




Infinite families of manifolds of positive k th-intermediate Ricci curvature with k small

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Abstract

Positive k th-intermediate Ricci curvature on a Riemannian n -manifold, to be denoted by $\text{Ric}_k > 0$, is a condition that interpolates between positive sectional and positive Ricci curvature (when $k = 1$ and $k = n - 1$ respectively). In this work, we produce many examples of manifolds of $\text{Ric}_k > 0$ with k small by examining symmetric and normal homogeneous spaces, along with certain metric deformations of fat homogeneous bundles. As a consequence, we show that every dimension $n \geq 7$ congruent to 3 mod 4 supports infinitely many closed simply connected manifolds of pairwise distinct homotopy type, all of which admit homogeneous metrics of $\text{Ric}_k > 0$ for some $k < n/2$. We also prove that each dimension $n \geq 4$ congruent to 0 or 1 mod 4 supports closed manifolds which carry metrics of $\text{Ric}_k > 0$ with $k \leq n/2$, but do not admit metrics of positive sectional curvature.

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A complete Riemannian manifold (M, g) is said to have *positive k th-intermediate Ricci curvature*, to be denoted by $\text{Ric}_k > 0$, if for every point $p \in M$ and every set of $k + 1$ orthonormal tangent vectors $x, e_1, \dots, e_k \in T_p M$, the sum of the sectional curvatures of the planes spanned by x, e_i is positive, i.e.

$$\sum_{i=1}^k \text{sec}_g(x, e_i) > 0.$$

If $n = \dim M$, then $\text{Ric}_1 > 0$ and $\text{Ric}_{n-1} > 0$ correspond to the classical conditions of positive sectional curvature $\text{sec} > 0$ and positive Ricci curvature $\text{Ric} > 0$, respectively. Note that $\text{Ric}_k > 0$ implies $\text{Ric}_l > 0$ for any $l \geq k$; thus, the smaller the k , the more restrictive the condition $\text{Ric}_k > 0$. We refer the reader to Sect. 2 for background, context, and a more general definition of Ric_k .

At the present state of knowledge, no purely topological obstructions are known for a closed manifold M to admit a metric of $\text{Ric}_k > 0$ with $k > 1$, except those already obstructing the existence of metrics of $\text{Ric} > 0$ or positive scalar curvature (e.g. Bonnet-Myers Theorem or index-theoretical results by Atiyah-Singer-Lichnerowicz-Hitchin; see [56]).

As the dimension of the manifold under consideration increases, the condition $\text{sec} > 0$ appears to be stronger and stronger. In fact, we only know of two dimensions supporting infinitely many simply connected closed manifolds of $\text{sec} > 0$ of pairwise distinct homotopy type (7 and 13, given by Eschenburg and Bazaikin spaces respectively). Furthermore, in dimensions above 24, the only known examples are compact rank one symmetric spaces. In contrast, the condition $\text{Ric} > 0$ appears to be weaker and weaker as the dimension grows. In particular, every dimension ≥ 4 is known to support infinitely many simply connected closed manifolds of $\text{Ric} > 0$ of pairwise distinct real homotopy type [41].

The main goal of this article is to generate examples of manifolds of $\text{Ric}_k > 0$. An elementary way to produce new examples is to take the Riemannian product of two manifolds $M^n = N_1 \times N_2$. However, such a product metric will be of $\text{Ric}_k > 0$ only for $k > \frac{n}{2}$, even if both factors are of $\text{sec} > 0$; see Proposition 2.2. In order to distinguish our examples from such trivial constructions, we put primary focus in this article on spaces M^n of $\text{Ric}_k > 0$ with $k \leq \frac{n}{2}$ and, more generally, on spaces of $\text{Ric}_k > 0$ where k is small compared to their dimension. (It is remarkable that the bound $k \leq \frac{n}{2}$ also shows up in various structural results on manifolds of $\text{Ric}_k > 0$, see [26, 54].) The following theorem is a particular instance of the results in this work.

Theorem A *Every dimension $n \geq 7$ congruent to 3 mod 4 supports infinitely many closed simply connected manifolds of pairwise distinct homotopy type, all of which admit homogeneous metrics of $\text{Ric}_k > 0$ for some $k < n/2$.*

The families of spaces referenced in Theorem A, which are generalized versions of the so-called Aloff–Wallach spaces from [1], are defined in (1.3) below. The reader can find the precise k for each of the spaces and a discussion about their topology in Theorem G and the paragraph following it, respectively. Besides the spaces in Theorem A, we construct many other manifolds of $\text{Ric}_k > 0$ for some $k < \frac{n}{2}$, including the unit tangent bundles or projectivized tangent bundles of compact rank one symmetric spaces.

A number of results is involved in our constructions, and they all can be found in Sect. 1; we believe each of them is of independent interest. Our methods generalize those that were used to construct all homogeneous spaces of $\text{sec} > 0$, whose classification was achieved in the 1970s. In particular, we will study curvature properties of symmetric spaces, normal homogeneous spaces, and certain homogeneous spaces that fiber over symmetric spaces via fat homogeneous bundles.

Because we study homogeneous metrics here, we have many isometric actions at our disposal. By taking appropriate quotients, we obtain many manifolds of $\text{Ric}_k > 0$ which, by Synge’s theorem, cannot carry metrics of $\text{sec} > 0$. Subsequently, we establish the following result, which we prove in Sect. 1.3.

Theorem B *Each dimension $n \geq 4$ congruent to 0 or 1 mod 4 supports (non-simply connected) closed manifolds which admit metrics of $\text{Ric}_k > 0$ for some $k \leq n/2$, but not metrics of $\text{sec} > 0$.*

1 Summary of results

In this section, we discuss the main results of this article, put them into context, and examine their implications. Specifically, we deal with normal homogeneous and symmetric spaces in Sect. 1.1, we discuss fat homogeneous bundles in Sect. 1.2, and we examine some important consequences in relation to non-simply connected examples, manifolds of $\text{Ric}_2 > 0$, and Riemannian submersions in Sect. 1.3.

1.1 Normal homogeneous and symmetric spaces

We start with the class of compact homogeneous spaces G/H with *finite fundamental group* and G connected. It is a classical result that *normal* homogeneous metrics on G/H (i.e. descending from a bi-invariant metric on G) are of $\text{Ric} > 0$; see e.g. the work of Nash [39, Proposition 3.4] or Berestovskii [6, Theorem 1]. As we shall see in Theorem C below, for most spaces, normal metrics are actually “more curved” than simply $\text{Ric} > 0$ from the perspective of positive k th-intermediate Ricci curvature.

First, we observe that for any two normal metrics g, g' on G/H , the minimum k for which g is of $\text{Ric}_k > 0$ agrees with the minimum k for which g' is of $\text{Ric}_k > 0$; see Corollary 3.3. Hence, in order to facilitate our discussion, we introduce the following notation:

$$b(G/H) := \min\{k \in \mathbb{N} : \text{Ric}_k > 0 \text{ for a normal homogeneous metric } g \text{ on } G/H\}.$$

Note that the integer $b(G/H)$ depends on the pair (G, H) rather than on the diffeomorphism type of G/H ; see Remarks 1.1 and 1.2 below. With this notation, the result of Nash-Berestovskii can be stated with the inequality $b(G/H) \leq \dim G/H - 1$. We extend this result by characterizing the equality case:

Theorem C *Let G/H be a normal homogeneous space for a semisimple Lie group G . Then $b(G/H) = \dim G/H - 1$ if and only if G/H locally splits off a factor isometric to a round 2-sphere.*

Theorem C can be seen as a classification of those homogeneous spaces G/H with $b(G/H) = \dim G/H - 1$. In general, it would be a large undertaking to determine $b(-)$ for every homogeneous space. Indeed, only the opposite and most restrictive case, $b(G/H) = 1$, is fully known. Berger classified all pairs (G, H) for which G/H admits a normal homogeneous metric of $\sec > 0$ [7], with an omission corrected by Wilking [50]. The rather short list consists of manifolds diffeomorphic to compact rank one symmetric spaces and three other spaces: the so-called Berger spaces $B^7 := \text{SO}_5/\text{SO}_3$ and $B^{13} := \text{SU}_5/\text{Sp}_2\text{U}_1$, and Wilking’s space $(\text{SO}_3 \times \text{SU}_3)/\text{U}_2$. For descriptions of these spaces, see, for example, [53].

Remark 1.1 Wilking’s positively curved normal homogeneous space $(\text{SO}_3 \times \text{SU}_3)/\text{U}_2$ is diffeomorphic to the Aloff–Wallach space $W_{1,1}^7 := \text{SU}_3/S(\text{U}_1^{1,1} \times \text{U}_1)$; see (1.2) for a description of $\text{U}_1^{1,1}$. However, $W_{1,1}^7$ does not admit a positively curved normal homogeneous metric (i.e. descending from a bi-invariant metric on SU_3). Thus, $b(W_{1,1}^7) > 1 = b((\text{SO}_3 \times \text{SU}_3)/\text{U}_2)$.

There are two subclasses of homogeneous spaces, though, for which it is manageable to determine $b(-)$; namely symmetric spaces and spaces of the form $(G \times G \times G)/\Delta G$. We will discuss symmetric spaces here. For the spaces $(G \times G \times G)/\Delta G$, see the discussion around Theorem E below.

In Sect. 4 we determine $b(G/K)$ for every symmetric space G/K of compact type. In particular, we list the values $b(G/K)$ for the irreducible symmetric spaces in Table 3 at the end of this article. Very recently, during the writing of this paper, Amann, Quast and Zarei obtained the same result independently, and used it to study the higher connectedness of symmetric spaces [3].

In order to determine $b(G/K)$ for each symmetric space G/K , the basic observation is that $b(G/K) = \max_{x \in \mathfrak{p} \setminus \{0\}} \dim Z_{\mathfrak{p}}(x)$, where \mathfrak{p} is the orthogonal complement of \mathfrak{k} in \mathfrak{g} (the Lie algebras of K and G , respectively), and $Z_{\mathfrak{p}}(x)$ is the centralizer of x in \mathfrak{p} . The centralizers that may have maximal dimension determine some particularly nice class of totally geodesic submanifolds of G/K . By standard theory of root systems, one can calculate the dimensions of such totally geodesic submanifolds, thus deriving the value of $b(G/K)$. The proof in [3] is, in essence, equivalent to ours, the main difference being that we make use of the Dynkin diagrams instead of the explicit description of the roots.

We now summarize the main facts of our study of symmetric spaces. Throughout this article, $G_p^+(\mathbb{R}^{p+q}) := \text{SO}_{p+q}/(\text{SO}_p \times \text{SO}_q)$ denotes the oriented real Grassmannians, and $G_p(\mathbb{C}^{p+q}) := \text{SU}_{p+q}/S(\text{U}_p \times \text{U}_q)$, $G_p(\mathbb{H}^{p+q}) := \text{Sp}_{p+q}/(\text{Sp}_p \times \text{Sp}_q)$ the complex and quaternionic Grassmannians.

Theorem D *Let $M = G/K$ be a symmetric space of compact type. If M is irreducible, then $b(M)$ is given in Table 3 at the end of this article. If M is reducible, and $\tilde{M} = M_1 \times \cdots \times M_s$ is the decomposition into irreducible factors of the universal cover of M , then $b(M) = \max\{b(M_j) + \dim M - \dim M_j : j = 1, \dots, s\}$. Moreover:*

- (a) $b(M) \geq 2 \text{rank } M - 1$, with equality if and only if M has rank one or its universal cover is SU_3/SO_3 , $G_2^+(\mathbb{R}^5)$, G_2/SO_4 or a finite product of 2-spheres. In particular, $b(M)$ is never equal to 2.

- (b) $b(M) \leq \dim M - 3$ if and only if M does not locally split off a 2-sphere, a 3-sphere, or a Wu space SU_3/SO_3 .
- (c) $b(M) \leq (\dim M)/2$ if and only if M is irreducible and either
 - (i) M has rank 1 or 2 but $M \not\cong SU_3/SO_3$, or
 - (ii) M is isomorphic to one of the following spaces of rank 3 or 4: $G_3(\mathbb{C}^6)$, $G_3(\mathbb{H}^6)$, E_7/E_6U_1 , SO_{12}/U_6 , SO_{14}/U_7 , and any space with Dynkin diagram of F_4 type (i.e. F_4/Sp_3Sp_1 , E_6/SU_6SU_2 , $E_7/Spin_{12}Sp_1$, E_8/E_7Sp_1 , F_4).
- (d) $b(M) \leq (\dim M)/3$ if and only if either M is of rank 1 but $M \not\cong \mathbb{S}^2$, or $M \cong G_2$.
- (e) All irreducible M with $3 \leq b(M) \leq 6$ are listed in Table 1 in Sect. 4.
- (f) If M has rank 2, the value of $b(M)$ is given in Table 2 in Sect. 4.

We remark that if we consider general homogeneous (non-symmetric) metrics g and replace $b(M)$ with the minimal k for which g satisfies $Ric_k > 0$, then Items (a) to (d) in Theorem D would not hold. For example, the rank 2 reducible symmetric space $\mathbb{S}^3 \times \mathbb{S}^3$ admits a homogeneous metric of $Ric_2 > 0$; see Remark 1.2. We also note that Item (b) does not hold for normal metrics on homogeneous spaces. Indeed, in Sect. 3.4, we will observe that every odd dimension $2n + 3 \geq 7$ contains infinitely many homogeneous spaces G/H of distinct homotopy type with $b(G/H) = \dim G/H - 2$ that do not locally split off any factor. Moreover, building upon the results in [16], we conclude that for many of them, the moduli space of metrics of $Ric_{\dim G/H - 2} > 0$ has infinitely many connected components.

We now discuss normal metrics on homogeneous spaces of the form $(G \times G \times G)/\Delta G$, where G is a compact semisimple Lie group. The motivation for considering these spaces comes from a construction of Burkhard Wilking, shared with the third author in a personal communication, that the Cheeger deformation of the product of round metrics on $\mathbb{S}^3 \times \mathbb{S}^3$ via the diagonal action by $\Delta \mathbb{S}^3 < \mathbb{S}^3 \times \mathbb{S}^3$ is of $Ric_2 > 0$. We refer the reader unfamiliar with Cheeger deformations to [56, Section 2]. Cheeger deformations do not generally result in homogeneous metrics, however in this particular case Wilking’s metric on $\mathbb{S}^3 \times \mathbb{S}^3$ is isometric to a normal homogeneous metric on $(\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\Delta \mathbb{S}^3$. His construction generalizes as follows.

Theorem E *For any compact semisimple Lie group G , it holds that:*

$$b\left(\frac{G \times G \times G}{\Delta G}\right) = 2b(G).$$

Consequently, $G \times G$ admits a metric of $Ric_k > 0$ for $k = 2b(G)$ which is left-invariant and invariant under right ΔG -diagonal multiplication.

We will explore some consequences of the resulting metric on $G \times G$ in Sect. 1.3. Observe that the spaces $M := G^3/\Delta G$ for which $b(M) \leq (\dim M)/2$ are precisely those for which $b(G) \leq (\dim G)/2$. From Theorem D, we know that the only Lie groups satisfying $b(G) \leq (\dim G)/2$ are the simple ones of rank ≤ 2 and F_4 . The particular case $G = SU_2 \cong \mathbb{S}^3$ is very remarkable, in that it yields a normal metric of $Ric_2 > 0$.

Remark 1.2 The homogeneous space $(\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\Delta\mathbb{S}^3$ is the only simply connected one we know of that satisfies $b(-) = 2$; compare to Item (a) in Theorem D. Observe that normal homogeneous metrics on $\mathbb{S}^3 \times \mathbb{S}^3$ do not satisfy $\text{Ric}_2 > 0$. Indeed, they are products of round metrics, and hence $b(\mathbb{S}^3 \times \mathbb{S}^3) = \dim \mathbb{S}^3 + 1 = 4$ (cf. Proposition 3.5).

1.2 Fat homogeneous bundles

After the compact rank one symmetric spaces and the Berger spaces B^7, B^{13} , the next examples of closed manifolds of $\text{sec} > 0$ were constructed by Wallach in 1972 [46]; namely the spaces $W^6 := \text{SU}_3/\text{U}_1^2, W^{12} := \text{Sp}_3/\text{Sp}_1^3$ and $W^{24} := \text{F}_4/\text{Spin}_8$ now known as the *Wallach flag manifolds*. Their common feature is that they arise as total spaces of *fat homogeneous bundles* over compact rank one symmetric spaces. We refer the reader unfamiliar with fat homogeneous bundles to Sect. 5 for definitions and references. Throughout this article we understand a metric on G/H to be homogeneous if it is G -invariant, i.e. our definition depends on the pair (G, H) .

