Journal of Pure and Applied Algebra 213 (2009) 2244–2261

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Preunits and weak crossed products

J.M. Fernández Vilaboa [a], R. González Rodríguez [b,*], A.B. Rodríguez Raposo [a]

[a] Departamento de Álxebra, Universidad de Santiago de Compostela, E-15771 Santiago de Compostela, Spain

[b] Departamento de Matemática Aplicada II, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36310 Vigo, Spain

ARTICLE INFO

Article history:
Received 23 October 2008
Received in revised form 19 January 2009
Available online 2 May 2009
Communicated by S. Donkin

MSC:
18D10
16W30

ABSTRACT

Using the notion of a preunit and the properties of idempotent morphisms, we give a general notion of a crossed product of an algebra A and an object V both living in a monoidal category $\mathcal{C}$. We endow $A\otimes V$ with a multiplication and an idempotent morphism, whose image inherits the multiplication. Sufficient conditions for these multiplications to be associative are given. If the product on $A\otimes V$ has a preunit, the related idempotent is given in terms of the preunit, and its image has an algebra structure. A characterization of crossed products with preunit is given, and it is used to recover classical examples of crossed products and to study crossed products in weak contexts. Finally crossed products of an algebra by a weak bialgebra are recovered using this theory.

1. Introduction

Crossed products of an algebra A by a Hopf algebra H, both living in a category of K-vector spaces, were introduced independently by Blattner, Cohen and Montgomery [1] and by Doi and Takeuchi [2]. Such crossed products consist of an algebra structure on $A\otimes H$ whose multiplication $\mu_{A\sharp_\sigma H}$ is given in terms of an action $\varphi_A : A\otimes H \to A$ and a twisted normal two-cocycle $\sigma_A : H\otimes H \to A$. The unit of this product is given by $\eta_{A\sharp_\sigma H} = \eta_A\otimes\eta_H$, for η_A and η_H the unit of A and the unit of H respectively. The necessary and sufficient conditions are given in [1,2] for $\mu_{A\sharp_\sigma H}$ to be associative and also for $\eta_{A\sharp_\sigma H}$ to be the unit. Moreover a crossed product of an algebra by a Hopf algebra is used to characterize Galois extensions of Hopf algebras that satisfy the normal basis condition, that is, cleft extensions. The generalization of crossed products to braided Hopf algebras is due to Majid, and it turned out to be an important tool in the study of quantum geometry (see for example [3–5]). In particular, Galois extensions of Hopf algebras can be seen as non-commutative principal bundles being cleft extensions trivial principal bundles.

In order to obtain a more general setting to study principal bundles, Brzeziński and Majid define in [6] the so-called entwining structures, where they replace the Hopf algebra structure by an algebra and a coalgebra related by an entwining morphism. In this new context Brzeziński and Hajac [7] define the concept of coalgebra-Galois extension, and Brzeziński in [8] gives a general theory to study crossed products. He replaces the Hopf algebra H by an arbitrary vector space V, and the weak action φ_A is generalized by a linear map $\tilde{\psi} : V\otimes A \to A\otimes V$ and the two-cocycle is replaced by $\tilde{\sigma} : V\otimes V \to A\otimes V$. These morphisms, under some necessary and sufficient conditions, are used to obtain an associative multiplication on $A\otimes V$. Moreover a unit can be defined, provided that there exists a morphism $e : K \to V$ that must satisfy certain equalities with respect to $\tilde{\psi}$ and $\tilde{\sigma}$. This unit is given by $\eta_{A\otimes V} = \eta_A\otimes e$, and it is the last ingredient to obtain an algebra structure on $A\otimes V$. This general crossed product was used by Bespalov and Drabant in [9] to obtain crossed products without a cocycle.

* Corresponding address: Universidad de Vigo, Departamento de Matemática Aplicada II, Campus Universitario Lagoas-Marcosende, E.T.S.E. Teleomunicaci E-36310 Vigo, Spain. Tel.: +34 986812150; fax: +34 986812116.

E-mail addresses: alvila@usc.es (J.M. Fernández Vilaboa), rgon@dma.uvigo.es (R. González Rodríguez), abraposo@edu.xunta.es (A.B. Rodríguez Raposo).

0022-4049/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2009.04.005

Moreover it generalizes the classical crossed products of Hopf algebras, of braided Hopf algebras and was also used by J.A. Guccione and J.J. Guccione in [10] to develop a theory of braided Hopf crossed products which include the classical ones and the Ore extension type.

Although Brzeziński's general crossed product supposes a very useful and general tool to study a wide class of crossed products, it turned out that it was not enough general to cover more recent generalizations of Hopf algebras, as weak Hopf algebras, given by Böhm, Nill and Szlachányi in [11], and the more general setting of weak entwining structures due to Caeneepel and De Groot [12]. These weak cases of Hopf algebras and of entwining structures imply that the properties of the units of the algebras and the counits of the coalgebras are weaker than that in the classical cases. These weaker properties reveal the existence of some idempotent morphisms that in the non-weak cases were identities, and in particular when one tries to extend crossed products to weak contexts one finds that these idempotent morphisms appear in a very natural way (see for example [13]). In [14] we give a general theory of crossed products where we combine the ideas in [8] and the use of the idempotents in [13,15]. We define a product on $A \otimes V$, for an algebra $A = (A, \mu_A, \eta_A)$ and V an object both in a strict monoidal category $\mathcal{C}$ where every idempotent splits, by means of two morphisms $\psi_V^A : V \otimes A \to A \otimes V$ and $\sigma_V^A : V \otimes V \to A \otimes V$ that must satisfy some conditions that generalize the classical ones. Associated to these morphisms we obtain an idempotent

$$\nabla_{A\otimes V} = (\mu_A \otimes V) \circ (A \otimes (\psi_V^A \circ (V \otimes \eta_A))) : A \otimes V \to A \otimes V$$

which plays the role of the identity in the non-weak case. Now we obtain not only a multiplication on $A \otimes V$ but on the image of the idempotent $\nabla_{A\otimes V}$. If we also ask for the existence of a morphism $\eta_V : K \to V$, where K is the base object in the category $\mathcal{C}$, we obtain under certain conditions a weak crossed product system with unity, that permits us to give a unit for the product on the image of the idempotent $\nabla_{A\otimes V}$, and hence to endow it with a unital algebra structure. In [14,16] we use our theory to obtain a crossed product related to a weak C-cleft extension for a weak entwining structure (A, C, ψ), where A is an algebra, C a coalgebra and $\psi : C \otimes A \to A \otimes C$ the weak entwining morphism (see [12] for weak entwining structures and [17] for weak C-cleft extensions), provided that there exists $\eta_C : K \to C$ and that the cleaving morphism satisfies $f \circ \eta_A = \eta_C$. In the Hopf algebra case this equality does not suppose any restriction, but in the weak case we cannot assure it for all weak cleft extension, not even in the simplest case of weak Hopf algebras. Then the theory developed in [14] was suitable and valid for the cleft extensions associated to projections of weak Hopf algebras (see [13,18,19]) but not complete, as it did not cover all the general cases.

Following the relation of the existence of a preunit and an idempotent morphism on a non-unital algebra given in [12] by Caenepeel and De Groot, and the relation of weak contexts with idempotent morphisms, we found that the key to define a weak general crossed product is to replace the morphism $\eta_V : K \to V$ by a morphism $\nu : K \to A \otimes V$ that will play the role of the preunit. Using this idea we obtain in this paper a general theory of crossed products that recovers the one in [14] and the non-weak ones. To present this theory we give in the first section a survey of preliminary results where we recall the definition of a preunit and its relation with idempotent morphisms. In the second section we give, first of all, the notion of a weak crossed product, where we introduce the morphisms ψ_V^A, σ_V^A and the idempotent $\nabla_{A\otimes V}$. Then using classical twisted and cocycle conditions we give an associative product on $A \otimes V$ and on the image of $\nabla_{A\otimes V}$. The next step is to introduce a morphism $\nu : K \to A \otimes V$, that permits us to obtain a weak crossed product with preunit. In Theorem 3.11 we give necessary and sufficient conditions to obtain a left A-linear multiplication on $A \otimes V$ with preunit ν, or equivalently to obtain a weak crossed product with preunit. This crossed product with preunit permits us to endow the image of the idempotent $\nabla_{A\otimes V}$ with an algebra structure, and then to recover the classical cases of crossed products. In particular we obtain the weak crossed products for weak bialgebras given in [20] as an example of our general weak crossed products.

Throughout the paper $\mathcal{C}$ denotes a strict monoidal category with tensor product $\otimes$ and base object K. Given objects A, B, D and a morphism $f : B \to D$, we write $A \otimes f$ for $id_A \otimes f$ and $f \otimes A$ for $f \otimes id_A$. Also we assume all idempotent splits, i.e., for every morphism $\nabla_Y : Y \to Y$, such that $\nabla_Y = \nabla_Y \circ \nabla_Y$, there exist an object Z and morphisms $i_Y : Z \to Y$ and $p_Y : Y \to Z$ satisfying $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$.

As for prerequisites, the reader is expected to be familiar with the notions of algebra (monoid), coalgebra (comonoid), module and comodule in the monoidal setting. Given an algebra A and a coalgebra C, we let $\eta_A : K \to A$, $\mu_A : A \otimes A \to A$, $\varepsilon_D : D \to K$, and $\delta_D : D \to D \otimes D$ denote the unity, the product, the counity, and the coproduct respectively. Also, if A, B are algebras, $f : A \to B$ is an algebra morphism if $f \circ \eta_A = \eta_B$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$. In a dual form we have the notion of coalgebra morphism.

2. Preunits and products

Definition 2.1. Let $\nabla_A : A \to A$ be an idempotent morphism. Fix $\mathrm{Im}(\nabla_A) = \bar{A}$, and $i_A : \bar{A} \to A$ and $p_A : A \to \bar{A}$ the injection and the projection associated to ∇_A.

If there exists a morphism $\mu_A : A \otimes A \to A$ that $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, i.e., it is an associative product on A, then we say that it is normalized with respect to ∇_A (or in general that it is normalized) if it satisfies:

$$\nabla_A \circ \mu_A = \mu_A = \mu_A \circ (\nabla_A \otimes \nabla_A).$$

Lemma 2.2. *Suppose that $\nabla_A : A \to A$ is an idempotent morphism as above, and suppose that $\mu_{\bar{A}} : \bar{A} \otimes \bar{A} \to \bar{A}$ is an associative product on $\bar{A}$. Then there exists an associative product $\mu_A : A \otimes A \to A$ normalized with respect to ∇_A such that $\mu_{\bar{A}} = p_A \circ \mu_A \circ (i_A \otimes i_A)$.*

Proof. Take $\mu_A = i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A)$ and compute:

$$\begin{aligned}\mu_A \circ (\mu_A \otimes A) &= i_A \circ \mu_{\bar{A}} \circ (\mu_{\bar{A}} \otimes \bar{A}) \circ (p_A \otimes p_A \otimes p_A)\\ &= i_A \circ \mu_{\bar{A}} \circ (\bar{A} \otimes \mu_{\bar{A}}) \circ (p_A \otimes p_A \otimes p_A)\\ &= \mu_A \circ (A \otimes \mu_A)\end{aligned}$$

by virtue of the definition of μ_A and the associativity of $\mu_{\bar{A}}$. Now as $\nabla_A \circ i_A = i_A$:

$$\nabla_A \circ \mu_A = \nabla_A \circ i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A) = i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A) = \mu_A.$$

In a similar way, and as $p_A \circ \nabla_A = p_A$:

$$\mu_A \circ (\nabla_A \otimes \nabla_A) = i_A \circ \mu_{\bar{A}} \circ ((p_A \circ \nabla_A) \otimes (p_A \circ \nabla_A)) = i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A) = \mu_A,$$

and hence μ_A is indeed a normalized product. Finally,

$$\mu_{\bar{A}} = p_A \circ i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A) \circ (i_A \otimes i_A) = p_A \circ \mu_A \circ (i_A \otimes i_A). \quad \square$$

Then if we have an idempotent morphism and an associative product on its image, we can recover an associative product defined on the whole original object, but we cannot say anything about a possible unit for this product, even if the image of the idempotent is an algebra. To study this cases we need the concept of a preunit introduced by Caenepeel and De Groot in [12]:

Definition 2.3. Let A be an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ an associative product. The morphism $\nu : k \to A$ is a *preunit* for μ_A if

$$\mu_A \circ (A \otimes \nu) = \mu_A \circ (\nu \otimes A) = \mu_A \circ (A \otimes (\mu_A \circ (\nu \otimes \nu))).$$

Observe that if (A, η_A, μ_A) is an algebra, then $\nu = \eta_A$ is a preunit for μ_A.

