ON CONSTRUCTION OF SOLUTIONS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT. One of the effective methods for finding exact solutions of differential equations is the method based on the operator representation of solutions. The essence of this method is to construct a series, whose members are the relevant iteration operators acting to some classes of sufficiently smooth functions. This method is widely used in the papers of Bondarenko for construction of solutions of differential equations of the integer order. In this paper, the operator method is applied to construct solutions of linear differential equations with constant coefficients and Caputo fractional derivatives. Then fundamental solutions are used to obtain the unique solution of the Cauchy problem, where the initial conditions are given in terms of the unknown function and its derivatives of integer order.

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1. Introduction

The theory and applications of fractional differential equations received in the last 10-15 years a lot of research attention from both pure mathematicians and engineers (see, for instance, [3, 7],[Introduction]). There exist several different definitions of fractional differentiation. In this paper we consider the Caputo approach to the notion of the fractional derivative.

Let m be a positive integer and $m-1<\alpha\le m$. The Caputo fractional derivative of order α is defined as (see [8])

$$D^\alpha f(t) := I^{m-\alpha}\frac{d^m}{dt^m}f(t),\ t>0,$$

where

$$I^\beta f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \ t>0, \ \beta>0,$$

is the Riemann-Liouville fractional integral of order β.

If $\beta \to 0$, then it is easy to verify, that $I^\beta f(t) \to f(t)$ almost everywhere [10]. Therefore we may define $I^0 f(t) := f(t)$, which leads for $\alpha = m$ to the equality

$$D^m f(t) = \frac{d^m}{dt^m} f(t).$$

Consider a homogenous linear fractional differential equation

$$D^\alpha y(t) - a_1 D^{\alpha-1} y(t) - ... - a_{m-1} D^{\alpha-(m-1)} y(t) - a_m y(t) = 0, \ t > 0 \tag{1}$$

with constant real coefficients a_j, $j = 1, ..., m$ and $m > 1$. If $m = 1$, i.e. $0 < \alpha \leq 1$, then we have the equation $D^\alpha y(t) - a_1 y(t) = 0$.

When $\alpha = m$ equation (1) coincides with the ordinary linear differential equation and for this equation the construction of fundamental solutions and the solution of the Cauchy problem with the initial data

$$y^{(n)}(0) := \frac{d^n}{dt^n} y(0) = b_n, \ n = 0, 1, ..., m-1, \tag{2}$$

are well known. This fundamental theory, based on the characteristic equation

$$\lambda^m - a_1 \lambda^{m-1} - \ldots - a_{m-1} \lambda - a_m = 0,$$

can be found in any textbook on differential equations. The main goal of the present paper is to construct the solution of the Cauchy problem (1), (2) and to obtain the fundamental solutions of equation (1). For this purpose we modify and use the technique based on the method of operator algorithms introduced in [1] and then developed by V.V. Karachik [4] for ordinary differential equations.

There have been several fundamental works on the fractional (ordinary and partial) differential equations so far (see, for example, [3], [8], [10], [12], [13]). These works are an introduction to the theory of the fractional differential equations and provide a systematic understanding of the fractional calculus such as the existence and the uniqueness, some analytical methods for solving fractional differential equations. We note that the Cauchy problem for equation (1) and even for more general equations has been studied by various authors using other methods (see, for example, [2], [3], [5]-[14]). The commonly used methods among these are the Banach fixed point method followed by the reduction of the Cauchy problem for fractional differential equations to a Volterra integral equation of second kind, the Greens function method, the power series and Yu. I. Babenko's symbolic method, Laplace, Fourier and Mellin transforms and the modification of the Mikusinski operational calculus, the Adomian decomposition method. A survey of these methods can be found in [5], [12], [13].

The main idea of the method, which we make use in the present paper is based on the properties of the normed system of functions and consists on the following.

Let us introduce the notations

$$L_1 = D^{\alpha},\ L_2 = a_1 D^{\alpha-1} + ... + a_{m-1} D^{\alpha-(m-1)} + a_m,$$

and $\mathbb{R}_+ = (0, +\infty)$. Then equation (1) can be written as $L_1 y(t) = L_2 y(t)$, $t \in \mathbb{R}_+$.

