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PhD Dissertation

NUMERICAL ANALYSIS AND SIMULATIONS OF SOME PROBLEMS WITH DAMAGE IN SOLID MECHANICS

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Resumen

Esta memoria de tesis es el resultado de la investigación llevada a cabo durante los últimos cinco años en el Departamento de Matemática Aplicada de la Universidad de Santiago de Compostela bajo la dirección de los profesores Jose Ramón Fernández García y Juan Manuel Viaño Rey. A lo largo del presente trabajo se describen una serie de problemas en mecánica de sólidos cuyo nexo común es la consideración de un modelo de daño del material.

Exceptuando el primero de los problemas mecánicos que se estudiarán a lo largo de este trabajo, el resto de los problemas que nos encontraremos son problemas de contacto. Prácticamente cualquier proceso, tanto en el ámbito industrial como en la vida diaria, involucra fenómenos de contacto entre cuerpos deformables, como lo son el clásico ejemplo de una rueda con la carretera o simplemente cuando caminamos. Debido a esto, dichos fenómenos han sido profundamente estudiados y analizados desde, incluso, el antiguo Egipto, donde ya entonces lubricaban los transportes de las monumentales piedras que utilizaban para la construcción. A lo largo de la historia se ha ido profundizando en el conocimiento de estos fenómenos, como así lo reflejan los experimentos de *Leonardo da Vinci* en el siglo XV para medir fuerzas de rozamiento haciendo deslizar bloques sobre planos inclinados, o los estudios teóricos de Euler sobre el rozamiento allá por el siglo XVIII.

El estudio matemáticamente formal de estos problemas, sin embargo, no llegó hasta el siglo XX, en que *Signorini* describió el contacto entre cuerpos deformables. Duvaut y Lions aportaron con sus trabajos resultados de existencia y unicidad de solución

para formulaciones variacionales de algunos procesos de contacto, así como algoritmos para la resolución numérica de los mismos. En la actualidad son muchos los esfuerzos que las matemáticas dirigen al estudio de estos problemas, pudiendo decir que existe en la actualidad una teoría matemática de mecánica de contacto. Si bien se ha avanzado mucho en el entendimiento de este tipo de problemas, siguen surgiendo necesidades que obligan a considerar cada vez modelos más complejos, creándose así una retroalimentación que sigue enriqueciendo este campo del conocimiento. Los primeros modelos que se estudiaron fueron modelos estáticos con materiales elásticos. Sin embargo, la observación de las propiedades mecánicas de materiales como pastas o gomas puso en evidencia las carencias de este tipo de modelización material. Estos modelos fueron evolucionando hacia otros dependientes del tiempo, incorporando nuevos comportamientos materiales y, en la actualidad, se estudian por ejemplo, el acoplamiento entre las deformaciones y la temperatura, teniendo en cuenta otros efectos como la adhesión, el desgaste, la fatiga o el daño.

A principios del siglo XIX se comenzaron a estudiar los factores que podían variar las propiedades de aquellos materiales que soportaban cargas cíclicas. Así, a raíz de varios accidentes ferroviarios provocados por a fallos en las vías, se constató que dichos materiales, debido a la sucesión de estados de tensión, estaban “cansados” o “fatigados”. Este efecto de fatiga del material está producido por la aparición y crecimiento de fracturas microscópicas internas que derivan también en una reducción de su capacidad de carga, esto es, un reblandecimiento del mismo. Este efecto es el conocido como daño material. Surge desde entonces la necesidad de estudiar y tratar de predecir lo más fielmente posible este comportamiento, para así garantizar la fiabilidad y durabilidad de los componentes de cualquier estructura.

La simulación numérica comienza a jugar hoy en día un papel fundamental en los procesos industriales. La reducción de tiempos y costes en las fases iniciales de diseño de los productos es una necesidad en pos de la competitividad dentro de los distintos sectores (automoción, naval, aeronáutico, ...), y es ahí donde los métodos numéricos juegan ya en la actualidad un papel preponderante. Una correcta simulación de los procesos mediante técnicas numéricas adecuadas puede permitir la reducción del tiempo de diseño y el número de prototipos necesarios para la validación de los productos en la fase previa a la producción en serie. Es necesaria una elección correcta de los esquemas y algoritmos numéricos para garantizar así tanto la fiabilidad de los resultados como la obtención de los mismos en márgenes de tiempo razonables.

El modelo de daño considerado fue derivado por Frémond y Nedjar en [39, 40] del principio del trabajo virtual. La nueva idea consiste en la introducción de la función de *daño* $\zeta = \zeta(\boldsymbol{x}, t)$, que es el cociente entre el módulo de elasticidad del material dañado y el módulo de elasticidad cuando el material está libre de daño, esto es,

$$\zeta = \zeta(\boldsymbol{x}, t) = \frac{E_{ef}}{E},$$

donde E y E_{ef} son el módulo de elasticidad del material y el módulo de elasticidad efectivo, respectivamente. Se puede observar, por la definición anterior, que la función de daño está restringida a tomar valores entre 0 y 1. Cuando $\zeta = 1$ el material está libre de daño, cuando $\zeta = 0$ el material está completamente dañado y cuando $0 < \zeta < 1$ existe un daño parcial y el sistema ve reducida su capacidad de carga con respecto a la original.

La consideración de dicho modelo implica que con respecto al problema mecánico clásico se considere la nueva variable global, la función de daño ζ, junto con su correspondiente expresión evolutiva, que ha sido deducida del principio de trabajo virtual en [36]. Dicha forma evolutiva se presentará de dos formas distintas: a través de la ecuación en derivadas parciales

$$\dot{\zeta} - \kappa \Delta \zeta = \phi(\boldsymbol{u}, \zeta),$$

o a través de la inclusión diferencial

$$\dot{\zeta} - \kappa \Delta \zeta + \partial \psi_{[0,1]}(\zeta) \ni \phi(\boldsymbol{u}, \zeta).$$

En las anteriores expresiones se observa que existe un término de difusión, que indica la influencia existente entre las zonas con distinta densidad de microfracturas. La función ϕ es la denominada función fuente de daño, de la que depende la contribución del estado del proceso para la creación de daño y que se asume dependiente de las deformaciones y del propio daño. Esta función fuente se considerará lo suficientemente general para que pueda presentar contribuciones dinstintas según las deformaciones se deban a tensión o a compresión, como se verá en el Capítulo 2. Puede observarse también que no existe restricción alguna acerca del signo de $\dot{\zeta}$, lo que significa que las microfracturas responsables del daño pueden cerrarse y por lo lo tanto el material puede recuperarse del daño. La consideración de la hipótesis de daño irreversible implicaría introducir también, en la expresión evolutiva del daño, el término subdiferencial $\partial\psi_{(-\infty,0]}(\dot{\zeta})$, pero dicho modelo no será considerado en el presente trabajo. Otros modelos de daño más complejos podrían incorporar, por ejemplo, dependencia con respecto a la velocidad de deformación, o en el caso de un problema termomecánico, contribuciones debido al campo de temperaturas o incluso a la velocidad a que se realiza dicho cambio de temperatura.

Estas expresiones son equivalentes en el sentido de que la única diferencia entre ambas es el término subdiferencial, término que acota los posibles valores de la variable ζ entre 0 y 1. Cuando se utiliza la primera opción, es necesario proveer al problema de hipótesis adicionales necesarias para que el valor del daño se mantenga en el intervalo deseado. Usando el modelo que incluye el término subdiferencial las soluciones que se obtienen son menos regulares, pero la inclusión de este término facilita la obtención de soluciones globales en el intervalo temporal.

Un tema de interés en modelos que tienen cuenta el daño material es el comportamiento de las soluciones cuando se obtiene un daño completo, esto es, $\zeta = 0$ en algún punto. Sin embargo, con los modelos que se consideran aquí, cuando la variable ζ se aproxima al valor 0 el material estaría saturado de microfracturas y modelizarlo según las leyes materiales aquí expuestas dejaría de tener sentido.

Surge por lo tanto una problema acoplado entre el proceso mecánico y la evolución del daño.

Casi la totalidad de los problemas desarrollados en este trabajo son problemas de contacto, lo que les confiere un comportamiento inherentemente no lineal con las dificultades que ello implica. Dependiendo del tipo de proceso de contacto que se

considere podemos dividir los problemas en dos grandes grupos:
Contacto bilateral. Fenómeno en el que el contacto es mantenido, esto es, no existe separación de los cuerpos a lo largo del proceso de estudio.
Contacto unilateral. Es posible la separación de los cuerpos involucrados, la parte de la frontera que está en contacto o separada en cada instante es desconocida.

Dentro de este último grupo, atendiendo a las hipótesis sobre la componente normal del proceso de contacto, tenemos:
Contacto rígido. Proceso en el que se supone que debido a la gran diferencia de rigidez entre los materiales, uno de los cuerpos involucrados no presenta posibilidad de ser deformado.
Contacto deformable. Fenómeno de contacto en el que los cuerpos objeto del estudio son susceptibles de verse deformados.

Según las hipótesis que se consideren en el contacto en la dirección tangencial a la frontera, los problemas se dividen en: *Contacto sin rozamiento.* Proceso en el que se desprecian los efectos de la fricción, y por tanto los desplazamientos tangenciales son libres. *Contacto con rozamiento.* En este caso los desplazamientos tangenciales se ven restringidos por una ley de rozamiento.

Cada conjunto concreto de condiciones de contacto implica unas características particulares, y por lo tanto se debe realizar un tratamiento diferente del problema en cada caso, tanto desde el punto de vista del análisis matemático como desde el análisis numérico y la consiguiente resolución numérica, y sólo puntualmente se pueden estudiar varias condiciones distintas dentro de un mismo marco.

Otro de los elementos diferenciadores entre los sucesivos problemas mecánicos que se plantean es la ley material o ley constitutiva que se emplea en cada uno de ellos. Si bien en los primeros modelos que se estudiaron se emplearon leyes materiales de tipo elástico, al ir avanzando las herramientas del análisis, nuevos comportamientos han ido incorporándose como objeto de estudio. Así, en la presente tesis se consideran procesos donde intervienen materiales con las siguientes características:
Material elástico. Comportamiento material en el que las tensiones internas depen-

den exclusivamente de las deformaciones instantáneas, y además, dicha deformación es reversible.
Material viscoelástico. Aparte de la respuesta de tipo elástico, el estado de tensión del material está afectado también por la velocidad de deformación. De este tipo de materiales consideraremos dos tipos concretos: materiales de tipo Kelvin-Voigt y materiales con memoria. En el primero de estos el efecto viscoso viene dado por la velocidad instantánea de deformación, mientras que en el caso de los materiales con memoria el efecto viscoso viene dado por la historia completa de deformaciones del material.
Material elasto-viscoplástico. Este tipo de materiales se caracteriza fundamentalmente por la posibilidad de aparición de deformaciones permanentes.

El primer paso dentro del tratamiento completo de cada uno de los problemas consiste en un análisis puramente matemático o funcional del mismo. Para ello se obtiene una formulación variacional o abstracta del problema en cuestión y se realiza el estudio de su existencia y unicidad de solución. A lo largo del trabajo, dicho estudio se realizará mediante dos técnicas distintas, dependiendo de cada caso. Así, en algunos casos será posible realizar la demostración a través de argumentos de punto fijo en espacios de Banach similares a otros ya utilizados en problemas de contacto. En otros casos es necesario echar mano de la teoría de operadores pseudomonótonos, que si bien implica unas demostraciones extremadamente técnicas, permite probar en algunos casos la existencia de soluciones muy regulares.

Una vez probada la existencia y unicidad de solución, el interés radica en obtener un esquema numérico y un método de resolución numérica eficientes. Para ello, en cada caso se propondrá un esquema completamente discretizado para la aproximación numérica de la solución del problema y se realizará el correspondiente análisis numérico del mismo. Este análisis consiste en la obtención de estimaciones del error que, bajo adecuadas condiciones de regularidad, nos permitan asegurar el buen comportamiento del esquema discretizado escogido.

Una vez realizado este estudio numérico teórico, se introducirá un algoritmo eficiente para la resolución numérica basado en el método de los elementos finitos para la discretización espacial y de diferencias finitas para discretizar las derivadas temporales. Nos encontraremos en todos los casos con un problema desacoplado debido a la elección que se realiza de los términos que se declaran de forma explícita. Esto permitirá resolver por separado la evolución del daño y la evolución del problema mecánico.

Se obtendrán varios tipos de problemas discretos, en forma de ecuaciones o inecuaciones variacionales (lineales o no lineales). Aquellos sistemas de ecuaciones que no deriven en un sistema lineal serán resueltos mediante un algoritmo de penalización-dualidad que ya fue utilizado satisfactoriamente en otros problemas de contacto. En aquellos problemas discretos cuyo término no lineal venga dado por los efectos de la fricción, será necesario primero realizar una regularización de la ley de rozamiento para poder aplicar el citado algoritmo. Para cada problema se presentará un test de convergencia numérica y algunos ejemplos ilustrativos en problemas bidimensionales.

Aunque se presentará el desarrollo completo de cada uno de los problemas planteados, el objetivo principal de esta tesis se centra en el análisis numérico y la resolución numérica de los problemas discretos.

Este trabajo se encuentra estructurado en tres capítulos.

Capítulo 1. En este primer capítulo se realiza una introducción formal de todos los aspectos que intervienen en los sucesivos problemas individuales que se irán presentando. Así, se comienza con una descripción general de las distintas características mecánicas que puede ofrecer un material, presentando las leyes constitutivas más generales, esto es, leyes elásticas, leyes viscoelásticas y elasto-viscoplásticas. Posteriormente, se introducen el concepto y el modelo de daño material y las modificaciones que provocan sobre las anteriores leyes de comportamiento. Se describen a continuación las características propias de los procesos de contacto, realizando un recorrido general sobre aquellas leyes de contacto más comunes tanto en el ámbito académico como en el de la ingeniería. El capítulo finaliza con una sección dedicada a la introducción de notaciones y a la presentación de algunos resultados teóricos que serán utilizados durante el estudio matemático y numérico de los problemas desarrollados en los capítulos posteriores.

Capítulo 2. Los problemas estudiados en este capítulo se engloban dentro del marco de los procesos cuasiestáticos, esto es, procesos en donde los efectos de la inercia son despreciables. Debido a esta suposición, se prescinde del término de inercia en la ecuación del movimiento. A lo largo de este capítulo se consideran las leyes materiales o leyes constitutivas descritas en el capítulo previo. Así, en la Sección 2.1 se presenta un problema mecánico con ley constitutiva de tipo elástico. Este primer problema es el único en el que no se consideran condiciones de contorno de contacto y por lo tanto es un problema de tipo desplazamiento-tracción. En la subsección correspondiente a los ejemplos numéricos se realiza una comparativa exitosa en la predicción del mecanismo de colapso de un puente con respecto a los resultados obtenidos en [28]. En la Sección 2.2 se presenta un problema de contacto bilateral con la clásica ley de rozamiento de Tresca para una ley material viscoelástica de tipo Kelvin-Voigt. Para la resolución numérica de este problema se propone una regularización de la ley de rozamiento y la posterior resolución del problema discreto a través de un algoritmo de tipo penalización-dualidad como alternativa a la resolución directa del problema original. Se realiza una comparativa entre la resolución directa con un algoritmo de tipo Uzawa y la mencionada ley regularizada y se constata que se obtiene un procedimiento cualitativamente más eficiente teniendo en cuenta la precisión y los tiempos de cálculo. En la Sección 2.3 se analiza un problema de contacto unilateral sin rozamiento con un obstáculo deformable para una ley material de tipo viscoelástico con memoria larga. Dicho comportamiento deformable se expresa a través de una condición de respuesta normal. Finalmente, en la Sección 2.4 se estudia otro problema de contacto bilateral con rozamiento, en este caso para una ley de tipo elasto-viscoplástico y dos leyes de rozamiento distintas; la ley de Tresca y una versión de la ley de Coulomb, que se estudian bajo el mismo esquema. Los resultados presentados en este capítulo están publicados o sometidos para publicación en [10, 13, 14, 18, 19].

Capítulo 3. A lo largo de este capítulo se describen tres problemas de contacto con leyes de comportamiento viscoelástico de tipo Kelvin-Voigt. El primero de ellos, desarrollado en la Sección 3.1, consiste en un proceso de contacto unilateral sin rozamiento con una condición de respuesta normal. El modelo presentado en la Sección 3.2 consiste en un fenómeno de contacto bilateral en el que se consideran los efectos del rozamiento por medio de la ley de Tresca. Por último, en la Sección 3.3, se presenta una modelización distinta de la influencia del daño sobre la respuesta del material. Mientras que en los anteriores problemas se suponía que el daño afectaba únicamente a la respuesta elástica del material, en este caso también la respuesta viscosa del material se verá modificada por la evolución del daño. Además, en este problema se supondrá también que el daño influye en las condiciones de contacto, pues la función de respuesta normal depende de la variable ζ. Los resultados correspondientes a este capítulo pueden ser consultados en [12, 16, 17].

Finalmente se presentan algunas conclusiones. Como conclusión general, vistos los resultados obtenidos en las distintas secciones, podemos destacar la viabilidad de la consideración del citado modelo de daño. Una investigación posterior podría ir encaminada al estudio de este modelo con otras leyes materiales, como por ejemplo de tipo piezoeléctrico, o a la dependencia de otras magnitudes, como por ejemplo, la temperatura.

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Introduction

Contact problems and the effects which involve have been known and studied since, at least, ancient Egypt (egyptians lubricated the stone blocks to transport them, as it is drawn in ancient pictures), and so, some of the most famous researchers along history studied this kind of problems. Even in the 15^{th} century *Leonardo da Vinci* had experimented with frictional contact problems by measuring the friction effect of different blocks sliding on a slope. He already found that the friction was proportional to the weight of the blocks and aparently independent of the contact area.

The first mathematical research about an effect derived from a contact process, particularly friction, was made by Euler in 1748 [31, 32], who studied the frictional coefficients solving the equation of motion for a block on a slope. Moreover, it was *Signorini* [66] in 1933 who first described the concept of contact between deformable bodies from a mathematical point of view as a variational formulation.

However, the *Mathematical Theory of Contact Mechanics* does not really emerge until the monograph by Duvaut and Lions [29], where the variational formulations of contact problems and some results on existence and uniqueness of weak solutions were provided, as well as the numerical resolution of some of these problems.

Initially, the mechanical problems considered in the literature involved elastic materials and static processes. Theories were improved and began to incorporate evolutive problems and different material behaviours, such as viscous effects, plastic deformations, memory, etc. Also, many different classes of contact conditions have been studied, such as rigid of deformable contacts, frictionless or different ways of frictional processes, etc.

Furthermore, more recently, other effects which can eventually have a decisive influence on contact problems have been and are currently being studied. Perhaps the most studied one is the thermal effect, which can have influence not only on the mechanical behaviour of the material, but it is also coupled with the contact process as frictional effects are related with heat generation. However, other effects such as the loss of material, or wear, on the contact surface, or the adhesion of the contact surface are being considered nowadays.

It is experimentally known that some materials, such as concrete and different metals, suffer a decrease in their stiffness when they are under continuous situations of internal stresses. This effect is known as *damage* of the material, and it can be included in contact problems by introducing an additional internal variable in their evolution law, which is coupled with the mechanical formulation in order to consider the combined problem.

This PhD Thesis is mainly oriented to the numerical analysis and simulation of some problems with damage, although theoretical results for the weak problems are also stated and details of their proofs provided. We present a variety of individual problems, where the most important material behaviours have been considered, as well as the most famous contact conditions that can be found in literature. Depending on the material behaviour and the properties considered on the contact interfaces, the variational formulations vary and each problem needs to be analyzed individually.

All the problems presented along this work are focused in studying only one of the bodies that take part in the contact process. For the second body in contact we will be interested only in its geometry and the type of boundary condition which arises when the contact occurs. This way, in each case our physical problem is in the situation which can be observed in Figure 1. However, we notice that in the elastic damage problem presented in Section 2.1, contact is not assumed and therefore $\Gamma_C = \emptyset$ there.

We observe that the body occupies a volume denoted by Ω, and that its boundary Γ

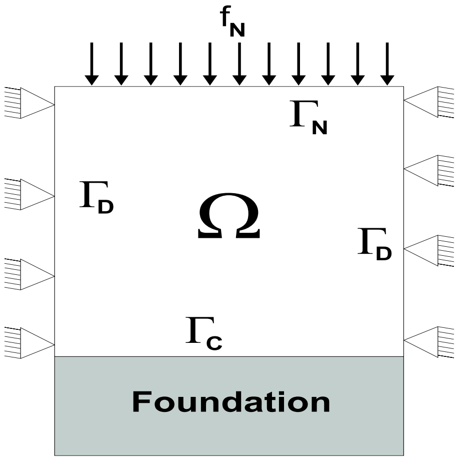

Figure 1: Physical setting of the contact problem

is divided into three disjoint parts: Γ_D, Γ_N and Γ_C. In each of these parts different boundary conditions are considered. This way, on Γ_D restrictions over the displacements are imposed, on Γ_N a density of surface tractions, $\boldsymbol{f}_N$, acts and Γ_C is the part of the boundary that may eventually come into contact with an obstacle or foundation. Also depicted in Figure 1, $\boldsymbol{f}_B$ is a density of volume forces which acts in the body, such as gravity or magnetic influence.

These situations are described, from a mathematical point of view, as evolutive boundary problems, where the *motion equation*

$$\rho \ddot{\boldsymbol{u}} = Div \boldsymbol{\sigma} + \boldsymbol{f}_B$$

is involved. Here, ρ is the mass density, $\ddot{\boldsymbol{u}}$ is the acceleration field and $\boldsymbol{\sigma}$ is the stress field. Moreover, in order to describe the different possibilities in the mechanical properties of the material the body is made of, a *constitutive law* must be provided, and in order to take into account the effect of the damage, its expression should be coupled with the material definition.

The work is structured as follows. The different mechanical concepts will be introduced in Chapter 1. In Section 1.1 the most important material behaviours are presented, characterized mathematically by their general *constitutive law*. In Section 1.2 the damage model is detailed and its effect over the different constitutive laws discussed, in Section 1.3 the mathematical description of the different possibilities of contact conditions are introduced and finally, in Section 1.4, some notation, as well as some theoretical results which will be often used, are provided.

Chapter 2 deals with quasistatic problems, that is, processes where inertia terms are neglected. Four different models corresponding to several constitutive relations are discussed. In Section 2.1 we consider a mechanical problem with damage in elasticity where no contact was supposed. The rest of the models presented in this chapter deal with contact problems, but the different constitutive relations and hypothesis on the behaviour of the contact interface make a need to analyze them separately. This way, Sections 2.2 and 2.3 deal with *short memory* and *long memory* viscoelasticty, respectively, while in Section 2.4 an elastic-viscoplastic behaviour is presented. All the problems are mathematically analyzed and numerically solved. The results provided in this chapter have been recently published or submitted for publication (see [10, 13, 14, 18, 19]).

In Chapter 3 three dynamic viscoelastic contact problems are studied. In Sections 3.1 and 3.2, the unilateral frictionless and the bilateral frictional cases, respectively, are presented. Finally, in Section 3.3 the frictionless process is again considered but with a different influence of the damage on the behaviour of both the material and the contact interface. These results have been recently published or submitted for publication (see [12, 16, 17]).

Variational analysis of the models including existence and uniqueness results of weak solutions are presented in every case. Furthermore, fully discrete numerical schemes based on the finite element method and finite differences are proposed to approximate the solutions to the problems. These schemes are numerically analyzed, obtaining for each case error estimates. We also present optimal error estimates under additional regularity assumptions on the solutions. At the end of each section, some numerical examples, obtained by implementing the algorithm discussed in the previous sections, are provided in order to demonstrate the accuracy and the behaviour of the fully discrete approximations.

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Chapter 1

Preliminaries

In this Chapter we introduce all the concepts required to describe and to analyze the different problems that are studied along this work. This way, in Section 1.1 the four different constitutive laws that will be considered are introduced, and in Section 1.2 the model of damage is detailed and we also present the different possibilities of influence of the damage over the material response. In Section 1.3 the different contact conditions are provided and, finally, in Section 1.4 we present some theoretical results that will be employed in the analysis of the different problems.

1.1 Material behaviour

Let $[0,T]$, $T>0$, be the time interval of interest and let us suppose that the studied body initially occupies a volume $\overline{\Omega} \subset \mathbb{R}^d$, where Ω is an open and bounded domain with boundary $\Gamma = \partial\Omega$ and $d = 1,2,3$ is the dimension of the space considered. The boundary Γ is divided into three disjoint parts which are denoted as Γ_D, Γ_N and Γ_C, where Γ_D is supposed to have positive measure ($meas(\Gamma_D) > 0$). The body will be considered fixed (displacements vanish) on Γ_D, a density of volume forces $\boldsymbol{f}_B$ will be acting on $\Omega \times [0,T]$ and a density of surface tractions $\boldsymbol{f}_N$ will be considered on $\Gamma_N \times [0,T]$.

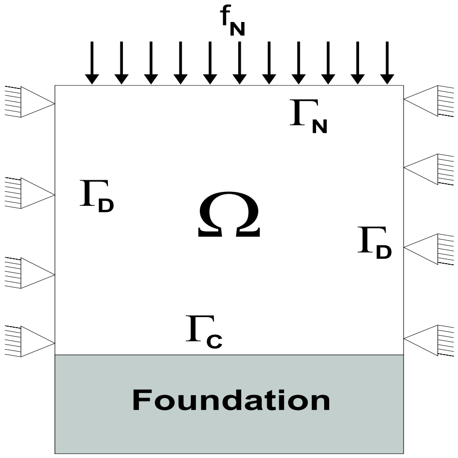

Figure 1.1: Physical setting

In order to obtain an easier reading, some notation is presented. In a d-dimensional space, let $\mathbb{S}^d$ be the space of second order symmetric tensors in $\mathbb{R}$, that is,

$$\mathbb{S}^d = \{\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{R}^{d\times d}; \quad \tau_{ij} = \tau_{ji}, \quad 1 \le i,j \le d\},$$

and we will denote by " $\cdot$ " and $|\cdot|$ the inner product and euclidean norm in $\mathbb{S}^d$ and $\mathbb{R}^d$.

The specific behaviour of a material is characterized by its constitutive law, which we understand as a relation between the stress tensor $\boldsymbol{\sigma}$, the strain tensor $\boldsymbol{\varepsilon}$ and their time derivatives $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$. Of course, other variables may take part and modify these relations, such as temperature, damage, etc.

Along this work, four different material behaviours will be considered. The first one, used in Section 2.1, describes an elastic behaviour, and this relation is given by

$$\boldsymbol{\sigma} = \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})),$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^{d}, \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and $\mathcal{E}$ is a nonlinear real-valued function named *elasticity operator* ($d = 1, 2, 3$ is the dimension of the problem). We note that here, and in the sequel, $\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ is a short hand notation for $\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)))$, where $\boldsymbol{x}$ is a generic material point and t an arbitrary time. The particular case of linear elasticity, described by *Hooke* in the 17^{th} century, is written, on components, as

$$\sigma_{ij} = \mathcal{E}_{ijkl}\varepsilon_{kl}(\boldsymbol{u}),$$

where $\mathcal{E}_{ijkl}$ are the elasticity components. Remark that for elastic materials the current stresses depend exclusively on the strains at that exact time instant.

Remark 1.1. *If these components $\mathcal{E}_{ijkl}$ do not depend on the point, the material is said to be* homogeneous. *In other case it is said to be* nonhomogeneous. *Moreover, if the tensor $\mathcal{E}(\boldsymbol{x})$ is invariant with respect to rotations of the coordinate system, the material is said to be* isotropic *at the point $\boldsymbol{x}$. Otherwise the material is said to be* anisotropic.

In Section 2.2, as well as in all the problems described in Chapter 3, a *Kelvin-Voigt* or *short memory* viscoelastic law is used. Here the state of internal stresses depends not only on the strains but also on the velocity that the deformation occurs; that is, it presents both elastic and viscous responses. The general form is given by (see, e.g., [29, 54]),

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1.1}$$

where $\mathcal{A}$ is the *viscosity operator* and $\mathcal{E}$ represents the *elasticity operator*. Here, and in what follows, a dot above a variable will represent its derivative with respect to the time variable.

This viscosity term depends on the velocity, and it is local in time, so it represents a *short memory*. Long memory viscoelastic terms can be found also in literature, and have the form (see [29, 54]),

$$\int_0^t \mathcal{G}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds,$$

where the function $\mathcal{G}$ depends on time depend on time and so the stresses are affected by their history. A constitutive law including this term will be used in Section 2.3, with the form

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds.$$

In order to model other effects like the possible permanent (plastic) deformations, the *elastic-viscoplastic* law is defined by the relation (see [26, 45]),

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})),$$

where again $\mathcal{E}$ is the elasticity operator and $\mathcal{G}$ is a nonlinear viscoplastic operator, which depends on both the stress and the strain tensors. With this type of constitutive law, which will be used in Section 2.4, the body is allowed to demonstrate elastic, viscous and plastic behaviours.

Remark 1.2. *We point out that here and in the sequel, in the equations involving the constitutive relations (*$\boldsymbol{u}, \mathcal{E}, \mathcal{A}, \mathcal{G}$*), we assume implicit dependence on the spatial variable* $\boldsymbol{x}$*. As an example, the complete writing of expression (1.1) would be:*

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\boldsymbol{x},t))) + \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t))), \qquad (\boldsymbol{x},t) \in \Omega \times (0,T).$$

1.2 Damage

The constitutive laws presented above have been studied in many works during the last twenty years. These equations may eventually be modified by a number of effects involved in deformation processes such as the *material damage*.

There is a number of materials, such as concrete, marble, some metals and even the human bones, which suffer an observed decrease in their load carrying capacity due to the appearance and growth of internal microfractures. This process is caused by accumulation of strains in metals, modifications of the intermolecular connections in organic materials, decohesion on minerals, etc.

The topic has decisive importance in industrial and civil engineering, as it affects the reliability and useful life spam of the components of the structures. Due to this fact, there exists an important engineering literature devoted to damage (see, e.g., [1, 3, 5, 27, 55, 59, 61, 64, 67, 68]), although only some recent models containing it have been studied from a mathematical point of view.

The first publication relative to the mechanics of damage is from Kachanov in 1958 ([46]), who introduced a continuous variable for the damage studying the breaking up of some metals in the one-dimensional case. This was followed by Lemaitre and Chaboche ([54]), Leckie and other authors in the $70's$ generalizing the case to isotropic three-dimensional problems. Particularly, Lemaitre and Chaboche define an inner variable which represents the surface density of microcavities in the material.

In general, two types of damage are usually considered in literature: *brittle damage* and *fatigue damage*. Brittle damage is caused by the growth of microscopic cracks in the material and fatigue damage is associated with the accumulation of damage after states of loading and unloading.

A general new model which allows for taking into account both processes were derived by Frémond and Nedjar ([39, 40]), from the virtual work principle (for full details see [36]). The idea is that damage results from microscopic motions, and the power of these microscopic motions must be taken into account for a predictive theory.

In order to describe in the macroscopic level the effect of microfractures, an internal variable $\zeta = \zeta(\boldsymbol{x}, t)$, called *damage function*, is introduced as the volume fraction of microvoids. In an isotropic homogeneous elastic material this variable corresponds to the ratio between the effective elastic modulus of the material E_{eff} and the elastic modulus of the original damage-free material E,

$$\zeta(\boldsymbol{x}, t) = \frac{E_{eff}}{E}.$$

As we can deduce from this definition, ζ takes values between 0 and 1 (when $\zeta = 1$ there is no damage, when $\zeta = 0$ the material is fully damaged and when $0 < \zeta < 1$ there is partial damage). This may be enforced in the damage evolution equation using the indicator function $I_{[0,1]}(\zeta)$ of the interval $[0, 1]$ and its subdifferential $\partial I_{[0,1]}(\zeta)$. The evolution of the microcracks responsible for the damage has been derived from the principle of virtual power in [36] and it has the form,

$$\dot{\zeta} - \kappa \Delta \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta). \tag{1.2}$$

Here κ is the *damage diffusion constant*, and indicates the degree of influence of areas with accumulation of damage on neighbouring zones. Finally, ϕ is the *damage source function*, which gives the contribution of the state of the material to the evolution of the damage through the strains and the damage itself.

Remark 1.3. *When damage is considered in elastic-viscoplastic problems, the state of the material does not depend exclusively on the strains, but also on the stresses, and in these cases the damage source function is supposed to depend also on the stress tensor; that is,*

$$\phi = \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

The expression (1.2) has no restrictions about the sign of $\dot{\zeta}$, so it is supposed that microcracks may close and the material recovers from the damage (the so-called self-mending). There are some cases where the process is considered irreversible, and so the restriction $\dot{\zeta} \leq 0$ has to be imposed. In that case the evolution equation has the form (see [37]),

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) + \partial I_{(-\infty,0]}(\dot{\zeta}) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

The damage source function used in [39, 40] was

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = \lambda_D(\frac{1-\zeta}{\zeta}) - \frac{1}{2}\lambda_u \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}) - \lambda_w,$$

where λ_D, λ_u and λ_w are process parameters. The second term on the right-hand side is proportional to the strain energy and λ_w is an energy threshold for initiation of damage, that is, damage only grows when this threshold is exceeded, i.e. when

$$\lambda_D(\frac{1-\zeta}{\zeta}) - \frac{1}{2}\lambda_u \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}) \geq \lambda_w.$$

We note that the term $\frac{1}{2}\lambda_u \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})$ is quadratic, and for large strains it becomes unmanageable mathematically. Nevertheless under very large strains the damage model becomes inadequate since our problems assume small deformations theory. Therefore we truncate it by introducing the following truncation function,

$$\Psi_{q^*} : \boldsymbol{\tau} \mapsto \Psi_{q^*}(\boldsymbol{\tau}) = \begin{cases} |\boldsymbol{\tau}|^2 & \text{if } |\boldsymbol{\tau}|^2 \leq q^*, \\ q^* & \text{otherwise,} \end{cases}$$

which represents no loss of generality. For technical reasons let us also introduce a truncation operator $\eta_* : \mathbb{R} \mapsto [\zeta_*, 1]$ given by,

$$\eta_*(r) = \begin{cases} r & \text{if } \zeta_* < r < 1, \\ 1 & \text{if } r \geq 1, \\ \zeta_* & \text{if } r \leq \zeta_*, \end{cases}$$

for a value $\zeta_* << 1$, which leads to the expression

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = \lambda_D \left(\frac{1-\eta_*(\zeta)}{\eta_*(\zeta)}\right) - \frac{1}{2}\lambda_u \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \lambda_w. \tag{1.3}$$

The purpose of using η_* is to give the Lipschitz character to the term $(1-\zeta)/\zeta$ and it allows to obtain global regular solutions. Starting with an initial condition ζ_0 such that $\zeta_* < \zeta_0 \leq 1$, during at least a short time period, the solution satisfies the same inequality, and so it is also solution to the problem without the truncation η_*.

Both compression and tension may or may not contribute equally to the evolution of material damage. If we want to model different contributions, we must note that compression is associated with the negative eigenvalues of the strain matrix $\boldsymbol{\varepsilon}(\boldsymbol{u})$, while tension is related to the positive eigenvalues.

We may write the strain tensor as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \sum_{i=1}^{d} \lambda_i \boldsymbol{v}_i \otimes \boldsymbol{v}_i,$$

where $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ is the orthonormal basis of the corresponding eigenvectors and $\lambda_1, \ldots, \lambda_d$ are real eigenvalues. Now we can define the positive and negative parts of the strain tensor as follows

$$\boldsymbol{\varepsilon}(\boldsymbol{u})^+ \equiv \sum_{i=1}^{d} \lambda_i^+ \boldsymbol{v}_i \otimes \boldsymbol{v}_i, \qquad \boldsymbol{\varepsilon}(\boldsymbol{u})^- = \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u})^+,$$

where we use the standard notation $\lambda^+ = (\lambda + |\lambda|)/2$ and $\lambda^- = (\lambda - |\lambda|)/2$, for $\lambda \in \mathbb{R}$.

Then, we can replace the term $\frac{1}{2}\lambda_u \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ by

$$\frac{1}{2}[\lambda_u^- \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u})^-) + \lambda_u^+ \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u})^+)]$$

and therefore we obtain the final form of the damage source function,

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = \lambda_D \left(\frac{1-\zeta}{\eta_*(\zeta)} \right) - \frac{1}{2}\lambda_u^- \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u})^-) - \frac{1}{2}\lambda_u^+ \Psi_{q^*}(\boldsymbol{\varepsilon}(\boldsymbol{u})^+) - \lambda_w. \qquad (1.4)$$

The subgradient term is a convenient way to have ζ taking values between 0 and 1, and it allows to obtain global weak solutions. However, this technique implies several difficulties obtaining estimates and it represents a drop in the regularity for the solutions of the problem.

Without the subgradient term it is possible, taking suitable hypothesis under the damage source function, to demonstrate that the solution satisfies $\zeta \in [0, 1]$ for some time. The analysis of these modified problems is done through the theory of pseudomonotone operators, developed in [51], and improved regularity can be obtained.

Thus, a second possibility for the damage evolution is to employ the following partial differential equation,

$$\dot{\zeta} - \kappa \Delta \zeta = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)). \tag{1.5}$$

In order to obtain the well-posedness of the problem, a boundary condition for the damage field must be considered. Along all this work we will consider that there is no damage influx through the boundary and so, the normal derivative of ζ vanishes; that is,

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0,$$

where $\boldsymbol{\nu}$ denotes the outward unit normal vector to Γ. This will be the boundary condition in all the problems that will be described along this work, although other possibilities could be considered. For example, in [15], as ζ is a variable defined over the whole domain Ω, one can assume that the boundary keeps undamaged, and a Dirichlet condition was assumed,

$$\zeta(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Gamma = \partial\Omega.$$

1.2.1 Modified constitutive laws

As mentioned above, the constitutive relations described are the general relations which model the material behaviour on a simple mechanical process. Now we will see how *damage* modifies them.

We pointed out that for an homogeneous and isotropic linear elastic material, the value of the damage variable is equal to the ratio between the effective elastic modulus of the material and the elastic modulus of the damage-free material. Then, we have

$$\boldsymbol{\sigma} = \zeta \mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}),$$

when the elasticity function is linear and

$$\boldsymbol{\sigma} = \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta),$$

in the case that a nonlinear elasticity function is considered.

When dealing with *Kelvin-Voigt* viscoelastic materials, two models are considered in the literature. The first one, used for example in [12, 19, 22, 42, 43, 71], supposes that the elastic response of the material is affected by ζ, while the viscous response is supposed damage-independent; that is,

$$\boldsymbol{\sigma} = \mathcal{A}\left(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\right) + \mathcal{E}\left(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta\right).$$

In other cases, such as in [38], the damage is supposed also to modify the viscous response of the material, so the constituve law used is

$$\boldsymbol{\sigma} = \zeta\mathcal{A}\left(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\right) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

We note that this second model for damageable viscoelastic materials, with the damage field affecting the viscous part, involves aditional dificulties to both the variational and numerical analyses of the problem.

The same consideration may be made for viscoelastic materials with long memory. In Section 2.3 the following relation is considered,

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))\, ds,$$

where only the memory is supposed to suffer the damage influence. For the case assuming that also the elastic operator is modified by the damage there are no results in the literature yet.

Finally, we present the different possibilities for elastic-viscoplastic materials. In [13, 20, 34, 71] the following relation was considered

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

Depending on the particular elastic-viscoplastic function $\mathcal{G}$ considered, the damage may appear in form of loss of stiffness or developing permanent strains.

The second model for damageable elastic-viscoplastic materials involves an elastic response which is also affected by the damage field. In [15, 50], the following constitutive law was considered,

$$\dot{\boldsymbol{\sigma}} = \overset{\cdot}{\overbrace{\zeta\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})}} + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

1.3 Contact

An optimal design of technical systems strongly depends on the precision in the modelling of contact interfaces which connect different parts of those systems. These are, for instance, the classical example of the tyres with the road, composite structures, robotics, orthodontic implants, brackets, etc. Some general references on modelling, variational and numerical analysis on contact problems are [42, 47, 57, 65, 71].

Due to the highly nonlinear character of these processes, a great accuracy is needed to solve the mechanical behaviour on the contact interfaces, and so, different approaches have been developed over the history to represent the behaviour on the contact area.

Since the real surfaces which appear in engineering and other applications are far away from idealized geometrical surfaces, and the description of contact processes by boundary conditions needs to address lots of kinds of surfaces (which may or may not be clean or lubricated, and can be smooth or rough), a great number of different contact conditions exists and the used one will depend on the particular process that is currently being considered.

In order to describe these different possibilities, we define the *normal* and *tangential* components of the variables involved. We denote by u_ν and $\boldsymbol{u}_\tau$ the *normal* and *tangential* components, respectively, of the displacement field $\boldsymbol{u}$ on the contact boundary, Γ_C, given by

$$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}.$$

Similarly, the *normal* and *tangential* components of the velocity field $\boldsymbol{v} = \dot{\boldsymbol{u}}$ are defined on the contact boundary by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* components of the stress field $\boldsymbol{\sigma}$ on the contact boundary; that is,

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

To describe the contact of a body with an obstacle we need to consider the normal approach and the tangential processes. By *normal contact condition* we understand a condition which describes the normal approach, that is, a relation involving the normal components of the displacement and stress fields. By a *friction condition* we understand a condition which describes the tangential process, i.e., a relation involving the tangential stress $\boldsymbol{\sigma}_\tau$ and the tangential velocity $\dot{\boldsymbol{u}}_\tau$.

We begin by showing the *normal contact conditions*. There are two different possibilities: *bilateral contact* and *unilateral contact*.

Bilateral contact. It describes the situation when contact between the body and the foundation is always maintained. It is possible to found processes with this kind of condition, for example, in moving parts and components of mechanical equipments. As there is no loss of contact, there is no gap. We denote by g the gap between the obstacle and the contact boundary Γ_C, measured in the outward normal direction to the boundary, and so, $g = 0$ and

$$u_\nu = 0.$$

Unilateral contact. In this case the process accepts states of contact and separation, and this state will depend on the specific forces, geometries, etc of each situation. Inside this group there are several possibilities, depending on the behaviour of the foundation, that is, depending if it is rigid or deformable. Let us describe briefly the most used.

Signorini contact condition. This condition is used when the foundation is assumed to be perfectly rigid, and so no penetration is allowed during the process. It is written in the complementary form

$$u_\nu - g \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu(u_\nu - g) = 0.$$

If $u_\nu < g$ there is no contact between the contact and the foundation, so there is no reaction from the foundation, that is, $\sigma_\nu = 0$. If $u_\nu = g$ there exists contact and the foundation exerts a normal compression force $\sigma_\nu \leq 0$ on the body.

Normal compliance condition. It describes a deformable foundation. It is assumed that a reactive normal traction exists and it depends on the amount of the penetration of the asperities of the body into the space initially occupied by the obstacle. The normal stress σ_ν satisfies the condition

$$-\sigma_\nu = p_\nu(u_\nu - g),$$

where p_ν is a prescribed function such that $p_\nu(r) = 0$ if $r \leq 0$. The quantity $u_\nu - g$, when positive, represents the penetration of the body into the foundation. An example of the general form of the function p_ν is (see [48]),

$$p_\nu(r) = c_\nu(r_+)^m,$$

where c_ν is a positive constant, and $r_+ = max\{0, r\}$ is the positive part of r. Let us remark that the Signorini's nonpenetration conditions are obtained formally in the limit $c_\nu \mapsto \infty$. It is also possible to find in the literature the following kind of normal compliance condition (see [69]),

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if} \quad r \leq \alpha, \\ c_\nu \alpha & \text{if} \quad r > \alpha, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface. This condition means that when the penetration is very large, that is, when it exceeds a limit α, the obstacle opens or disintegrates offering no aditional resistance to the penetration.

This condition was first introduced in [58], and since then, it has been used by a number of authors (see e.g. [47, 48] and the references therein).

Normal damped response. In the case that we deal with granular or wet surfaces, the reaction of the foundation appears related to the normal velocity, that is,

$$-\sigma_\nu = p_{ndr}(\dot{u}_\nu),$$

where p_{ndr} is a nonnegative prescribed function which vanishes when its argument is nonpositive. One may use, as an example,

$$-\sigma_\nu = c_{ndr}(\dot{u}_\nu)_+^m,$$

or

$$-\sigma_\nu = c_{ndr}(|\dot{u}_\nu|)^m,$$

in the case that the normal reaction is active also when the surface element is moving away the foundation. Several kind of functions p_{ndr} have been employed in the literature:

- $p_{ndr}(r) = \beta r_+ + p_0$,

  to simulate a foundation covered by a thin lubricant layer (see [11, 35, 69]).

- $p_{ndr}(r) = \kappa |r|^{q-1} r$,

  for $0 < q < 1$ and $\kappa \geq 0$, this condition is named *viscous* contact, and it was used, for instance, in [69].

- $p_{ndr} = S$

  where S is a given positive function. This type of contact conditions, where the normal stresses are prescribed, has been considered in [29, 63].

We turn now to describe the conditions in the tangential direction. Although friction has been investigated by many authors along the history, many frictional phenomena are not fully understood yet. This comes through the fact that the frictional behaviour takes part at the atomic level, that is, an interaction of chemical, electro-magnetic and mechanical processes. There are two possibilities.

Frictionless contact condition. The first approach on the tangential processes, as an approximation of more realistic conditions, is as follows,

$$\boldsymbol{\sigma}_\tau = \mathbf{0}.$$

This is an idealization of the process, since even fully lubricated surfaces generate resistance to tangential movement. However, good approximations can be obtained in some cases.

Frictional contact laws.

We recall below the most used.

Tresca's friction law

This friction condition is expressed in the following form:

$$\left.\begin{array}{l} |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \exists\, \lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau, \end{array}\right\}$$

where g is a given constant that represents the friction bound, that is, the maximun magnitude of the friction force. When the strict inequality holds, the material point supports no tangential movement and we say that it is in the *stick* zone. When the equality holds, the relative movement appears and we say that the material point is in the *slip* zone. The boundary of these zones is unknown a priori, so such a boundary is a *free boundary.*

In certain applications, when loads are light or the friction is large, even when the load is very large and the area where there is no contact is small, the friction bound behaves as a constant, and this condition makes a good approximation. Studies on this frictional condition can be seen in [29, 63].

Coulomb's friction law

The classical Coulomb's law assumes that the friction bound is proportional to the magnitude of the normal stress; that is,

$$\left.\begin{array}{l} |\boldsymbol{\sigma}_\tau| \le \mu |\sigma_\nu|, \\ |\boldsymbol{\sigma}_\tau| < \mu |\sigma_\nu| \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = \mu |\sigma_\nu| \Rightarrow \exists\, \lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau. \end{array}\right\}$$

μ represents the *coefficient of friction*, which is assumed here as a constant but eventually depends on the surface roughness, the relative sliding velocity, the temperature, etc.

Generalized Coulomb's friction law

We consider now

$$\left.\begin{array}{l}|\boldsymbol{\sigma}_\tau| \le \mu p_\tau(|\sigma_\nu|),\\ |\boldsymbol{\sigma}_\tau| < \mu p_\tau(|\sigma_\nu|) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0,\\ |\boldsymbol{\sigma}_\tau| = \mu p_\tau(|\sigma_\nu|) \Rightarrow \exists\,\lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau.\end{array}\right\}$$

Here, p_τ is a nonnegative function, and $\mu \ge 0$ denotes a coefficient of friction. We remark that this law contains the previous ones as particular cases. In the case that p_τ is a constant, this law becomes Tresca's friction law, and if p_τ is the identity, we recover the Coulomb's friction law.

Other particular and interesting cases are included into this general law, it is the case where the wear and the hardness of the surface are taken into account. It consists of using the function (see [73]),

$$p_\tau(r) = [r(1-\delta r)]_+.$$

This law means that when the normal stress is large (when it exceeds the value $\frac{1}{\delta}$), the surface disintegrates and offers no resistance to the motion.

Viscous friction law

All the previous friction laws are characterized by the existence of stick zones in those places where the *friction bound* is not achieved. However, slip appears in lubricated surfaces, even with low tangential stresses. This phenomenom can be modeled by the *viscous friction law* where the friction force depends exclusively on the tangential velocity, that is

$$-\boldsymbol{\sigma}_\tau = p_\tau(\dot{\boldsymbol{u}}_\tau),$$

where p_τ is a prescribed vector-valued function. As an example, the function

$$p_\tau(\boldsymbol{v}) = \mu|\boldsymbol{v}|^{m-1}\boldsymbol{v}$$

was employed in the case when the surface was lubricated with a thin layer of a non-Newtonian fluid (see, e.g., [11, 70]).

1.4 Theoretical results and notation

An important number of the variational problems and almost all the discrete problems which will appear along this PhD thesis will consist of Elliptic Variational Inequalities (EVI). Along the work, we will refer to the simplest and most important kinds of them, namely, following the terminology of [41], EVI of the *first kind* and EVI of the *second kind.*

We introduce some notation in order to describe these problems.

Notation:

- V: real Hilbert space.
- V': dual space of V.
- $b(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ is a bilinear, continuous and V-elliptic form on $V \times V$.
- $L : V \mapsto \mathbb{R}$ is a linear continuous form on V.
- K is a closed convex nonempty subset of V.
- $j(\cdot) : V \mapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous convex (l.s.c.) and proper functional.

With this notation, these two different classes of EVI's consist of

EVI of the first kind Find $\boldsymbol{u} \in K$ such that $\boldsymbol{u}$ is a solution to the problem

$$b(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) \geq L(\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in K. \tag{1.6}$$

EVI of the second kind Find $\boldsymbol{u} \in V$ such that $\boldsymbol{u}$ is a solution to the problem

$$b(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq L(\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in V. \tag{1.7}$$

Problems (1.6) and (1.7) have a unique solution (see [41]).

In order to demonstrate the existence and uniqueness of the solution to discretized problems, we will use the following result, which is known as the *Lax-Milgram Theorem* (a proof of this result can be seen, for example, in [41], p. 322).

Theorem 1.1. *We consider*

1. *a real Hilbert space X with scalar product $(\cdot,\cdot)_X$;*
2. *a bilinear form $b : X \times X \mapsto \mathbb{R}$, continuous, and X-elliptic (i.e., $\exists \alpha > 0$ such that $b(\boldsymbol{u},\boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|_X^2, \quad \forall \boldsymbol{v} \in X$);*
3. *a linear continuous functional $L : X \mapsto \mathbb{R}$.*

Then, the following problem: Find $\boldsymbol{u} \in X$ such that

$$b(\boldsymbol{u},\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in X,$$

has a unique solution.

Analyzing the numerical schemes for the variational inequalities we will need to apply the following two discrete versions of the Gronwall's inequality, whose complete proofs can be found in [33].

Lemma 1.1. *Assume that $\{g_n\}_{n=0}^N$ and $\{e_n\}_{n=0}^N$ are two sequences of nonnegative numbers satisfying*

$$\begin{aligned} &e_0 \leq c\, g_0, \\ &e_n \leq c g_n + c \sum_{j=1}^{n} k_j e_j, \quad n = 1, \ldots, N, \end{aligned}$$

where $\{k_j\}_{j=1}^N$ is a sequence of positive numbers such that $0 \leq c k_j \leq \frac{1}{2}, \quad 1, \ldots, N$. Then,

$$\max_{0 \leq n \leq N} e_n \leq C \max_{0 \leq n \leq N} g_n,$$

where $C = c(1 + cTe^{2cT})$, *with* $T = \displaystyle\sum_{j=1}^{N} k_j$.

Lemma 1.2. *Assume that* $\{g_n\}_{n=0}^N$ *and* $\{e_n\}_{n=0}^N$ *are two sequences of nonnegative real numbers satisfying*

$$\begin{aligned} &e_0 \le c\, g_0, \\ &e_n \le c g_n + c \sum_{j=1}^{n} k_j e_{j-1}, \quad n = 1, \dots, N, \end{aligned}$$

where $\{k_j\}_{j=1}^N$ *is a sequence of positive numbers. Then*

$$\max_{0 \le n \le N} e_n \le C \max_{0 \le n \le N} g_n,$$

where $C = c(1 + cTe^{cT})$, *with* $T = \displaystyle\sum_{j=1}^{N} k_j$.

Now we present some notation that will be used in the following chapters during the analysis of the different problems.

For each real Banach space X we denote its norm by $\| \cdot \|_X$ and we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le +\infty$, $k \ge 1$ and we denote by $\mathcal{C}([0,T];X)$ and $\mathcal{C}^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, respectively. Moreover, if X_1 and X_2 are real Banach spaces, then $X_1 \times X_2$ denotes its product space, where the canonic norm is induced, and is denoted by $\| \cdot \|_{X_1 \times X_2}$.