Wallach’s construction can be stated as follows: if $H < K < G$ is a nested triple of compact Lie groups such that the induced bundle $K/H \rightarrow G/H \rightarrow G/K$ is fat and G/K is a rank one symmetric space, then G/H admits a homogeneous metric of $\text{sec} > 0$ [56, Proposition 4.3]. This approach was subsequently used by Aloff and Wallach [1] to construct homogeneous metrics of $\text{sec} > 0$ on the spaces $W_{p,q}^7$; see (1.3). This family completes the list of simply connected homogeneous manifolds admitting a metric of $\text{sec} > 0$, up to diffeomorphism [53]. Observe that Wallach’s construction results in metrics with “more” curvature than the corresponding normal metrics since, by Berger’s classification, none of the spaces $W^6, W^{12}, W^{24}, W_{p,q}^7$ admit normal homogeneous metrics of $\text{sec} > 0$, with a subtle exception in the case of $W_{1,1}^7$; see Remark 1.1.

Wallach’s construction has been generalized to several other curvature conditions (such as almost positive curvature [51], quasi-positive curvature [29, 44] or strongly positive curvature [11]). Here, we present the following generalization of Wallach’s result:

Theorem F *Let $K/H \rightarrow G/H \rightarrow G/K$ be a fat homogeneous bundle with G/K a symmetric space. Then G/H admits a homogeneous metric of $\text{Ric}_k > 0$ for $k = b(G/K)$.*

Fat homogeneous bundles are very rigid; indeed, a full classification of them was established by Bérard-Bergery [5]. In Sect. 5.2, we examine fat homogeneous bundles whose base space G/K is symmetric and satisfies $b(G/K) \leq (\dim G/K)/2$. By Theorem F, the corresponding total spaces G/H admit homogeneous metrics of $\text{Ric}_k > 0$ for some $k < (\dim G/H)/2$. Theorem 5.4 provides a list of such total spaces G/H and the k for which $\text{Ric}_k > 0$. Here we only discuss in detail the necessary cases for the proof of Theorem A. The following triples induce fat homogeneous bundles:

$$S(\text{U}_1^{p,q} \times \text{U}_{n-1}) < S(\text{U}_2 \times \text{U}_{n-1}) < \text{SU}_{n+1}, \quad n \geq 2. \tag{1.1}$$

The inclusions are induced by standard diagonal block embeddings, except for the first one, where p, q are coprime integers with $pq > 0$, and

$$U_1^{p,q} := \{(z^p, z^q) : z \in U_1\} < U_1 \times U_1. \tag{1.2}$$

The corresponding bundles have the Grassmannian $G_2(\mathbb{C}^{n+1}) := SU_{n+1}/S(U_2 \times U_{n-1})$ as base space, and the respective total spaces are the *generalized Aloff–Wallach spaces*

$$W_{p,q}^{4n-1} := SU_{n+1}/S(U_1^{p,q} \times U_{n-1}). \tag{1.3}$$

Note that $W_{1,1}^{4n-1}$ equals the unit tangent bundle $T^1\mathbb{C}P^n$. Applying Theorem F, in combination with the value of $b(G_2(\mathbb{C}^{n+1}))$ from Table 2 in Sect. 4, we obtain the first row of Item (a) in Theorem G below.

The last four rows of Item (a) in Theorem G below correspond to other infinite series of manifolds that arise as total spaces of fat homogeneous bundles (let $n \geq 2$):

- the projectivized tangent bundle $\mathbb{P}_{\mathbb{C}}T\mathbb{C}P^n := SU_{n+1}/S(U_1 \times U_1 \times U_{n-1})$ of $\mathbb{C}P^n$,
- $SU_{n+1}/(SU_2 \times SU_{n-1})$, which can be described as the total space of a circle bundle over $G_2(\mathbb{C}^{n+1})$ whose Euler class is a generator of $H^2(G_2(\mathbb{C}^{n+1}); \mathbb{Z}) = \mathbb{Z}$,
- the projectivized tangent bundle $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^n := Sp_{n+1}/(Sp_1^2 \times Sp_{n-1})$ of $\mathbb{H}P^n$, and
- the unit tangent bundle $T^1S^n := SO_{n+1}/SO_{n-1}$ of the sphere S^n .

Observe that $\mathbb{P}_{\mathbb{C}}T\mathbb{C}P^2$ and $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^2$ coincide with the Wallach spaces W^6 and W^{12} (and moreover recall that the projectivized tangent bundle of the Cayley plane coincides with W^{24}). We refer to Sect. 5.2 and Theorem 5.4 for the corresponding constructions and values, from where Item (b) below can be likewise extracted.

Theorem G *The following hold:*

- (a) *Each homogeneous space G/H in the following table carries a homogeneous metric of $\text{Ric}_k > 0$ for the corresponding value of k , where $n \geq 2$:*

G/H	$\dim G/H$	$(n \neq 3)$	k	$(n = 3)$
$W_{p,q}^{4n-1}$	$4n-1$	$2n-3$		4
$\mathbb{P}_{\mathbb{C}}T\mathbb{C}P^n$	$4n-2$	$2n-3$		4
$SU_{n+1}/(SU_2 \times SU_{n-1})$	$4n-3$	$2n-3$		4
$\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^n$	$8n-4$	$4n-7$		6
T^1S^n	$2n-1$		$n-1$	

- (b) *The following G/H carry homogeneous metrics of $\text{Ric}_k > 0$ for some $k \leq (\dim G/H)/3$: $W_{p,q}^{15}$, $SU_6/S(U_2 \times Sp_2)$, G_2/SU_2^\pm , G_2/U_2^\pm , $E_6/Spin_{10}$, SO_{10}/SU_5 , $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^3$ and $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^4$.*

The cohomology ring of $W_{p,q}^{4n-1}$ can be computed using standard techniques, see Sect. 5.3. In particular, $W_{p,q}^{4n-1}$ is simply connected and $H^{2n}(W_{p,q}^{4n-1}; \mathbb{Z})$ is finite of

order equal to $|p^n + p^{n-1}q + \dots + pq^{n-1} + q^n|$; see Lemma 5.3 or [51, p. 119]. Hence by varying p, q appropriately, we obtain infinitely many spaces of pairwise distinct homotopy type. This, along with the first row of Theorem G (a), implies Theorem A. We remark that Wilking (resp. Tapp) constructed on each $W_{p,q}^{4n-1}$ with $pq < 0$ (resp. $pq \neq 0$) inhomogeneous metrics of positive sectional curvature on an open dense subset [51] (resp. of non-negative sectional curvature and positive at a point [44]).

The unit tangent bundle $T^1\mathbb{S}^n$ is especially interesting in the cases $n = 3, 7$, since the corresponding spheres \mathbb{S}^n are parallelizable and hence $T^1\mathbb{S}^n$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^n$. Observe that the corresponding metrics of $\text{Ric}_k > 0$ with $k = 2, 6$ given by Theorem G improve the standard product metrics, which are only of $\text{Ric}_k > 0$ for $k = 4, 8$, respectively. The case $n = 3$ is somehow exceptional since the metric given by the fat bundle construction is only of $\text{Ric}_3 > 0$ (cf. Theorem 5.4). However, $T^1\mathbb{S}^3 = \text{SO}_4/\text{SO}_2$ is $\text{Sp}_1 \times \text{Sp}_1$ -equivariantly diffeomorphic to the homogeneous space $(\mathbb{S}^3 \times \mathbb{S}^3)/\Delta\mathbb{S}^1$, which does admit a metric of $\text{Ric}_2 > 0$, as it will be discussed in Remark 1.3; see Remark 5.2 for details.

The spaces G_2/SU_2^- and G_2/U_2^- appearing in Item (b) carry metrics of $\text{Ric}_3 > 0$ by Theorem 5.4, and are diffeomorphic to the unit tangent bundle $T^1\mathbb{S}^6$ (for which the homogeneous metric given by Theorem G is “only” of $\text{Ric}_5 > 0$) and to $\text{G}_2^+(\mathbb{R}^7)$, respectively [28]. The spaces SU_3/SO_3 , $\text{G}_2^+(\mathbb{R}^5)$, $\text{G}_2^+(\mathbb{R}^7)$, and G_2/SO_4 are the only (simply connected) irreducible symmetric spaces of rank > 1 we know of to admit metrics of $\text{Ric}_3 > 0$; see Table 1 in Sect. 4. It is an open question whether there exists an irreducible symmetric space of rank > 1 which admits a metric of $\text{Ric}_2 > 0$, much less of $\text{sec} > 0$ (see [17] for the construction of a metric on $\text{G}_2^+(\mathbb{R}^7)$ of $\text{sec} > 0$ on an open dense subset).

1.3 Isometric quotients and manifolds of $\text{Ric}_2 > 0$

The metrics considered in the previous subsections have considerably large isometry groups, and hence their quotients constitute a large class of examples. It is straightforward to see that quotient spaces of large enough dimension inherit metrics of positive intermediate Ricci curvature by O’Neill’s curvature equations for Riemannian submersions; see Corollary 2.3. Also recall that Synge’s Theorem states that if a closed manifold is orientable, not simply connected, and even-dimensional, or if it is non-orientable and odd-dimensional, then it cannot admit a metric of $\text{sec} > 0$. So by taking appropriate quotients of previous examples, we will now establish Theorem B using Synge’s Theorem.

Proof of Theorem B For any $n \geq 3$, the Grassmannian $\text{G}_2(\mathbb{R}^{n+2}) = \text{SO}_{n+2}/\text{S}(\text{O}_2 \times \text{O}_n)$ of (non-oriented) 2-planes in \mathbb{R}^{n+2} is an irreducible symmetric space of dimension $2n$ and rank 2. It is a \mathbb{Z}_2 -quotient of the oriented 2-plane Grassmannian $\text{G}_2^+(\mathbb{R}^{n+2})$. By Theorem D (cf. Table 2), its symmetric metric is of $\text{Ric}_k > 0$, for $k = n = (\dim \text{G}_2(\mathbb{R}^{n+2}))/2$, whenever $n \geq 3$. It is well known that $\text{G}_2(\mathbb{R}^{n+2})$ is orientable if and only if n is even (this follows, for example, from a characterization of the orientability of homogeneous spaces [35, p. 275]). Thus, for each even $n \geq 4$, the orientable but non-simply connected manifold $\text{G}_2(\mathbb{R}^{n+2})$ cannot admit a Riemannian

metric of $\text{sec} > 0$, by Synge’s Theorem. This shows Theorem B for dimensions ≥ 8 that are multiples of 4.

The unit tangent bundle $T^1\mathbb{S}^n = \text{SO}_{n+1}/\text{SO}_{n-1}$ induces a Riemannian covering over the projectivized tangent bundle $\mathbb{P}_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n := \text{SO}_{n+1}/\text{S}(\text{O}_1^2 \times \text{O}_{n-1})$ of $\mathbb{R}\mathbb{P}^n$. The latter inherits a homogeneous metric of $\text{Ric}_k > 0$ with the same k as $T^1\mathbb{S}^n$, namely $k = n - 1 < (\dim \mathbb{P}_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n)/2$. Recall that $\mathbb{P}_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n$ is non-orientable if n is odd since it fits into a smooth fiber bundle $\mathbb{R}\mathbb{P}^{n-1} \rightarrow \mathbb{P}_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n \rightarrow \mathbb{R}\mathbb{P}^n$ whose fiber is non-orientable. Thus $\mathbb{P}_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n$ does not admit metrics of $\text{sec} > 0$ by Synge’s Theorem. This proves Theorem B for dimensions ≥ 5 congruent to 1 mod 4.

The case of dimension 4 in Theorem B follows from the metric of $\text{Ric}_2 > 0$ on $\text{SO}_3 \times \text{SO}_3$ resulting from Theorem E. The quotient of $\text{SO}_3 \times \text{SO}_3$ under the left action by the product subgroup $\text{O}_2 \times \text{O}_2$ is diffeomorphic to $\mathbb{R}\mathbb{P}^2 \times \mathbb{R}\mathbb{P}^2$. Then $\mathbb{R}\mathbb{P}^2 \times \mathbb{R}\mathbb{P}^2$ inherits a metric of $\text{Ric}_2 > 0$ by Corollary 2.3, but cannot admit a metric of $\text{sec} > 0$. □

Remark 1.3 The Wilking metric of $\text{Ric}_2 > 0$ on $\mathbb{S}^3 \times \mathbb{S}^3$ is left-invariant and right $\Delta\mathbb{S}^3$ -invariant, see Theorem E. By Corollary 2.3 it induces metrics of $\text{Ric}_2 > 0$ on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \times \mathbb{S}^3$, by taking the quotients of the left action of a maximal torus of $\mathbb{S}^3 \times \mathbb{S}^3$, and of the right $\Delta\mathbb{S}^1$ -action, respectively. The manifolds $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{S}^3$, and $\mathbb{S}^3 \times \mathbb{S}^3$ are the only simply connected ones we know of to admit metrics of $\text{Ric}_2 > 0$, besides those of $\text{sec} > 0$.

The approach followed above can be extended to other homogeneous spaces of low dimensions to generate more examples of manifolds admitting metrics of $\text{Ric}_k > 0$ but not $\text{sec} > 0$. Specifically, one can consider bi-invariant metrics on a compact simple Lie group G , as well as the homogeneous metrics on G^2 given in Theorem E. Both have large groups of isometries acting freely, and thus give rise to many isometric quotients. We can, for instance, consider subgroups H of G or G^2 of the form $H = S \cdot \Gamma$, where S is a closed connected subgroup with $\text{rank } S < \text{rank } G$, and Γ is a finite subgroup of a torus $\mathbb{T} \subset G$ orthogonal to S . If we restrict to quotients G/H or G^2/H of even dimension, and take subgroups Γ such that H has at least 3 connected components, then the quotients cannot admit metrics of $\text{sec} > 0$ by Synge’s Theorem. In order to obtain quotients of $\text{Ric}_k > 0$ where k is at most half of their dimensions, by Theorem D we have to restrict to simple Lie groups of $\text{rank} \leq 2$ and to F_4 , and their squares. Thus, for example, taking S a (possibly trivial) even-dimensional torus in the groups $G = \text{SU}_3, \text{Sp}_2, G_2, F_4$ with bi-invariant metrics, we obtain examples in dimensions 8, 10, 14, 50 and 52. We can argue similarly with squares G^2 endowed with the metrics from Theorem E, thus obtaining examples in many even dimensions $\leq 104 = 2 \dim F_4$. The case of the group $F_4 \times F_4$ is rather interesting, in that it allows to show that each finite subgroup of the 8-torus can be realized as the fundamental group of a closed n -manifold of $\text{Ric}_k > 0$ for some $k < n/2$. Indeed, $F_4 \times F_4$ carries a homogeneous metric of $\text{Ric}_{44} > 0$ (by Theorem E and Table 3). Taking the quotient under any finite subgroup Γ of a maximal torus \mathbb{T}^8 we obtain a manifold of dimension 104 with fundamental group equal to Γ and a metric of $\text{Ric}_{44} > 0$.

We conclude this subsection with an observation regarding Riemannian submersions and the Wilking metric of $\text{Ric}_2 > 0$ on $\mathbb{S}^3 \times \mathbb{S}^3$ discussed above Theorem E. The right action by $\Delta\mathbb{S}^3$ on $\mathbb{S}^3 \times \mathbb{S}^3$ induces a metric on $(\mathbb{S}^3 \times \mathbb{S}^3)/\Delta\mathbb{S}^3$ that inherits

the left action by $\mathbb{S}^3 \times \mathbb{S}^3$. As such it is a homogeneous metric (i.e. $\mathbb{S}^3 \times \mathbb{S}^3$ -invariant), which must be normal since $(\mathbb{S}^3 \times \mathbb{S}^3)/\Delta\mathbb{S}^3$ is an irreducible symmetric pair. Under the natural diffeomorphism $(\mathbb{S}^3 \times \mathbb{S}^3)/\Delta\mathbb{S}^3 \cong \mathbb{S}^3$ (see Remark 3.7) the metric becomes bi-invariant on \mathbb{S}^3 , which is round and hence submerges onto a metric \bar{g} of constant curvature on \mathbb{S}^2 . Altogether, we have a Riemannian submersion from Wilking's metric g :

$$(\mathbb{S}^3 \times \mathbb{S}^3, g) \rightarrow (\mathbb{S}^2, \bar{g}) \quad \text{with } \text{Ric}_2(g) > 0. \quad (1.4)$$

Riemannian submersions between closed manifolds satisfying certain curvature bounds are challenging to construct, and it is expected that there might be topological obstructions for their existence. In this context, the Petersen–Wilhelm Conjecture suggests that any Riemannian submersion $M \rightarrow B$ defined on a positively curved manifold M with fibers F must have $\dim F < \dim B$; see [2, 21, 22, 43] for further information. Although this conjecture is still open, it is natural to ask whether one should expect an analogous fiber dimension bound for Riemannian submersions defined on manifolds with $\text{Ric}_k > 0$. More precisely, for each $k \geq 1$ one could ask if there exists a positive number $f(k)$ such that if $\text{Ric}_k(M) > 0$, then $\dim F < f(k) \dim B$. In this language, the Petersen–Wilhelm Conjecture suggests that $f(1)$ exists and is equal to 1. The submersion in (1.4) shows that if $f(2)$ exists, then its value is strictly greater than 2.