A system of functions $\{f_k(t)\}_{k=0}^{\infty}$ is called to be f-normed with respect to operator L_1 in the domain $\mathbb{R}_+$, if the equations $L_1 f_0(t) = f(t)$, and $L_1 f_k(t) = f_{k-1}(t)$ hold everywhere in $\mathbb{R}_+$ (see [4]). In case of $f(t) \equiv 0$, the system $\{f_k(t)\}$ is called 0-normed with respect to L_1.

Now let the system $\{f_k(t)\}_{k=0}^{\infty}$ be 0-normed with respect to L_1 in the domain $\mathbb{R}_+$ and satisfy the following two conditions everywhere in $\mathbb{R}_+$:

(i) $L_1 L_2 f_k(t) = L_2 L_1 f_k(t), k = 1, 2, ...$

(ii) the series

$$y(t) = \sum_{k=0}^{\infty} L_2^k f_k(t) \tag{3}$$

converges and allows term-wise application of L_1.

Then it is easy to verify that the function defined in (3) is a solution of (1). Indeed,

$$L_1 y(t) = \sum_{k=1}^{\infty} L_2^k L_1 f_k(t) = \sum_{k=1}^{\infty} L_2^k f_{k-1}(t) = L_2 \sum_{k=0}^{\infty} L_2^k f_k(t) = L_2 y(t).$$

If we consider, instead of a 0-normed system, a f-normed system, then we may construct solutions of non-homogeneous equations.

We note also that a similar method was used to solve the Cauchy problem for the equation $(D^{\alpha,\beta} - \lambda)^N y(t) = f(t)$ in [11], where $D^{\alpha,\beta}$ is the generalized Riemann-Liouville fractional derivative introduced by R. Hilfer (see [2]).

2. Homogenous equations

In this section we consider equation (1) and construct a 0-normed system of functions with respect to the operator L_1. Based on these functions we find the fundamental system of solutions of equation (1).

For $s = 0, 1, ..., m-1$ we introduce the following system of functions

$$f_{s,k}(t) = \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}, \ k = 0, 1, 2, ... \qquad (4)$$

First we show that for any s the system $\{f_{s,k}(t)\}_{k=0}^{\infty}$ is 0-normed with respect to L_1 and, from Section 1, it satisfies conditions (i) and (ii).

Lemma 1. *For any $s = 0, 1, ..., m-1$ the system of functions $\{f_{s,k}(t)\}_{k=0}^{\infty}$ is 0-normed with respect to L_1 in the domain $\mathbb{R}_+$, i.e. for all $t \in \mathbb{R}_+$*

$$D^\alpha f_{s,0}(t) = 0, \quad D^\alpha f_{s,k}(t) = f_{s,k-1}(t), \ k \geq 1,$$

Proof. Obviously $D^\alpha t^s = 0$ for all $s = 0, 1, ..., m-1$. Therefore $D^\alpha f_{s,0}(t) = 0$ for these s.

Let $k \geq 1$. Then by the definition of derivatives D^α one has

$$\begin{aligned} D^\alpha t^{\alpha k+s} &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^{\alpha k+s} d\tau \\ &= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-1))}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \tau^{\alpha k+s-m} d\tau \\ &= \frac{(\alpha k+s)\cdots(\alpha k+s-(m-1))}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-m)}{\Gamma(\alpha k+s+1-\alpha)} t^{\alpha k+s-\alpha} \\ &= \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s}. \end{aligned}$$

Thus

$$D^\alpha t^{\alpha k+s} = \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s}. \qquad (5)$$

Therefore

$$D^\alpha f_{s,k}(t) = \frac{1}{\Gamma(\alpha k+s+1)} \frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha(k-1)+s+1)} t^{\alpha(k-1)+s} = f_{s,k-1}(t).$$

□

Lemma 2. *For any $s = 0, 1, .., m-1$, $k \geq 1$ and all $t \in \mathbb{R}_+$ one has $L_1 L_2 f_{s,k}(t) = L_2 L_1 f_{s,k}(t)$, i.e.*

$$D^\alpha D^{\alpha-j} f_{s,k}(t) = D^{\alpha-j} D^\alpha f_{s,k}(t), \ j = 1, ..., m-1. \qquad (6)$$