The following functional spaces are introduced.

$$\begin{aligned} &Y = L^2(\Omega), \quad H = [L^2(\Omega)]^d, \\ &B = H^1(\Omega) = \{\phi \in L^2(\Omega); \ \ \frac{\partial \phi}{\partial x_i}, \quad i = 1, \dots, d \in L^2(\Omega)\}, \quad H_1 = [H^1(\Omega)]^d, \\ &Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in [L^2(\Omega)]^{d \times d}; \quad \tau_{ij} \ = \ \tau_{ji}, \quad 1 \le i,j \le d\}, \end{aligned}$$

The spaces H, H_1 and Q are Hilbert spaces provided with the canonic inner products

$$\begin{aligned} &(\boldsymbol{u}, \boldsymbol{v})_H = \sum_{i=1}^{d} \int_\Omega u_i v_i d\boldsymbol{x}, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \sum_{i,j=1}^{d} \int_\Omega \sigma_{ij} \tau_{ij} d\boldsymbol{x}, \\ &(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \end{aligned}$$

and their respective associated norms $\|\cdot\|_H, \|\cdot\|_{H_1}$ and $\|\cdot\|_Q$.

The space of admissible displacement functions will be denoted by V, but its definition changes depending on if we consider *unilateral* or *bilateral* contact; that is,

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \,; \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\},$$

or

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \,; \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D, \quad v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_C\},$$

respectively, where the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,$$

is considered. We note that since $meas(\Gamma_D) > 0$, it follows from Korn's inequality that there exists a positive constant C such that $|\boldsymbol{\varepsilon}(\boldsymbol{u})|_Q \geq C\|\boldsymbol{v}\|_V$. Thus, $\|\cdot\|_V$ and $\|\cdot\|_{H_1}$ are equivalent norms on V and so $(V, \|\cdot\|_V)$ is a Hilbert space (see [60]).

We recall that for all $\boldsymbol{\sigma} \in [\mathcal{C}^1(\overline{\Omega})]^{d\times d}_{sym}$, we have $\text{Div}\boldsymbol{\sigma} \in H$ (Div denotes the divergence operator $\text{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$), and the following *Green's* formula holds

$$(\boldsymbol{\sigma\nu}, \boldsymbol{v})_{[L^2(\Gamma)]^d} = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\text{Div}\boldsymbol{\sigma}, \boldsymbol{v})_H \quad \forall \boldsymbol{v} \in H_1, \tag{1.8}$$

which will be used to obtain the variational formulations of the problems considered in this work.

Finally, as some of the variational inequalities presented along this work will be formulated as subdifferential inclusions, let us recall some definitions. Let X be a Hilbert space and $\psi : X \to (-\infty, \infty]$. The function ψ is said to be *subdifferentiable* in $\boldsymbol{u} \in X$ if $\psi(\boldsymbol{u}) < +\infty$ and there exists $G(\boldsymbol{u}) \in X$ such that

$$\psi(\boldsymbol{v}) \geq \psi(\boldsymbol{u}) + (G(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u})_X \qquad \forall \boldsymbol{v} \in X. \tag{1.9}$$

The element $G(\boldsymbol{u})$ is known as the *subgradient* of ψ in $\boldsymbol{u}$. The set of subgradients of ψ in $\boldsymbol{u} \in X$ is named *subdifferential* of ψ in $\boldsymbol{u}$, and is denoted by $\partial\psi(\boldsymbol{u})$.

Chapter 2

Quasistatic problems with damage

This chapter deals with quasistatic processes, that is, situations where the system configuration and the external forces and tractions vary slowly in time in such a way that the accelerations in the system are rather small, so that the inertia term can be neglected.

Four different models will be analyzed, according to four different material behaviours and contact conditions. In the first model, studied along Section 2.1, an elastic constitutive relation is considered, and has the peculiarity that no contact was supposed. The second model consists of a frictional problem in *Kelvin-Voigt* viscoelasticity. The third model considers also a viscoelastic constituve law, but with the difference that the viscous effect comes as the result of an integral term whose contributions are the previous deformed configurations (that is, the “memory” of the material is taken into account). The last problem is related to an elastic-viscoplastic material behaviour.

In the model described along Section 2.3 a unilateral frictionless contact condition is considered, with a normal compliance condition modelling the normal contact, while models presented in Sections 2.2 and 2.4 deal with bilateral frictional contact conditions.

The variational formulations of these problems have new terms coming from the contact conditions. In each case, a function j will be defined in order to represent these contact terms. In the viscoelastic with long memory problem, the discrete variational problem leads to a well-known variational inequality which is numerically solved using a penalty-duality algorithm defined in [6] and whose application for an elastic problem is detailed in [75].

Both second and fourth problems presented below are frictional problems and these will lead to a different kind of variational inequality, since, in this case, the contact term j is not smooth and the previous numerical method can not be applied. An Uzawa-type algorithm can be used (see [23]) but we have also considered a penalization over the friction condition, leading to a new equation which can be solved by the penalty-duality algorithm. Good comparison results are achieved concerning the CPU time, improving those obtained with Uzawa's algorithm.

Moreover, in Sections 2.2 and 2.4 the damage model used is that one including the subdifferential term, described in (1.2). This will change both the numerical analysis and the numerical resolution of the damage problem performed in Section 2.1.

The results detailed in the present Chapter have been recently published or submitted for publication in [10, 13, 14, 18, 19].

2.1 Quasistatic elastic damage problem

In this section we present a model to describe the quasistatic evolution of damage in an elastic body. The body, which occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, has an outer surface $\partial\Omega = \Gamma$ supposed to be sufficiently smooth. Also Γ_D and Γ_N are disjoint subsets of $\partial\Omega$ whose union equals $\partial\Omega$, Γ_D has positive surface measure $(meas(\Gamma_D) > 0)$) and $\boldsymbol{\nu}$ denotes the outward unit normal vector to Γ. The material is assumed elastic with a constitutive law

$$\boldsymbol{\sigma} = \zeta \mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where ζ is the damage field and $\mathcal{E}$ is a prescribed linear function. Since contact is not assumed, the boundary conditions consist only of surface tractions and restrictions over the displacements $(\Gamma_C = \emptyset)$. Denoting by $\boldsymbol{f}_B$ and $\boldsymbol{f}_N$ the density of volume and surface forces, respectively, the classical form of the problem is as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, and a damage field $\zeta : \Omega \times [0, T] \to \mathbb{R}$, such that

$$-\mathrm{Div}\, \boldsymbol{\sigma} = \boldsymbol{f}_B \quad \text{in} \quad \Omega \times (0, T), \tag{2.1}$$

$$\boldsymbol{\sigma} = \eta_*(\zeta)\, \mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega \times (0, T), \tag{2.2}$$

$$\dot{\zeta} - \kappa \Delta \zeta = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{in} \quad \Omega \times (0, T), \tag{2.3}$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N \times (0, T). \tag{2.6}$$

$$\zeta(0) = \zeta_0 \quad \text{in} \quad \Omega \tag{2.7}$$

Let us remember now the physical meaning of expressions which compose Problem P. Expression (2.1) is the equilibrium equation which governs the deformation process on a quasistatic problem. Expression (2.2) is the constitutive law of an elastic material when the damage of the material is taken into account. Equation (2.3) is the damage evolution equation, which models the variation on the mechanical properties of the material, and (2.4)–(2.6) are the boundary conditions over the damage field, displacements and stresses (see Chapter 1 for further details concerning the notation).

Before we continue with the abstract formulation, we list the assumptions on the problem data, and let us introduce the space of admissible displacements V defined by

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\, \boldsymbol{v} = \mathbf{0} \quad \text{on} \quad \Gamma_D\}.$$

As $meas(\Gamma_D) > 0$, Korn's inequality holds and so there exists a positive constant C_K depending only on the domain Ω such that $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \geq C_K \|\boldsymbol{v}\|_{[H^1(\Omega)]^d}$, for $\boldsymbol{v} \in V$. Let us define also the inner product on V

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \tag{2.8}$$

and its associated norm $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q$.

The *elasticity tensor* $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is linear and satisfies:

$$\left.\begin{array}{l} \text{(a) There exists } C_{\mathcal{E}} > 0 \text{ such that} \\ \qquad\qquad \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) \cdot \boldsymbol{\tau} \geq C_{\mathcal{E}} |\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{E}\left(\boldsymbol{x}, \boldsymbol{\tau}\right) \text{ is measurable and bounded } \forall \boldsymbol{\tau} \in \mathbb{S}^d. \end{array}\right\} \tag{2.9}$$

The *damage source function* $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left.\begin{array}{l} \text{(a) } |\phi\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \zeta_1\right) - \phi\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \zeta_2\right)| \leq L_\phi \left(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2|\right) \\ \qquad\qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \;\; \zeta_1, \zeta_2 \in \mathbb{R}, \quad \text{a.e. } \; \boldsymbol{x} \in \Omega. \\ \text{(b) The function } \; \boldsymbol{x} \to \phi\left(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta\right) \text{ is measurable and bounded} \\ \qquad\qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \quad \zeta \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \to \phi\left(\boldsymbol{x}, 0, 0\right) \text{ belongs to } L^2(\Omega). \end{array}\right\} \tag{2.10}$$

Here, L_ϕ is the Lipschitz constant. Also we assume that for $0 < \zeta_* < 1$,

$$\phi\left(\boldsymbol{\varepsilon}, \zeta\right) \leq 0 \;\text{ if }\; \zeta \geq 1, \quad \phi\left(\boldsymbol{\varepsilon}, \zeta\right) \geq 0 \text{ if } \zeta \leq \zeta_*. \tag{2.11}$$

The first of these assumptions states that the source term for damage is nonpositive whenever $\zeta = 1$. This makes perfect physical sense because it says nothing more that there is nothing which can cause the damage to become larger than 1. The second one is assumed for convenience but it could be omitted. If we leave it out, it might result in only local solutions to the problem being obtained.

The density of body forces and the tractions are assumed to satisfy

$$\boldsymbol{f}_B \in \mathcal{C}\left([0,T]\,;V\right), \quad \boldsymbol{f}_N \in \mathcal{C}\left([0,T]\,;[H^{3/2}\left(\Gamma_N\right)]^d\right), \tag{2.12}$$

and we define the element $\boldsymbol{f}(t) \in V$ by

$$\left(\boldsymbol{f}\left(t\right), \boldsymbol{v}\right)_V = \left(\boldsymbol{f}_B\left(t\right), \boldsymbol{v}\right)_H + \left(\boldsymbol{f}_N\left(t\right), \boldsymbol{v}\right)_{[L^2(\Gamma_N)]^d} \quad \forall \boldsymbol{v} \in V. \tag{2.13}$$

Also, let us assume that

$$\zeta_0 \in B, \quad \zeta_0\left(\boldsymbol{x}\right) \in (\zeta_*, 1], \quad 1 > \zeta_* > 0, \tag{2.14}$$

$$L\zeta_0 \in Y, \quad \nabla\left(\kappa\Delta\zeta_0 + \phi\left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}_0\right), \eta_*\left(\zeta_0\right)\right)\right) \in H, \tag{2.15}$$

where $L : B \to B'$ is defined as

$$< L\zeta, \xi >= \int_\Omega \nabla\zeta \cdot \nabla\xi d\boldsymbol{x},$$

$\boldsymbol{u}_0$ are the initial displacements, obtained as the solution to the set of equations (2.1), (2.2), (2.5)–(2.7) at time instant $t = 0$, and

$$\boldsymbol{f} \in L^2\left(0,T;H\right) \cap L^\infty\left(0,T;V'\right), \quad \dot{\boldsymbol{f}} \in L^\infty\left(0,T;V'\right). \tag{2.16}$$

We turn now to obtain a variational formulation of this problem. In this way let us suppose $(\boldsymbol{u}, \zeta)$ to be regular functions satisfying (2.1)–(2.6) and let us take $\boldsymbol{w} \in V$ and $t \in (0,T)$. Plugging (2.2) into (2.1) and using the Green's formula (1.8) with that expression, we have

$$(\eta^*(\zeta(t))\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (\boldsymbol{f}_B(t), \boldsymbol{w})_V + (\boldsymbol{\sigma}(t)\boldsymbol{\nu}, \boldsymbol{w})_{[L^2(\Gamma)]^d}.$$

Since $\boldsymbol{w} \in V$, we know that $\boldsymbol{w} = \boldsymbol{0}$ over Γ_D, so taking into account (2.13), we finally obtain

$$(\eta^*(\zeta(t))\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (\boldsymbol{f}(t), \boldsymbol{w})_V.$$

In the same way let us consider now expression (2.3) and multiply it by $\xi \in B$. Applying again (1.8) and using (2.4) we get

$$(\dot{\zeta}(t), \xi)_Y + a(\zeta(t), \xi) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi)_Y,$$

where $a : H^1(\Omega) \times H^1(\Omega) \mapsto \mathbb{R}$ is the bilinear form

$$a(\xi, \psi) = \kappa \int_\Omega \nabla\xi \nabla\psi d\boldsymbol{x}, \quad \xi, \psi \in B. \tag{2.17}$$

Accordingly to this, every regular solution $(\boldsymbol{u}, \zeta)$ to Problem P verifies the following variational formulation.

Problem VP**.** *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$*, and a damage field* $\zeta : [0, T] \to B$*, such that* $\zeta(0) = \zeta_0$ *and for a.e.* $t \in [0, T]$*,*

$$(\eta^*(\zeta(t))\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = (\boldsymbol{f}(t), \boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V, \tag{2.18}$$

$$(\dot{\zeta}(t), \xi)_Y + a(\zeta(t), \xi) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi)_Y \quad \forall \xi \in B. \tag{2.19}$$

2.1.1 An existence and uniqueness result

The existence of a unique solution to Problem VP is stated in the following theorem, which was demonstrated in [52].

Theorem 2.1. *Assume that (2.9)-(2.16) hold and* $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$*. Then there exists a unique solution to Problem* VP *with the following regularity:*

$$\begin{aligned}
&\zeta \in L^\infty(0, T; W^{2,6}(\Omega)) \cap L^\infty(0, T; H^3(\Omega)) \cap \mathcal{C}([0, T]; H^2(\Omega)), \\
&\dot{\zeta} \in L^\infty(0, T; B) \cap L^2(0, T; H^2(\Omega)), \quad \ddot{\zeta} \in L^2(0, T; Y), \\
&\boldsymbol{u} \in L^\infty\left(0, T; [H^3(\Omega)]^d\right) \cap \mathcal{C}\left([0, T]; [H^r(\Omega)]^d\right), \quad \dot{\boldsymbol{u}} \in L^2(0, T; V),
\end{aligned}$$

with $r > 5/2$*. This solution satisfies* $\zeta(t)(\boldsymbol{x}) \in [\zeta_*, 1]$ *a.e. for each* $t \in [0, T]$.

Proof

We only indicate the main facts of the proof. Details can be found in [52].

First, under less stringent assumptions over the data, and through the Schauder-fixed-point theorem, the existence and uniqueness of solution to Problem VP is demonstrated. Then, taking into account assumptions (2.11) and (2.14), it can be proved, through some fundamental comparison theorems which apply to semilinear parabolic equations, that the solution obtained is the same if we replace $\eta_\star(\zeta)$ with ζ in the definitions of both the elasticity operator and the damage source function.

Finally, using the assumptions $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$ and (2.15), and the regularity conditions required for the applied forces, the regularity indicated in the theorem is obtained. This last step is done obtaining the solution to the first part of the problem as a limit of solutions to a regularized problem whose solutions have all the estimates desired. ■

2.1.2 Numerical analysis

Now we consider and analyze a fully discrete approximation for solving problems (2.18)–(2.19). We also notice that some of the results presented here and in the following section will appear in [14].

The discretization of Problem VP will be done in two steps. First we consider two finite dimensional spaces $V^h \subset V$ and $B^h \subset B$, approximating the spaces of admissible displacements and damage functions V and B, respectively (with $h > 0$ being the spatial discretization parameter).

To discretize the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$ and let k be the time step size, $k = T/N$.

Here, and in the rest of this manuscript, for a continuous function $f(t)$, let us denote $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

The fully discrete approximation of Problem VP, based on the forward Euler scheme to discretize the time derivatives, is as follows.

Problem VP^{hk}. *Find a discrete displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete damage field* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset B^h$ *such that for all* $\xi^h \in B^h$, $\boldsymbol{w}^h \in V^h$,

$$(\delta\zeta_n^{hk}, \xi^h)_Y + a(\zeta_n^{hk}, \xi^h) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h)_Y, \quad n = 1, 2, \ldots, N \tag{2.20}$$

$$(\eta^*(\zeta_n^{hk})\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V, \quad n = 0, 1, 2, \ldots, N \tag{2.21}$$

where ζ_0^{hk} *is an appropriate approximation of the initial condition* ζ_0.

We can state the following result.

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold. Then, there exists a unique solution to Problem* VP^{hk} *such that* $\boldsymbol{u}^{hk} \subset V^h$ *and* $\zeta^{hk} \subset B^h$.

Proof

Let us suppose that, for $n \in \{1, 2, \ldots, N\}$, $\boldsymbol{u}_{n-1}^{hk}$ and ζ_{n-1}^{hk} are known. We must prove that there exists a unique $(\boldsymbol{u}_n^{hk}, \zeta_n^{hk})$ satisfying (2.20)–(2.21). First, we will see it for ζ_n^{hk}.

Now, B^h will be our Hilbert space. We rewrite expression (2.20) as follows,

$$(\zeta_n^{hk}, \xi^h)_Y + ka(\zeta_n^{hk}, \xi^h) = k(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h)_Y + (\zeta_{n-1}^{hk}, \xi^h)_Y \quad \forall \xi^h \in B^h,$$

where the terms on the right-hand side are known. Let us define the bilinear form $b : B^h \times B^h \mapsto \mathbb{R}$ as

$$b(\zeta_1, \zeta_2) = (\zeta_1, \zeta_2)_Y + ka(\zeta_1, \zeta_2) \quad \forall \zeta_1, \zeta_2 \in B^h.$$

It is straightforward to check that function b is bilinear and continuous, so we only have to check the coercivity. It follows that

$$\begin{aligned} b(\zeta, \zeta) &= (\zeta, \zeta)_Y + ka(\zeta, \zeta) = \|\zeta\|_Y^2 + k\kappa\|\nabla\zeta\|_H^2 \\ &\geq \min\{1, k\kappa\}\|\zeta\|_B^2. \end{aligned}$$

Also, we define the linear operator $L : B^h \mapsto \mathbb{R}$ as

$$L(\zeta) = k(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \zeta)_Y + (\zeta_{n-1}^{hk}, \zeta)_Y \quad \forall \zeta \in B^h.$$

With these functionals, we are under the assumptions of Lax-Milgram theorem, so we can deduce that there exists a unique ζ_n^{hk} verifying (2.20).

For the case of the displacements problem (2.21), we apply the same argument. The Hilbert space to consider is V^h. In this case we already know ζ_n^{hk}, so the bilinear form is defined as

$$b(\boldsymbol{u}_1, \boldsymbol{u}_2) = (\eta^*(\zeta_n^{hk})\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1)), \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_Q \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V^h.$$

Due to assumption (2.9), we can assure that

$$b(\boldsymbol{v}, \boldsymbol{v}) \geq \zeta_* C_{\mathcal{E}} \|\boldsymbol{v}\|_V^2, \quad \forall \boldsymbol{v} \in V^h.$$

Finally, defining L as

$$L(\boldsymbol{v}) = (\boldsymbol{f}_n, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V^h,$$

we are again under the hypothesis of Lax-Milgram theorem, and so we have obtained that $\boldsymbol{u}_n^{hk}$ and ζ_n^{hk} are the unique solution to Problem VP^{hk}. ■

Our interest lies now in estimating the numerical errors $\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V$ and $\|\zeta_n - \zeta_n^{hk}\|_Y$. In this way, first of all, we rewrite equation (2.18) at time $t = t_n$ for all $\boldsymbol{w} = \boldsymbol{w}^h \in V^h$ and we substract it to equation (2.21) to obtain

$$(\eta^*(\zeta_n)\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \eta^*(\zeta_n^{hk})\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = 0 \quad \text{for all} \quad \boldsymbol{w}^h \in V^h.$$

Then, we have

$$\begin{aligned}
&(\eta^*(\zeta_n)\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \eta^*(\zeta_n^{hk})\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q \\
&\quad = (\eta^*(\zeta_n)\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \eta^*(\zeta_n^{hk})\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h))_Q \quad \text{for all } \boldsymbol{w}^h \in V^h.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&(\eta^*(\zeta_n)\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \eta^*(\zeta_n^{hk})\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\
&\quad = (\eta^*(\zeta_n^{hk})[\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))], \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\
&\qquad +([\eta^*(\zeta_n) - \eta^*(\zeta_n^{hk})]\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \text{for all } \boldsymbol{w} \in V,
\end{aligned}$$

we can write

$$\begin{aligned}
&\eta^*(\zeta_n^{hk})(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q = \\
&\quad -([\eta^*(\zeta_n) - \eta^*(\zeta_n^{hk})]\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) \\
&\quad +\eta^*(\zeta_n^{hk})(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h) \\
&\quad +([\eta^*(\zeta_n) - \eta^*(\zeta_n^{hk})]\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h)) \quad \text{for all } \boldsymbol{w} \in V.
\end{aligned}$$

Using now the property $\eta^*(\zeta_n),\ \eta^*(\zeta_n^{hk}) \geq \zeta_*$ and the regularity $\boldsymbol{u} \in L^\infty(0,T;[H^3(\Omega)]^d)$ (which implies that $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(0,T;[L^\infty(\Omega)]^{d\times d})$), after easy algebraic manipulations we obtain

$$\begin{aligned}\zeta_*\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leq c\|\zeta_n - \zeta_n^{hk}\|_Y \|\boldsymbol{u}\|_{L^\infty(0,T;[H^3(\Omega)]^d)}\Big(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V\Big)\\ +c\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V\|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V \quad \text{for all } \boldsymbol{w}^h \in V^h.\end{aligned}$$

Applying now the Cauchy's inequality

$$ab \leq \epsilon a^2 + (1/4\epsilon)b^2, \quad \text{for} \quad \epsilon > 0, \tag{2.22}$$

it leads to the following estimate for the displacement field,

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leq c\Big(\|\zeta_n - \zeta_n^{hk}\|_Y^2 + \|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V^2\Big). \tag{2.23}$$

We turn now to obtain an error estimate for the damage field. We will denote by $\phi_n = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n)$ and $\phi_{n-1}^{hk} = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk})$. In this way we take equation (2.20) and substract it to equation (2.19) at time $t = t_n$. It follows that

$$\begin{aligned}&(\dot\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk})_Y + a(\zeta_n - \zeta_n^{hk}, \zeta_n - \zeta_n^{hk}) - (\phi_n - \phi_{n-1}^{hk}, \zeta_n - \zeta_n^{hk})_Y\\ &\quad = (\dot\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \xi^h)_Y + a(\zeta_n - \zeta_n^{hk}, \zeta_n - \xi^h) - (\phi_n - \phi_{n-1}^{hk}, \zeta_n - \xi^h)_Y,\end{aligned}$$

for all $\xi = \xi^h \in B^h$. As we know that $\|\phi_n - \phi_{n-1}^{hk}\|_Y \leq c(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y)$, after some algebra we get

$$\begin{aligned}&(\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk})_Y + c\|\nabla(\zeta_n - \zeta_n^{hk})\|_H^2\\ &\quad \leq c\big(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y\big)\big(\|\zeta_n - \zeta_n^{hk}\|_Y + \|\zeta_n - \xi^h\|_Y\big)\\ &\qquad +\|\dot\zeta_n - \delta\zeta_n\|_Y\big(\|\zeta_n - \zeta_n^{hk}\|_Y + \|\zeta_n - \xi^h\|_Y\big) + \|\nabla(\zeta_n - \zeta_n^{hk})\|_H\|\zeta_n - \xi^h\|_B\\ &\qquad +(\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \xi^h)_Y.\end{aligned}$$

Since

$$(\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk})_Y \geq \frac{1}{2k}\Big[\|\zeta_n - \zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2\Big],$$

using again (2.22) we obtain

$$\begin{aligned}&\|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\|\nabla(\zeta_n - \zeta_n^{hk})\|_H^2\\ &\quad \leq ck\Big(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y^2 + \|\dot\zeta_n - \delta\zeta_n\|_Y^2 + \|\zeta_n - \xi^h\|_B^2\Big)\\ &\qquad +c(\zeta_n - \zeta_n^{hk} - (\zeta_{n-1} - \zeta_{n-1}^{hk}), \zeta_n - \xi^h)_Y + \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2,\end{aligned}$$

for all $\xi^h \in B^h$. By induction we obtain that

$$\begin{aligned}
\|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \leq ck \sum_{j=1}^{n} \Big(\|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2 + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 \\
+ \|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 + \|\zeta_j - \xi_j^h\|_B^2 \Big) + \|\zeta_0 - \zeta_0^{hk}\|_Y^2 \\
+ c\sum_{j=1}^{n} (\zeta_j - \zeta_j^{hk} - (\zeta_{j-1} - \zeta_{j-1}^{hk}), \zeta_j - \xi_j^h)_Y \quad \forall \xi^h = \{\xi_j^h\}_{j=0}^n \subset B^h. \qquad (2.24)
\end{aligned}$$

Now, taking into account that

$$\begin{aligned}
&\sum_{j=1}^{n} (\zeta_j - \zeta_j^{hk} - (\zeta_{j-1} - \zeta_{j-1}^{hk}), \zeta_j - \xi_j^h)_Y \\
&\qquad = (\zeta_n - \zeta_n^{hk}, \zeta_n - \xi_n^h)_Y + (\zeta_0 - \zeta_0^{kh}, \zeta_1 - \xi_1^h)_Y \\
&\qquad\quad + \sum_{j=1}^{n-1} (\zeta_j - \zeta_j^{hk}, \zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)_Y \\
&\qquad \leq \epsilon \|\zeta_n - \zeta_n^{hk}\|_Y^2 + c\|\zeta_n - \xi_n^h\|_Y^2 + c\|\zeta_0 - \zeta_0^{hk}\|_Y^2 + c\|\zeta_1 - \xi_1^h\|_Y^2 \\
&\qquad\quad + k\sum_{j=1}^{n-1} \|\zeta_j - \zeta_j^{hk}\|_Y^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2, \qquad (2.25)
\end{aligned}$$

where $\epsilon > 0$ is assumed sufficiently small, combining (2.23) and (2.24), it leads to the following estimate,

$$\begin{aligned}
\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \leq ck \sum_{j=1}^{n} \Big(\|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \\
+ \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + \|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 \\
+ \|\zeta_j - \xi_j^h\|_B^2 \Big) + c\|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + c\|\zeta_1 - \xi_1^h\|_Y^2 + c\|\zeta_n - \xi_n^h\|_Y^2 + c\|\zeta_0 - \zeta_0^{hk}\|_Y^2 \\
+ c\frac{1}{k}\sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2, \qquad (2.26)
\end{aligned}$$

for all $\xi^h = \{\xi_j^h\}_{j=0}^n \subset B^h$ and $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$. Let us define now, for $n =$

$1, \ldots, N$, the following quantities,

$$\begin{aligned}
e_n &= \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2,\\
g_n &= k\sum_{j=1}^{n} \big(\|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2\\
&\quad + \|\zeta_j - \xi_j^h\|_B^2\big) + \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + \|\zeta_1 - \xi_1^h\|_Y^2 + \|\zeta_n - \xi_n^h\|_Y^2\\
&\quad + \|\zeta_0 - \zeta_0^{hk}\|_Y^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2.
\end{aligned}$$

With this new notation, inequality (2.26) may be written as

$$e_n \le c g_n + c\sum_{j=1}^{n} k e_{j-1}, \quad n = 1, \ldots, N.$$

Let $e_0 = \|\boldsymbol{u}_0 - \boldsymbol{u}_0^{kh}\|_V^2 + \|\zeta_0 - \zeta_0^{hk}\|_Y^2 = g_0$. Then, we can apply the discrete version of Gronwall's inequality shown in Lemma 1.2 and we obtain the following theorem.

Theorem 2.3. *Let the assumptions of Theorem 2.1 hold. There exists a constant $c > 0$, independent of h and k, such that for all $\{\zeta_j^h\}_{j=0}^N \subset B^h$ and $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$,*

$$\begin{aligned}
&\max_{0\le n\le N} \{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\} + k\sum_{j=1}^{N} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2\\
&\quad\le ck\sum_{j=1}^{N} \Big(\|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 + \|\zeta_j - \xi_j^h\|_B^2\Big)\\
&\qquad + c\frac{1}{k}\sum_{j=1}^{N-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2 \qquad (2.27)\\
&\qquad + c\max_{0\le n\le N} \|\zeta_n - \xi_n^h\|_Y^2 + c\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + c\|\zeta_0 - \zeta_0^{hk}\|_Y^2.
\end{aligned}$$

The inequality (2.27) is the basis for the convergence analysis. As an example, let Ω be a polyhedral domain and denote by $\mathcal{T}^h$ a finite element triangulation of Ω compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D and Γ_N. Let V^h and B^h be defined by

$$V^h = \{\boldsymbol{w}^h \in [\mathcal{C}(\overline{\Omega})]^d \ ; \ \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d, \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \mathbf{0} \text{ on } \Gamma_D\}, \qquad (2.28)$$

$$B^h = \{\xi^h \in \mathcal{C}(\overline{\Omega}) \ ; \ \xi^h_{|Tr} \in P_1(Tr), \quad \forall Tr \in \mathcal{T}^h\}, \qquad (2.29)$$

where $P_1(Tr)$ represents the space of polynomial functions of global degree less or equal to 1 in Tr, and assume that the discrete initial condition ζ_0^h is defined by

$$\zeta_0^{hk} = \pi^h \zeta_0, \tag{2.30}$$

where $\pi^h : \mathcal{C}(\overline{\Omega}) \to B^h$ is the standard finite element interpolation operator (see, e.g., [25]).

The following result is obtained, which establishes the linear convergence of the algorithm with respect to the discretization parameters h and k.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold, and the regularity stated there satisfied for the case of Lipschitz boundaries. Then, the numerical algorithm introduced in Problem VP^{hk} is linearly convergent; that is, there exists $c > 0$, independent of h and k, such that,*

$$\max_{0\leq n\leq N}\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y\} \leq c(h+k). \tag{2.31}$$

Proof

First, we have

$$\delta\zeta_j - \dot{\zeta}_j = \frac{1}{k}\int_{t_{j-1}}^{t_j} (\dot{\zeta}(t) - \dot{\zeta}(t_j))dt = \frac{1}{k}\int_{t_{j-1}}^{t_j}\int_{t_j}^{t} \ddot{\zeta}(s)dsdt.$$

Then, it is easy to see that

$$k\sum_{j=1}^{N} \|\delta\zeta_j - \dot{\zeta}_j\|_Y^2 \leq k^2 \|\ddot{\zeta}\|^2_{L^2(0,T;Y)}.$$

By other part, it is straightforward to check that

$$k\sum_{j=1}^{N} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 \leq ck^2\|\boldsymbol{u}\|_{H^1(0,T;V)}, \quad k\sum_{j=1}^{N} \|\zeta_j - \zeta_{j-1}\|_Y^2 \leq ck^2\|\zeta\|_{H^1(0,T;Y)}.$$

We need to estimate the errors provided by the approximation of the finite element spaces V^h and B^h. Since $\boldsymbol{u} \in \mathcal{C}([0,T];[H^2(\Omega)]^d)$ and $\zeta \in \mathcal{C}([0,T];H^2(\Omega))$, we have (see [25]),

$$\max_{1\leq n\leq N}\inf_{\boldsymbol{w}_n^h\in V^h} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V \leq ch\|\boldsymbol{u}\|_{\mathcal{C}([0,T];[H^2(\Omega)]^d)}, \tag{2.32}$$

$$\max_{0\leq n\leq N}\inf_{\xi_n^h\in B^h} \|\zeta_n - \xi_n^h\|_B \leq ch\|\zeta\|_{\mathcal{C}([0,T];H^2(\Omega))}, \tag{2.33}$$

and from the definition of operator π^h we obtain

$$\inf_{w_0^h \in V^h} \|\boldsymbol{u}_0 - \boldsymbol{w}_0^h\|_V \leq \|\boldsymbol{u}_0 - \Pi^h \boldsymbol{u}_0\|_V \leq ch\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d},$$
$$\|\zeta_0 - \zeta_0^h\|_Y \leq ch^2\|\zeta_0\|_{H^2(\Omega)},$$

where

$$\Pi^h \boldsymbol{v} = (\pi^h v_i)_{i=1}^d. \tag{2.34}$$

Taking $\xi_j^h = \pi^h \zeta(t_j) = \pi^h \zeta_j$, we obtain that

$$(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h) = (\zeta_{j+1} - \zeta_j) - \pi^h(\zeta_{j+1} - \zeta_j) = k(\delta\zeta_{j+1} - \pi^h \delta\zeta_{j+1}),$$

and, as $\delta\zeta_{j+1} = \dfrac{1}{k}\displaystyle\int_{t_j}^{t_{j+1}} \dot{\zeta}(t)dt$, we find that

$$\|\delta\zeta_{j+1}\|_B^2 \leq \frac{1}{k}\int_{t_j}^{t_{j+1}} \|\dot{\zeta}(t)\|_B^2 dt. \tag{2.35}$$

From estimations (2.33) and (2.35) we deduce that

$$\|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \leq ckh^2 \int_{t_j}^{t_{j+1}} \|\dot{\zeta}(t)\|_B^2 dt,$$

and then,

$$\frac{1}{k}\sum_{j=1}^{N-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \leq ch^2\|\dot{\zeta}\|_{L^2(0,T;B)}^2.$$

which concludes the proof of (2.31). ■

2.1.3 Numerical examples

In order to verify the behaviour of the numerical method described above, some numerical experiments have been performed in two-dimensional problems. In all the examples presented, the elasticity tensor was chosen as the $2D$ plane-stress elasticity tensor,

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{Er}{1 - r^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1 + r}\tau_{\alpha\beta} \quad \forall \boldsymbol{\tau} \in \mathbb{S}^2, \tag{2.36}$$

where $\alpha, \beta = 1, 2$, E and r are the Young's modulus and the Poisson's ratio, respectively, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

The finite element spaces V^h and B^h are given by (2.28) and (2.29), respectively, and damage source function ϕ is given by expression (1.4) with constants $q^* = 1000$ and $\zeta_* = 0.01$.

Numerical resolution

As it was described above in order to prove Theorem 2.2, for each time instant $t_n, n \in \{1, \ldots, N\}$, the first step is to obtain the damage field ζ_n^{hk}. Expression (2.20) is equivalent to a linear system, and due to the properties of symmetry and coercivity for the bilinear form b defined in that proof, Cholesky's method was used to solve it.

Once ζ_n^{hk} is calculated, it is introduced into (2.21). In practice, it is obtained that $\zeta_n^{hk} > \zeta_*$ (the program will stop when that value is reached), and therefore, equation (2.21) leads to another linear system which is solved also using Cholesky's method.

First example: numerical convergence.

As a first example, a sequence of numerical solutions based on uniform partitions of both the time interval and the domain $\Omega = [0,5] \times [0,5] - [2,3] \times [2,3]$ have been performed in order to see the numerical behaviour of the scheme.

The physical setting of the example is depicted in Figure 2.1 (left-hand side). Γ_D is the inner boundary $\{2,3\} \times [2,3] \cup [2,3] \times \{2,3\}$, and so the displacement field vanishes there. The outer surface Γ_N has been considered traction-free ($\Gamma_N = \{0,5\} \times [0,5] \cup [0,5] \times \{0,5\}$, with $\boldsymbol{f}_N = \mathbf{0}$).

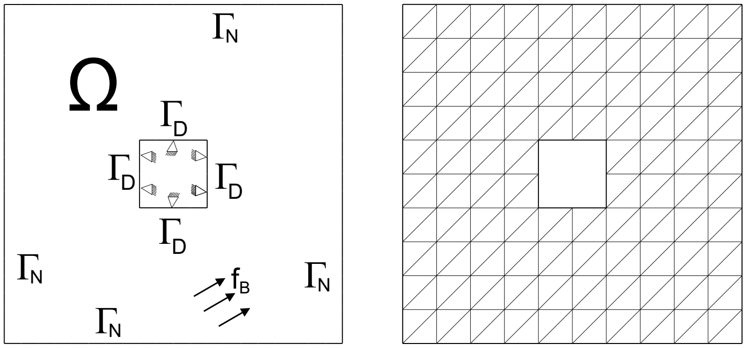

Figure 2.1: Example 1: Physical setting and mesh for n=10.

The numerical solution corresponding to $n = 160$ subdivisions on each outer side of the square (see the right-hand side of Figure 2.1 for the case $n = 10$), and $k = 0.0005$

$n \downarrow k \rightarrow$	0.05	0.02	0.01	0.005	0.002	0.001
5	7.385e-2	7.000e-2	6.875e-2	6.811e-2	6.774e-2	6.761e-2
10	4.842e-2	4.196e-2	3.979e-2	3.872e-2	3.809e-2	3.788e-2
20	3.441e-2	2.632e-2	2.367e-2	2.238e-2	2.163e-2	2.138e-2
40	2.681e-2	1.736e-2	1.420e-2	1.264e-2	1.172e-2	1.143e-2
80	2.259e-2	1.205e-2	8.550e-3	6.849e-3	5.876e-3	5.573e-3

Table 2.1: Example 1: Numerical errors for some n and k.

has been considered as the "exact" solution in order to compute the numerical errors given by

$$\max_{0\le n\le N}\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\zeta_n-\zeta_n^{hk}\|_Y\}.$$

The following data have been employed in the simulations:

$$\begin{aligned}
&T=1\,s,\quad \boldsymbol{f}_B(\boldsymbol{x},t)=(50,25)t\ \ N/m^3,\quad \boldsymbol{f}_N=\boldsymbol{0}\ \ N/m^2,\\
&E=10000\ \ N/m^2,\quad r=0.33,\\
&\lambda_D=0.1,\quad \lambda_u^+=1000,\quad \lambda_u^-=100,\quad \lambda_w=0,\\
&\boldsymbol{u}_0=\boldsymbol{0}\ \ m,\quad \zeta_0(\boldsymbol{x})=1\quad \forall \boldsymbol{x}\in\Omega.
\end{aligned}$$

In Table 2.1 the numerical errors obtained for some n and k are shown. The evolution of the error with respect to the parameter $k+h$ is plotted in Figure 2.2 (here, $h=\frac{5\sqrt{2}}{n}$). As we can see, the linear convergence of the algorithm is clearly observed.

Second example: damage due to tension or compression

In this second example, we considered the problem depicted in Figure 2.3 where only tension or only compression have been supposed to contribute to damage. In this case only surface tractions have been considered, tangential to the external boundary and in clockwise direction.

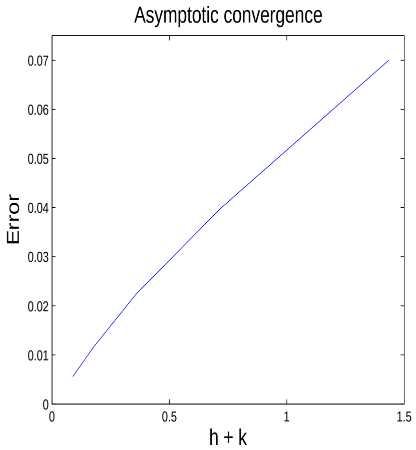

Figure 2.2: Example 1: Evolution of the numerical error with respect to $k + h$.

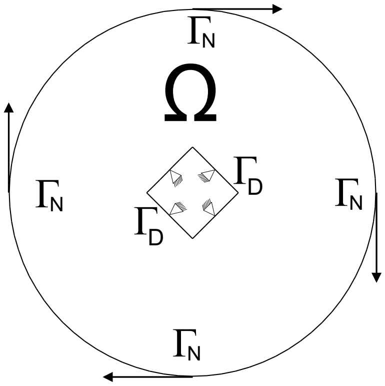

Figure 2.3: Example 2: Damage due only to tension or compression.

The data used in the simulations are the following:

$$
\begin{aligned}
&\Omega = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \quad \text{such that} \quad |\boldsymbol{x}| < 4, \quad |x_1| + |x_2| > 1\}\\
&T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N(x, y, t) = (5y, -5x)t\,N/m^2,\\
&E = 10000\,N/m^2, \quad r = 0.33, \quad \lambda_D = 0.1, \quad \lambda_w = 0,\\
&\boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega.
\end{aligned}
$$

First, only tension is assumed to contribute to the damage and so we employed the values $\lambda_u^+ = 1000$ and $\lambda_u^- = 0$ for the rates of tension and compression, respectively. In Figure 2.4 both the von Mises stress norm and the damage field are plotted at final time $t = 1\,s$ and over the deformed configuration.

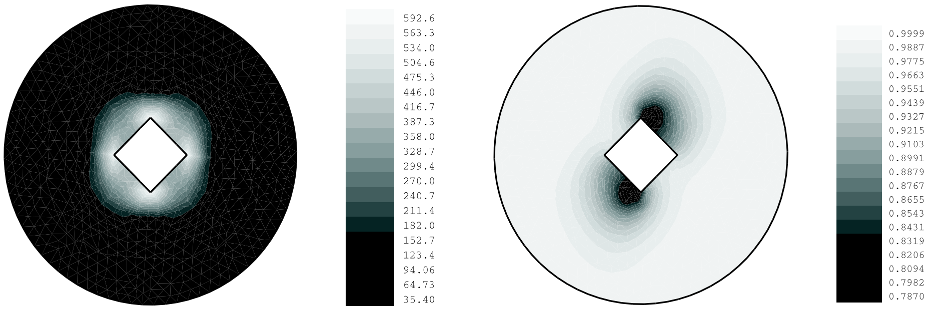

Figure 2.4: Example 2: von Mises stress norm and damage field at final time for a pure tension contribution.

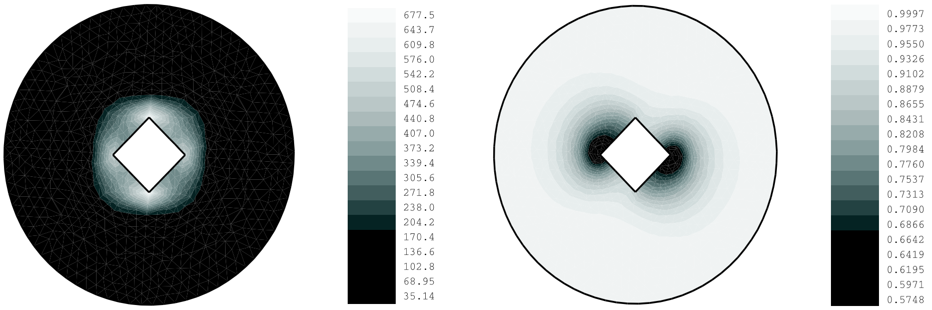

Figure 2.5: Example 2: von Mises stress norm and damage field at final time for a pure compression contribution.

Secondly, only compression is assumed now to contribute to the damage and so we employed the values $\lambda_u^+ = 0$ and $\lambda_u^- = 1000$. In Figure 2.5 both the von Mises stress norm and the damage field are plotted at final time $t = 1\,s$ and over the deformed configuration.

If we compare Figures 2.4 and 2.5, we observe that there are no qualitative changes for the stresses, while important differences can be appreciated in the results concerning the damage distribution, which could lead in important differences on the mechanical behaviour.

Third example: A masonry arch bridge

In this last example of the section we consider a body whose geometry represents an arch bridge (see Figure 2.6). There is an extensive literature concerning the topic of bridges collapse mechanisms (see e.g. [44, 62]). In particular, in [28] the study of load carrying capabilities of masonry arch bridges was performed. The aim of this example is to study the collapse mechanism of an stone arch bridge by comparison with the method proposed in [28]. The results presented below will appear in [10].

A stone arch bridge consists of stone blocks and the mortar joints. Blocks have high strength in compression and low strength in tension while mortar has generally low strength. In [44], Heyman indroduced the collapse mechanism method while several computational methods for the evaluation of the limit load of a masonry arch followed. In this example two models are used for the calculation of the failure load and mechanism. Their predictions on a bridge with a concentrated force applied at the quarter span of the arch, which is probably the worst position of the live load (see [44]), are compared.

This example consists on the observation of the collapse mechanism and the estimation of the ultimate failure load of a stone arch bridge, and the comparison with the result obtained in [28] with a multi - part elastic model with unilateral contact frictional interfaces. According to the contact model, contact interfaces simulating potential cracks are considered and their opening or sliding indicates crack initiation.

In [28] the author consider the following modelling procedure. The bridge is divided by a number of interfaces perpendicular to the center line of the arch (see Figure 2.6). Each of those interfaces simulates the interface between two stone blocks. Unilateral contact conditions have been considered to govern the behaviour over those interfaces, which implies that in the normal direction to the boundary no tension forces can be transmitted. The behaviour in the tangential direction corresponds to the friction effects, and so, sliding may or may not occur, depending on the particular situation.

The calculation of the ultimate load is based on the explotation of the solvability conditions for linear complementarity problems and variational inequalities. Each block of the bridge has its boundary divided into two parts, Γ_C on the interfaces and Γ_N on the rest of the boundary. As $\Gamma_D = \emptyset$, every block between two interfaces may develop rigid body displacements, which must be compatible with the constraints

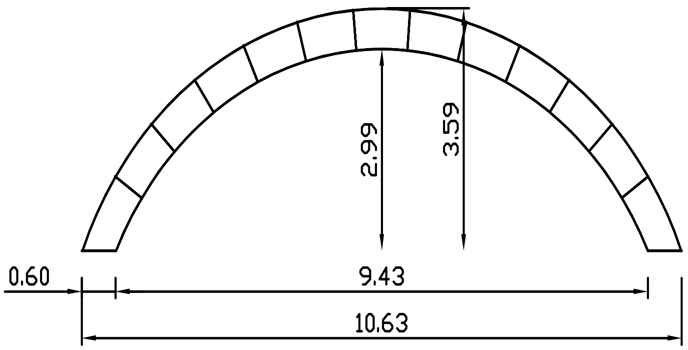

Figure 2.6: Geometry of the bridge

of the unilateral contact problem, otherwise no solution exists. In other words an equilibrium configuration may or may not exist, depending on the geometry of the structure and the direction of the applied loading. Collapse occurs at the load step of the load incrementation where solvability is first violated.

In our damage model, collapse occurs at the loading step for which a whole cross-section of the bridge reaches the lower bound for the damage ζ_*.

A plane stress model of stone arch bridge without fill is considered, as shown in Figure 2.6. Loading includes self - weight and a concentrated load at the quarter span of the bridge. Young's modulus is 5GPa, Poisson' s ratio 0.49 and density 2200Kg/m^3.

For the contact analysis, the finite element model used in [28] consists of 3036 quadrilateral, four - node, bilinear finite elements with two translational degrees of freedom per node. A typical finite element length is 0.05m. A load increment equal to 0.02KN is chosen in the iterative incremental procedure. The friction coefficient is chosen equal to 0.6. This value is high enough to prevent sliding. Therefore a direct comparison of the results with the ones provided by the damage model is possible. In addition, a relatively large number of interfaces, equal to 40, has been considered for the arch as the exact number of interfaces along the bridge's geometry tends to be meaningless in case many interfaces are used (see [28]).

The damage model consists of 8400 triangular Lagrange finite elements with two translational degrees of freedom per node, and a total of 4515 nodes. Furthermore, the following data for the damage model have been used: $k = 0.001$s (time step size), $\kappa = 10^{-2}$, $\zeta_* = 0.01$, $\lambda_D = 2 \times 10^{-3}$, $\lambda_u = 5 \times 10^9$, $\lambda_w = 7 \times 10^{-4}$.

The failure load for the contact model of 40 interfaces is 87.14KN. This limit load compares well with the one obtained by the damage model, which is equal to 90 KN. In Figure 2.7 the force - displacement diagrams of the two methods are compared. Moreover, both method predict the same, well-known, mechanism of collapse. A four hinges mechanism arise in case of a quarter span load. The same conclusion arises from both experimental research ([62]) and the classical collapse mechanism method of Heyman [44]. The above results are schematically shown in Figures 2.8 and 2.9. In 2.9 also the damage field over the deformed configuration (deformations multiplied by a factor 50) is plotted.

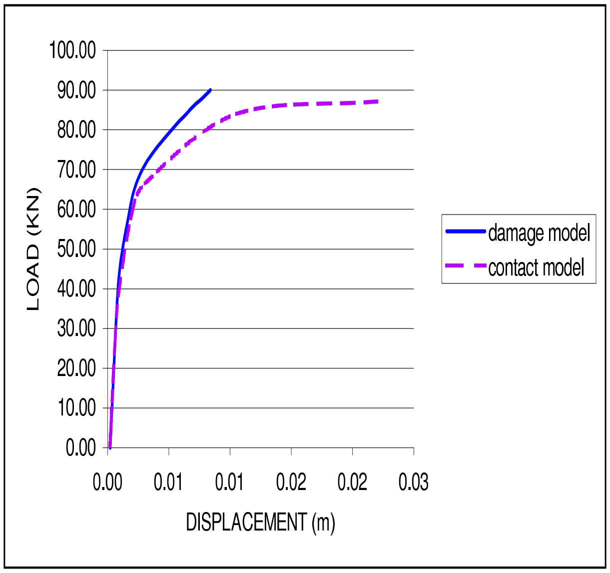

Figure 2.7: Force-displacement diagram

The collapse mechanism of both contact and damage models at failure coincides with the one predicted by the classical collapse mechanism method of Heyman. The failure load obtained by the two methods is almost identical as well. However, there is a divergence in the force - displacement diagrams where the damage model has an ascending branch at failure. Further investigation, including for example parameter identification, will possibly lead to better comparison results.

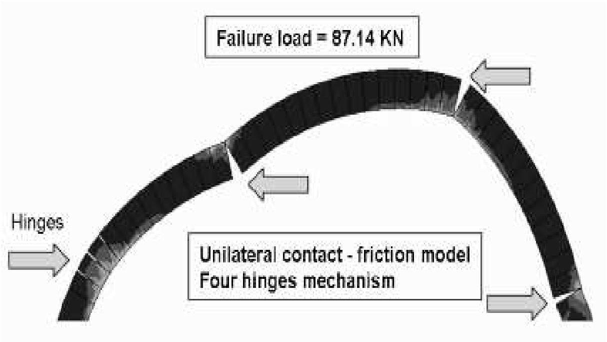

Figure 2.8: Failure of the bridge

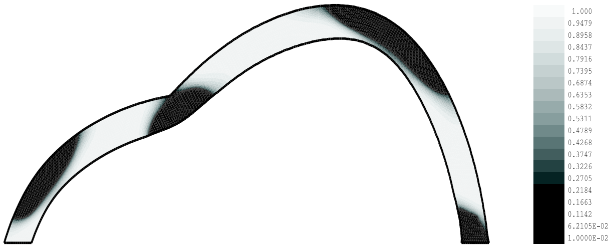

Figure 2.9: Failure of the bridge

2.2 A frictional contact problem in viscoelasticity

In this section a bilateral frictional contact problem is studied. We obtain a variational formulation of the mechanical problem and state the existence of a unique weak solution. The variational inequality for the discrete problem is numerically analyzed and solved. The behaviour of the viscoelastic material is given by the *Kelvin-Voigt* constitutive law (1.1), and the contact phenomenom is modeled by Tresca's law (see section 1.3).

The unilateral version of this problem, where the normal contact is modelled with a normal compliance condition, was studied in [43] and our analysis is based on the results obtained there. The results of this section are published in [19].

The mathematical description of the problem is as follows (see Chapter 1 for details concerning the notation).

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, and a damage field* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *such that,*

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B = \boldsymbol{0} \quad \text{in} \ \ \Omega \times (0,T), \tag{2.37}$$

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T), \tag{2.38}$$

$$\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T), \tag{2.39}$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{2.40}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{2.41}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on} \ \ \Gamma_N \times (0,T), \tag{2.42}$$

$$\left.\begin{array}{l} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \text{ there exists } \lambda > 0 \\ \qquad\qquad \text{such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau \end{array}\right\} \ \text{on} \quad \Gamma_C \times (0,T), \tag{2.43}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in} \quad \Omega. \tag{2.44}$$

We notice that in (2.39) the interval $[0,1]$ was replaced by $[\zeta_*, 1]$ since when the damage field is less than this lower value $\zeta_* > 0$, then the microcraks are too dense and modelling the material as viscoelastic ceases to make sense.