Remark 2.4. Note that if A is an object in $\mathcal{C}$ with an associative product $\mu_A : A \otimes A \to A$ and preunit ν, then $\nu' = \mu_A \circ (\nu \otimes \nu)$ is also a preunit. Furthermore $\mu_A \circ (\nu' \otimes \nu') = \nu'$. Henceforth we will consider that ν satisfies this condition as it does not suppose any loss of generality.

The existence of a preunit is very closely related to the existence of an idempotent morphism as is claimed in the next proposition:

Proposition 2.5. *Let A an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ an associative product. Then the following assertions hold:*

(i) *If $\nu : k \to A$ is a preunit for μ_A, the morphism $\nabla_A^\nu : A \to A$ given by*

$$\nabla_A^\nu = \mu_A \circ (A \otimes \nu)$$

is idempotent, multiplicative and such that $\mu_A \circ (A \otimes \nabla_A^\nu) = \nabla_A^\nu \circ \mu_A$.

(ii) *If $\nu : k \to A$ is a preunit for μ_A, then $(\bar{A}, \eta_{\bar{A}}, \mu_{\bar{A}})$ is an algebra, where $\bar{A} = \mathrm{Im}(\nabla_A^\nu)$, p_A^ν and i_A^ν are the morphisms such that $\nabla_A^\nu = i_A^\nu \circ p_A^\nu$, $p_A^\nu \circ i_A^\nu = id_{\bar{A}}$, $\mu_{\bar{A}} = p_A^\nu \circ \mu_A \circ (i_A^\nu \otimes i_A^\nu)$ and $\eta_{\bar{A}} = p_A^\nu \circ \nu$.*

(iii) *If $\nabla_A : A \to A$ is an idempotent morphism and $\bar{A} = \mathrm{Im}(\nabla_A)$ is an algebra, then the morphism $\mu_A = i_A \circ \mu_{\bar{A}} \circ (p_A \otimes p_A)$ is an associative product on A with preunit $\nu = i_A \circ \eta_{\bar{A}}$. Moreover $\nabla_A = \nabla_A^\nu$.*

(iv) *If ν is a preunit for A, then $\nabla_A^\nu = id_A$ if and only if ν is a unit (that is, if A is an algebra).*

Proof. To prove that ∇_A^ν is an idempotent morphism just consider the equality

$$\mu_A \circ (A \otimes (\mu_A \circ (\nu \otimes \nu))) = \mu_A \circ (A \otimes \nu)$$

and the associativity of μ_A and compute:

$$\nabla_A^\nu \circ \nabla_A^\nu = \mu_A \circ ((\mu_A \circ (A \otimes \nu)) \otimes \nu) = \mu_A \circ (A \otimes (\mu_A \circ (\nu \otimes \nu))) = \mu_A \circ (A \otimes \nu) = \nabla_A^\nu.$$

Hence ∇_A^ν is idempotent.

Now using the properties of the preunit and the associativity of μ_A we obtain:

$$\begin{aligned}\mu_A \circ (\nabla_A^\nu \otimes \nabla_A^\nu) &= \mu_A \circ (A \otimes \mu_A) \circ (A \otimes (\mu_A \circ (\nu \otimes A)) \otimes \nu)\\ &= \mu_A \circ (\mu_A \otimes (\mu_A \circ (\nu \otimes \nu)))\\ &= \nabla_A^\nu \circ \mu_A,\end{aligned}$$

and then ∇_A^{ν} is multiplicative. On the other hand, equality $\nabla_A^{\nu}\circ\mu_A=\mu_A\circ(A\otimes\nabla_A^{\nu})$ follows by the associativity of μ_A and implies that ∇_A^{ν} is of left A-modules.

To prove the second assertion of the proposition first consider $\bar{A}=\mathrm{Im}(\nabla_A^{\nu})$ and define $\mu_{\bar{A}}=p_A^{\nu}\circ\mu_A\circ(i_A^{\nu}\otimes i_A^{\nu})$. This expression gives an associative product on $\bar{A}$. Indeed:

$$\begin{aligned}\mu_{\bar{A}}\circ(\mu_{\bar{A}}\otimes\bar{A}) &= p_A^{\nu}\circ\mu_A\circ((\nabla_A^{\nu}\circ\mu_A)\otimes A)\circ(i_A^{\nu}\otimes i_A^{\nu}\otimes i_A^{\nu})\\ &= p_A^{\nu}\circ\mu_A\circ(A\otimes\mu_A)\circ(i_A^{\nu}\otimes i_A^{\nu}\otimes i_A^{\nu})\\ &= p_A^{\nu}\circ\mu_A\circ(A\otimes(\nabla_A^{\nu}\circ\mu_A))\circ(i_A^{\nu}\otimes i_A^{\nu}\otimes i_A^{\nu})\\ &= \mu_{\bar{A}}\circ(\bar{A}\otimes\mu_{\bar{A}})\end{aligned}$$

by the multiplicativity of ∇_A^{ν}, equality $\nabla_A^{\nu}\circ i_A^{\nu}=i_A^{\nu}$ and the associativity of μ_A. Hence the product $\mu_{\bar{A}}$ is associative. Now compute:

$$\begin{aligned}\mu_{\bar{A}}\circ(\bar{A}\otimes\eta_{\bar{A}}) &= p_A^{\nu}\circ\mu_A\circ(i_A^{\nu}\otimes i_A^{\nu})\circ(\bar{A}\otimes(p_A^{\nu}\circ\nu))\\ &= p_A^{\nu}\circ\mu_A\circ(A\otimes\nabla_A^{\nu})\circ(i_A^{\nu}\otimes\nu)\\ &= p_A^{\nu}\circ\nabla_A^{\nu}\circ\nabla_A^{\nu}\circ i_A^{\nu}\\ &= p_A^{\nu}\circ i_A^{\nu}\\ &= id_{\bar{A}}\end{aligned}$$

using the definition of ∇_A^{ν}, that it is of left A-modules and equalities $p_A^{\nu}\circ\nabla_A^{\nu}=p_A^{\nu}$ and $\nabla_A^{\nu}\circ i_A^{\nu}=i_A^{\nu}$. On the other hand:

$$\begin{aligned}\mu_{\bar{A}}\circ(\eta_{\bar{A}}\otimes\bar{A}) &= p_A^{\nu}\circ\mu_A\circ(i_A^{\nu}\otimes A)\circ((p_A^{\nu}\otimes\nu)\otimes i_A^{\nu})\\ &= p_A^{\nu}\circ\mu_A\circ(\nabla_A^{\nu}\otimes A)\circ(\nu\otimes i_A^{\nu})\\ &= p_A^{\nu}\circ\nabla_A^{\nu}\circ\mu_A\circ(\nu\otimes A)\circ i_A^{\nu}\\ &= p_A^{\nu}\circ\nabla_A^{\nu}\circ\nabla_A^{\nu}\circ i_A^{\nu}\\ &= id_{\bar{A}}\end{aligned}$$

by similar arguments to the previous ones and by the properties of the preunit.

To show the third assertion let $\nabla_A: A\to A$ be an idempotent morphism such that $(\bar{A},\mu_{\bar{A}},\eta_{\bar{A}})$ is an algebra, and define $\mu_A=i_A\circ\mu_{\bar{A}}\circ(p_A\otimes p_A)$. This morphism gives an associative product on A, as is shown in Lemma 2.2.

Now to check that ν is a preunit compute:

$$\begin{aligned}\mu_A\circ(\nu\otimes A) &= i_A\circ\mu_{\bar{A}}\circ((p_A\circ i_A\circ\eta_{\bar{A}})\otimes\bar{A})\circ p_A\\ &= i_A\circ\mu_{\bar{A}}\circ(\eta_{\bar{A}}\otimes p_A)\\ &= \nabla_A\\ &= i_A\circ\mu_{\bar{A}}\circ(p_A\otimes\eta_{\bar{A}})\\ &= i_A\circ\mu_{\bar{A}}\circ(\bar{A}\otimes(p_A\circ i_A\circ\eta_{\bar{A}}))\circ p_A\\ &= \mu_A\circ(A\otimes\nu)\end{aligned}$$

just by using that $p_A\circ i_A=id_{\bar{A}}$ and that $\eta_{\bar{A}}$ is the unit from $\mu_{\bar{A}}$. Using again these two conditions:

$$\begin{aligned}\mu_A\circ(A\otimes(\mu_A\circ(\nu\otimes\nu))) &= i_A\circ\mu_{\bar{A}}\circ(p_A\otimes(p_A\circ i_A))\circ(A\otimes(\mu_{\bar{A}}\circ(\eta_{\bar{A}}\otimes\eta_{\bar{A}})))\\ &= i_A\circ\mu_{\bar{A}}\circ(p_A\otimes\mu_{\bar{A}})\circ(A\otimes\eta_{\bar{A}}\otimes\eta_{\bar{A}})\\ &= i_A\circ p_A\\ &= \nabla_A\end{aligned}$$

so ν is a preunit for μ_A. Observe that by these calculations we have also obtained that $\nabla_A=\mu_A\circ(A\otimes\nu)=\nabla_A^{\nu}$.

Finally if $\nabla_A^{\nu}=id_A$ then $A=\bar{A}$ and as the unit in $\bar{A}$ is given by $p_A\circ\nu$, we obtain that ν is a unit, as in this case $p_A=id_A$. Conversely if ν is a unit it is clear that $\nabla_A^{\nu}=id_A$. □

3. General weak crossed products

The following section is devoted to define a weak crossed product structure on $A\otimes V$, for an algebra A and an object V in the category $\mathcal{C}$. The idea that we followed is to consider an associative multiplication on $A\otimes V$ such that when it is endowed with a preunit we can define an idempotent morphism whose image has an algebra structure in a natural way. If in particular the preunit is $\eta_A\otimes\eta_V$, where $\eta_V: k\to V$ is a morphism in the base category, and K is the base object, we recover the crossed product described in [14]. Of course, if we have not only a non-unitary algebra but an algebra on $A\otimes V$ we recover the general crossed product described in [8,9] and at last the crossed product of a Hopf algebra by an algebra.

Moreover, it is possible to give a general crossed coproduct structure of a coalgebra C by an object V in $\mathcal{C}$ (see the final Appendix). The idea to obtain this crossed coproduct is to define a non-counital coalgebra structure on $C\otimes V$ such that if it has a precounit we will manage to define a coalgebra structure on a subobject of $C\otimes V$ given as the image of an idempotent morphism related to the precounit. This general crossed coproduct theory has as particular instances some crossed coproducts that appear in the context of Hopf algebras [21] and in general in [8,9].

Throughout the rest of the section we will consider that the left A-module structures on $A\otimes V$ and $A\otimes V\otimes A\otimes V$, for A an algebra and V an object in the base category $\mathcal{C}$, are given by

$$\varphi_{A\otimes V}=\mu_A\otimes V,\qquad \varphi_{A\otimes V\otimes A\otimes V}=\mu_A\otimes V\otimes A\otimes V.$$

Lemma 3.1. *Let A be an algebra and V be an object in $\mathcal{C}$. Suppose that there exists a morphism $\psi_V^A: V\otimes A\to A\otimes V$ such that the following equality holds*

$$(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\psi_V^A\otimes A)=\psi_V^A\circ(V\otimes\mu_A). \tag{1}$$

The morphism $\nabla_{A\otimes V}: A\otimes V\to A\otimes V$ defined by

$$\nabla_{A\otimes V}=(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(A\otimes V\otimes\eta_A) \tag{2}$$

is idempotent. Moreover, $\nabla_{A\otimes V}$ satisfies that

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)=(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V}),$$

that is, $\nabla_{A\otimes V}$ is a left A-module morphism.