Proof. Since $m-1-j<\alpha-j\le m-j$, then by the definition of derivatives we have

$$D^{\alpha-j}t^{\alpha k+s}=I^{m-j-(\alpha-j)}\frac{d^{m-j}}{dt^{m-j}}t^{\alpha k+s}=I^{m-\alpha}\frac{d^{m-j}}{dt^{m-j}}t^{\alpha k+s}=$$

$$\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{d^{m-j}}{d\tau^{m-j}}\tau^{\alpha k+s}d\tau=$$

$$\frac{(\alpha k+s)\cdots(\alpha k+s-(m-j-1))}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\tau^{\alpha k+s-(m-j)}d\tau=$$

$$\frac{(\alpha k+s)\cdots(\alpha k+s-(m-j-1))}{\Gamma(m-\alpha)}\frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-(m-j))}{\Gamma(\alpha k+s+1-(\alpha-j))}\times$$

$$t^{\alpha k+s-(\alpha-j)}=\frac{\Gamma(\alpha k+s+1)}{\Gamma(\alpha k+s+1-(\alpha-j))}t^{\alpha k+s-(\alpha-j)}.$$

Thus

$$D^{\alpha-j}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j))}t^{\alpha k+s-(\alpha-j)}.$$

Therefore

$$D^{\alpha}D^{\alpha-j}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j))}\frac{1}{\Gamma(m-\alpha)}\times$$

$$\int_0^t(t-\tau)^{m-\alpha-1}\frac{d^m}{d\tau^m}\tau^{\alpha k+s-(\alpha-j)}d\tau.$$

If the number $\alpha k+s-(\alpha-j)\le m-1$ and it is integer, i.e. if $k=1$ and $s\in\{0,1,...,m-j-1\}$, then

$$D^{\alpha}D^{\alpha-j}f_{s,k}(t)=0. \tag{7}$$

Otherwise one has

$$D^{\alpha}D^{\alpha-j}f_{s,k}(t)=\frac{(\alpha k+s-(\alpha-j))\cdots(\alpha k+s-(\alpha-j)-(m-1))}{\Gamma(\alpha k+s+1-(\alpha-j))\Gamma(m-\alpha)}\times$$

$$\int_0^t(t-\tau)^{m-\alpha-1}\tau^{\alpha k+s-(\alpha-j)-m}d\tau,$$

and the integral can be written as

$$\frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-(\alpha-j)-m)}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}.$$

Therefore, if $k\ge 1$ and $s\notin\{0,1,...,m-j-1\}$, then

$$D^{\alpha}D^{\alpha-j}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}. \tag{8}$$

On the other hand, from (5) we have

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=\frac{1}{\Gamma(\alpha(k-1)+s+1)}D^{\alpha-j}t^{\alpha(k-1)+s}.$$

Obviously, if $k=1$ and $s\in\{0,1,...,m-j-1\}$, then

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=0. \tag{9}$$

Otherwise one has

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=\frac{1}{\Gamma(\alpha(k-1)+s+1)\Gamma(m-\alpha)}\times$$

$$\int_0^t(t-\tau)^{m-\alpha-1}\frac{d^{m-j}}{d\tau^{m-j}}\tau^{\alpha(k-1)+s}d\tau=$$

$$\frac{(\alpha k-\alpha+s)\cdots(\alpha k-\alpha+s-(m-j-1))}{\Gamma(\alpha(k-1)+s+1)\Gamma(m-\alpha)}\times$$

$$\int_0^t(t-\tau)^{m-\alpha-1}\tau^{\alpha k-\alpha+s-(m-j)}d\tau.$$

The last integral has the form

$$\frac{\Gamma(m-\alpha)\Gamma(\alpha k+s+1-\alpha-(m-j))}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}.$$

Therefore, if $k\geq 1$ and $s\notin\{0,1,...,m-j-1\}$, then

$$D^{\alpha-j}D^{\alpha}f_{s,k}(t)=\frac{1}{\Gamma(\alpha k+s+1-(\alpha-j)-\alpha)}t^{\alpha k+s-(\alpha-j)-\alpha}. \tag{10}$$

Comparing the equalities (7) with (9) and (8) with (10) we deduce the equality (6). □

According to (3) we introduce the following m functions

$$y_s(t)=\sum_{k=0}^{\infty}L_2^k f_{s,k}(t)=$$

$$\sum_{k=0}^{\infty}(a_1D^{\alpha-1}+\cdots+a_{m-1}D^{\alpha-(m-1)}+a_m)^k\frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)},\ t\geq 0, \tag{11}$$

where $s=0,1,...,m-1$.