Let us consider the set of admissible displacements as

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \, ; \, \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \quad v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_C\},$$

with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,$$

and also suppose that

$$\boldsymbol{f}_B \in \mathcal{C}([0,T]; H), \quad \boldsymbol{f}_N \in \mathcal{C}([0,T]; [L^2(\Gamma_N)]^d). \tag{2.45}$$

Let us define the element $\boldsymbol{f}(t) \in V$ as

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_B(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d},$$

let the friction bound $g : \Gamma_C \to [0, +\infty)$ be given such that

$$g \in L^\infty(\Gamma_C), \tag{2.46}$$

and the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \zeta_0 \in \mathcal{K}, \tag{2.47}$$

where $\mathcal{K}$ represents the set of admissible damage functions

$$\mathcal{K} = \{\zeta \in B\,;\, \zeta_* \leq \zeta(\boldsymbol{x}) \leq 1 \quad \text{a.e.} \quad \boldsymbol{x} \in \Omega\}. \tag{2.48}$$

We now provide a variational formulation of problem (2.37)–(2.44), which is needed for the numerical discretization of the problem. In this way let us assume that the mechanical problem has a solution $(\boldsymbol{u}, \zeta)$, smooth enough so that the calculations below are meaningful and let us make the following assumptions on the data.

The *viscosity operator* $\mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}) \in \Omega \times \mathbb{S}^d \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\left.\begin{array}{l}
\text{(a) There exists } C_{\mathcal{A}}, > 0 \text{ such that} \\
\quad |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}_2)| \leq C_{\mathcal{A}}\, |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| \\
\quad \forall \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\quad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq m_{\mathcal{A}}\, |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2|^2 \\
\quad \forall\, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \in \Omega \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\tau}) \text{ is Lebesgue measurable on } \Omega, \\
\quad \forall \boldsymbol{\tau} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \in \Omega \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \quad \text{belongs to} \quad Q.
\end{array}\right\} \tag{2.49}$$

The *elasticity operator* $\mathcal{E} : (\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \in \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \in \mathbb{S}^d$ satisfies:

$$\left.\begin{array}{l}
\text{(a) There exists } C_{\mathcal{E}} > 0 \text{ such that} \\
\quad |\mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}_1, \zeta_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}_2, \zeta_2)| \leq C_{\mathcal{E}} \left(|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| + |\zeta_1 - \zeta_2|\right) \\
\quad \forall\, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \in \Omega \mapsto \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \text{ is Lebesgue measurable on } \Omega \\
\quad \forall \boldsymbol{\tau} \in \mathbb{S}^d,\ \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \in \Omega \mapsto \mathcal{E}(\boldsymbol{x}, \boldsymbol{0}, 0) \quad \text{belongs to} \quad Q.
\end{array}\right\} \tag{2.50}$$

The *damage source founction* $\phi : (\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \in \Omega \times \mathbb{S}^d \times \mathbb{R} \to \phi(\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \in \mathbb{R}$ satisfies:

$$\left.\begin{array}{l}
\text{(a) There exists } C_\phi > 0 \ \text{ such that} \\
\qquad |\phi(\boldsymbol{x}, \boldsymbol{\tau}_1, \zeta_1) - \phi(\boldsymbol{x}, \boldsymbol{\tau}_2, \zeta_2)| \le C_\phi\,(|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| + |\zeta_1 - \zeta_2|) \\
\qquad \forall\, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d, \ \ \forall \zeta_1, \zeta_2 \in \mathbb{R}, \qquad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \in \overline{\Omega} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\tau}, \zeta) \ \text{ is Lebesgue measurable} \\
\qquad \text{on } \Omega, \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \ \ \forall \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \in \Omega \mapsto \phi(\boldsymbol{x}, \boldsymbol{0}, 0) \quad \text{belongs to} \quad Y.
\end{array}\right\} \tag{2.51}$$

Let $\boldsymbol{v} \in V$ be arbitrary. We multiply equation (2.37) by $(\boldsymbol{v} - \boldsymbol{u})$, integrate over Ω and apply Green's formula to obtain

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q = \int_\Omega \boldsymbol{f}_B \cdot (\boldsymbol{v} - \boldsymbol{u}) d\boldsymbol{x} + \int_\Gamma \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) da.$$

Using boundary conditions (2.41) and (2.42) we deduce the equality

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V + \int_{\Gamma_C} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) da, \tag{2.52}$$

and over the contact boundary Γ_C,

$$\boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) = \sigma_\nu (v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau).$$

Moreover, let $j : V \to \mathbb{R}$ be the friction functional

$$j(\boldsymbol{v}) = \int_{\Gamma_C} g |\boldsymbol{v}_\tau|\, dS \qquad \forall \boldsymbol{v} \in V. \tag{2.53}$$

Now, from (2.43) we find that

$$\boldsymbol{\sigma}_\tau(t) \cdot \dot{\boldsymbol{u}}_\tau = -g |\dot{\boldsymbol{u}}_\tau(t)|, \quad \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau \ge -g |\boldsymbol{v}_\tau|. \tag{2.54}$$

Putting (2.54) into (2.52) and taking into account (2.53) we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}(t)) \ge (\boldsymbol{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{w} \in V. \tag{2.55}$$

Expression (2.38) is equivalent to the following

$$\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)) - \dot{\zeta}(t) + \kappa \triangle \zeta(t) \in \partial I_{[\zeta_*, 1]}(\zeta(t)), \quad \forall t \in [0, T], \tag{2.56}$$

so, aplying the definition of subdifferential (1.9)to the functional $I_{[\zeta_*,1]}$ we obtain (see, for example, [7, 74] for further details),

$$(\xi-\zeta(t),\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t))-\dot{\zeta}(t)+\kappa\triangle\zeta(t))_Y\leq 0,\quad \forall\xi\in\mathcal{K}.$$

Then, applying the Green's formula to the term $(\xi-\zeta(t),\kappa\triangle\zeta(t))_Y$, taking into account the condition (2.42) and the definition (2.17), the following evolutionary variational inequality is obtained

$$(\dot{\zeta}(t),\xi-\zeta(t))_Y+a(\zeta(t),\xi-\zeta(t))\geq(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t)),\ \xi-\zeta(t))_Y\qquad\forall\xi\in\mathcal{K}.$$

This expression, together with (2.38) and (2.55), constitutes the variational formulation of the mechanical problem **P**.

Problem VP. *Find a displacement field* $\boldsymbol{u}:[0,T]\to V$, *a stress field* $\boldsymbol{\sigma}:[0,T]\to Q$, *and a damage field* $\zeta:[0,T]\to\mathcal{K}$, *such that* $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\zeta(0)=\zeta_0$ *and for a.e.* $t\in[0,T]$,

$$\boldsymbol{\sigma}(t)=\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))+\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t)),\tag{2.57}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}-\dot{\boldsymbol{u}}(t)))_Q+j(\boldsymbol{w})-j(\dot{\boldsymbol{u}}(t))\geq(\boldsymbol{f}(t),\boldsymbol{w}-\dot{\boldsymbol{u}}(t))_V\quad\forall\boldsymbol{w}\in V,\tag{2.58}$$

$$\begin{aligned}&(\dot{\zeta}(t),\xi-\zeta(t))_Y+a(\zeta(t),\xi-\zeta(t))\\&\qquad\geq(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t)),\ \xi-\zeta(t))_Y\quad\forall\xi\in\mathcal{K}.\end{aligned}\tag{2.59}$$

2.2.1 An existence and uniqueness result

The existence of a unique solution to Problem **VP** and its regularity are summarized in the following theorem.

Theorem 2.4. *Assume that* (2.45)–(2.51) *hold. If the initial conditions are chosen in such a way that* $\boldsymbol{u}_0\in V$ *and* $\zeta_0\in\mathcal{K}$, *then Problem* **VP** *has a unique solution with the following regularity*

$$\boldsymbol{u}\in\mathcal{C}^1([0,T];V),\qquad\zeta\in H^1(0,T;Y)\cap L^2(0,T;B).\tag{2.60}$$

Proof

The proof of this theorem is based on classical results for elliptic and parabolic variational inequalities and fixed-point arguments, details being similar to those used in [43] and [71]. Therefore, we sketch below the main steps of its proof.

First, for any $\boldsymbol{\eta} \in \mathcal{C}([0,T];Q)$ and $\theta \in \mathcal{C}([0,T];Y)$, we consider the following intermediate problems.

Problem$\mathbf{VP^1_\eta}$. *Find a displacement field* $\boldsymbol{u}_\eta : [0,T] \to V$ *such that* $\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0$, *and for a.e.* $t \in [0,T]$ *and* $\boldsymbol{w} \in V$,

$$(\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t))) + \boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{w} - \dot{\boldsymbol{u}}_\eta(t)))_Q + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}_\eta(t)) \geq (\boldsymbol{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_\eta(t))_V.$$

Problem$\mathbf{VP^2_\theta}$. *Find a damage field* $\zeta_\theta : [0,T] \to \mathcal{K}$ *such that* $\zeta_\theta(0) = \zeta_0$ *and for a.e.* $t \in (0,T)$ *and* $\xi \in \mathcal{K}$,

$$(\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t))_Y + a(\zeta_\theta(t), \xi - \zeta_\theta(t)) \geq (\theta(t), \xi - \zeta_\theta(t))_Y.$$

Secondly, from [43] (see Proposition 4.1, p. 385), we obtain that Problem $\mathbf{VP^1_\eta}$ has a unique solution $\boldsymbol{u}_\eta \in \mathcal{C}^1([0,T];V)$. Moreover, using standard results for parabolic variational inequalities (see [4]), it follows that Problem $\mathbf{VP^2_\theta}$ has a unique solution $\zeta_\theta \in H^1(0,T;Y) \cap L^2(0,T;B)$.

Let $\Lambda : \mathcal{C}([0,T];Q \times Y) \to \mathcal{C}([0,T];Q \times Y)$ be defined by

$$\Lambda(\boldsymbol{\eta}, \theta) = (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta)), \zeta_\theta), \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta), \zeta_\theta)) \quad \forall (\boldsymbol{\eta}, \theta) \in \mathcal{C}([0,T];Q \times Y).$$

Using properties (2.50) and (2.51), it follows that the operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in \mathcal{C}([0,T];Q \times Y)$.

Finally, let $\boldsymbol{u}_{\eta^*}$ and ζ_{θ^*} be the solutions to the variational problems $\mathbf{VP^1_\eta}$ and $\mathbf{VP^2_\theta}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and $\theta = \theta^*$, respectively. Then, $\{\boldsymbol{u}_{\eta^*}, \zeta_{\theta^*}\}$ is the unique solution to Problem **VP** which satisfies (2.60). ■

2.2.2 Numerical analysis

Now we will analyze a numerical scheme to solve numerically Problem **VP** and obtain error estimates on the approximate solutions.

First of all, for convenience, we rewrite the variational problem **VP** in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ given by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)ds + \boldsymbol{u}_0. \tag{2.61}$$

Then, Problem **VP** can be written in the following equivalent form. **Problem VP^{vel}.** *Find a velocity field $\boldsymbol{v} : [0,T] \to V$, and a damage field $\zeta : [0,T] \to \mathcal{K}$ such that $\zeta(0) = \zeta_0$ and for a.e. $t \in [0,T]$, $\boldsymbol{w} \in V$ and $\xi \in \mathcal{K}$,*

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}(t))), \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}(t)))_Q + j(\boldsymbol{w}) - j(\boldsymbol{v}(t)) &\geq (\boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}(t))_V \\ &-(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}(t)))_Q, \end{aligned} \tag{2.62}$$

$$(\dot{\zeta}(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t)) \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_Y, \tag{2.63}$$

where $\boldsymbol{u}(t)$ is defined by (2.61).

The discretization of (2.62) and (2.63) will be done in two steps. First, we consider arbitrary general finite dimensional spaces $V^h \subset V$ and $B^h \subset B$ in order to approximate the spaces V and B, respectively. Let $\mathcal{K}^h = \mathcal{K} \cap B^h$. Here, $h > 0$ denotes the spatial discretization parameter.

Remark 2.1. *In the numerical simulations described in the following subsection, V^h and B^h consist of continuous and piecewise affine functions; that is,*

$$\begin{aligned} V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{w}^h_{|_T} \in [P_1(T)]^d \quad \forall T \in \mathcal{T}^h, \\ \boldsymbol{w}^h = \boldsymbol{0} \;\; on \;\; \Gamma_D, \quad w^h_{\nu|_C} = 0 \quad \forall C \in \theta^h\}, \end{aligned} \tag{2.64}$$

$$B^h = \{\xi^h \in C(\overline{\Omega}) \; ; \; \xi^h_{|_T} \in P_1(T), \quad \forall T \in \mathcal{T}^h\}, \tag{2.65}$$

where Ω is assumed to be a polygonal domain, $\mathcal{T}^h$ denotes a finite element triangulation of $\overline{\Omega}$, θ^h is the $(d-1)$-dimensional triangulation induced by $\mathcal{T}^h$ on Γ_C and $P_1(T)$ represents the space of polynomial functions of global degree less or equal to 1 in T.

To discretize the time derivatives, we consider a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$ and let k be the time step size, $k = T/N$. Let also $\boldsymbol{u}_0^h \in V^h$ and $\zeta_0^h \in \mathcal{K}^h$ be suitable approximations of the initial conditions $\boldsymbol{u}_0$ and ζ_0, respectively.

Then, the fully discrete approximation of Problem $\mathbf{VP}^{vel}$ is as follows.

Problem $\mathbf{VP}^{hk}$. *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete damage field* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$, *such that* $\zeta_0^{hk} = \zeta_0^h$ *and for all* $\xi^h \in \mathcal{K}^h$, $\boldsymbol{w}^h \in V^h$ *and* $n = 1, 2, \ldots, N$,

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y, \tag{2.66}$$

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + j(\boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk})& \\ \geq (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk})_V,&\end{aligned} \tag{2.67}$$

where the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *is defined by*

$$\boldsymbol{u}_n^{hk} = \sum_{j=1}^n k\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h = \boldsymbol{u}_{n-1}^{hk} + k\boldsymbol{v}_n^{hk} \quad \text{for } n = 1, 2, \ldots, N, \quad \boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h. \tag{2.68}$$

Using classical results on variational inequalities (see [41]) we deduce the existence and uniqueness of the solution to Problem $\mathbf{VP}^{hk}$, which we state as follows.

Theorem 2.5. *Assume that the conditions of Theorem* 2.4 *hold. Then, there exists a unique solution to Problem* $\mathbf{VP}^{hk}$ *such that* $\boldsymbol{u}^{hk} \subset V^h$ *and* $\zeta^{hk} \subset \mathcal{K}^h$.

Our interest lies now in estimating the errors $\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V$ and $\|\zeta_n - \zeta_n^{hk}\|_Y$. The following additional regularity conditions on the solution to Problem **VP** are assumed,

$$\zeta \in \mathcal{C}([0,T]; H^2(\Omega)) \cap H^2(0,T;Y), \tag{2.69}$$

which also implies that $\zeta_0 \in H^2(\Omega)$. Moreover, we notice that, under these assumptions, variational inequalities (2.62) and (2.63) are satisfied for all $t \in [0,T]$.

We are now ready to do the error estimation for the damage. We write expression (2.63) at time $t = t_n$ with $\xi = \zeta_n^{hk}$, and add it to expression (2.66) with $\xi^h = \xi_n^h \in \mathcal{K}^h$.

Adding and substracting some terms in order to symmetrize the expressions, we obtain

$$\begin{aligned}
&(\delta\zeta_n-\delta\zeta_n^{hk},\zeta_n-\zeta_n^{hk})_Y+a(\zeta_n-\zeta_n^{hk},\zeta_n-\zeta_n^{hk})\\
&\quad\leq(\delta\zeta_n-\dot{\zeta}_n,\zeta_n-\zeta_n^{hk})_Y+(\delta\zeta_n-\delta\zeta_n^{hk},\zeta_n-\xi_n^h)_Y\\
&\quad+a(\zeta_n-\zeta_n^{hk},\zeta_n-\xi_n^h)-(\delta\zeta_n,\zeta_n-\xi_n^h)_Y-a(\zeta_n,\zeta_n-\xi_n^h)\\
&\quad+(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n),\zeta_n-\xi_n^h)_Y+(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n)-\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\xi_n^h-\zeta_n^{hk})_Y. \qquad (2.70)
\end{aligned}$$

The left-hand side of (2.70) may be bounded as

$$(\delta\zeta_n-\delta\zeta_n^{hk},\zeta_n-\zeta_n^{hk})_Y\geq\frac{1}{2k}(\|\zeta_n-\zeta_n^{hk}\|_Y^2-\|\zeta_{n-1}-\zeta_{n-1}^{hk}\|_Y^2). \qquad (2.71)$$

Using (2.71) and (2.70) and taking the inequality for $j=1,\ldots,n$, by adding terms we get

$$\begin{aligned}
&\frac{1}{2}\|\zeta_n-\zeta_n^{hk}\|_Y^2-\frac{1}{2}\|\zeta_0-\zeta_0^{hk}\|_Y^2+k\sum_{j=1}^{n}a(\zeta_j-\zeta_j^{hk},\zeta_j-\zeta_j^{hk})\\
&\qquad\leq k\sum_{j=1}^{n}(\delta\zeta_j-\dot{\zeta}_j,\zeta_j-\zeta_j^{hk})_Y+(\zeta_n-\zeta_n^{hk},\zeta_n-\xi_n^h)_Y\\
&\qquad-\sum_{j=1}^{n-1}(\zeta_j-\zeta_j^{hk},(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h))_Y-(\zeta_0-\zeta_0^{hk},\zeta_1-\xi_1^h)_Y\\
&\qquad+k\sum_{j=1}^{n}a(\zeta_j-\zeta_j^{hk},\zeta_j-\xi_j^h)\\
&\qquad+k\sum_{j=1}^{n}[(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j),\zeta_j-\xi_j^h)_Y-(\delta\zeta_j,\zeta_j-\xi_j^h)_Y-a(\zeta_j,\zeta_j-\xi_j^h)]\\
&\qquad+k\sum_{j=1}^{n}(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}),\zeta_{j-1}^{hk}),\xi_j^h-\zeta_j^{hk})_Y. \qquad (2.72)
\end{aligned}$$

Then, applying Cauchy's formula (2.22), we obtain the inequality

$$\begin{aligned}
&\|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \\
&\quad \leq c\Big\{\|\zeta_0 - \zeta_0^h\|_Y^2 + k\sum_{j=1}^{n} \|\delta\zeta_j - \dot{\zeta}_j\|_Y \|\zeta_j - \zeta_j^{hk}\|_Y \\
&\qquad + \|\zeta_n - \zeta_n^{hk}\|_Y \|\zeta_n - \xi_n^h\|_Y + \|\zeta_0 - \zeta_0^{hk}\|_Y \|\zeta_1 - \xi_1^h\|_Y \\
&\qquad + \sum_{j=1}^{n-1} \|\zeta_j - \zeta_j^{hk}\|_Y \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y \\
&\qquad + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H \|\nabla(\zeta_j - \xi_j^h)\|_H \\
&\qquad + k\sum_{j=1}^{n} \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \|\zeta_j - \xi_j^h\|_Y \\
&\qquad + k\sum_{j=1}^{n} (\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V + \|\zeta_j - \zeta_{j-1}^{hk}\|_Y)(\|\zeta_j - \zeta_j^{hk}\|_Y + \|\zeta_j - \xi_j^h\|_Y)\Big\}. \qquad (2.73)
\end{aligned}$$

Now, from (2.69) we have that $\dot{\zeta} \in \mathcal{C}([0,T];Y)$, and therefore,

$$\|\zeta_j - \zeta_{j-1}^{hk}\|_Y^2 \leq 2(\|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + k^2\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}^2). \qquad (2.74)$$

Taking into account (2.74) and inequality (2.22), we obtain from (2.73),

$$\begin{aligned}
&\|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \\
&\quad \leq c\Big\{\|\zeta_0 - \zeta_0^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 + k\sum_{j=1}^{n} \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 \\
&\qquad + k\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 + k^2\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}^2 + k\sum_{j=1}^{n} \|\zeta_j - \xi_j^h\|_B^2 + \|\zeta_n - \xi_n^h\|_Y^2 \\
&\qquad + \frac{1}{k}\sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + k\sum_{j=1}^{n} \|\delta\zeta_j - \dot{\zeta}_j\|_Y^2 \\
&\qquad + k\sum_{j=1}^{n} \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y\Big\}. \qquad (2.75)
\end{aligned}$$

Now, we will obtain an error estimates for the velocity field. First, we consider variational inequality (2.62) at time $t = t_n$ and for $\boldsymbol{w} = \boldsymbol{v}_n^{hk}$,

$$(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \leq j(\boldsymbol{v}_n^{hk}) - j(\boldsymbol{v}_n) + (\boldsymbol{f}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_V \\ -(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q. \tag{2.76}$$

Given $\{\boldsymbol{w}_j^h\}_{j=1}^n \subset V^h$, taking $\boldsymbol{w}^h = \boldsymbol{w}_n^h$ in (2.67) it follows that

$$-(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \leq j(\boldsymbol{w}_n^h) - j(\boldsymbol{v}_n^{hk}) - (\boldsymbol{f}_n, \boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk})_V \\ -(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}))_Q, \tag{2.77}$$

for all $\boldsymbol{w}_n^h \in V^h$. Substracting (2.76) and (2.77), we obtain for all $\boldsymbol{w}_n^h \in V^h$,

$$(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \leq j(\boldsymbol{w}_n^h) - j(\boldsymbol{v}_n) + (\boldsymbol{f}_n, \boldsymbol{v}_n - \boldsymbol{w}_n^h)_V \\ -(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}))_Q \\ -(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h))_Q. \tag{2.78}$$

From (2.50) it follows that

$$(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ = (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ +(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_{n-1}^{hk}) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ \leq c(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y)\|\boldsymbol{w}\|_V \quad \forall \boldsymbol{w} \in V.$$

Applying the Cauchy's inequality and using properties (2.49) and (2.46), from (2.78) we have,

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq c\Big(\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y^2 \\ +\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V\Big) \quad \forall \boldsymbol{w}_n^h \in V^h.$$

Keeping in mind that $nk \leq T$, $\forall n = 0, 1, \ldots, N$, we find that

$$\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \leq 2\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + 2\|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \\ = 2\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + 2\Big\|\int_0^{t_{j-1}} \boldsymbol{v}(s)ds + \boldsymbol{u}_0 - \sum_{l=1}^{j-1} k\boldsymbol{v}_j^{hk} - \boldsymbol{u}_0^h\Big\|_V^2 \\ \leq 2\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + 4\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + 8\Big\|\int_0^{t_{j-1}} \boldsymbol{v}(s)ds - \sum_{l=1}^{j-1} k\boldsymbol{v}_l\Big\|_V^2 \\ +8\|\sum_{l=1}^{j-1} k\boldsymbol{v}_l - \sum_{l=1}^{j-1} k\boldsymbol{v}_l^{hk}\|_V^2, \tag{2.79}$$

and we define by I_n the integration error

$$I_n = \Big\| \int_0^{t_{n-1}} \boldsymbol{v}(s)ds - \sum_{j=1}^{n-1} k\boldsymbol{v}_j \Big\|_V .$$

Then, it follows that

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq c\Big(I_n^2 + \sum_{j=1}^{n-1} k\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V^2 + \|\zeta_n - \zeta_{n-1}\|_Y^2$$

$$+\|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 \Big). \quad (2.80)$$

Now we are going to use use Lemma 1.1. In this way, we rewrite estimate (2.75) in the form

$$\|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^{n} \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \leq ck\sum_{j=1}^{n} \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2$$

$$+ck\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 + cA_n, \quad (2.81)$$

and estimate (2.80) as follows

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq c\big(k\sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V^2 + \|\zeta_n - \zeta_{n-1}\|_Y^2$$

$$+\|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2 + B_n\Big), \quad (2.82)$$

with A_n and B_n being the corresponding groups of terms,

$$A_n = \|\zeta_0 - \zeta_0^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 + k\sum_{j=1}^{n} \|\zeta_j - \xi_j^h\|_B^2$$

$$+\frac{1}{k}\sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + k^2\|\zeta\|_{H^2(0,T;Y)}^2 + \|\zeta_n - \xi_n^h\|_Y^2$$

$$+k\sum_{j=1}^{n} \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \|\zeta_j - \xi_j^h\|_Y, \quad (2.83)$$

$$B_n = \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + I_n^2 \quad (2.84)$$

From (2.81) we have,

$$\|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2 \leq ck\sum_{j=1}^{n-1} \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2$$

$$+ck\sum_{j=1}^{n-1} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 + cA_{n-1}. \quad (2.85)$$

Now let e_n be the numerical errors given by

$$e_n = \|\zeta_n - \zeta_n^{hk}\|_Y^2 + ck\sum_{j=1}^{n} \|(\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2.$$

Combining (2.81), (2.82) and (2.85) we have,

$$\begin{aligned} e_n \le{}& ck\sum_{j=1}^{n} \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + ck\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \\ &+ck\sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + c\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V^2 + c\|\zeta_n - \zeta_{n-1}\|_Y^2 \\ &+cA_{n-1} + cA_n + cB_n. \end{aligned} \tag{2.86}$$

Using (2.79) we obtain,

$$\begin{aligned} k\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \le{}& ck\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 \\ &+ck\sum_{j=1}^{n} \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + ck\sum_{j=1}^{n} I_j^2 + ck^2\sum_{j=1}^{n}(\sum_{l=1}^{j-1} \|\boldsymbol{v}_l - \boldsymbol{v}_l^{hk}\|_V^2) \\ \le{}& ck\sum_{j=1}^{n} \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + ck\sum_{j=1}^{n} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 \\ &+c\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + c\max_{1\le n\le N} I_n^2. \end{aligned} \tag{2.87}$$

From assumptions (2.60) and (2.69) we have

$$\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V \le k\|\dot{\boldsymbol{u}}\|_{C([0,T];V)} = k\|\boldsymbol{v}\|_{C([0,T];V)}, \tag{2.88}$$

$$\|\zeta_n - \zeta_{n-1}\|_Y \le k\|\dot{\zeta}\|_{C([0,T];Y)} \tag{2.89}$$

Using (2.87), (2.88) and (2.89) in (2.86) we get,

$$\begin{aligned} e_n \le{}& ck\sum_{j=1}^{n} \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + ck\sum_{j=1}^{n} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 \\ &+ck^2\|\boldsymbol{v}\|_{C([0,T];V)}^2 + ck^2\|\dot{\zeta}\|_{C([0,T];Y)}^2 + cA_{n-1} + cA_n + cB_n \\ &+c\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + c\max_{1\le n\le N} I_n^2. \end{aligned}$$

Now, we can apply a discrete version of Gronwall's inequality (see Lemma 1.1), so the following error estimates are obtained.

Theorem 2.6. *Let the assumptions of Theorem 2.4 still hold. Let us assume the additional regularity condition* (2.69) *for the damage field. Then, the following error estimate is obtained for all* $\{\xi_j^h\}_{j=0}^N \subset \mathcal{K}^h$ *and* $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$,

$$\begin{align}
&\max_{0\leq n\leq N}\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\}+k\sum_{j=1}^N\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2 \nonumber\\
&\quad\leq c\Big\{\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\zeta_0-\zeta_0^h\|_Y^2+k^2\|\boldsymbol{v}\|_{\mathcal{C}([0,T];V)}^2 \tag{2.90}\\
&\qquad+\max_{1\leq n\leq N}I_n^2+k\sum_{j=1}^N\|\zeta_j-\xi_j^h\|_B^2+k^2\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}^2+\max_{0\leq n\leq N}\|\zeta_n-\xi_n^h\|_Y^2 \nonumber\\
&\qquad+\frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k^2\|\zeta\|_{H^2(0,T;Y)}^2 \nonumber\\
&\qquad+k\sum_{j=1}^N\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\|\zeta_j-\xi_j^h\|_Y \nonumber\\
&\qquad+\max_{0\leq n\leq N}\{\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V^2\}\Big\}. \tag{2.91}
\end{align}$$

These estimates are the basis for the study of the error analysis. As an example, let us consider V^h and B^h be defined by (2.64) and (2.65), respectively. Then, we have a first error estimate result.

Corollary 2.2. Let the assumptions of Theorem 2.4 still hold. Let the discrete initial conditions be chosen as

$$\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0,\quad \zeta_0^h=\pi^h\zeta_0,$$

where π^h and Π^h were defined in (2.30) and (2.34). Under the following additional regularity conditions,

$$\begin{align}
&\boldsymbol{u}\in H^2(0,T;V)\cap\mathcal{C}^1([0,T];[H^2(\Omega)]^d), \nonumber\\
&\zeta\in\mathcal{C}([0,T];H^2(\Omega)\cap H^2(0,T;Y),\quad \dot{\zeta}\in L^2(0,T;B), \tag{2.92}
\end{align}$$

we have the following error estimates,

$$\max_{0\leq n\leq N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\zeta_n-\zeta_n^{hk}\|_Y\Big\}\leq c(h^{1/2}+k).$$

Proof

The above error estimates are obtained taking into account the approximation results of the variational spaces V and B by the finite element spaces V^h and B^h, respectively. We recall that (see [25])

$$\inf_{w_n^h \in V^h} \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V \le ch\|\boldsymbol{v}_n\|_{[H^2(\Omega)]^d},$$
$$\inf_{\xi_n^h \in B^h} \|\zeta_n - \xi_n^h\|_Y \le ch^2\|\zeta_n\|_{H^2(\Omega)}, \quad \inf_{\xi_n^h \in B^h} \|\zeta_n - \xi_n^h\|_B \le ch\|\zeta_n\|_{H^2(\Omega)}.$$

It is easy to check that

$$\max_{1\le n\le N} I_n \le ck\|\boldsymbol{u}\|_{H^2(0,T;V)}.$$

By other hand, taking into account (2.51) and (2.92), it follows that

$$\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \le \|\dot{\zeta}_j - \delta\zeta_j\|_Y + \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j) - \dot{\zeta}_j + \kappa\Delta\zeta_j\|_Y$$
$$\le c(k\|\zeta\|_{H^2(0,T;Y)} + 1 + \|\boldsymbol{u}\|_{\mathcal{C}([0,T];V)} + \|\zeta\|_{\mathcal{C}([0,T];H^2(\Omega)} + \|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}).$$

Finally, we have, as in Corollary 2.1, that

$$\frac{1}{k}\sum_{j=1}^{N-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \le ch^2,$$

which concludes the proof. ■

We notice that estimates (2.2) can be improved if we assume stronger regularity conditions. Integrating by parts equilibrium equation (2.37) and proceeding as in the proof of Theorem 2.4, the following error estimates are obtained.

Theorem 2.7. *Let the assumptions of Theorem 2.6 still hold. Let us assume the additional regularity condition* (2.69) *for the damage field and also that*

$$\boldsymbol{\sigma\nu} \in \mathcal{C}([0,T];[L^2(\Gamma)]^2). \qquad (2.93)$$

Then, the following error estimates are obtained for all $\{\xi_j^h\}_{j=0}^N \subset \mathcal{K}^h$ *and* $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset$

V^h,

$$
\begin{aligned}
&\max_{0\le n\le N}\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\}+k\sum_{j=1}^{N}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\quad\le c\Big\{\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\zeta_0-\zeta_0^h\|_Y^2+k^2\|\boldsymbol{v}\|^2_{\mathcal{C}([0,T];V)}\\
&\qquad+\max_{1\le n\le N}I_n^2+k\sum_{j=1}^{N}\|\zeta_j-\xi_j^h\|_B^2+k^2\|\dot\zeta\|^2_{\mathcal{C}([0,T];Y)}+\max_{0\le n\le N}\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad+\frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k^2\|\zeta\|^2_{H^2(0,T;Y)}\\
&\qquad+k\sum_{j=1}^{N}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\\
&\qquad+\max_{0\le n\le N}\|(\boldsymbol{v}_n)_\tau-(\boldsymbol{w}_n^h)_\tau\|_{[L^2(\Gamma_C)]^d}+\max_{0\le n\le N}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V^2\Big\}.
\end{aligned}
$$

Thus, if we assume the following additional regularity assumptions,

$$\dot{\boldsymbol{u}}_\tau\in\mathcal{C}([0,T];[H^2(\Gamma_C)]^d),\tag{2.94}$$

we obtain the following corollary which states the linear convergence of the algorithm depending on the discretization parameters h and k.

Corollary 2.3. Let the assumptions of Corollary 2.2 still hold. Under the additional regularity conditions (2.93) and (2.94), we have the following error estimate,

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\zeta_n-\zeta_n^{hk}\|_Y\Big\}\le c(h+k).\tag{2.95}$$

The proof of Corollary 2.3 is similar to that of Corollary 2.2. We only need to keep in mind that (see [25])

$$\inf_{w_n^h\in V^h}\|(\boldsymbol{v}_n)_\tau-(\boldsymbol{w}_n^h)_\tau\|_{[L^2(\Gamma_C)]^d}\le ch^2\|(\boldsymbol{v}_n)_\tau\|_{[H^2(\Gamma_C)]^d}.$$

2.2.3 Numerical resolution of two-dimensional problems

Let $n\in\{1,\dots,N\}$ and assume that $\boldsymbol{u}_{n-1}^{hk}$ and ζ_{n-1}^{hk} are known. First, from Problem $\mathbf{VP}^{hk}$ we obtain that the discrete damage field is the unique solution to the following problem,

$$(\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + ka(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geq k(\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y$$
$$+(\zeta_{n-1}^{hk}, \xi^h - \zeta_n^{hk})_Y \quad \forall \xi^h \in \mathcal{K}^h. \tag{2.96}$$

Problem (2.96) is a classical first-kind variational inequality which has been solved using a penalty-duality algorithm introduced in [6].

Secondly, the discrete velocity field is obtained solving the following variational inequality,

$$(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + j(\boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}) \geq (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk})_V$$
$$-(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q \quad \forall \boldsymbol{w}^h \in V^h. \tag{2.97}$$

We note that problem (2.97) is a second-kind variational inequality. The aim of this subsection is to present and to compare two numerical algorithms to solve it involving two-dimensional problems. The first one is based on the Uzawa's algorithm, described in [23] and used it to solve frictional contact problems in elasticity. We will show now that these results can be extended to the viscoelastic case.

It is easy to check that (2.97) is a discrete variational inequality which satisfies the conditions required in [23]. There, in order to improve the speed of convergence, an inexact Uzawa algorithm was employed using a preconditionner of the stiffness matrix associated with $\boldsymbol{\mathcal{A}}$. Here, in order to simplify the calculations, we apply the Uzawa's algorithm. Thus, the numerical algorithm applied to solve (2.97) is as follows.

Let Λ be the following subset of $L^\infty(\Gamma_C)$,

$$\Lambda = \{\mu \in L^\infty(\Gamma_C)\, ;\, \|\mu\|_{L^\infty(\Gamma_C)} \leq 1\},$$

and define the projection mapping $P_\Lambda : L^\infty(\Gamma_C) \to \Lambda$ by the formula

$$P_\Lambda \mu = \frac{\mu}{sup\{1, \mu\}}.$$

Assume that $\lambda_0^h \in \Lambda$ and $\boldsymbol{v}_0^{hk} \in V^h$ are given, and let $\rho > 0$ a constant. The sequence

$\{(\boldsymbol{v}_s^{hk}, \lambda_s^{hk})\}_{s\geq 1} \subset V^h \times \Lambda$ is computed inductively by

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{s+1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V - (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q \\
&\qquad\qquad -(\lambda_s^h, w_\tau^h)_{L^2(\Gamma_C)} \quad \forall \boldsymbol{w}^h \in V^h, \\
&\lambda_{s+1}^h = P_\Lambda(\lambda_s^h + \rho g v_{s+1,\tau}^{hk}),
\end{aligned} \tag{2.98}$$

where

$$w_\tau^h = \boldsymbol{w}^h \cdot \boldsymbol{\tau}, \quad v_{s+1,\tau}^{hk} = \boldsymbol{v}_{s+1}^{hk} \cdot \boldsymbol{\tau} \quad \text{on} \quad \Gamma_C.$$

In [23], the following convergence result was proved.

Theorem 2.8. *Let* $0 < \rho < \dfrac{2}{c_0}$, *where* $c_0 > 0$ *is defined in such a way that*

$$\|g\, \boldsymbol{v}_\tau\|_{[L^2(\Gamma_C)]^d} \leq c_0(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall \boldsymbol{v} \in V.$$

Then, the algorithm (2.98) *converges:*

$$\lim_{s\to+\infty} \|\boldsymbol{v}_s^{hk} - \boldsymbol{v}_n^{hk}\|_V = 0.$$

The second procedure is based on a penalization of the friction law which was introduced in [19]. The main idea is to substitute the Tresca's law by the following equation,

$$-\sigma_\tau = \Phi_\mu(\dot{u}_\tau), \tag{2.99}$$

where, for $0 < \mu$,

$$\Phi_\mu(r) = \begin{cases} -g & \text{if} \quad r < -\mu, \\ \dfrac{g}{\mu} r & \text{if} \quad r \in [-\mu, \mu], \\ g & \text{if} \quad r > \mu, \end{cases}$$

and $\sigma_\tau = \boldsymbol{\sigma}_\tau \cdot \boldsymbol{\tau} = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\tau}$ and $\dot{u}_\tau = \dot{\boldsymbol{u}}_\tau \cdot \boldsymbol{\tau} = \dot{\boldsymbol{u}} \cdot \boldsymbol{\tau}$ are the respective tangential projections of the shear stress and tangential velocity field,

Using (2.99) instead of (2.43), the following nonlinear variational equation is obtained for the velocity field

$$\begin{aligned}
&(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_\mu(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + \int_{\Gamma_C} \Phi_\mu(\boldsymbol{v}_\mu \cdot \boldsymbol{\tau}) w_\tau \, da = (\boldsymbol{f}(t), \boldsymbol{w})_V \\
&\qquad -(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \zeta_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \forall \boldsymbol{w} \in V,
\end{aligned} \tag{2.100}$$

where $\boldsymbol{u}_\mu(t) = \int_0^t \boldsymbol{v}_\mu(s)\,ds + \boldsymbol{u}_0$ and we used the notation $w_\tau = \boldsymbol{w}\cdot\boldsymbol{\tau}$ for all $\boldsymbol{w} \in V$.

From [41] it follows that problem (2.100) is equivalent to the following second-kind variational inequality,

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_\mu(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}-\boldsymbol{v}_\mu(t)))_Q + j_\mu(\boldsymbol{w}) - j_\mu(\boldsymbol{v}_\mu(t)) &\geq (\boldsymbol{f}(t), \boldsymbol{w}-\boldsymbol{v}_\mu(t))_V \\ -(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \zeta_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}-\boldsymbol{v}_\mu(t)))_Q &\quad \forall \boldsymbol{w} \in V, \end{aligned} \tag{2.101}$$

where $j_\mu : V \to \mathbb{R}$ is defined by

$$j_\mu(\boldsymbol{w}) = \int_{\Gamma_C} \mathcal{F}_\mu(w_\tau)\,da, \quad \mathcal{F}_\mu(r) = \begin{cases} -gr - g\dfrac{\mu}{2} & \text{if} \quad r < -\mu, \\ \dfrac{g}{2\mu} r^2 & \text{if} \quad r \in [-\mu, \mu], \\ gr - g\dfrac{\mu}{2} & \text{if} \quad r > \mu. \end{cases}$$

Let us consider the following fully discrete variational problem associated with (2.101),

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_\mu^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h-\boldsymbol{v}_\mu^{hk}))_Q + j_\mu(\boldsymbol{w}^h) - j_\mu(\boldsymbol{v}_\mu^{hk}) &\geq (\boldsymbol{f}_n, \boldsymbol{w}^h-\boldsymbol{v}_\mu^{hk})_V \\ -(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h-\boldsymbol{v}_\mu^{hk}))_Q &\quad \forall \boldsymbol{w}^h \in V^h, \end{aligned} \tag{2.102}$$

where the subscript $_n$ has been removed in order to simplify the writing. We notice that (2.102) has a unique solution $\boldsymbol{v}_\mu^{hk} \in V^h$ (see [41]), and it can be calculated using the penalty-duality algorithm applied to solve (2.96).

From [41] we have that

$$\lim_{\mu\to 0} \|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_\mu^{hk}\|_V = 0. \tag{2.103}$$

It is easy to check that if $\boldsymbol{v}_n^{hk}$ and $\boldsymbol{v}_\mu^{hk}$ are the solutions to (2.97) and (2.102), respectively, then we have

$$\|\boldsymbol{v}_\mu^{hk}\|_V + \|\boldsymbol{v}_n^{hk}\|_V \leq c(1 + \|\boldsymbol{u}_{n-1}^{hk}\|_V), \tag{2.104}$$

where we used that

$$\begin{aligned}\|\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk})\|_Q &\leq \|\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}) - \mathcal{E}(\boldsymbol{0}, 0)\|_Q + \|\mathcal{E}(\boldsymbol{0}, 0)\|_Q \\ &\leq c(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})\|_Q + \|\zeta_{n-1}^{hk}\|_Y) + c \leq c(1 + \|\boldsymbol{u}_{n-1}^{hk}\|_V).\end{aligned}$$

Taking $\boldsymbol{w}^h = \boldsymbol{v}_\mu^{hk}$ in (2.97) and $\boldsymbol{w}^h = \boldsymbol{v}^{hk}$ in (2.102), and substracting both inequalities, we find that

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_\mu^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk} - \boldsymbol{v}_\mu^{hk}))_Q \\ \leq j_\mu(\boldsymbol{v}_n^{hk}) - j(\boldsymbol{v}_n^{hk}) + j(\boldsymbol{v}_\mu^{hk}) - j_\mu(\boldsymbol{v}_\mu^{hk}).\end{aligned}$$

We also have

$$j(\boldsymbol{w}^h) - j_\mu(\boldsymbol{w}^h) \leq C\mu \|\boldsymbol{w}^h\|_V \quad \forall \boldsymbol{w}^h \in V^h.$$

Under the assumptions of Corollary 2.3, we obtain

$$\|\boldsymbol{u}_{n-1}^{hk}\|_V \leq \|\boldsymbol{u}\|_{\mathcal{C}([0,T];V)} + c(h+k),$$

and therefore,

$$\begin{aligned}\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_\mu^{hk}\|_V^2 &\leq c\mu[\|\boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{v}_\mu^{hk}\|_V] \\ &\leq c\mu(1 + \|\boldsymbol{u}_{n-1}^{hk}\|_V) \leq c\mu(h + k + \|\boldsymbol{u}\|_{\mathcal{C}([0,T];V)}).\end{aligned} \tag{2.105}$$

Convergence (2.103) is now deduced.

2.2.4 Numerical examples

Three numerical examples have been performed including two-dimensional problems. The first one consists of a comparison between the numerical resolution with Uzawa's algorithm and that obtained with the penalization over the friction condition. The second one is a numerical verification of the linear convergence of the algorithm with respect to the discretization parameters and the third one is an example of stick-slip problem.

In all the examples the following elasticity tensor was employed,

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = \zeta \Phi(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{2.106}$$

where $\mathcal{B}$ is the two-dimensional elasticity tensor under the plane stress hypothesis, already described in (2.36), and $\Phi : \mathbb{S}^d \to \mathbb{S}^d$ is a truncation operator defined by

$$\forall \boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in \mathbb{S}^d, \quad (\Phi(\boldsymbol{\tau}))_{ij} = \begin{cases} L & \text{if} \quad \tau_{ij} > L, \\ \tau_{ij} & \text{if} \quad \tau_{ij} \in [-L, L], \\ -L & \text{if} \quad \tau_{ij} < -L. \end{cases} \tag{2.107}$$

Here, $L > 0$ is a given constant. We notice that the existence of L is justified taking into account that the small displacement theory is used. In all the examples, the following values have been used,

$$L = 1000, \quad E = 5000\, N/m^2, \quad r = 0.3.$$

As a viscosity operator, it has been considered the tensor $\mathcal{A} = 10^{-2}\mathcal{B}$.

The damage source function used was that described in (1.3) with the values

$$\zeta_* = 0.01, \quad q^* = 1000, \quad \lambda_D = 0.1, \quad \lambda_u = 1000, \quad \lambda_w = 0.$$

First example: comparison of the two numerical methods

In order to compare the rate of convergence of both methods of resolution explained above, as a first example, a domain $\Omega = (0,1) \times (0,1)$ is considered with the physical situation depicted in Figure 2.10 during one second (i.e. $T = 1\, s$). Several uniform partitions for both the time interval and the domain, dividing Ω into $2n^2$ triangles, have been performed. Note that the discretization parameter is $h = \frac{\sqrt{2}}{n}$ and the number of degrees of freedom is $2(n+1)^2$.

At each time step, the penalty parameter $\varepsilon = 10^{-9}$ was employed in the approximation of the Tresca's condition which led to (2.100) (and then, $\boldsymbol{v}_n^{hk,\varepsilon} \approx \boldsymbol{v}_n^{hk}$). The stop criterium is a relative stop criterium with $\delta = 10^{-6}$ for both algorithms; that is

$$\frac{\|\boldsymbol{v}_{s+1}^{hk} - \boldsymbol{v}_s^{hk}\|_V}{\|\boldsymbol{v}_s^{hk}\|_V} \leq \delta,$$

where $\{\boldsymbol{v}_s^{hk}\}_{s\geq 0}$ represents the computed solution.

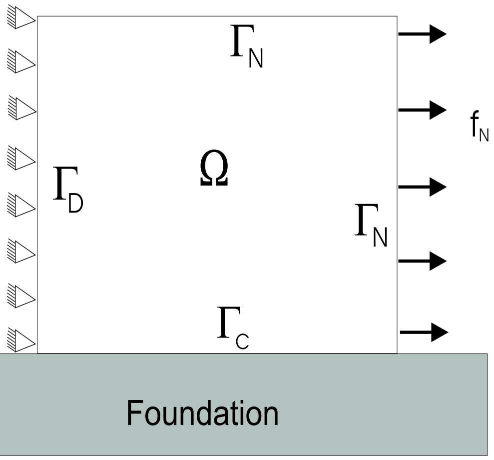

Figure 2.10: A first example: comparison between the algorithms.

$n \downarrow k \rightarrow$	0.02	0.01	0.005	0.002
16	5	8	17	43
32	91	399	737	1929
64	740	4459	8732	21499
128	69516	318045	—	—

Table 2.2: Test **1**: CPU time (seconds) with Uzawa's algorithm.

The following data have been used:

$$T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N(\boldsymbol{x},t) = (100,0)e^t\,N/m^2,$$
$$\boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \zeta_0 = 1, \quad g = 5\,N/m^2.$$

In Tables 2.2 and 2.3, the CPU time (in seconds) for different discretization parameters is shown (the CPU time for $n = 128$ and $k = 0.005$, 0.002 was too large using Uzawa's algorithm). We can appreciate that the penalty-duality algorithm provides much better results than Uzawa's algorithm.

$n \downarrow k \rightarrow$	0.02	0.01	0.005	0.002
16	5	11	17	46
32	27	92	173	365
64	185	537	947	1874
128	1185	3296	5907	13592

Table 2.3: Test **1**: CPU time (seconds) with the penalty-duality algorithm.

$n \downarrow k \rightarrow$	0.02	0.01	0.005	0.002	0.001
8	1.441e-2	1.110e-2	1.017e-2	9.566e-3	9.363e-3
16	1.411e-2	6.719e-3	5.761e-3	5.146e-3	4.939e-3
32	1.352e-2	4.517e-3	3.548e-3	2.923e-3	2.708e-3
64	1.305e-2	3.324e-3	2.344e-3	1.694e-3	1.440e-3
128	1.273e-2	2.651e-3	1.673e-3	1.042e-3	8.262e-4

Table 2.4: Test **2**: Numerical errors for different k and h.

Second example: numerical convergence

We now want to verify the linear convergence of the numerical scheme proposed. In order to do that, we consider the physical problem of the previous example and compare the solutions (those obtained using the penalty-duality algorithm) with that obtained for $n = 256$ and $k = 5 \times 10^{-4}$. The numerical errors (in H^1-norm) are depicted in Table 2.4.

The linear convergence of the method is obtained, as it can be seen in Figure 2.11.

Third example: stick-slip case

As a third example, a physical setting, similar to that of the above test, has been considered during a time interval of 5 seconds (i.e., $T = 5$ s). The boundary Γ is now defined by $\Gamma_D = \emptyset$, $\Gamma_N = \{0, 4\} \times (0, 4)$ and $\Gamma_C = [0, 4] \times \{0, 4\}$, and the body

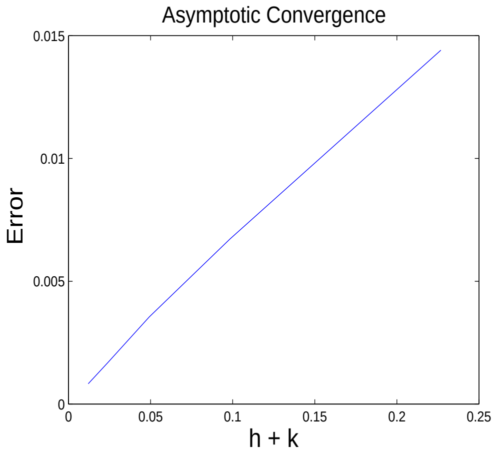

Figure 2.11: Example **2**: Asymptotic convergence.

is acted upon by volume forces whose direction oscillates periodically ($\boldsymbol{f}_B(x,y,t) = (100,0)sin(\frac{\pi t}{2})\,N/m^3$), and it is traction-free on Γ_N.

The friction bound g is now given by $g(x,y) = g_1$ if $y = 0$ and $g(x,y) = g_2$ if $y = 4$, where g_1 and g_2 are friction coefficients for the contact boundary on $[0,4]\times\{0\}$ and $[0,4]\times\{4\}$, respectively. In this example, values $g_1 = 45\,N/m^2$ and $g_2 = 1000\,N/m^2$ were employed.

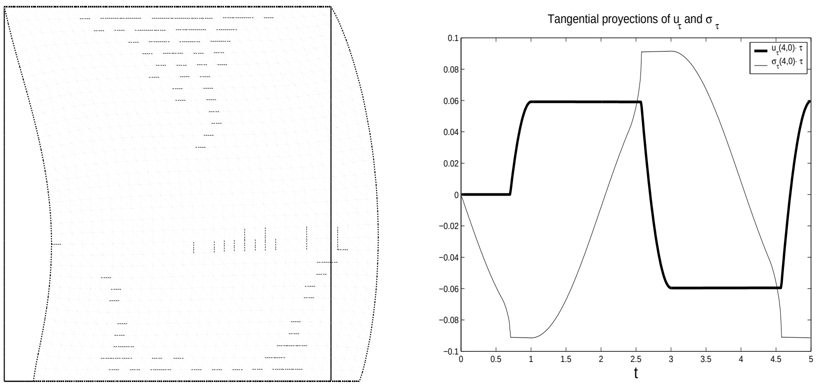

Figure 2.12: Example **2**: Deformed configuration (multiplied by 3) at time $t = 5$ s and horizontal displacements and tangential stresses at $\boldsymbol{x} = (4,0)$.

The deformed mesh at final time and the initial configuration are shown in Figure 2.12 (left-hand side). We notice the movement of the lower horizontal boundary, while the upper one remains clamped (because the tangential stresses do not reach the friction bound on the upper boundary). On the right-hand side, the respective tangential projections of the displacements and shear stresses (divided by an adequate factor) are plotted at point $\boldsymbol{x} = (4, 0)$. It is possible to observe the absence of movement until the friction bound is reached.

Finally, in Figure 2.13, the von Mises stress norm and the damage field at final time $t = 5$ are plotted on the deformed configuration. The maximum stresses are located on the upper boundary due to the clamping condition, while on the lower one stresses appear due to the friction. As we expected, the damage is concentrated on the most stressed areas, the contact boundaries.

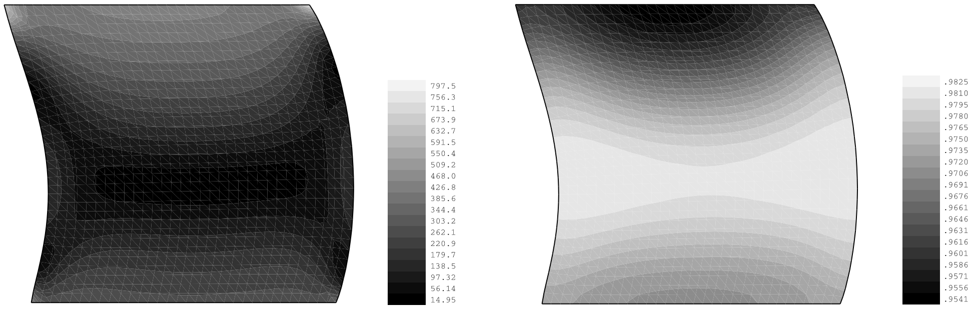

Figure 2.13: Example **2**: von Mises stress norm and damage field at final time.

2.3 A contact problem in viscoelasticity with long memory

The second problem we introduce in this chapter deals with another viscoelastic problem. Moreover, in this case the viscous effect is not related with the velocity of the deformations, but with the previous strains; that is, the last deformed configurations of the body. The main differences with the previous problem are not only in the constitutive law, but also on the contact conditions. In this case, a frictionless process has been considered, and the obstacle is supposed to be deformable with a reaction force that depends on the amount of the penetration; that is, a normal compliance condition has been employed. We have also considered equation (1.5) for the damage evolution.