Proof. First note that due to the associativity of μ_A, $\nabla_{A\otimes V}$ is a morphism of left A-modules, that is:

$$\varphi_{A\otimes V}\circ(A\otimes\nabla_{A\otimes V})=(\mu_A\otimes V)\circ(A\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)=\nabla_{A\otimes V}\circ\varphi_{A\otimes V}.$$

Besides, and as a consequence of (1) we obtain:

$$\nabla_{A\otimes V}\circ\psi_V^A=\psi_V^A. \tag{3}$$

Indeed:

$$\begin{aligned}\nabla_{A\otimes V}\circ\psi_V^A&=(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\psi_V^A\otimes\eta_A)\\&=\psi_V^A\circ(V\otimes(\mu_A\circ(A\otimes\eta_A)))\\&=\psi_V^A.\end{aligned}$$

This equality and the left A-linearity of $\nabla_{A\otimes V}$ yield that $\nabla_{A\otimes V}$ is idempotent, as:

$$\begin{aligned}\nabla_{A\otimes V}\circ\nabla_{A\otimes V}&=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes(\psi_V^A\circ(V\otimes\eta_A)))\\&=(\mu_A\otimes V)\circ(A\otimes(\nabla_{A\otimes V}\circ\psi_V^A\circ(V\otimes\eta_A)))\\&=(\mu_A\otimes V)\circ(A\otimes(\psi_V^A\circ(V\otimes\eta_A)))\\&=\nabla_{A\otimes V}.\quad\square\end{aligned}$$

3.2. From now on we consider quadruples $(A,V,\psi_V^A,\sigma_V^A)$ where A, V and ψ_V^A satisfy the conditions of Lemma 3.1 and $\sigma_V^A: V\otimes V\to A\otimes V$ is a morphism in $\mathcal{C}$. Also, for the idempotent morphism $\nabla_{A\otimes V}$ defined in Lemma 3.1 we denote by $A\times V$ the image of $\nabla_{A\otimes V}$, and by $i_{A\otimes V}: A\times V\to A\otimes V$ and $p_{A\otimes V}: A\otimes V\to A\times V$ the injection and the projection respectively.

Definition 3.3. We say that $(A,V,\psi_V^A,\sigma_V^A)$ satisfies the twisted condition if

$$(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\sigma_V^A\otimes A)=(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A). \tag{4}$$

Proposition 3.4. *Let $(A,V,\psi_V^A,\sigma_V^A)$ satisfying the twisted condition. Then the following equalities hold:*

$$(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nabla_{A\otimes V})=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V), \tag{5}$$

$$\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nabla_{A\otimes V}\otimes V)=\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\sigma_V^A). \tag{6}$$

Proof. The proof for equality (5) is the following:

$$\begin{aligned}
&(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nabla_{A\otimes V}) \\
&= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes ((\mu_A \otimes V) \circ (A \otimes (\psi_V^A \circ (V \otimes \eta_A))))) \\
&= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \circ (\psi_V^A \otimes (\psi_V^A \circ (V \otimes \eta_A))) \\
&= (\mu_A \otimes V) \circ (A \otimes \mu_A \otimes V) \circ (A \otimes A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \circ (\psi_V^A \otimes (\psi_V^A \circ (V \otimes \eta_A))) \\
&= (\mu_A \otimes V) \circ (A \otimes \mu_A \otimes V) \circ (A \otimes A \otimes \psi_V^A) \circ (A \otimes \sigma_V^A \otimes A) \circ (\psi_V^A \otimes V \otimes \eta_A) \\
&= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V).
\end{aligned}$$

The first equality follows by definition, the second one by (1), the third one by the associativity of μ_A, the fourth one by the twisted condition and finally the fifth one by the associativity of μ_A.

On the other hand, the proof for identity (6) is

$$\begin{aligned}
&\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nabla_{A\otimes V} \otimes V) \\
&= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes (\psi_V^A \circ (V \otimes \eta_A)) \otimes V) \\
&= (\mu_A \otimes V) \circ (A \otimes (\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ ((\psi_V^A \circ (V \otimes \eta_A)) \otimes V))) \\
&= (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \circ (A \otimes V \otimes (\nabla_{A\otimes V} \circ (\eta_A \otimes V))) \\
&= (\mu_A \otimes V) \circ (A \otimes ((\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A))) \circ (A \otimes V \otimes V \otimes \eta_A) \\
&= (\mu_A \otimes V) \circ (A \otimes (\nabla_{A\otimes V} \circ \sigma_V^A)) \\
&= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (V \otimes \sigma_V^A),
\end{aligned}$$

where to derive the first equation we used the definition of $\nabla_{A\otimes V}$, the second one is a consequence of the left A-linearity of $\nabla_{A\otimes V}$ and the third one is a consequence of (5). The fourth one follows by equality $\nabla_{A\otimes V} \circ (\eta_A \otimes V) = \psi_V^A \circ (V \otimes \eta_A)$ and the fifth one by (4). The last one is a consequence of the linear properties of $\nabla_{A\otimes V}$. □

The following definition contains the notion of crossed product system introduced in [14].

Definition 3.5. An algebra A and and object V together with two morphisms

$$\psi_V^A : V \otimes A \to A \otimes V, \qquad \sigma_V^A : V \otimes V \to A \otimes V$$

is called a crossed product system if it satisfies (1), (5) and (6).

Obviously, every crossed product system is an example of the quadruples considered in this paper and, by Proposition 3.4, we obtain that every quadruple $(A, V, \psi_V^A, \sigma_V^A)$ satisfying the twisted condition is a crossed product system. As a consequence we have that under twisted conditions equalities (5) and (6) are redundant in the definition of crossed product system.

Definition 3.6. We say that $(A, V, \psi_V^A, \sigma_V^A)$ satisfies the cocycle condition if

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A). \tag{7}$$

Proposition 3.7. *For a quadruple $(A, V, \psi_V^A, \sigma_V^A)$ that satisfies the twisted and the cocycle conditions there exists a morphism $\tau_V^A : V \otimes V \to A \otimes V$ such that $(A, V, \psi_V^A, \tau_V^A)$ satisfies the equality $\nabla_{A\otimes V} \circ \tau_V^A = \tau_V^A$, the twisted and the cocycle conditions.*

Proof. Take $\tau_V^A = \nabla_{A\otimes V} \circ \sigma_V^A$. Then (1) holds for $(A, V, \psi_V^A, \tau_V^A)$ and $\nabla_{A\otimes V} \circ \tau_V^A = \tau_V^A$.

The twisted condition follows by:

$$\begin{aligned}
(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\tau_V^A \otimes A) &= (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A) \\
&= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A) \\
&= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nabla_{A\otimes V}) \circ (V \otimes \psi_V^A) \\
&= (\mu_A \otimes V) \circ (A \otimes \tau_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A)
\end{aligned}$$

where we used Lemma 3.1 to obtain the first equality, in the second one we applied (4), the third one is a consequence of (3) and the last one of (5) and the left A-linearity of $\nabla_{A\otimes V}$.

Finally, using equality (5), the cocycle condition for $(A, V, \psi_V^A, \sigma_V^A)$ and Eq. (6):

$$\begin{aligned}
(\mu_A \otimes V) \circ (A \otimes \tau_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \tau_V^A) &= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A) \\
&= \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V) \\
&= (\mu_A \otimes V) \circ (A \otimes \tau_V^A) \circ (\tau_V^A \otimes V)
\end{aligned}$$

and hence the cocycle condition is satisfied for $(A, V, \psi_V^A, \tau_V^A)$. □

By virtue of Proposition 3.7 we will consider from now on, and without loss of generality, that

$$\nabla_{A\otimes V}\circ\sigma_V^A=\sigma_V^A$$

for all quadruples $(A, V, \psi_V^A, \sigma_V^A)$.

In the following proposition sufficient conditions are given for $A\times V$ and $A\otimes V$ to be objects with an associative product.

Proposition 3.8. *For the quadruple* $(A, V, \psi_V^A, \sigma_V^A)$ *define the product*

$$\mu_{A\otimes V}=(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V) \tag{8}$$

and let $\mu_{A\times V}$ *be the product*

$$\mu_{A\times V}=p_{A\otimes V}\circ\mu_{A\otimes V}\circ(i_{A\otimes V}\otimes i_{A\otimes V}). \tag{9}$$

Then if ψ_V^A *and* σ_V^A *satisfy the twisted and the cocycle condition,* $\mu_{A\otimes V}$ *is an associative product that it is normalized with respect to* $\nabla_{A\otimes V}$. *As a consequence* $\mu_{A\times V}$ *is also an associative product.*

Proof. If $(A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted condition we know that $(A, V, \psi_V^A, \sigma_V^A)$ is a crossed product system. Then, the associativity of $\mu_{A\otimes V}$ and $\mu_{A\times V}$ follows by Proposition 2.7 of [14].

On the other hand, as a consequence of (1) we obtain:

$$(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\nabla_{A\otimes V}\otimes A)=(\mu_A\otimes V)\circ(A\otimes\psi_V^A). \tag{10}$$

To prove this assertion compute:

$$\begin{aligned}(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\nabla_{A\otimes V}\otimes A)&=(\mu_A\otimes V)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\psi_V^A\otimes A)))\circ(A\otimes V\otimes\eta_A\otimes A)\\&=(\mu_A\otimes V)\circ(A\otimes(\psi_V^A\circ(V\otimes\mu_A)\circ(V\otimes\eta_A\otimes A)))\\&=(\mu_A\otimes V)\circ(A\otimes\psi_V^A).\end{aligned}$$

As $\nabla_{A\otimes V}$ is a morphism of left A-modules we obtain:

$$\nabla_{A\otimes V}\circ\mu_{A\otimes V}=(\mu_A\otimes V)\circ(\mu_A\otimes(\nabla_{A\otimes V}\circ\sigma_V^A))\circ(A\otimes\psi_V^A\otimes V).$$

Hence for $\nabla_{A\otimes V}\circ\sigma_V^A=\sigma_V^A$ equality $\nabla_{A\otimes V}\circ\mu_{A\otimes V}=\mu_{A\otimes V}$ is satisfied.

Finally consider

$$\begin{aligned}\mu_{A\otimes V}\circ(\nabla_{A\otimes V}\otimes\nabla_{A\otimes V})&=\mu_{A\otimes V}\circ(A\otimes V\otimes\nabla_{A\otimes V})\\&=(\mu_A\otimes V)\circ(\mu_A\otimes(\nabla_{A\otimes V}\circ\sigma_V^A))\circ(A\otimes\psi_V^A\otimes V)\end{aligned}$$

as a consequence of (10) and (5). Then if $\nabla_{A\otimes V}\circ\sigma_V^A=\sigma_V^A$ we have

$$\mu_{A\otimes V}\circ(\nabla_{A\otimes V}\otimes\nabla_{A\otimes V})=\mu_{A\otimes V}$$

and the product is normalized. □

Definition 3.9. If $(A, V, \psi_V^A, \sigma_V^A)$ satisfies (4), i.e., the twisted condition, and (7), that is, the cocycle condition we say that $(A\otimes V, \mu_{A\otimes V})$ is a weak crossed product.

Our next task is to characterize weak crossed products with preunit.

Remark 3.10. Let A be an algebra and V an object in $\mathcal{C}$. If $\mu_{A\otimes V}$ is an associative product defined in $A\otimes V$ with preunit ν and such that it is left A-linear and normalized with respect to $\nabla_{A\otimes V}^{\nu}$ the morphism

$$\beta_\nu : A\to A\otimes V,\ \ \beta_\nu=(\mu_A\otimes V)\circ(A\otimes\nu)$$

is multiplicative and left A-linear. Indeed, the left A-linearity is a consequence of the associativity of μ_A, and to prove that it is multiplicative compute:

$$\begin{aligned}\mu_{A\otimes V}\circ(\beta_\nu\otimes\beta_\nu)&=(\mu_A\otimes V)\circ(A\otimes\mu_{A\otimes V})\circ(A\otimes\nu\otimes((\mu_A\otimes V)\circ(A\otimes\nu)))\\&=(\mu_A\otimes V)\circ(A\otimes\mu_{A\otimes V})\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\nu))\otimes\nu)\\&=(\mu_A\otimes V)\circ(\mu_A\otimes(\mu_{A\otimes V}\circ(\nu\otimes\nu)))\\&=\beta_\nu\circ\mu_A\end{aligned}$$

using in the first equality that $\mu_{A\otimes V}$ is left A-linear, in the second one the properties of the preunit, in the third one using again that $\mu_{A\otimes V}$ is of left A-modules and the last one is a consequence of Remark 2.4.