Note that it is well known, (see [8], page 12), that for the Gamma function the asymptotic estimation

$$\Gamma(x+1)=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left\{1+O\left(\frac{1}{x}\right)\right\},$$

holds as $x\to\infty$.

Theorem 1. *Series (11) converges uniformly on any segment $[0, T]$. When $t \in \mathbb{R}_+$ one may differentiate series (11) term-wise at any natural order and apply operator L_1 term-wise.*

Proof. If $m = 1$, i.e. $0 < \alpha \leq 1$, then the statement of the theorem follows from the asymptotic estimation of the Gamma function.

Let us assume $m > 1$ and denote $\varepsilon = \alpha - (m-1)$ and $a = \max\{|a_1|, ..., |a_m|\}$. Then it is not hard to verify that

$$\big|(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1} D^{\varepsilon} + a_m)^k \frac{t^{\alpha k + s}}{\Gamma(\alpha k + s + 1)}\big| \leq$$

$$a^k (D^{m-2+\varepsilon} + \cdots + D^{\varepsilon} + 1)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k + s + 1)}.$$

Note that $\sum_{i_1+\cdots+i_m=k} \left(\begin{array}{c} k \\ i_1 ... i_m \end{array} \right) = m^k$.

Therefore

$$\begin{aligned} (D^{m-2+\varepsilon} + \cdots + D^{\varepsilon} + 1)^k f_{s,k}(t) &= \\ \sum_{i_1+\cdots+i_m=k} \left(\begin{array}{c} k \\ i_1 ... i_m \end{array} \right) D^{(m-2+\varepsilon)i_1} \cdots D^{\varepsilon i_{m-1}} f_{s,k}(t) &= \\ \sum_{n=0}^{(m-2)k} \sum_{(m-2)i_1+(m-3)i_2 \cdots + i_{m-2}=n} \left(\begin{array}{c} k \\ i_1 ... i_m \end{array} \right) D^n D^{\varepsilon(i_1+\cdots+i_{m-1})} f_{s,k}(t) &\leq \\ m^k \sum_{n=0}^{(m-2)k} D^n \sum_{j=0}^{k} D^{\varepsilon j} f_{s,k}(t), \end{aligned}$$

since the corresponding derivatives of $f_{s,k}(t)$ are positive and

$$D^{j+\varepsilon} f_{s,k}(t) = D^j D^{\varepsilon} f_{s,k}(t).$$

Let $(D-1)g_{s,k}(t) = f_{s,k}(t)$ and $(D^{\varepsilon} - 1)h_{s,k}(t) = g_{s,k}(t)$, i.e.

$$g_{s,k}(t) = \int_0^t e^{\tau} f_{s,k}(t-\tau) d\tau$$

and

$$h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^{\varepsilon}) g_{s,k}(t-\tau) d\tau,$$

where $E_{\varepsilon,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varepsilon k+\mu)}$ is the Mittag-Leffler function (see [10]). Then $f_{s,k}(t)=(D-1)(D^{\varepsilon}-1)h_{s,k}(t)$, and therefore

$$\begin{aligned} m^k\sum_{n=0}^{(m-2)k}D^n\sum_{j=0}^{k}D^{\varepsilon j}f_{s,k}(t) &= m^k(D^{(m-2)k+1}-1)(D^{\varepsilon(k+1)}-1)h_{s,k}(t)\\ &= m^k(D^{(\alpha-1)k+1+\varepsilon}-D^{(m-2)k+1}-D^{\varepsilon(k+1)}+1)h_{s,k}(t). \end{aligned}$$