The complete mathematical description of the problem is as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, and a damage field* $\zeta : \Omega \times [0,T] \to \mathbb{R}$ *such that* $\zeta(0) = \zeta_0$ *in* Ω *and,*

$$-\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_B \quad \text{in} \quad \Omega \times (0,T), \tag{2.108}$$

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))\,ds \ \text{ in } \Omega, \quad \text{a.e. } t \in (0,T), \tag{2.109}$$

$$\dot{\zeta} - \kappa \,\triangle\zeta = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T), \tag{2.110}$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{2.111}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T), \tag{2.112}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{2.113}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \quad -\sigma_\nu = p_\nu(u_\nu - g) \quad \text{on} \quad \Gamma_C \times (0,T). \tag{2.114}$$

In this case, since the contact is unilateral, the set of admissible displacement functions does not include restrictions provided by the contact conditions, so we consider,

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \,;\ \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\}.$$

We suppose that the body forces and surface tractions satisfy

$$\boldsymbol{f}_B \in C([0,T];H), \quad \boldsymbol{f}_N \in C([0,T];L^2(\Gamma_N)]^d), \tag{2.115}$$

and define the function $\boldsymbol{f} : [0,T] \to V$ as

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = (\boldsymbol{f}_B(t),\boldsymbol{v})_H + (\boldsymbol{f}_N(t),\boldsymbol{v})_{[L^2(\Gamma_N)]^d} \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T].$$

Notice that conditions (2.115) imply the regularity

$$\boldsymbol{f} \in C([0,T];V).$$

We also assume that for the initial condition we have,

$$\zeta_0 \in B, \quad \zeta_* \leq \zeta_0(\boldsymbol{x}) \leq 1 \quad \text{a.e.} \quad \boldsymbol{x} \in \Omega. \tag{2.116}$$

In the study of the mechanical problem (2.108)–(2.114), we assume that the elasticity tensor $\mathcal{E} : (\boldsymbol{x},\boldsymbol{\tau}) \in \Omega \times \mathbb{S}^d \to \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) = (e_{ijkl}(\boldsymbol{x})\tau_{kl}) \in \mathbb{S}^d$ satisfies

$$\left.\begin{array}{l}
\text{(a) The mapping } \boldsymbol{x} \to \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) \quad \text{is Lebesgue measurable and bounded in } \Omega,\\
\qquad \forall \boldsymbol{\tau} \in \mathbb{S}^d.\\
\text{(b) } \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau})\cdot\boldsymbol{\sigma} = \boldsymbol{\tau}\cdot\mathcal{E}(\boldsymbol{x},\boldsymbol{\sigma}), \quad \forall \boldsymbol{\tau},\boldsymbol{\sigma} \in \mathbb{S}^d, \quad \text{a.e. in } \Omega.\\
\text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that } \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau})\cdot\boldsymbol{\tau} \geq m_{\mathcal{E}}\,|\boldsymbol{\tau}|^2\\
\qquad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right\} \tag{2.117}$$

The viscoelastic long memory operator $\mathcal{G} : \Omega \times [0,T] \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\left.\begin{array}{l}
\text{(a) There exists } M_{\mathcal{G}} > 0 \text{ such that}\\
\qquad \|\mathcal{G}(\boldsymbol{x},t,\boldsymbol{\tau}_1,\zeta_1) - \mathcal{G}(\boldsymbol{x},t,\boldsymbol{\tau}_2,\zeta_2)\| \leq M_{\mathcal{G}}\left(|\boldsymbol{\tau}_1-\boldsymbol{\tau}_2| + |\zeta_1-\zeta_2|\right)\\
\qquad \forall\, t \in [0,T],\ \boldsymbol{\tau}_1,\boldsymbol{\tau}_2 \in \mathbb{S}^d,\ \zeta_1,\zeta_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Omega.\\
\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},t,\boldsymbol{\tau},\zeta) \text{ is Lebesgue measurable in } \Omega,\\
\qquad \forall\, t \in [0,T],\ \boldsymbol{\tau} \in \mathbb{S}^d,\ \zeta \in \mathbb{R}.\\
\text{(c) The mapping } t \mapsto \mathcal{G}(\boldsymbol{x},t,\boldsymbol{\tau},\zeta) \text{ is continuous in } [0,T],\\
\qquad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d,\ \zeta \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Omega.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},t,\boldsymbol{0},0) \quad \text{belongs to } Q \quad \forall\, t \in [0,T].
\end{array}\right\} \tag{2.118}$$

The damage source function $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left.\begin{array}{l}
\text{(a) There exists } L_\phi > 0 \text{ such that} \\
\quad |\phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \zeta_2)| \le L_\phi \left(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2|\right) \\
\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \zeta_1, \zeta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta) \text{ is Lebesgue measurable on } \Omega, \\
\quad \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \quad \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{0}, 0) \text{ belongs to } Y. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta) \text{ is bounded } \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \zeta \in \mathbb{R}. \\
\text{(e) } \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta) \le 0 \text{ if } \zeta \ge 1, \quad \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \zeta) \ge 0 \text{ if } \zeta \le \zeta_*.
\end{array}\right\} \tag{2.119}$$

The normal compliance function $p_\nu : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+$ verifies

$$\left.\begin{array}{l}
\text{(a) There exists } L_p > 0 \text{ such that} \\
\quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_p\, |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_C, \\
\quad \forall r \in \mathbb{R}. \\
\text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)) \cdot (r_1 - r_2) \ge 0 \quad \forall\, r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(d) } p_\nu(\boldsymbol{x}, r) = 0 \quad \text{for all } r \le 0, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C.
\end{array}\right\} \tag{2.120}$$

We turn now to derive a variational formulation of the mechanical problem P. To that end, we assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \zeta)$ are regular functions satisfying (2.108)–(2.114). Applying a Green's formula we obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (-Div\boldsymbol{\sigma}(t), \boldsymbol{v})_H + (\boldsymbol{\sigma}(t)\boldsymbol{\nu}, \boldsymbol{v})_{[L^2(\Gamma)]^d}$$

for all $\boldsymbol{v} \in V$. We have, from (2.112), (2.113) and (2.114),

$$(\boldsymbol{\sigma}(t)\boldsymbol{\nu}, \boldsymbol{v})_{[L^2(\Gamma)]^d} = (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d} - j(\boldsymbol{u}(t), \boldsymbol{v}),$$

where the function $j : V \times V \mapsto \mathbb{R}$ is given by

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} p_\nu(u_\nu - g)\, v_\nu \, da \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{2.121}$$

Notice that the integral (2.121) is well defined by (2.120). For the damage equation we use the same arguments presented in Section 2.1 for the elastic problem, and therefore, we obtain the complete variational formulation of the problem, which is as follows.

Problem VP. *Find a displacement field $\boldsymbol{u} : [0, T] \to V$, a stress field $\boldsymbol{\sigma} : [0, T] \to Q$, and a damage field $\zeta : [0, T] \to B$ such that $\zeta(0) = \zeta_0$, and for a.e. $t \in [0, T]$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(t - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))\, ds, \tag{2.122}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t), \boldsymbol{w}) = (\boldsymbol{f}(t), \boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V, \tag{2.123}$$

$$(\dot{\zeta}(t), \xi)_Y + a(\zeta(t), \xi) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)),\ \xi)_Y \quad \forall \xi \in B. \tag{2.124}$$

2.3.1 An existence and uniqueness result

The existence of a unique solution to Problem VP is proved using some results for parabolic variational equations and Banach fixed point arguments.

Theorem 2.9. *Assume that (2.115)–(2.120) hold. Then, there exists a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \zeta)$ to Problem VP with the following regularity:*

$$\boldsymbol{u} \in \mathcal{C}([0, T]; V),\ \ \boldsymbol{\sigma} \in \mathcal{C}([0, T]; Q),\ \ \zeta \in W^{1,2}(0, T; Y) \cap L^2(0, T; B).$$

Moreover, the damage field ζ satisfies $\zeta(t, \boldsymbol{x}) \in [\zeta_*, 1]$ a.e. $t \in [0, T]$ and $\boldsymbol{x} \in \Omega$.

Proof

As a first step towards an existence and uniqueness result, we introduce the operator $A : V \to V$ defined as follows,

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{v}, \boldsymbol{w}) \quad \forall\, \boldsymbol{v},\, \boldsymbol{w} \in V.$$

From (2.108) and (2.120), we claim that A is a strongly monotone and Lipschitz continuous operator. Now, let $\theta \in \mathcal{C}([0, T]; Y)$ be a given auxiliary function and consider the following variational problem.

Problem VP_θ**.** *Find* $\boldsymbol{u}_\theta : [0,T] \to V$ *and* $\zeta_\theta : [0,T] \to B$ *such that* $\zeta_\theta(0) = \zeta_0$*, and for a.e.* $t \in [0,T]$*,*

$$(A\boldsymbol{u}_\theta(t), \boldsymbol{w})_V + (\int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_\theta(s)), \zeta_\theta(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

$$= (\boldsymbol{f}(t), \boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V, \tag{2.125}$$

$$(\dot{\zeta}_\theta(t), \xi)_Y + a(\zeta_\theta(t), \xi) = (\theta, \xi)_Y \quad \forall \xi \in B. \tag{2.126}$$

By using some arguments of evolutionary variational equations (see, e.g., [56]), it follows that there exists a unique solution to (2.126) satisfying $\zeta_\theta(0) = \zeta_0$ and the regularity,

$$\zeta_\theta \in W^{1,2}(0,T;Y) \cap L^2(0,T;B).$$

Now, let $\boldsymbol{\mu}_\theta \in \mathcal{C}([0,T];Q)$ be a given auxiliary function. There exists a unique $\boldsymbol{\eta}_\theta \in \mathcal{C}([0,T];V)$ such that

$$(\boldsymbol{\eta}_\theta(t), \boldsymbol{w})_V = (\boldsymbol{\mu}_\theta(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \forall \boldsymbol{w} \in V,\ t \in [0,T]. \tag{2.127}$$

We consider the following variational problem.

Problem $VP_{\theta,\eta}$**.** *Find* $\boldsymbol{u}_{\theta,\eta} : [0,T] \to V$ *such that*

$$(A\boldsymbol{u}_{\theta,\eta}(t), \boldsymbol{v})_V + (\boldsymbol{\eta}_\theta(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, t \in [0,T]. \tag{2.128}$$

By using standard arguments of variational equations, we deduce that there exists a unique solution to Problem $VP_{\theta,\eta}$ satisfying

$$\boldsymbol{u}_{\theta,\eta} \in \mathcal{C}([0,T];V).$$

We introduce the operator $\Lambda_\theta : \mathcal{C}([0,T];Q) \to \mathcal{C}([0,T];Q)$ given by

$$\Lambda_\theta(\boldsymbol{\mu}_\theta)(t) = \int_0^t \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\theta,\eta}(s)), \zeta_\theta(s))ds \quad \forall t \in [0,T],$$

where $\boldsymbol{\eta}_\theta$ is defined as in (2.127), and $\boldsymbol{u}_{\theta,\eta}$ and ζ_θ are the solutions to problems $VP_{\theta,\eta}$ and (2.126) with $\zeta_\theta(0) = \zeta_0$, respectively.

Now, given $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{C}([0,T];Q)$, we have

$$\begin{aligned}
&\|\Lambda_\theta(\boldsymbol{\mu}_1)(t) - \Lambda_\theta(\boldsymbol{\mu}_2)(t)\|_Q \\
&\qquad \leq \int_0^t \|\mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\theta,\eta_1}(s)), \zeta_\theta(s)) - \mathcal{G}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\theta,\eta_2}(s)), \zeta_\theta(s))\|_Q ds \\
&\qquad \leq M_{\mathcal{G}} \int_0^t \|\boldsymbol{u}_{\theta,\eta_1}(s) - \boldsymbol{u}_{\theta,\eta_2}(s)\|_V ds \quad \forall t \in [0,T]. \qquad (2.129)
\end{aligned}$$

By taking in (2.128) $\boldsymbol{\eta}_\theta = \boldsymbol{\eta}_1$ and $\boldsymbol{v} = \boldsymbol{u}_{\theta,\eta_2} - \boldsymbol{u}_{\theta,\eta_1}$ and then $\boldsymbol{\eta}_\theta = \boldsymbol{\eta}_2$ and $\boldsymbol{v} = \boldsymbol{u}_{\theta,\eta_1} - \boldsymbol{u}_{\theta,\eta_2}$ and doing some algebra, we find that

$$\|\boldsymbol{u}_{\theta,\eta_1}(s) - \boldsymbol{u}_{\theta,\eta_2}(s)\|_V \leq c\|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V = c\|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_Q. \qquad (2.130)$$

Using (2.130) in (2.129), we obtain

$$\|\Lambda_\theta(\boldsymbol{\mu}_1)(t) - \Lambda_\theta(\boldsymbol{\mu}_2)(t)\|_Q \leq c \int_0^t \|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_Q ds \quad \forall t \in [0,T],$$

where here and in what follows, c is a positive constant whose value may change from line to line.

By reiterating the previous expression we find that there exists a number $p \in \mathbb{N}$ such that

$$\|\Lambda_\theta^p(\boldsymbol{\mu}_1)(t) - \Lambda_\theta^p(\boldsymbol{\mu}_2)(t)\|_Q \leq \frac{c^pT^p}{p!}\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{C}([0,T];Q)} \quad \forall t \in [0,T],$$

and $\frac{c^pT^p}{p!} < 1$. Thus, applying the Banach fixed point theorem, it follows that there exists a unique $\boldsymbol{\mu}^* \in \mathcal{C}([0,T];Q)$ such that $\Lambda_\theta(\boldsymbol{\mu}^*)(t) = \boldsymbol{\mu}^*(t)$ for all $t \in [0,T]$. As a consequence, $\boldsymbol{u}_\theta = \boldsymbol{u}_{\theta,\eta^*} \in \mathcal{C}([0,T];V)$ is the unique solution to (2.125) and the pair $(\boldsymbol{u}_\theta, \zeta_\theta)$ is the unique solution to Problem VP_θ.

We introduce the operator $\Theta : \mathcal{C}([0,T];Y) \to \mathcal{C}([0,T];Y)$ defined by

$$\Theta(\theta)(t) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta(t)), \zeta_\theta(t)) \quad \forall t \in [0,T],$$

where the pair $(\boldsymbol{u}_\theta, \zeta_\theta)$ is the solution to Problem VP_θ.

Now, given $\theta_1, \theta_2 \in \mathcal{C}([0,T];Y)$, we find that

$$\begin{aligned}
\|\Theta(\theta_1)(t) - \Theta(\theta_2)(t)\|_Y^2 &= \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\theta_1}(t), \zeta_{\theta_1}(t)) - \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\theta_2}(t), \zeta_{\theta_2}(t))\|_Y^2 \\
&\leq L_\phi^2(\|\boldsymbol{u}_{\theta_1}(t) - \boldsymbol{u}_{\theta_2}(t)\|_V^2 + \|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2) \quad \forall t \in [0,T]. \qquad (2.131)
\end{aligned}$$

On the other hand, by taking in (2.126) $\theta = \theta_1$ and $\xi = \zeta_{\theta_2}(t) - \zeta_{\theta_1}(t)$ and then $\theta = \theta_2$ and $\xi = \zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 + a(\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t), \zeta_{\theta_1}(t) - \zeta_{\theta_2}(t))\\ &\quad = (\theta_1(t) - \theta_2(t), \zeta_{\theta_1}(t) - \zeta_{\theta_2}(t))_Y.\end{aligned}$$

By integrating in $[0, t]$, we obtain

$$\begin{aligned}&\frac{1}{2}\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 + \int_0^t a(\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s), \zeta_{\theta_1}(s) - \zeta_{\theta_2}(s))ds\\ &= \int_0^t (\theta_1(s) - \theta_2(s), \zeta_{\theta_1}(s) - \zeta_{\theta_2}(s))_Y ds.\end{aligned}$$

Thus,

$$\frac{1}{2}\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 \leq \int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_Y^2 ds + \int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 ds.$$

By using the Gronwall's lemma, we find that

$$\|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_Y^2 \leq c\int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 ds. \tag{2.132}$$

Also, by a similar argument to that of (2.129), it follows that

$$\|\boldsymbol{u}_{\theta_1}(t) - \boldsymbol{u}_{\theta_2}(t)\|_V^2 \leq c\int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_Y^2 ds. \tag{2.133}$$

Summarizing, using (2.132) and (2.133) in (2.131), it leads to the following inequality,

$$\|\Theta(\theta_1)(t) - \Theta(\theta_2)(t)\|_Y^2 \leq c\int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 ds.$$

Reiterating the previous expression and proceeding as we did with the operator Λ_θ, we find that there exists a $\theta^* \in \mathcal{C}([0, T]; Y)$ which is the unique fixed point of Θ. We define $\boldsymbol{u} = \boldsymbol{u}_{\theta^*}$, $\zeta = \zeta_{\theta^*}$ and $\boldsymbol{\sigma}$ by (2.122). Then, $(\boldsymbol{u}, \boldsymbol{\sigma}, \zeta)$ is a solution to Problem VP. The uniqueness is a consequence of the fixed point theorem.

Finally, we need to prove that $\zeta \in [\zeta_*, 1]$. This is based on the following comparison theorem (see [50] for details).

Theorem 2.10. *Let the damage source function* ϕ *and the initial damage field* ζ_0 *satisfy assumptions (2.119) and (2.116), respectively. Then, the solution,* ζ*, to*

$$(\dot{\zeta}(t), \xi)_Y + a(\zeta(t), \xi) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)),\ \xi)_Y \quad \forall\, \xi \in B, \quad \zeta(0) = \zeta_0,$$

satisfies $\zeta\,(t, \boldsymbol{x}) \in [\zeta_*, 1]$ *a.e.* $t \in [0, T]$ *and* $\boldsymbol{x} \in \Omega$.

This concludes the proof of Theorem 2.9. ■

2.3.2 Numerical analysis

In this subsection we consider a numerical scheme for solving Problem VP and derive some error estimates on the approximate solutions.

As in previous sections, we use two finite dimensional spaces $V^h \subset V$ and $B^h \subset B$ to approximate the spaces V and B, respectively, and for the time derivatives, we consider a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, denoting by $k = T/N$ the time step size.

Remark 2.2. *In the two-dimensional numerical simulations presented in the next subsection,* V^h *and* B^h *are composed of continuous and piecewise affine functions; that is,*

$$V^h = \{\boldsymbol{v}^h \in [\mathcal{C}(\overline{\Omega})]^2\ ;\ \boldsymbol{v}^h|_{\mathcal{T}} \in [P_1(\mathcal{T})]^2 \ \ \forall \mathcal{T} \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \boldsymbol{0} \ \ \text{on} \ \ \Gamma_D\}, \tag{2.134}$$

$$B^h = \{\xi^h \in \mathcal{C}(\overline{\Omega})\ ;\ \xi^h|_{\mathcal{T}} \in P_1(\mathcal{T}) \quad \forall \mathcal{T} \in \mathcal{T}^h\}, \tag{2.135}$$

where Ω *is assumed to be a polyhedral domain and we denote by* $\mathcal{T}^h$ *a regular family of triangulations of* $\overline{\Omega}$ *compatible with the partition of the boundary* $\Gamma = \partial\Omega$ *into* Γ_D*,* Γ_N *and* Γ_C.

Let ζ_0^h be an appropriate approximation of the initial condition ζ_0. A fully discrete approximation of Problem VP, based on the forward Euler scheme, is the following.

Problem VP^{hk}**.** *Find a discrete displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete damage field* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset B^h$ *such that* $\zeta_0^{hk} = \zeta_0^h$ *and, for* $n = 1, \ldots, N$,

$$(\delta\zeta_n^{hk}, \xi^h)_Y + a(\zeta_n^{hk}, \xi^h) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}),\ \xi^h)_Y \quad \forall\, \xi^h \in B^h, \tag{2.136}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V$$

$$-\Big(\sum_{j=1}^n k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\Big)_Q \quad \forall \boldsymbol{w}^h \in V^h, \tag{2.137}$$

where $\boldsymbol{u}_0^{hk} \in V^h$ *is given.*

Remark 2.3. *We notice that the discrete "initial condition"* $\boldsymbol{u}_0^{hk}$ *for Problem* VP^{hk} *is the solution to the nonlinear variational equation*

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_0^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_0, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h.$$

Using classical arguments of nonlinear variational equations (see [41]), we obtain the existence of a unique solution to Problem VP^{hk}.

Theorem 2.11. *Assume that* (2.115)–(2.120) *hold. Then there exists a unique solution* $(\boldsymbol{u}^{hk}, \zeta^{hk}) \subset V^h \times B^h$ *to Problem* VP^{hk}.

Proof

In a time step $n \in 1, \ldots, N$, let us define the bilinear form $b : B^h \times B^h \mapsto \mathbb{R}$ as follows

$$b(w_1, w_2) = \frac{1}{k}(w_1, w_2)_Y + a(w_1, w_2),$$

and a linear functional $L : B^h \mapsto \mathbb{R}$ given by

$$L(w) = (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), w)_Y + \frac{1}{k}(\zeta_{n-1}^{hk}, w)_Y.$$

Now, our discrete problem is a variational inequality of the type described, in (1.6), and taking K to be B^h, we get that the damage equation (2.136) has a unique solution ζ_n^{hk}. Using the same arguments, but defining now $b : V^h \times V^h \mapsto \mathbb{R}$ as

$$b(\boldsymbol{w}_1, \boldsymbol{w}_2) = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{w}_1), \boldsymbol{\varepsilon}(\boldsymbol{w}_2))_Q,$$

and $L : V^h \mapsto \mathbb{R}$ as

$$L(\boldsymbol{w}) = (\boldsymbol{f}_n, \boldsymbol{w})_V - (\sum_{j=1}^{n} k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

we have another variational inequality, in this case of type described in (1.7) with $V = V^h$ and the functional $j(\cdot)$ is defined in an appropriate way from the contact functional $j(\cdot,\cdot)$ (see [75]). Therefore, we deduce the existence and uniqueness for $\boldsymbol{u}_n^{hk}$. As $(\boldsymbol{u}_0^{hk}, \zeta_0^h)$ are known and n is arbitrary, in $\{1, \ldots, n\}$, we have obtained the desired result. ■

We assume the following additional regularity of the damage field,

$$\zeta \in \mathcal{C}^1([0,T]; Y) \cap \mathcal{C}([0,T]; B). \tag{2.138}$$

Our aim now is to estimate the numerical errors $\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}$ and $\zeta_n - \zeta_n^{hk}$. We will use the estimation of the damage field obtained in Section 2.1. For the displacement field, let us plug equation (2.122) at time $t = t_n$ into equation (2.123) and, substracting it to equation (2.137), it follows that

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_n, \boldsymbol{w}^h) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h)\\ &\quad+\Big(\int_0^{t_n} \mathcal{G}(t_n - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \sum_{j=1}^{n} k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\Big)_Q = 0,\end{aligned}$$

for all $\boldsymbol{w}^h \in V^h$. Therefore, we have

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}))_Q + j(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\\ &\quad+\Big(\int_0^{t_n} \mathcal{G}(t_n - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \sum_{j=1}^{n} k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\Big)_Q\\ &\quad= (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h))_Q + j(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{w}^h) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{w}^h) \qquad (2.139)\\ &\quad+\Big(\int_0^{t_n} \mathcal{G}(t_n - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \sum_{j=1}^{n} k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{w}^h)\Big)_Q.\end{aligned}$$

From the property (2.120), it is easy to check that

$$\begin{aligned}&j(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{u}_n^{hk}) \geq 0,\\ &j(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{w}^h) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n - \boldsymbol{w}^h) \leq c\|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V,\end{aligned}$$

and we have

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} \mathcal{G}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds - k\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}),\zeta^{hk}_{j-1})\\
&= \int_{t_{n-1}}^{t_n} \mathcal{G}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds - k\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1})\\
&\qquad +k\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1}) - k\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}),\zeta^{hk}_{j-1})\\
&= \int_{t_{n-1}}^{t_n} [\mathcal{G}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s)) - \mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1})]ds\\
&\qquad +k[\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1}) - \mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}),\zeta^{hk}_{j-1})].
\end{aligned}$$

Using the properties (2.118) we can write

$$\begin{aligned}
&\|\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1}) - \mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}),\zeta^{hk}_{j-1})\|_Q\\
&\qquad \le ck(\|\boldsymbol{u}_{j-1}-\boldsymbol{u}^{hk}_{j-1}\|_V + \|\zeta_{j-1}-\zeta^{hk}_{j-1}\|_Y).
\end{aligned}$$

Therefore, adding terms, it is straightforward to obtain that

$$\begin{aligned}
&\|\int_0^{t_n} \mathcal{G}(s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds - \sum_{j=1}^n k\mathcal{G}(t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{j-1}),\zeta^{hk}_{j-1})\|_Q\\
&\qquad \le c\sum_{j=1}^n k[\|\boldsymbol{u}_{j-1}-\boldsymbol{u}^{hk}_{j-1}\|_V + \|\zeta_{j-1}-\zeta^{hk}_{j-1}\|_Y] + I_{\mathcal{G},n},
\end{aligned} \tag{2.140}$$

where $I_{\mathcal{G},n}$ is the integration error given by

$$I_{\mathcal{G},n} = \Big\| \int_0^{t_n} \mathcal{G}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds - \sum_{j=1}^n k\mathcal{G}(t_n-t_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}),\zeta_{j-1})\Big\|_Q.$$

Using the properties (2.117)-(2.118), applying repeatedly Cauchy's inequality (2.22), and taking into account that

$$\Big(k\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}^{hk}_j\|_V\Big)^2 \le ck\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}^{hk}_j\|_V^2, \tag{2.141}$$

from (2.139) we get

$$\begin{aligned}
\|\boldsymbol{u}_n-\boldsymbol{u}^{hk}_n\|_V^2 \le c\Big(&\sum_{j=1}^n k[\|\boldsymbol{u}_{j-1}-\boldsymbol{u}^{hk}_{j-1}\|_V^2 + \|\zeta_{j-1}-\zeta^{hk}_{j-1}\|_Y^2]\\
&+I^2_{\mathcal{G},n} + \|\boldsymbol{u}_n-\boldsymbol{w}^h\|_V^2\Big) \quad \forall \boldsymbol{w}^h \in V^h.
\end{aligned} \tag{2.142}$$

Combining equations (2.24), (2.25) and (2.142), it leads to the following estimate for all $\boldsymbol{w}^h \in V^h$ and $\{\xi_j^h\}_{j=1}^n \subset B^h$,

$$\begin{aligned}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^n \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \le ck\sum_{j=1}^n \Big(\|\boldsymbol{u}_{j-1} - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \\ +\|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + \|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 \\ +\|\zeta_j - \xi_j^h\|_B^2\Big) + cI_{\mathcal{G},n}^2 + c\|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V^2 + c\|\zeta_n - \xi_n^h\|_Y^2 + c\|\zeta_1 - \xi_1^h\|_Y^2 \\ +c\frac{1}{k}\sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2 + c\|\zeta_0 - \zeta_0^h\|_Y^2.\end{aligned}$$

We define

$$\begin{aligned}e_n &= \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^n \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2, \\ g_n &= k\sum_{j=1}^n \Big(\|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 \\ &\quad +\|\zeta_j - \xi_j^h\|_B^2\Big) + I_{\mathcal{G},n}^2 + \|\boldsymbol{u}_n - \boldsymbol{w}^h\|_V^2 + \|\zeta_n - \xi_n^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 \\ &\quad +\frac{1}{k}\sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2 + \|\zeta_0 - \zeta_0^h\|_Y^2,\end{aligned}$$

and applying Lemma 1.2 we obtain the following main error estimates result.

Theorem 2.12. *Assume that* (2.115)–(2.120) *and the additional regularity (2.138) hold. Let* $(\boldsymbol{u}, \zeta) \in V \times B$ *and* $(\boldsymbol{u}^{hk}, \zeta^{hk}) \subset V^h \times B^h$ *be the respective solutions to Problems* VP *and* VP^{hk}*. Then, we have the following error estimates for all* $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=1}^N \subset V^h$ *and* $\{\xi_j^h\}_{j=1}^N \subset B^h$,

$$\begin{aligned}&\max_{0\le n\le N}\{\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\} + k\sum_{j=1}^N \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \\ &\le ck\sum_{j=1}^N \Big(\|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 + \|\zeta_j - \xi_j^h\|_B^2\Big) \\ &+c\max_{0\le n\le N} I_{\mathcal{G},n}^2 + c\max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + c\max_{0\le n\le N} \|\zeta_n - \xi_n^h\|_Y^2 \\ &+c\frac{1}{k}\sum_{j=1}^{N-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_Y^2 + c\|\zeta_0 - \zeta_0^h\|_Y^2 + c\|\zeta_1 - \xi_1^h\|_V^2, \quad (2.143)\end{aligned}$$

where $\boldsymbol{u}_0^{hk} \in V^h$ *is given.*

Error estimates (2.143) are the basis for the analysis of the convergence rate of the algorithm. As an example, let Ω be a polyhedral domain and denote by $\mathcal{T}^h$ a regular triangulation of Ω compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D, Γ_N and Γ_C. Let V^h and B^h be defined by (2.134) and (2.135), respectively, and assume that the discrete initial condition ζ_0^h is

$$\zeta_0^h = \pi^h \zeta_0,$$

with π^h defined as in (2.30).

The following corollary is then derived, establishing the linear convergence of the algorithm with respect to the discretization parameters h and k.

Corollary 2.4. Let the assumptions of Theorem 2.12 hold. Under the following additional regularity conditions

$$\begin{aligned}
&\boldsymbol{u} \in \mathcal{C}([0,T];[H^2(\Omega)]^d) \cap W^{1,\infty}(0,T;V),\\
&\zeta \in H^2(0,T;Y) \cap \mathcal{C}([0,T];H^2(\Omega)) \cap H^1(0,T;B),
\end{aligned}$$

the numerical algorithm introduced in Problem VP^{hk} is linearly convergent; that is, there exists $c > 0$, independent of h and k, such that,

$$\max_{0 \le n \le N} \Big\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y \Big\} \le c(h+k). \tag{2.144}$$

Proof

We note that the expression in Theorem 2.12 is quite similar to that obtained in Theorem 2.6, where the main difference is the term $I^2_{\mathcal{G},n}$, so let us bound it.

We have

$$\begin{aligned}
&\| \int_0^{t_n} \mathcal{G}(t_n - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \sum_{j=1}^n k\mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}), \zeta_{j-1})\|_Q\\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|\mathcal{G}(t_n - s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \mathcal{G}(t_n - t_{j-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}), \zeta_{j-1})\|_Q\\
&\le \sum_{j=1}^n \int_{t_{j-1}}^{t_j} c(|t_n - s - (t_n - t_{j-1})| + \|\boldsymbol{u}(s) - \boldsymbol{u}_{j-1})\|_V + \|\zeta(s) - \zeta_{j-1}\|_Y)ds.
\end{aligned}$$

Since

$$\begin{aligned}\|\boldsymbol{u}(s)-\boldsymbol{u}_{j-1}\| &\leq \int_{t_{j-1}}^{s}\|\dot{\boldsymbol{u}}(z)\|dz\\ &\leq \|\Big(\int_{t_{j-1}}^{s}1dz\Big)^{1/2}\Big(\int_{t_{j-1}}^{s}\|\dot{\boldsymbol{u}}(z)^2dz\Big)^{1/2}\leq\sqrt{k}\|\dot{\boldsymbol{u}}\|_{L^2(0,T;V)},\end{aligned}$$

it follows that

$$\|\zeta(s)-\zeta_{j-1}\|\leq\sqrt{k}\|\dot{\zeta}\|_{L^2(0,T;Y)},$$

and taking into account again (2.141), we easily obtain

$$I_{\mathcal{G},n}\leq ck(1+\|\dot{\boldsymbol{u}}\|_{L^2(0,T;V)}+\|\dot{\zeta}\|_{L^2(0,T;Y)}),$$

which concludes the proof of (2.144). ■

Numerical resolution

We notice that the numerical resolution of Problem VP^{hk} is as follows. First, the discrete “initial condition” $\boldsymbol{u}_0^{hk}$ is obtained as detailed in Remark 2.3, and the nonlinear variational equation can be solved by using a penalty-duality algorithm (see [6]).

Next, for each $n\in\{1,\dots,N\}$, the discrete damage field is obtained solving the discrete variational equation (2.136) by employing Cholesky's method since it leads to a linear system. Moreover, the discrete displacement field is the solution to the nonlinear variational equation (2.137). Again, the penalty-duality algorithm introduced in [6] is employed.

2.3.3 Numerical examples

In this subsection, we report some numerical results on two-dimensional test problems to show the performance of the numerical method discussed in the previous subsection. In the examples below the long memory operator $\mathcal{G}(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta)$ has the same form considered in the previous subsection for the elastic operator; that is,

$$\mathcal{G}(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta)=\zeta\Phi(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u})),$$

where $\mathcal{B}$ is the two-dimensional elasticity tensor under the plane stress hypothesis (see (2.36)) and the truncator operator Φ was given in (2.107). In all the examples, value $L = 1000$ for the truncator operator has been used.

Note that this choice of function $\mathcal{G}$ shows that more damage the material undergoes (ζ decreases), less memory effects arise and more the material behaves like a purely elastic material.

In the simulations below, the parameters of the damage source function λ_D, λ_u and λ_w were taken as

$$\lambda_D = 10^{-2}, \quad \lambda_u = 10, \quad \lambda_w = 0.$$

Also, the value $\kappa = 10^{-2}$ was chosen.

The following normal compliance function has been employed in the simulations,

$$p_\nu(r) = \frac{1}{\mu} r_+, \tag{2.145}$$

where $r_+ = \max\{0, r\}$ and μ is a positive constant which represents a deformability coefficient ($\mu = 10^{10}$ was taken).

Finally, we used linear elements for the finite element spaces V^h and B^h (see (2.134)–(2.135)).

First example: numerical convergence

In order to verify the numerical method studied above, a sequence of numerical solutions has been computed simulating the physical setting described in Figure 2.14. We considered uniform partitions of both the time interval and the spatial domain (where each side of the square $[0,1] \times [0,1]$ is divided into n equal parts) and compared them with the "exact solution", the one with $k = 0.0005$ and $n = 256$. The boundary $\Gamma_D = \{0\} \times [0,1]$ is supposed to be fixed, $\Gamma_C = [0,1] \times \{0\}$ is in frictionless contact with a deformable foundation and Γ_N is divided into two parts, $\{1\} \times [0,1]$, where a density of surface tractions f_N acts, and $[0,1] \times \{1\}$, which is traction-free. No volume forces are supposed to act in the body.

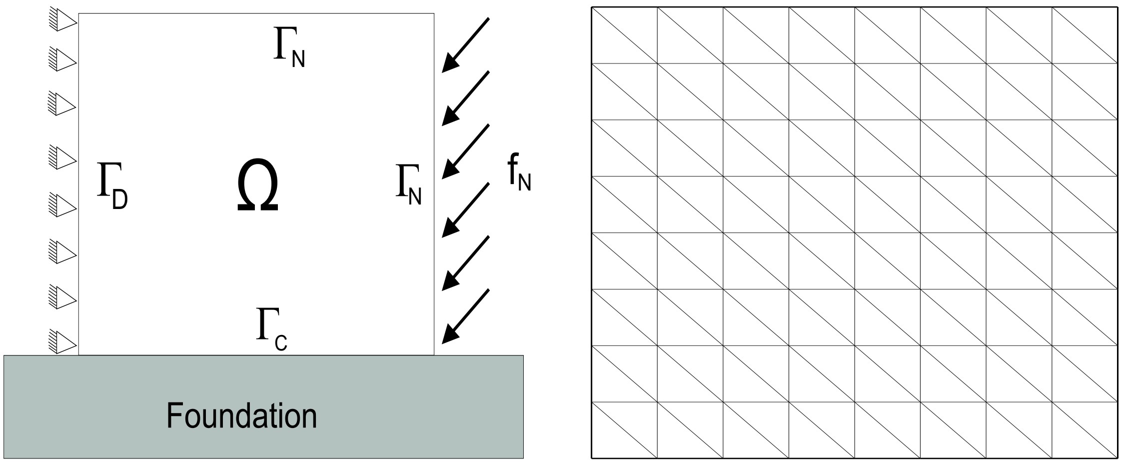

Figure 2.14: Example **1**: Physical setting and mesh for $n = 8$.

$n \downarrow k \rightarrow$	0.05	0.02	0.01	0.005	0.002	0.001
4	6.779e-2	6.665e-2	6.628e-2	6.610e-2	6.610e-2	6.596e-2
8	3.204e-2	3.032e-2	2.978e-2	2.951e-2	2.935e-2	2.930e-2
16	1.574e-2	1.353e-2	1.285e-2	1.254e-2	1.235e-2	1.299e-2
32	9.222e-3	6.560e-3	5.757e-3	5.390e-3	5.187e-3	5.123e-3
64	6.767e-3	3.793e-3	2.863e-3	2.429e-3	2.197e-3	2.128e-3
128	5.879e-3	2.682e-3	1.671e-3	1.186e-3	9.101e-4	8.289e-4

Table 2.5: Example **1**: Numerical errors obtained with differents n and k.

The following data have been used in this case:

$$T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N = (-10, -10)\,N/m^2,$$
$$E = 100N/m^2, \qquad r = 0.3, \qquad \zeta_* = 0.01, \qquad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega$$

The numerical errors given by

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y$$

are depicted in Table 2.5. Moreover, the linear convergence of the algorithm with respect to $h + k$ ($h = \frac{\sqrt{2}}{n}$), stated in Corollary 2.4, is clearly observed in Figure 2.15.

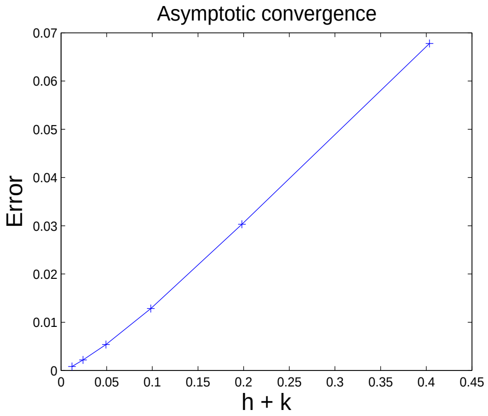

Figure 2.15: Example **1**: Evolution of the error with respect to $h + k$.

Second example: a compression problem

As a second example, we have considered the rectangular domain $[0, 2] \times [0, 3]$ depicted in Figure 2.16. Now, the body is fixed on $\Gamma_D = [0, 3] \times \{0\}$, the contact occurs on $\Gamma_C = \{0\} \times [0, 1]$ and surface tractions act on $\{2\} \times [0, 3]$, while $\{0\} \times [1, 2] \cup [0, 3] \times \{2\}$ is traction-free. The following data have been used in this example:

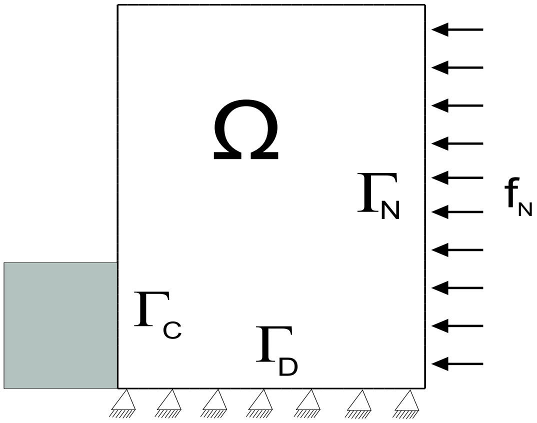

Figure 2.16: Example **2**: Physical setting.

$$T = 2\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N = (-30, 0)\,N/m^2,$$
$$E = 1000 N/m^2, \qquad r = 0.3, \qquad \zeta_* = 0.01, \qquad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega$$

In Figure 2.17 the von Mises stress norm is plotted over the deformed configuration at initial time (left-hand side) and at final time (right-hand side). Since a constant force was applied, the effect of the memory can be seen, because at final time the body tends to recover the stress-free position.

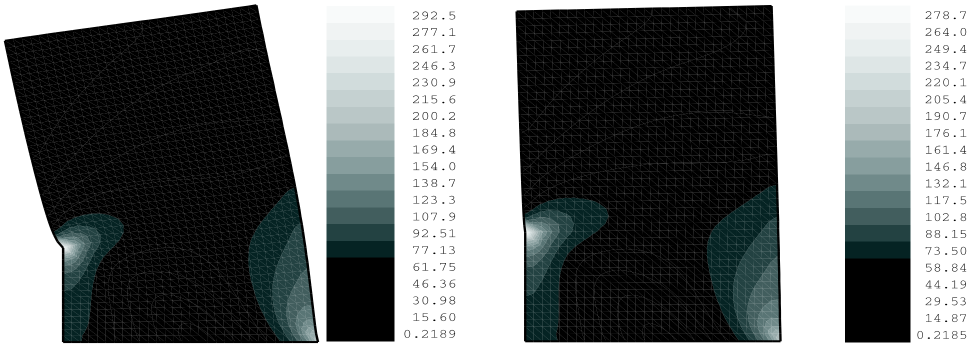

Figure 2.17: Example **2**: von Mises stress norm at initial and final times.

Also, in Figure 2.18 the damage field at final time, plotted over the deformed configuration, is shown. Again, the highest stressed areas coincide with the most damaged ones.

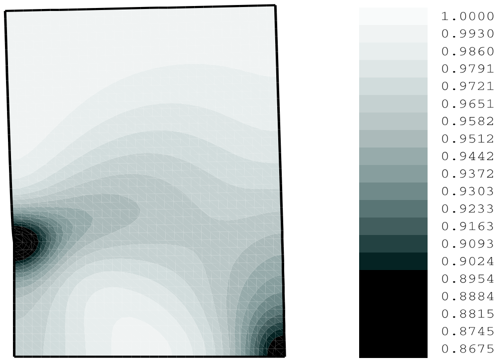

Figure 2.18: Example **2**: Damage field over the deformed configuration.

Third example: a beam in damageable contact

In this last example we have computed the setting shown in Figure 2.19 using a different memory function that in the previous examples,

$$\mathcal{G}(t-s,\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta)=-\zeta e^{-(t-s)}\Phi(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u})).$$

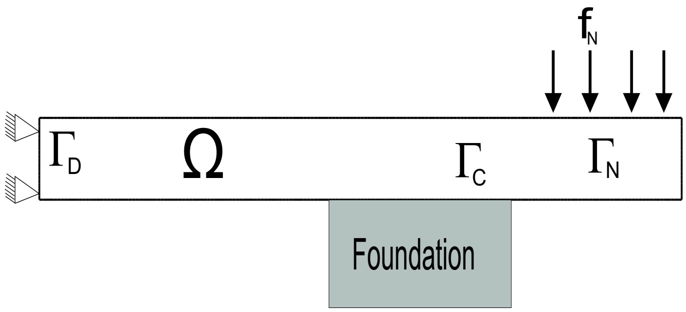

Figure 2.19: Example **3**: Physical setting.

In order to see the effect of this memory function, the applied surface tractions vanish during the time interval of interest; that is,

$$\boldsymbol{f}_N(t) = \begin{cases} (0,-10)\, N/m^2 & \text{if} \quad 0 < t \leq 0.5, \\ \mathbf{0}\, N/m^2 & \text{if} \quad 0.5 < t \leq 1. \end{cases}$$

For this example we have used the following data:

$$T = 1\, s, \quad \boldsymbol{f}_B = \mathbf{0}\, N/m^3,$$
$$E = 1000\, N/m^2, \qquad r = 0.3, \qquad \zeta_* = 0.01, \quad \zeta_0 = 1 \quad \forall \boldsymbol{x} \in \Omega$$

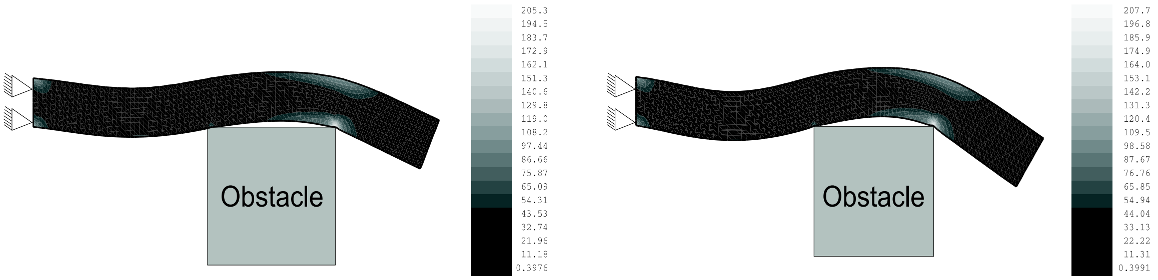

Figure 2.20: Example **3**: von Mises stress norm at times $t = 0$ and $t = 0.5$.

The von Mises stress norm is plotted at initial time and at time $t = 0.5$ over the corresponding deformed configurations in Figure 2.20. We notice that for this memory function the body has tendency to keep the nearest deformed configurations (in time). Due to this fact, at time $t = 0.5$, the forces are the same as at the initial time but the deformations have increased. Finally, we can observe in Figure 2.21 the von Mises stress norm (left-hand side) and the damage field (right-hand side) at final time. It is

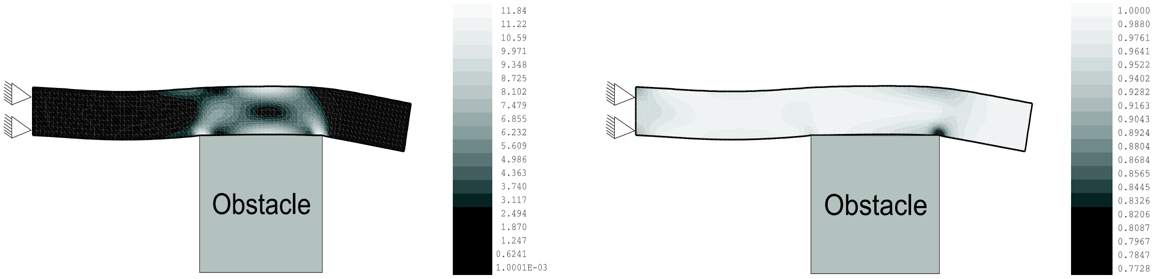

Figure 2.21: Example **3**: von Mises stress norm and damage field at final time $t = 1$.

possible to see that, although the forces have vanished, at final time the configuration is not the stress-free one (because of the influence of the memory trying to keep the previous deformations).

2.4 A frictional contact problem in elasto-viscoplasticity

The last problem that we are going to analyze in this chapter deals with an elastic-viscoplastic constitutive law. We consider a frictional contact process and assume that there is no loss of contact during the process, so the normal displacement u_ν vanishes on Γ_C. Two different friction laws, *Tresca's law* and a simplified version of the *Coulomb's law of dry friction*, will be considered, which lead to very similar variational formulations that can be analyzed using the same procedure. The results provided in this section have been recently published in [13].

The complete set of equations that fulfills the mechanical problem is as follows (see Section 2.2 for issues concerning the notations).

Problem P *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *and a damage field* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *such that,*

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0,T), \tag{2.146}$$

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T), \tag{2.147}$$

$$\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T), \tag{2.148}$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{2.149}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T), \tag{2.150}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{2.151}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \zeta(0) = \zeta_0 \quad \text{in} \quad \Omega. \tag{2.152}$$

To complete (2.146)–(2.152) we need to include the conditions on $\Gamma_C \times (0,T)$. We consider here two possibilities, which leads to two different mechanical problems which will be analyzed together.

First, we assume that the contact is bilateral and it is associated to Tresca's law of friction (see [2, 24, 42] and references therein). Therefore, we have

$$\left.\begin{array}{l} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \ \exists\, \lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau \end{array}\right\} \text{ on } \Gamma_C \times (0,T), \tag{2.153}$$

where g represents the friction bound. Secondly, we consider a simplified version of Coulomb's law of dry friction (see [29, 63] and references therein),

$$\left.\begin{array}{l} \sigma_\nu = S, \quad |\boldsymbol{\sigma}_\tau| \le \mu|\sigma_\nu|, \\ |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu| \Rightarrow \ \exists\, \lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau \end{array}\right\} \text{ on } \Gamma_C \times (0,T). \tag{2.154}$$

Thus, we denote by Problem $\mathbf{P}_1$ the above mechanical problem with frictional contact condition (2.153), and by Problem $\mathbf{P}_2$ that problem with condition (2.154).

In order to obtain the variational formulations of the above problems, we need to introduce additional notation and assumptions on the problem data.

If Problem $\mathbf{P}_1$ is considered, let us define the space of admissible displacements V by

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\, \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \quad v_\nu = 0 \quad \text{on} \quad \Gamma_C\},$$

or

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\, \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\}$$

for Problem $\mathbf{P}_2$. Moreover, let us recall that $\mathcal{K}$ is the following closed convex subset of B,

$$\mathcal{K} = \{\xi \in B\,;\, \zeta_* \le \xi \le 1 \quad \text{a.e. in} \quad \Omega\}.$$

The elastic tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a fourth-order symmetric positive definite positive tensor; that is,

$$\left.\begin{array}{l} \text{(a) The mapping } \boldsymbol{x} \in \Omega \to \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) \in \mathbb{S}^d \\ \qquad \text{is Lebesgue measurable and bounded.} \\ \text{(b) } \mathcal{E}(\boldsymbol{x})\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau},\boldsymbol{\sigma} \in \mathbb{S}^d. \\ \text{(c) There exists } C_{\mathcal{E}} > 0 \text{ such that} \\ \qquad\qquad \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge C_{\mathcal{E}}|\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d. \end{array}\right\} \tag{2.155}$$

The viscoplastic function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\left.\begin{array}{l}
\text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
\quad |\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \zeta_2)| \\
\qquad\qquad \le L_{\mathcal{G}} \left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2|\right) \\
\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \zeta_1, \zeta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \zeta) \text{ is a Lebesgue measurable function in } \Omega, \\
\qquad\qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \quad \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0) \text{ belongs to } Q.
\end{array}\right\} \tag{2.156}$$

The damage source function $\phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ verifies:

$$\left.\begin{array}{l}
\text{(a) There exists } L_{\phi} > 0 \text{ such that} \\
\quad |\phi(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \zeta_2)| \\
\qquad\qquad \le L_{\phi} \left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\zeta_1 - \zeta_2|\right) \\
\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \zeta_1, \zeta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \zeta) \text{ is a Lebesgue measurable function in } \Omega, \\
\qquad\qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \quad \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0) \text{ belongs to } Y.
\end{array}\right\} \tag{2.157}$$

The body forces and surface tractions have the regularity

$$\boldsymbol{f}_B \in W^{1,2}(0,T;H), \quad \boldsymbol{f}_N \in W^{1,2}(0,T;[L^2(\Gamma_N)]^2),$$

and let g, $S : \Gamma_C \to [0,+\infty)$ be given such that

$$g \in L^\infty(\Gamma_C),\ g \ge 0 \text{ a.e. on } \Gamma_C, \quad S \in L^\infty(\Gamma_C).$$

Using Riesz's representation theorem, let $\boldsymbol{f}(t) \in V$ be defined by the relation

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_B(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d} \quad \forall \boldsymbol{v} \in V,$$

in the case of Problem $\mathbf{P}_1$ or

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_B(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d} + (S, v_\nu)_{L^2(\Gamma_C)} \quad \forall \boldsymbol{v} \in V,$$

if Problem $\mathbf{P}_2$ is considered.

We denote by $j : V \to \mathbb{R}$ the functional

$$j(\boldsymbol{v}) = \int_{\Gamma_C} g|\boldsymbol{v}_\tau|\, da \quad \forall \boldsymbol{v} \in V, \tag{2.158}$$

for Problem $\mathbf{P}_1$, or

$$j(\boldsymbol{v}) = \int_{\Gamma_C} \mu|S||\boldsymbol{v}_\tau|\, da \quad \forall \boldsymbol{v} \in V, \tag{2.159}$$

if Problem $\mathbf{P}_2$ is considered.