Although β_ν is not an algebra morphism, because $A\otimes V$ is not an algebra, we have that $\beta_\nu\circ\eta_A=\nu$, and thus the morphism

$$\bar{\beta}_\nu=p_{A\otimes V}^{\nu}\circ\beta_\nu : A\to A\times V$$

is an algebra morphism, where $A \times V = \mathrm{Im}(\nabla^{\nu}_{A\otimes V})$ and $p^{\nu}_{A\otimes V}$ is the projection associated to $\nabla^{\nu}_{A\otimes V}$. Indeed, it is multiplicative because:

$$\begin{aligned}\mu_{A\times V} \circ (\bar{\beta_{\nu}} \otimes \bar{\beta_{\nu}}) &= p^{\nu}_{A\otimes V} \circ \mu_{A\otimes V} \circ (\nabla^{\nu}_{A\otimes V} \otimes \nabla^{\nu}_{A\otimes V}) \circ (\beta_{\nu} \otimes \beta_{\nu}) \\ &= p^{\nu}_{A\otimes V} \circ \nabla^{\nu}_{A\otimes V} \circ \mu_{A\otimes V} \circ (\beta_{\nu} \otimes \beta_{\nu}) \\ &= p^{\nu}_{A\otimes V} \circ \beta_{\nu} \circ \mu_A \\ &= \bar{\beta_{\nu}} \circ \mu_A.\end{aligned}$$

And as $\beta_{\nu} \circ \eta_A = \nu$ we have that:

$$\bar{\beta_{\nu}} \circ \eta_A = p^{\nu}_{A\otimes V} \circ \nu = \eta_{A\times V}$$

and hence $\bar{\beta_{\nu}}$ is a morphism of algebras.

In the following theorem we give a characterization of weak crossed products with preunit:

Theorem 3.11. *Let A be an algebra, V an object and $m_{A\otimes V} : A \otimes V \otimes A \otimes V \to A \otimes V$ a morphism of left A-modules. Then the following statements are equivalent:*

(i) *The product $m_{A\otimes V}$ is associative with preunit ν and normalized with respect to $\nabla^{\nu}_{A\otimes V}$.*

(ii) *There exist morphisms $\psi^A_V : V \otimes A \to A \otimes V$, $\sigma^A_V : V \otimes V \to A \otimes V$ and $\nu : k \to A \otimes V$ such that if $\mu_{A\otimes V}$ is the product defined in (8), the pair $(A \otimes V, \mu_{A\otimes V})$ is a weak crossed product with $m_{A\otimes V} = \mu_{A\otimes V}$ satisfying:*

$$(\mu_A \otimes V) \circ (A \otimes \sigma^A_V) \circ (\psi^A_V \otimes V) \circ (V \otimes \nu) = \nabla_{A\otimes V} \circ (\eta_A \otimes V) \tag{11}$$

$$(\mu_A \otimes V) \circ (A \otimes \sigma^A_V) \circ (\nu \otimes V) = \nabla_{A\otimes V} \circ (\eta_A \otimes V) \tag{12}$$

$$(\mu_A \otimes V) \circ (A \otimes \psi^A_V) \circ (\nu \otimes A) = \beta_{\nu}, \tag{13}$$

where β_{ν} is the morphism defined in Remark 3.10. In this case ν is a preunit for $\mu_{A\otimes V}$, the idempotent morphism of the weak crossed product $\nabla_{A\otimes V}$ is the idempotent $\nabla^{\nu}_{A\otimes V}$, and we say that $(A \otimes V, \mu_{A\otimes V})$ is a weak crossed product with preunit ν.

Proof. If $m_{A\otimes V}$ is an associative product with preunit ν define:

$$\psi^A_V = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \beta_{\nu}) \tag{14}$$

$$\sigma^A_V = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V) \tag{15}$$

where β_{ν} is the morphism introduced in Remark 3.10. First observe that as a consequence of being $m_{A\otimes V}$ of left A-modules and equality $\beta_{\nu} \circ \eta_A = \nu$:

$$\nabla_{A\otimes V} = m_{A\otimes V} \circ (A \otimes V \otimes \nu) = \nabla^{\nu}_{A\otimes V},$$

and then, by (i) of Proposition 2.5, the morphism $\nabla_{A\otimes V}$ is multiplicative and $m_{A\otimes V} \circ (A \otimes V \otimes \nabla_{A\otimes V}) = \nabla_{A\otimes V} \circ m_{A\otimes V}$.

Now, for being β_{ν} multiplicative:

$$\begin{aligned}(\mu_A \otimes V) \circ (A \otimes \psi^A_V) \circ (\psi^A_V \otimes A) &= m_{A\otimes V} \circ (\eta_A \otimes V \otimes (m_{A\otimes V} \circ (\beta_{\nu} \otimes \beta_{\nu}))) \\ &= m_{A\otimes V} \circ (\eta_A \otimes V \otimes (\beta_{\nu} \circ \mu_A)) \\ &= \psi^A_V \circ (V \otimes \mu_A)\end{aligned}$$

equality (1) is satisfied.

As $m_{A\otimes V}$ is a normalized product and $\nabla^{\nu}_{A\otimes V} = \nabla_{A\otimes V}$ we obtain that $\nabla_{A\otimes V} \circ \sigma^A_V = \sigma^A_V$. To complete the proof for $(A \otimes V, m_{A\otimes V})$ to be a weak crossed product just consider the associativity of $m_{A\otimes V}$, and using that it is a morphism of left A-modules and the properties of the preunit compute on the one hand:

$$\begin{aligned}&(\mu_A \otimes V) \circ (A \otimes \sigma^A_V) \circ (\psi^A_V \otimes V) \circ (V \otimes \sigma^A_V) \\ &= m_{A\otimes V} \circ (m_{A\otimes V} \otimes \eta_A \otimes V) \circ (\eta_A \otimes V \otimes ((\mu_A \otimes V) \circ (A \otimes \nu)) \otimes V) \circ (V \otimes (m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V))) \\ &= m_{A\otimes V} \circ (\eta_A \otimes V \otimes ((\mu_A \otimes V) \circ (A \otimes m_{A\otimes V}) \circ (A \otimes \nu \otimes \eta_A \otimes V))) \circ (V \otimes (m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V))) \\ &= m_{A\otimes V} \circ ((m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V)) \otimes \eta_A \otimes V)\end{aligned}$$

and on the other hand by similar arguments:

$$\begin{aligned}(\mu_A \otimes V) \circ (A \otimes \sigma^A_V) \circ (\sigma^A_V \otimes V) &= (\mu_A \otimes V) \circ (A \otimes (m_{A\otimes V} \circ (\eta_A \otimes V \otimes A \otimes V))) \\ &\quad \circ ((m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V)) \otimes \eta_A \otimes V) \\ &= m_{A\otimes V} \circ ((m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V)) \otimes \eta_A \otimes V).\end{aligned}$$

Thus $(A, V, \psi_V^A, \sigma_V^A)$ satisfies equality (7), i.e., the cocycle condition. The twisted condition follows by:

$$\begin{aligned}
&(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes m_{A\otimes V}) \circ (V \otimes \beta_\nu \otimes \eta_A \otimes V) \circ (V \otimes m_{A\otimes V}) \circ (V \otimes \eta_A \otimes V \otimes \beta_\nu) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \mu_A \otimes V) \circ (V \otimes A \otimes m_{A\otimes V}) \circ (V \otimes A \otimes \eta_A \otimes V \otimes \nu) \\
&\qquad \circ (V \otimes m_{A\otimes V}) \circ (V \otimes \eta_A \otimes V \otimes \beta_\nu) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes (\nabla_{A\otimes V} \circ m_{A\otimes V})) \circ (V \otimes \eta_A \otimes V \otimes \beta_\nu) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes m_{A\otimes V}) \circ (V \otimes \eta_A \otimes V \otimes \beta_\nu) \\
&\quad = (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A).
\end{aligned}$$

To check equality (11) calculate:

$$\begin{aligned}
&(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nu) \\
&\quad = m_{A\otimes V} \circ (m_{A\otimes V} \otimes \eta_A \otimes V) \circ (\eta_A \otimes V \otimes \beta_\nu \otimes V) \circ (V \otimes \nu) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \mu_A \otimes V) \circ (V \otimes A \otimes m_{A\otimes V}) \circ (V \otimes A \otimes \nu \otimes \eta_A \otimes V) \circ (V \otimes \nu) \\
&\quad = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \mu_A \otimes V) \circ (V \otimes A \otimes m_{A\otimes V}) \circ (V \otimes A \otimes \eta_A \otimes V \otimes \nu) \circ (V \otimes \nu) \\
&\quad = \nabla_{A\otimes V} \circ (\eta_A \otimes V)
\end{aligned}$$

where we used that $m_{A\otimes V}$ is associative, of left A-modules and the properties of the preunit. By similar deductions:

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nu \otimes V) = m_{A\otimes V} \circ (\nu \otimes \eta_A \otimes V) = \nabla_{A\otimes V} \circ (\eta_A \otimes V),$$

so (12) holds. Finally:

$$\begin{aligned}
(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\nu \otimes A) &= m_{A\otimes V} \circ (\nu \otimes \beta_\nu) \\
&= (\mu_A \otimes V) \circ (A \otimes (m_{A\otimes V} \circ (\nu \otimes \nu))) \\
&= \beta_\nu
\end{aligned}$$

and assertion (13) is verified.

Then it just remains proving that $m_{A\otimes V} = \mu_{A\otimes V}$. But using the associativity of $\mu_{A\otimes V}$, its left A-linearity and that it is normalized we have:

$$\begin{aligned}
\mu_{A\otimes V} &= (\mu_A \otimes V) \circ (A \otimes \mu_A \otimes V) \circ (A \otimes A \otimes m_{A\otimes V}) \circ (A \otimes A \otimes \eta_A \otimes V \otimes \eta_A \otimes V) \\
&\quad \circ (A \otimes m_{A\otimes V} \otimes V) \circ (A \otimes \eta_A \otimes V \otimes \beta_\nu \otimes V) \\
&= m_{A\otimes V} \circ (m_{A\otimes V} \otimes \eta_A \otimes V) \circ (A \otimes V \otimes \beta_\nu \otimes V) \\
&= m_{A\otimes V} \circ (A \otimes V \otimes m_{A\otimes V}) \circ (A \otimes V \otimes \beta_\nu \otimes \eta_A \otimes V) \\
&= m_{A\otimes V} \circ (A \otimes V \otimes \mu_A \otimes V) \circ (A \otimes V \otimes A \otimes m_{A\otimes V}) \circ (A \otimes V \otimes A \otimes \nu \otimes \eta_A \otimes V) \\
&= m_{A\otimes V} \circ (A \otimes V \otimes m_{A\otimes V}) \circ (A \otimes V \otimes A \otimes V \otimes \nu) \\
&= m_{A\otimes V} \circ (A \otimes V \otimes \nabla_{A\otimes V}) \\
&= \nabla_{A\otimes V} \circ m_{A\otimes V} \circ (A \otimes V \otimes \nabla_{A\otimes V}) \\
&= \nabla_{A\otimes V} \circ m_{A\otimes V} \circ (\nabla_{A\otimes V} \otimes \nabla_{A\otimes V}) \\
&= m_{A\otimes V}
\end{aligned}$$

where we also used the properties of the preunit. Then the pair $(A \otimes V, m_{A\otimes V})$ is a weak crossed product with preunit ν.

Conversely suppose that $(A \otimes V, m_{A\otimes V})$ is a weak crossed product satisfying (11)–(13) such that $m_{A\otimes V} = \mu_{A\otimes V}$. Therefore $m_{A\otimes V}$ is an associative normalized product. To see that ν is a preunit first compute:

$$\mu_{A\otimes V} \circ (A \otimes V \otimes \nu) = (\mu_A \otimes V) \circ (A \otimes (\nabla_{A\otimes V} \circ (\eta_A \otimes V))) = \nabla_{A\otimes V}$$

using equality (11).