After some routine calculation exhibited below we have the following estimate for $h_{s,k}(t)$:

$$\begin{aligned} h_{s,k}(t) &\le \frac{E_{\varepsilon,\varepsilon}(t^{\varepsilon})e^t}{\Gamma(\alpha k+s+1)}\int_0^t\tau^{\varepsilon-1}\int_0^{t-\tau}(t-\tau-p)^{\alpha k+s}dpd\tau\\ &= \frac{E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t}{\Gamma(\alpha k+s+2)}\int_0^t\tau^{\varepsilon-1}(t-\tau)^{\alpha k+s+1}d\tau\\ &= \frac{E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t}{\Gamma(\alpha k+s+2)}\frac{\Gamma(\varepsilon)\Gamma(\alpha k+s+2)}{\Gamma(\alpha k+s+2+\varepsilon)}t^{\alpha k+s+1+\varepsilon}\\ &= \frac{G(t)t^{\alpha k+s+1+\varepsilon}}{\Gamma(\alpha k+s+2+\varepsilon)}, \end{aligned}$$

where $G(t):=\Gamma(\varepsilon)E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})e^t$ is a bounded function in any segment $[0,T]$.

Let $N-1<\beta\le N$, and $0<N\le(m-1)k+2$ be an integer number. Let the integer k_0 be such that $k_0\,\varepsilon>1$. From here on in this Section it is assumed that $k\ge k_0$. We apply the operator D^{β} to the function $h_{s,k}(t)$. First we note

$$\frac{d^N}{dt^N}g_{s,k}(t)=\int_0^t e^{\tau}\frac{d^N}{dt^N}f_{s,k}(t-\tau)d\tau,$$

since all the corresponding derivatives of $f_{s,k}(t)$, up to order $N-1$, are zero at the origin. In the same way one has

$$\frac{d^N}{dt^N}h_{s,k}(t)=\int_0^t\tau^{\varepsilon-1}E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})\frac{d^N}{dt^N}g_{s,k}(t-\tau)d\tau.$$

Therefore

$$\begin{aligned} D^{\beta}h_{s,k}(t) &= I^{N-\beta}\frac{d^N}{dt^N}h_{s,k}(t)=\frac{1}{\Gamma(N-\beta)}\int_0^t(t-x)^{N-\beta-1}\frac{d^N}{dx^N}h_{s,k}(x)dx\\ &= \int_0^t\tau^{\varepsilon-1}E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})\big[\frac{1}{\Gamma(N-\beta)}\int_{\tau}^t(t-x)^{N-\beta-1}\frac{d^N}{dx^N}g_{s,k}(x-\tau)dx\big]d\tau\\ &= \int_0^t\tau^{\varepsilon-1}E_{\varepsilon,\varepsilon}(\tau^{\varepsilon})\big[\frac{1}{\Gamma(N-\beta)}\int_0^{t-\tau}(t-\tau-p)^{N-\beta-1}\frac{d^N}{dp^N}g_{s,k}(p)dp\big]d\tau. \end{aligned}$$

Thus

$$D^\beta h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon)(D^\beta g_{s,k})(t-\tau)d\tau,$$

or by using the same argument,

$$D^\beta h_{s,k}(t) = \int_0^t \tau^{\varepsilon-1} E_{\varepsilon,\varepsilon}(\tau^\varepsilon) \int_0^{t-\tau} e^p (D^\beta f_{s,k})(t-\tau-p)dpd\tau.$$

To prove Theorem 1 we estimate $D^\beta h_{s,k}(t)$. First, by direct calculation (see proof of Lemma 1), one has

$$D^\beta f_{s,k}(t) = D^\beta \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)} = \frac{t^{\alpha k+s-\beta}}{\Gamma(\alpha k+s+1-\beta)}.$$

Therefore, in a similar manner as we estimated $h_{s,k}(t)$, we have

$$D^\beta h_{s,k}(t) \leq \frac{G(t)t^{\alpha k+s+1+\varepsilon-\beta}}{\Gamma(\alpha k+s+2+\varepsilon-\beta)}.$$