Finally, let the initial data $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and ζ_0 be chosen in such a way that

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \quad \zeta_0 \in \mathcal{K}, \tag{2.160}$$

and verifying the following compatibility condition

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V. \tag{2.161}$$

At this point, we take $\boldsymbol{v} \in V$ an arbitrary test function. Multiplying equation (2.146) by $\boldsymbol{v} - \dot{\boldsymbol{u}}(t)$ and integrating by parts, we get

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_Q = \int_\Omega & \boldsymbol{f}_B(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) d\boldsymbol{x} \\ &+ \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) da.\end{aligned}$$

Using boundary conditions (2.150) and (2.151), we replace the right-hand side of the above expression by

$$(\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V$$

with the corresponding definition of $\boldsymbol{f}(t)$, depending on the problem that we are considering. On Γ_C we have

$$\begin{aligned}\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) &= \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau(t)) + \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \\ &= \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau - \boldsymbol{\sigma}_\tau(t) \cdot \dot{\boldsymbol{u}}_\tau(t) \\ &= \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau + g\|\dot{\boldsymbol{u}}_\tau(t)\| \\ &\geq g(\|\dot{\boldsymbol{u}}_\tau\| - \|\boldsymbol{v}_\tau\|),\end{aligned}$$

in the case that Problem $\mathbf{P}_1$ is considered or

$$\begin{aligned}\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)) &= \boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}_\tau-\dot{\boldsymbol{u}}_\tau(t))+S(v_\nu-\dot{u}_\nu(t))\\ &= \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau-\boldsymbol{\sigma}_\tau(t)\cdot\dot{\boldsymbol{u}}_\tau(t)+S(v_\nu-\dot{u}_\nu(t))\\ &= \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau+g\|\dot{\boldsymbol{u}}_\tau(t)\|+S(v_\nu-\dot{u}_\nu(t))\\ &\geq g(\|\dot{\boldsymbol{u}}_\tau\|-\|\boldsymbol{v}_\tau\|)+S(v_\nu-\dot{u}_\nu(t)),\end{aligned}$$

in the case of Problem $\mathbf{P}_2$.

Therefore, taking into account definitions (2.4)–(2.4) and (2.158)–(2.159), we finally have

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_Q\geq(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_V-j(\boldsymbol{v})+j(\dot{\boldsymbol{u}}(t))\quad\forall\boldsymbol{v}\in V.$$

The damage model used for this problem only differs with that used in Section 2.2 in the arguments of the damage source ϕ, so we follow the procedure obtained in that case to obtain,

$$\begin{aligned}&(\dot{\zeta}(t),\xi-\zeta(t))_Y+a(\zeta(t),\xi-\zeta(t))\\ &\qquad\geq(\phi(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t)),\ \xi-\zeta(t))_Y\quad\forall\xi\in\mathcal{K}.\end{aligned}$$

With these notations, the weak formulation for both problems $\mathbf{P}_1$ and $\mathbf{P}_2$ is as follows.

Problem VP. *Find a displacement field* $\boldsymbol{u}:[0,T]\to V$*, a stress field* $\boldsymbol{\sigma}:[0,T]\to Q$*, and a damage field* $\zeta:[0,T]\to\mathcal{K}$ *such that* $\boldsymbol{u}(0)=\boldsymbol{u}_0$*,* $\boldsymbol{\sigma}(0)=\boldsymbol{\sigma}_0$*,* $\zeta(0)=\zeta_0$ *and for a.e.* $t\in[0,T]$*,*

$$\dot{\boldsymbol{\sigma}}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))+\mathcal{G}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t))\quad\text{a.e.}\quad t\in(0,T),\tag{2.162}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_Q+j(\boldsymbol{v})-j(\dot{\boldsymbol{u}}(t))\geq(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_V\quad\forall\boldsymbol{v}\in V,\tag{2.163}$$

$$\begin{aligned}&(\dot{\zeta}(t),\xi-\zeta(t))_Y+a(\zeta(t),\xi-\zeta(t))\geq(\phi(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t)),\xi-\zeta(t))_Y\\ &\qquad\qquad\forall\xi\in\mathcal{K}.\end{aligned}\tag{2.164}$$

2.4.1 An existence and uniqueness result

Our main result in this subsection is the following.

Theorem 2.13. *Under the assumptions* (2.155)-(2.161)*, there exists a unique weak solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \zeta\}$ *to Problem* **VP** *with the following regularity*

$$\begin{aligned} &\boldsymbol{u} \in W^{1,2}(0,T;V), \quad \boldsymbol{\sigma} \in W^{1,2}(0,T;Q), \\ &\zeta \in W^{1,2}(0,T;Y) \cap L^2(0,T;B). \end{aligned} \tag{2.165}$$

Proof

It is done in three steps which will be developed below by using arguments similar to those applied in [70]. Since the modifications are straightforward, we skip the details.

i) Let $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \eta^2) \in L^2(0,T;Q\times Y)$. We define $\boldsymbol{z}^1_\eta$ by

$$\boldsymbol{z}^1_\eta(t) = \int_0^t \boldsymbol{\eta}^1(s)ds + \boldsymbol{z}^1_0, \tag{2.166}$$

where $\boldsymbol{z}^1_0$ is given by $\boldsymbol{z}^1_0 = \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)$.

Using a standard result (see [7], p.117), we obtain that there exists a unique solution $\boldsymbol{u}_\eta \in W^{1,2}(0,T;V)$ and $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;Q)$ to the following variational problem,

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \boldsymbol{z}^1_\eta(t) \quad \text{for all } t \in [0,T], \tag{2.167}$$

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}_\eta(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_V$$
$$\forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T), \tag{2.168}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{2.169}$$

Using now the fact that $\kappa \geq 0$, (2.160) and classical results on parabolic equations (see [4]), we conclude the existence of a unique $\zeta_\eta \in W^{1,2}(0,T;Y) \cap L^2(0,T;B)$ such that

$$\zeta_\eta(t) \in \mathcal{K} \quad \text{for all } t \in [0,T], \tag{2.170}$$

$$(\dot{\zeta}_\eta(t), \xi - \zeta_\eta(t))_Y + a(\zeta_\eta(t), \xi - \zeta_\eta(t)) \geq (\eta^2(t), \xi - \zeta_\eta(t))_Y$$
$$\forall \xi \in \mathcal{K}, \text{ a.e. } t \in (0,T), \tag{2.171}$$

$$\zeta_\eta(0) = \zeta_0. \tag{2.172}$$

ii) Next, we define the operator $\Lambda : L^2(0,T;Q\times Y) \to L^2(0,T;Q\times Y)$ by

$$\Lambda\boldsymbol{\eta}(t) = \Big(\mathcal{G}(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \zeta_\eta(t)),\ \phi(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \zeta_\eta(t))\Big), \tag{2.173}$$

for all $\boldsymbol{\eta} \in L^2(0,T;Q \times Y)$ and for all $t \in [0,T]$. Here, for every $\boldsymbol{\eta} \in L^2(0,T;Q \times Y)$, the triplet $\{\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta, \zeta_\eta\}$ denotes the unique solution to the variational problems (2.167)–(2.172) introduced in the first step.

We prove now that the operator Λ has a unique fixed point. To this end, we consider $\boldsymbol{\eta}_i = (\boldsymbol{\eta}_i^1, \eta_i^2) \in L^2(0,T;Q \times Y)$ for $i = 1, 2$, and let $t \in [0,T]$. Using the assumptions (2.155) and (2.156), we deduce that

$$\begin{aligned}||\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)||_Q \leq c\Big(&||\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)||_Q + ||\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)||_V \\ &+||\zeta_{\eta_1}(t) - \zeta_{\eta_2}(t)||_Y\Big).\end{aligned} \tag{2.174}$$

Here and below, c represents a generic positive constant whose value may change from line to line. From (2.166)–(2.172), after some algebraic manipulations we obtain the following three inequalities,

$$||\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)||_Q^2 \leq C\int_0^t ||\boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s)||_Q^2 ds, \tag{2.175}$$

$$||\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)||_V^2 \leq C\int_0^t ||\boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s)||_Q^2 ds, \tag{2.176}$$

$$||\zeta_{\eta_1}(t) - \zeta_{\eta_2}(t)||_Y^2 \leq C\int_0^t ||\boldsymbol{\eta}_1^2(s) - \boldsymbol{\eta}_2^2(s)||_Y^2 ds. \tag{2.177}$$

Using now (2.174)–(2.177), we conclude that the operator Λ defined in (2.173) is a contraction. By the Banach fixed point theorem, we obtain that this operator has a unique fixed point $\boldsymbol{\eta}^\star \in L^2(0,T;Q \times Y)$.

iii) We use (2.167)–(2.172) to prove that the triplet $\{\boldsymbol{u}_{\eta^\star}, \boldsymbol{\sigma}_{\eta^\star}, \zeta_{\eta^\star}\}$ is the unique solution to Problem **VP** satisfying (2.165), which concludes the proof. ■

2.4.2 Numerical analysis

Our interest lays now in numerically analyze a fully discrete approximation of Problem **VP**. Let $V^h \subset V, Q^h \subset Q$, and $B^h \subset B$ be three finite dimensional spaces approximating the spaces V, Q and B, respectively, and denote by $\mathcal{K}^h = \mathcal{K} \cap B^h$.

Remark 2.4. *Assume that Ω is a polygonal domain and let $\mathcal{T}^h$ be a regular finite element triangulation of the domain Ω compatible with the boundary partition $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$. We denote by θ^h the triangulation induced by $\mathcal{T}^h$ on Γ_C.*

In the two-dimensional numerical examples, we will consider the following variational spaces:

$$B^h = \{\xi^h \in \mathcal{C}(\overline{\Omega})\,;\, \xi^h_{|_T} \in P_1(T) \quad \forall T \in \mathcal{T}^h\}, \tag{2.178}$$

$$Q^h = \{\boldsymbol{\gamma}^h \in Q\,;\, \boldsymbol{\gamma}^h_{|_T} \in [P_0(T)]^{2\times 2} \quad \forall T \in \mathcal{T}^h\}, \tag{2.179}$$

in order to approximate the spaces B and Q, respectively, where $\mathcal{T}^h$ is a finite element triangulation. If Problem $\mathbf{P}_1$ is considered, the variational space V is approximated by

$$\begin{aligned} V^h = \{\boldsymbol{v}^h \in [\mathcal{C}(\overline{\Omega})]^2\,;\, \boldsymbol{v}^h_{|_T} \in [P_1(T)]^2 \quad \forall T \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \mathbf{0} \quad on \quad \Gamma_D,\\ v^h_{\nu|_C} = 0 \quad \forall C \in \theta^h\}, \end{aligned} \tag{2.180}$$

or

$$V^h = \{\boldsymbol{v}^h \in [\mathcal{C}(\overline{\Omega})]^2\,;\, \boldsymbol{v}^h_{|_T} \in [P_1(T)]^2 \quad \forall T \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \mathbf{0} \quad on \quad \Gamma_D\} \tag{2.181}$$

in the case of Problem $\mathbf{P}_2$.

Let $\mathcal{P}_{Q^h} : Q \to Q^h$ be the orthogonal projection operator defined through the relation

$$(\mathcal{P}_{Q^h}\boldsymbol{\gamma}, \boldsymbol{\gamma}^h)_Q = (\boldsymbol{\gamma}, \boldsymbol{\gamma}^h)_Q \quad \forall \boldsymbol{\gamma} \in Q,\, \boldsymbol{\gamma}^h \in Q^h. \tag{2.182}$$

We notice that this operator is nonexpansive:

$$\|\mathcal{P}_{Q^h}\boldsymbol{\gamma}\| \le \|\boldsymbol{\gamma}\|_Q \quad \forall \boldsymbol{\gamma} \in Q. \tag{2.183}$$

Let $\boldsymbol{u}^h_0$, $\boldsymbol{\sigma}^h_0$ and ζ^h_0 be appropriate approximations of the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and ζ_0, respectively. The fully discrete approximation is based on the forward Euler scheme and it has the following form.

Problem VP^{hk}. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}^{hk}_n\}_{n=0}^N \subset V^h$, a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}^{hk}_n\}_{n=0}^N \subset Q^h$, and a discrete damage field $\zeta^{hk} = \{\zeta^{hk}_n\}_{n=0}^N \subset \mathcal{K}^h$*

such that, for $n = 1, \ldots, N$,

$$\delta\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}) + \mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \tag{2.184}$$

$$(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk}))_Q + j(\boldsymbol{v}^h) - j(\delta\boldsymbol{u}_n^{hk}) \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk})_V \quad \forall \boldsymbol{v}^h \in V^h, \tag{2.185}$$

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geq (\phi(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y \quad \forall \xi^h \in \mathcal{K}^h, \tag{2.186}$$

where $\boldsymbol{u}_0^{hk}, \boldsymbol{\sigma}_0^{hk}$ *and* ζ_0^{hk} *are appropriate approximations of the initial conditions* $\boldsymbol{u}_0, \boldsymbol{\sigma}_0$ *and* ζ_0.

Remark 2.5. *We notice that plugging (2.184) into (2.185) and taking into account the definition of* $\mathcal{P}_{Q^h}$ *(2.182), we obtain the following variational inequality of second kind in the variable* $\delta\boldsymbol{u}_n^{hk}$ *for all* $\boldsymbol{v}^h \in V^h$,

$$\begin{aligned} k(\mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk}))_Q + j(\boldsymbol{v}^h) - j(\delta\boldsymbol{u}_n^{hk}) \\ \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk})_V - k(\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk}))_Q \\ -(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta\boldsymbol{u}_n^{hk}))_Q. \end{aligned} \tag{2.187}$$

It was solved applying similar ideas to those used in the previous problem, by regularizing the frictional term. Once $\delta\boldsymbol{u}_n^{hk}$ *is computed, we obtain* $\boldsymbol{u}_n^{hk}$ *and then, from (2.184) the discrete stress field* $\boldsymbol{\sigma}_n^{hk}$*. Finally, the damage field* ζ_n^{hk} *is obtained using a penalty-duality algorithm introduced in [6].*

Applying again the existence and uniqueness results for problems (1.6) and (1.7), we obtain the following theorem.

Theorem 2.14. *Let the assumptions of Theorem 2.13 still hold. Then, Problem* $\mathbf{VP}^{hk}$ *admits a unique solution* $\{\boldsymbol{u}^{hk}, \boldsymbol{\sigma}^{hk}, \zeta^{hk}\} \subset V^h \times Q^h \times \mathcal{K}^h$.

Proof

It is done using similar arguments to those applied in Theorem 2.5, by taking, in this case,

$$\begin{aligned} b(w_1, w_2) &= \frac{1}{k}(w_1, w_2)_Y + a(w_1, w_2), \\ L(w) &= (\phi(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), w)_Y + (\zeta_{n-1}^{hk}, w)_Y \end{aligned}$$

in order to apply the existence result for problem (1.6) to expression (2.186) and

$$\begin{aligned} b(\boldsymbol{w}_1, \boldsymbol{w}_2) &= k(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{w}_1), \boldsymbol{\varepsilon}(\boldsymbol{w}_2))_Q, \\ L(\boldsymbol{w}) &= (\boldsymbol{f}_n, \boldsymbol{w})_V - \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \end{aligned}$$

in order to apply the corresponding result for problem (1.7) to expression (2.187). ■

Now we proceed to derive error estimates for the discrete solution. First, we need to make the following additional regularity assumptions on the continuous solution:

$$\begin{aligned} &\boldsymbol{u} \in \mathcal{C}^1([0,T]; V), \quad \boldsymbol{\sigma} \in \mathcal{C}^1([0,T]; Q), \\ &\zeta \in \mathcal{C}([0,T]; H^2(\Omega)) \cap H^2(0,T;Y), \end{aligned} \tag{2.188}$$

which implies that $\boldsymbol{\sigma}_0 \in Q, \boldsymbol{u}_0 \in V$ and $\zeta_0 \in H^2(\Omega)$.

We use (2.184) recursively to get

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}) + \sum_{j=1}^{n} k\mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{n-1}^{hk}) + \boldsymbol{\sigma}_0^{hk} - \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \tag{2.189}$$

and we integrate (2.147) and use the initial conditions (2.152) to obtain

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{2.190}$$

Substracting now (2.189) from (2.190) at time $t = t_n$, we have

$$\begin{aligned} \boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} &= (I_Q - \mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0) + \mathcal{P}_{Q^h}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})) \\ &+ \mathcal{P}_{Q^h}\Big[\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s)) - \sum_{j=1}^{n} k\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{n-1}^{hk})\Big] \\ &+ \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}). \end{aligned}$$

We denote

$$r_n = \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\zeta_n - \zeta_n^{hk}\|_Y$$

and then, we find that

$$\|\int_{t_{n-1}}^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds-k\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk})\|_Q$$
$$\leq ckr_{n-1}+ck^2(\|\dot{\boldsymbol{u}}\|_{\mathcal{C}([0,T];V)}+\|\dot{\boldsymbol{\sigma}}\|_{\mathcal{C}([0,T];Q)}+\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}). \qquad (2.191)$$

Let us see it, we write

$$\int_{t_{n-1}}^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds-k\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk})$$
$$=\int_{t_{n-1}}^{t_n}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))-\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}),\zeta_{n-1})]ds$$
$$+k[\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}),\zeta_{n-1})-\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk})].$$

Using the assumption (2.156), we have

$$\|\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))-\mathcal{G}(\boldsymbol{\sigma}_{n-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}),\zeta_{n-1})\|_Q$$
$$\leq ck(\|\dot{\boldsymbol{u}}\|_{\mathcal{C}([0,T];V)}+\|\dot{\boldsymbol{\sigma}}\|_{\mathcal{C}([0,T];Q)}+\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)},$$

and

$$\|\mathcal{G}(\boldsymbol{\sigma}_{n-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}),\zeta_{n-1})-\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk})\|_Q\leq ckr_{n-1}.$$

We obtain the following estimation using inequality (2.191), assumptions (2.155) and the property (2.183),

$$\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q\leq\|(I_Q-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q+c\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+cr_0$$
$$+c\sum_{j=1}^{n}kr_{j-1}+ck(\|\dot{\boldsymbol{u}}\|_{\mathcal{C}([0,T];V)}+\|\dot{\boldsymbol{\sigma}}\|_{\mathcal{C}([0,T];Q)}+\|\dot{\zeta}\|_{\mathcal{C}([0,T];Y)}). \qquad (2.192)$$

Now we want to bound $\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V$, for $n=1,\dots,N$. We first integrate (2.162) and use the initial conditions to obtain

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\zeta(t))ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).$$

Now, we plug it into (2.162) at $t=t_n$, with $\boldsymbol{w}=\delta\boldsymbol{u}_n^{hk}$ and we get

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}))_V\leq(\boldsymbol{f}_n,\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk})+j(\delta\boldsymbol{u}_n^{hk})-j(\dot{\boldsymbol{u}}_n)$$
$$+(\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))ds,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}))_Q$$
$$+(\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}))_Q. \qquad (2.193)$$

Similarly, we rewrite expression (2.184) as follows

$$\boldsymbol{\sigma}_n^{hk} = k\mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}) + k\mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}) + \boldsymbol{\sigma}_{n-1}^{hk},$$

and plug it into (2.185) with an arbitrary $\boldsymbol{w}^h = \boldsymbol{w}_n^h \in V^h$, obtaining

$$\begin{aligned}
-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \delta\boldsymbol{u}_n^{hk}))_Q &\le (\boldsymbol{f}_n, \delta\boldsymbol{u}_n^{hk} - \boldsymbol{w}_n^h)_V + j(\boldsymbol{w}_n^h) - j(\delta\boldsymbol{u}_n^{hk})\\
&+\big(\sum_{j=1}^n k\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \delta\boldsymbol{u}_n^{hk})\big)_Q\\
&+(\boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \delta\boldsymbol{u}_n^{hk}))_Q. \qquad (2.194)
\end{aligned}$$

Using assumptions (2.155) we have

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}))_Q \ge \frac{1}{2k}(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 - \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2), \qquad (2.195)$$

and then, adding (2.194) and (2.193), after some algebra we obtain

$$\begin{aligned}
\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \le \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + ck\Big((\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n - \boldsymbol{w}_n^h))_Q\\
+(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}) + I_n, \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk} - \delta\boldsymbol{u}_n))_Q + R_n(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_n^h)\\
+(I_n + \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n - \boldsymbol{w}_n^h))_Q\Big),
\end{aligned}$$

where

$$\begin{aligned}
R_n(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_n^h) &= (\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \dot{\boldsymbol{u}}_n))_Q + j(\boldsymbol{w}_n^h) - j(\dot{\boldsymbol{u}}_n) - (\boldsymbol{f}_n, \boldsymbol{w}_n^h - \dot{\boldsymbol{u}}_n)_V, \qquad (2.196)\\
I_n &= \int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))ds - \sum_{j=1}^n k\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}), \zeta_{j-1}^{hk}). \qquad (2.197)
\end{aligned}$$

Using Cauchy's inequality (2.22) and the properties (2.155) and (2.156), we have

$$\begin{aligned}
&\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \le \|\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + ck\{r_0 + \|\delta\boldsymbol{u}_n - \boldsymbol{w}_n^h\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2\\
&+k\sum_{j=1}^n \|I_n\|_Q^2 + R_n(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_n^h) + (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}) + I_n, \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk} - \delta\boldsymbol{u}_n))_Q\}.
\end{aligned}$$

By an induction argument, taking into account that

$$\begin{aligned}
&k\sum_{j=1}^n(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk} - \delta\boldsymbol{u}_n))_Q\\
&= (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk} - (\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})))_Q \le c(r_0 + \epsilon\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2),
\end{aligned}$$

and the estimation,

$$\begin{aligned}
&k\sum_{j=1}^{n}(I_j,\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}-\delta\boldsymbol{u}_n))_Q\\
&\qquad=(I_1,\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}-\boldsymbol{u}_0))_Q+(I_n,\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}_n))_Q+\sum_{j=1}^{n-1}(I_j,\boldsymbol{\varepsilon}(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}))_Q\\
&\qquad\le c(r_0+\sum_{j=1}^{n}k[\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V^2+\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\zeta_j-\zeta_j^{hk}\|_Y^2]+k^2\\
&\qquad+\epsilon\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+k^2[\|\boldsymbol{\sigma}\|^2_{C^1([0,T];Q)}+\|\boldsymbol{u}\|^2_{C^1([0,T];V)}+\|\zeta\|^2_{C^1([0,T];Y)}),
\end{aligned}$$

we add the resulting expression to (2.192) to obtain

$$\begin{aligned}
&\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q\le c\{r_0+k\sum_{j=1}^{n}r_{j-1}+\|(I-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q^2\\
&\quad+k^2[\|\boldsymbol{\sigma}\|^2_{C^1([0,T];Q)}+\|\boldsymbol{u}\|_{C^1([0,T];V)}+\|\zeta\|^2_{C^1([0,T];Y)}]+\sum_{j=1}^{n}k\|\delta\boldsymbol{u}_j-\boldsymbol{w}_j^h\|_V^2\\
&\quad+k^2+\sum_{j=1}^{n}k[|R_j(\dot{\boldsymbol{u}}_j,\boldsymbol{w}_j^h)|+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V^2+\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\zeta_j-\zeta_j^{hk}\|_Y^2]\}.
\end{aligned}\tag{2.198}$$

The numerical analysis for the damage field is done in a similar way to that performed with viscoelastic materials. The only difference is that, in this case, the source damage function ϕ depends also on the stresses, and so, with respect to the expression (2.75), a new term $k\sum_{j=1}^{n}\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q^2$ appears on the right-hand side; that is,

$$\begin{aligned}
&\|\zeta_n-\zeta_n^{hk}\|_Y^2+k\sum_{j=1}^{n}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\quad\le c\Big\{\|\zeta_0-\zeta_0^{hk}\|_Y^2+\|\zeta_1-\xi_1^h\|_Y^2+k\sum_{j=1}^{n}\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2+k^2\|\dot\zeta\|^2_{C([0,T];Y)}\\
&\qquad+k\sum_{j=1}^{n}(\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2+\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q^2)+k\sum_{j=1}^{n}\|\zeta_j-\xi_j^h\|_B^2+\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k\sum_{j=1}^{n}\|\delta\zeta_j-\dot\zeta_j\|_Y^2\\
&\qquad+k\sum_{j=1}^{n}\|\phi(\boldsymbol{\sigma}_j,\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\Big\}.
\end{aligned}\tag{2.199}$$

Combining now (2.198) and (2.199), and using again a discrete version of Gronwall's inequality (see Lemma 1.1), we obtain the following error estimates result.

Theorem 2.15. *Let $\{\boldsymbol{u},\boldsymbol{\sigma},\zeta\}$ and $\{\boldsymbol{u}^{hk},\boldsymbol{\sigma}^{hk},\zeta^{hk}\}$ be the respective solutions to problems VP and VP^{hk}. Let the assumptions of Theorem 2.13 still hold. Under the additional regularity conditions (2.188), the following error estimates are obtained for all $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ and $\{\xi_j^h\}_{j=0}^N \subset \mathcal{K}^h$,*

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\Big\}+k\sum_{j=1}^N\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\qquad\le c\Big(e_0+\|\zeta_1-\xi_1^h\|_Y^2+\max_{0\le n\le N}\|(I-\mathcal{P}_{Q^h})(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_0)\|_Q^2+k^2\\
&\qquad\quad+k^2[\|\boldsymbol{u}\|^2_{\mathcal{C}^1([0,T];V)}+\|\boldsymbol{\sigma}\|^2_{\mathcal{C}^1([0,T];Q)}+\|\zeta\|^2_{\mathcal{C}^1([0,T];Y)}]+k\sum_{j=1}^N\|\zeta_j-\xi_j^h\|_B^2\\
&\qquad\quad+\frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+\max_{0\le n\le N}\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad\quad+k\sum_{j=1}^N\|\phi(\boldsymbol{\sigma}_j,\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\\
&\qquad\quad+\sum_{j=1}^N k[\|\delta\boldsymbol{u}_j-\boldsymbol{w}_j^h\|_V^2+|R_j(\dot{\boldsymbol{u}}_j,\boldsymbol{w}_j^h)|+\|\delta\zeta_j-\dot\zeta_j\|_Y^2]\Big).
\end{aligned}\tag{2.200}$$

We notice that from estimates (2.200) we obtain different rates of convergence, depending on the regularity assumed for the continuous solution. Thus, proceeding as in [20, 24], the following theorem is deduced.

Theorem 2.16. *Let the assumptions of Theorem 2.15 still hold and consider the finite element spaces B^h, Q^h and V^h defined by (2.178)–(2.181). Let the initial conditions $\boldsymbol{u}_0^{hk}$, $\boldsymbol{\sigma}_0^{hk}$ and ζ_0^{hk} be defined by*

$$\boldsymbol{u}_0^{hk}=\Pi^h\boldsymbol{u}_0,\quad \boldsymbol{\sigma}_0^{hk}=\mathcal{P}_{Q^h}\boldsymbol{\sigma}_0,\quad \zeta_0^{hk}=\pi^h\zeta_0.$$

where π^h and Π^h are defined in (2.30) and (2.34). We also assume that

$$\begin{aligned}
&\boldsymbol{u}\in\mathcal{C}^1([0,T];[H^2(\Omega)]^d),\quad \boldsymbol{\sigma}\in W^{1,\infty}(0,T;[H^1(\Omega)]^{d\times d}),\\
&\boldsymbol{\sigma\nu}\in\mathcal{C}([0,T];[L^2(\Gamma)]^d),\quad \dot\zeta\in L^2(0,T;B).
\end{aligned}$$

Then, we have the following error estimation

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q+\|\zeta_n-\zeta_n^{hk}\|_Y\Big\}$$
$$+k\sum_{j=1}^{N}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H\le c\Big(h^{3/4}+k+I^k(\dot{\boldsymbol{u}})^{1/2}\Big), \quad (2.201)$$

where

$$I^k(\dot{\boldsymbol{u}})=\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|\dot{\boldsymbol{u}}(t)-\dot{\boldsymbol{u}}_j\|_V^2\,dt.$$

Moreover, under the additional regularity assumption

$$\boldsymbol{u}_{\tau|C}\in\mathcal{C}^1(0,T;H^2(C))\quad\forall C\in\theta^h,$$

we can obtain the following improved error estimation,

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q+\|\zeta_n-\zeta_n^{hk}\|_Y\Big\}$$
$$+k\sum_{j=1}^{N}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H\le c\Big(h+k+I^k(\dot{\boldsymbol{u}})^{1/2}\Big). \quad (2.202)$$

Proof

The terms that come from the evolution equation for the damage are bounded in the same way than in Corollaries 2.1 and 2.2, so we will discuss only those which come from the displacements equation.

The condition $\boldsymbol{u}\in\mathcal{C}^1([0,T];[H^2(\Omega)]^d)$ implies that $\boldsymbol{u}_\tau\in\mathcal{C}^1([0,T];[H^{\frac{3}{2}}(\Gamma_C)]^d)$.

For each j, let $\Pi^h\dot{\boldsymbol{u}}_j$ be the piecewise linear interpolant of $\dot{\boldsymbol{u}}_j$. Therefore,

$$\begin{aligned}\|\delta\boldsymbol{u}_j-\Pi^h\dot{\boldsymbol{u}}_j\|_V&\le c(\|\delta\boldsymbol{u}_j-\dot{\boldsymbol{u}}_j\|_V+\|\dot{\boldsymbol{u}}_j-\Pi^h\dot{\boldsymbol{u}}_j\|_V)\\&\le\frac{c}{k}\int_{t_{j-1}}^{t_j}\|\dot{\boldsymbol{u}}(t)-\dot{\boldsymbol{u}}_j\|_V dt+ch\|\dot{\boldsymbol{u}}_j\|_{[H^2(\Omega)]^d},\end{aligned}$$

which leads to the following,

$$\sum_{j=1}^{N}k\|\delta\boldsymbol{u}_j-\Pi^h\dot{\boldsymbol{u}}_j\|_V^2\le cI^k(\dot{\boldsymbol{u}})+ch^2\|\dot{\boldsymbol{u}}\|_{\mathcal{C}([0,T];[H^2(\Omega)]^d)}.$$

Under the assumption

$$\boldsymbol{\sigma\nu}\in\mathcal{C}([0,T];[L^2(\Gamma)]^d),$$

we can follow a standard argument to obtain

$$\begin{aligned}
R_j(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_j^h) &= (\boldsymbol{\sigma}_j, \boldsymbol{\varepsilon}(\boldsymbol{w}_j^h - \dot{\boldsymbol{u}}_j))_Q + j(\boldsymbol{w}_j^h) - j(\dot{\boldsymbol{u}}_j) - (\boldsymbol{f}_j, \boldsymbol{w}_j^h - \dot{\boldsymbol{u}}_j)_V \\
&= \int_{\Gamma_C} (\boldsymbol{\sigma}_n)_\tau (\boldsymbol{w}_\tau^h - (\dot{\boldsymbol{u}}_n)_\tau) da + j(\boldsymbol{w}^h) - j(\dot{\boldsymbol{u}}_n) \\
&\le \int_{\Gamma_C} (|\boldsymbol{\sigma}_n)_\tau| + g)|\boldsymbol{w}_\tau^h - (\dot{\boldsymbol{u}}_n)_\tau| da \le c\|\boldsymbol{w}_\tau^h - (\dot{\boldsymbol{u}}_n)_\tau\|_{[L^2(\Gamma_C)]^d},
\end{aligned}$$

and so we get

$$R_j(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_j^h) \le ch^{\frac{3}{2}},$$

or

$$R_j(\dot{\boldsymbol{u}}_n, \boldsymbol{w}_j^h) \le ch^2$$

in the case that the regularity $\boldsymbol{u}_{\tau|C} \in H^1(0,T;H^2(C))$ is assumed. ■

Both estimates (2.201) and (2.202) involve the quantity $I^k(\dot{\boldsymbol{u}})$. Under the assumption

$$\boldsymbol{u} \in H^2(0,T;V),$$

we obtain that

$$I^k(\dot{\boldsymbol{u}}) \le ck^2\|\ddot{\boldsymbol{u}}\|^2_{L^2(0,T;V)},$$

and estimates (2.202) implies the linear convergence of the algorithm under the required regularity assumptions. We notice that the regularity condition in time is not too hard to be fulfilled since the processes are assumed to be quasistatic.

2.4.3 Numerical examples

Now, some numerical simulations involving two-dimensional problems are presented. In all the examples below the elasticity tensor $\mathcal{E}$ was taken as the two-dimensional plane stress elasticity tensor defined in (2.36).

The viscoplastic function is a version of the Maxwell function defined by

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) = (1-\zeta)\Phi(\boldsymbol{\sigma}),$$

being $\Phi : \mathbb{S}^d \to \mathbb{S}^d$ a truncation operator defined by

$$\forall \boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in \mathbb{S}^d, \quad (\Phi(\boldsymbol{\tau}))_{ij} = \begin{cases} L & \text{if} \quad \tau_{ij} > L, \\ \tau_{ij} & \text{if} \quad \tau_{ij} \in [-L, L], \\ -L & \text{if} \quad \tau_{ij} < -L, \end{cases}$$

and value $L = 1000$ was taken.

Also, $\epsilon = 10^{-9}$ was employed in the regularization of the frictional term.

First example: an academical test

In this first numerical example, the numerical convergence of the algorithm is verified by computing a sequence of numerical solutions, based on uniform partitions of the time interval and regular triangulations of the domain over the physical setting depicted in Figure 2.22 (left-hand side).

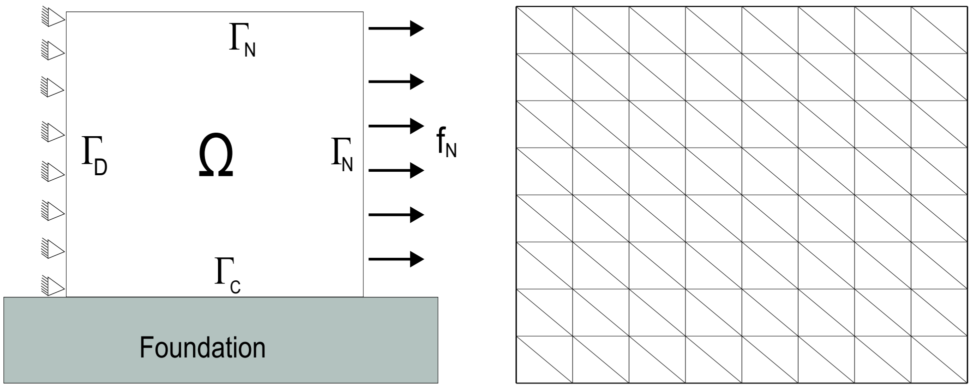

Figure 2.22: Example **1**: Physical setting and FE mesh for $n = 8$.

The domain $\Omega = (0,1) \times (0,1)$ is the cross-section of a three-dimensional elasto-viscoplastic body. On the part $\Gamma_D = \{0\} \times [0,1]$ the body is clamped and therefore, the displacement field vanishes there. Finally, horizontal tractions act on $\{1\} \times (0,1]$, the boundary part $(0,1) \times \{1\}$ is traction-free and contact with a rigid foundation is assumed on $\Gamma_C = [0,1] \times \{0\}$.

The following data have been used in the numerical simulations:

$$T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N(\boldsymbol{x},t) = (50,0)e^t\,N/m^2,$$
$$E = 1000\,N/m^2, \quad r = 0.3, \quad g = 0.7\,N/m^2,$$
$$\mu = 0.05, \quad S = 10\,N/m^2, \quad \boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \boldsymbol{\sigma}_0 = \boldsymbol{0}\,N/m^2, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega.$$

Our aim here is to show the numerical convergence of the algorithm for both problems $\mathbf{P}_1$ and $\mathbf{P}_2$. Therefore, several uniform partitions for the time interval and the domain, dividing Ω into $2n^2$ triangles, have been performed (the finite element mesh corresponding to $n = 8$ is plotted on the right-hand side of Figure 2.22). Note that the number of degrees of freedom is $2(n+1)^2$ and we used the solutions obtained with $n = 256$ and $k = 0.001$ as the "exact solutions". The numerical errors are given by

$$e_{hk} = \max_{0\leq n\leq N} \left\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q + \|\zeta_n - \zeta_n^{hk}\|_Y \right\}.$$

In Tables 2.6 and 2.7 these errors, obtained for some n and k, are shown. The numerical convergence of the algorithm is clearly observed. Moreover, in Figure 2.23 the numerical errors are depicted, with respect to $h + k$, for both problems. The calculations performed do not allow to observe the linear convergence stated in Theorem 2.16.

$n \downarrow k \rightarrow$	0.02	0.01	0.005	0.002
8	0.10709	0.10949	0.11073	0.11150
16	0.086661	0.088699	0.089865	0.090584
32	0.056016	0.054691	0.055179	0.055792
64	0.022069	0.020082	0.019149	0.018625
128	0.0086640	0.0056945	0.0044492	0.0038105

Table 2.6: Example **1**: Numerical errors for some n and k (Tresca's law).

Concerning Problem $\mathbf{P}_1$, in Figure 2.24 the von Mises stress norm and the damage field are plotted at time $t = 1$ over the deformed configuration for values $n = 64$ and $k = 0.002$. Due to the friction, the highest stressed area, and the corresponding most damaged one, is located at the right lower corner.

$n \downarrow k \rightarrow$	0.02	0.01	0.005	0.002
8	0.25021	0.20819	0.20990	0.21095
16	0.17420	0.17705	0.17868	0.17968
32	0.11018	0.10967	0.11032	0.11108
64	0.019688	0.017136	0.015941	0.015253
128	0.0087856	0.0049974	0.0033757	0.0025772

Table 2.7: Example **1**: Numerical errors for some n and k (Coulomb's law).

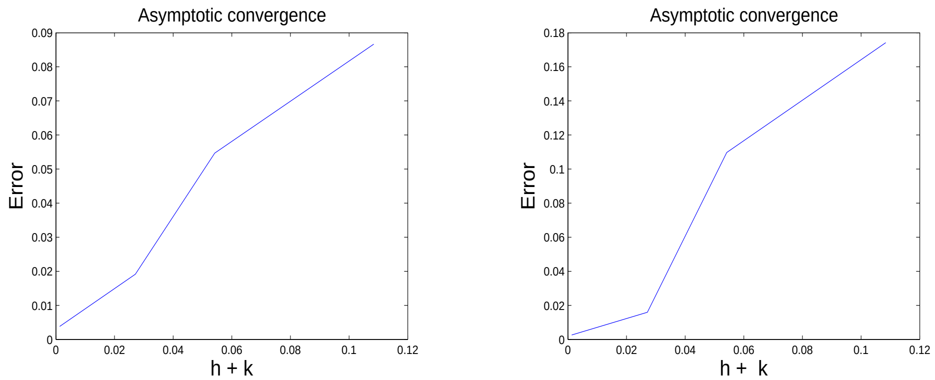

Figure 2.23: Example **1**: Evolution of the numerical errors e_{hk} with respect to $h+k$ for the Tresca's problem (left) and Coulomb's problem (right).

Using the same values, the von Mises stress norm and the damage field are shown, over the deformed configuration and at final time, in Figure 2.25 for Problem $\mathbf{P}_2$. We notice now that the highest stressed areas (and so, the most damaged ones) are located at the upper boundary and at the right lower corner.

Second example: Tresca's law with a large friction bound

In this last example, the problem depicted in Figure 2.26 has been considered. The Tresca's friction condition is employed with a friction bound very large, in such a way that it will not be achieved. This implies that, although we have not considered clamping conditions ($\Gamma_D = \emptyset$), we obtain the uniqueness of solution due to the fixation of the contact boundary.

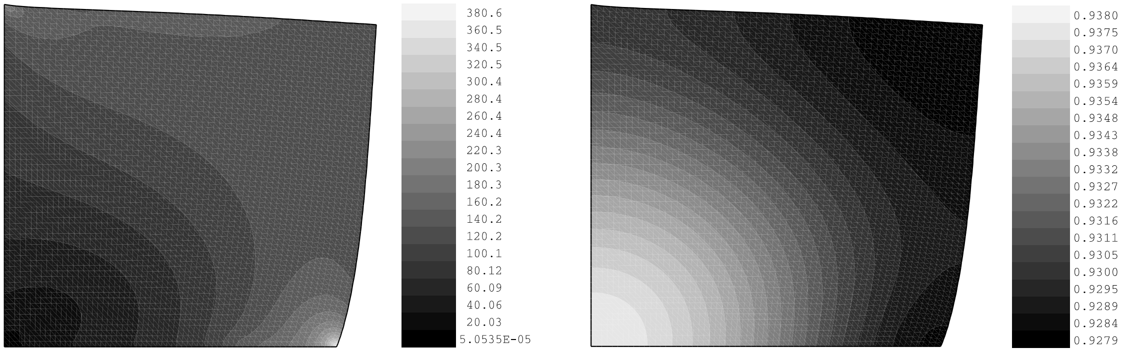

Figure 2.24: Example **1**: von Mises stress norm and damage field at $t = 1\,s$ (Tresca's law).

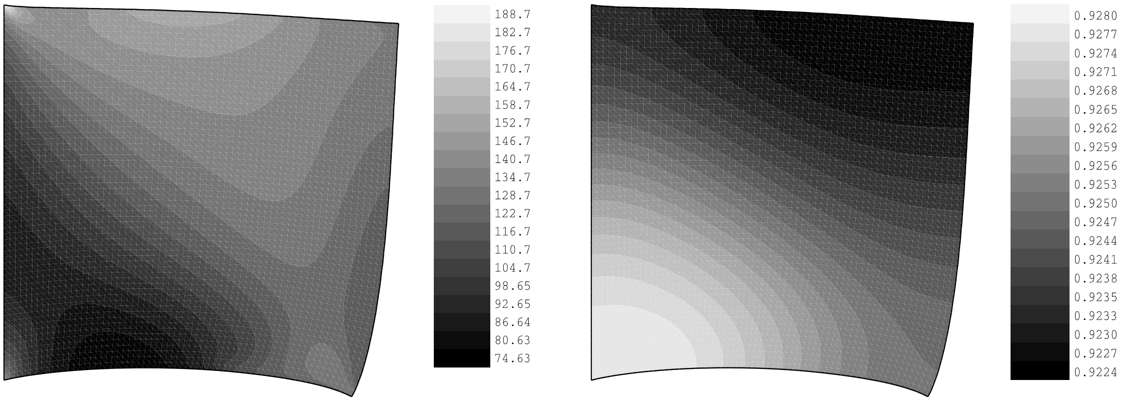

Figure 2.25: Example **1**: von Mises stress norm and damage field at $t = 1\,s$ (Coulomb's law).

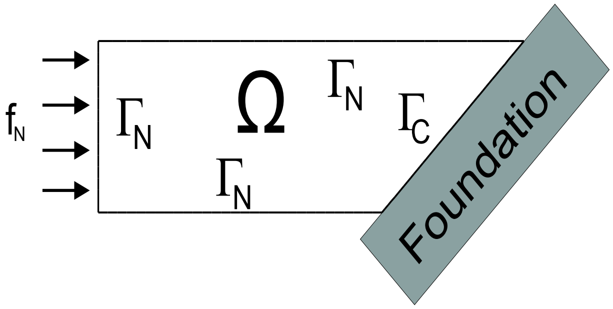

Figure 2.26: Example **2**: Tresca's law with a large friction bound.

The following data have been used:

$$T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N(\boldsymbol{x}, t) = (55, 0)e^t\,N/m^2,$$
$$E = 1000\,N/m^2, \quad r = 0.3, \quad g = 1000\,N/m^2,$$
$$\boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \boldsymbol{\sigma}_0 = \boldsymbol{0}\,N/m^2, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega.$$

The deformed mesh at final time and the initial configuration are plotted in Figure

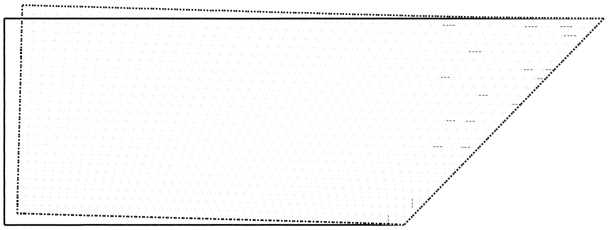

Figure 2.27: Example **2**: Deformed mesh at final time and the initial configuration.

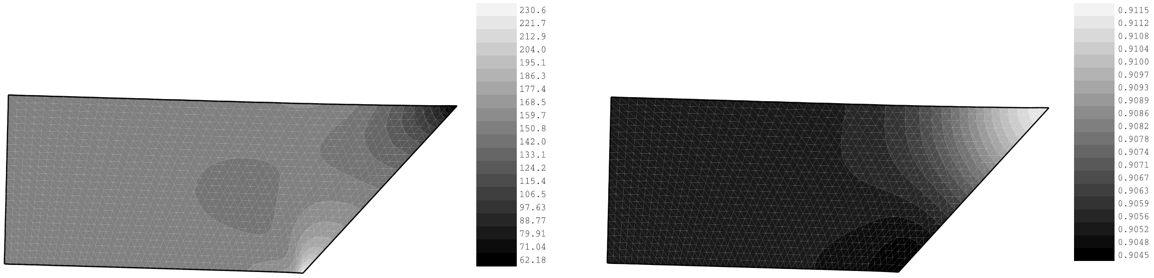

Figure 2.28: Example **2**: von Mises stress norm and damage field at $t = 1\,s$.

2.27. We notice that the contact boundary remains clamped. The von Mises stress norm and the damage field at final time are shown in Figure 2.28. As it was expected, the highest stressed region and the corresponding most damaged one, is located at the right lower contact corner with the foundation. On the right upper corner, since the body does not slip on the contact boundary, the stresses are lower and therefore the damage is smaller there.

Chapter 3

Dynamic viscoelastic contact problems with damage

Processes where bodies suffer fast forces or impacts depend strongly on the inertia effects. These problems involve an extra term in the motion equation which depends on mass properties and accelerations; that is

$$\rho(\boldsymbol{x},t)\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \mathrm{Div}\boldsymbol{\sigma}(\boldsymbol{x},t) + \boldsymbol{f}_B(\boldsymbol{x},t).$$

In this fourth chapter we analyze three dynamic contact problems arising in *Kelvin-Voigt* viscoelasticity. We have only employed this material behaviour because of the absence of theoretical results for the existence of a unique solution to the variational problems in the case that elastic-viscoplastic or viscoelastic with long memory constitutive laws.

The first two problems arise with two different cases of contact conditions: the first one is a unilateral frictionless contact condition with a normal compliance condition and in the second one we consider a frictional contact problem on a bilateral process. In the third problem we have assumed a material behaviour which consists of a different influence of the damage function over the constitutive law. In this case, the damage function affects not only to the elastic response of the material, but also to the viscous part and to the contact condition. These facts deal with a different treatment on the numerical analysis and, mainly, on the variational analysis of the problem.

Because of the analogies between these three problems, due to the fact that all of them are dynamic processes with similar constitutive laws, we list now some assumptions over the problem data that will hold on the whole chapter.

The results detailed in the present chapter have been recently published or submitted for publication in [12, 16, 17].

In the study of the mechanical problems, we assume that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left.\begin{array}{l}
\text{(a) There exists } C_{\mathcal{A}} > 0 \text{ such that} \\
\quad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\tau}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\tau}_2)\| \le C_{\mathcal{A}}\,|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| \quad \forall \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d, \\
\quad \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\quad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}}\,|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \\
\quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega, \\
\quad \forall \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \quad \text{is an element of } Q.
\end{array}\right\} \qquad (3.1)$$

The elasticity operator $\mathcal{E} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ verifies:

$$\left.\begin{array}{l}
\text{(a) There exists } M_{\mathcal{E}} > 0 \text{ such that} \\
\quad \|\mathcal{E}(\boldsymbol{x},\boldsymbol{\xi}_1,\zeta_1) - \mathcal{E}(\boldsymbol{x},\boldsymbol{\xi}_2,\zeta_2)\| \le M_{\mathcal{E}}\,(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\zeta_1 - \zeta_2|) \\
\quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \zeta_1, \zeta_2 \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{E}(\boldsymbol{x},\boldsymbol{\xi},\zeta) \text{ is Lebesgue measurable on } \Omega, \\
\quad \forall \boldsymbol{\xi} \in \mathbb{S}^d,\ \zeta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{E}(\boldsymbol{x},\boldsymbol{0},0) \quad \text{is an element of } Q.
\end{array}\right\} \qquad (3.2)$$

The damage source function $\phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left.\begin{array}{l}\text{(a) There exists } L_\phi > 0 \ \text{ such that} \\ \qquad |\phi(\boldsymbol{x},\boldsymbol{\varepsilon}_1,\zeta_1) - \phi(\boldsymbol{x},\boldsymbol{\varepsilon}_2,\zeta_2)| \leq L_\phi\,(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|) \\ \qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \zeta_1,\zeta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x},\boldsymbol{\varepsilon},\zeta) \ \text{ is Lebesgue measurable on } \Omega, \\ \qquad \forall\,\boldsymbol{\varepsilon} \in \mathbb{S}^d, \quad \zeta \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \phi(\boldsymbol{x},\boldsymbol{0},0) \quad \text{belongs to } Y. \end{array}\right\} \tag{3.3}$$

We suppose that the mass density satisfies

$$\rho \in L^\infty(\Omega) \quad \text{and there exists } \rho_* > 0 \text{ such that } \rho(\boldsymbol{x}) \geq \rho_* \ \text{ a.e. } \ \boldsymbol{x} \in \Omega. \tag{3.4}$$

In the following we use a new inner product on the space H,

$$((\boldsymbol{u},\boldsymbol{v}))_H = (\rho\,\boldsymbol{u},\boldsymbol{v})_H \quad \forall \boldsymbol{u},\,\boldsymbol{v} \in H,$$

and let the associated norm be

$$|||\boldsymbol{v}|||_H = ((\boldsymbol{v},\boldsymbol{v}))_H^{1/2}.$$

By assumption (3.4), the two norms $|||\cdot|||_H$ and $\|\cdot\|_H$ are equivalent on H.

As in the previous chapters, V will denote the set of admissible displacement functions, depending its definition on the particular contact conditions that are provided for each problem. Again, as it happened for quasistatic contact problems, the following inner product is considered on V,

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,$$

and, due to the Korn's inequality, $(V,\|\cdot\|_V)$ is a Hilbert space.

Moreover, the inclusion mapping of $(V,\|\cdot\|_V)$ into $(H,|||\cdot|||_H)$ is continuous and dense. Let $(V',\|\cdot\|_{V'})$ be the dual space of V. Identifying H with its own dual, we can write

$$V \subset H = H' \subset V'.$$

We use the notation $\langle \cdot, \cdot \rangle_{V' \times V}$ to represent the duality pairing between V' and V. We have

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V' \times V} = ((\boldsymbol{u}, \boldsymbol{v}))_H \qquad \forall \boldsymbol{u} \in H,\ \boldsymbol{v} \in V. \tag{3.5}$$

We will also denote by $\mathcal{K}$ the set of admissible damage functions,

$$\mathcal{K} = \{\xi \in B;\ \zeta_* \leq \xi \leq 1 \text{ a.e. in } \Omega\}. \tag{3.6}$$

In all the numerical examples presented in this chapter the elasticity operator $\mathcal{E}$ has the same form used in viscoelastic materials, already presented in (2.106), with a truncation value $L = 1000$. Moreover, the viscosity operator $\mathcal{A}$ is defined as $\mathcal{A} = \frac{1}{m}\mathcal{B}$, being m a positive factor which modifies the weight of the viscous response in the equation and that varies between the different examples. We also notice that the damage source function ϕ is given by (1.3) (with $q^* = 1000$), where λ_D, λ_u and λ_w are process parameters defined differently for each example. The normal compliance function p_ν, given by (2.145), is also employed.

3.1 A dynamic frictionless contact problem in viscoelasticity

The first problem of this chapter deals with a frictionless contact process. The contact conditions are the same that those used in Section 2.3 in viscoelasticity with long memory; that is, the contact is with a deformable foundation, modeled with a normal compliance condition and without friction. The whole mechanical formulation of this problem is as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *and a damage field* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *such that*

$$\rho\,\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_B \quad \text{in } \Omega \times (0,T), \tag{3.7}$$

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in } \Omega \times (0,T), \tag{3.8}$$

$$\dot{\zeta} - \kappa\,\triangle\zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in } \Omega \times (0,T), \tag{3.9}$$

$$\frac{\partial\zeta}{\partial\boldsymbol{\nu}} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.10}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T), \tag{3.11}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on } \Gamma_N \times (0,T), \tag{3.12}$$

$$-\sigma_\nu = p_\nu(u_\nu - g) \quad \text{on } \Gamma_C \times (0,T), \tag{3.13}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_C \times (0,T), \tag{3.14}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{3.15}$$

As usual, we consider the space of admissible displacement functions given by

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \ ; \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_D\}.$$

The normal compliance function $p_\nu : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+$ verifies

$$\left.\begin{array}{l}
\text{(a) There exists } L_\nu > 0 \text{ such that} \\
\qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu\,|r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_C, \\
\qquad \forall r \in \mathbb{R}. \\
\text{(c) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)) \cdot (r_1 - r_2) \ge 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(d) } p_\nu(\boldsymbol{x}, r) = 0 \quad \text{for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_C.
\end{array}\right\} \tag{3.16}$$

The body forces and surface tractions have the following regularity,

$$\boldsymbol{f}_B \in \mathcal{C}([0,T];H), \quad \boldsymbol{f}_N \in \mathcal{C}([0,T];[L^2(\Gamma_N)]^d). \tag{3.17}$$

Finally, the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in H, \quad \zeta_0 \in \mathcal{K}, \tag{3.18}$$

where $\mathcal{K}$ is given by (2.48).