Structure of the article

In Sect. 3 we consider normal homogeneous spaces and give the proofs of Theorems C and E. Section 4 is devoted to the computation of $b(G/K)$ for any symmetric space G/K of compact type, and contains the proof of Theorem D. In Sect. 5, we review fat homogeneous bundles, we provide the proof of Theorem F, we discuss various families of fat homogeneous bundles, from where Theorem G follows, and we prove the topological claims in Theorem A. Finally, we include Table 3 with the values of $b(G/K)$ for all irreducible compact symmetric spaces G/K at the end of the article.

2 Background and context

In this section, we give a general overview of basic concepts related to intermediate Ricci curvature. Our goal is to compensate for the fact that there is limited existing literature on intermediate Ricci curvature. At the end of this section, we also include a discussion of other different intermediate curvature conditions that others have studied.

2.1 Definition and basic constructions

Here, we provide a more general definition of Ric_k , namely as a map defined on flags that, when it is required to be positive, restricts to the definition given at the beginning of this article. To that end, let $\text{Fl}(1, k + 1; TM)$ denote the *partial flag bundle* consisting of signature- $(1, k + 1)$ flags tangent to M . In other words, elements of $\text{Fl}(1, k + 1; TM)$ are pairs $(\mathcal{V}^1, \mathcal{V}^{k+1})$ such that \mathcal{V}^1 is a 1-dimensional subspace of a $(k + 1)$ -dimensional subspace \mathcal{V}^{k+1} of a tangent space T_pM for some $p \in M$. Given a Riemannian metric on M , let ∇ and R denote the associated Levi-Civita connection and curvature tensor, respectively. For any vector $x \in T_pM$, the *directional curvature operator* or *Jacobi operator* $R_x : T_pM \rightarrow T_pM$ is then defined by

$$R_x(y) := R(y, x)x = \nabla_Y \nabla_X X - \nabla_X \nabla_Y X - \nabla_{[Y, X]} X,$$

where X and Y are any extensions of x and y to smooth vector fields, respectively.

Definition 2.1 Suppose (M, g) is a Riemannian manifold and $k \in \{1, \dots, \dim M - 1\}$. Given $p \in M$, a $(k + 1)$ -dimensional subspace $\mathcal{V}^{k+1} \subset T_pM$, and a vector $x \in \mathcal{V}^{k+1}$, let $R_x^{\mathcal{V}^{k+1}}$ denote the directional curvature operator restricted to \mathcal{V}^{k+1} composed with the orthogonal projection onto \mathcal{V}^{k+1} . Then $\text{Ric}_k : \text{Fl}(1, k + 1; TM) \rightarrow \mathbb{R}$ is defined by

$$\text{Ric}_k(\mathcal{V}^1, \mathcal{V}^{k+1}) := \text{trace}(R_x^{\mathcal{V}^{k+1}}) = \sum_{i=1}^k \sec(x, e_i),$$

where x is any unit vector in the line \mathcal{V}^1 , and e_1, \dots, e_k is any orthonormal basis for the orthogonal complement of \mathcal{V}^1 in \mathcal{V}^{k+1} .

Given a signature- $(1, 2, n)$ flag $\mathcal{V}^1 \subset \mathcal{V}^2 \subset \mathcal{V}^n$ tangent to an n -manifold, notice that $\text{Ric}_1(\mathcal{V}^1, \mathcal{V}^2)$ is the sectional curvature of the 2-plane \mathcal{V}^2 , and $\text{Ric}_{n-1}(\mathcal{V}^1, \mathcal{V}^n)$ is the Ricci curvature $\text{Ric}(x, x)$ for either choice of unit vector $x \in \mathcal{V}^1$. Furthermore, notice that this notion of Ric_k is well-defined because the value of $\text{trace}(R_x^{\mathcal{V}^{k+1}}) = \sum_{i=1}^k \sec(x, e_i)$ is independent of the choice of unit vector $x \in \mathcal{V}^1 \subset \mathcal{V}^{k+1}$ or orthonormal vectors $e_1, \dots, e_k \in \mathcal{V}^{k+1} \cap (\mathcal{V}^1)^\perp$.

Using this general definition, one can consider manifolds for which Ric_k is constant on $\text{Fl}(1, k + 1; TM)$. On the one hand, it is well known that manifolds of constant Ric_1 are characterized as (open parts of) space forms. On the other hand, manifolds of constant Ric_{n-1} (where n is the dimension) are precisely Einstein manifolds and have been extensively investigated; however their classification remains open, even in the homogeneous case. Rather surprisingly, it is not difficult to show that manifolds of constant Ric_k with $k < n - 1$ must have constant Ric_1 , see [40, Section 5].

Regarding positive k th-intermediate Ricci curvature, Riemannian products give us an elementary way of constructing manifolds of $\text{Ric}_k > 0$ from known examples. Since the Levi-Civita connection of a Riemannian product splits as the sum of the Levi-Civita of both factors, it is straightforward to check the following:

Proposition 2.2 *Suppose (M_1, g_1) and (M_2, g_2) are Riemannian manifolds such that $\text{Ric}_{k_i}(M_i, g_i) > 0$ for some $k_i \in \{1, \dots, \dim(M_i) - 1\}$. Then their Riemannian product $(M_1 \times M_2, g_1 + g_2)$ is of $\text{Ric}_k > 0$ for $k = \max\{k_1 + \dim(M_2), k_2 + \dim(M_1)\}$.*

Riemannian submersions give another source of examples of manifolds with $\text{Ric}_k > 0$. An important subclass in this article is given by quotients of free, isometric actions by compact Lie groups.

It follows from O'Neill's horizontal curvature equation that a Riemannian submersion $\pi : (M, g_M) \rightarrow (B, g_B)$ satisfies $\sec_B(d\pi(x), d\pi(y)) \geq \sec_M(x, y)$ for all orthonormal vectors x, y in the horizontal distribution $(\ker d\pi)^\perp$; see [23] for further information. Thus, the following is an immediate consequence:

Corollary 2.3 *Suppose $\pi : (M, g_M) \rightarrow (B, g_B)$ is a Riemannian submersion. If $\text{Ric}_k > 0$ on the horizontal distribution in M for some $k \in \{1, \dots, \dim(B) - 1\}$, then $\text{Ric}_k(B, g_B) > 0$.*

2.2 Related results and curvature conditions

We close this section by highlighting some existing work on manifolds of $\text{Ric}_k > 0$ and related curvature conditions. Various classical results for manifolds with lower bounds on sectional or Ricci curvature have been extended appropriately to the context of $\text{Ric}_k > 0$ by several authors. Synge's Theorem was generalized by Wilhelm [49], Heintze–Karcher's Inequality by Chahine [14], and Gromoll–Meyer's Theorem and Cheeger–Gromoll's Soul Theorem on open manifolds by Shen [42]. Moreover, various comparison results have been established by Guijarro and Wilhelm in the series of papers [24–26], while the third named author is carrying out a systematic study of manifolds of $\text{Ric}_k > 0$ under the presence of isometric actions by large Lie groups [36, 38]. We note that Wilking's Connectedness Principle, a foundational tool for studying isometric group actions on manifolds of positive sectional curvature, also generalizes to the setting of $\text{Ric}_k > 0$ [52, Remark 2.4].

Now given a unit vector $x \in T_p M$, let x^\perp denote the orthogonal complement of $\text{span}\{x\}$ in $T_p M$. Then the restriction of the directional curvature operator R_x to x^\perp is a self-adjoint endomorphism. Verdiani and Ziller studied manifolds for which the sum of the k smallest eigenvalues of $R_x|_{x^\perp}$ is positive or non-negative in [45]. One advantage of defining Ric_k as we have in this section is that one can easily see this condition is equivalent to Ric_k being positive or non-negative as a consequence of the Min-Max Theorem applied to $R_x|_{x^\perp}$:

Proposition 2.4 *On any Riemannian manifold, Ric_k is positive (resp. non-negative) if and only if the sum of the k smallest eigenvalues of the operator $R_x|_{x^\perp}$ (counted with multiplicity) is positive (resp. non-negative) for all unit vectors x .*

On the other hand, there has been active research on certain curvature conditions that interpolate between the classical positive Ricci and positive scalar curvature. These conditions have often been called k -Ricci curvature in the literature, which should not be mistaken with the curvature conditions under consideration in this work. We refer to the recent preprints [15, 47] for further information on the topic.

Beyond this condition, many researchers have investigated other notions of partially positive curvature. For example, Gromov proposed the study of the so-called positive p -curvature, which interpolates between positive sectional curvature when $p = n - 2$ and positive scalar curvature when $p = 0$, where n is the dimension of the manifold. For more information, see [33] or [31, Section 2.2]. For a second example, a Riemannian manifold (M, g) is said to have p -positive curvature operator if the sum of the p smallest eigenvalues of the curvature operator $\mathfrak{R}: \Lambda^2 TM \rightarrow \Lambda^2 TM$ is positive. Böhm and Wilking proved that a metric with 2-positive curvature operator evolves through Ricci flow to a metric of constant positive sectional curvature, thus showing that such spaces are spherical space forms [12].

3 Normal homogeneous spaces

This section is devoted to the investigation of the positive intermediate Ricci curvature condition in the context of normal homogeneous spaces. In particular, in Sect. 3.1 we derive various general properties of the smallest k for which normal homogeneous spaces satisfy $\text{Ric}_k > 0$. In Sect. 3.2 we briefly discuss Nash–Berestovskii’s result on homogeneous spaces with positive Ricci curvature, and prove Theorem C. In Sect. 3.3 we consider certain homogeneous spaces with diagonal isotropy groups, and prove Theorem E. Finally, as an interesting example, in Sect. 3.4 we discuss some curvature and topological properties of certain circle bundles over a product of two complex projective spaces.

3.1 Positive intermediate Ricci curvature on normal homogeneous spaces

Let G be a compact connected Lie group and H a closed subgroup of G , and denote their Lie algebras by \mathfrak{g} and \mathfrak{h} respectively. Endow the homogeneous space G/H with a normal homogeneous Riemannian metric. Let \mathfrak{p} be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to the $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} that defines the normal homogeneous metric on G/H .

For each non-zero $x \in \mathfrak{p}$, let us define $b(G/H, x)$ as the smallest integer b for which the following holds: for every set of orthonormal vectors $y_1, \dots, y_b \in \mathfrak{p}$ orthogonal to x , there is some $i \in \{1, \dots, b\}$ such that $[x, y_i] \neq 0$. Then we define

$$b(G/H) = \max_{x \in \mathfrak{p} \setminus \{0\}} b(G/H, x).$$

Note that this definition of $b(G/H)$ agrees with the definition given in Sect. 1.1. The equivalence immediately follows from the well-known curvature formula for normal homogeneous metrics; see e.g. [23, Corollary 2.4.1]:

$$\sec(x, y) = \frac{1}{4} |[x, y]_{\mathfrak{p}}|^2 + |[x, y]_{\mathfrak{h}}|^2, \quad \text{for orthonormal } x, y \in \mathfrak{p}.$$

As mentioned in Sect. 1.1, $b(G/H)$ may depend on the actual pair (G, H) under consideration; see Remarks 1.1 and 1.2. However, as we will see in Corollary 3.3, once the pair (G, H) is fixed, the number $b(G/H)$ does not depend on the normal homogeneous metric chosen on G/H .

The next proposition gives an upper bound of $b(G/H, x)$ by the codimension in \mathfrak{p} of the orbit $\text{Ad}(H)x$. Here, $\text{Ad}(H)x$ is the orbit through x of the restriction of the adjoint representation of G to H . Observe that $\text{Ad}(H)x \subset \mathfrak{p}$ can be identified with the orbit of the isotropy representation of G/H through the tangent vector $x \in \mathfrak{p} \cong T_{eH}(G/H)$. The bound we prove is sharp when $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$, i.e. when (G, H) is a symmetric pair. More precisely,

$$b(G/H) \leq \max_{x \in \mathfrak{p} \setminus \{0\}} \text{codim}_{\mathfrak{p}} \text{Ad}(H)x,$$

with equality if (G, H) is a symmetric pair. Of course, the maximum can be taken over the unit sphere of \mathfrak{p} .

Proposition 3.1 *Let G/H be a normal homogeneous space, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ the corresponding reductive decomposition. Then $b(G/H, x) = \dim Z_{\mathfrak{p}}(x)$, where $Z_{\mathfrak{p}}(x) = \{y \in \mathfrak{p} : [x, y] = 0\}$ is the centralizer of x in \mathfrak{p} , and hence*

$$b(G/H) = \max_{x \in \mathfrak{p} \setminus \{0\}} \dim Z_{\mathfrak{p}}(x).$$

Moreover, $Z_{\mathfrak{p}}(x)$ is a linear subspace of $\nu_x(\text{Ad}(H)x)$, the normal space at x to the orbit $\text{Ad}(H)x \subset \mathfrak{p}$.

Proof Fix $x \in \mathfrak{p} \setminus \{0\}$. If there is a set of orthonormal vectors $y_1, \dots, y_b \in \mathfrak{p}$ perpendicular to x and such that $[x, y_i] = 0$ for all i , then $\text{span}\{x, y_1, \dots, y_b\} \subset Z_{\mathfrak{p}}(x)$, and hence $b < \dim Z_{\mathfrak{p}}(x)$. Thus $b(G/H, x) \geq \dim Z_{\mathfrak{p}}(x)$. Consider now a set of orthonormal vectors $y_1, \dots, y_{\dim Z_{\mathfrak{p}}(x)} \in \mathfrak{p}$ perpendicular to x . If $[x, y_i] = 0$ for all i , then $\text{span}\{x, y_1, \dots, y_{\dim Z_{\mathfrak{p}}(x)}\} \subset Z_{\mathfrak{p}}(x)$, whence $\dim Z_{\mathfrak{p}}(x) + 1 \leq \dim Z_{\mathfrak{p}}(x)$, a contradiction. Thus there must exist y_i such that $[x, y_i] \neq 0$. Therefore, $b(G/H, x) \leq \dim Z_{\mathfrak{p}}(x)$, which yields the desired equality.

The tangent space to the orbit $\text{Ad}(H)x$ at x is given by $[\mathfrak{h}, x]$. Let $y \in \mathfrak{p}$. By the $\text{Ad}(G)$ -invariance of the inner product on \mathfrak{g} , we have $\langle [x, y], T \rangle = \langle y, [T, x] \rangle$ for any $T \in \mathfrak{h}$. Thus, $[x, y]_{\mathfrak{h}} = 0$ (where the subscript denotes orthogonal projection) if and only if y is orthogonal to the $\text{Ad}(H)$ -orbit through x at x , i.e. $y \in \nu_x(\text{Ad}(H)x)$. Similarly, for any $z \in \mathfrak{p}$ we have $\langle [x, y], z \rangle = \langle y, [z, x] \rangle$. Therefore, $[x, y]_{\mathfrak{p}} = 0$ if and only if y is orthogonal to $[\mathfrak{p}, x]$. All in all, $[x, y] = 0$ if and only if $y \in \nu_x(\text{Ad}(H)x) \cap [\mathfrak{p}, x]^{\perp}$, where $[\mathfrak{p}, x]^{\perp}$ denotes the orthogonal complement of $[\mathfrak{p}, x]$ in \mathfrak{g} . This shows that $Z_{\mathfrak{p}}(x) = \nu_x(\text{Ad}(H)x) \cap [\mathfrak{p}, x]^{\perp} \subset \nu_x(\text{Ad}(H)x)$. \square

Remark 3.2 An immediate consequence (which also follows by a standard Riemannian submersion argument) is that for a nested triple $H < K < G$ of compact Lie groups, we have $b(G/K) \leq b(G/H)$. Denote by $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ their Lie algebras and fix a bi-invariant metric on G . For any $x \in \mathfrak{k}^{\perp}$ we have $Z_{\mathfrak{k}^{\perp}}(x) \subset Z_{\mathfrak{h}^{\perp}}(x)$, which implies

$b(G/K, x) \leq b(G/H, x)$, and thus

$$\begin{aligned} b(G/K) &= \max_{x \in \mathfrak{k}^\perp \setminus \{0\}} b(G/K, x) \leq \max_{x \in \mathfrak{k}^\perp \setminus \{0\}} b(G/H, x) \\ &\leq \max_{x \in \mathfrak{h}^\perp \setminus \{0\}} b(G/H, x) = b(G/H). \end{aligned}$$

Corollary 3.3 *The number $b(G/H)$ is independent of the normal homogeneous Riemannian metric on G/H .*

Remark 3.4 Corollary 3.3 is obvious if G is simple, since all bi-invariant metrics on G (and hence all normal metrics on G/H) are isometric up to scaling by Schur’s lemma. Likewise, it clearly holds if G/H is isotropy irreducible (cf. [9, Proposition 7.91]).