Making use of this estimate and the one of $h_{s,k}(t)$ we easily obtain

$$\big|(D^{(\alpha-1)k+1+\varepsilon} - D^{(m-2)k+1} - D^{\varepsilon(k+1)} + 1)h_{s,k}(t)\big| \leq (D^{(\alpha-1)k+1+\varepsilon}+1)h_{s,k}(t) \leq$$

$$\frac{G(t)t^{k+s}}{\Gamma(k+s+1)} + \frac{G(t)t^{\alpha k+s+1+\varepsilon}}{\Gamma(\alpha k+s+2+\varepsilon)}.$$

Therefore the asymptotic estimation of the Gamma function implies:

$$\sum_{k=k_0}^{\infty} \big|(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1}D^\varepsilon + a_m)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}\big| \leq$$

$$G(t)\left\{ t^s \sum_{k=k_0}^{\infty} \frac{(amt)^k}{\Gamma(k+s+1)} + t^{s+1+\varepsilon} \sum_{k=k_0}^{\infty} \frac{(am)^k t^{\alpha k}}{\Gamma(\alpha k+s+2+\varepsilon)} \right\} < \infty.$$

Thus series (11) converges uniformly on any segment $[0, T]$.

Moreover, if $t \in \mathbb{R}_+$, then it is not hard to verify that

$$\sum_{k=k_0+n}^{\infty} \big|\frac{d^n}{dt^n}\big[\big(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1}D^\varepsilon + a_m\big)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}\big]\big| \leq$$

$$G(t)\frac{d^n}{dt^n}\left\{ t^s \sum_{k=k_0+n}^{\infty} \frac{(amt)^k}{\Gamma(k+s+1)} + t^{s+1+\varepsilon} \sum_{k=k_0+n}^{\infty} \frac{(am)^k t^{\alpha k}}{\Gamma(\alpha k+s+2+\varepsilon)} \right\} < \infty,$$

which implies the convergence of the series on the left hand side.

Hence, when $t \in \mathbb{R}_+$ one may differentiate series (11) term-wise at any natural order.

Similarly, for any $t \in \mathbb{R}_+$ one obtains

$$\sum_{k=k_0+m}^{\infty} \left| L_1 \big(a_1 D^{m-2+\varepsilon} + \cdots + a_{m-1} D^{\varepsilon} + a_m\big)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)} \right| \leq$$

$$G(t) \left\{ \sum_{k=k_0+m}^{\infty} \frac{t^{s-\alpha}(amt)^k}{\Gamma(k+s-\alpha+1)} + \sum_{k=k_0+m}^{\infty} \frac{t^{s-\alpha+1+\varepsilon}(am)^k t^{\alpha k}}{\Gamma(\alpha k+s-\alpha+2+\varepsilon)} \right\} < \infty,$$

i.e. the series on the left hand side converges. Hence, when $t \in \mathbb{R}_+$ we can apply operator L_1 term-wise to (11).

□

As consequence of the previous assertions, we conclude that for each value of $s = 0, 1, ..., m-1$ functions (11) are solutions of equation (1).

Theorem 2. *Functions $y_s(t)$, $s = 0, 1, ..., m-1$ are linearly independent on any segment $[t_1, t_2] \subset \mathbb{R}_+$.*

Proof. We prove the theorem by contradiction. Let us assume that functions $y_s(t)$, $s = 0, 1, ..., m-1$ are not linearly independent on some segment $[t_1, t_2] \subset \mathbb{R}_+$, i.e. there exist constants C_s, not all of them are equal to zero, such that

$$\varphi(t) := \sum_{s=0}^{m-1} C_s y_s(t) = 0, \;\; t \in [t_1, t_2].$$

According to Theorem 1, functions $y_s(t)$ are power series, converging in $\mathbb{R}_+$. Hence function $\varphi(t)$ is a power series too, converging in $\mathbb{R}_+$. Therefore $\varphi(t) = 0, t \in [t_1, t_2]$ implies the equality $\varphi(t) = 0, \;\; t \in \mathbb{R}_+$ and, in particular, $\varphi(0) = 0$.

Now it is easy to verify that $0 = \sum_{s=0}^{m-1} C_s y_s(0) = C_0$. Hence $\sum_{s=1}^{m-1} C_s y_s(t) = 0$, $t \in \mathbb{R}_+$.