We define the function $\boldsymbol{f} : [0,T] \to V'$ and the functional $j : V \times V \to \mathbb{R}$ by the following equations:

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V' \times V} = (\boldsymbol{f}_B(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{[L^2(\Gamma_N)]^d} \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T], \tag{3.19}$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} p_\nu(u_\nu - g)\, v_\nu \, da \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.20}$$

Notice that integral (3.20) is well defined by (3.16), and conditions (3.17) imply the regularity

$$\boldsymbol{f} \in \mathcal{C}([0,T];V'). \tag{3.21}$$

Now it is time to provide a variational formulation of problem P. To that end, we assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \zeta)$ are regular functions satisfying (3.7)–(3.15) and let $\boldsymbol{v} \in V$, $t \in [0,T]$. Multiplying (3.7) by $\boldsymbol{v}$, using Green's formula (1.8) and (3.5) we obtain

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v} \rangle_{V'V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = \int_\Omega \boldsymbol{f}_B(t) \cdot \boldsymbol{v} d\boldsymbol{x} + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v} d\Gamma.$$

With the boundary conditions (3.11)–(3.14) and (3.17) we have

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v} \rangle_{V'V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V'V} + \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v} d\Gamma, \tag{3.22}$$

and taking into account now (3.20), we find that

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v} \rangle_{V'V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{u}(t), \boldsymbol{v}) = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_{V'V} \quad \forall \boldsymbol{v} \in V. \tag{3.23}$$

For the damage evolution, we perform the same procedure than that used in Sections 2.2 and 2.4, and therefore the variational formulation of the mechanical problem P is the following.

Problem VP**.** *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q$*, and a damage field* $\zeta : [0,T] \to \mathcal{K}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \tag{3.24}$$

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} \rangle_{V' \times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} \quad \forall \boldsymbol{w} \in V, \tag{3.25}$$

$$\begin{aligned} &(\dot{\zeta}(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t)) \\ &\qquad \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)),\ \xi - \zeta(t))_Y \quad \forall \xi \in \mathcal{K}, \end{aligned} \tag{3.26}$$

for almost any $t \in (0,T)$, and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0. \tag{3.27}$$

3.1.1 An existence and uniqueness result

The existence of a unique solution to Problem VP is proved based on results for variational equations, variational inequalities and Banach fixed point. The proof is detailed below.

We assume in what follows that (3.1)–(3.4) and (3.16)–(3.18) hold. As a first step towards an existence and uniqueness result, let $\boldsymbol{\eta} \in L^2(0,T;V')$ and $\theta \in L^2(0,T;Y)$ be given and consider the following variational problems.

Problem VP^1_η**.** *Find a displacement field* $\boldsymbol{u}_\eta : [0,T] \to V$ *such that*

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{w} \rangle_{V' \times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + \langle \boldsymbol{\eta}(t), \boldsymbol{w} \rangle_{V' \times V} = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} \\ &\qquad \forall \boldsymbol{w} \in V,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.28}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0. \tag{3.29}$$

Problem VP^2_θ**.** *Find a damage field* $\zeta_\theta : [0,T] \to B$ *such that*

$$\begin{aligned} &\zeta_\theta(t) \in \mathcal{K}, \quad (\dot{\zeta}_\theta(t), \xi - \zeta_\theta(t))_Y + a(\zeta_\theta(t), \xi - \zeta_\theta(t)) \\ &\qquad \geq (\theta(t),\ \xi - \zeta_\theta(t))_Y \quad \forall \xi \in \mathcal{K},\ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.30}$$

$$\zeta(0) = \zeta_0. \tag{3.31}$$

Proceeding as in [21], we can show that Problem VP^1_η has a unique solution $\boldsymbol{u}_\eta$ satisfying

$$\boldsymbol{u}_\eta \in W^{1,2}(0,T;V) \cap \mathcal{C}^1([0,T];H), \quad \ddot{\boldsymbol{u}}_\eta \in L^2(0,T;V').$$

It follows from [4] that Problem VP^2_θ has a unique solution ζ_θ satisfying

$$\zeta_\theta \in W^{1,2}(0,T;Y) \cap L^2(0,T;B).$$

Consequently, for all $t \in [0,T]$ we may define the element

$$\Lambda(\boldsymbol{\eta},\theta)(t) = (\Lambda^1(\boldsymbol{\eta},\theta)(t), \Lambda^2(\boldsymbol{\eta},\theta)(t)) \in V' \times Y$$

given by

$$\langle \Lambda^1(\boldsymbol{\eta},\theta)(t), \boldsymbol{w}\rangle_{V'\times V} = (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \zeta_\theta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}_\eta(t), \boldsymbol{w}) \quad \forall \boldsymbol{w} \in V, \tag{3.32}$$

$$\Lambda^2(\boldsymbol{\eta},\theta)(t) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \zeta_\theta(t)). \tag{3.33}$$

We have the following result.

Lemma 3.1. *For all $(\boldsymbol{\eta},\theta) \in L^2(0,T;V'\times Y)$ the function $\Lambda(\boldsymbol{\eta},\theta) : [0,T] \to V' \times Y$ is continuous. Moreover, there exists a unique element $(\boldsymbol{\eta}^*,\theta^*) \in L^2(0,T;V'\times Y)$ such that $\Lambda(\boldsymbol{\eta}^*,\theta^*) = (\boldsymbol{\eta}^*,\theta^*)$.*

Proof

Let $(\boldsymbol{\eta},\theta) \in L^2(0,T;V'\times Y)$ and let t_1, $t_2 \in [0,T]$. From (3.2), (3.16) and (3.32) it follows that

$$\begin{aligned}
&\|\Lambda^1(\boldsymbol{\eta},\theta)(t_1) - \Lambda^1(\boldsymbol{\eta},\theta)(t_2)\|_{V'} \\
&\quad \le \|\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t_1)), \zeta_\theta(t_1)) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t_2)), \zeta_\theta(t_2))\|_Q \\
&\qquad + c_0 \|p_\nu(u_{\eta\nu}(t_1) - g) - p_\nu(u_{\eta\nu}(t_2) - g)\|_{L^2(\Gamma_C)} \\
&\quad \le (L_{\mathcal{E}} + c_0)\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V + \|\zeta_\theta(t_1) - \zeta_\theta(t_2)\|_Y.
\end{aligned} \tag{3.34}$$

By the regularities of $\boldsymbol{u}_\eta$ and ζ_θ, we deduce that $\Lambda^1(\boldsymbol{\eta},\theta) \in \mathcal{C}([0,T];V')$.

From (3.33) and (3.3) it follows that

$$\|\Lambda^2(\boldsymbol{\eta},\theta)(t_1) - \Lambda^2(\boldsymbol{\eta},\theta)(t_2)\|_Y \le L_\phi(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V + \|\zeta_\theta(t_1) - \zeta_\theta(t_2)\|_Y). \tag{3.35}$$

Then, $\Lambda^2(\boldsymbol{\eta},\theta) \in \mathcal{C}([0,T];Y)$. We conclude that $\Lambda(\boldsymbol{\eta},\theta) \in \mathcal{C}([0,T];V' \times Y)$.

Let $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2) \in L^2(0,T;V' \times Y)$, $t \in [0,T]$. We use the notation $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_{\eta_i} = \boldsymbol{v}_i$, $\zeta_{\theta_i} = \zeta_i$ for $i = 1,2$. Arguments similar to those employed in the proof of (3.34) and (3.35) yield

$$\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{V'\times Y} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V + \|\zeta_1(t) - \zeta_2(t)\|^2_Y\Big). \tag{3.36}$$

Here and below, c denotes a generic positive constant whose value may change from line to line. Since $\boldsymbol{u}_i(t) = \int_0^t \boldsymbol{v}_i(s)ds + \boldsymbol{u}_0$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V \le c\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|^2_V ds. \tag{3.37}$$

Moreover, from (3.28) we obtain

$$\langle \dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V'\times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_1)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2))_Q + \langle \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V'\times V} = 0 \quad \text{a.e. in } (0,T).$$

We integrate this relation with respect to t and use the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$ and (3.1)(b) to find

$$m_{\mathcal{A}}\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|^2_V ds \le -\int_0^t \langle \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\rangle_{V'\times V} ds.$$

Then, using Cauchy's inequality (2.22), we have

$$\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|^2_V ds \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|^2_{V'} ds. \tag{3.38}$$

Similarly, from (3.30) and (3.31) we get

$$(\dot{\zeta}_1(t) - \dot{\zeta}_2(t), \zeta_1(t) - \zeta_2(t))_Y + a(\zeta_1(t) - \zeta_2(t), \zeta_1(t) - \zeta_2(t)) \le (\theta_1(t) - \theta_2(t), \zeta_1(t) - \zeta_2(t))_Y \quad \text{a.e. in } (0,T).$$

Since $a(\zeta_1(t)-\zeta_2(t),\zeta_1(t)-\zeta_2(t))\geq 0$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\zeta_1(t)-\zeta_2(t)\|_Y^2\leq\|\theta_1(t)-\theta_2(t)\|_Y\|\zeta_1(t)-\zeta_2(t)\|_Y \quad \text{for a.e. } t\in(0,T),$$

and therefore,

$$\begin{aligned}\|\zeta_1(t)-\zeta_2(t)\|_Y^2&\leq 2\int_0^t\|\theta_1(s)-\theta_2(s)\|_Y\|\zeta_1(s)-\zeta_2(s)\|_Y ds\\&\leq\frac{1}{2\alpha}\int_0^t\|\theta_1(s)-\theta_2(s)\|_Y^2+2\alpha\int_0^t\|\zeta_1(s)-\zeta_2(s)\|_Y^2 ds.\end{aligned}$$

By Gronwall's lemma, it follows that

$$\|\zeta_1(t)-\zeta_2(t)\|_Y^2\leq c\int_0^t\|\theta_1(s)-\theta_2(s)\|_Y^2 ds, \quad \text{for a.e.} \quad t\in(0,T). \tag{3.39}$$

Combining (3.36) and (3.39), we obtain

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{V'\times Y)}^2\\&\qquad\leq c\int_0^t\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|_{V'\times Y}^2 ds,\end{aligned}$$

and reiterating this inequality n times it leads to

$$\begin{aligned}&\|\Lambda^n(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^n(\boldsymbol{\eta}_2,\theta_2)(t)\|_{L^2(0,T;V'\times Y)}^2\\&\qquad\leq\frac{(cT)^n}{n!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|_{L^2(0,T;V'\times Y)}^2.\end{aligned}$$

This implies that for n sufficiently large, Λ^n is a contraction in the Banach space $L^2(0,T;V'\times Y)$. Therefore, Λ has a unique fixed point. ■

We now state and prove the following existence and uniqueness result.

Theorem 3.1. *Assume that* (3.1)–(3.4) *and* (3.16)–(3.18) *hold. Then there exists a unique solution* $(\boldsymbol{u},\boldsymbol{\sigma},\zeta)$ *to Problem* VP*. Moreover, the solution satisfies*

$$\boldsymbol{u}\in W^{1,2}(0,T;V)\cap\mathcal{C}^1([0,T];H), \quad \ddot{\boldsymbol{u}}\in L^2(0,T;V'), \tag{3.40}$$

$$\boldsymbol{\sigma}\in L^2(0,T;Q), \quad \text{Div }\boldsymbol{\sigma}\in L^2(0,T;V'), \tag{3.41}$$

$$\zeta\in W^{1,2}(0,T;Y)\cap L^2(0,T;B). \tag{3.42}$$

Proof

Existence. Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0,T; V' \times Y)$ be the fixed point of Λ. Denote by $\boldsymbol{u}^*$ the solution to (3.28) and (3.29) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and let ζ^* be the solution to (3.30) and (3.31) for $\theta = \theta^*$. Define $\boldsymbol{\sigma}^* = \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}^*)) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}^*), \zeta^*)$. Using (3.32), (3.33) and keeping in mind that $\Lambda^1(\boldsymbol{\eta}^*, \theta^*) = \eta^*$, $\Lambda^2(\boldsymbol{\eta}^*, \theta^*) = \theta^*$, we find that the triplet $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*, \zeta^*)$ is a solution to (3.24)–(3.27). The regularities (3.40) and (3.42) follow from that of the solutions to problems VP^1_η and VP^2_θ, respectively. Moreover, from (3.40), (3.1) and (3.2) it follows that $\boldsymbol{\sigma}^* \in L^2(0,T;Q)$. Letting $\boldsymbol{w} \in [\mathcal{C}_0^\infty(\Omega)]^d$ in (3.24) and using (3.19) and (3.20), we have

$$\rho \ddot{\boldsymbol{u}}(t) = \text{Div } \boldsymbol{\sigma}(t) + \boldsymbol{f}_B(t) \quad \text{a.e.} \quad t \in (0,T).$$

Now, the assumptions (3.4), (3.17), the regularity (3.40) and the previous equality imply that Div $\boldsymbol{\sigma} \in L^2(0,T;V')$.

Uniqueness. The uniqueness statement is a straightforward consequence of the unique fixed point of the operator Λ given by (3.32)–(3.33). ■

We conclude that, under assumptions (3.1)–(3.4) and (3.16)–(3.18), Problem P has a unique *weak solution* with regularity (3.40)–(3.42).

3.1.2 Numerical analysis

We consider now a numerical scheme for Problem VP. We assume the following additional regularity of the displacement and damage fields,

$$\left.\begin{array}{l} \boldsymbol{u} \in \mathcal{C}^1([0,T];V), \quad \ddot{\boldsymbol{u}} \in \mathcal{C}([0,T];V'), \\ \zeta \in H^2(0,T;Y) \cap \mathcal{C}([0,T];H^2(\Omega)), \end{array}\right\} \tag{3.43}$$

which also implies that $\dot{\boldsymbol{u}} \in \mathcal{C}([0,T];V)$. For convenience, we rewrite variational problem VP in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$. We can recover the displacements from the velocity by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)ds + \boldsymbol{u}_0 \quad t \in [0,T]. \tag{3.44}$$

Then, Problem VP can be written in the following equivalent form.

Problem VP^{eq}**.** *Find a velocity field* $\boldsymbol{v} : [0,T] \to V$ *and a damage field* $\zeta : [0,T] \to \mathcal{K}$ *such that*

$$\langle \dot{\boldsymbol{v}}(t), \boldsymbol{w} \rangle_{V' \times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

$$+ j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} \quad \forall \boldsymbol{w} \in V, \tag{3.45}$$

$$(\dot{\zeta}(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t))$$

$$\geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_Y \quad \forall \xi \in \mathcal{K}, \tag{3.46}$$

for almost any $t \in (0,T)$, and

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0. \tag{3.47}$$

We use two finite dimensional spaces $V^h \subset V$ and $B^h \subset B$ to approximate the spaces V and B, respectively. Let us denote by $\mathcal{K}^h \subset B^h$ a convex set which approximates the convex set $\mathcal{K}$ ($\mathcal{K}^h = \mathcal{K} \cap B^h$).

Let $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and ζ_0^h be appropriate approximations of the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and ζ_0, respectively. A fully discrete approximation of Problem VP^{eq}, based on the forward Euler scheme, is the following (see the previous chapter for details concerning the notation).

Problem VP^{hk}**.** *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete damage field* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$ *such that*

$$((\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h))_H + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q$$

$$+ j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) = \langle \boldsymbol{f}_n, \boldsymbol{w}^h \rangle_{V' \times V} \quad \forall \boldsymbol{w}^h \in V^h, \tag{3.48}$$

$$(\delta \zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk})$$

$$\geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y \quad \forall \xi^h \in \mathcal{K}^h, \tag{3.49}$$

for $n = 1, 2, \ldots, N$, *and*

$$\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h, \quad \zeta_0^{hk} = \zeta_0^h. \tag{3.50}$$

Here, the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ is defined by

$$\boldsymbol{u}_n^{hk} = \sum_{j=1}^{n} k\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad n = 1, \dots N. \tag{3.51}$$

Using classical arguments on variational inequalities (see [41]), we can deduce the existence of a unique solution to Problem VP^{hk}.

Theorem 3.2. *Assume that* (3.1)–(3.4) *and (3.16)–(3.18) hold. Then there exists a unique solution* $(\boldsymbol{v}^{hk}, \zeta^{hk}) \subset V^h \times \mathcal{K}^h$ *to Problem* VP^{hk}.

We turn now to an estimation of the errors $\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}$, $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$ and $\zeta_n - \zeta_n^{hk}$. Writing (3.45) at time $t = t_n$ and taking $\boldsymbol{w} = \boldsymbol{w}_n^h \in V^h \subset V$, we obtain

$$\begin{aligned}
&\langle \dot{\boldsymbol{v}}_n, \boldsymbol{w}_n^h \rangle_{V' \times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\
&\qquad + j(\boldsymbol{u}_n, \boldsymbol{w}_n^h) = \langle \boldsymbol{f}_n, \boldsymbol{w}_n^h \rangle_{V' \times V} \quad \forall \boldsymbol{w}_n^h \in V^h.
\end{aligned} \tag{3.52}$$

Substracting (3.48) from (3.52), we have for all $\boldsymbol{w}_n^h \in V^h$,

$$\begin{aligned}
&\langle \dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h \rangle_{V' \times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\
&\qquad + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\
&\qquad + j(\boldsymbol{u}_n, \boldsymbol{w}_n^h) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}_n^h) = 0,
\end{aligned}$$

and hence,

$$\begin{aligned}
&((\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h))_H + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\
&\quad = \langle \delta \boldsymbol{v}_n - \dot{\boldsymbol{v}}_n, \boldsymbol{w}_n^h \rangle_{V' \times V} + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\
&\qquad + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}_n^h) - j(\boldsymbol{u}_n, \boldsymbol{w}_n^h).
\end{aligned}$$

Write $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} = (\boldsymbol{v}_n - \boldsymbol{w}_n^h) + (\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk})$ and use the above equation to obtain

$$\begin{aligned}
&((\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_H + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\qquad = ((\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}_n^h))_H + T_n,
\end{aligned} \tag{3.53}$$

where T_n is defined as,

$$\begin{aligned}T_n &= (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h))_Q + \langle \delta \boldsymbol{v}_n - \dot{\boldsymbol{v}}_n, \boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk} \rangle_{V' \times V} \\ &\quad + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}))_Q \\ &\quad + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}) - j(\boldsymbol{u}_n, \boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}).\end{aligned}$$

Using the inequality

$$|j(\boldsymbol{u}_1, \boldsymbol{v}) - j(\boldsymbol{u}_2, \boldsymbol{v})| \leq c\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^2(\Gamma_C)} \|\boldsymbol{v}\|_{L^2(\Gamma_C)} \leq c\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V,$$

which comes from definition (3.20) and (3.16), we find that

$$\begin{aligned}|T_n| &\leq c\, \big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + \|\delta \boldsymbol{v}_n - \dot{\boldsymbol{v}}_n\|_{V'} \|\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}\|_V \\ &\qquad + (\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y) \|\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}\|_V\big].\end{aligned}$$

Using Cauchy's inequality (2.22) with $\epsilon = \frac{1}{2}$ we obtain

$$\begin{aligned}((\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_H &= \frac{1}{k}((\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_H - \frac{1}{k}((\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_H \\ &\geq \frac{1}{k}|||\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}|||_H^2 - \frac{1}{k}|||\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}|||_H |||\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}|||_H \\ &\geq \frac{1}{2k}(|||\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}|||_H^2 - |||\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}|||_H^2). \qquad (3.54)\end{aligned}$$

Thus, from (3.53), (using (3.1 b)), we obtain

$$\begin{aligned}&|||\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}|||_H^2 - |||\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}|||_H^2 + c\, k\, \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \\ &\quad \leq 2\, ((\, (\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{w}_n^h\,))_H \\ &\qquad + c\, k\, \big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + \|\delta \boldsymbol{v}_n - \dot{\boldsymbol{v}}_n\|_{V'} \|\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}\|_V \\ &\qquad + (\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y) \|\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}\|_V\big].\end{aligned}$$

To proceed further, we use the triangle inequality

$$\|\boldsymbol{w}_n^h - \boldsymbol{v}_n^{hk}\|_V \leq \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V$$

and Cauchy's inequality (2.22) with $\epsilon > 0$ small enough to include terms with the form $k\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2$ on the left-hand side of the inequality. In this way we obtain the following estimation, valid for $n = 1, \dots, N$:

$$\begin{aligned}
&|||\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}|||_H^2 - |||\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}|||_H^2 + c\,k\,\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \\
&\quad \le 2\,(((\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{w}_n^h\,))_H + c\,k\,\big[\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 \\
&\quad + \|\delta\boldsymbol{v}_n - \dot{\boldsymbol{v}}_n\|_{V'}^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_Y^2\big].
\end{aligned}$$

An induction argument leads to

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + c\sum_{j=1}^n k\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 \le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 \\
&\quad + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 + \|\zeta_j - \zeta_{j-1}^{hk}\|_Y^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2\Big) \\
&\quad + c\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + 2\sum_{j=1}^n(((\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}) - (\boldsymbol{v}_{j-1} - \boldsymbol{v}_{j-1}^{hk}), \boldsymbol{v}_j - \boldsymbol{w}_j^h\,))_H,
\end{aligned}$$

where the equivalence of the two norms $\|\cdot\|_H$ and $|||\cdot|||_H$ over H is used. Notice that

$$\begin{aligned}
\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\|_V^2 \le \Big(&\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + I_j^2 \\
&+ \sum_{l=1}^{j-1} k^2\|\boldsymbol{v}_l - \boldsymbol{v}_l^{hk}\|_V^2\Big),
\end{aligned}$$

where I_j is the following integration error

$$I_j = \Big\| \int_0^{t_{j-1}} \boldsymbol{v}(s)ds - \sum_{l=1}^{j-1} k\boldsymbol{v}_l \Big\|_V.$$

Then, using the last two inequalities we obtain

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \sum_{j=1}^n k\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 \\
&\quad \le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 + I_j^2 + \sum_{l=1}^{j-1} k\|\boldsymbol{v}_l - \boldsymbol{v}_l^{hk}\|_V^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 \\
&\qquad + \|\zeta_j - \zeta_{j-1}\|_Y^2 + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2\Big) \\
&\qquad + c\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + c\,\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 \\
&\qquad + 2\sum_{j=1}^n((\boldsymbol{v}_j - \boldsymbol{v}_j^{hk} - (\boldsymbol{v}_{j-1} - \boldsymbol{v}_{j-1}^{hk}), \boldsymbol{v}_j - \boldsymbol{w}_j^h))_H. \qquad (3.55)
\end{aligned}$$

Since we are using the same damage model than that studied in Section 2.2, we use the estimation (2.75) previously obtained. Also, keeping in mind that

$$\begin{aligned}
&\sum_{j=1}^{n}((\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h))_H\\
&\quad=((\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h))_H-((\boldsymbol{v}_0-\boldsymbol{v}_0^h,\boldsymbol{v}_1-\boldsymbol{w}_1^h))_H\\
&\qquad\quad+\sum_{j=1}^{n-1}((\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H\\
&\quad\le\gamma\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+c\,\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+c\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2\\
&\qquad\quad+c\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H,
\end{aligned}\tag{3.56}$$

where $\gamma>0$ is a parameter chosen to be small enough, and combining (3.55) and (2.75), we find that

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^{n}k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2+k\sum_{j=1}^{n}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\quad\le c\sum_{j=1}^{n}k\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_{V'}^2+I_j^2+\sum_{l=1}^{j-1}k\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2\\
&\qquad+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2\Big)+c\Big\{\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2\\
&\qquad+\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+\|\zeta_0-\zeta_0^h\|_Y^2+\|\zeta_1-\xi_1^h\|_Y^2+\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+k\sum_{j=1}^{n}\|\zeta_j-\xi_j^h\|_B^2+k\sum_{j=1}^{n-1}\|\zeta_j-\zeta_j^{hk}\|_Y^2\\
&\qquad+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k^2+k\sum_{j=1}^{n}\|\delta\zeta_j-\dot{\zeta}_j\|_Y^2\\
&\qquad+k\sum_{j=1}^{n}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\\
&\qquad+\frac{1}{k}\sum_{j=1}^{n}\|(\boldsymbol{v}_j-\boldsymbol{w}_j^{hk})-(\boldsymbol{v}_{j-1}-\boldsymbol{w}_{j-1}^{hk})\|_H^2\Big\}.
\end{aligned}\tag{3.57}$$

Defining now, for $n=1,2,\dots N$,

$$e_n=\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+k\sum_{j=1}^{n}\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2+k\sum_{j=1}^{n}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2,$$

$$
\begin{aligned}
g_n = k\sum_{j=1}^{n}&\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2\Big)\\
&+\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|\zeta_0 - \zeta_0^h\|_Y^2\\
&+\|\zeta_1 - \xi_1^h\|_Y^2 + \|\zeta_n - \xi_n^h\|_Y^2 + k^2 + k\sum_{j=1}^{n}\|\zeta_j - \xi_j^h\|_B^2\\
&+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + k\sum_{j=1}^{n}\|\delta\zeta_j - \dot{\zeta}_j\|_Y^2\\
&+k\sum_{j=1}^{n}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y\\
&+\frac{1}{k}\sum_{j=1}^{n-1}\|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2,
\end{aligned}
$$

estimates (3.57) establish that

$$
e_n \le c g_n + c\sum_{j=1}^{n} k e_{j-1}, \quad n = 1, \dots, N.
$$

Let $e_0 = \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\zeta_0 - \zeta_0^h\|_Y^2 = g_0$. We can then apply a discrete version of Gronwall's lemma (see Lemma 1.2) to obtain the following.

Lemma 3.2. Assume that (3.1)–(3.4) and (3.16)–(3.18) hold. Then, under the additional regularity assumption (3.43), the following error estimates hold for all $\{\boldsymbol{w}_j^h\}_{j=1}^N \subset V^h$ and $\{\xi_j^h\}_{j=1}^N \subset \mathcal{K}^h$,

$$
\begin{aligned}
\max_{0\le n\le N}&\Big\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\Big\}\\
&\quad +k\sum_{j=1}^{N}(\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2)\\
&\le c\Big\{\sum_{j=1}^{N} k\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2\Big)\\
&\quad +\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \max_{0\le n\le N}\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2\\
&\quad +\|\zeta_0 - \zeta_0^h\|_Y^2 + \max_{0\le n\le N}\|\zeta_n - \xi_n^h\|_Y^2 + k^2\\
&\quad +\frac{1}{k}\sum_{j=1}^{N-1}\|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + k\sum_{j=1}^{N}\|\delta\zeta_j - \dot{\zeta}_j\|_Y^2
\end{aligned}
$$

$$+k\sum_{j=1}^{N}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y$$

$$+k\sum_{j=1}^{N}\|\zeta_j-\xi_j^h\|_B^2+\frac{1}{k}\sum_{j=1}^{N-1}\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2\Big\}. \tag{3.58}$$

The error relation (3.58) is a basis for error estimation. Assume that Ω is a polyhedral domain and denote by $\{\mathcal{T}^h\}$ a regular family of triangulations of $\overline{\Omega}$ compatible with the partition of the boundary $\Gamma=\partial\Omega$ into Γ_D, Γ_N and Γ_C. Let the spaces V^h and B^h consist of continuous and piecewise affine functions,

$$V^h=\{\boldsymbol{v}^h\in[\mathcal{C}(\overline{\Omega})]^d\ ;\ \boldsymbol{v}^h|_{\mathcal{T}}\in[P_1(\mathcal{T})]^d,\quad \mathcal{T}\in\mathcal{T}^h,\quad \boldsymbol{w}^h=\mathbf{0}\ \text{ on }\ \Gamma_D\}, \tag{3.59}$$

$$B^h=\{\xi^h\in\mathcal{C}(\overline{\Omega})\ ;\ \xi^h|_{\mathcal{T}}\in P_1(\mathcal{T}),\quad \mathcal{T}\in\mathcal{T}^h\}, \tag{3.60}$$

and define the following convex subset of B^h,

$$\mathcal{K}^h=\{\xi^h\in B^h\ ;\ \zeta_*\le\xi^h\le 1\}. \tag{3.61}$$

The following corollary which states the linear convergence of the algorithm is obtained.

Corollary 3.1. Let assumptions (3.1)–(3.4) and (3.16)–(3.18) hold and let the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and ζ_0^h be defined by interpolation:

$$\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0,\quad \boldsymbol{v}_0^h=\Pi^h\boldsymbol{v}_0,\quad \zeta_0^h=\pi^h\zeta_0,$$

where π^h and Π^h were given in (2.30) and (2.34). Under the following additional solution regularity

$$\boldsymbol{u}\in W^{2,2}(0,T;V)\cap\mathcal{C}^1([0,T];[H^2(\Omega)]^d),\quad \dddot{\boldsymbol{u}}\in L^2(0,T;V'), \tag{3.62}$$

$$\zeta\in\mathcal{C}([0,T];H^2(\Omega))\cap W^{2,2}(0,T;Y)\cap W^{1,2}(0,T;B), \tag{3.63}$$

the fully discrete scheme is linearly convergent; that is,

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\zeta_n-\zeta_n^{hk}\|_Y\Big\}$$

$$+\,k\sum_{j=1}^{N}\big(\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V+\|\nabla(\zeta_j-\zeta_j^{hk})\|_H\big)\le c(h+k).$$

Proof

The only new term which needs to be bounded is the expression which comes from the inertia term $k\sum_{j=1}^{N}\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_{V'}$, since the remaining terms were considered in the previous chapter. We have

$$\delta\boldsymbol{v}_j-\dot{\boldsymbol{v}}_j=\frac{1}{k}\int_{t_{j-1}}^{t_j}(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}(t_j))dt=\frac{1}{k}\int_{t_{j-1}}^{t}\ddot{\boldsymbol{v}}(s)dsdt,$$

and therefore,

$$k\sum_{j=1}^{N}\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_{V'}\leq ck\|\ddot{\boldsymbol{v}}\|_{L^2(0,T;V')}.$$

■

3.1.3 Numerical examples

Numerical resolution

Let $n\in\{1,\ldots,N\}$. The damage equation (3.49) is solved with the same procedure that it was applied in the previous chapter for viscoelastic and elastic-viscoplastic materials.

Secondly, we must solve the nonlinear variational equation,

$$\begin{aligned}(\rho\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+k(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q&=-k(\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q\\&-kj(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{w}^h)+k(\boldsymbol{f}_n,\boldsymbol{w}^h)_V+(\boldsymbol{v}_{n-1}^{hk},\boldsymbol{w}^h)_H\quad\forall\boldsymbol{w}^h\in V^h.\end{aligned}\tag{3.64}$$

Since, in practice, $\mathcal{A}$ is linear ($(\mathcal{A}(\boldsymbol{x})(\boldsymbol{\tau}))_{ij}=(A\boldsymbol{\tau})_{ij}=a_{ijkl}\tau_{kl}$, $1\leq i,j\leq d$), then (3.64) is equivalent to a linear system and its resolution implies no difficulty. It was solved by using Cholesky's method. Finally, the displacement field $\boldsymbol{u}_n^{hk}$ is updated using (3.51).

First example: numerical convergence

As it was done for all the previous problems, we analyzed with the same procedure the asymptotic behaviour of the numerical resolution for this problem. Thus, we considered the square domain $(0,6) \times (0,6)$ in the physical situation depicted in Figure 3.1 and we solve it for different time and spatial discretization parameters.

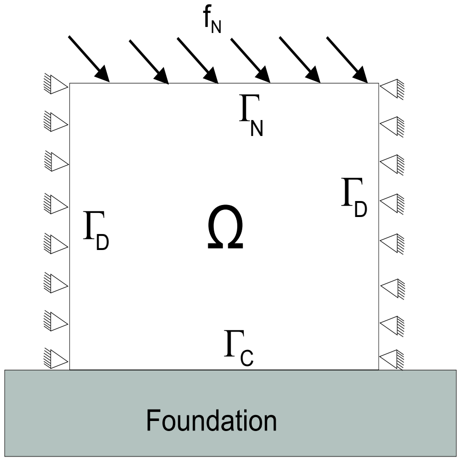

Figure 3.1: Example **1**: Contact problem with a deformable foundation.

The following data were used in the simulations:

$$T = 1s, \quad q^* = 1, \quad \boldsymbol{f}_B = \boldsymbol{0}\, N/m^3, \quad \boldsymbol{f}_N(x_1, x_2, t) = (20, -20)\, t\, N/m^2,$$
$$\mu = 0.5 \times 10^{-4}\, N/m^2, \quad E = 1000\, N/m^2, \quad r = 0.3, \quad \rho = 1\, Kg/m^3,$$
$$\kappa = 1, \quad \zeta^* = 0.01, \quad \lambda_D = 10^{-2}, \quad \lambda_u = 10^3, \quad \lambda_w = 0,$$
$$\boldsymbol{u}_0 = \boldsymbol{0}\, m, \quad \boldsymbol{v}_0 = \boldsymbol{0}\, m/s, \quad \zeta_0 = 1,$$

and also $\mathcal{A} = \mathcal{B}/10$ was considered.

The solution obtained for discretization parameters $n = 256$ and $k = 5 \times 10^{-4}$ was taken as "exact" solution and compared with the rest of solutions in order to see the performance of the numerical scheme. The results are presented in Table 3.1 Moreover, the evolution of the error with respect to the parameter $k + h$ is plotted in Figure 3.2. As we can see, the linear convergence of the algorithm is obtained (see Corollary 3.1).

A second example: compression with a deformable obstacle

As a second example, we considered the physical situation depicted in Figure 3.3. The viscoelastic body $\Omega = (0,3) \times (0,1)$ was clamped on its left boundary $\Gamma_D = \{0\} \times [0,1]$,

$n \downarrow k \rightarrow$	0.05	0.02	0.01	0.005	0.002	0.001
4	5.992e-2	5.231e-2	5.077e-2	5.000e-2	4.953e-2	4.938e-2
8	4.059e-2	2.756e-2	2.576e-2	2.486e-2	2.436e-2	2.420e-2
16	2.864e-2	1.543e-2	1.339e-2	1.254e-2	1.211e-2	1.198e-2
32	1.944e-2	9.986e-3	7.562e-4	6.585e-3	6.162e-3	6.057e-3
64	2.145e-2	7.685e-3	4.898e-3	3.667e-3	3.642e-3	3.039e-3
128	2.094e-2	6.675e-3	3.715e-3	2.270e-3	1.548e-3	1.408e-3

Table 3.1: Example **1**: Numerical errors.

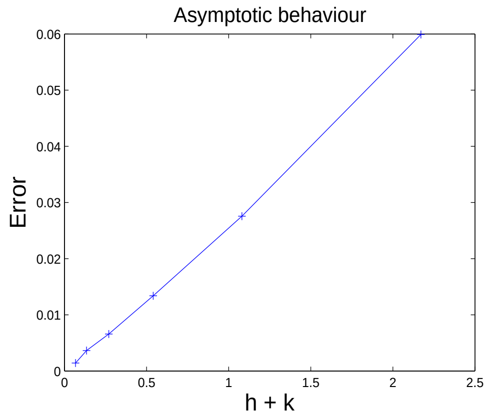

Figure 3.2: Example **1**: Asymptotic convergence.

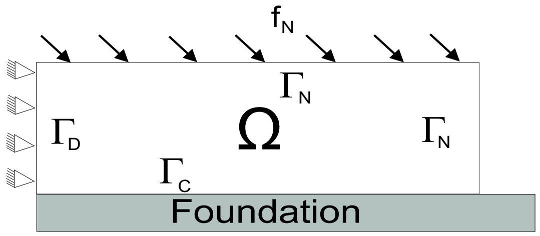

Figure 3.3: Example **2**: Contact problem with a deformable foundation.

and a density of traction forces acted on $\Gamma_N = (0, 3) \times \{1\}$. No volume forces acted in Ω and we assumed that the body was in frictionless contact with a deformable foundation on $\Gamma_C = [0, 3] \times \{0\}$. Here, again, $g = 0$.

The following data were used in the simulations:

$$T = 1\,s,\quad q^* = 1,\quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3,\quad \boldsymbol{f}_N(x_1, x_2, t) = (4.5, -4.5)\sin 2\pi t\,N/m^2,$$

$$\mu = 0.5\times 10^{-4}\,N/m^2,\quad E = 100\,N/m^2,\quad r = 0.3,\quad \rho = 1Kg/m^3,$$

$$\kappa = 1,\quad \zeta_* = 0.01,\quad \lambda_D = 10^{-2},\quad \lambda_u = 10^3,\quad \lambda_w = 0,$$

$$\boldsymbol{u}_0 = \boldsymbol{0}\,m,\quad \boldsymbol{v}_0 = \boldsymbol{0}\,m/s.$$

The initial damage field ζ_0 was assumed to be 0.1 in the area $(1.3, 1.7)\times(0, 1)$ and then increasing until 1 near to the boundary.

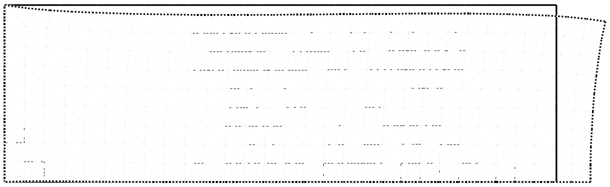

Figure 3.4: Example **2**: Deformations (amplified by 4) at final time $t = 1$ s and initial configuration.

The numerical results are presented in Figures 3.4 and 3.5 for the value $k = 0.01$ of the time discretization parameter. The deformed mesh at final time (amplified by a factor 4) and the initial boundary are shown in Figure 3.4. No penetration of the body into the obstacle is produced because of large deformability coefficient employed ($\frac{1}{\mu}$). The von Mises stress norm and the damage field in the deformed configuration at final time are plotted in Figure 3.5 (left-hand side and right-hand side, respectively). We notice that the highest stresses and the more damaged areas are located near to the clamped part. However, if contact is not considered, it is well-known that the stress (and so the damage) concentrates around the middle part of the domain because of bending effects.

A third example: deformable contact of an L-shaped domain

As a third example, we considered an L-shaped body which was submitted to the action of traction forces on its upper horizontal boundary. The body was clamped

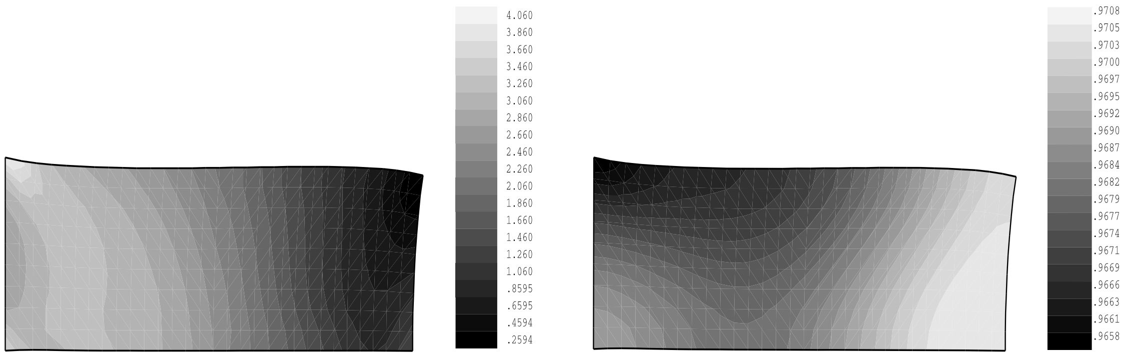

Figure 3.5: Example **2**: von Mises stress norm and damage field at final time.

on its lower horizontal boundary and an obstacle was assumed to be in contact, i.e. $g = 0$ on Γ_C (see Figure 3.6).

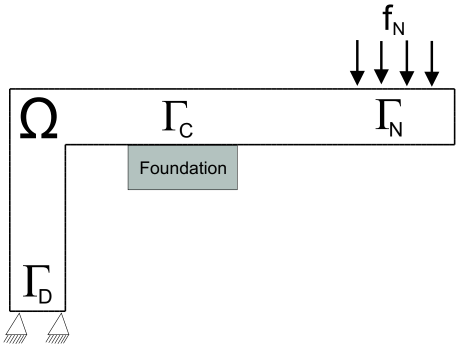

Figure 3.6: Example **3**: Contact problem of an L-shaped domain.

The following data were used in the simulations:

$$T = 10.5\,s, \quad q^* = 1, \quad \boldsymbol{f}_B = \mathbf{0}\,N/m^3, \quad \boldsymbol{f}_N(x_1, x_2, t) = (0, -7)|\sin \pi t|\,N/m^2,$$
$$\mu = 0.5 \times 10^{-5}\,N/m^2, \quad E = 10000\,N/m^2, \quad r = 0.3, \quad \rho = 5 \times 10^{-2} Kg/m^3,$$
$$\kappa = 1 \quad \lambda_D = 5 \times 10^{-4}, \quad \lambda_u = 10^3, \quad \lambda_w = 0, \quad \zeta_* = 0.01,$$
$$\boldsymbol{u}_0 = \mathbf{0}\,m, \quad \boldsymbol{v}_0 = \mathbf{0}\,m/s, \quad \zeta_0 = 1.$$

In order to show the influence of the damage in the evolution of the process, two simulations were done: the first one assuming that the material keeps undamaged (that is, $\zeta(\boldsymbol{x}, t) = 1$ for all $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$), and the second one when the damage takes place. Again, value $k = 0.01$ was employed for the time step.

The deformed mesh (amplified by 10) at final time and the initial configuration are plotted in Figure 3.7 (the undamaged case on the left-hand side and the damaged one on the right-hand side). Moreover, in Figure 3.8 the von Mises norm for the strains are shown for the respective undamaged and damaged processes. Clearly, inclusion of the damage effect leads to an increase of the magnitude of the deformation field.

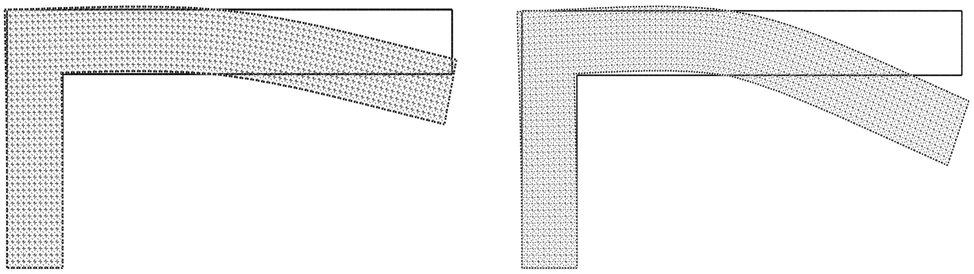

Figure 3.7: Example **3**: Deformed mesh (amplified by 10) and initial configuration (undamaged and damaged cases).

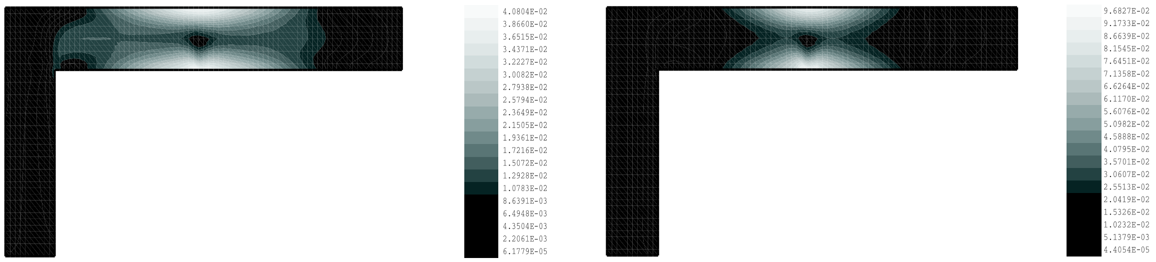

Figure 3.8: Example **3**: Strains at final time for the undamaged and damaged cases.

3.2 A dynamic frictional contact problem in viscoelasticity

This second dynamic contact problem deals with a frictional process. The procedure for the mathematical analysis changes with respect to the previous problem. In this case we follow an argument based on pseudomonotone operators theory (see [51]), obtaining improved regularity of the solutions. The mechanical problem consists now of a bilateral contact problem with Tresca's friction law modelling the tangential behaviour, as it was already described in Section 2.2, and its mathematical description is as follows (see Chapter 2 and the previous section for details concerning the notation).

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, and a damage field* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *such that,*

$$\begin{aligned}
&\rho\, \ddot{\boldsymbol{u}} = \operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B \quad \text{in} \;\; \Omega \times (0,T),\\
&\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T),\\
&\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{in} \quad \Omega \times (0,T),\\
&\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T),\\
&\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T),\\
&\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{on} \;\; \Gamma_N \times (0,T),\\
&u_\nu = 0 \quad \text{on} \quad \Gamma_C \times (0,T)\,,\\
&\left.\begin{aligned}
&|\boldsymbol{\sigma}_\tau| \le g,\\
&|\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0},\\
&|\boldsymbol{\sigma}_\tau| = g \Rightarrow \text{there exists } \lambda > 0\\
&\qquad\qquad \text{such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau,
\end{aligned}\right\} \quad \text{on} \quad \Gamma_C \times (0,T),\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0 \quad \text{in} \quad \Omega.
\end{aligned}$$

Let V be the following closed subspace of H_1,

$$V = \{\boldsymbol{v} \in H_1 \;;\; \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_D, \quad v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \;\text{ on } \Gamma_C\}.$$

Define the operators $A : V \to V'$ and $G : V \times B \to V'$ by

$$\begin{aligned} \langle A\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} &= \int_\Omega \mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{v})\right) \cdot \boldsymbol{\varepsilon}\left(\boldsymbol{w}\right) d\boldsymbol{x}, \\ \langle G(\boldsymbol{u}, \zeta), \boldsymbol{w}\rangle_{V'\times V} &= \int_\Omega \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}) d\boldsymbol{x}. \end{aligned}$$

For convenience and to save on notation, we can assume $\rho = 1$. The body forces and surface tractions satisfy

$$\boldsymbol{f}_B \in \mathcal{C}([0,T]; H), \quad \boldsymbol{f}_N \in \mathcal{C}([0,T]; [L^2(\Gamma_N)]^d); \tag{3.65}$$

and the friction bound verifies

$$g \in L^\infty(\Gamma_C). \tag{3.66}$$

To avoid cluttering the presentation with extra symbols, we assume $g = 1$ from now on and we supress the dependence on $\boldsymbol{x}$.

We turn now to a variational formulation.

To that end, let $L : B \to B'$ be the linear operator,

$$\langle L\zeta_1, \zeta_2\rangle_{B'\times B} = \kappa \int_\Omega \nabla\zeta_1 \cdot \nabla\zeta_2 \, d\boldsymbol{x} \quad \zeta_1,\, \zeta_2 \in B.$$

Next, by (3.65) we can define the element $\boldsymbol{f}(t) \in V'$ given by

$$\langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}_B(t), \boldsymbol{w})_H + (\boldsymbol{f}_N(t), \boldsymbol{w})_{[L^2(\Gamma_N)]^d} \quad \forall \boldsymbol{w} \in V,$$

and conclude that $\boldsymbol{f} \in \mathcal{C}([0,T]; V')$.

With this preparation, we can now give the abstract formulation of the problem. As mentioned above, we will let $\rho = 1$ to simplify the presentation. Then it is routine to verify that an appropriate abstract formulation of Problem **P** is to find $\boldsymbol{v}$ and ζ satisfying $\boldsymbol{v} \in L^2(0,T;V)$, $\dot{\boldsymbol{v}} \in L^2(0,T,V'), \zeta \in L^2\left(0,T;B\right), \dot{\zeta} \in L^2\left(0,T;B'\right)$ and the equations,

$$\begin{aligned} &a.) && \dot{\boldsymbol{v}} + A\boldsymbol{v} + G(\boldsymbol{u},\zeta) + \gamma_T^*\boldsymbol{\xi} = \boldsymbol{f} \quad \text{in} \quad L^2(0,T;V'), \\ &b.) && \dot{\zeta} + \kappa L\zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{in} \quad L^2(0,T;B'), \\ &c.) && \zeta\left(0\right) = \zeta_0 \in B, \quad \boldsymbol{v}\left(0\right) = \boldsymbol{v}_0 \in V, \\ &d.) && \boldsymbol{u}\left(t\right) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}\left(s\right) ds, \quad \text{in } [0,T], \quad \boldsymbol{u}_0 \in V, \end{aligned}$$

where

$$\langle \gamma_T^* \boldsymbol{\xi}, \boldsymbol{w} \rangle_{V' \times V} = \int_0^T \int_{\Gamma_C} \boldsymbol{\xi} \cdot \boldsymbol{w}_\tau d\sigma dt$$

and

$$\boldsymbol{\xi}(t) \in \partial (g \psi (\boldsymbol{v}_\tau)) \text{ a.e., with } \psi(\boldsymbol{v}) = |\boldsymbol{v}| .$$

We will need the following compatibility conditions for the initial data:

$$\zeta_0(\boldsymbol{x}) \in [\zeta_*, 1], \quad L\zeta_0 \in Y, \tag{3.67}$$

$$\boldsymbol{f}(0) - (A\boldsymbol{v}_0 + G(\boldsymbol{u}_0, \zeta_0)) \in H, \tag{3.68}$$

and also the following regularity on the applied loads:

$$\boldsymbol{f}, \dot{\boldsymbol{f}} \in L^2(0, T; V'). \tag{3.69}$$

3.2.1 An existence and uniqueness result

Under these above conditions, we will prove the following existence theorem which contains some regularity conditions.

Theorem 3.3. *Assume that (3.1)-(3.4) and (3.65)-(3.69) hold. There exists a unique solution to*

$$\begin{aligned}
&a) && \dot{\boldsymbol{v}} + A\boldsymbol{v} + G(\boldsymbol{u}, \zeta) + \gamma_T^* \boldsymbol{\xi} = \boldsymbol{f} \ \textit{in } L^2(0, T; V'),\\
&b) && \dot{\zeta} + \kappa L\zeta + \zeta_1 = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)), \quad \textit{in } L^2(0, T; B')\\
&c) && \zeta(0) = \zeta_0, \ \boldsymbol{v}(0) = \boldsymbol{v}_0,\\
&d) && \boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)\, ds, \ \boldsymbol{u}_0 \in V,\\
&e) && \boldsymbol{\xi} \in \partial (g \psi (\boldsymbol{v}_\tau)),\\
& && \textit{and for all } \lambda \in L^2(0, T; B),\\
&f) && \int_0^T \langle \zeta_1, \lambda - \zeta \rangle_{B'B}\, dt \le \int_0^T \int_\Omega \left(I_{[\zeta_*,1]}(\lambda) - I_{[\zeta_*,1]}(\zeta) \right) d\boldsymbol{x}\, dt,
\end{aligned} \tag{3.70}$$

which has the regularity $\zeta \in L^\infty(0, T; H^2(\Omega)) \cap \mathcal{C}([0, T]; H^r(\Omega))$ *for* $2 > r > 3/2, \dot{\zeta} \in L^\infty(0, T; Y) \cap L^2(0, T; B)$, $\boldsymbol{v}, \dot{\boldsymbol{v}} \in L^\infty(0, T; H), \cap L^2(0, T; V), \zeta_1 \in L^2(0, T; B')$.