Proof of Corollary 3.3 Let $\mathfrak{g} = \bigoplus_{i=0}^k \mathfrak{g}_i$, where \mathfrak{g}_0 is the abelian factor, and $\mathfrak{g}_i, i \geq 1$, are simple ideals. Fix an $\text{Ad}(G)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . By Schur’s lemma, any other $\text{Ad}(G)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} satisfies $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_i \times \mathfrak{g}_i} = s_i \langle \cdot, \cdot \rangle$, for some $s_i > 0$ and each $i \geq 1$, and $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_0 \times \mathfrak{g}_0} = \langle \langle A \cdot, \cdot \rangle \rangle$, for some linear automorphism A of \mathfrak{g}_0 . Since each $\mathfrak{g}_i, i \geq 1$, is simple, we have $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, which together with the $\text{Ad}(G)$ -invariance implies $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_i \times \mathfrak{g}_j} = 0$ for each $i, j, i \neq j$. Let \mathfrak{m} (respectively \mathfrak{p}) be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to $\langle \cdot, \cdot \rangle$ (resp. with respect to $\langle \cdot, \cdot \rangle$). Consider the linear isomorphism $\varphi: \mathfrak{m} \rightarrow \mathfrak{p}$ given by $\varphi(\sum_i x_i) = A^{-1}x_0 + \sum_{i \geq 1} s_i^{-1}x_i$, where $x_i \in \mathfrak{g}_i$. It is well defined, since for any $T = \sum_i T_i \in \mathfrak{h}, T_i \in \mathfrak{g}_i$, we have $\langle \varphi(\sum_i x_i), T \rangle = \langle A^{-1}x_0 + \sum_{i \geq 1} s_i^{-1}x_i, \sum_i T_i \rangle = \langle \langle \sum_i x_i, T \rangle \rangle = 0$.

Let $x = \sum_i x_i, y = \sum_i y_i \in \mathfrak{m}, x_i, y_i \in \mathfrak{g}_i$. Then $y \in Z_{\mathfrak{m}}(x)$ if and only if $0 = [y, x] = \sum_i [y_i, x_i]$, which is equivalent to $[y_i, x_i] = 0$ for all i . Similarly, $\varphi(y) \in Z_{\mathfrak{p}}(\varphi(x))$ if and only if $0 = [A^{-1}y_0 + \sum_{i \geq 1} s_i^{-1}y_i, A^{-1}x_0 + \sum_{i \geq 1} s_i^{-1}x_i] = \sum_{i \geq 1} s_i^{-2}[y_i, x_i]$, which again is equivalent to $[y_i, x_i] = 0$ for all i . Therefore, $y \in Z_{\mathfrak{m}}(x)$ if and only if $\varphi(y) \in Z_{\mathfrak{p}}(\varphi(x))$. Since φ is an isomorphism, it follows that $\dim Z_{\mathfrak{m}}(x) = \dim Z_{\mathfrak{p}}(\varphi(x))$, and then by Proposition 3.1, we get that $b(G/H, x) = b(G/H, \varphi(x))$. Therefore, $b(G/H)$ does not depend on the choice of $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} . □

For normal homogeneous spaces that are products, one can easily prove the following (cf. Proposition 2.2):

Proposition 3.5 *Let $G/H = G_1/H_1 \times \dots \times G_s/H_s$ be a normal homogeneous space. Then $b(G/H) = \max\{b(G_j/H_j) + \dim G/H - \dim G_j/H_j : j = 1, \dots, s\}$.*

3.2 The case $b(G/H) = \dim G/H - 1$ and the proof of Theorem C

Nash [39] and Berestovskii [6] proved that for a compact Lie group G and a closed subgroup H of G , the following are equivalent:

- (i) the fundamental group $\pi_1(G/H)$ is finite,
- (ii) there is a compact semisimple Lie group acting transitively on G/H ,
- (iii) G/H admits a metric of positive Ricci curvature,

- (iv) any normal metric on G/H has positive Ricci curvature (i.e. $b(G/H) \leq \dim G/H - 1$).

Since we are interested in spaces admitting metrics of positive Ricci curvature, we will assume from now on that the normal homogeneous space G/H is such that G is semisimple.

Remark 3.6 Suppose G is semisimple and hence G/H has finite fundamental group. Then its universal covering space carries a natural homogeneous structure which can be described as \tilde{G}/\tilde{H} , where \tilde{G} is the universal cover of G and \tilde{H} is the connected Lie subgroup of \tilde{G} with Lie algebra \mathfrak{h} . Since $b(-)$ depends on the corresponding Lie algebras and these remain unchanged under coverings, it clearly follows that $b(G/H) = b(\tilde{G}/\tilde{H})$.

We will now prove Theorem C, which states that, if the De Rham decomposition of the universal cover of G/H does not contain any \mathbb{S}^2 -factor, then the previous conditions (i)-(iv) are also equivalent to $b(G/H) \leq \dim G/H - 2$. Thus, Theorem C serves as a gap theorem for normal homogeneous metrics of positive intermediate Ricci curvature.

Remark 3.7 In the proof of Theorem C, we use the computation of $b(G)$ for any semisimple G , which is carried out in Sect. 4 below. Actually we shall compute $b((G \times G)/\Delta G)$, but it holds that $b(G) = b((G \times G)/\Delta G)$, as any normal homogeneous metric (i.e. bi-invariant metric) on G is isometric to a normal homogeneous metric (i.e. symmetric metric) on $(G \times G)/\Delta G$, via the identification $G \cong (G \times G)/\Delta G$ given by $g \mapsto [g, 1]$. See also Proposition 3.9 (b) below.

Proof of Theorem C Let G/H be a normal homogeneous space for a semisimple Lie group G . In view of Remark 3.6, we can assume that both G and G/H are simply connected, and hence the proof of Theorem C reduces to showing that $b(G/H) = \dim G/H - 1$ if and only if G/H splits off a factor isometric to a round \mathbb{S}^2 .

To prove sufficiency, suppose G/H is isometric to a Riemannian product $\mathbb{S}^2 \times M$ for some M . This, by De Rham decomposition theorem for naturally reductive spaces (see [30, §X.5]), implies that $G/H = G_1/H_1 \times G_2/H_2$ for closed subgroups $H_i < G_i < G, i \in \{1, 2\}$, where $G_1/H_1 \cong \mathbb{S}^2$ and $G_2/H_2 \cong M$ are endowed with normal homogeneous metrics. Then, by Proposition 3.5:

$$b(G/H) = \max\{b(\mathbb{S}^2) + \dim M, \dim \mathbb{S}^2 + b(M)\} \geq \dim M + 1 = \dim G/H - 1.$$

It follows that $b(G/H) = \dim G/H - 1$ by Nash’s result referenced above.

For the necessity, assume $b(G/H) = \dim G/H - 1$. By Proposition 3.1, there is $x \in \mathfrak{p}$ such that $\dim Z_{\mathfrak{p}}(x) = \dim G/H - 1$ and $\dim \text{Ad}(H)x \in \{0, 1\}$. We treat these cases separately.

Assume first that there exists $x \in \mathfrak{p}$ such that $\dim Z_{\mathfrak{p}}(x) = \dim G/H - 1$ and $\text{Ad}(H)x$ has dimension 1. Then, there exists a normal subgroup N of H such that the induced action of H/N on $\text{Ad}(H)x$ is effective. Since $\text{Ad}(H)x$ is 1-dimensional, we must have $\dim H/N = 1$, as only 1-dimensional Lie groups can act effectively on a 1-dimensional manifold. Since \mathfrak{h} is a compact (and hence reductive) Lie algebra, it

splits as an orthogonal direct sum $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathfrak{n}$, where \mathfrak{n} denotes the Lie algebra of N . Let T be a generator of the $\mathfrak{so}(2)$ -factor. Note that $[T, x]$ spans $T_x(\text{Ad}(H)x)$, and by Proposition 3.1 and dimension reasons, $Z_{\mathfrak{p}}(x) = [T, x]^{\perp}$ is the orthogonal complement of $[T, x]$ in \mathfrak{p} . Then $Z_{\mathfrak{g}}(x) = \mathfrak{n} \oplus [T, x]^{\perp}$. Indeed, no linear combination of T and $[T, x]$ centralizes x , as $\langle [rT + [T, x], x], T \rangle = -\langle [T, x], [T, x] \rangle \neq 0$, $r \in \mathbb{R}$, and $[T, x] \neq 0$. But then $b(G) \geq \dim Z_{\mathfrak{g}}(x) = \dim(\mathfrak{n} \oplus Z_{\mathfrak{p}}(x)) = \dim G - 2$. Theorem D (b) and Remark 4.1 imply that \mathfrak{g} splits as an orthogonal direct sum $\mathfrak{g} = \mathfrak{su}_2 \oplus \mathfrak{g}'$, for some compact semisimple Lie algebra \mathfrak{g}' , and where the \mathfrak{su}_2 -factor is precisely $\mathfrak{su}_2 = \mathbb{R}x \oplus Z_{\mathfrak{g}}(x)^{\perp} = \text{span}\{T, x, [T, x]\}$. Then $\mathfrak{so}_2 = \mathbb{R}T \subset \mathfrak{su}_2$ and $\mathfrak{n} \subset \mathfrak{g} \cap \mathfrak{su}_2^{\perp} = \mathfrak{g}'$. By assumption G is simply connected and H is connected, thus $G = \text{SU}_2 \times G'$ and $H = \text{SO}_2 \times N$, where G' is the connected Lie subgroup of G with Lie algebra \mathfrak{g}' . Therefore $G/H = (\text{SU}_2/\text{SO}_2) \times (G'/N) = \mathbb{S}^2 \times (G'/N)$.

Now assume that there exists $x \in \mathfrak{p}$, $x \neq 0$, such that $\dim Z_{\mathfrak{p}}(x) = \dim G/H - 1$ and $\dim \text{Ad}(H)x = 0$. Hence $b(G) \geq \dim Z_{\mathfrak{g}}(x) = \dim \mathfrak{h} \oplus Z_{\mathfrak{p}}(x) = \dim G - 1$, which is impossible by Remark 4.1. Thus we get a contradiction, and this case is not possible. □

In the previous proof, an important step was to show that if $b(G/H) = \dim G/H - 1$, then $b(G) \geq \dim G - 2$. This can be obtained as a consequence of the following result, which, given any homogeneous space G/H , provides an upper bound for $b(G/H)$ in terms of $b(G)$, whose value will be computed explicitly for any compact Lie group G in Sect. 4. We also state a straightforward lower bound for $b(G/H)$ in terms of the ranks of G and H , which generalizes an observation of Berger [7, Proposition 6.1]. For a broader generalization of Berger’s observation, see [37, Corollary G].

Proposition 3.8 *Let G/H be a normal homogeneous space. Then*

$$\text{rank } G - \text{rank } H \leq b(G/H) \leq \frac{1}{2}(b(G) + \dim G/H - \dim H).$$

Proof Let $x \in \mathfrak{p} \setminus \{0\}$. Denote by H_x the isotropy group at x of the restriction of the adjoint representation of G to H . In particular, $\text{Ad}(H)x = H/H_x$. Since $Z_{\mathfrak{p}}(x)$ and the Lie algebra of H_x are orthogonal subspaces of \mathfrak{g} , and both commute with x , we have:

$$\begin{aligned} \dim Z_{\mathfrak{g}}(x) &\geq \dim Z_{\mathfrak{p}}(x) + \dim H_x = \dim Z_{\mathfrak{p}}(x) + \dim H - \dim \text{Ad}(H)x \\ &= \dim Z_{\mathfrak{p}}(x) + \dim H - (\dim \mathfrak{p} - \dim \nu_x(\text{Ad}(H)x)) \\ &\geq 2 \dim Z_{\mathfrak{p}}(x) + \dim H - \dim G/H, \end{aligned}$$

where in the last inequality we have used Proposition 3.1. Taking the maximum over all $x \in \mathfrak{p} \setminus \{0\}$ in both sides of the inequality, we obtain the upper bound for $b(G/H)$ in the statement. The lower bound follows from the fact that any maximal abelian subalgebra \mathfrak{t} of \mathfrak{h} can be extended to a maximal abelian subalgebra $\mathfrak{s} \supset \mathfrak{t}$ of \mathfrak{g} . Indeed, if $x \in \mathfrak{s} \cap \mathfrak{t}^{\perp} \subset \mathfrak{p}$, $x \neq 0$, then $\mathfrak{s} \cap \mathfrak{t}^{\perp} \subset Z_{\mathfrak{p}}(x)$, and hence $b(G/H) \geq \dim Z_{\mathfrak{p}}(x) \geq \dim \mathfrak{s} \cap \mathfrak{t}^{\perp} = \text{rank } G - \text{rank } H$. □

3.3 Diagonal subgroups and the proof of Theorem E

In this subsection we will consider normal metrics on homogeneous spaces of the form $(K \times G)/\Delta K$, where $K < G$ are compact Lie groups and ΔK is the diagonal embedding of K into the product $K \times G$. Note that G is diffeomorphic to $(K \times G)/\Delta K$ by the identification $g \mapsto [1, g]$. Our study is motivated by the fact, proved in Proposition 3.9 below, that normal metrics on $(K \times G)/\Delta K$ are “at least as curved” as normal metrics (i.e. bi-invariant metrics) on G . Theorem E, which we will prove below, shows that in some cases normal metrics on $(K \times G)/\Delta K$ are indeed “more curved” than bi-invariant metrics on G .

Proposition 3.9 *Let G be a compact semisimple Lie group, and K a closed subgroup of G . Let $\Delta K = \{(k, k) : k \in K\} < K \times G$. Then, we have:*

- (a) $\max\{b(K), b(G/K)\} \leq b((K \times G)/\Delta K) \leq b(G)$.
- (b) *If $\text{rank } K = \text{rank } G$, then $b((K \times G)/\Delta K) = b(G)$.*
- (c) *Any homogeneous metric on $(K \times G)/\Delta K$ is isometric to a left-invariant and right K -invariant metric on G .*

Proof Let $\langle \cdot, \cdot \rangle$ denote a bi-invariant metric on G . We endow $K \times G$ with the bi-invariant product metric $\langle \cdot, \cdot \rangle|_{\mathfrak{k} \oplus \mathfrak{g}}$. We consider the reductive decomposition $\Delta \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{k} \oplus \mathfrak{g}$, where $\Delta \mathfrak{k} = \{(w, w) : w \in \mathfrak{k}\}$ and $\mathfrak{p} = \{(-y_{\mathfrak{k}}, y) : y \in \mathfrak{g}\}$. Here, $y_{\mathfrak{k}}$ denotes orthogonal projection of y onto \mathfrak{k} . Let $x = (-y_{\mathfrak{k}}, y), x' = (-y'_{\mathfrak{k}}, y') \in \mathfrak{p}$ be arbitrary. Then $[x, x'] = 0$ if and only if $[y_{\mathfrak{k}}, y'_{\mathfrak{k}}] = 0 = [y, y']$. In other words, $x' \in Z_{\mathfrak{p}}(x)$ if and only if $y'_{\mathfrak{k}} \in Z_{\mathfrak{k}}(y_{\mathfrak{k}})$ and $y' \in Z_{\mathfrak{g}}(y)$. Since the projection $\pi_{\mathfrak{g}} : \mathfrak{p} \rightarrow \mathfrak{g}$ onto the second factor of $\mathfrak{k} \oplus \mathfrak{g}$ is a linear isomorphism, we get that $\dim Z_{\mathfrak{p}}(x) \leq \dim Z_{\mathfrak{g}}(\pi_{\mathfrak{g}}(x))$, for any $x \in \mathfrak{p}$, where $\pi_{\mathfrak{g}}(x) \neq 0$ if $x \neq 0$. By Proposition 3.1, this implies that $b((K \times G)/\Delta K) \leq b(G)$, which shows the second inequality in (a).