Now check:

$$\begin{aligned}
\mu_{A\otimes V} \circ (\nu \otimes A \otimes V) &= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (((\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\nu \otimes A)) \otimes V) \\
&= (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ ((\nabla_{A\otimes V} \otimes V) \circ (\beta_\nu \otimes V)) \\
&= (\mu_A \otimes V) \circ (A \otimes ((\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nu \otimes V))) \\
&= (\mu_A \otimes V) \circ (A \otimes (\nabla_{A\otimes V} \circ (\eta_A \otimes V))) \\
&= \nabla_{A\otimes V},
\end{aligned}$$

by properties (12), (13) and the left A-linearity of $\nabla_{A\otimes V}$. Finally:

$$\begin{aligned}\mu_{A\otimes V}\circ(A\otimes V\otimes(\mu_{A\otimes V}\circ(\nu\otimes\nu))) &= \mu_{A\otimes V}\circ((\mu_{A\otimes V}\circ(A\otimes V\otimes\nu))\otimes\nu)\\ &= \mu_{A\otimes V}\circ(\nabla_{A\otimes V}\otimes\nu)\\ &= \nabla_{A\otimes V}\circ\nabla_{A\otimes V}\\ &= \nabla_{A\otimes V}\end{aligned}$$

as a consequence of equality $\mu_{A\otimes V}\circ(A\otimes V\otimes\nu)=\nabla_{A\otimes V}$ and the idempotent character of $\nabla_{A\otimes V}$. Hence we obtain that ν is a preunit and $\nabla_{A\otimes V}=\nabla^{\nu}_{A\otimes V}$. □

As a corollary of Theorem 3.11 and Proposition 2.5 we obtain:

Corollary 3.12. *If $(A\otimes V,\mu_{A\otimes V})$ is a weak crossed product with preunit ν, then $A\times V$ is an algebra with the product defined in (9) and unit $\eta_{A\times V}=p_{A\otimes V}\circ\nu$.*

Remark 3.13. Note that if ν is not only a preunit but also a unit, $\nabla_{A\otimes V}=id_{A\otimes V}$ and $A\otimes V$ is an algebra. Conversely if $(A\otimes V,\mu_{A\otimes V})$ is a weak crossed product with preunit ν and $\nabla_{A\otimes V}=id_{A\otimes V}$ then ν is a unit and $A\otimes V$ is an algebra.

Remark 3.14. In this section we develop a theory of crossed products in $A\otimes V$. In a symmetric way it is possible to obtain the same results for $V\otimes A$. In this case we must work with morphisms $\psi_A^V: A\otimes V\to V\otimes A$, $\sigma_A^V: V\otimes V\to V\otimes A$ and the associated idempotent, $\nabla_{V\otimes A}: V\otimes A\to V\otimes A$, is defined by

$$\nabla_{V\otimes A}=(V\otimes\mu_A)\circ((\psi_A^V\circ(\eta_A\otimes V))\otimes A). \tag{16}$$

The induced product $\mu_{V\otimes A}: V\otimes A\otimes V\otimes A\to V\otimes A$ is

$$\mu_{V\otimes A}=(V\otimes\mu_A)\circ(\sigma_A^V\otimes\mu_A)\circ(V\otimes\psi_A^V\otimes A) \tag{17}$$

and the preunit is a morphism $\nu: K\to V\otimes A$.

Weak crossed products provide a general setting to study several examples of crossed products. Here we recall some of the most important ones.

Example 3.15. The first example was developed by Brzeziński in [8] working in a category $\mathcal{C}$ of K-vector spaces where in this case K denotes a field. Suppose that A is an algebra and V an object in $\mathcal{C}$ equipped with a morphism $e: K\to V$ and suppose also that there exist morphisms $\tilde{\psi}: V\otimes A\to A\otimes V$ and $\tilde{\sigma}: V\otimes V\to A\otimes V$ that satisfy condition (1) and equalities (4), (7) and

(a) $\tilde{\psi}\circ(e\otimes A)=A\otimes e$,
(b) $\tilde{\psi}\circ(V\otimes\eta_A)=\eta_A\otimes V$,
(c) $\tilde{\sigma}\circ(e\otimes V)=\tilde{\sigma}\circ(V\otimes e)=\eta_A\otimes V$.

Then $(A,V,\psi_V^A=\tilde{\psi},\sigma_V^A=\tilde{\sigma})$ induces a weak crossed product $(A\otimes V,\mu_{A\otimes V})$ with preunit $\nu=\eta_A\otimes e$ and $\nabla_{A\otimes V}=id_{A\otimes V}$. Indeed, conditions (1), (4) and (7) are satisfied by the definition of crossed product. Moreover as $\tilde{\psi}\circ(V\otimes\eta_A)=\eta_A\otimes V$ it is clear that $\nabla_{A\otimes V}=id_{A\otimes V}$ and then $\nabla_{A\otimes V}\circ\tilde{\sigma}=\tilde{\sigma}$ holds. Now compute:

$$(\mu_A\otimes V)\circ(A\otimes\tilde{\sigma})\circ(\tilde{\psi}\otimes V)\circ(V\otimes\nu)=\tilde{\sigma}\circ(V\otimes e)=\eta_A\otimes V,$$

thus (11) holds. Condition (12) is straightforward and (13) follows by:

$$(\mu_A\otimes V)\circ(A\otimes\tilde{\psi})\circ(\nu\otimes A)=\tilde{\psi}\circ(e\otimes A)=A\otimes e=\beta_\nu.$$

Then, as a corollary of Theorem 3.11, we obtain Proposition 2.1 of [8], i.e., the conditions which allow one to build an algebra structure, with unit $\nu=\eta_A\otimes e$ and product $\mu_{A\otimes V}$, on a tensor product of an algebra A and a vector space V.

Of course the last results remain valid if $\mathcal{C}$ is a category of modules over a commutative ring and then, as a particular instance of the crossed product constructed in this example, we obtain the crossed products defined by Blattner, Cohen, Montgomery, Doi and Takeuchi. Also, the twisted tensor products or matched pairs, studied by Cap, Schichl, Vanzura and Tambara [22,23] are examples of Brzeziński's crossed product. On the other hand, this notion of crossed products is needed in the theory of braided Hopf crossed products developed by J.A. Gucccione and J.J. Guccione in [10], which includes the classical type (Blattner–Cohen–Montgomery, Doi–Takeuchi) and the automorphism Ore extension type. Finally, Brzeziński's theory can be generalized, in a straightforward form, to the context of braided monoidal categories, and then, we obtain as examples, the crossed products by braided groups defined by Majid and Bespalov in [5,24,25,9].

Example 3.16. The second example was introduced by Caenepeel and De Groot in [12]. Let K be a commutative ring and let $\mathcal{C}$ be the category of modules over K. Let A and B be algebras in $\mathcal{C}$, and let $R : B\otimes A \to A\otimes B$ be a morphism that satisfies:

(a) $R\circ(\mu_B\otimes A) = (A\otimes\mu_B)\circ(R\otimes B)\circ(B\otimes R)$
(b) $R\circ(B\otimes\mu_A) = (\mu_A\otimes B)\circ(A\otimes R)\circ(R\otimes B)$
(c) $R\circ(\eta_B\otimes A) = (\mu_A\otimes B)\circ(A\otimes(R\circ(\eta_B\otimes\eta_A)))$
(d) $R\circ(B\otimes\eta_A) = (A\otimes\mu_B)\circ((R\circ(\eta_B\otimes\eta_A))\otimes B)$.

Under these circumstances, the morphism $\mu_{A\otimes B} : A\otimes B\otimes A\otimes B \to A\otimes B$ given by

$$\mu_{A\otimes B} = (\mu_A\otimes\mu_B)\circ(A\otimes R\otimes B)$$

is an associative product with preunit $\nu = \nabla_{A\otimes B}\circ(\eta_A\otimes\eta_B)$. By virtue of Theorem 3.11, as $\mu_{A\otimes B}$ is left A-linear, this product is induced by a weak crossed product structure.

Observe that in this case the morphism $\beta_\nu : A\to A\otimes B$ is given by

$$\beta_\nu = (\mu_A\otimes B)\circ(A\otimes\eta_A\otimes\eta_B) = A\otimes\eta_B.$$

Therefore:

$$\psi_B^A = \mu_{A\otimes B}\circ(\eta_A\otimes B\otimes A\otimes\eta_B) = R$$

and

$$\sigma_B^A = \mu_{A\otimes B}\circ(\eta_A\otimes B\otimes\eta_A\otimes B) = R\circ(\mu_B\otimes\eta_A).$$

Note that in this case

$$\nabla_{A\otimes B} = (\mu_A\otimes B)\circ(A\otimes(R\circ(B\otimes\eta_A))) = \mu_{A\otimes B}\circ(A\otimes\eta_B\otimes\eta_A\otimes B)$$

and then $A\times B\neq A\otimes B$.

The following example, that gives a more general weak crossed product, was introduced in [14] for objects living in a strict monoidal category $\mathcal{C}$ where every idempotent morphism splits.

Example 3.17. Let $(A, V, \psi_V^A, \sigma_V^A)$ be a crossed product system. If there exists a morphism $\eta_V : k\to V$ that satisfies

$$\nabla_{A\otimes V}\circ(A\otimes\eta_V) = \psi_V^A\circ(\eta_V\otimes A)$$

we say that $(A, V, \psi_V^A, \sigma_V^A)$ is a crossed product system with unit. If moreover it satisfies

$$\nabla_{A\otimes V}\circ\sigma_V^A\circ(\eta_V\otimes V) = \nabla_{A\otimes V}\circ\sigma_V^A\circ(V\otimes\eta_V) = \nabla_{A\otimes V}\circ(\eta_A\otimes V)$$

the crossed product system is said to be normal. As in the previous cases $\nabla_{A\otimes V}$ denotes the idempotent morphism $\nabla_{A\otimes V} = (\mu_A\otimes V)\circ(A\otimes(\psi_V^A\circ(V\otimes\eta_A))) : A\otimes V\to A\otimes V$.

Suppose that the normal crossed product system with unit satisfies the twisted and the cocycle conditions. Set $\nu = \nabla_{A\otimes V}\circ(\eta_A\otimes\eta_V)$. Then $(A\otimes V, \mu_{A\otimes V})$ is a weak crossed product with preunit ν. We just have to prove the properties related to the preunit, as the rest of them are assumed.

To prove (11) compute:

$$\begin{aligned}(\mu_A\circ V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nu) &= (\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\eta_A\otimes\eta_V)\\ &= (\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ((\nabla_{A\otimes V}\circ(\eta_A\otimes V))\otimes\eta_V)\\ &= (\mu_A\otimes V)\circ(\eta_A\otimes(\sigma_V^A\circ(V\otimes\eta_V)))\\ &= \nabla_{A\otimes V}\circ(\eta_A\otimes V)\end{aligned}$$

using (5) and equation $\nabla_{A\otimes V}\circ\sigma_V^A = \sigma_V^A$ (we can assume this condition without loss of generality) to obtain the first equality, the second one is derived using $\nabla_{A\otimes V}\circ(\eta_A\otimes V) = \psi_V^A\circ(V\otimes\eta_A)$, the third one is a consequence of (6) and the fourth one of the normality condition. Equality (12) is straightforward using (6) and the normal property, and (13) follows by:

$$\begin{aligned}(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\nu\otimes A) &= (\mu_A\otimes V)\circ(\eta_A\otimes(\psi_V^A\circ(\eta_V\otimes A)))\\ &= \nabla_{A\otimes V}\circ(A\otimes\eta_V)\\ &= \nabla_{A\otimes V}\circ((\mu_A\circ(A\otimes\eta_A))\otimes\eta_V)\\ &= \beta_\nu\end{aligned}$$

using equality (10), the unit condition and the left A-linearity of $\nabla_{A\otimes V}$.