Proof

The first step is to consider an approximate problem involving penalization and regularization. Define

$$j_\varepsilon : \mathbb{R} \to \mathbb{R}, \qquad j_\varepsilon(r) \equiv \begin{cases} \varepsilon^{-1}(r-1) \text{ if } r > 1, \\ 0 \text{ if } \zeta_* \le r \le 1, \\ \varepsilon^{-1}(r-\zeta_*) \text{ if } r < \zeta_*, \end{cases}$$

$$J_\varepsilon : \mathbb{R} \to \mathbb{R}, \qquad J_\varepsilon(s) \equiv \int_{\zeta_*}^{s} j_\varepsilon(r)\, dr,$$

$$\Phi : \mathbb{R}^d \to \mathbb{R}, \qquad \Phi_\varepsilon(\boldsymbol{v}) \equiv \sqrt{\varepsilon^2 + |\boldsymbol{v}|^2},$$

and then,

$$\nabla\Phi_\varepsilon(\boldsymbol{v}) = \frac{\boldsymbol{v}}{\sqrt{\varepsilon^2 + |\boldsymbol{v}|^2}}.$$

Also, let R denote the Riesz map from E to E' and from V to V'. We use the same symbol for each because it will be clear from the context what is wanted. Then our regularized and penalized problem, $\mathcal{P}_\varepsilon$ is of the following form for $\zeta_0 \in B$ and $\boldsymbol{v}_0 \in V$,

$$\begin{array}{ll} a.) & \overbrace{((I+\varepsilon R)\,\boldsymbol{v})}^{\cdot} + A\boldsymbol{v} + G(\boldsymbol{u},\zeta) + \gamma_T^*\nabla\Phi_\varepsilon(\boldsymbol{v}_\tau) = \boldsymbol{f} \text{ in } L^2(0,T;V'), \\ b.) & \overbrace{((I+\varepsilon R)\,\zeta)}^{\cdot} + \kappa L\zeta + j_\varepsilon(\zeta) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)), \text{ in } L^2(0,T;B') \\ c.) & (I+\varepsilon R)\,\zeta(0) = (I+\varepsilon R)\,\zeta_0,\ (I+\varepsilon R)\,\boldsymbol{v}(0) = (I+\varepsilon R)\,\boldsymbol{v}_0, \\ c.) & \boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)\, ds,\ \boldsymbol{u}_0 \in V. \end{array} \tag{3.71}$$

It is possible to prove, using standard techniques, that there exists a solution to Problem $\mathcal{P}_\varepsilon$ described above. This has the regularity

$$\zeta, \dot{\zeta} \in L^2(0,T;B),\ \boldsymbol{v}, \dot{\boldsymbol{v}} \in L^2(0,T;V).$$

The next task is to obtain estimates for the solutions to Problem $\mathcal{P}_\varepsilon$. Under the assumptions (3.67–3.69) and

$$\boldsymbol{v}_{0\tau} = \boldsymbol{0},$$

which is made for the sake of simplicity in presentation, the following estimate is obtained

$$
\begin{aligned}
&\int_{\Omega} J_{\varepsilon}\left(\zeta\left(t\right)\right)d\boldsymbol{x}+\int_{0}^{t}\|\dot{\zeta}\left(s\right)\|_{Y}^{2}ds+\varepsilon\int_{0}^{t}\langle R\dot{\zeta},\dot{\zeta}\rangle_{B'B}ds+\frac{\kappa}{2}\int_{\Omega}|\nabla\zeta\left(t\right)|^{2}d\boldsymbol{x}\\
&\quad+\|\zeta\left(t\right)\|_{Y}^{2}+\int_{0}^{t}\|\zeta\left(s\right)\|_{B}^{2}ds+\|\boldsymbol{v}\left(t\right)\|_{H}^{2}+\varepsilon\|\boldsymbol{v}\left(t\right)\|_{V}^{2}\\
&\quad+\int_{0}^{t}||\boldsymbol{v}\left(s\right)||_{V}^{2}\,ds\leq C,
\end{aligned}
$$

where C is independent of ε. Taking differences and doing more fairly standard estimates leads to the final estimate,

$$
\begin{aligned}
&||\zeta||_{L^{\infty}(0,T;H^{2}(\Omega))}+\|\dot{\zeta}\left(t\right)\|_{Y}^{2}+\int_{0}^{t}\|\dot{\zeta}(s)\|_{B}^{2}ds\\
&+\langle\left(I+\varepsilon R\right)\dot{\boldsymbol{v}}\left(t\right),\dot{\boldsymbol{v}}\left(t\right)\rangle_{L^{2}(0,T;V')\times L^{2}(0,T;V)}+\int_{0}^{t}||\dot{\boldsymbol{v}}\left(s\right)||_{V}^{2}\,ds\\
&+\int_{\Omega}J_{\varepsilon}\left(\zeta\left(t\right)\right)d\boldsymbol{x}+||\boldsymbol{u}\left(t\right)||_{V}^{2}+\|\boldsymbol{v}\left(t\right)\|_{H}^{2}+\int_{0}^{t}||\boldsymbol{v}||_{V}^{2}\,ds\leq C.
\end{aligned}
\tag{3.72}
$$

Now we recall the following two important theorems found in [72].

Theorem 3.4. *Let* $p\geq 1,\ q>1,\ X_1\subseteq X_2\subseteq X_3$ *with compact inclusion map* $X_1\to X_2$ *and continuous inclusion map* $X_2\to X_3$*, and let*

$$S_R=\{\boldsymbol{u}\in L^p(0,T;X_1)\,;\,\dot{\boldsymbol{u}}\in L^q(0,T;X_3),||\boldsymbol{u}||_{L^p(0,T;X_1)}+||\dot{\boldsymbol{u}}||_{L^q(0,T;X_3)}<R\}.$$

Then S_R *is precompact in* $L^p(0,T;X_2)$.

Theorem 3.5. *Let* X_1,X_2 *and* X_3 *be as above and let*

$$S_{RT}=\{\boldsymbol{u}\,;\,||\boldsymbol{u}(t)||_{X_1}+||\dot{\boldsymbol{u}}||_{L^q(0,T;X_3)}\leq R,\quad t\in[0,T]\},$$

for some $q>1$. *Then* S_{RT} *is precompact in* $\mathcal{C}(0,T;X_2)$.

Estimate (3.72) along with (3.71b) shows $j_{\varepsilon}(\zeta^{\varepsilon})$ is bounded in $L^2(0,T;B')$. Also $\nabla\Phi_{\varepsilon}\left(\boldsymbol{v}_{\tau}\right)$ is bounded in $L^{\infty}\left(0,T;[L^{\infty}\left(\Gamma_C\right)]^d\right)$ independently of ε. This follows directly from its definition. From (3.72) it follows

$$\varepsilon R\dot{\boldsymbol{v}}^{\varepsilon}\to 0 \text{ strongly in } L^2(0,T;V'),$$

$$\varepsilon R\dot{\zeta}^{\varepsilon}\to 0 \text{ strongly in } L^2(0,T;B').$$

Recall also that $\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}^\varepsilon), \eta_*(\zeta^\varepsilon))$ is bounded in $L^\infty(0,T;Y)$. Therefore, using Sobolev imbedding theorems along with Theorems 3.4 and 3.5 there exists a subsequence, still denoted by ε such that as $\varepsilon \to 0$,

$$\begin{aligned}
\zeta^\varepsilon &\to \zeta \text{ weak } * \text{ in } L^\infty\left(0,T;H^2(\Omega)\right), && (3.73)\\
\dot{\zeta}^\varepsilon &\to \dot{\zeta} \text{ weak } * \text{ in } L^\infty(0,T;Y), && (3.74)\\
\dot{\zeta}^\varepsilon &\to \dot{\zeta} \text{ weakly in } L^2(0,T;B), && (3.75)\\
\zeta^\varepsilon &\to \zeta \text{ strongly in } \mathcal{C}(0,T;H^r(\Omega)),\ 2 > r > \frac{3}{2}, && (3.76)\\
\zeta^\varepsilon(\boldsymbol{x},t) &\to \zeta(\boldsymbol{x},t) \text{ uniformly on } [0,T]\times\Omega, && (3.77)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} \text{ weak } * \text{ in } L^\infty(0,T;H), && (3.78)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} \text{ weakly in } L^2(0,T;V'), && (3.79)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} \text{ strongly in } L^2(0,T;V), && (3.80)\\
\boldsymbol{v}^\varepsilon(\boldsymbol{x},t) &\to \boldsymbol{v}(\boldsymbol{x},t) \text{ for a.e. } (\boldsymbol{x},t)\in\Gamma_C\times[0,T], && (3.81)\\
\dot{\boldsymbol{v}}^\varepsilon &\to \dot{\boldsymbol{v}} \text{ weak } * \text{ in } L^\infty(0,T;H), && (3.82)\\
\boldsymbol{u}^\varepsilon &\to \boldsymbol{u} \text{ weak } * \text{ in } L^\infty(0,T;V), && (3.83)\\
j_\varepsilon(\zeta^\varepsilon) &\to \zeta_1 \text{ weakly in } L^2(0,T;B'), && (3.84)\\
\nabla\Phi_\varepsilon(\boldsymbol{v}_\tau) &\to \boldsymbol{\xi} \text{ weak } * \text{ in } L^\infty\left(0,T;[L^\infty(\Gamma_C)]^d\right). && (3.85)
\end{aligned}$$

It follows from (3.72) and Fatou's lemma

$$\zeta(\boldsymbol{x},t) \in [\zeta_*,1] \text{ a.e. } \boldsymbol{x}\in\Omega. \tag{3.86}$$

To see this, note that $\liminf_{\varepsilon\to 0} J_\varepsilon(\zeta^\varepsilon) \geq I_{[\zeta_*,1]}(\zeta)$.

Note (3.76) follows from the Sobolev imbedding theorem which implies (3.77). It remains to pass to the limit in $\mathcal{P}_\varepsilon$ as $\varepsilon \to 0$.

For a positive number λ, new dependent variables ζ_λ and $\boldsymbol{v}_\lambda$ are defined by

$$\boldsymbol{v}_\lambda(t)e^{\lambda t} = \boldsymbol{v}(t), \quad \zeta_\lambda(t)e^{\lambda t} = \zeta(t). \tag{3.87}$$

Then, in terms of these new dependent variables, Problem $\mathcal{P}_\varepsilon$ is of the form

$$\begin{aligned}
&\overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda)}^{\cdot} + \lambda\,(I+\varepsilon R)\,\boldsymbol{v}_\lambda + A\boldsymbol{v}_\lambda \\
&\qquad + e^{-\lambda(\cdot)}G\,(\boldsymbol{u},\zeta) + e^{-\lambda(\cdot)}\gamma_T^*\nabla\Phi_\varepsilon\,(\boldsymbol{v}_\tau) = e^{-\lambda(\cdot)}\boldsymbol{f} \text{ in } L^2(0,T;V'), \\
&\overbrace{((I+\varepsilon R)\,\zeta_\lambda)}^{\cdot} + \lambda\,(I+\varepsilon R)\,\zeta_\lambda + \kappa L\zeta_\lambda + e^{-\lambda(\cdot)}j_\varepsilon(\zeta) \\
&\qquad - e^{-\lambda(\cdot)}\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\eta_*(\zeta)) = 0, \\
&(I+\varepsilon R)\,\zeta_\lambda\,(0) = (I+\varepsilon R)\,\zeta_0,\ (I+\varepsilon R)\,\boldsymbol{v}_\lambda\,(0) = (I+\varepsilon R)\,\boldsymbol{v}_0, \\
&\boldsymbol{u}\,(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}\,(s)\,ds,\ \boldsymbol{u}_0 \in V, \\
&\zeta_\lambda, \dot{\zeta}_\lambda \in L^2(0,T;B), \quad \boldsymbol{v}_\lambda, \dot{\boldsymbol{v}}_\lambda \in L^2(0,T;V).
\end{aligned} \tag{3.88}$$

In the above, where ζ or $\boldsymbol{v}$ occurs, we can regard it as being an expression in terms of ζ_λ or $\boldsymbol{v}_\lambda$ according to (3.87). By placing a subscript of λ on all the functions in the above convergences and the subscripted variables being given by (3.87), all the resulting convergences of (3.73) - (3.85) continue to be valid. We write the system in (3.88) as an implicit first order evolution equation, restoring the superscript of ε as follows.

$$\begin{aligned}
&\overbrace{\left(\begin{pmatrix} (I+\varepsilon R) & 0 \\ 0 & (I+\varepsilon R) \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_\lambda^\varepsilon \\ \zeta_\lambda^\varepsilon \end{pmatrix}\right)}^{\cdot} + \begin{pmatrix} \lambda\boldsymbol{v}_\lambda^\varepsilon + A\boldsymbol{v}_\lambda^\varepsilon + e^{-\lambda(\cdot)}G\,(\boldsymbol{u}^\varepsilon,\zeta^\varepsilon) \\ \lambda\zeta_\lambda^\varepsilon + \kappa L\zeta_\lambda^\varepsilon - e^{-\lambda(\cdot)}\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}^\varepsilon),\eta_*(\zeta^\varepsilon)) \end{pmatrix} \\
&\qquad + \begin{pmatrix} \lambda\varepsilon R\boldsymbol{v}_\lambda^\varepsilon + e^{-\lambda(\cdot)}\gamma_T^*\nabla\Phi_\varepsilon\,(\boldsymbol{v}_\tau^\varepsilon) \\ \lambda\varepsilon R\zeta_\lambda^\varepsilon + e^{-\lambda(\cdot)}j_\varepsilon(\zeta^\varepsilon) \end{pmatrix} = \begin{pmatrix} e^{-\lambda(\cdot)}\boldsymbol{f} \\ 0 \end{pmatrix}
\end{aligned} \tag{3.89}$$

along with the initial condition,

$$\begin{pmatrix} (I+\varepsilon R) & 0 \\ 0 & (I+\varepsilon R) \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_\lambda^\varepsilon \\ \zeta_\lambda^\varepsilon \end{pmatrix}(0) = \begin{pmatrix} (I+\varepsilon R) & 0 \\ 0 & (I+\varepsilon R) \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_0 \\ \zeta_0 \end{pmatrix}.$$

For simplicity of reference, we write equation (3.89) in the form

$$\overbrace{(\mathcal{B}_\varepsilon\boldsymbol{y}^\varepsilon)}^{\cdot} + \mathcal{C}\,(\boldsymbol{y}^\varepsilon) + \mathcal{S}_\varepsilon\,(\boldsymbol{y}^\varepsilon) = \mathbf{F}.$$

The operator $\mathcal{C}$ is monotone hemicontinuous and bounded provided λ is large enough. The proof of the following result can be seen in [16].

Lemma 3.3. Whenever λ is large enough, the operator $\mathcal{C} : L^2(0,T;V) \times L^2(0,T;B) \to L^2(0,T;V') \times L^2(0,T;B')$ is monotone, hemicontinuous, and bounded.

Now we want to pass to a limit as $\varepsilon \to 0$. First consider the term

$$\left\langle \overbrace{(\mathcal{B}_\varepsilon \boldsymbol{y}^\varepsilon)}^{\cdot}, \boldsymbol{y}^\varepsilon - \boldsymbol{y} \right\rangle .$$

This is a sum of two terms, $\left\langle \overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda^\varepsilon)}^{\cdot}, \boldsymbol{v}_\lambda^\varepsilon - \boldsymbol{v}_\lambda \right\rangle_{L^2(0,T;V')\times L^2(0,T;V)}$ and $\left\langle \overbrace{((I+\varepsilon R)\,\zeta_\lambda^\varepsilon)}^{\cdot}, \zeta_\lambda^\varepsilon - \zeta_\lambda \right\rangle_{L^2(0,T;B')\times L^2(0,T;B)}$. Consider the first of these,

$$\left\langle \overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda^\varepsilon)}^{\cdot}, \boldsymbol{v}_\lambda^\varepsilon - \boldsymbol{v}_\lambda \right\rangle_{L^2(0,T;V')\times L^2(0,T;V)} \geq \left\langle \overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda)}^{\cdot}, \boldsymbol{v}_\lambda^\varepsilon - \boldsymbol{v}_\lambda \right\rangle_{L^2(0,T;V')\times L^2(0,T;V)} .$$

Similar reasoning yields

$$\left\langle \overbrace{((I+\varepsilon R)\,\zeta_\lambda^\varepsilon)}^{\cdot}, \zeta_\lambda^\varepsilon - \zeta_\lambda \right\rangle_{L^2(0,T;B')\times L^2(0,T;B)} \geq \left\langle \overbrace{((I+\varepsilon R)\,\zeta_\lambda)}^{\cdot}, \zeta_\lambda^\varepsilon - \zeta_\lambda \right\rangle_{L^2(0,T;B')\times L^2(0,T;B)} .$$

Now

$$S_\varepsilon(\boldsymbol{y}^\varepsilon) = \begin{pmatrix} \lambda\varepsilon R \boldsymbol{v}_\lambda^\varepsilon + e^{-\lambda(\cdot)}\gamma_T^* \nabla\Phi_\varepsilon(\boldsymbol{v}_\tau^\varepsilon) \\ \lambda\varepsilon R \zeta_\lambda^\varepsilon + e^{-\lambda(\cdot)} j_\varepsilon(\zeta^\varepsilon) \end{pmatrix}$$

and this converges weakly in $L^2(0,T;V') \times L^2(0,T;B')$ to

$$\begin{pmatrix} e^{-\lambda(\cdot)}\gamma_T^* \boldsymbol{\xi} \\ e^{-\lambda(\cdot)}\zeta_1 \end{pmatrix} = S(\boldsymbol{y})$$

by (3.84) and (3.85). Therefore, from these observations and the monotonicity of $\nabla\Phi_\varepsilon$ and j_ε along with (3.86), keeping in mind that

$$\left\langle \overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda)}^{\cdot}, \boldsymbol{v}_\lambda^\varepsilon - \boldsymbol{v}_\lambda \right\rangle_{L^2(0,T;V')\times L^2(0,T;V)} + \left\langle \overbrace{((I+\varepsilon R)\,\zeta_\lambda)}^{\cdot}, \zeta_\lambda^\varepsilon - \zeta_\lambda \right\rangle_{L^2(0,T;B')\times L^2(0,T;B)}$$
$$+ \langle \mathcal{C}(\boldsymbol{y}^\varepsilon), \boldsymbol{y}^\varepsilon - \boldsymbol{y} \rangle \leq \langle \mathbf{F}, \boldsymbol{y}^\varepsilon - \boldsymbol{y} \rangle ,$$

and taking lim sup of both sides, and letting $\mathbf{z} = (\boldsymbol{w}, \psi)^T$, this yields

$$\limsup_{\varepsilon\to 0} \langle \mathcal{C}(\boldsymbol{y}^\varepsilon), \boldsymbol{y}^\varepsilon - \boldsymbol{y} \rangle \leq 0,$$
$$\left\langle \overbrace{((I+\varepsilon R)\,\zeta_\lambda)}^{\cdot}, \zeta_\lambda^\varepsilon - \zeta_\lambda \right\rangle_{L^2(0,T;B)\times L^2(0,T;B)} + \left\langle \overbrace{((I+\varepsilon R)\,\boldsymbol{v}_\lambda)}^{\cdot}, \boldsymbol{v}_\lambda^\varepsilon - \boldsymbol{v}_\lambda \right\rangle_{L^2(0,T;V')\times L^2(0,T;V)}$$
$$+ \langle \mathcal{C}(\boldsymbol{y}^\varepsilon), \boldsymbol{y}^\varepsilon - \boldsymbol{z} \rangle + \langle S_\varepsilon(\boldsymbol{y}^\varepsilon), \boldsymbol{y}^\varepsilon - \boldsymbol{z} \rangle \leq \langle \mathbf{F}, \boldsymbol{y}^\varepsilon - \boldsymbol{z} \rangle .$$

Since $\mathcal{C}$ has been shown to be monotone, hemicontinuous and bounded, it follows this operator is pseudomonotone. Therefore, we can take $\liminf_{\varepsilon\to 0}$ of both sides and conclude

$$\langle \dot{\boldsymbol{y}}, \boldsymbol{y}-\boldsymbol{z}\rangle + \langle \mathcal{A}\left(\boldsymbol{y}\right), \boldsymbol{y}-\boldsymbol{z}\rangle + \langle S(\boldsymbol{y}), \boldsymbol{y}-\boldsymbol{z}\rangle \leq \langle \mathbf{F}, \boldsymbol{y}-\boldsymbol{z}\rangle$$

for all $\mathbf{z} \in L^2(0,T;V)\times L^2(0,T;B)$. Since $\mathbf{z}$ is arbitrary, this shows

$$\dot{\boldsymbol{y}} + \mathcal{C}\left(\boldsymbol{y}\right) + S(\boldsymbol{y}) = \mathbf{F}.$$

Now, in terms of the original operators, this yields the existence of a solution to the following problem,

$$\begin{array}{ll}
a) & \dot{\boldsymbol{v}}_\lambda + \lambda\boldsymbol{v}_\lambda + A\left(\boldsymbol{v}_\lambda\right) + e^{-\lambda(\cdot)}G\left(\boldsymbol{u},\zeta\right) + e^{-\lambda(\cdot)}\gamma_T^*\boldsymbol{\xi} = e^{-\lambda(\cdot)}\boldsymbol{f}, \\
b) & \dot{\zeta}_\lambda + \lambda\zeta_\lambda + \kappa L\zeta_\lambda + e^{-\lambda(\cdot)}\zeta_1 = e^{-\lambda(\cdot)}\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\eta_*(\zeta)), \\
c) & \zeta_\lambda\left(0\right) = \zeta_0,\ \boldsymbol{v}_\lambda\left(0\right) = \boldsymbol{v}_0, \\
d) & \boldsymbol{u}\left(t\right) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}\left(s\right)ds,\ \boldsymbol{u}_0 \in V, \\
e) & \zeta_\lambda, \dot{\zeta}_\lambda \in L^2(0,T;B),\quad \boldsymbol{v}_\lambda, \dot{\boldsymbol{v}}_\lambda \in L^2(0,T;V).
\end{array}$$

Then, removing the λ subscripts using (3.87) we have obtained a solution to

$$\begin{array}{ll}
a) & \dot{\boldsymbol{v}} + A\boldsymbol{v} + G\left(\boldsymbol{u},\zeta\right) + \gamma_T^*\boldsymbol{\xi} = \boldsymbol{f} \\
b) & \dot{\zeta} + \kappa L\zeta + \zeta_1 = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\eta_*(\zeta)), \\
c) & \zeta\left(0\right) = \zeta_0,\ \boldsymbol{v}\left(0\right) = \boldsymbol{v}_0, \\
d) & \boldsymbol{u}\left(t\right) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}\left(s\right)ds,\ \boldsymbol{u}_0 \in V, \\
e) & \zeta, \dot{\zeta} \in L^2(0,T;B),\quad \boldsymbol{v}, \dot{\boldsymbol{v}} \in L^2(0,T;V).
\end{array} \tag{3.90}$$

having the regularity implied from the above list of convergences.

It only remains to consider the properties of $\boldsymbol{\xi}$ and ζ_1. First, recall $\zeta(\boldsymbol{x},t) \in [\zeta_*,1]$ a.e. and so $\int_\Omega I_{[\zeta_*,1]}(\zeta(\boldsymbol{x},t))d\boldsymbol{x} = 0$. Also, letting $\lambda \in L^2\left(0,T;B\right)$,

$$\int_0^T \langle j_\varepsilon\left(\zeta^\varepsilon\right), \lambda - \zeta^\varepsilon\rangle\, dt \leq \int_0^T\int_\Omega \left(J_\varepsilon\left(\lambda\right) - J_\varepsilon\left(\zeta^\varepsilon\right)\right)d\boldsymbol{x}dt \leq \int_0^T\int_\Omega J_\varepsilon\left(\lambda\right)d\boldsymbol{x}dt$$

and passing to the limit by using $\zeta^\varepsilon \to \zeta$ strongly in $\mathcal{C}\left(0,T;H^r\left(\Omega\right)\right)$, it yields, for all $\lambda \in L^2(0,T;B)$,

$$\int_0^T \langle \zeta_1, \lambda - \zeta\rangle\, dt \leq \int_0^T\int_\Omega I_{[\zeta_*,1]}\left(\lambda\right)d\boldsymbol{x}dt = \int_0^T\int_\Omega \left(I_{[\zeta_*,1]}\left(\lambda\right) - I_{[\zeta_*,1]}\left(\zeta\right)\right)d\boldsymbol{x}dt.$$

Next consider $\boldsymbol{\xi}$. For $\boldsymbol{w} \in L^2(0,T;V)$,

$$\int_0^T \int_{\Gamma_C} \nabla\Phi_\varepsilon\left(\boldsymbol{v}_\tau^\varepsilon\right) \cdot \left(\boldsymbol{w}_\tau - \boldsymbol{v}_\tau^\varepsilon\right) d\sigma dt \le \int_0^T \int_{\Gamma_C} \left(\Phi_\varepsilon\left(\boldsymbol{w}_\tau\right) - \Phi_\varepsilon\left(\boldsymbol{v}_\tau^\varepsilon\right)\right) d\sigma dt$$

and by (3.144) or (3.136) we can pass to the limit to obtain

$$\int_0^T \int_{\Gamma_C} \boldsymbol{\xi} \cdot \left(\boldsymbol{w}_\tau - \boldsymbol{v}_\tau\right) d\sigma dt \le \int_0^T \int_{\Gamma_C} \left(g\left|\mathbf{w}_\tau\right| - g\left|\boldsymbol{v}_\tau\right|\right) d\sigma dt$$

and then $\boldsymbol{\xi} \in \partial\left(g\Psi\left(\boldsymbol{v}_\tau\right)\right)$ a.e. where $\Psi\left(\boldsymbol{v}\right) = |\boldsymbol{v}|$. This proves existence for Theorem 3.3. It only remains to prove uniqueness.

Suppose $(\zeta_i, \boldsymbol{v}_i)$ are two solutions with ζ_{1i} and $\boldsymbol{\xi}_i$ the corresponding subgradient functions. Then, from (3.90b), if follows that

$$\begin{aligned} &\frac{1}{2}\|\zeta_1(t) - \zeta_2(t)\|_Y^2 + \kappa\int_0^t ||\zeta_1 - \zeta_2||_B^2\, ds \\ \le\ & \kappa\int_0^t \|\zeta_1 - \zeta_2\|_Y^2 ds + C\int_0^t\int_\Omega \left(\left|\boldsymbol{\varepsilon}\left(\boldsymbol{u}_1\right) - \boldsymbol{\varepsilon}\left(\boldsymbol{u}_2\right)\right| + |\zeta_1 - \zeta_2|\right)|\zeta_2 - \zeta_2|\, d\boldsymbol{x}ds, \end{aligned}$$

and therefore, an application of Gronwall's inequality and other simple manipulations lead to an inequality of the form

$$\begin{aligned} \|\zeta_1(t) - \zeta_2(t)\|_Y^2 + \int_0^t ||\zeta_1 - \zeta_2||_B^2\, ds \ &\le\ C\int_0^t ||\boldsymbol{u}_1 - \boldsymbol{u}_2||_V^2\, ds \\ &\le\ C\int_0^t\int_0^s ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, drds. \quad (3.91) \end{aligned}$$

Next consider (3.90a). Using (3.2) this leads to

$$\begin{aligned} &\frac{1}{2}\,||\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)||_V^2 + m_{\mathcal{A}}\int_0^t ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, ds - K\int_0^t \left(\|\zeta_1 - \zeta_2\|_Y + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\right) \\ &\quad\times\left(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V\right) ds + \int_0^t\int_{\Gamma_C} \left(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\right)\cdot\left(\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\right) d\sigma ds \le 0. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\,||\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)||_V^2 + m_{\mathcal{A}}\int_0^t ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, ds + \int_0^t\int_{\Gamma_C} \left(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\right)\cdot\left(\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\right) d\sigma ds \\ &\le C\int_0^t \|\zeta_1 - \zeta_2\|_Y^2 ds + \frac{m_{\mathcal{A}}}{2}\int_0^t ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, ds + C\int_0^t ||\boldsymbol{u}_1 - \boldsymbol{u}_2||_V\, ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V\, ds \end{aligned}$$

and, adjusting the constants and using the monotonicity of $\partial\psi$ where $\psi\left(\boldsymbol{v}\right) = |\boldsymbol{v}|$, we have

$$\begin{aligned} ||\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)||_V^2 + \int_0^t ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, ds &\le C\int_0^t \|\zeta_1 - \zeta_2\|_Y^2 ds + C\int_0^t ||\boldsymbol{u}_1 - \boldsymbol{u}_2||_V^2\, ds \\ &\le C\int_0^t\int_0^s ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, drds + C\int_0^t ||\zeta_1 - \zeta_2||_B^2\, ds. \end{aligned}$$

Therefore, from (3.91) we can adjust the constants again and obtain

$$||\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)||_V^2 + \int_0^t ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, ds \leq C \int_0^t \int_0^s ||\boldsymbol{v}_1 - \boldsymbol{v}_2||_V^2\, drds,$$

which establishes $\boldsymbol{v}_1 = \boldsymbol{v}_2$ after an application of Gronwall's inequality. From (3.91) this also shows $\zeta_1 = \zeta_2$, which proves uniqueness. ■

3.2.2 Numerical analysis

In this subsection we introduce a finite element algorithm for solving problem (3.70) and obtain an error estimate on the approximate solutions.

For convenience, problem (3.70) will be again rewritten in the following equivalent form, in terms of the velocity field, and using the notations and functionals given in the previous section and in Section 2.2.

Problem VP. *Find a velocity field* $\boldsymbol{v} : [0,T] \to V$*, and a damage field* $\zeta : [0,T] \to \mathcal{K}$*, such that* $\boldsymbol{v}(0) = \boldsymbol{v}_0$*,* $\zeta(0) = \zeta_0$ *and for a.e.* $t \in [0,T]$*,*

$$\begin{aligned} \langle \dot{\boldsymbol{v}}(t), \boldsymbol{w} - \boldsymbol{v}(t)\rangle_{V'\times V} + (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}(t))), \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}(t)))_Q + j(\boldsymbol{w}) - j(\boldsymbol{v}(t)) \\ \geq \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}(t)\rangle_{V'\times V} - (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}(t)))_Q, \quad (3.92) \end{aligned}$$

$$(\zeta'(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t)) \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_Y, \quad (3.93)$$

for all $\boldsymbol{w} \in V$ *and* $\xi \in \mathcal{K}$*.*

Let us consider now two finite dimensional spaces $V^h \subset V$ and $B^h \subset B$, approximating the spaces V and B, respectively, and let also $\mathcal{K}^h = \mathcal{K} \cap B^h$.

In this subsection, no summation is assumed over a repeated index and c denotes a positive constant which is independent of the discretization parameters h and k.

The fully discrete approximation of Problem **VP**, based on the forward Euler scheme, is as follows.

Problem VP^{hk}. *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete damage field* $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$*, such that* $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$*,* $\zeta_0^{hk} = \zeta_0^h$ *and for all* $\xi^h \in \mathcal{K}^h$*,*

$\boldsymbol{w}^h \in V^h$ and $n = 1, 2, \ldots, N$,

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y, \tag{3.94}$$

$$\begin{aligned}(\delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk})_H &+ (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + j(\boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}) \\ &\geq \langle \boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk}\rangle_{V'\times V} - (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q, \end{aligned}\tag{3.95}$$

where the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ is defined by

$$\boldsymbol{u}_n^{hk} = \sum_{j=1}^n k\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad \text{for } n = 1, \ldots, N, \quad \boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \tag{3.96}$$

and the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and ζ_0^h are appropriate approximations of the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and ζ_0, respectively.

Using standard arguments for variational inequalities (see [41]), we deduce the existence and uniqueness of solution to Problem $\mathbf{VP}^{hk}$, which we state as follows.

Theorem 3.6. *Assume that the conditions of Theorem* 3.3 *hold. Then, there exists a unique solution to Problem $\boldsymbol{VP}^{hk}$ such that $\boldsymbol{v}^{hk} \subset V^h$ and $\zeta^{hk} \subset \mathcal{K}^h$.*

Let us now estimate the numerical errors $\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_H$, $\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H$ and $\|\zeta_n - \zeta_n^{hk}\|_Y$.

The following regularity conditions on the solution to Problem $\mathbf{VP}$ are assumed,

$$\boldsymbol{u} \in \mathcal{C}^1([0,T]; V) \cap \mathcal{C}^2([0,T]; H), \quad \zeta \in \mathcal{C}([0,T]; H^2(\Omega)) \cap \mathcal{C}^1([0,T]; Y), \tag{3.97}$$

which also implies that $\dot{\boldsymbol{u}} \in \mathcal{C}([0,T]; V)$, $\boldsymbol{u}_0, \boldsymbol{v}_0 \in V$ and $\zeta_0 \in H^2(\Omega)$.

Taking (3.93) at time $t = t_n$ with $\xi = \zeta_n^{hk}$, and adding it to (3.94) with $\xi^h = \xi_n^h \in \mathcal{K}$, after some algebra (see Section 2.2 for details), the following estimate for the damage field is obtained,

$$\begin{aligned}
&\|\zeta_n-\zeta_n^{hk}\|_Y^2+k\sum_{j=1}^{n}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\quad\le c\Big\{\|\zeta_0-\zeta_0^h\|_Y^2+\|\zeta_1-\xi_1^h\|_Y^2+k\sum_{j=1}^{n}\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2\\
&\qquad+k\sum_{j=1}^{n}\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2+k^2\|\dot\zeta\|_{C(0,T;Y)}^2+k\sum_{j=1}^{n}\|\zeta_j-\xi_j^h\|_B^2+\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k\sum_{j=1}^{n}\|\delta\zeta_j-\dot\zeta_j\|_Y^2\\
&\qquad+k\sum_{j=1}^{n}\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\Big\},
\end{aligned}\tag{3.98}$$

for all $\{\xi_j^h\}_{j=1}^n\subset\mathcal{K}^h$. Now, we will obtain an error estimate for the velocity field. First, let us consider the variational inequality (3.92) at time $t=t_n$ and for $\boldsymbol{w}=\boldsymbol{v}_n^{hk}$,

$$\begin{aligned}
(\dot{\boldsymbol{v}}_n,\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q&\le j(\boldsymbol{v}_n^{hk})-j(\boldsymbol{v}_n)\\
&+\langle\boldsymbol{f}_n,\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\rangle_{V'\times V}+(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}_n))_Q,
\end{aligned}\tag{3.99}$$

and the corresponding discrete variational inequality (3.95) as follows,

$$\begin{aligned}
&(-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H-(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\\
&\quad\le j(\boldsymbol{w}^h)-j(\boldsymbol{v}_n^{hk})-\langle\boldsymbol{f}_n,\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}\rangle_{V'\times V}\\
&\qquad-(\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H-(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}^h))_Q\\
&\qquad+(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}))_Q\quad\text{for all }\boldsymbol{w}^h\in V^h.
\end{aligned}\tag{3.100}$$

Adding (3.99) and (3.100), it leads to the following inequality,

$$\begin{aligned}
&(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n))-\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\\
&\quad\le j(\boldsymbol{w}^h)-j(\boldsymbol{v}_n)+\langle\boldsymbol{f}_n,\boldsymbol{v}_n-\boldsymbol{w}^h\rangle_{V'\times V}+(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}_n))_Q\\
&\qquad-(\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)-(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}^h))_Q\\
&\qquad+(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}))_Q\quad\text{for all }\boldsymbol{w}^h\in V^h.
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n)),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}_n))+(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}_n^h-\boldsymbol{v}_n^{hk}))=\\
&(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n)-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}_n))+\\
&(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\zeta_n^{hk})-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),\zeta_{n-1}^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}_n^h)-(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n),\zeta_n),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}_n^h)\\
&\le c(\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\|_V+\|\zeta_n-\zeta_{n-1}^{hk}\|_Y)(\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_n\|+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|)_V+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}_n^h))_Q&=(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n))-\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}_n^h))_Q-(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{w}_n^h)\\
&\le\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V,\\
(\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)&=(\delta\boldsymbol{v}_n^{hk}-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)+(\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H\\
&\le(\delta\boldsymbol{v}_n^{hk}-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\delta\boldsymbol{v}_n-\dot{\boldsymbol{v}}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\dot{\boldsymbol{v}}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H\\
&\le(\delta\boldsymbol{v}_n^{hk}-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+c(\|\delta\boldsymbol{v}_n-\dot{\boldsymbol{v}}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V^2+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V^2),\\
|j(\boldsymbol{v}_n)-j(\boldsymbol{w}_n^h)|&\le\|\boldsymbol{v}_n-\boldsymbol{w}_n^h|_V,
\end{aligned}$$

using regularities (3.97), we obtain the following estimate,

$$\begin{aligned}
(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2\le c\Big(&\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2\\
&+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V+\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\|_V^2+\|\zeta_n-\zeta_{n-1}^{hk}\|_Y^2\\
&+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}$$

Taking into account the inequality

$$\begin{aligned}
(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H&\\
&\ge\frac{1}{k}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\frac{1}{k}\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H\\
&\ge\frac{1}{2k}(\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2),
\end{aligned}$$

we find that

$$\begin{aligned}
\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+k\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2\le ck\Big(&\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V^2\\
&+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V+\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\|_V^2+\|\zeta_n-\zeta_{n-1}^{hk}\|_Y^2\Big)+\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\\
&+c(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}),\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H\quad\text{for all }\boldsymbol{w}_n^h\in V^h.
\end{aligned}$$

Therefore, by induction we obtain

$$\begin{aligned}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2 &\le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2\\ &+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2+\|\zeta_j-\zeta_{j-1}^{hk}\|_Y^2\Big)\\ &+c\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2,\end{aligned}$$

for all $\{\boldsymbol{w}_j^h\}_{j=0}^n\subset V^h$. Since $\boldsymbol{u}\in\mathcal{C}^1([0,T];V)$, it follows that

$$\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2\le c\Big(\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+I_j^2+\sum_{l=1}^{j-1}k\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2\Big), \quad (3.101)$$

where I_j is the integration error

$$I_j=\Big\|\int_0^{t_{j-1}}\boldsymbol{v}_j(s)ds-\sum_{l=1}^{j-1}k\boldsymbol{v}_l\Big\|_V.$$

Then, using the above two inequalities we have,

$$\begin{aligned}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2 &\le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+I_j^2\\ &+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\zeta_j-\zeta_{j-1}\|_Y^2\\ &+\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2+\sum_{l=1}^{j-1}k\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2\Big)+c(\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2)\\ &+c\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H \text{ for all } \{\boldsymbol{w}_j^h\}_{j=0}^n\subset V^h. \quad (3.102)\end{aligned}$$

Keeping in mind (3.56), we have

$$\begin{aligned}&\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\ &\le\epsilon\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+c\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+c\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2\\ &+c\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H,\end{aligned}$$

where $\epsilon>0$ can be assumed small enough, and combining (3.102) and (3.98) and using also (3.101), we finally obtain

$$
\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \sum_{j=1}^n k\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^n \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \\
&\quad \le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_Y^2 + \|\zeta_j - \zeta_{j-1}\|_V^2 \\
&\quad +\|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 + \sum_{l=1}^{j-1} k\|\boldsymbol{v}_l - \boldsymbol{v}_l^{hk}\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V\Big) \\
&\quad +c\Big\{\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 + \|\zeta_0 - \zeta_0^h\|_Y^2 \\
&\quad +\|\zeta_1 - \xi_1^h\|_Y^2 + \|\zeta_n - \xi_n^h\|_Y^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + k\sum_{j=1}^n \|\zeta_j - \xi_j^h\|_B^2 \\
&\quad +k\sum_{j=1}^{n-1} \|\zeta_j - \zeta_j^{hk}\|_Y^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \\
&\quad +k\sum_{j=1}^n \|\delta\zeta_j - \dot{\zeta}_j\|_Y^2 + \frac{1}{k}\sum_{j=1}^n \|(\boldsymbol{v}_j - \boldsymbol{w}_j^{hk}) - (\boldsymbol{v}_{j-1} - \boldsymbol{w}_{j-1}^{hk})\|_H^2\Big\} \\
&\quad +k\sum_{j=1}^n \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y. \qquad (3.103)
\end{aligned}
$$

Define now

$$
\begin{aligned}
e_n &= \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + k\sum_{j=1}^n \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 + k\sum_{j=1}^n \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2, \\
g_n &= k\sum_{j=1}^n \Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_{V'}^2 + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 + \|\zeta_j - \zeta_{j-1}\|_Y^2 + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_Y^2 \\
&\quad +\|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|\Big) + \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 \\
&\quad +\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|\zeta_0 - \zeta_0^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 + \|\zeta_n - \xi_n^h\|_Y^2 \\
&\quad +k\sum_{j=1}^n \|\zeta_j - \xi_j^h\|_B^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \\
&\quad +k\sum_{j=1}^n \|\delta\zeta_j - \dot{\zeta}_j\|_Y^2 + \frac{1}{k}\sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 \\
&\quad +k\sum_{j=1}^n \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y
\end{aligned}
$$

and using again Lemma 1.2, we obtain the following error estimates result.

Theorem 3.7. *Let the assumptions of Theorem 3.3 hold. Let us assume the regularity conditions (3.97) for the displacement and damage fields. Then, the following* *error estimates are obtained for all* $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ *and* $\{\xi_j^h\}_{j=0}^N \subset \mathcal{K}^h$*,*

$$\begin{aligned}
&\max_{0\le n\le N} \Big\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\Big\} + \sum_{j=1}^N k\Big[\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 + \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2\Big] \\
&\qquad \le c\sum_{j=1}^N k\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + I_j^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V + \|\zeta_j - \zeta_{j-1}\|_Y^2 \\
&\qquad\quad +\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2\Big) + c\Big\{\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\zeta_0 - \zeta_0^h\|_Y^2 \\
&\qquad\quad +k\sum_{j=1}^N \|\zeta_j - \xi_j^h\|_E^2 + \max_{0\le n\le N}\{\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|\zeta_n - \xi_n^h\|_Y^2\} \\
&\qquad\quad +\frac{1}{k}\sum_{j=1}^{N-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 \\
&\qquad\quad +k\sum_{j=1}^N \|\delta\zeta_j - \dot{\zeta}_j\|_Y^2 + \frac{1}{k}\sum_{j=1}^{N-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 \\
&\qquad\quad +k\sum_{j=1}^N \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta\zeta_j + \kappa\Delta\zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y\Big\}. \qquad (3.104)
\end{aligned}$$

We notice that the above error estimates are the basis for the analysis of the convergence rate of the algorithm. Thus, let Ω be a polyhedral domain and denote by $\mathcal{T}^h$ a triangulation of Ω compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D, Γ_N and Γ_C. Let V^h and B^h be defined by

$$V^h = \{\boldsymbol{v}^h \in [\mathcal{C}(\overline{\Omega})]^d \ ; \ \boldsymbol{v}^h|_{\mathcal{T}} \in [P_1(\mathcal{T})]^d \ \ \forall \mathcal{T} \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \boldsymbol{0} \ \text{ on } \ \Gamma_D\}, \qquad (3.105)$$

$$B^h = \{\xi^h \in \mathcal{C}(\overline{\Omega}) \ ; \ \xi^h|_{\mathcal{T}} \in P_1(\mathcal{T}) \quad \forall \mathcal{T} \in \mathcal{T}^h\}, \qquad (3.106)$$

respectively, and assume that initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and ζ_0^h are obtained by

$$\boldsymbol{u}_0^h = \Pi^h\boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \Pi^h\boldsymbol{v}_0, \quad \zeta_0^h = \pi^h\zeta_0,$$

where π^h and Π^h were defined in previous sections. Moreover, we make an additional assumption on the regularity of the solution,

$$\begin{aligned}
&\boldsymbol{u} \in \mathcal{C}^1([0,T]; [H^2(\Omega)]^d) \cap H^2(0,T;V) \cap H^3(0,T;H), \\
&\zeta \in H^1(0,T;B) \cap H^2(0,T;Y).
\end{aligned} \qquad (3.107)$$

Finally, keeping in mind the approximation properties of the finite element spaces V^h and B^h (see [25]), we obtain the following corollary which states the linear convergence of the algorithm.

Corollary 3.2. *Let V^h and B^h be the finite element spaces defined by (3.105)–(3.106). Let the assumptions of Theorem* 3.7 *and the additional regularity conditions* (3.107) *hold. Then, there exists a positive constant c, independent of h and k, such that*

$$\max_{0\le n\le N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_H+\|\zeta_n-\zeta_n^{hk}\|_Y\right\}\le c(h^{1/2}+k).$$

Moreover, if we also assume that

$$\boldsymbol{\sigma}\in\mathcal{C}([0,T];[H^1(\Omega)]^{d\times d}),\quad \boldsymbol{u}_{|\Gamma_C}\in\mathcal{C}^1([0,T];[H^2(\Gamma_C)]^d),\tag{3.108}$$

then the fully discrete scheme is linearly convergent; that is, there exists a positive constant c, independent of h and k, such that

$$\max_{0\le n\le N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_H+\|\zeta_n-\zeta_n^{hk}\|_Y\right\}\le c(h+k).$$

The first part or the corollary is directly obtained using the estimates already presented in previous sections. The second part of this corollary is obtained integrating by parts the equilibrium equation in Problem **P** and using the regularity (3.108), procedure which was detailed in Section 2.4 in the study of an elastic-viscoplastic contact problem.

3.2.3 Numerical examples

Example 1: numerical convergence

In this first example we show the numerical behaviour of the scheme. So we computed a sequence of numerical solutions, based on uniform partitions of both the spatial domain $[0,1]\times[0,1]$, subdividing each side of the domain into n parts (see the right-hand side of Figure 3.9 for the case $n=8$), and the time interval. The physical setting is depicted on the left-hand side of Figure 3.9. On the part $\Gamma_D=\{1\}\times[0,1]$ the body was clamped, and so the displacement field vanished there; on the part $\Gamma_N=\{0\}\times[0,1]$ the body was acted upon by surface tractions and on the part $\Gamma_C=[0,1]\times\{0\}$ the body was in bilateral contact with a rigid foundation, while the rest of the boundary was traction-free.

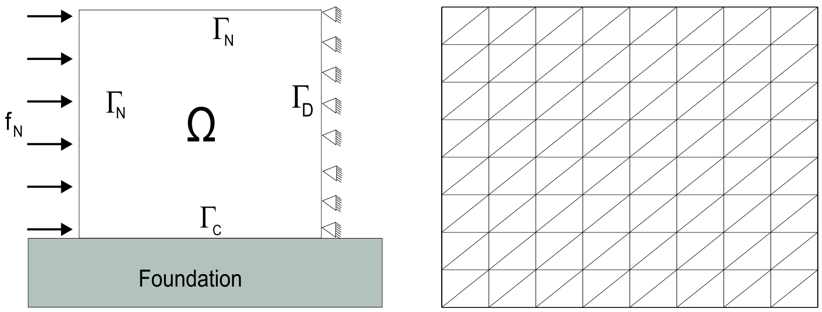

Figure 3.9: Example 1: Physical setting and mesh for n=8.

n/k	0.02	0.01	0.005	0.002	0.001
4	0.08150	0.07909	0.07805	0.07750	0.07733
8	0.06456	0.06154	0.06015	0.05937	0.05912
16	0.03666	0.03286	0.03107	0.03008	0.02977
32	0.01808	0.01356	0.01125	0.009858	0.009405
64	0.01290	0.008293	0.005945	0.004524	0.004049

Table 3.2: Example 1: Numerical errors for various n and k.

The numerical solution corresponding to $n = 128$ and $k = 0.0005$ was taken as the "exact" solution to compute the numerical errors given by

$$\max_{0\leq n\leq N} \Big\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_H + \|\zeta_n - \zeta_n^{hk}\|_Y \Big\}. \tag{3.109}$$

The following data were used in the computations:

$$\begin{aligned}
&T = 1\,s, \quad \boldsymbol{f}_B = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{f}_N(\boldsymbol{x},t) = (200,0)(e^t-1)\,N/m^2,\\
&E = 10^4\,N/m^2, \quad r = 0.3, \quad \rho = 1\,Kg/m^3, \quad \mathcal{A} = \mathcal{B}/100,\\
&\kappa = 1, \quad \lambda_D = 0, \quad \lambda_u = 1000, \quad \lambda_w = 0, \quad \zeta_* = 0.01,\\
&g = 5\,N/m^2, \quad \boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \boldsymbol{v}_0 = \boldsymbol{0}\,m/s, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \boldsymbol{x}\in\Omega.
\end{aligned}$$

In Table 3.2 the numerical errors, defined by (3.109) and obtained with some n and k, are shown. We notice that the numerical convergence of the algorithm is observed although the calculations performed do not allow to observe the convergence rate stated in Corollary 3.2 ($h \approx \sqrt{2}/n$).

Example 2: two different friction bounds

In this second example the problem depicted in Figure 3.10 was simulated ($\Omega = (0,4)\times(0,2)$), with two different friction bounds depending on the respective part of Γ_C (g_1 on the upper boundary $[0,4]\times\{2\}$ and g_2 on $[0,4]\times\{0\}$). The friction bound on the lower part of the boundary is very large, in such a way that it will not be achieved. This implies that, although we have not considered clamping conditions ($\Gamma_D = \emptyset$), we obtain the uniqueness of solution because the body is essentially fixed on the bottom contact boundary.

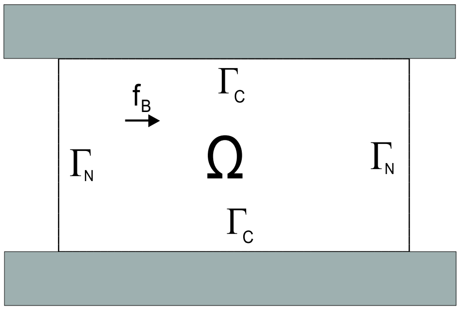

Figure 3.10: Example 2: Physical setting.

For computations, we used the following data:

$$
\begin{aligned}
&T = 3\,s, \quad \boldsymbol{f}_B(\boldsymbol{x},t) = (40,0)sin(\tfrac{3}{2}t\pi)\,N/m^3, \quad \boldsymbol{f}_N = \mathbf{0}\,N/m^2,\\
&E = 10000\,N/m^2, \quad r = 0.3, \quad \rho = 1\ Kg/m^3, \quad \mathcal{A} = \mathcal{B}/100,\\
&\kappa = 1, \quad \lambda_D = 0, \quad \lambda_u = 1000, \quad \lambda_w = 0, \quad \zeta_* = 0.01,\\
&g_1 = 1\,N/m^2, \quad g_2 = 1000\,N/m^2, \quad \boldsymbol{u}_0 = \mathbf{0}\,m, \quad \boldsymbol{v}_0 = \mathbf{0}\,m/s, \quad \zeta_0(\boldsymbol{x}) = 1\ \forall \boldsymbol{x} \in \Omega.
\end{aligned}
$$

In Figure 3.11 the deformed mesh (amplified 5 times) at final time and the initial configuration are shown. We notice that the lower part of the contact boundary remains fixed because of the size of the friction bound.

The von Mises stress norm and the damage field at final time are plotted in Figure 3.12. We observe, as expected, the correspondence between the more stressed areas and the most damaged ones.

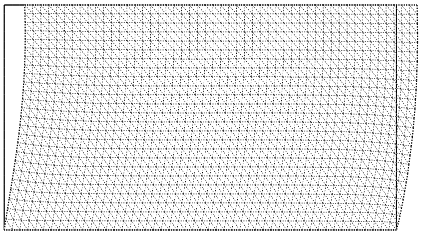

Figure 3.11: Example 2: Deformed mesh (multiplied by 5) at final time and the initial configuration.

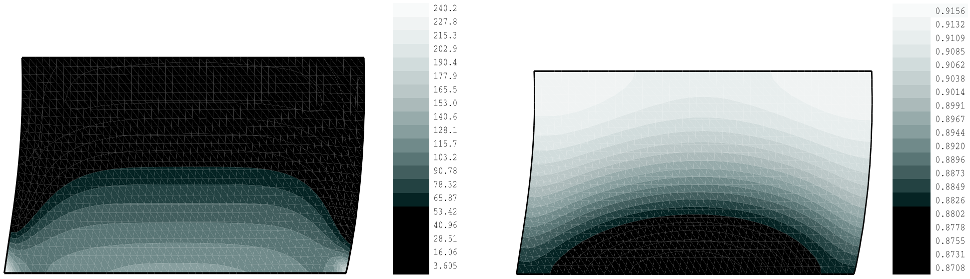

Figure 3.12: Example 2: von Mises stress norm and damage field at final time.

Example 3: contact with an sloping plane

As a last numerical example, we consider a rectangle over an sloping plane whose movement is restricted by an obstacle, see Figure 3.13 (left-hand side).

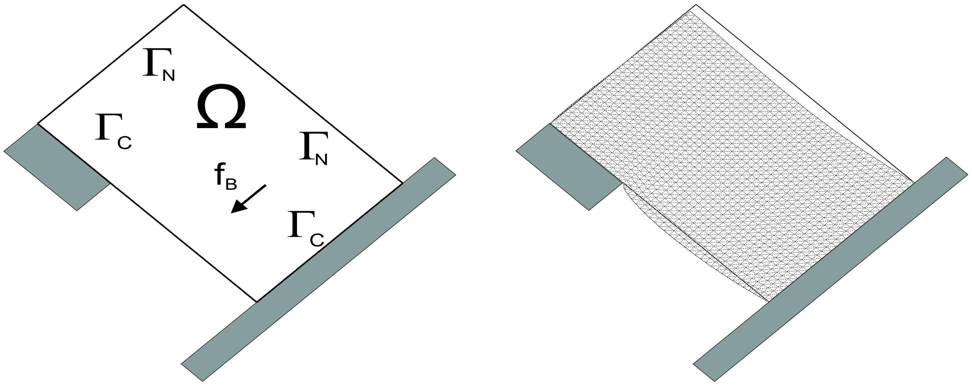

Figure 3.13: Example 3: Physical setting and deformed configuration at final time.