For the first inequality in (a), let $w \in \mathfrak{k} \setminus \{0\}$ be such that $b(K) = \dim Z_{\mathfrak{k}}(w)$. Then $(-w, w) \in \mathfrak{p} \setminus \{0\}$, and $Z_{\mathfrak{p}}(-w, w) \supset \{(-w', w') : w' \in Z_{\mathfrak{k}}(w)\}$, from where $b((K \times G)/\Delta K) \geq \dim Z_{\mathfrak{p}}(-w, w) \geq \dim Z_{\mathfrak{k}}(w) = b(K)$. Similarly, let $y \in \mathfrak{p}' := \mathfrak{g} \cap \mathfrak{k}^{\perp}, y \neq 0$, be such that $b(G/K) = \dim Z_{\mathfrak{p}'}(y)$. Then $(0, y) \in \mathfrak{p} \setminus \{0\}$, $Z_{\mathfrak{p}}(0, y) \supset \{(0, y') : y' \in Z_{\mathfrak{p}'}(y)\}$, and hence $b((K \times G)/\Delta K) \geq \dim Z_{\mathfrak{p}}(0, y) \geq \dim Z_{\mathfrak{p}'}(y) = b(G/K)$. This completes the prove of Item (a).

Now assume that $\text{rank } K = \text{rank } G$. Let \mathfrak{t} be a maximal abelian subalgebra of \mathfrak{k} , which by assumption is also a maximal abelian subalgebra of \mathfrak{g} . Since centralizers are conjugated under the adjoint representation, and \mathfrak{t} intersects any orbit of the adjoint representation of G , there is $y \in \mathfrak{t} \setminus \{0\}$ such that $\dim Z_{\mathfrak{g}}(y) = \max_{z \in \mathfrak{g} \setminus \{0\}} \dim Z_{\mathfrak{g}}(z) = b(G)$. Define $x := (-y, y) = (-y_{\mathfrak{k}}, y) \in \mathfrak{p} \setminus \{0\}$. Note that if $y' \in Z_{\mathfrak{g}}(y)$, then $0 = [y, y'] = [y, y'_{\mathfrak{k}}] + [y, y'_{\mathfrak{g} \cap \mathfrak{k}^{\perp}}]$, which implies $[y, y'_{\mathfrak{k}}] = 0$. Hence, $Z_{\mathfrak{p}}(x) = \{(-y'_{\mathfrak{k}}, y') : y' \in Z_{\mathfrak{g}}(y)\}$, which has dimension $\dim Z_{\mathfrak{g}}(y) = b(G)$. Therefore $b((K \times G)/\Delta K) \geq \dim Z_{\mathfrak{p}}(x) = b(G)$, from where (b) follows.

Finally, note that the natural diffeomorphism $(K \times G)/\Delta K \cong G$ given by $[k, g] \mapsto gk^{-1}$ sends the canonical left $(K \times G)$ -action on $(K \times G)/\Delta K$ given by $(k, g) \cdot [a, b] = [ka, gb]$ to the two-sided $(K \times G)$ -action on G given by $(k, g) \cdot b = gbk^{-1}$. Thus, any homogeneous metric on $(K \times G)/\Delta K$ is isometric to a metric on G which is left G -invariant and right K -invariant. This proves (c). □

We will now focus on the particular case considered in Theorem E, namely that of the homogeneous spaces of the form $(K \times K \times K)/\Delta K$, where K is a compact Lie group, and $\Delta K = \{(k, k, k) : k \in K\}$. Theorem E asserts that $b(K^3/\Delta K) = 2b(K) < b(K^2)$ if K is semisimple, thus showing that both inequalities in Proposition 3.9 (a) are strict in this particular case.

Proof of Theorem E Let $H := \Delta K < K^3 =: G$, and consider the homogeneous space $G/H = K^3/\Delta K$. We will prove that $b(G/H) = 2b(K)$. The corresponding Lie algebras are $\mathfrak{g} = \mathfrak{k}^3$ and $\mathfrak{h} = \{(v, v, v) : v \in \mathfrak{k}\}$. On $G = K^3$ we consider the bi-invariant metric given by the product metric of a fixed bi-invariant metric on K . Then clearly $\mathfrak{p} := \mathfrak{h}^\perp = \{(w, z, -w - z) : w, z \in \mathfrak{k}\}$. The Lie bracket splits factor-wise and we can compute the centralizer $\dim Z_{\mathfrak{p}}(x)$ of an arbitrary vector $x = (w, z, -w - z) \in \mathfrak{p}$ by considering the following subcases:

$$\begin{aligned} w \neq 0, z = 0, & \quad [(w, 0, -w), (w', z', -w' - z')] = 0 \Leftrightarrow w', z' \in Z_{\mathfrak{k}}(w), \\ w = 0, z \neq 0, & \quad [(0, z, -z), (w', z', -w' - z')] = 0 \Leftrightarrow w', z' \in Z_{\mathfrak{k}}(z), \\ w, z \neq 0, & \quad [(w, z, -w - z), (w', z', -w' - z')] = 0 \Rightarrow w' \in Z_{\mathfrak{k}}(w), z' \in Z_{\mathfrak{k}}(z). \end{aligned}$$

In the first two cases we have $\dim Z_{\mathfrak{p}}(x) = 2 \dim Z_{\mathfrak{k}}(w)$ and $\dim Z_{\mathfrak{p}}(x) = 2 \dim Z_{\mathfrak{k}}(z)$, respectively, whereas in the third case we have $\dim Z_{\mathfrak{p}}(x) \leq \dim Z_{\mathfrak{k}}(w) + \dim Z_{\mathfrak{k}}(z)$. Thus, by Proposition 3.1 we get

$$b(G/H) = \max_{x \in \mathfrak{p} \setminus \{0\}} \dim Z_{\mathfrak{p}}(x) = 2 \max_{w \in \mathfrak{k} \setminus \{0\}} \dim Z_{\mathfrak{k}}(w) = 2b(K),$$

from where the first claim in Theorem E holds.

Finally, the claim that K^2 admits a metric of $\text{Ric}_k > 0$ for $k = 2b(K)$ which is left-invariant and invariant under right ΔK -diagonal multiplication is a direct consequence of the existence of a (normal) homogeneous metric on $K^3/\Delta K$ with $\text{Ric}_{2b(K)} > 0$, along with Proposition 3.9 (c). □

3.4 Circle bundles over projective spaces

Here, we discuss properties of a family of homogeneous spaces $M_{k,l}^{p,q}$ that are circle bundles over products of complex projective spaces. We follow the notation in the article by Wang and Ziller [48].

Consider the product $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ with $1 \leq p \leq q$. The Hopf action on each factor yields a free $U_1 \times U_1$ -action on $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$. Let k, l be coprime integers and consider the subgroup $U_1^{l,-k}$ from (1.2) in Sect. 1.2. The corresponding quotient is a simply connected homogeneous space [48, p. 231], which we denote by

$$M_{k,l}^{p,q} := \mathbb{S}^{2p+1} \times_{U_1^{l,-k}} \mathbb{S}^{2q+1} = \frac{U_{p+1} \times U_{q+1}}{U_1^{l,-k} \times U_p \times U_q},$$

where here $U_1^{l,-k}$ is embedded in the $U_1 \times U_1$ given by the U_1 -factors in the block diagonal embedding $(U_1 \times U_p) \times (U_1 \times U_q) < U_{p+1} \times U_{q+1}$. The 7-dimensional

spaces $M_{k,l}^{1,2}$ are commonly known as *Witten spaces*. The space $M_{k,l}^{p,q}$ can also be seen as the total space of a circle bundle over $\mathbb{C}\mathbb{P}^p \times \mathbb{C}\mathbb{P}^q$ with Euler class $k\alpha_1 + l\alpha_2$, where α_1, α_2 denote the generators of $H^2(\mathbb{C}\mathbb{P}^p; \mathbb{Z}), H^2(\mathbb{C}\mathbb{P}^q; \mathbb{Z})$, respectively.

It is a routine calculation to check that, for $kl \neq 0$, normal metrics for the above homogeneous description satisfy

$$b(M_{k,l}^{p,q}) = 2q + 1, \quad \text{with } \dim M_{k,l}^{p,q} = 2p + 2q + 1.$$

Here we are mostly interested in the case $p = 1$, where one can alternatively prove that $b(M_{k,l}^{1,q}) = 2q + 1$ as follows: Proposition 3.5 in combination with Remark 3.2 implies

$$2q + 1 = b(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^q) \leq b(M_{k,l}^{1,q}) \leq b(\mathbb{S}^3 \times \mathbb{S}^{2q+1}) = 2q + 2.$$

However, the value $2q + 2$ is ruled out by Theorem C since, by the homogeneous description above, it follows that $M_{k,l}^{1,q}$ does not split off isometrically a round 2-sphere. Now recall that $H^4(M_{k,l}^{1,q}; \mathbb{Z}) = \mathbb{Z}_{l^2}$ if $q > 1$ [48, Proposition 2.1]. By varying q, l we obtain infinitely many homogeneous spaces G/H of distinct homotopy type with $b(G/H) = \dim G/H - 2$ in every odd dimension $2q + 3 \geq 7$, as claimed in Sect. 1.1 (discussion below Theorem D).

Observe that if $l, q > 1$ then $M_{k,l}^{1,q}$ is not even homotopy equivalent to a product $\mathbb{S}^2 \times X$ [48, p. 233]. The homotopy sequence associated to $\mathbb{S}^3 \times \mathbb{S}^{2q+1} \rightarrow M_{k,l}^{1,q}$ shows that $\pi_d(M_{k,l}^{1,q}) = \pi_d(\mathbb{S}^2) \oplus \pi_d(\mathbb{S}^{2q+1})$ for any d . If $M_{k,l}^{1,q}$ was homotopy equivalent to $\mathbb{S}^2 \times X$, it would follow that the homotopy groups of X equal those of \mathbb{S}^{2q+1} , and the same conclusion would hold for the homology and cohomology groups by the Hurewicz Theorem and Poincaré duality. But then the Künneth formula for cohomology would yield that $H^4(\mathbb{S}^2 \times X; \mathbb{Z})$ vanishes, contradicting that $H^4(M_{k,l}^{1,q}; \mathbb{Z}) = \mathbb{Z}_{l^2}$.

The spaces $M_{k,l}^{p,q}$ have very interesting properties and have been used to construct examples exhibiting various kinds of phenomena. For instance, fixing l, q and letting k vary in \mathbb{Z} gives only finitely many diffeomorphism types among the family $M_{k,l}^{1,q}$ [48, p. 217]. Equivalently, there is an infinite subfamily of $M_{k,l}^{1,q}$, all of which are diffeomorphic. Taking one of them as a reference (say $M := M_{k_0,l}^{1,q}$) one can pull-back the normal metric on each of the spaces of the family to M via the corresponding diffeomorphism, thus defining infinitely many elements in the moduli space $\mathcal{M}_{\text{Ric}_{\dim M-2} > 0}(M)$ of metrics of $\text{Ric}_{\dim M-2} > 0$ on M . It is obvious that there are natural inclusions

$$\mathcal{M}_{\text{Ric}_{\dim M-2} > 0}(M) \subset \mathcal{M}_{\text{Ric} > 0}(M) \subset \mathcal{M}_{\text{scal} > 0}(M)$$

into the moduli spaces of metrics of $\text{Ric} > 0$ and positive scalar curvature, respectively.

Dessai et al. [16] used the Kreck-Stolz invariant to show that, for q even and for many odd values of l , the classes of metrics belong to different path components of $\mathcal{M}_{\text{scal} > 0}(M)$ and concluded that the same holds for $\mathcal{M}_{\text{Ric} > 0}(M)$. Our

observation above shows that they actually belong to different path components of $\mathcal{M}_{\text{Ric}_{\dim M - 2} > 0}(M)$.

4 Symmetric spaces

The purpose of this section is to determine, for each symmetric space G/K of compact type, the lowest k for which the symmetric metric on G/K satisfies $\text{Ric}_k > 0$. Thus, in Sect. 4.1 we review the main tools needed and derive the recipe for the calculation of $b(G/K)$. Then, in Sect. 4.2 we discuss further properties of the values $b(G/K)$, from where Theorem D in Sect. 1.1 will follow. We refer the reader to the recent work [3] for an alternative discussion of the problem addressed in this section.

4.1 Root space decomposition and recipe to compute $b(G/K)$

Let M be a (not necessarily irreducible) symmetric space of compact type. Let (G, K) be a symmetric pair representing M and consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{p} is the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Killing form $B_{\mathfrak{g}}$ of \mathfrak{g} . As usual, we can identify \mathfrak{p} with the tangent space T_oM , where o is a fixed point of K . We also have $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Recall from Proposition 3.1 that for each non-zero $x \in \mathfrak{p}$, we have

$$b(M, x) = \dim Z_{\mathfrak{p}}(x) = \dim\{y \in \mathfrak{p} : [x, y] = 0\}.$$

Our purpose is then to determine

$$b(M) = \max_{x \in \mathfrak{p} \setminus \{0\}} b(M, x)$$

for each symmetric space M . We will use a direct approach by means of the classical theory of root systems; for more information on the restricted root decomposition in the compact setting, see [34, Chapter VI]. We note that the value $b(-)$ is clearly invariant under finite quotients and coverings, i.e., $b(M) = b(\tilde{M})$, where \tilde{M} is the universal cover of M . Moreover, all symmetric pairs (G, K) representing M give rise to the same value $b(M) = b(G/K)$.

Fix $x \in \mathfrak{p}$, $x \neq 0$. Consider a maximal abelian subspace \mathfrak{a} of \mathfrak{p} containing x . Let us put $r := \dim \mathfrak{a} = \text{rank } M$. The endomorphisms $\text{ad}(Y) \in \text{End}(\mathfrak{g})$, $Y \in \mathfrak{a}$, form a commuting family of $B_{\mathfrak{g}}$ -skew-adjoint operators, and hence diagonalize simultaneously. Moreover, the self-adjoint operators $\text{ad}(Y)^2$, $Y \in \mathfrak{a}$, leave both \mathfrak{k} and \mathfrak{p} invariant. For any linear form $\lambda \in \mathfrak{a}^*$, let us consider

$$\begin{aligned} \mathfrak{k}_{\lambda} &= \{X \in \mathfrak{k} : \text{ad}(Y)^2 X = -\lambda(Y)^2 X \text{ for all } Y \in \mathfrak{a}\}, \\ \mathfrak{p}_{\lambda} &= \{X \in \mathfrak{p} : \text{ad}(Y)^2 X = -\lambda(Y)^2 X \text{ for all } Y \in \mathfrak{a}\}. \end{aligned}$$

Then $\mathfrak{p}_0 = \mathfrak{a}$, $\mathfrak{p}_{\lambda} = \mathfrak{p}_{-\lambda}$, $\mathfrak{k}_{\lambda} = \mathfrak{k}_{-\lambda}$, and we have the bracket relations $[\mathfrak{k}_{\lambda}, \mathfrak{k}_{\mu}] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}$, $[\mathfrak{k}_{\lambda}, \mathfrak{p}_{\mu}] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}$, and $[\mathfrak{p}_{\lambda}, \mathfrak{p}_{\mu}] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}$, for any $\lambda, \mu \in \mathfrak{a}^*$.

Let us denote by $\Sigma = \{\lambda \in \mathfrak{a}^* : \lambda \neq 0, p_\lambda \neq 0\}$ the set of roots of $M \cong G/K$ with respect to $\mathfrak{a} \subset \mathfrak{p}$. It is well-known that Σ constitutes a (possibly non-reduced) root system on the dual space of \mathfrak{a} . Moreover, we have the $B_{\mathfrak{g}}$ -orthogonal decompositions

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma^+} \mathfrak{p}_\lambda,$$

where $\Sigma^+ \subset \Sigma$ is any choice of positive roots (i.e. Σ is the disjoint union of Σ^+ and $-\Sigma^+$, and if the sum of two positive roots is a root, then it is also positive).