Observe that if we also consider $\psi_V^A\circ(V\otimes\eta_A) = \eta_A\otimes V$, we are in the conditions of Example 3.15. Hence normal crossed product systems with unit that satisfy the twisted and the weak cocycle conditions are a generalization of crossed products described in [8,9].

Crossed product systems have been used in [14] to obtain a crossed product induced by a weak cleft extension [17], so the classical results of Blattner and Montgomery [26] and of Doi and Takeuchi [2] that relate cleft extensions and crossed products are generalized to weak cleft extensions of weak entwining structures. Also the crossed products defined in this example covers the products associated to projections of weak Hopf algebras (see [13,16,19]). The study of these products was the main motivation of [16] where the authors introduced an associative product in $A \otimes H$ with preunit obtained composing $\eta_A \otimes \eta_H$ with a suitable idempotent $\nabla_{A\otimes V}$, where H is a weak Hopf algebra and A an H-comodule algebra living in a strict monoidal category with splitting of idempotents. To generalize Blattner and Montgomery and Doi and Takeuchi theorem using this crossed product it is necessary to impose the condition $h \circ \eta_H = \eta_A$ on the associated cleaving morphism h (Theorem 2.7 of [16]). Recall that in the Hopf algebra case this property does not suppose any restriction. In the weak Hopf algebra case it is not possible to prove this assertion in general, and hence it seems necessary to change the definition of weak crossed product to obtain one whose preunit is not $\nabla_{A\otimes V} \circ (\eta_A \otimes \eta_H)$ necessarily. In [20] this new crossed product is defined and in the final section of this paper we will see that it is a particular instance of the general theory of weak crossed product developed in this paper.

Example 3.18. Let $\mathcal{C}$ be a category. The category of endofunctors of $\mathcal{C}$ is a strict monoidal category with the composition of functors, denoted by $\circledcirc$, as the product and the identity functor as the unit. We denote this category by $End(\mathcal{C})$. The morphisms in $End(\mathcal{C})$ are natural transformations between endofunctors and we denote the composition (the vertical composition) of these morphisms by $\circ$. The tensor product of morphisms in $End(\mathcal{C})$ is defined by the horizontal composition of natural transformations and in this paper is denoted by the same symbol used for the composition of functors (see [27] for the details of the horizontal and vertical compositions). Given objects T, S, H and a morphism $\tau : S \to H$, we write $T \circledcirc \tau$ for $id_T \circledcirc \tau$ and $\tau \circledcirc T$ for $\tau \circledcirc id_T$ where id_T denotes the identity morphism for the object T.

A monad on $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta_T : id_{\mathcal{C}} \to T$ (where $id_{\mathcal{C}}$ denotes the identity functor on $\mathcal{C}$) and $\mu_T : T^2 = T \circledcirc T \to T$. These are required to fulfill the following conditions

(a) $\mu_T \circ (T \circledcirc \eta_T) = \mu_T \circ (\eta_T \circledcirc T) = id_T$ (as natural transformations between T and T).
(b) $\mu_T \circ (T \circledcirc \mu_T) = \mu_T \circ (\mu_T \circledcirc T)$ (as natural transformations between $T^3 = T \circledcirc T \circledcirc T$ and T).

Then, a monad on $\mathcal{C}$ can alternatively be defined as an algebra (monoid) in the strict monoidal category $End(\mathcal{C})$.

The notion of wreath was introduced by Lack and Street in [28]. A monad T in $\mathcal{C}$ is a wreath if there exist an object in $S \in End(\mathcal{C})$ and morphisms in $End(\mathcal{C})$, $\lambda : T \circledcirc S \to S \circledcirc T$, $\tau : id_{\mathcal{C}} \to S \circledcirc T$ and $\upsilon : S \circledcirc S \to S \circledcirc T$ satisfying the following conditions:

(a) $(S \circledcirc \mu_T) \circ (\lambda \circledcirc T) \circ (T \circledcirc \lambda) = \lambda \circ (\mu_T \circledcirc S)$,
(b) $\lambda \circ (\eta_T \circledcirc S) = S \circledcirc \eta_T$,
(c) $(S \circledcirc \mu_T) \circ (\tau \circledcirc T) = (S \circledcirc \mu_T) \circ (\lambda \circledcirc T) \circ (T \circledcirc \tau)$,
(d) $(S \circledcirc \mu_T) \circ (\upsilon \circledcirc T) \circ (S \circledcirc \lambda) \circ (\lambda \circledcirc S) = (S \circledcirc \mu_T) \circ (\lambda \circledcirc T) \circ (T \circledcirc \upsilon)$,
(e) $(S \circledcirc \mu_T) \circ (\upsilon \circledcirc T) \circ (S \circledcirc \upsilon) = (S \circledcirc \mu_T) \circ (\upsilon \circledcirc T) \circ (S \circledcirc \lambda) \circledcirc (\upsilon \circledcirc S)$,
(f) $(S \circledcirc \mu_T) \circ (\upsilon \circledcirc T) \circ (S \circledcirc \tau) = S \circledcirc \eta_T = (S \circledcirc \mu_T) \circ (\upsilon \circledcirc T) \circ (S \circledcirc \lambda) \circ (\tau \circledcirc S)$.

If we put $\psi_T^S = \lambda$ and $\sigma_T^S = \upsilon$, we obtain that $(T, S, \psi_T^S, \sigma_T^S)$ is a crossed product system where the associated idempotent defined in (16) is $\nabla_{S\circledcirc T} = id_{S\circledcirc T}$ because λ satisfies the identity (b). Then, the product induced by a wreath (wreath product) defined by

$$\mu_{S\circledcirc T} = (S \circledcirc \mu_T) \circ (\upsilon \circledcirc \mu_T) \circ (S \circledcirc \lambda \circledcirc T)$$

is the one defined in (17) and it is associative because it satisfies (d) (twisted condition) and (e) (cocycle condition). The preunit (in this case is a unit) is $\eta_{S\circledcirc T} = \tau$.

Note that, in this case we do not need that every idempotent splits because the associated idempotent $\nabla_{S\circledcirc T} = id_{S\circledcirc T}$. In any case it is easy to show that if every idempotent splits in $\mathcal{C}$, every idempotent splits in $End(\mathcal{C})$.

Therefore wreath products are examples of weak crossed products with trivial idempotent.

As in the case of weak crossed products it is possible to introduce a theory of wreaths working with a monad T an object in $S \in End(\mathcal{C})$ and morphisms in $End(\mathcal{C})$, $\lambda : S \circledcirc T \to T \circledcirc S$, $\tau : id_{\mathcal{C}} \to T \circledcirc S$ and $\upsilon : S \circledcirc S \to T \circledcirc S$ satisfying the convenient conditions. In this case the wreath product is defined in $T \circledcirc S$. For example, if we are in the conditions of Example 3.15 we have that the monad $T = A \otimes -$ is a wreath where $S = V \otimes -$, $\lambda = \tilde{\psi} \otimes -$, $\tau = \eta_A \otimes e \otimes -$, $\upsilon = \tilde{\sigma} \otimes -$. Then, all the examples of crossed products listed in Example 3.15 are instances of wreath products. We can find a similar construction in [29] although wreathes are not mentioned there.

4. Weak crossed products and weak bialgebras

In this section we relate weak crossed products and crossed products of an algebra A by a weak bialgebra H. Note that in this case we are using not only any object V in the category but a very special one, a weak bialgebra, that has a very rich structure. Due to this structure we will consider a left A-linear right H-colinear multiplication on $A \otimes H$, that will permit us to obtain explicit formulae for ψ_H^A and σ_H^A.

Let $\mathcal{C}$ be a strict symmetric monoidal category with natural isomorphism of symmetry c where every idempotent splits. Remember that by weak Hopf algebras we understand the monoidal generalization of the notion introduced in [11] in a category of vector spaces. Recall that a weak bialgebra H is an algebra-coalgebra $(H, \eta_H, \mu_H, \varepsilon_H, \delta_H)$ (η_H is the unit, μ_H is the product, ε_H is counit and δ_H is the coproduct) in $\mathcal{C}$ such that the following axioms hold:

(a) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,
(b) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,
(c) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) = (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

If moreover, there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying:

(d-1) $\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
(d-2) $\mu_H \circ (\lambda_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
(d-3) $\mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H = \lambda_H$.

we will say that H is a weak Hopf algebra.

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative, leaves the unit η_H and the counit ε_H invariant and if we define the endomorphisms of H, Π_H^L (target morphism), Π_H^R (source morphism), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by $\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$, $\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$, $\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H)$, and $\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$, it is straightforward to show that they are idempotent.

Let A be an algebra and H a weak bialgebra $\mathcal{C}$. Put

$$\delta_{H\otimes H} = (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) : H \otimes H \to H \otimes H \otimes H \otimes H.$$

Consider the following:

- There exists a measuring, that is, a morphism $\varphi_A : H \otimes A \to A$ such that:

$$\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A). \tag{18}$$

- There exists a morphism $\sigma : H \otimes H \to A$ a morphism such that:

$$\sigma \circ ((\mu_H \circ (H \otimes \Pi_H^R)) \otimes H) = \sigma \circ (H \otimes (\mu_H \circ (\Pi_H^R \otimes H))). \tag{19}$$

- The morphism σ is a weak 2-cocycle, that is, it fulfills

$$\begin{aligned} \sigma &= \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes \sigma) \circ (\delta_H \otimes H) \\ &= \mu_A \circ (\sigma \otimes (\varphi_A \circ (\mu_H \otimes \eta_A))) \circ \delta_{H\otimes H} \end{aligned} \tag{20}$$

  and satisfies the 2-cocycle condition

$$\mu_A \circ (\varphi_A \otimes \sigma) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H})) = \mu_A \circ (A \otimes \sigma) \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes H). \tag{21}$$

- There exists $\nu : k \to A \otimes H$ that satisfies:

$$(A \otimes \delta_H) \circ \nu = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\nu \otimes (\delta_H \circ \eta_H)) \tag{22}$$

  and the weak 2-cocycle is normal with respect to ν:

$$\begin{aligned} \varphi_A \circ (H \otimes \eta_A) &= \mu_A \circ (\varphi_A \otimes \sigma) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes \nu) \\ &= \mu_A \circ (A \otimes \sigma) \circ (\nu \otimes H). \end{aligned} \tag{23}$$

- The normal weak 2-cocycle satisfies the twisted condition:

$$\begin{aligned} &\mu_A \circ (A \otimes \sigma) \circ (((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes H) \\ &\circ (H \otimes ((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A))) = \mu_A \circ (A \otimes \varphi_A) \circ (((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes A). \end{aligned} \tag{24}$$

- The morphism β_ν satisfies the equality:

$$\beta_\nu = (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A))) \circ (\nu \otimes A). \tag{25}$$

Recall from [20] that, under the previous conditions, the product $\mu_{A\sharp_\sigma H} : A \otimes H \otimes A \otimes H \to A \otimes H$ defined by:

$$\mu_{A\sharp_\sigma H} = (\mu_A \otimes \mu_H) \circ (\mu_A \otimes \sigma \otimes H \otimes H) \circ (A \otimes \varphi_A \otimes \delta_{H\otimes H}) \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H).$$

is associative, with preunit ν and normalized with respect to the idempotent morphism

$$\nabla_{A\otimes V}^{\nu} = (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes \eta_A))).$$

The product $\mu_{A\sharp_\sigma H}$ will be called the weak crossed product of the algebra A by the weak bialgebra H.

Moreover, $\mu_{A\sharp_\sigma H}$ is a morphism of left A-modules where $\varphi_A = \mu_A \otimes H$, $\varphi_{A\otimes H\otimes A\otimes H} = \varphi_A \otimes A \otimes H$ and it satisfies the identity

$$\rho_{A\otimes H} \circ \mu_{A\otimes H} = (\mu_{A\otimes H} \otimes H) \circ \rho_{A\otimes H\otimes A\otimes H} \tag{26}$$

where $\rho_{A\otimes H} = A \otimes \delta_H$ and

$$\rho_{A\otimes H\otimes A\otimes H} = (A \otimes H \otimes A \otimes H \otimes \mu_H) \circ (A \otimes H \otimes c_{H,A\otimes H} \otimes H) \circ (\rho_{A\otimes H} \otimes \rho_{A\otimes H}).$$

The following result is the monoidal version of Lemma 3.8 of [20] and shows the transcendence of condition (26).