The following data were employed in this example:

$$
\begin{aligned}
&T = 1\,s, \quad \boldsymbol{f}_B(\boldsymbol{x},t) = -(20\sqrt{2}, 20\sqrt{2})(e^t - 1)\,N/m^3, \quad \boldsymbol{f}_N = \boldsymbol{0}\,N/m^2,\\
&E = 10000\,N/m^2, \quad r = 0.3, \quad \rho = 1\;Kg/m^3, \quad \mathcal{A} = \mathcal{B}/100,\\
&\kappa = 1, \quad \lambda_D = 0, \quad \lambda_u = 1000, \quad \lambda_w = 0, \quad \zeta_* = 0.01,\\
&g = 1\,N/m^2, \quad \boldsymbol{u}_0 = \boldsymbol{0}\,m, \quad \boldsymbol{v}_0 = \boldsymbol{0}\,m/s, \quad \zeta_0(\boldsymbol{x}) = 1 \quad \forall \boldsymbol{x} \in \Omega.
\end{aligned}
$$

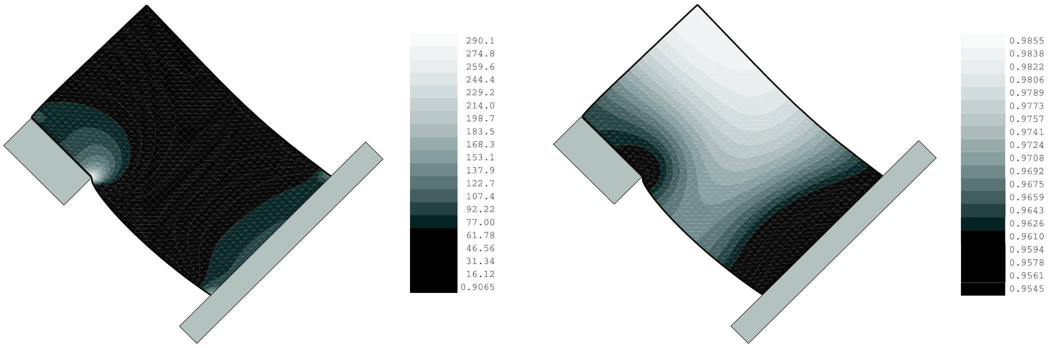

Figure 3.14: Example 3: von Mises stress norm and damage field at final time.

The deformed mesh at final time and the initial configuration are shown in Figure 2.26 (right-hand side), and the von Mises stress norm as well as the damage field are plotted in Figure 3.14. We notice again that the contacting zones are the more stressed areas and therefore, the most damaged ones.

3.3 A frictionless dynamic fully damageable contact problem

In this last section of the manuscript, we consider a viscoelastic constitutive law modified in such a way that not only the elastic response, but also the viscous one, are affected by the damage field; that is,

$$\boldsymbol{\sigma} = \zeta \mathcal{A} \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

The contact conditions here are similar to that presented in the first section of this chapter, and therefore, a frictionless process with a reactive foundation is considered. The only difference is that the material damage is supposed to affect also the contact pressure in the following way,

$$\sigma_\nu = -\eta_*(\zeta) p_\nu(u_\nu - g).$$

The problem of existence and uniqueness of solution to this problem was studied in [53] (where also the adhesion on the contact boundary is considered), with both models of damage (that one with the indicator function and that one without it) which have been considered in this work. Obviously, for the problem without the indicator term, more regularity is obtained for the solution. A short description of the proof will be presented.

The classical form of the mechanical problem is as follows (again, the notations defined in the previous sections will be employed).

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, *and a damage field* $\zeta : \Omega \times [0,T] \to [\zeta_*, 1]$ *such that*

$$\rho\, \ddot{\boldsymbol{u}} = \operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B \quad \text{in} \quad \Omega \times (0,T), \tag{3.110}$$

$$\boldsymbol{\sigma} = \eta_*(\zeta) \mathcal{A} \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{in} \quad \Omega \times (0,T), \tag{3.111}$$

$$\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[\zeta_*,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{in} \quad \Omega \times (0,T), \tag{3.112}$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{3.113}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T), \tag{3.114}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{3.115}$$

$$-\sigma_\nu = \eta_*(\zeta) p_\nu(u_\nu - g) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{3.116}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_C \times (0,T), \tag{3.117}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \zeta(0) = \zeta_0 \quad \text{in} \quad \Omega. \tag{3.118}$$

The assumptions over the normal compliance function are the same that those required in Sections 2.3 and 3.1, which are listed in (3.16). The space of admissible displacements for this case is defined as

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \; ; \; \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\}.$$

The body forces and the surface tractions are assumed to satisfy

$$\boldsymbol{f}_B \in \mathcal{C}([0,T]; H), \qquad \boldsymbol{f}_N \in \mathcal{C}([0,T]; V), \tag{3.119}$$

and this allows us to define the element $\boldsymbol{f}(t) \in V'$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V} = (\boldsymbol{f}_B(t), \boldsymbol{w})_H + (\boldsymbol{f}_N(t), \boldsymbol{w})_{[L^2(\Gamma_N)]^d} \quad \forall \boldsymbol{w} \in V,$$

and thus, we have

$$\boldsymbol{f} \in \mathcal{C}([0,T]; V').$$

We assume also that the initial conditions satisfy

$$\boldsymbol{u}_0 \in V, \qquad \boldsymbol{v}_0 \in H, \qquad \zeta_0 \in \mathcal{K}, \tag{3.120}$$

where $\mathcal{K}$ is given by (2.48). In order to describe the abstract formulation of the problem we need the following operators. First, let the *contact* operator $J : V \times \mathbb{R} \mapsto V'$ be given by

$$\langle J(\boldsymbol{u}, \zeta), \boldsymbol{w} \rangle_{V'V} \equiv \int_{\Gamma_C} \eta_*(\zeta) p_\nu(\boldsymbol{u}_\nu - g) w_\nu \, dS. \tag{3.121}$$

Let also the viscosity operator $A : V \times Y \mapsto V'$ and the elasticity operator $G : V \times Y \mapsto V'$ be given by

$$\langle A(\boldsymbol{v}, \zeta), \boldsymbol{w} \rangle_{V'V} \equiv (\eta_* \zeta \mathcal{A} \boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \tag{3.122}$$

$$\langle G(\boldsymbol{v}, \zeta), \boldsymbol{w} \rangle_{V'V} \equiv (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v}), \zeta), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \tag{3.123}$$

respectively.

We define the operator $L : B \mapsto B'$ by

$$\langle L\zeta, \xi \rangle_{B'B} \equiv \kappa \int_\Omega \nabla\zeta \cdot \nabla\xi \, d\boldsymbol{x}, \quad \zeta, \xi \in B. \tag{3.124}$$

The abstract variational formulation of Problem **P** is the following.

Problem VP. *Find a velocity field* $\boldsymbol{v} : [0,T] \to V$*, and a damage field* $\zeta : [0,T] \to \mathcal{K}$*, such that*

$$\dot{\boldsymbol{v}} + A(\boldsymbol{v}, \zeta) + G(\boldsymbol{u}, \zeta) + J(\boldsymbol{u}, \zeta) = \boldsymbol{f} \quad \text{in } L^2(0,T;V'), \tag{3.125}$$

$$\begin{aligned}(\dot{\zeta}, \xi - \zeta)_{L^2(0,T;Y)} &+ \langle L\zeta, \xi - \zeta \rangle_{L^2(0,T;B)\times L^2(0,T;B)} \\ &\geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)), \xi - \zeta)_{L^2(0,T;Y)}, \quad \text{for all } \xi \in L^2(0,T;\mathcal{K}),\end{aligned} \tag{3.126}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in V, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \in H, \quad \zeta(0) = \zeta_0 \in \mathcal{K}. \tag{3.127}$$

3.3.1 An existence and uniqueness result

We have the following result.

Theorem 3.8. *Assume that (3.1)–(3.4), (3.16) and (3.119)–(3.120) hold. Then, for each* $0 < T < \infty$*, Problem VP has a solution and it satisfies*

$$\begin{aligned}&\zeta, \dot{\zeta} \in L^2(0,T;Y), \qquad \zeta \in L^\infty(0,T;B), \\ &\boldsymbol{v} \in L^2(0,T;V) \cap \mathcal{C}([0,T];H), \qquad \dot{\boldsymbol{v}} \in L^2(0,T;V').\end{aligned} \tag{3.128}$$

If, in addition, a solution satisfies $|\boldsymbol{\varepsilon}(\boldsymbol{v})| \in L^\infty(0,T;L^\infty(\Omega))$*, then it is unique.*

The proof is based on a regularized and time-retarded sequence of approximate problems, we indicate the main steps, details can be seen in [53].

Proof

We obtain a solution to Problem VP by considering a penalized problem which also involves time retarding. We use the operator $\Psi_* : \mathbb{R} \mapsto \mathbb{R}$ given by

$$\Psi_*(r) \quad = \quad \begin{cases} |r-\zeta_*|^2 \text{ if } r < \zeta_*, \\ 0 \text{ if } \zeta_* \leq r \leq 1, \\ |r-1|^2 \text{ if } r > 1, \end{cases}$$

which regularizes the indicator function $I_{[\zeta_*,1]}$, and let $P_{\mathcal{K}} : Y \mapsto Y = Y'$ be defined by

$$\langle P_{\mathcal{K}}(\zeta), \xi \rangle_{Y'Y} = \int_\Omega \Psi'_*(\zeta)\xi \, d\boldsymbol{x}.$$

The regularized and penalized approximate problem is defined as follows, for $\varepsilon > 0$:

Problem VP^{rp}**.** *Find a velocity field* $\boldsymbol{v} : [0,T] \to V$*, and a damage field* $\zeta : [0,T] \to \mathcal{K}$*, such that* $\boldsymbol{v}(0) = \boldsymbol{v}_0$*,* $\zeta(0) = \zeta_0$ *and*

$$\dot{\boldsymbol{v}} + A(\boldsymbol{v},\zeta) + G(\boldsymbol{u},\zeta) + J(\boldsymbol{u},\zeta) = \boldsymbol{f} \quad \text{ in } L^2(0,T;V'), \tag{3.129}$$

$$\overbrace{((\varepsilon\mathcal{R}+I)\zeta)}^{\cdot} + L\zeta + \frac{1}{\varepsilon}P_{\mathcal{K}}(\zeta) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{ in } L^2(0,T;B'), \tag{3.130}$$

where $\mathcal{R}$ denotes the Riesz map from B to B', and for the sake of simplicity we did not label the variables with ε.

We also define the time retardation as follows. Let $r > 0$, and for a function $\varphi \in L^2(0,T;L^2(\Gamma_C))$, we set

$$\varphi_r(t) \quad = \quad \begin{cases} \varphi(t-r) & \text{if } t > r, \\ \varphi(0) & \text{if } t \leq r, \end{cases}$$

and consider the penalized and time-retarded problem.

Problem VP^{rp}**.** *Find a velocity field* $\boldsymbol{v} : [0,T] \to V$*, and a damage field* $\zeta : [0,T] \to \mathcal{K}$*, such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$*,* $\boldsymbol{v}(0) = \boldsymbol{v}_0$*,* $\zeta(0) = \zeta_0$ *and*

$$\dot{\boldsymbol{v}} + A(\boldsymbol{v},\zeta_r) + G(\boldsymbol{u},\zeta_r) + J(\boldsymbol{u},\zeta_r) = \boldsymbol{f} \quad \text{ in } \quad L^2(0,T;V'), \tag{3.131}$$

$$\overbrace{((\varepsilon\mathcal{R}+I)\zeta)}^{\cdot} + L\zeta + \frac{1}{\varepsilon}P_{\mathcal{K}}(\zeta) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)) \quad \text{ in } \quad L^2(0,T;B'). \tag{3.132}$$

It is not difficult to show that, for r small enough, there exists a solution to Problem VP^{rp} (see [53]).

Let us suppose also for the initial data to satisfy the compatibility condition

$$L\zeta_0 - \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta_0)) \in Y. \tag{3.133}$$

Let us denote by $\mathcal{V}$ and $\mathcal{B}$ Banach spaces satisfying: $V \subset \mathcal{V}$, V is dense in $\mathcal{V}$, the embedding is compact, and the trace map from $\mathcal{V}$ to $L^2(\Gamma_C)$ is continuous; $B \subset \mathcal{B}$, B is dense in $\mathcal{B}$, the embedding is compact, and the trace map from $\mathcal{B}$ to $L^2(\Gamma_C)$ is continuous.

We need to pass to the limits as r and ε tend to 0. Using *a priori* estimates and through some compacity results, denoting by $\boldsymbol{v}_\varepsilon$ and ζ_ε the solutions to Problem VP^{rp} it follows that there exists a subsequence, denoted by ε, such that as $\varepsilon \mapsto 0$ the following convergence results hold:

$$\begin{aligned}
\dot{\boldsymbol{v}}^\varepsilon &\rightarrow \dot{\boldsymbol{v}} \text{ weakly in } L^2(0,T;V), && (3.134)\\
\boldsymbol{v}^\varepsilon &\rightarrow \boldsymbol{v} \text{ weak } * \text{ in } L^\infty(0,T;H), && (3.135)\\
\boldsymbol{v}^\varepsilon &\rightarrow \boldsymbol{v} \text{ weakly in } L^2\left(0,T;V\right), && (3.136)\\
\zeta^\varepsilon &\rightarrow \zeta \text{ weak } * \text{ in } L^\infty\left(0,T;B\right), && (3.137)\\
\zeta^\varepsilon &\rightarrow \zeta \text{ weakly in in } L^2\left(0,T;B\right), && (3.138)\\
\dot{\zeta}^\varepsilon &\rightarrow \dot{\zeta} \text{ weakly in } L^2\left(0,T;Y\right), && (3.139)\\
\dot{\zeta}^\varepsilon &\rightarrow \dot{\zeta} \text{ strongly in } \mathcal{C}\left([0,T];\mathcal{B}\right), && (3.140)\\
\boldsymbol{v}^\varepsilon &\rightarrow \boldsymbol{v} \text{ strongly in } L^2\left(0,T;H\right), && (3.141)\\
\boldsymbol{u}^\varepsilon &\rightarrow \boldsymbol{u} \text{ strongly in } \mathcal{C}\left([0,T];\mathcal{V}\right), && (3.142)\\
\varepsilon\mathcal{R}\dot{\zeta}^\varepsilon &\rightarrow 0 \text{ strongly in } L^2(0,T;H^{-1}(\Omega)), && (3.143)\\
\varepsilon\zeta^\varepsilon &\rightarrow 0 \text{ strongly in } L^\infty(0,T;B). && (3.144)
\end{aligned}$$

When the compatibility condition (3.133) holds, we have, in adition to (3.134)–

(3.144), that

$$\dot{\zeta}^{\varepsilon} \mapsto \dot{\zeta} \quad \text{weakly in } L^2(0,T;B), \tag{3.145}$$

$$\dot{\zeta}^{\varepsilon} \mapsto \dot{\zeta} \quad \text{weak } * \, L^{\infty}(0,T;Y). \tag{3.146}$$

As it is done in the proof of existence to Problem VP^{rp}, a monotonicity argument is used to pass to the limit in the abstract problem. Again, an exponential shift to define new dependent variables is used for which standard techniques for passing to the limit are easier to apply. In this way, for a large positive number λ, we define θ and $\boldsymbol{w}$ by

$$\theta(t) = e^{-\lambda t}\zeta(t), \qquad \boldsymbol{w}(t) = e^{-\lambda t}\boldsymbol{v}(t),$$

and so the initial conditions are as above,

$$\boldsymbol{w}(0) = \boldsymbol{v}_0 \in H, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in V, \qquad \theta(0) = \theta_0 \in \mathcal{K}.$$

In terms of these new variables, the new problem consists of finding $\boldsymbol{w} \in L^2(0,T;V)$, $\dot{\boldsymbol{w}} \in L^2(0,T;H^{-1}(\Omega))$, $\theta \in L^2(0,T;B)$ and $\dot{\theta} \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\begin{aligned} &\dot{\boldsymbol{w}} + \lambda\boldsymbol{w} + e^{-\lambda(\cdot)}A(e^{\lambda(\cdot)}\boldsymbol{w}, (e^{\lambda(\cdot)}\theta)_{\varepsilon}) + e^{-\lambda(\cdot)}G(\boldsymbol{u}, (e^{\lambda(\cdot)}\theta)_{\varepsilon}) \qquad (3.147)\\ &\quad + e^{-\lambda(\cdot)}J(\boldsymbol{u}, (e^{\lambda(\cdot)}\theta)_{\varepsilon}) = e^{-\lambda(\cdot)}\boldsymbol{f} \quad \text{in } L^2(0,T;V'),\\ &\overbrace{((\varepsilon\mathcal{R}+I)\theta)}^{\cdot} + \lambda(\varepsilon\mathcal{R}+I)\theta + L\theta\\ &\quad + e^{-\lambda(\cdot)}\frac{1}{\varepsilon}P_{\mathcal{K}}(e^{\lambda(\cdot)}\theta) = e^{-\lambda(\cdot)}\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(e^{\lambda(\cdot)}\theta)) \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \qquad (3.148) \end{aligned}$$

It is possible to prove, from the theory of pseudomonotone operators, the existence of solution to problem (3.147)–(3.148)

The proof of the uniqueness of the solution, provided that $|\boldsymbol{\varepsilon}(\boldsymbol{v})| \in L^{\infty}(0,T;[L^{\infty}(\Omega)]^{d\times d})$, is done using some a priori estimations and the Gronwall's inequality. ■

Without the subgradient term in (3.112), global existence can be obtained, but without the guarantee that $\zeta \in (\zeta_*, 1]$. However, if the initial condition lies in an open subinterval of $[\zeta_*, 1]$ then for some time the solution will have this property.

Theorem 3.9. *Let* $\zeta_0 \in \mathcal{K}$, $\zeta_* + \delta \le \zeta_0 \le 1 - \delta$, *for some small* $0 < \delta$, $\boldsymbol{u}_0, \boldsymbol{v}_0 \in V$, *and suppose that the following compatibility conditions hold,*

$$\boldsymbol{v}_1 \equiv \boldsymbol{f}(0) - (A(\boldsymbol{v}_0, \zeta_0) + G(\boldsymbol{u}_0, \zeta_0) + J(\boldsymbol{u}_0, \zeta_0)) \in H, \tag{3.149}$$

$$L\zeta_0 - \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \eta_*(\zeta_0)) \in B. \tag{3.150}$$

Assume, in addition, that $\boldsymbol{f}$ *and* $\dot{\boldsymbol{f}}$ *lie in* V'. *Then, there exist* $0 < T^*$ *and a unique solution* $(\boldsymbol{u}, \boldsymbol{v}, \zeta)$ *of the abstract problem,*

$$\dot{\boldsymbol{v}} + A(\boldsymbol{v}, \zeta) + G(\boldsymbol{u}, \zeta) + J(\boldsymbol{u}, \zeta) = \boldsymbol{f} \quad \textit{in } L^2(0, T^*; V'), \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0,$$

$$\dot{\zeta} + L\zeta = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \eta_*(\zeta)), \qquad \zeta(0) = \zeta_0.$$

The solution satisfies,

$$\boldsymbol{u} \in W^{2,2}(0, T^*; V) \cap \mathcal{C}^2([0, T^*]; H), \quad \dddot{\boldsymbol{u}} \in L^2(0, T^*; V'), \tag{3.151}$$

$$\zeta \in W^{1,2}(0, T^*; B) \cap \mathcal{C}^1([0, T^*]; Y), \quad \ddot{\zeta} \in L^2(0, T^*; H^{-1}(\Omega)). \tag{3.152}$$

Proof

The proof of this theorem is based on similar arguments than the previous one (with the subgradient term), but improved *a priori* estimates can be obtained (details can be seen in [53]). We note that here there is no gap between uniqueness and existence, which exists for the weaker solutions above. ■

3.3.2 Numerical analysis

In order to introduce now a finite element algorithm for the approximate solutions and derive error estimates we have to assume some stronger regularity of the solutions,

$$\boldsymbol{u} \in \mathcal{C}^1([0, T]; V \cap [W^{1,\infty}(\Omega)]^d) \cap L^2(0, T; V) \cap \mathcal{C}^2([0, T]; H), \tag{3.153}$$

$$\zeta \in \mathcal{C}([0, T]; H^2(\Omega)) \cap H^2(0, T; Y) \cap H^1(0, T; B), \tag{3.154}$$

and, in particular, it follows that

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) \in \mathcal{C}([0, T]; [L^\infty(\Omega)]^{d\times d}). \tag{3.155}$$

We recall the variational formulation of our problem, that is obtained with the same process that was used in Section 3.1, and it is equivalent to Problem VP.

Problem VP^{eq}. *Find a velocity field $\boldsymbol{v} : [0,T] \to V$, and a damage field $\zeta : [0,T] \to \mathcal{K}$, such that $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\zeta(0) = \zeta_0$ and for $t \in [0,T]$,*

$$(\rho\dot{\boldsymbol{v}}(t), \boldsymbol{w})_H + (\zeta(t)\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}(t))), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$
$$+ j(\zeta(t), \boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'V}, \tag{3.156}$$

$$(\dot{\zeta}(t), \xi - \zeta(t))_Y + a(\zeta(t), \xi - \zeta(t))$$
$$\geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_Y, \tag{3.157}$$

for all $\boldsymbol{w} \in V$ and $\xi \in \mathcal{K}$, where $j : \mathcal{K} \times V \times V \mapsto \mathbb{R}$ is the contact functional given by

$$j(\zeta, \boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_C} \zeta(t) p_\nu(u_\nu(t) - g) w_\nu \, da \quad \forall \zeta \in \mathcal{K},\ \boldsymbol{v}, \boldsymbol{w} \in V.$$

Let $V^h \subset V$ and $B^h \subset B$ approximating the spaces V and B and let $\mathcal{K}^h = \mathcal{K} \cap B^h$. In terms of the velocity field, our fully discrete approximation of Problem VP^{hk} is the following (see the previous sections for issues concerning the notation and the definitions).

Problem VP^{hk}. *Given $\boldsymbol{u}_0^h, \boldsymbol{v}_0^h \in V^h$, and $\zeta_0^h \in \mathcal{K}^h$, find a discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and a discrete damage field $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$, such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\zeta_0^{hk} = \zeta_0^h$, and for $n = 1, 2, \ldots, N$,*

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_Y + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk})$$
$$\geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_Y, \tag{3.158}$$

$$(\rho\delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\zeta_n^{hk}\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q$$
$$+ j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}^h) = \langle \boldsymbol{f}_n, \boldsymbol{w}^h\rangle_{V'V}, \tag{3.159}$$

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_{n-1}^{hk} + k\boldsymbol{v}_n^{hk}, \tag{3.160}$$

for all $\xi^h \in \mathcal{K}^h$ and $\boldsymbol{w}^h \in V^h$.

Using classical results on nonlinear variational inequalities the existence of a unique solution to Problem VP^{hk} is obtained.

Theorem 3.10. *Assume that the conditions of Theorem* 3.8 *hold. Then, there exists a unique solution to Problem* VP^{hk}.

Let us estimate now the numerical errors. In order to simplify the reading, we assume without loss of generality that $\rho = 1$ and $g = 0$.

First, writing (3.156) at time t_n and choosing $\boldsymbol{w} = \boldsymbol{w}_n^h \in V^h \subset V$, it yields

$$\begin{aligned}(\dot{\boldsymbol{v}}_n, \boldsymbol{w}_n^h)_H + (\zeta_n \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q &+ \big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h)\big)_Q \\ +j(\zeta_n, \boldsymbol{u}_n, \boldsymbol{w}_n^h) &= \langle \boldsymbol{f}_n, \boldsymbol{w}_n^h \rangle_{V'V},\end{aligned} \tag{3.161}$$

for $\boldsymbol{w}_n^h \in V^h$. Subtracting (3.159) from (3.161) and using $\boldsymbol{w}^h = \boldsymbol{w}_n^h$, we find

$$\begin{aligned}(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}_n^h)_H &+ (\zeta_n \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h))_Q \\ &+\Big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}_n^h)\Big)_Q \\ &+j(\zeta_n, \boldsymbol{u}_n, \boldsymbol{w}_n^h) - j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w}_n^h) = 0.\end{aligned}$$

Therefore, for $\boldsymbol{w}_n^h \in V^h$ it follows that

$$\begin{aligned}&(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + (\zeta_n \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\ &\quad +\Big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\Big)_Q \\ &\quad +j(\zeta_n, \boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) \\ &= (\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H + (\zeta_n \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h))_Q \\ &\quad +\Big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{w}_n^h)\Big)_Q \\ &\quad +j(\zeta_n, \boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{w}_n^h) - j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n - \boldsymbol{w}_n^h).\end{aligned}$$

Next, we write

$$\begin{aligned}&(\zeta_n \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ &= ((\zeta_n - \zeta_n^{hk}) \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n)) - \zeta_n^{hk} \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \\ &(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ &= (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_n) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \zeta_{n-1}^{hk}) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}, \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q.\end{aligned}$$

Since $\zeta_{n-1}^{hk} \in \mathcal{C}(\overline{\Omega})$ and $\zeta_{n-1}^{hk} \leq 1$, the fact that $\boldsymbol{u} \in \mathcal{C}^1([0,T];[W^{1,\infty}(\Omega)]^d)$ and the inclusion $[W^{1,\infty}(\Omega)]^d \subset [\mathcal{C}(\overline{\Omega})]^d$ imply

$$\begin{aligned}
&|j(\zeta_n, \boldsymbol{u}_n, \boldsymbol{w}) - j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w})| \\
&\qquad = |j(\zeta_n - \zeta_{n-1}^{hk}, \boldsymbol{u}_n, \boldsymbol{w}) + j(\zeta_{n-1}^{hk}, \boldsymbol{u}_n, \boldsymbol{w}) - j(\zeta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{w})| \\
&\qquad \leq c(\|\boldsymbol{u}\|_{\mathcal{C}([0,T];[W^{1,\infty}(\Omega)]^d)}\|\zeta_n - \zeta_{n-1}^{hk}\|_{L^2(\Gamma_C)}\|w_\nu\|_{L^2(\Gamma_C)} \\
&\qquad\qquad + \|(\boldsymbol{u}_n)_\nu - (\boldsymbol{u}_{n-1}^{hk})_\nu\|_{L^2(\Gamma_C)}\|w_\nu\|_{L^2(\Gamma_C)}) \\
&\qquad \leq (\|\zeta_n - \zeta_{n-1}^{hk}\|_B + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V)\|\boldsymbol{w}\|_V.
\end{aligned}$$

Straightforward calculations together with (3.155) lead to

$$\begin{aligned}
&(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + m_{\mathcal{A}}\zeta_*\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq c(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V \\
&\quad + \|\zeta_n - \zeta_{n-1}^{hk}\|_B)\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + c\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H(\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V) \\
&\quad + c(\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_B)\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V \\
&\quad + c(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\zeta_n - \zeta_n^{hk}\|_Y)\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V + c\|\zeta_n - \zeta_n^{hk}\|_Y\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V \\
&\quad + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H \quad \forall \boldsymbol{w}_n^h \in V^h.
\end{aligned}$$

Using now Cauchy's inequality (2.22) (for $\epsilon > 0$ small enough to pass terms of the form $\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V$ to the left-hand side of the inequality), we obtain

$$\begin{aligned}
&(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq c\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 \\
&\qquad + \|\zeta_n - \zeta_{n-1}^{hk}\|_B^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 \\
&\qquad + \frac{1}{k}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H\Big) \quad \forall \boldsymbol{w}^h \in V^h.
\end{aligned}$$

Taking into account the inequality

$$\begin{aligned}
(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H &\geq \frac{1}{k}\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \frac{1}{k}\|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H \\
&\geq \frac{1}{2k}(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2),
\end{aligned}$$

it yields

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + k\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq ck\Big(\|\zeta_n - \zeta_n^{hk}\|_Y^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 \\
&\qquad + \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_B^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2\Big) + \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2 \\
&\qquad + c(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H.
\end{aligned}$$

By induction we deduce that

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2\\
&\qquad\le c\sum_{j=1}^n k\Big(\|\zeta_j-\zeta_j^{hk}\|_Y^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2\\
&\qquad+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2+\|\zeta_j-\zeta_{j-1}^{hk}\|_B^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2\Big)\\
&\qquad+c\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2.
\end{aligned}$$

Next, since

$$\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_B^2=\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2+\|\nabla(\zeta_{j-1}-\zeta_{j-1}^{hk})\|_H^2,$$

we find that

$$\begin{aligned}
\|\zeta_j-\zeta_{j-1}^{hk}\|_B^2 &\le c\Big(\|\zeta_j-\zeta_{j-1}\|_B^2+\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2\\
&+\|\nabla(\zeta_{j-1}-\zeta_{j-1}^{hk})\|_H^2\Big),
\end{aligned}$$

and also we have (see (2.79)),

$$\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}^{hk}\|_V^2\le c(\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+I_j^2+\sum_{l=1}^{j-1}k^2\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2),$$

where I_j is the integration error

$$I_j=\Big\|\int_0^{t_{j-1}}\boldsymbol{v}_j(s)ds-\sum_{l=1}^{j-1}k\boldsymbol{v}_l\Big\|_V.$$

The two inequalities yield

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2\\
&\qquad\le c\sum_{j=1}^n k\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+I_j^2+\sum_{l=1}^{j-1}k\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2+\|\zeta_j-\zeta_j^{hk}\|_Y^2\\
&\qquad+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\zeta_j-\zeta_{j-1}\|_B^2+\|\zeta_{j-1}-\zeta_{j-1}^{hk}\|_Y^2\\
&\qquad+\|\nabla(\zeta_{j-1}-\zeta_{j-1}^{hk})\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2\Big)\\
&\qquad+c(\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2)\\
&\qquad+c\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H. \qquad (3.162)
\end{aligned}$$

We consider again the estimates for the damage field (2.75) and add it to (3.162), obtaining, taking into account the estimation (3.56), the following expression

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\sum_{j=1}^n k\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2+k\sum_{j=1}^n\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\\
&\quad\leq c\sum_{j=1}^n k\Big(\|\zeta_j-\zeta_j^{hk}\|_Y^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_{V'}^2+I_j^2+\sum_{l=1}^{j-1}k\|\boldsymbol{v}_l-\boldsymbol{v}_l^{hk}\|_V^2+\|\nabla(\zeta_j-\zeta_{j-1}^{hk})\|_Y^2\\
&\qquad+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2\Big)+c\Big\{\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2\\
&\qquad+\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+\|\zeta_0-\zeta_0^h\|_Y^2+\|\zeta_1-\xi_1^h\|_Y^2+\|\zeta_n-\xi_n^h\|_Y^2\\
&\qquad+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+k\sum_{j=1}^n\|\zeta_j-\xi_j^h\|_{B)}^2+k\sum_{j=1}^{n-1}\|\zeta_j-\zeta_{j-1}\|_B^2\\
&\qquad+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2+k^2+k\sum_{j=1}^n\|\delta\zeta_j-\dot{\zeta}_j\|_Y^2\\
&\qquad+k\sum_{j=1}^n\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\\
&\qquad+\frac{1}{k}\sum_{j=1}^n\|(\boldsymbol{v}_j-\boldsymbol{w}_j^{hk})-(\boldsymbol{v}_{j-1}-\boldsymbol{w}_{j-1}^{hk})\|_H^2\Big\}.
\end{aligned}$$

Let us denote by

$$e_n=\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2+\sum_{j=1}^n k[\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2]$$

the numerical errors and

$$\begin{aligned}
g_n=\sum_{j=1}^n k\Big\{&\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+I_j^2+\|\zeta_j-\zeta_{j-1}\|_B^2\\
&+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\zeta_j-\xi_j^h\|_B^2+\|\delta\zeta_j-\dot{\zeta}_j\|_Y^2+\\
&\|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_Y\cdot\|\zeta_j-\xi_j^h\|_Y\Big\}\\
&+\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+\|\zeta_1-\xi_1^h\|_Y^2+\|\zeta_n-\xi_n^h\|_Y^2+\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2\\
&+e_0+k^2+\frac{1}{k}\sum_{j=1}^n\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk})\|_H^2\\
&+\frac{1}{k}\sum_{j=1}^{n-1}\|(\zeta_{j+1}-\xi_{j+1}^h)-(\zeta_j-\xi_j^h)\|_Y^2,
\end{aligned}$$

where $e_0=\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\zeta_0-\zeta_0^h\|_Y^2$ is an error on the approximation of the initial conditions.

We can easily verify that the following expression yields

$$e_n \le c(g_n + \sum_{j=1}^{n} k e_j).$$

Therefore, using the discrete version of Gronwall's inequality (see Lemma 1.1), we have

$$\max_{1\le n\le N} e_n \le c \max_{1\le n\le N} g_n,$$

and therefore it follows the following error estimates.

Theorem 3.11. *Under the assumptions of Theorem* 3.10 *and the regularity conditions (3.153) and (3.154), the following error estimates hold for all* $\{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ *and* $\{\xi_j^h\}_{j=0}^N \subset \mathcal{K}^h$,

$$\begin{aligned}
\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2 \Big\} + \sum_{j=1}^{N} k \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2 \\
+ \sum_{j=1}^{N} k \|\nabla(\zeta_j - \zeta_j^{hk})\|_H^2 \le c \sum_{j=1}^{n} k \Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + I_j^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}\|_V^2 \\
+ \|\zeta_j - \zeta_{j-1}\|_B^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 \Big) + c \Big\{ \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 \\
+ \|\zeta_0 - \zeta_0^h\|_Y^2 + \|\zeta_1 - \xi_1^h\|_Y^2 \\
+ k \sum_{j=1}^{N} \|\zeta_j - \xi_j^h\|_B^2 + \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|\zeta_n - \xi_n^h\|_Y^2 + \|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 \\
+ \frac{1}{k} \sum_{j=1}^{N-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Y^2 + k \sum_{j=1}^{N} \|\delta \zeta_j - \dot{\zeta}_j\|_Y^2 \\
+ \frac{1}{k} \sum_{j=1}^{N-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + k^2 \\
+ k \sum_{j=1}^{N} \|\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_j), \zeta_j) - \delta \zeta_j + \kappa \Delta \zeta_j\|_Y \cdot \|\zeta_j - \xi_j^h\|_Y \Big\}. \qquad (3.163)
\end{aligned}$$

Let V^h and B^h, be defined by

$$V^h = \{\boldsymbol{v}^h \in [\mathcal{C}(\overline{\Omega})]^d \; ; \; \boldsymbol{v}^h|_{\mathcal{T}} \in [P_1(\mathcal{T})]^d, \quad \mathcal{T} \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \boldsymbol{0} \quad \text{on } \Gamma_D\},$$

$$B^h = \{\xi^h \in \mathcal{C}(\overline{\Omega}) \; ; \; \xi^h|_{\mathcal{T}} \in P_1(\mathcal{T}), \quad \mathcal{T} \in \mathcal{T}^h\},$$

and define the following convex subset of B^h,

$$\mathcal{K}^h = \{\xi^h \in B^h \ ; \ \zeta_* \leq \xi^h \leq 1\}.$$

Assume that initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$ and ζ_0^h are obtained by

$$\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \Pi^h \boldsymbol{v}_0, \quad \zeta_0^h = \pi^h \zeta_0,$$

where π^h and Π^h were introduced in (2.30) and (2.34). We make an additional assumption on the regularity of the solution,

$$\boldsymbol{v} \in \mathcal{C}([0,T];[H^2(\Omega)]^d), \quad \ddot{\boldsymbol{v}} \in L^1(0,T;H). \tag{3.164}$$

The following result is obtained from estimates (3.163).

Corollary 3.3. Let the assumptions of Theorem 3.11 and the regularity condition (3.164) hold. Then the fully discrete scheme is linearly convergent; that is, there exists a positive constant c such that

$$\max_{0\leq n\leq N} \left\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_H + \|\zeta_n - \zeta_n^{hk}\|_Y \right\} \leq c(h+k).$$

All the terms involved have already been bounded in previous sections, and therefore, this result is straightforward.

3.3.3 Numerical examples

Example 1: numerical convergence

The physical setting is depicted in Figure 3.15, where $\Omega = (0,6)\times(0,6)$ is the cross-section of a three-dimensional rectangular body clamped on $x_1 = 0$ and $x_1 = 6$, since $\Gamma_D = \{0\}\times(0,6)\cup\{6\}\times(0,6)$. The traction force $\boldsymbol{f}_N$ acts on $\Gamma_N = (0,6)\times\{6\}$ and the contact surface is $\Gamma_C = (0,6)\times\{0\}$.

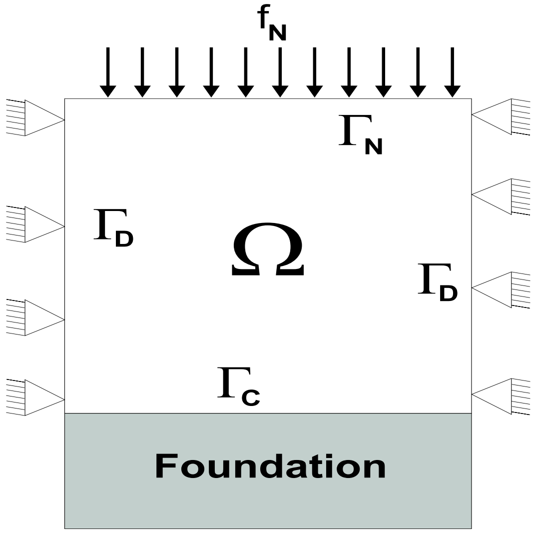

Figure 3.15: Example 1: The setting

The following data have been used in the numerical simulations:

$$T = 1\, s, \quad q^* = 1000, \quad \boldsymbol{f}_B = \mathbf{0}\, N/m^3, \quad \boldsymbol{f}_N = (0, -10t)\, N/m^2,$$
$$c_\nu = 2 \times 10^3\, N/m^3, \quad E = 100\, N/m^2, \quad r = 0.3, \quad \rho = 2700\, kg/m^3,$$
$$\kappa = 1, \quad \lambda_D = 0, \quad \lambda_u = 2000, \quad \lambda_w = 0, \quad \zeta_* = 0.01,$$
$$\zeta_0 = 1, \quad \boldsymbol{u}_0 = \mathbf{0}\, m, \quad \boldsymbol{v}_0 = \mathbf{0}\, m/s, \quad \mathcal{A} = \mathcal{B}.$$

We note that c_ν is large so that the foundation is quite rigid, and the problem is “close” to the Signorini problem for a perfectly rigid foundation.

Our aim here is to show the numerical convergence of the algorithm. Therefore, several uniform partitions of both the time interval and the domain, dividing Ω into $2n^2$ triangles, have been performed. We note that the number of degrees of freedom is $2(n+1)^2$ and we used the solution obtained with $n = 256$ and $k = 0.0005$ as the “exact solution”.

The numerical errors in the L^2-norm are depicted in Table 3.3.3. The decrease in the errors with respect to the parameter $k+h$ is plotted in Figure 3.16, by taking the values $n = 4, 8, 16, 32, 64, 128$ and the respective values $k = 0.1, 0.02, 0.01, 0.005, 0.002, 0.001$. As it can be seen, the algorithm seems to converge linearly.

$n \downarrow k \rightarrow$	0.1	0.02	0.01	0.005	0.002	0.001
4	8.774	9.001	9.071	9.106	9.128	9.135
8	5.207	4.534	4.455	4.416	4.393	4.385
16	3.209	2.604	2.644	2.679	2.707	2.717
32	2.592	1.655	1.581	1.539	1.500	1.485
64	2.051	0.8555	0.7666	0.7454	0.7448	0.7575
128	1.950	0.6457	0.5282	0.4853	0.4676	0.4633

Table 1. Example 1: Numerical errors (x100).

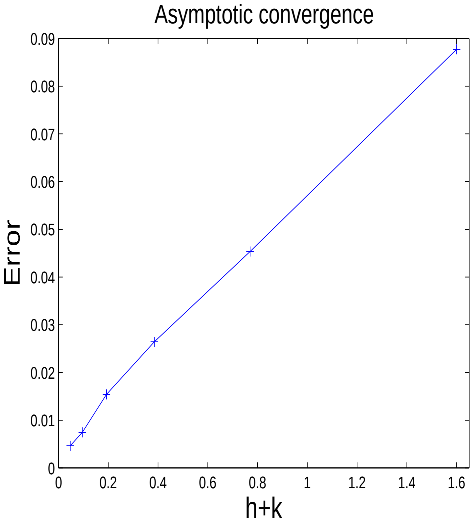

Figure 3.16: Example 1: Evolution of the error with respect to $k + h$.

Example 2: damage evolution inside the domain and near to its boundary

We consider the same setting as in the previous example with the following data:

$$\begin{aligned}
&T = 7.5\,s, \quad q^* = 1000, \quad \boldsymbol{f}_B = \mathbf{0}\,N/m^3, \quad \boldsymbol{f}_N = (0, -2.5\times 10^8 t)\,\sin \pi\, t\,N/m^2,\\
&c_\nu = 10^9\,N/m^3, \quad E = 7.1\times 10^9\,N/m^2, \quad r = 0.33, \quad \rho = 2700\,kg/m^3,\\
&\kappa = 1, \quad \lambda_D = 0, \quad \lambda_u = 100, \quad \lambda_w = 0, \quad \zeta_* = 0.01,\\
&\boldsymbol{u}_0 = \mathbf{0}\,m, \quad \boldsymbol{v}_0 = \mathbf{0}\,m/s, \quad \mathcal{A} = 10^{-6}\mathcal{B}.
\end{aligned}$$

Our aim is to study the damage as it tends to the limiting value ζ_* and the corresponding behaviour of the system.

First, we use the initial damage given by

$$\zeta_0(x_1,x_2)=\begin{cases} 0.1 \quad \text{if} \quad 2.5\le x_1,x_2\le 3.5, \\ 1 \quad \text{elsewhere.} \end{cases}$$

That is, the damaged area is located at the center of the domain.

In Figure 3.17 (left) we depict the damage field, at different times, on the line segment $[0,6]\times\{x_2=3\}$. It is found that there is self-mending in the center, and after some time the maximal damage (i.e., the minimum point of the damage field ζ) is located at the edge points $\boldsymbol{x}=(0,6)$ and $(6,6)$, by symmetry. The evolution of the damage field at the point $\boldsymbol{x}=(0,6)$ is shown (right) until the time in which the limiting value ζ_* is achieved. The damage oscillates since the applied traction does, but the minima at the edges are decreasing and the limiting value is reached in finite time.

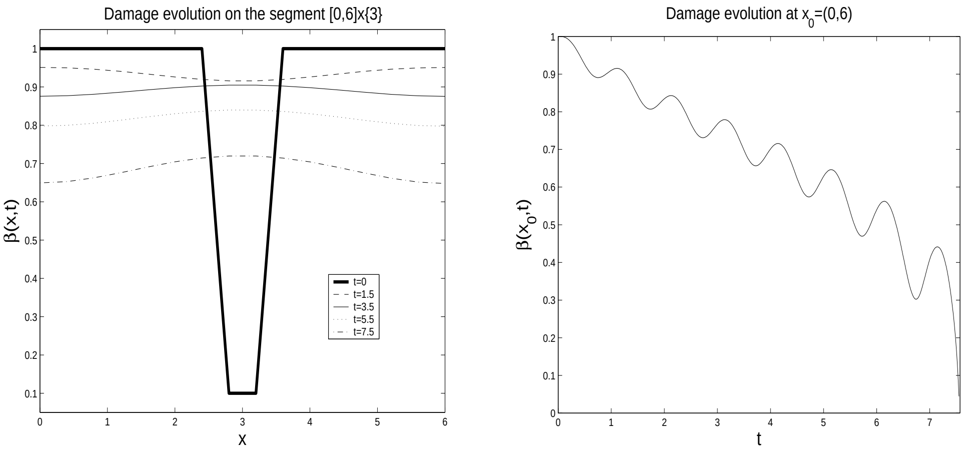

Figure 3.17: Example 2: Damage field at different times on the segment $x_2=3$ and the evolution of the damage at $\boldsymbol{x}=(0,6)$.

Secondly, when ζ_0 is given by

$$\zeta_0(x_1,x_2)=\begin{cases} 0.1 \quad \text{if} \quad 2.5\le x_1\le 3.5,\ 0\le x_2\le 1, \\ 1 \quad \text{elsewhere,} \end{cases}$$

we show in Figure 3.18 (left) the damage field at different times on the line segment $[0,6]\times\{x_2=0\}$. As above, after some time the maximal damage is located at the edge points $\boldsymbol{x}=(0,6)$ or $(6,6)$. The evolution of the damage field at these points, until the limiting value ζ_* is achieved, is plotted on the right.

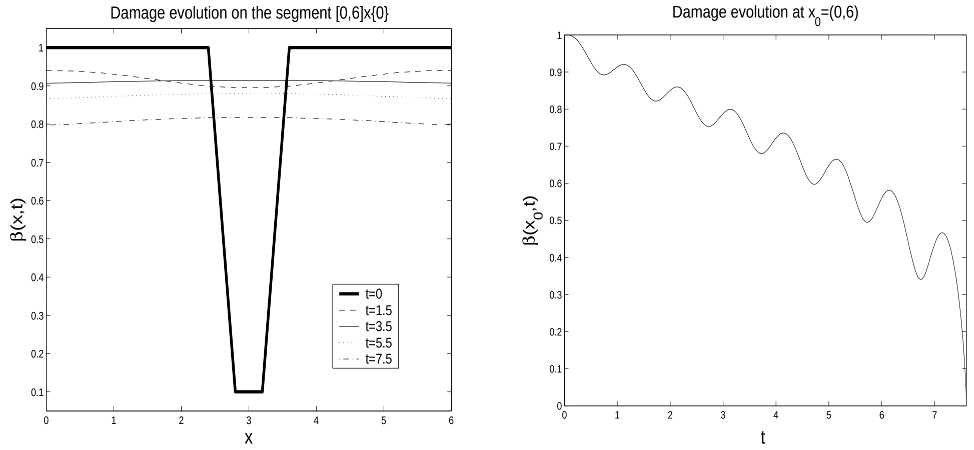

Figure 3.18: Example 2: Damage field at different times on the segment $x_2 = 0$ and the evolution of the damage at $\boldsymbol{x} = (0, 6)$.

We conclude that, in these examples, allowing for self-mending caused the rapid reduction in the damage from its initial high value in the center, while the damage developed to the edges. A more detailed investigation of the dependence of self-mending on the system parameters might be of interest.

Example 3: damageable contact of a viscoelastic wrench

The final example simulates a viscoelastic wrench. The setting is depicted in Figure 3.19. The wrench is in contact with a hard screw (not shown) which has a large stiffness coefficient, and is acted upon by a normal cyclic traction $\boldsymbol{f}_N$ on the indicated portion of Γ_N.

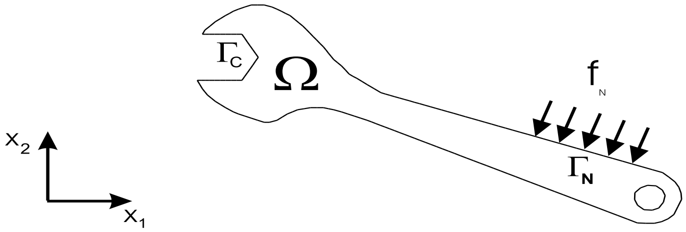

Figure 3.19: Example 3: A viscoelastic wrench in contact with a rigid screw.

The following data were used in the simulations:

$$\begin{aligned}
&T = 6.5\,s, \quad \boldsymbol{f}_B = \mathbf{0}\,N/m^3, \quad \boldsymbol{f}_N = (-10^5, -10^5)|\sin \pi t|\,N/m^2, \quad \rho = 2700\,kg/m^3,\\
&c_\nu = 10^9\,N/m^3, \quad \lambda_D = 0, \quad \lambda_u = 1.5 \times 10^3, \quad \lambda_w = 0, \quad r = 0.33,\\
&E = 7.1 \times 10^9\,N/m^2, \quad \zeta_* = 0.01, \quad q^* = 1000, \quad \kappa = 1, \quad \zeta_0 = 1,\\
&\boldsymbol{u}_0 = \mathbf{0}\,m, \quad \boldsymbol{v}_0 = \mathbf{0}\,m/s, \quad \mathcal{A} = 10^{-6} \times \mathcal{B}.
\end{aligned}$$

We depict in Figure 3.20 the deformed mesh (amplified by 2) at final time T. As can be seen, the normal compliance contact condition caused a negligible amount of interpenetration between the wrench and the screw.

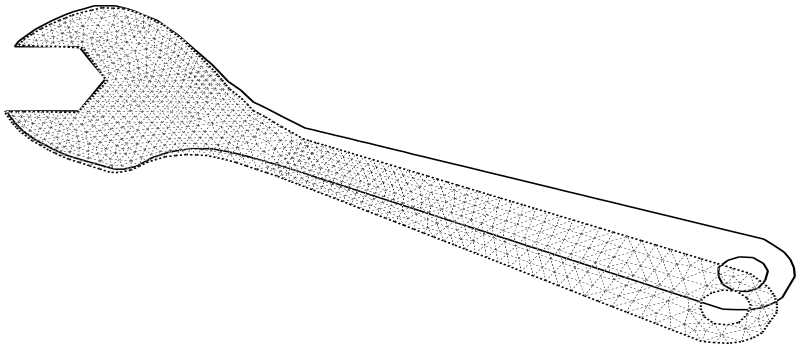

Figure 3.20: Example 3: the deformed mesh (x2) at final time T.

In Figure 3.21 the von Mises stress norm is plotted at final time T. The highest stress area is located, as expected, at the central region where the wrench bends.

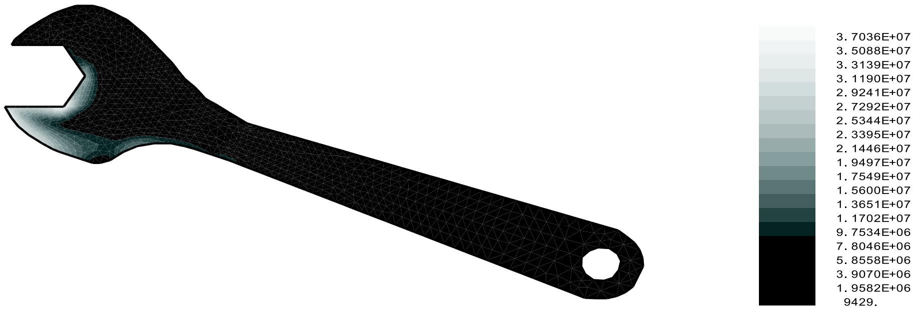

Figure 3.21: Example 3: von Mises stress norm at final time T.

Finally, the damage field at final time T in the deformed configuration is shown in Figure 3.22. We note that the highest damage coincides with the highest stress.

Also, in addition to the main region of stress, some material damage developed at the bottom of the neck of the wrench, and on a part of the contact surface.

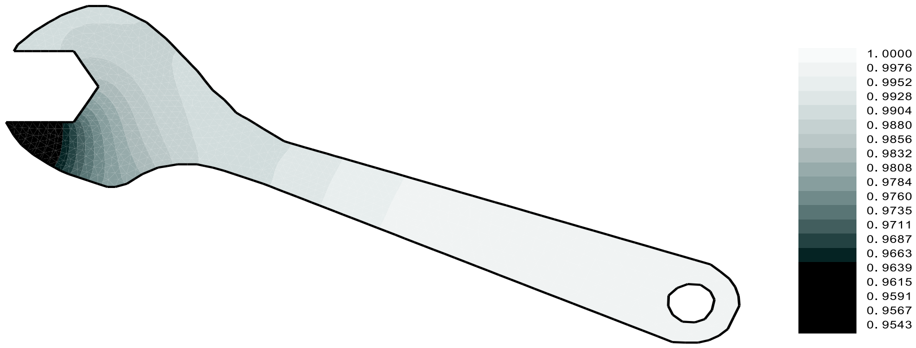

Figure 3.22: Example 3: damage field in the deformed configuration at final time.

Conclusions

Along this memory, a number of different mechanical problems with damage are completely analyzed, from their clasical or continuous description until their numerical simulation on two-dimensional scenaries. In all the cases an abstract or variational formulation is derived and well-posedness of the problems, in terms of existence and uniqueness of weak solution, are obtained. For each case, a fully discretized numerical scheme based on the finite element method is proposed and error estimates obtained.

The characteristic common properties of the problems studied in Chapters 2 and 3 are the material damage and the contact boundary conditions. We make use of general techniques commonly used on the mathematical theory of contact mechanics for both the mathematical and numerical analysis of the different problems.

Both quasistatic and dynamic problems have been considered and, in the case of quasistatic processes, the most common material behaviours were considered, that is, elasticity, viscoelasticity and viscoplasticity. In dynamic processes only the viscoelastic case was considered, because of the absence of theoretical results for the existence of solution in the viscoplastic or viscoelastic with long memory cases.

The damage model used along the work introduces several complications in the mechanical models, specially in those studied in Sections 2.1 and 3.3, which were solved by the use of the theory of pseudomonotone operators on the mathematical analysis. It is also remarkable the numerical resolution for frictional problems, which implies a qualitative improvement in terms of CPU cost.

Future steps in this research line should be the combined study of other effects which modify the material behaviour, such as temperature, piezoelectricity, wear or adhesion, and their mutual interaction with the damage model.

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