Select $\Lambda = \{\alpha_1, \dots, \alpha_r\} \subset \Sigma$ a set of simple roots for the root system Σ such that x belongs to the closed Weyl chamber determined by the relations $\alpha_i \geq 0$, for all $i \in \{1, \dots, r\}$. Then Λ determines a set of positive roots $\Sigma^+ = \{\sum_{i=1}^r a_i \alpha_i \in \Sigma : a_i \in \mathbb{Z}_{\geq 0}\}$. Note that if $\lambda = \sum_{i=1}^r a_i \alpha_i \in \Sigma$ is an arbitrary root (where of course the coefficients $a_i \in \mathbb{Z}$ are either all non-negative or all non-positive), then $\lambda(x) = 0$ if and only if λ is a linear combination of roots in $\Phi = \Phi_x = \{\alpha_i \in \Lambda : \alpha_i(x) = 0\}$. Therefore, if we denote by $\Sigma_\Phi^+ = \Sigma^+ \cap \text{span } \Phi$ the set of positive roots spanned by Φ , it follows from the definition of the subspaces \mathfrak{p}_λ that

$$Z_p(x) = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma_\Phi^+} \mathfrak{p}_\lambda. \tag{4.1}$$

This is easily checked to be a Lie triple system (i.e. $[[Z_p(x), Z_p(x)], Z_p(x)] \subset Z_p(x)$), and hence $Z_p(x) \cong T_o F_\Phi$ is the tangent space of a totally geodesic submanifold F_Φ containing $o \in M$. As a totally geodesic submanifold of a symmetric space of compact type, F_Φ is a symmetric space of non-negative curvature. Indeed, in this case, F_Φ is (up to finite quotients) of the form $B_\Phi \times \mathbb{T}^{r-|\Phi|}$, where B_Φ is the compact semisimple factor with associated Lie triple system $(\bigoplus_{\lambda \in \Phi} Y_\lambda) \oplus (\bigoplus_{\lambda \in \Sigma_\Phi^+} \mathfrak{p}_\lambda)$, where $Y_\lambda \in \mathfrak{a}$ is determined by $B_{\mathfrak{g}}(Y_\lambda, Y) = \lambda(Y)$ for all $Y \in \mathfrak{a}$, and $\mathbb{T}^{r-|\Phi|}$ is the Euclidean factor with Lie triple system $\bigcap_{\lambda \in \Phi} \ker \lambda \subset \mathfrak{a}$.

Since our aim is to determine $b(M) = \max_{x \in \mathfrak{p} \setminus \{0\}} \dim Z_p(x)$, in view of (4.1) we can restrict our attention to maximal proper subsets Φ of Λ . Indeed, if x is such that $\Phi_x = \Lambda$, this implies that $x = 0$, since Λ is a basis of the dual space of \mathfrak{a} ; and if $|\Phi_x| \leq r - 2$, then we can consider $\Phi' \subsetneq \Lambda$ properly containing Φ_x , and take a non-zero $x' \in \bigcap_{\alpha_i \in \Phi'} \ker \alpha_i$, which would then satisfy $Z_p(x) \subset Z_p(x')$ by (4.1).

Thus, let us put $\Phi = \Phi^k = \Lambda \setminus \{\alpha_k\}$, for each $k \in \{1, \dots, r\}$. Then we can choose $x = Y^k$, where Y^k is the vector in \mathfrak{a} such that $\alpha_i(Y^k) = \delta_{ik}$, for each $i \in \{1, \dots, r\}$. We have to determine $\dim Z_p(x) = \dim B_{\Phi^k} \times \mathbb{T}^1$ for each k , and then $b(M)$ will be the maximum of all these values, where k runs through $\{1, \dots, r\}$. This is an easy combinatorial problem that can be done with the help of the Dynkin diagram of the root system Σ of M along with the multiplicities of the roots (i.e. the dimensions of each \mathfrak{p}_λ). The approach is the following. In the Dynkin diagram of M delete the node corresponding to α_k (and any edge connected to it). This gives rise to a new Dynkin diagram which, together with the multiplicities attached to each one of the remaining nodes, must then correspond in a unique way to some symmetric space of compact type

(up to finite quotients and coverings). Such symmetric space is precisely B_{Φ^k} . Finally, we observe that, for each symmetric space M and each $k \in \{1, \dots, r\}$, $\dim B_{\Phi^k}$ takes the value of a quadratic polynomial in k with positive leading coefficient. Thus its maximum is achieved at $k = 1$ or $k = r$, from where we directly obtain $b(M)$ by just adding 1 to such maximum. All this study can be done case by case for each irreducible symmetric space; see Table 3 at the end of the article.

We note that Table 3 contains some redundancies stemming from the isomorphisms existing between some low dimensional symmetric spaces. These isomorphisms can be consulted in [27, pp. 519-520], but can also be derived from Table 3, as the Dynkin diagram and the multiplicities of the simple roots determine the simply connected compact symmetric space.

4.2 Discussion of particular cases and proof of Theorem D

We now discuss the values $b(M)$ in various relevant cases. Theorem D will follow from this discussion.

First, observe that the first claim in Theorem D just refers to Table 3 for the values of $b(M)$ of irreducible symmetric spaces M , whereas the second claim follows from a direct application of Proposition 3.5 to the De Rham decomposition $\tilde{M} = M_1 \times \dots \times M_s$ of the universal cover of M into irreducible factors.

Let us prove Item (a) in Theorem D. We start by noting that $b(M) = 1$ if and only if M is of rank one, but if M has higher rank, then $b(M) \geq 2 \operatorname{rank} M - 1 \geq 3$. Indeed, for any maximal proper $\Phi \subset \Lambda$ we have $\dim \bigoplus_{\lambda \in \Sigma_{\Phi}^+} \mathfrak{p}_{\lambda} \geq |\Phi| = r - 1$; then from (4.1) it follows that

$$b(M) \geq \dim \mathfrak{a} + \dim \bigoplus_{\lambda \in \Sigma_{\Phi}^+} \mathfrak{p}_{\lambda} \geq r + (r - 1) = 2r - 1. \tag{4.2}$$

Assume $b(M) = 2 \operatorname{rank} M - 1$. Then, in view of (4.2), for any maximal proper $\Phi \subset \Lambda$ we must have $|\Sigma_{\Phi}^+| = r - 1$ (and hence $\Sigma_{\Phi}^+ = \Phi$) and $\dim \mathfrak{p}_{\lambda} = 1$ for all $\lambda \in \Sigma_{\Phi}^+$. This means that, for any maximal proper $\Phi \subset \Lambda$, B_{Φ} is a symmetric space of type $(A_1)^{r-1}$ (equivalently, when one removes a node from the Dynkin diagram of M , one obtains a totally disconnected graph with no double nodes) and with all multiplicities equal to 1. If $r = 1$, this is trivially satisfied. If M is irreducible, it easily follows from the Dynkin diagrams and multiplicities in Table 3 that M must be isometric to SU_3/SO_3 , $G_2^+(\mathbb{R}^5)$ or G_2/SO_4 . If M is reducible, all factors have to be of type A_1 with multiplicity 1, that is, M is covered by a product of 2-spheres. All in all, this proves Theorem D (a).

Now assume $b(M) = \dim M - 1$. Then, there exists $x \in \mathfrak{a}$ such that $\dim Z_{\mathfrak{p}}(x) = \dim \mathfrak{p} - 1$. By (4.1) there is exactly one positive root $\alpha \in \Sigma^+$ such that \mathfrak{p}_{α} is not contained in (and is orthogonal to) $Z_{\mathfrak{p}}(x)$, and we have $\dim \mathfrak{p}_{\alpha} = 1$ for such root. Thus, $\{\alpha\} = \Sigma^+ \setminus \Sigma_{\Phi}^+$. By the properties of root systems, Σ is then reducible with a rank one factor (the span of α). Thus, the universal cover of M has a rank one factor, which must necessarily be isometric to \mathbb{S}^2 , since $\dim \mathfrak{p}_{\alpha} = 1$. The converse is direct from Proposition 3.5.

Let us consider the case $b(M) = \dim M - 2$. Then $\dim Z_{\mathfrak{p}}(x) = \dim \mathfrak{p} - 2$ for some $x \in \mathfrak{a}$, and by (4.1) we deduce that either there is exactly one $\alpha \in \Sigma^+ \setminus \Sigma_{\Phi}^+$, and it satisfies $\dim \mathfrak{p}_{\alpha} = 2$, or there are two different roots α, β such that $\{\alpha, \beta\} = \Sigma^+ \setminus \Sigma_{\Phi}^+$ and both with multiplicity 1. In the first case, similarly as above, the universal cover of M splits off a factor isometric to $SU_2 = S^3$. Let us assume we are in the second case. If the roots in $\Sigma^+ \setminus \text{span}\{\alpha, \beta\}$ are perpendicular to $\text{span}\{\alpha, \beta\}$, then Σ is reducible with a rank two factor of type A_2 (the span of α and β); since the multiplicities of the roots of this factor are all 1, M locally splits off a factor isometric to SU_3/SO_3 . On the contrary, if there is a root in $\Sigma^+ \setminus \text{span}\{\alpha, \beta\}$ not perpendicular to $\text{span}\{\alpha, \beta\}$, we can assume without restriction of generality that there exists $\lambda \in \Sigma \setminus \text{span}\{\alpha, \beta\}$ not perpendicular to α , and also $\langle \lambda, \alpha \rangle > 0$. Then, again by the properties of root systems, we have that $\lambda - \alpha \in \Sigma$, but $\lambda - \alpha \notin \Sigma_{\Phi}$ (since $\lambda \in \Sigma_{\Phi}$ and $\alpha \notin \Sigma_{\Phi}$). This would imply that $\lambda - \alpha \in \{\alpha, \beta, -\alpha, -\beta\}$, which contradicts $\lambda \notin \text{span}\{\alpha, \beta\}$. The discussion in this paragraph and the previous one implies Theorem D (b).

Remark 4.1 In the particular case where M is a compact semisimple Lie group G , the discussion in the previous two paragraphs implies that $b(G) = \dim G - 1$ is impossible, whereas $b(G) = \dim G - 2$ if and only if \mathfrak{g} is a Lie algebra direct sum of the form $\mathfrak{su}_2 \oplus \mathfrak{g}'$, for some compact semisimple \mathfrak{g}' . Even more, if for some $x \in \mathfrak{g}$ we have $\dim Z_{\mathfrak{g}}(x) = \dim \mathfrak{g} - 2$, then $(\mathbb{R}x \oplus Z_{\mathfrak{g}}(x)^{\perp}) \oplus (Z_{\mathfrak{g}}(x) \cap x^{\perp})$ yields precisely the splitting $\mathfrak{su}_2 \oplus \mathfrak{g}'$.

In order to prove Items (c) and (d) of Theorem D, we first show that if $b(M) \leq (\dim M)/2$, then M is irreducible. Indeed, by Proposition 3.5, $b(M) = \max\{b(M_j) - \dim M_j\} + \dim M$, where $\tilde{M} = M_1 \times \dots \times M_s$. Then $b(M) \leq (\dim M)/2$ is equivalent to $\min\{\dim M_j - b(M_j)\} \geq (\dim M)/2$, from where we get $\dim M_j \geq (\dim M)/2$ for all $j \in \{1, \dots, s\}$. If M is reducible, this necessarily implies $\tilde{M} = M_1 \times M_2$ with $\dim M_1 = \dim M_2$. But then, assuming without restriction of generality that $b(M_1) \leq b(M_2)$, we would have $\dim M_1 = (\dim M)/2 \geq b(M) = b(M_2) + \dim M_1 > \dim M_1$, which yields a contradiction. The rest of claims in Theorem D (c)-(d) follow from elementary case-by-case calculations making use of the values of $\dim M$ and $b(M)$ stated in Table 3 at the end of the article.

For convenience of the reader, we list the simply connected irreducible symmetric spaces M of rank at least 2 with $b(M) \leq 6$ in Table 1 below, as referenced in Theorem D (e).

In the case of rank 2 symmetric spaces, Proposition 3.1 implies that $b(M)$ is the highest codimension in T_oM of an orbit of the isotropy representation of $M = G/K$, restricted to the unit sphere of T_oM . Such restrictions of isotropy representations of rank 2 symmetric spaces give rise to all cohomogeneity one actions on round spheres (up to orbit equivalence), or equivalently, to all homogeneous isoparametric families of hypersurfaces on round spheres [8, §2.9.6]. Any such action has exactly two singular orbits, and the remaining ones are hypersurfaces of the sphere with $g \in \{1, 2, 3, 4, 6\}$ constant principal curvatures and multiplicities m_i satisfying $m_i = m_{i+2}$ (indices modulo g). (The integer g agrees with the number $|\Sigma^+|$ of positive roots of M , and the m_i are the multiplicities of such roots.) The codimension (in the sphere) of a singular orbit coincides with $m_i + 1$, for some $i \in \{1, \dots, g\}$. Hence, $b(M) = 2 + \max_{i \in \{1, \dots, g\}} m_i$. In Table 2, all simply connected compact semisimple

Table 1 Irreducible symmetric spaces with $3 \leq b(M) \leq 6$

M	$\dim M$	$b(M)$
SU_3/SO_3	5	3
$G_2^+(\mathbb{R}^5)$	6	
G_2/SO_4	8	
SU_3	8	4
$G_2^+(\mathbb{R}^6)$	8	
Sp_2	10	
G_2	14	
$\overline{G}_2^+(\mathbb{R}^7)$	10	5
$G_2(\mathbb{C}^5)$	12	
SU_4/SO_4	9	6
$G_2^+(\mathbb{R}^8)$	12	
SU_6/Sp_3	14	
$G_2(\mathbb{H}^4)$	16	

Table 2 Symmetric spaces of rank two

g	Multiplicities	M	$\dim M$	$b(M)$
2	$(k, l - k - 2)$	$\mathbb{S}^{k+1} \times \mathbb{S}^{l-k-1}$	l	$\max\{k + 2, l - k\}$
3	1	SU_3/SO_3	5	3
3	2	SU_3	8	4
3	4	SU_6/Sp_3	14	6
3	8	E_6/F_4	26	10
4	(2, 2)	Sp_2	10	4
4	(4, 5)	SO_{10}/U_5	20	7
4	$(1, k - 2)$	$G_2^+(\mathbb{R}^{k+2}), k \geq 3$	$2k$	k
4	$(2, 2k - 3)$	$G_2(\mathbb{C}^{k+2}), k \geq 2$	$4k$	$\max\{4, 2k - 1\}$
4	$(4, 4k - 5)$	$G_2(\mathbb{H}^{k+2}), k \geq 2$	$8k$	$\max\{6, 4k - 3\}$
4	(9, 6)	$E_6/Spin_{10}U_1$	32	11
6	(1, 1)	G_2/SO_4	8	3
6	(2, 2)	G_2	14	4

symmetric spaces M of rank 2 are shown, together with their dimensions, the number g of principal curvatures and the multiplicities of the corresponding homogeneous hypersurfaces, from where one directly obtains $b(M)$.

5 Fat homogeneous bundles: metrics and topology

The aim of this section is to investigate the k th-intermediate Ricci curvature of the total spaces of certain fat bundles, as well as the topology of some of them. In Sect. 5.1

we recall the notion of fat homogeneous bundle and prove Theorem F. In Sect. 5.2 we discuss various families of fat homogeneous bundles and obtain Theorem 5.4 as a consequence, which in particular implies Theorem G. In Sect. 5.3 we compute the topology of the generalized Aloff-Wallach spaces.

5.1 Metrics on fat homogeneous bundles

The notion of *fatness* was introduced by Weinstein to study Riemannian submersions with totally geodesic fibers and positive vertical curvatures; see the discussion in [23, Section 2.8] or [20, 55]. In this work we are only interested in the case where the submersion is given by a *homogeneous bundle*. Recall that a homogeneous bundle is the bundle $K/H \rightarrow G/H \rightarrow G/K$ resulting from a nested triple of compact Lie groups $H < K < G$. Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be the respective Lie algebras. We fix a bi-invariant metric $\langle \cdot, \cdot \rangle$ on G and use it to define:

$$\mathfrak{m} := \mathfrak{h}^\perp \cap \mathfrak{k}, \quad \mathfrak{p} := \mathfrak{k}^\perp.$$

Recall that there are canonical identifications

$$T_{eH}K/H \simeq \mathfrak{m}, \quad T_{eK}G/K \simeq \mathfrak{p}, \quad T_{eH}G/H \simeq \mathfrak{h}^\perp = \mathfrak{m} \oplus \mathfrak{p}.$$

With this notation, there is the following characterization of fat homogeneous bundles which we take as a definition (see [23, Proposition 2.8.3]).

Definition 5.1 A homogeneous bundle is fat if $[x, y] \neq 0$ for all non-zero $x \in \mathfrak{m}$ and $y \in \mathfrak{p}$.