Lemma 4.1. *Let A be an algebra and H a weak bialgebra, and suppose that $\mu_{A\otimes H} : A \otimes H \otimes A \otimes H \to A \otimes H$ is an associative normalized product with preunit ν.*

Define

$$\varphi_A = (A \otimes \varepsilon_H) \circ \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \beta_\nu)$$

and

$$\sigma = (A \otimes \varepsilon_H) \circ \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H).$$

Then $\mu_{A\otimes H}$ satisfies (26) if and only if the following conditions hold:

(i) $\mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \beta_\nu) = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$
(ii) $\mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H) = (\sigma \otimes \mu_H) \circ \delta_{H\otimes H}$.

In any of these cases, σ satisfies (19) and the next equation is satisfied:

$$\mu_{A\otimes H} \circ (A \otimes (\mu_H \circ (H \otimes \Pi_H^R)) \otimes A \otimes H) = \mu_{A\otimes H} \circ (A \otimes H \otimes A \otimes \mu_H) \circ (A \otimes H \otimes (c_{H,A} \circ (\Pi_H^R \otimes A)) \otimes H). \tag{27}$$

Now we can prove the main theorem of this subsection that shows the equivalence between a general crossed product and a weak crossed product of an algebra by a weak bialgebra:

Theorem 4.2. *Let A be an algebra, H a weak bialgebra and $\nu : k \to A \otimes H$ a morphism satisfying (22). Then the following statements are equivalent:*

(i) *There exist morphisms $\psi_H^A : H \otimes A \to A \otimes H$ and $\sigma_H^A : H \otimes H \to A \otimes H$ such that $(A \otimes H, \mu_{A\otimes H})$ is a weak crossed product with preunit ν and $\mu_{A\otimes H}$ is a morphism of left A-modules satisfying (26).*
(ii) *There exist a measuring $\varphi_A : H \otimes A \to A$ and a weak normal 2-cocycle $\sigma : H \otimes H \to A$ satisfying (19), the twisted condition and $\mu_{A\sharp_\sigma H}$ is an associative product with preunit ν and normalized with respect to the idempotent morphism*

$$\nabla_{A\otimes V}^\nu = (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes \eta_A))).$$

In this case morphism β_ν is given by (25).

Proof. Suppose that $(A \otimes H, \mu_{A\otimes H})$ is a weak crossed product. From the proof of Theorem 3.11 we have that $\sigma_H^A = \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H)$ and $\psi_H^A = \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \beta_\nu)$, where β_ν is the morphism given in Remark 3.10. Define

$$\varphi_A = (A \otimes \varepsilon_H) \circ \psi_H^A = (A \otimes \varepsilon_H) \circ \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \beta_\nu) \tag{28}$$

and

$$\sigma = (A \otimes \varepsilon_H) \circ \sigma_H^A = (A \otimes \varepsilon_H) \circ \mu_{A\otimes H} \circ (\eta_A \otimes H \otimes \eta_A \otimes H). \tag{29}$$

By virtue of Lemma 4.1, it happens that

$$\sigma_H^A = (\sigma \otimes \mu_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \delta_H)$$

and moreover condition (19) holds.

Now using again Lemma 4.1:

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A).$$

As a consequence of this equality φ_A is a measuring. Indeed:

$$\begin{aligned}\varphi_A \circ (H \otimes \mu_A) &= (A \otimes \varepsilon_H) \circ \psi_H^A \circ (H \otimes \mu_A)\\ &= (A \otimes \varepsilon_H) \circ (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes A)\\ &= \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)\end{aligned}$$

where we also used equality (1). To prove that σ is a weak 2-cocycle first compute

$$\begin{aligned}&\mu_A\circ(\varphi_A\otimes\sigma)\circ(H\otimes c_{H,A}\otimes H)\circ(\delta_H\otimes((\sigma\otimes\mu_H)\circ\delta_{H\otimes H}))\\&\quad=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\sigma_H^A)\\&\quad=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\sigma_H^A\otimes H)\\&\quad=\mu_A\circ(A\otimes\sigma)\circ(((\sigma\otimes\mu_H)\circ\delta_{H\otimes H})\otimes H)\end{aligned}$$

using equality (29), so we obtain (21). Now consider

$$\begin{aligned}\mu_A\circ(\sigma\otimes(\varphi_A\circ(\mu_H\otimes\eta_A)))\circ\delta_{H\otimes H}&=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ((\psi_H^A\circ(H\otimes\eta_A))\otimes H)\\&=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ((\nabla_{A\otimes H}\circ(\eta_A\otimes H))\otimes H)\\&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ\sigma_H^A\\&=\sigma\end{aligned}$$

that follows by the definition of $\nabla_{A\otimes H}$, Eq. (6) and equality $\sigma_H^A=\nabla_{A\otimes H}\circ\sigma_H^A$. Moreover

$$\begin{aligned}\mu_A\circ((\varphi_A\circ(H\otimes\eta_A))\otimes\sigma)\circ(\delta_H\otimes H)&=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes(\psi_H^A\circ(H\otimes\eta_A)))\\&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ\sigma_H^A\\&=\sigma\end{aligned}$$

by similar computations. Hence condition (20) is satisfied and σ is a weak 2-cocycle. The twisted condition is a consequence of the expressions for ψ_H^A and σ_H^A in terms of φ_A and σ, and of equality (4):

$$\begin{aligned}&\mu_A\circ(A\otimes\varphi_A)\circ(((\sigma\otimes\mu_H)\circ\delta_{H\otimes H})\otimes A)\\&\quad=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\psi_H^A)\circ(\sigma_H^A\otimes A)\\&\quad=(A\otimes\varepsilon_H)\circ(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\psi_H^A)\\&\quad=\mu_A\circ(A\otimes\sigma)\circ((((\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A))\otimes H)\circ(H\otimes((\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A))).\end{aligned}$$

Equality (23) is proved using (11) and (12) by the same techniques, and we obtain that σ is a normal cocycle. Indeed:

$$\begin{aligned}\varphi_A\circ(H\otimes\eta_A)&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ(\eta_A\otimes H)\\&=(\mu_A\otimes\varepsilon_H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\nu)\\&=\mu_A\circ(\varphi_A\otimes\sigma)\circ(H\otimes c_{H,A}\otimes H)\circ(\delta_H\otimes\nu)\end{aligned}$$

and

$$\begin{aligned}\varphi_A\circ(H\otimes\eta_A)&=(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}\circ(\eta_A\otimes H)\\&=(\mu_A\otimes\varepsilon_H)\circ(A\otimes\sigma_H^A)\circ(\nu\otimes H)\\&=\mu_A\circ(A\otimes\sigma)\circ(\nu\otimes H).\end{aligned}$$

Finally condition (25) is equivalent to (13) by virtue of the expression for ψ_H^A. Hence $\mu_{A\sharp_\sigma H}$ is an associative product with preunit ν and normalized with respect to the idempotent morphism $\nabla_{A\otimes V}^{\nu}$.

Conversely, in light of Theorem 3.11 there exist $\psi_H^A:H\otimes A\to A\otimes H$ and $\sigma_H^A:H\otimes H\to A\otimes H$ such that $(A\otimes H,\mu_{A\otimes H})$ is a weak crossed product with preunit ν where $\mu_{A\otimes H}=\mu_{A\sharp_\sigma H}$.

Note that by the construction of ψ_H^A in the proof of Theorem 3.11 and as $\mu_{A\sharp_\sigma H}$ is of right H-comodules:

$$\begin{aligned}\psi_H^A&=\mu_{A\sharp_\sigma H}\circ(\eta_A\otimes H\otimes\beta_\nu)\\&=(\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A).\end{aligned}$$

And Lemma 4.1 yields:

$$\begin{aligned}\sigma_H^A&=\mu_{A\sharp_\sigma H}\circ(\eta_A\otimes H\otimes\eta_A\otimes H)\\&=(\sigma\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H).\quad\square\end{aligned}$$

Acknowledgements

The authors were supported by Xunta de Galicia (Project: PGIDT07PXB322079PR), Ministerio de Educación (Projects: MTM2007-62427, MTM2006-14908-CO2-01), and FEDER. Moreover, they would like to express their gratitude to the anonymous referee for the helpful comments and interesting questions that helped in improving this article.

Appendix

In this section we present the coalgebra version for the results contained in the previous sections. The proofs are similar to the algebra case and we leave this work for the patient reader.

Definition A.1. Let $\Gamma_C : C \to C$ be an idempotent morphism for an object C in $\mathcal{C}$. Fix $\mathrm{Im}(\Gamma_C) = \bar{C}$, and $i_C : \bar{C} \to C$ and $p_C : C \to \bar{C}$ the injection and the projection associated to Γ_C.

If there exists a morphism $\delta_C : C \to C \otimes C$ that $(\delta_C \otimes C) \circ \delta_C = (C \otimes \delta_C) \circ \delta_C$, i.e., it is a coassociative coproduct on C, then we say that it is conormalized with respect to Γ_C (or in general that it is conormalized) if it satisfies:

$$(\Gamma_C \otimes \Gamma_C) \circ \delta_C = \delta_C = \delta_C \circ \Gamma_C.$$

Lemma A.2. *Suppose that $\Gamma_C : C \to C$ is an idempotent morphism as above, and suppose that $\delta_{\bar{C}} : \bar{C} \to \bar{C} \otimes \bar{C}$ is a coassociative coproduct on $\bar{C}$. Then there exists a coassociative coproduct $\delta_C : C \to C \otimes C$ conormalized with respect to Γ_C such that $\delta_{\bar{C}} = (p_C \otimes p_C) \circ \delta_C \circ i_C$.*

Definition A.3. Let C be an object in $\mathcal{C}$ and $\delta_C : A \to C \otimes C$ a coassociative coproduct. The morphism $\upsilon : C \to K$ is a *precounit* for δ_C if

$$(\upsilon \otimes C) \circ \delta_C = (C \otimes \upsilon) \circ \delta_C = (((\upsilon \otimes \upsilon) \circ \delta_C) \otimes C) \circ \delta_C.$$

Observe that if $(C, \varepsilon_C, \delta_C)$ is a coalgebra, then $\upsilon = \varepsilon_C$ is a precounit for δ_C.

Remark A.4. Note that if C is an object in $\mathcal{C}$ with a coassociative coproduct $\delta_C : C \to C \otimes C$ and precounit υ, then $\upsilon' = (\upsilon \otimes \upsilon) \circ \delta_C$ is also a precounit. Furthermore $(\upsilon' \otimes \upsilon') \circ \delta_C = \upsilon'$. Henceforth we will consider that υ satisfies this condition as it does not suppose any loss of generality.

Proposition A.5. *Let C be an object in $\mathcal{C}$ and $\delta_C : C \to C \otimes C$ a coassociative coproduct. Then the following assertions hold:*

(i) *If $\upsilon : C \to K$ is a precounit for δ_C, the morphism $\Gamma_C^{\upsilon} : C \to C$ given by*

$$\Gamma_C^{\upsilon} = (\upsilon \otimes C) \circ \delta_C$$

is idempotent, comultiplicative and such that $(C \otimes \Gamma_C^{\upsilon}) \circ \delta_C = \delta_C \circ \Gamma_C^{\upsilon}$.

