overline{G}(s)\, w(s)\right) \|y_1 - y_2\|. \end{aligned}$$

Notice that $K \sum_{s=0}^{b+1} \overline{G}(s)\, w(s) < 1$. Thus, T is a contraction mapping. By the contraction mapping principle it follows that T has a unique fixed point. Hence, FBVP (1.1) has a unique solution. Moreover, if $f(t,0) \equiv 0$ on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$, then clearly, $y(t) \equiv 0$ is a solution and by uniqueness of solutions, FBVP (1.1) has no nontrivial solutions. □

THEOREM 5.2. *Assume that*

$$\lim_{|x| \to 0} \left\{ \max_{t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}} \left[\frac{|f(t,x)|}{|x|}\right] \right\} = 0 \tag{5.2}$$

and $f(t,0) \not\equiv 0$ *on* $t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. *Then FBVP* (1.1) *has at least one nontrivial solution.*

P r o o f. Set $k = \frac{1}{\sum_{s=0}^{b+1} \overline{G}(s) w(s)}$. From (5.2), we have that there exists a constant $C_1 > 0$ such that $|f(t,x)| \le kC_1$ for $t \in [\upsilon-1, \upsilon+b1]_{\mathbb{N}_{\upsilon-1}}$ and x with $|x| \le C_1$.

Consider $\Omega := \{y \in X \mid \|y\| \leq C_1\}$. Then from (5.1) and Lemma 3.13 we have

$$\begin{aligned} |(Ty)(t)| &= \left|\sum_{s=0}^{b+1} G(t,s)\, w(s)\, f(s+\upsilon-1, y(s+\upsilon-1))\right| \\ &\leq \sum_{s=0}^{b+1} |G(t,s)|\, w(s)\, |f(s+\upsilon-1, y(s+\upsilon-1))| \\ &\leq kC_1 \sum_{s=0}^{b+1} \overline{G}(s)\, w(s) = C_1. \end{aligned}$$

Hence, $\|Ty\| \leq C_1$. By Schauder's fixed point Theorem, operator T has at least one fixed point in Ω, which is a nontrivial solution of FBVP (1.1). □

6. **Examples**

In the end of this paper we present some examples to illustrate our main results.

EXAMPLE 5.1. Let $\overline{a}$ satisfying condition (A). Consider the FBVP (1.1) with $f(t,x) = K \arctan x + g(t)$, where $0 < K < \left(\sum_{s=0}^{b+1} \overline{G}(s)\, w(s)\right)^{-1}$ and $g(t) \in \mathbb{R}$, with $g \not\equiv 0$ on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. One can easily check that f satisfies the Lipschitz condition in x, i.e.,

$$|f(t,x_1) - f(t,x_2)| \leq K|x_1 - x_2| \text{ for } (t,x_1),(t,x_2) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R}.$$

From Theorem 5.1 the considered FBVP has at least one solution. Moreover, the solution is nontrivial since $f(t,0) \not\equiv 0$ on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.

EXAMPLE 5.2. Suppose that $\overline{a}$ satisfies condition (A). Consider the FBVP (1.1) with $f(t,x) = x^{\alpha} + g(t)$, where $\alpha > 1$ and $g(t) \in \mathbb{R}$, with $g \not\equiv 0$ on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. It is obvious that condition (5.2) is fulfilled. So, from Theorem 5.2 the considered FBVP has at least one nontrivial solution since $f(t,0) \not\equiv 0$ on $t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.

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