We will now describe the metrics that will be used in the proof of Theorem F. For any $t > 0$, consider the bi-invariant product metric $(K \times G, t\langle \cdot, \cdot \rangle|_{\mathfrak{k}} + \langle \cdot, \cdot \rangle)$. The quotient with respect to the action by right multiplication $k_1(k_2, g) = (k_2k_1^{-1}, gk_1^{-1})$, $g \in G, k_1, k_2 \in K$, of the diagonal subgroup $\Delta K < K \times G$ is a normal homogeneous metric on $(K \times G)/\Delta K$, denoted by $\langle \cdot, \cdot \rangle_t$. Note that Proposition 3.9 applies in this case. Hence, the natural diffeomorphism $(K \times G)/\Delta K \cong G$ given by $[k, g] \mapsto gk^{-1}$ sends the action by left multiplication $(k_1, g_1)[k_2, g_2] = [k_1k_2, g_1g_2]$ to the $(K \times G)$ -action on G given by $(k_1, g_1)g_2 = g_1g_2k_1^{-1}$. In particular, the push-forward of the left $(K \times G)$ -invariant metric $\langle \cdot, \cdot \rangle_t$ on $(K \times G)/\Delta K$ results in a left-invariant and right K -invariant metric on G , which we also denote by $\langle \cdot, \cdot \rangle_t$. This discussion can be summarized in:

$$(K \times G, t\langle \cdot, \cdot \rangle|_{\mathfrak{k}} + \langle \cdot, \cdot \rangle) \rightarrow ((K \times G)/\Delta K, \langle \cdot, \cdot \rangle_t) \cong (G, \langle \cdot, \cdot \rangle_t),$$

where the first arrow is a Riemannian submersion.

The right action by H on $(G, \langle \cdot, \cdot \rangle_t)$ is by isometries since $H < K$, and hence $\langle \cdot, \cdot \rangle_t$ induces a metric q_t on G/H such that the quotient map

$$(G, \langle \cdot, \cdot \rangle_t) \rightarrow (G/H, q_t) \tag{5.1}$$

is a Riemannian submersion. Since $\langle \cdot, \cdot \rangle_t$ is left-invariant and the left G -action on G commutes with the right H -action, it follows that q_t is G -invariant, i.e. q_t is homogeneous on G/H . Observe that $\langle \cdot, \cdot \rangle_t$ and q_t are of $\text{sec} \geq 0$ by O’Neill’s formula. We will need the following result showing that, in some situations, $\langle \cdot, \cdot \rangle_t$ has fewer zero-curvature planes than the bi-invariant metric $\langle \cdot, \cdot \rangle$; see [19, Satz 231] or [56, Lemma 4.2].

Theorem 5.2 (Eschenburg) *Assume (G, K) is a compact symmetric pair (i.e. $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$). Then, for any $t > 0$, the vectors $x, y \in \mathfrak{g}$ span a zero-curvature plane with respect to $\langle \cdot, \cdot \rangle_t$ if and only if the following is satisfied:*

$$[x, y] = [x_{\mathfrak{k}}, y_{\mathfrak{k}}] = [x_{\mathfrak{p}}, y_{\mathfrak{p}}] = 0.$$

We are ready to prove Theorem F, which we restate here in a more precise manner:

Theorem 5.3 *Let $H < K < G$ be a fat homogeneous bundle with G/K a symmetric space. Then, for any $t > 0$, the homogeneous metric q_t defined on G/H as in (5.1) has $\text{Ric}_k > 0$ for $k = b(G/K)$.*

Proof We argue by contradiction. Suppose that there are orthonormal vectors

$$\{x, y^1, \dots, y^k\} \subset \mathfrak{m} \oplus \mathfrak{p} \simeq T_{eH}(G/H)$$

satisfying $\sum \text{sec}_{q_t}(x, y^i) \leq 0$. Since q_t is of $\text{sec} \geq 0$ it follows that $\text{sec}_{q_t}(x, y^i) = 0$ for all i .

Observe that, being $(G, \langle \cdot, \cdot \rangle_t)$ of $\text{sec} \geq 0$ and $(G, \langle \cdot, \cdot \rangle_t) \rightarrow (G/H, q_t)$ a Riemannian submersion, it follows that the planes spanned by x, y^i in $\mathfrak{m} \oplus \mathfrak{p} \subset \mathfrak{g}$ are also flat with respect to $\langle \cdot, \cdot \rangle_t$. Since G/K is symmetric, by Eschenburg’s Theorem 5.2 we have that for all i ,

$$[x, y^i] = [x_{\mathfrak{k}}, y_{\mathfrak{k}}^i] = [x_{\mathfrak{p}}, y_{\mathfrak{p}}^i] = 0.$$

The identities $[x_{\mathfrak{p}}, y_{\mathfrak{p}}^i] = 0$ together with the assumption $b(G/K) = k$ imply that either:

- there is i_0 such that $x_{\mathfrak{p}}, y_{\mathfrak{p}}^{i_0}$ are linearly dependent, or
- the vectors $y_{\mathfrak{p}}^1, \dots, y_{\mathfrak{p}}^k$ are linearly dependent.

In the second case we can rearrange the orthonormal vectors y^1, \dots, y^k so that $y_{\mathfrak{p}}^1 = 0$. Thus, in either case we can assume that there is i_0 such that $x_{\mathfrak{p}}, y_{\mathfrak{p}}^{i_0}$ are linearly dependent.

Now we look at the identity $[x_{\mathfrak{k}}, y_{\mathfrak{k}}^{i_0}] = 0$. Recall that $x, y^{i_0} \in \mathfrak{m} \oplus \mathfrak{p}$ and $\mathfrak{m} = \mathfrak{h}^\perp \cap \mathfrak{k}$, hence $x_{\mathfrak{k}} = x_{\mathfrak{m}}$ and $y_{\mathfrak{k}}^{i_0} = y_{\mathfrak{m}}^{i_0}$, and in particular $[x_{\mathfrak{m}}, y_{\mathfrak{m}}^{i_0}] = 0$. Since the bundle is fat, it follows from [5, Lemma 6] that $x_{\mathfrak{m}}, y_{\mathfrak{m}}^{i_0}$ must be linearly dependent as well.¹ Hence

¹ Note that Bérard–Bergery uses a different notation with respect to ours: we interchange H with K and \mathfrak{m} with \mathfrak{p} .

we can rearrange the vectors x, y^{i_0} so that either $x \in \mathfrak{p}$ and $y^{i_0} \in \mathfrak{m}$ or $x \in \mathfrak{m}$ and $y^{i_0} \in \mathfrak{p}$. In either case, the last condition $[x, y^{i_0}] = 0$ together with Definition 5.1 of fatness imply that x, y^{i_0} are linearly dependent, which is a contradiction. \square

5.2 Examples

The classification of fat homogeneous bundles was achieved in [5], it was revisited in [55, Section 3] and some cases were used in [20, Section 6]. In this article we consider various such bundles, which can be divided in three different sets.

The first set consists of those bundles induced by the following nested inclusions, where p, q are coprime integers with $pq > 0$ and $n \geq 2$:

$$\begin{aligned} S(U_1^{p,q} \times U_{n-1}) &< S(U_2 \times U_{n-1}) < SU_{n+1}, \\ Sp_1 \times Sp_1 \times Sp_{n-1} &< Sp_2 \times Sp_{n-1} < Sp_{n+1}, \\ S(U_2 \times Sp_2) &< S(U_2 \times U_4) < SU_6, \\ SO_2 \times Spin_7 &< SO_2 \times SO_8 < SO_{10}. \end{aligned}$$

These inclusions correspond to (D.29), (C.23), (C.21) and (C.24) from [5, Section 7] and to (E.6), (E.5.c), (E.5.a) and (E.5.b) from [55, Section 3], respectively. The two first rows are also discussed in [20, Section 6]. The inclusions are given by standard diagonal block embeddings except for the first one (explained in Sect. 1.2), $Sp_2 < SU_4$ (equivalent to the standard embedding $Spin_5 < Spin_6$ via the corresponding isomorphisms) and $Spin_7 < SO_8$ (induced by the irreducible representation of $Spin_7$ on \mathbb{R}^8). The base spaces of the corresponding fat bundles are $G_2(\mathbb{C}^{n+1})$, $G_2(\mathbb{H}^{n+1})$, $G_2(\mathbb{C}^6)$ and $G_2^+(\mathbb{R}^{10})$, respectively. The total space $Sp_{n+1}/(Sp_1^2 \times Sp_{n-1})$ corresponding to the second row equals the projectivized tangent bundle of $\mathbb{H}P^n$, denoted by $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^n$, and the low-dimensional case $\mathbb{P}_{\mathbb{H}}T\mathbb{H}P^2$ is by definition the Wallach flag manifold W^{12} .

The second set comes from the following source. Recall that a compact irreducible symmetric space G/K is Hermitian if K locally splits off a circle factor $K = K'U_1$ (see [27, p. 518] for their classification). The nested inclusion $K' < K'U_1 < G$ induces a fat homogeneous (circle) bundle; see [5, Section 4] or [55, p. 17]. From Theorem D we know that the Hermitian spaces of rank ≥ 2 and $b(G/K) \leq (\dim G/K)/2$ are

$$G_2^+(\mathbb{R}^{n+1}), G_2(\mathbb{C}^{n+1}), G_3(\mathbb{C}^6), SO_{10}/U_5, SO_{12}/U_6, SO_{14}/U_7, E_6/Spin_{10}U_1, E_7/E_6U_1.$$

In the case of $G_2^+(\mathbb{R}^{n+1})$, the total space SO_{n+1}/SO_{n-1} of the corresponding fat bundle equals the unit tangent bundle $T^1\mathbb{S}^n$. As mentioned in Sect. 1.2, the Euler class of $SU_{n+1}/(SU_2 \times SU_{n-1}) \rightarrow G_2(\mathbb{C}^{n+1})$ is a generator of $H^2(G_2(\mathbb{C}^{n+1}); \mathbb{Z}) = \mathbb{Z}$; this follows from [4, Lemma 12.2] since both spaces are simply connected.

The third set is similar to the previous one. Recall that a compact irreducible symmetric space G/K is quaternionic if K locally splits off an Sp_1 factor $K = K'Sp_1$ and the isotropy representation of this Sp_1 factor is equivalent to the standard Hopf action on \mathbb{H}^n (see [9, Chapter 14.E] for their classification). The bundle $G/K' \rightarrow G/K'Sp_1$ is fat and principal, with the group L acting being Sp_1 or SO_3 ; see [5, Section 8, Cas (B)] or [55, p. 17]. Moreover, the associated 2-sphere homogeneous bundle

$G/K' \times_L \mathbb{S}^2 \rightarrow G/K' \text{Sp}_1$ is also fat, see Cas (A) and Cas (B) of Sections 7 and 8 in [5] or [55, Proposition 2.22]. The quaternionic spaces of rank ≥ 2 and $b(G/K) \leq (\dim G/K)/2$ are

$G_2(\mathbb{C}^{n+1}), G_2^+(\mathbb{R}^6), F_4/\text{Sp}_3\text{Sp}_1, E_6/\text{SU}_6\text{SU}_2, E_7/\text{Spin}_{12}\text{Sp}_1, E_8/E_7\text{Sp}_1$ and G_2/SO_4 .

For each such G/K there is exactly one associated Sp_1 bundle (and the corresponding 2-sphere bundle) as above, except in the case G_2/SO_4 [5, Section 7, Cas (A) and Cas (B)]. The group SO_4 has two normal subgroups SU_2^\pm corresponding to the image of $\text{SU}_2 \times \{1\}$ and $\{1\} \times \text{SU}_2$ under the covering $\text{SU}_2 \times \text{SU}_2 \rightarrow \text{SO}_4$, and both SU_2^\pm can be enlarged to subgroups $\text{SU}_2^\pm < \text{U}_2^\pm < \text{SO}_4$; see [28, pp. 1–2]. In the case of $G_2(\mathbb{C}^{n+1})$, the total space equals $W_{1,1}^{4n-1} \cong T^1\mathbb{C}\mathbb{P}^n$ and the associated 2-sphere bundle equals the projectivized tangent bundle $\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$ of $\mathbb{C}\mathbb{P}^n$.

We remark that $\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$ is not only related to $W_{1,1}^{4n-1}$ but to any of the spaces $W_{p,q}^{4n-1}$. In fact, for any p, q there is a triple

$$S(\text{U}_1^{p,q} \times \text{U}_{n-1}) < S(\text{U}_1 \times \text{U}_1 \times \text{U}_{n-1}) < \text{SU}_{n+1} \tag{5.2}$$

inducing a circle bundle $W_{p,q}^{4n-1} \rightarrow \mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$. For our purposes it is not relevant whether these circle bundles are fat, but they will be very useful for topological computations in Sect. 5.3.

The following theorem is obtained by applying Theorem F to the fat bundles above, in combination with the values $b(-)$ from Table 3 at the end of the article. In order to keep track of the spaces in the theorem, observe that each table corresponds to the respective set of spaces discussed above, with the exception of $\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$ being included in the first set (since, as explained in the previous paragraph, it is associated to any of the $W_{p,q}^{4n-1}$).

Theorem 5.4 *Each homogeneous space G/H in the following tables carries a homogeneous metric of $\text{Ric}_k > 0$ for the corresponding value of k (we restrict to $n \geq 2$ and distinguish between the cases $n \neq 3$ and $n = 3$ where it corresponds).*

Moreover, for each G/H in the third table (one whose top-left entry is $6/(2 \times \text{SU}_2)$) the associated 2-sphere bundle $G/H \times_L \mathbb{S}^2$ with $L = \text{Sp}_1$ or SO_3 carries a homogeneous metric of $\text{Ric}_k > 0$ for the same k .

G/H	$\dim G/H$	k	
		$(n \neq 3)$	$(n = 3)$
$W_{p,q}^{4n-1}$	$4n-1$	$2n-3$	4
$\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$	$4n-2$	$2n-3$	4
$\mathbb{P}_{\mathbb{H}}T\mathbb{H}\mathbb{P}^n$	$8n-4$	$4n-7$	6
$\text{SU}_6/S(\text{U}_2 \times \text{Sp}_2)$	21		7
$\text{SO}_{10}/(\text{SO}_2 \times \text{Spin}_7)$	23		8

G/H	$\dim G/H$	k	
		$(n \neq 3)$	$(n = 3)$
$T^1\mathbb{S}^n$	$2n-1$	$n-1$	3
$SU_{n+1}/(SU_2 \times SU_{n-1})$	$4n-3$	$2n-3$	4
$SU_6/(SU_3 \times SU_3)$	19		9
SO_{10}/SU_5	21		7
SO_{12}/SU_6	31		15
SO_{14}/SU_7	43		21
$E_6/Spin_{10}$	33		11
E_7/E_6	55		27

G/H	$\dim G/H$	k
$SO_6/(SO_2 \times SU_2)$	11	4
F_4/Sp_3	31	13
E_6/SU_6	43	19
$E_7/Spin_{12}$	67	31
E_8/E_7	115	55
G_2/SU_2^\pm	11	3

Remark 5.2 As stated in Theorem G, $T^1\mathbb{S}^3 = SO_4/SO_2$ actually admits an SO_4 -homogeneous metric of $Ric_2 > 0$. Such metric is induced from the one obtained in $Sp_1 \times Sp_1$ by applying Theorem E to $G = Sp_1$. Indeed, let $\pi : Sp_1 \times Sp_1 \rightarrow SO_4$ be the Lie group covering map that sends a pair (p, q) of unit quaternions to the orthogonal map $\pi(p, q)$ of \mathbb{R}^4 given by $\pi(p, q)v := pvq^{-1}$, for each $v \in \mathbb{R}^4 \cong \mathbb{H} = \text{span}_{\mathbb{R}}\{1, i, j, k\}$. The image under π of the diagonal circle subgroup $\Delta\mathbb{S}^1 = \{(e^{i\theta}, e^{i\theta}) : \theta \in \mathbb{R}\} < Sp_1 \times Sp_1$ leaves $\text{span}_{\mathbb{R}}\{1, i\} \subset \mathbb{H}$ pointwise fixed, and hence $\pi(\Delta\mathbb{S}^1) = \{\text{diag}(1, 1, A) : A \in SO_2\} < SO_4$. Thus, π induces a diffeomorphism $[\pi] : (Sp_1 \times Sp_1)/\Delta\mathbb{S}^1 \rightarrow SO_4/SO_2$ which is $Sp_1 \times Sp_1$ -equivariant, since π is a Lie group homomorphism. Therefore, the submersion metric on $(Sp_1 \times Sp_1)/\Delta\mathbb{S}^1$ of $Ric_2 > 0$ determines an SO_4 -invariant metric of $Ric_2 > 0$ on $SO_4/SO_2 = T^1\mathbb{S}^3$.