(ii) *If $\upsilon : C \to K$ is a precounit for δ_C, then $(\bar{C}, \varepsilon_{\bar{C}}, \delta_{\bar{C}})$ is a coalgebra, where $\bar{C} = \mathrm{Im}(\Gamma_C^{\upsilon})$, p_C^{υ} and i_C^{υ} are the morphisms such that $\Gamma_C^{\upsilon} = i_C^{\upsilon} \circ p_C^{\upsilon}$, $p_C^{\upsilon} \circ i_C^{\upsilon} = id_{\bar{C}}$, $\delta_{\bar{C}} = (p_C^{\upsilon} \otimes p_C^{\upsilon}) \circ \delta_C \circ i_C^{\upsilon}$ and $\varepsilon_{\bar{C}} = \upsilon \circ i_C^{\upsilon}$.*

(iii) *If $\Gamma_C : C \to C$ is an idempotent morphism and $\bar{C} = \mathrm{Im}(\Gamma_C)$ is a coalgebra, then the morphism $\delta_C = (p_C \otimes p_C) \circ \delta_{\bar{C}} \circ i_C$ is a coassociative coproduct on C with precounit $\upsilon = \varepsilon_{\bar{C}} \circ p_C$. Moreover $\Gamma_C = \Gamma_C^{\upsilon}$.*

(iv) *If υ is a precounit for C, then $\Gamma_C^{\upsilon} = id_C$ if and only if υ is a counit (that is, if C is a coalgebra).*

Let C be a coalgebra and V an object in $\mathcal{C}$. Throughout the rest of the section we will consider that the left C-comodule structures on $C \otimes V$ and $C \otimes V \otimes C \otimes V$ are given by

$$\rho_{C\otimes V} = \delta_C \otimes V, \qquad \rho_{C\otimes V\otimes C\otimes V} = \delta_C \otimes V \otimes C \otimes V.$$

Lemma A.6. *Let C be a coalgebra and V an object. Suppose that there exists a morphism $\chi_V^C : C \otimes V \to V \otimes C$ such that the following equality holds*

$$(\chi_V^C \otimes C) \circ (C \otimes \chi_V^C) \circ (\delta_C \otimes V) = (V \otimes \delta_C) \circ \chi_V^C. \tag{30}$$

The morphism $\Gamma_{C\otimes V} : C \otimes V \to C \otimes V$ defined by

$$\Gamma_{C\otimes V} = (C \otimes V \otimes \varepsilon_C) \circ (C \otimes \chi_V^C) \circ (\delta_C \otimes V) \tag{31}$$

is idempotent. Moreover, $\Gamma_{C\otimes V}$ is a left C-comodule morphism.

From now on we consider quadruples $(C, V, \chi_V^C, \tau_V^C)$ where C, V and χ_V^C satisfy the conditions of Lemma A.6 and $\tau_V^C : C \otimes V \to V \otimes V$ is a morphism in $\mathcal{C}$. For the morphism $\Gamma_{C\otimes V}$ defined in the previous Lemma we denote by $C\square V$ the image of $\Gamma_{C\otimes V}$ and by $i_{C\otimes V} : C\square V \to C \otimes V$, $p_{C\otimes V} : C \otimes V \to C\square V$ the injection and the projection associated to the idempotent.

Definition A.7. We say that $(C, V, \chi_V^C, \tau_V^C)$ satisfies the twisted condition if

$$(\tau_V^C \otimes C) \circ (C \otimes \chi_V^C) \circ (\delta_C \otimes V) = (V \otimes \chi_V^C) \circ (\chi_V^C \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V). \tag{32}$$

The quadruple $(C, V, \chi_V^C, \tau_V^C)$ satisfies the cocycle condition if

$$(\tau_V^C \otimes V) \circ (V \otimes \tau_V^C) \circ (\delta_C \otimes V) = (V \otimes \tau_V^C) \circ (\chi_V^C \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V). \tag{33}$$

Proposition A.8. *Let $(C, V, \chi_V^C, \tau_V^C)$ satisfying the twisted condition. Then the following equalities hold:*

$$(V \otimes \Gamma_{C\otimes V}) \circ (\chi_V^C \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V) = (\chi_V^C \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V) \circ \Gamma_{C\otimes V}, \tag{34}$$

$$(\Gamma_{C\otimes V} \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V) \circ \Gamma_{C\otimes V} = (\Gamma_{C\otimes V} \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V). \tag{35}$$

Proposition A.9. *For a quadruple $(C, V, \chi_V^C, \tau_V^C)$ that satisfies the twisted and the cocycle conditions there exists a morphism $\omega_V^C : C \otimes V \to V \otimes V$ such that $(C, V, \chi_V^C, \omega_V^C)$ satisfies the equality $\omega_V^C \circ \Gamma_{C\otimes V} = \omega_V^C$, the twisted and the cocycle conditions.*

By virtue of Proposition A.9 we will consider from now on, and without loss of generality, that $\tau_V^C \circ \Gamma_{C\otimes V} = \tau_V^C$ for all quadruple $(C, V, \chi_V^C, \tau_V^C)$.

Proposition A.10. *For a quadruple $(C, V, \chi_V^C, \tau_V^C)$ define the coproduct*

$$\delta_{C\otimes V} = (C \otimes \chi_V^C \otimes V) \circ (\delta_C \otimes \tau_V^C) \circ (\delta_C \otimes V) \tag{36}$$

and let $\delta_{C\square V}$ be the coproduct

$$\delta_{C\square V} = (p_{C\otimes V} \otimes p_{C\otimes V}) \circ \delta_{C\otimes V} \circ i_{C\otimes V}. \tag{37}$$

Then if the twisted and the cocycle condition hold, $\delta_{C\otimes V}$ is a coassociative coproduct that it is normalized with respect to $\Gamma_{C\otimes V}$. As a consequence $\delta_{C\square V}$ is also a coassociative coproduct.

Definition A.11. If $(C, V, \chi_V^C, \tau_V^C)$ is a weak crossed coproduct system that satisfies (32), i.e., the twisted condition, and (33), that is, the cocycle condition we say that $(C \otimes V, \delta_{C\otimes V})$ is a weak crossed coproduct.

Remark A.12. If $\delta_{C\otimes V}$ is a coassociative coproduct defined in $C \otimes V$ with precounit υ and such that it is left C-colinear and normalized with respect to $\Gamma_{C\otimes V}^{\upsilon}$, the morphism

$$\gamma_\upsilon : C \otimes V \to C, \ \ \gamma_\upsilon = (C \otimes \upsilon) \circ (\delta_C \otimes V)$$

is comultiplicative and left C-colinear. Although γ_υ is not a coalgebra morphism, because $C \otimes V$ is not a coalgebra, we have that $\varepsilon_C \circ \gamma_\upsilon = \upsilon$, and as a consequence the morphism

$$\bar{\gamma_\upsilon} = \gamma_\upsilon \circ i_{C\otimes V}^{\upsilon} : C\square V \to C$$

is a coalgebra morphism.

In the following theorem we give a characterization of weak crossed coproducts with precounit:

Theorem A.13. *Let C be a coalgebra, V an object and $\Delta_{C\otimes V} : C \otimes V \to C \otimes V \otimes C \otimes V$ a morphism of left C-comodules. Then the following statements are equivalent:*

(i) *The coproduct $\Delta_{C\otimes V}$ is coassociative with precounit υ and normalized with respect to $\Gamma_{C\otimes V}^{\upsilon}$.*
(ii) *There exist morphisms $\chi_V^C : C \otimes V \to V \otimes C$, $\tau_V^C : C \otimes V \to V \otimes V$ and $\upsilon : C \otimes V \to K$ such that if $\delta_{C\otimes V}$ is the coproduct defined in (36), the pair $(C \otimes V, \delta_{C\otimes V})$ is a weak crossed coproduct with $\Delta_{C\otimes V} = \delta_{C\otimes V}$ satisfying:*

$$(V \otimes \upsilon) \circ (\chi_V^C \otimes V) \circ (C \otimes \tau_V^C) \circ (\delta_C \otimes V) = (\varepsilon_C \otimes V) \circ \Gamma_{C\otimes V} \tag{38}$$

$$(\upsilon \otimes V) \circ (C \otimes \tau_C^V) \circ (\delta_C \otimes V) = (\varepsilon_C \otimes V) \circ \Gamma_{C\otimes V} \tag{39}$$

$$(\upsilon \otimes C) \circ (C \otimes \chi_V^C) \circ (\delta_C \otimes V) = \gamma_\upsilon, \tag{40}$$

where γ_υ is the morphism defined in Remark A.12. In this case υ is a precounit for $\delta_{C\otimes V}$, the idempotent morphism of the weak crossed coproduct $\Gamma_{C\otimes V}$ is the idempotent $\Gamma_{C\otimes V}^{\upsilon}$, and we say that $(C \otimes V, \delta_{C\otimes V})$ is a weak crossed coproduct with precounit υ.

Corollary A.14. *If $(C \otimes V, \delta_{C\otimes V})$ is a weak crossed coproduct with precounit υ, then $C\square V$ is a coalgebra with the coproduct defined in (37) and counit $\varepsilon_{C\square V} = \upsilon \circ i_{C\otimes V}$.*

References

[1] R. Blattner, M. Cohen, S. Montgomery, Crossed products and inner actions of Hopf algebras, Trans. Amer. Math. Soc. 298 (1986) 671–711.
[2] Y. Doi, M. Takeuchi, Cleft comodule algebras for a bialgebra, Comm. Algebra 14 (1986) 801–817.
[3] S. Majid, Physics for algebrists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra 130 (1990) 17–64.

[4] S. Majid, More examples of bicrossproduct and double cross product of Hopf algebras, Israel J. Math. 72 (1990) 133–148.
[5] S. Majid, Crossed products by braided groups and bosonization, J. Algebra 163 (1994) 165–190.
[6] T. Brzezinśki, S. Majid, Coalgebra bundles, Comm. Math. Phys. 191 (1998) 467–492.
[7] T. Brzezinśki, P.M. Hajac, Coalgebra extensions and algebra coextensions of Galois type, Comm. Algebra 27 (1999) 1347–1367.
[8] T. Brzeziński, Crossed products by a coalgebra, Comm. Algebra 25 (1997) 3551–3575.
[9] Y. Bespalov, B. Drabant, Cross product bialgebras II, J. Algebra 240 (2001) 445–504.
[10] J.A. Guccione, J.J. Guccione, Theory of braided Hopf crossed products, J. Algebra 261 (2003) 54–101.
[11] G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I: Integral theory and C^*-structures, J. Algebra 221 (1999) 385–438.
[12] S. Caenepeel, E. De Groot, Modules over weak entwining structures, Contemp. Math. 267 (2000) 31–54.
[13] J.N. Alonso Álvarez, R. González Rodríguez, Crossed products for weak Hopf algebras with coalgebra splitting, J. Algebra 281 (2004) 731–752.
[14] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Crossed products in weak contexts, Appl. Cat. Struct. (2009), in press (doi:10.1007/s10485-008-9139-2).
[15] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, M.P. López López, E. Villanueva Novoa, Weak Hopf algebras with projection and weak smash bialgebra structures, J. Algebra 269 (2003) 701–725.
[16] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Crossed products and cleft extensions of weak Hopf algebras (submitted for publication).
[17] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Weak C-cleft extensions, weak entwining structures and weak Hopf algebras, J. Algebra 284 (2005) 679–704.
[18] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, Yetter–Drinfeld modules and projections of weak Hopf algebras, J. Algebra 315 (2007) 396–418.
[19] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, Weak Hopf algebras and weak Yang–Baxter operators, J. Algebra 320 (2008) 2101–2143.
[20] A.B. Rodríguez Raposo, Crossed products for weak Hopf algebras, Comm. Algebra (2008) (in press).
[21] S. Dăscălescu, S. Raianu, Y. Zhang, Finite Hopf Galois extensions, crossed coproducts and duality, J. Algebra 178 (1995) 400–413.
[22] A. Cap, H. Schichl, J. Vanzura, On twisted tensor products of algebras, Comm. Algebra 23 (1995) 4701–4735.
[23] D. Tambara, The endomorphism bialgebra of an algebra, J. Fac. Univ. Tokyo Sect. IA 37 (1990) 425–456.
[24] Y. Bespalov, Crossed modules and quantum groups in braided categories, Appl. Cat. Struct. 5 (1997) 155–204.
[25] Y. Bespalov, B. Drabant, Cross product bialgebras I, J. Algebra 219 (1999) 466–505.
[26] R. Blattner, S. Montgomery, Crossed products and Galois extensions of Hopf algebras, Pacific J. Math. 137 (1989) 37–54.
[27] S. Maclane, Categories for the Working Mathematician, in: G.T.M, vol. 5, Springer-Verlag, 1971.
[28] S. Lack, R. Street, The formal theory of monads II, J. Pure Appl. Algebra 175 (2002) 243–265.
[29] R. Wisbauer, Algebras vs coalgebras, Appl. Cat. Struct. 16 (2008) 255–295.