We finish with some topological comments. The spaces G_2/SU_2^+ and G_2/U_2^+ have the rational cohomology ring (but not the integral cohomology ring) of \mathbb{S}^{11} and $\mathbb{C}P^5$ respectively; see [28] and [10, pp. 195-196]. The integral cohomology rings of several homogeneous spaces of exceptional Lie groups have been computed in [18, Section 5.1].

5.3 Topology

Here we compute various topological properties of the spaces $W_{p,q}^{4n-1}$. Throughout this section all cohomology groups are understood to be taken with integer coefficients. As in Sect. 1.2, p, q are assumed to be coprime integers with $pq > 0$ (although all computations also work for the case $pq < 0$). The main tool will be the Gysin

sequence associated to the following circle bundles induced from the triples (5.2):

$$\pi_{p,q} : W_{p,q}^{4n-1} \rightarrow \mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n.$$

The cohomology ring of $\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n$ was computed by Borel in [13, Proposition 31.1]:

$$H^*(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n) = \frac{S(x_1) \otimes S(x_2) \otimes S(x_3, \dots, x_{n+1})}{S^+(x_1, \dots, x_{n+1})},$$

where $S(-)$ denotes symmetric polynomials in the corresponding variables (of degree 2) and $S^+(-) \subset S(-)$ denotes those of positive degree. In particular, $H^*(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n)$ has no torsion and is concentrated in even degrees, with non-trivial groups given by:

$$H^{2k}(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n) = \begin{cases} \mathbb{Z}^{k+1}, & \text{if } k < n, \\ \mathbb{Z}^{2n-k}, & \text{if } n \leq k \leq 2n - 1. \end{cases}$$

The cohomology ring can be rewritten as follows:

$$H^*(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n) = \frac{\mathbb{Z}[x, y]}{\begin{cases} x^n + x^{n-1}y + \dots + xy^{n-1} + y^n = 0 \\ xy(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1}) = 0 \end{cases}}, \quad (5.3)$$

where x, y are of degree 2. Moreover, as explained in [32, pp. 473-474] for the case $n = 2$, the generators x, y can be chosen so that the Euler class $e(\pi_{p,q})$ of $\pi_{p,q}$ equals

$$e(\pi_{p,q}) = -qx + py \in H^2(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n).$$

With this information at hand one can fully compute the cohomology ring of $W_{p,q}^{4n-1}$ using the Gysin sequence, which gives exact sequences

$$0 \rightarrow H^{2k-1}(W_{p,q}^{4n-1}) \rightarrow H^{2k-2}(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n) \xrightarrow{e \cup} H^{2k}(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n) \rightarrow H^{2k}(W_{p,q}^{4n-1}) \rightarrow 0.$$

For our purposes it will sufficient to look at cohomology groups of degree $\leq 2n$. Multiplication by the Euler class $e \cup$ is injective for $2k \leq 2n$, thus it follows that the cohomology groups $H^{2k-1}(W_{p,q}^{4n-1})$ vanish. Consequently we get isomorphisms

$$H^{2k}(W_{p,q}^{4n-1}) = \frac{H^{2k}(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n)}{e \cup H^{2k-2}(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n)}. \quad (5.4)$$

For $2k < 2n$ there are no relations among the elements $x^k, x^{k-1}y, \dots, xy^{k-1}, y^k$ from (5.3). Since p, q are coprime with $pq \neq 0$ it follows that the quotient (5.4) is isomorphic to \mathbb{Z} .

The case $2k = 2n$ is bit more involved, so let us have a closer look at the map $e\cup$. Using the first relation in (5.3) we can give the following bases to the spaces involved:

$$H^{2n-2}(\mathbb{P}_C T\mathbb{C}\mathbb{P}^n) = \langle x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1} \rangle \cong \mathbb{Z}^n,$$

$$H^{2n}(\mathbb{P}_C T\mathbb{C}\mathbb{P}^n) = \langle x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n \rangle \cong \mathbb{Z}^n.$$

The $n \times n$ matrix representing the map $e\cup$ with respect to these bases equals:

$$T_n(p, q) := \begin{pmatrix} p+q & -q & 0 & \dots & 0 & 0 \\ q & p & -q & \dots & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & 0 & 0 & \dots & p & -q \\ q & 0 & 0 & \dots & 0 & p \end{pmatrix}.$$

The quotient in (5.4) for $k = n$ is a finite group whose order equals the absolute value of the determinant of the matrix $T_n(p, q)$. We denote the latter by $\tau_n(p, q)$:

$$\tau_n(p, q) := |\det T_n(p, q)| = |p^n + p^{n-1}q + \dots + pq^{n-1} + q^n|.$$

Table 3 For each irreducible symmetric space M , we list the symmetric spaces B_{Φ^k} , where $\Phi^k = \Delta \setminus \{\alpha_k\}$, along with $b(M) = 1 + \max\{\dim B_{\Phi^k} : 1 \leq k \leq r\}$ and the index k_{\max} of the a simple root $\alpha_{k_{\max}}$ for which the maximum is achieved. The symmetric spaces M and B_{Φ^k} are listed up to finite quotients and coverings

Dynkin diagram	Type	Multiplicities	M	$\dim M$	$B_{\Phi^k}, \dim B_{\Phi^k}$	k_{\max}	$b(M)$
	A_1	n	S^{n+1}	$n+1$	{point}, 0	1	1
	A_r	$1, \dots, 1$	SU_{r+1}/SO_{r+1}	$\frac{1}{2}r(r+3)$	$SU_k/SO_k \times SU_{r-k+1}/SO_{r-k+1},$ $\frac{1}{2}((k-1)(k+2) + (r-k)(r-k+3))$	1	$\frac{1}{2}r(r+1)$
	A_r	$2, \dots, 2$	SU_{r+1}	$r(r+2)$	$SU_k \times SU_{r-k+1},$ $k^2 - 1 + (r-k)(r-k+2)$	1	r^2
	A_r	$4, \dots, 4$	SU_{2r+2}/Sp_{r+1}	$r(2r+3)$	$SU_{2k}/Sp_k \times SU_{2r-2k+2}/Sp_{r-k+1},$ $(k-1)(2k+1) + (r-k)(2r-2k+3)$	1	$2r^2 - r$
	A_2	8, 8	E_6/F_4	26	$S^3, 9$	1	10
	B_r	$1, \dots, 1, n$	$G_r^+(\mathbb{R}^{2r+n}), n \geq 1$	$r(r+n)$	$SU_k/SO_k \times G_{r-k}^+(\mathbb{R}^{2r-2k+n}),$ $\frac{1}{2}(k-1)(k+2) + (r-k)(r-k+n)$	1	$r^2 + (n-2)(r-1)$
	B_r	$2, \dots, 2, 2$	SO_{2r+1}	$r(2r+1)$	$SU_k \times SO_{2r-2k+1},$ $k^2 - 1 + (r-k)(2r-2k+1)$	1	$2r^2 - 3r + 2$
	C_r	$1, \dots, 1, 1$	Sp_r/U_r	$r(r+1)$	$SU_k/SO_k \times Sp_{r-k}/U_{r-k},$ $\frac{1}{2}(k-1)(k+2) + (r-k)(r-k+1)$	1	$r^2 - r + 1$
	C_r	$2, \dots, 2, 1$	$G_r(\mathbb{C}^{2r})$	$2r^2$	$SU_k \times G_{r-k}(\mathbb{C}^{2r-2k}),$ $k^2 - 1 + 2(r-k)^2$	1	$2r^2 - 4r + 3 (r \geq 3)$ $r^2 (r \leq 2)$
	C_r	$2, \dots, 2, 2$	Sp_r	$r(2r+1)$	$SU_k \times Sp_{r-k},$ $k^2 - 1 + (r-k)(2r-2k+1)$	1	$2r^2 - 3r + 2$
	C_r	$4, \dots, 4, 1$	SO_{4r}/U_{2r}	$2r(2r-1)$	$SU_{2k}/Sp_k \times SO_{4r-4k}/U_{2r-2k},$ $(k-1)(2k+1) + 2(r-k)(2r-2k-1)$	1	$4r^2 - 10r + 7 (r \geq 4)$ $2r^2 - r (r \leq 3)$
	C_r	$4, \dots, 4, 3$	$G_r(\mathbb{H}^{2r})$	$4r^2$	$SU_{2k}/Sp_k \times G_{r-k}(\mathbb{H}^{2r-2k}),$ $(k-1)(2k+1) + 4(r-k)^2$	1	$4r^2 - 8r + 5 (r \geq 3)$ $2r^2 - r (r \leq 2)$
	C_3	8, 8, 1	E_7/E_6U_1	54	$G_2^+(\mathbb{R}^{13}), 20 (k=1)$ $S^3 \times S^2, 11 (k=2)$ $E_6/F_4, 26 (k=3)$	3	27
	D_r	$1, \dots, 1, 1, 1$	$G_r^+(\mathbb{R}^{2r})$	r^2	$SU_k/SO_k \times G_{r-k}^+(\mathbb{R}^{2r-2k}),$ $\frac{1}{2}(k-1)(k+2) + (r-k)^2$	1	$r^2 - 2r + 2 (r \geq 4)$ $\frac{1}{2}(r^2 + r) (r = 3)$
	D_r	$2, \dots, 2, 2, 2$	SO_{2r}	$r(2r-1)$	$SU_k \times SO_{2r-2k},$ $k^2 - 1 + (r-k)(2r-2k-1)$	r	$2r^2 - 5r + 4 (r \geq 4)$ $r^2 (r = 3)$
	BC_r	$2, \dots, 2, 2n[1]$	$G_r(\mathbb{C}^{2r+n}), n \geq 1$	$2r(r+n)$	$SU_k \times G_{r-k}(\mathbb{C}^{2r-2k+n}),$ $k^2 - 1 + 2(r-k)(r-k+n)$	1	$2r^2 + 2(n-2)r + 3 - 2n$
	BC_r	$4, \dots, 4, 4n[3]$	$G_r(\mathbb{H}^{2r+n}), n \geq 1$	$4r(r+n)$	$SU_{2k}/Sp_k \times G_{r-k}(\mathbb{H}^{2r-2k+n}),$ $(k-1)(2k+1) + 4(r-k)(r-k+n)$	1	$4r^2 + 4(n-2)r + 5 - 4n$
	BC_r	$4, \dots, 4, 4[1]$	SO_{4r+2}/U_{2r+1}	$2r(2r+1)$	$SU_{2k}/Sp_k \times SO_{4r-4k+2}/U_{2r-2k+1},$ $(k-1)(2k+1) + (r-k)(4r-4k+2)$	1	$4r^2 - 6r + 3$
	BC_2	6, 8[1]	$E_6/Spin_{10}U_1$	32	$\mathbb{C}P^3, 10 (k=1)$ $S^7, 7 (k=2)$	1	11
BC_1	8[7]	$F_4/Spin_9$	16	{point}, 0	1	1	

Table 3 continued

Dynkin diagram	Type	Multiplicities	M	$\dim M$	k	$B_{\Phi,k}$	$\dim B_{\Phi,k}$	k_{\max}	$b(M)$	
	G_2	1, 1	G_2/SO_4	8	1, 2	S^2	2	1	3	
	G_2	2, 2	G_2	14	1, 2	SU_2	3	1	4	
	F_4	1, 1, 1, 1	F_4/Sp_3Sp_1	28	2, 3	Sp_3/U_3	12			
						$S^2 \times SU_3/SO_3$	7	1	13	
						$G_3^+(\mathbb{R}^7)$	12			
						$G_3(\mathbb{C}^6)$	18			
	F_4	1, 1, 2, 2	E_6/SU_6SU_2	40	2	$S^2 \times SU_3$	10			
					3	$SU_2 \times SU_3/SO_3$	8	1	19	
					4	$G_3^+(\mathbb{R}^8)$	15			
					1	SO_{12}/U_6	30			
					2	$S^2 \times SU_6/Sp_3$	16			
					3	$SU_3/SO_3 \times S^5$	10	1	31	
	F_4	1, 1, 4, 4	$E_7/Spin_{12}Sp_1$	64	3	$G_3^+(\mathbb{R}^{10})$	21			
					4	E_7/E_6U_1	54			
					2	$S^2 \times E_6/F_4$	28			
	F_4	1, 1, 8, 8	E_8/E_7Sp_1	112	3	$SU_3/SO_3 \times S^9$	14	1	55	
					4	$G_3^+(\mathbb{R}^{14})$	33			
					1	Sp_3	21			
					2, 3	$SU_2 \times SU_3$	11	1	22	
					4	SO_7	21			
		E_6	1, ..., 1	$E_6/(Sp_4/\mathbb{Z}_2)$	42	1, 6	$G_4^+(\mathbb{R}^{10})$	25		
						3, 5	$S^2 \times SU_6/SO_6$	16	1	26
					4	$(SU_3/SO_3)^2 \times S^2$	12			
					2	SU_6/SO_6	20			
					1, 6	SO_{10}	45			
E_6		2, ..., 2	E_6	78	3, 5	$SU_2 \times SU_5$	27			
					4	$SU_3^2 \times SU_2$	19	1	46	
					2	SU_6	35			
		E_7	1, ..., 1	$E_7/(Sp_4/\mathbb{Z}_2)$	70	1	$E_7/(SU_8/\mathbb{Z}_2)$	42		
						2	SU_7/SO_7	27		
					3	$S^2 \times G_4^+(\mathbb{R}^{10})$	27			
					4	$SU_3/SO_3 \times SU_4/SO_4$	19	1	43	
					5	$S^2 \times SU_3/SO_3 \times SU_4/SO_4$	16			
					6	$S^2 \times SU_6/SO_6$	22			
					7	$G_6^+(\mathbb{R}^{12})$	36			
					1	E_6	78			
					2	SU_7	48			
					3	$SU_2 \times SO_{10}$	48			
	E_7	2, ..., 2	E_7	133	4	$SU_3 \times SO_{10}$	32	1	79	
					5	$SU_2 \times SU_3 \times SU_4$	26			
					6	$SU_2 \times SU_6$	38			
					7	SO_{12}	66			
		E_8	1, ..., 1	$E_7/(SU_8/\mathbb{Z}_2)$	128	1	$E_7/(SU_8/\mathbb{Z}_2)$	70		
						2	SU_8/SO_8	35		
						3	$S^2 \times E_6/(Sp_4/\mathbb{Z}_2)$	44		
						4	$SU_3/SO_3 \times G_4^+(\mathbb{R}^{10})$	30	1	71
						5	$SU_4/SO_4 \times SU_4/SO_4$	23		
						6	$S^2 \times SU_5/SO_5 \times SU_3/SO_3$	21		
					7	$SU_7/SO_7 \times S^2$	29			
					8	$G_7^+(\mathbb{R}^{14})$	49			
					1	E_7	133			
					2	SU_8	63			
	E_8	2, ..., 2	E_8	248	3	$SU_2 \times E_6$	81			
					4	$SU_3 \times SO_{10}$	53	1	134	
					5	$SU_4 \times SU_5$	39			
					6	$SU_2 \times SU_5 \times SU_3$	35			
					7	$SU_7 \times SU_2$	51			
					8	SO_{14}	91			

Observe that $\tau_n(p, q) > 0$ if $pq > 0$. Let us summarize the discussion above.

Lemma 5.3 *The manifolds $W_{p,q}^{4n-1}$ are simply connected and the first $2n$ cohomology groups equal:*

$$H^{2k}(W_{p,q}^{4n-1}) = \begin{cases} \mathbb{Z}, & \text{if } k \text{ even and } < 2n, \\ \Gamma_{\tau_n(p,q)}, & \text{if } k = 2n, \\ 0, & \text{if } k \text{ odd and } < 2n, \end{cases}$$

where $\Gamma_{\tau_n(p,q)}$ is some finite group of order $\tau_n(p, q)$.

The fundamental group of $W_{p,q}^{4n-1}$ can be computed using [4, Lemma 12.2] in terms of the Euler class: since p, q are coprime, the element $e(\pi_{p,q}) = -qx + py$ is indivisible in $H^2(\mathbb{P}_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n)$ and hence the total space $W_{p,q}^{4n-1}$ is simply connected. The order of $H^{2n}(W_{p,q}^{4n-1})$ can be extracted from Wilking's article [51, p. 119]. He also observed that $\tau_n(p, q) = 0$ may occur if one allows $pq < 0$. For the case $n = 2$ the reader can find further details in the article of Kreck and Stolz [32, pp. 473-474].

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Declarations

Conflict of interest On behalf of all authors, the corresponding author David González-Álvarez states that there is no conflict of interest.

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