If we differentiate this equality, taking into account that $y_s^{(1)}(t) = y_{s-1}(t)$, we have that

$$\sum_{s=1}^{m-1} C_s y_{s-1}(t) = \sum_{s=0}^{m-2} C_{s+1} y_s(t) = 0.$$

Using the same reasoning as above, we have consistently $C_1 = 0, C_2 = 0, ..., C_{m-1} = 0$. Thus we arrive to a contradiction and we deduce the linear independence of functions $y_s(t)$, $s = 0, 1, ..., m-1$, on $[t_1, t_2]$. □

Definition 1. *The linearly independent functions $y_s(t)$, $s = 0, 1, ..., m-1$, are called the fundamental system of solutions of equation (1).*

Example 1. *Let m be a positive integer and $m-1<\alpha\le m$. Suppose that $a_j=0, j=1,2,...,m-1$, and $a_m=\lambda\neq 0$.*

Then equation (1) has the form

$$D^\alpha y(t)-\lambda y(t)=0,$$

and according to Theorem 1 the following functions

$$y_s(t)=\sum_{k=0}^{\infty}L_2^k\frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}=$$

$$t^s\sum_{k=0}^{\infty}\frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+s+1)}=t^sE_{\alpha,s+1}(\lambda t^\alpha),\ s=0,1,...,m-1,$$

are the fundamental system of solutions, where $E_{\alpha,s+1}(\lambda t^\alpha)$ is the Mittag-Leffler function.

3. The fundamental matrix and the Cauchy problem

In this section we consider the Cauchy problem (1), (2) and find its solution. Note that existence and uniqueness of solutions of the Cauchy problem, even for more general equations than (1), were proved by many authors (see, for example, [7]).

Let $y_s(t)$, $s=0,1,...,m-1$ be the fundamental system, defined above.

Definition 2. *The following matrix*

$$Y(t)=\begin{pmatrix} y_0(t) & y_1(t) & ... & y_{m-1}(t)\\ y_0^{(1)}(t) & y_1^{(1)}(t) & ... & y_{m-1}^{(1)}(t)\\ ... & ... & ... & ...\\ y_0^{(m-1)}(t) & y_1^{(m-1)}(t) & ... & y_{m-1}^{(m-1)}(t)\end{pmatrix}$$

is called the fundamental matrix of equation (1).

Based on this matrix one can easily find the solution of the Cauchy problem. Indeed, if $y(t)=\sum_{s=0}^{m-1}C_sy_s(t)$ is a solution of (1), then $y^{(n)}(t)=\sum_{s=0}^{m-1}C_sy_s^{(n)}(t)$ and therefore one has

$$\begin{pmatrix} y(t)\\ y^{(1)}(t)\\ ...\\ y^{(m-1)}(t)\end{pmatrix}=Y(t)\begin{pmatrix} C_0\\ C_1\\ ...\\ C_{m-1}\end{pmatrix}.$$

Thus, if the vector $\mathbf{C}=(C_0,...,C_{m-1})^T$ satisfies the equation $Y(0)\mathbf{C}=\mathbf{b}$, where $\mathbf{b}=(b_0,...,b_{m-1})^T$, then $y(t)$ is the solution of the Cauchy problem (1), (2). In other words, if we choose $\mathbf{C} = Y^{-1}(0)\mathbf{b}$, then the solution of the Cauchy problem has the form

$$y(t) = (Y^{-1}(0)\mathbf{b}, \mathbf{y}_F(t)), \tag{12}$$

where $\mathbf{y}_F(t) = (y_0(t), ..., y_{m-1}(t))$.

Obviously, in order to ensure the existence of the solution defined in (12) one should verify that $\det Y(0) \neq 0$.

Proposition 1. $\det Y(0) \neq 0$.

Proof. The lemma will be deduced by contradiction. For this let us assume that $\det Y(0) = 0$. In this case there exists a constant vector $\mathbf{C} = (C_0, ..., C_{m-1})^T$, such that not all coordinates C_s are zero and $Y(0)\mathbf{C} = \mathbf{0}$, where $\mathbf{0}$ is zero vector. This implies that $y(t) = \sum_{s=0}^{m-1} C_s y_s(t)$ is the solution of equation (1) with the initial data $y^{(n)}(0) = 0$, $n = 0, 1, ..., m-1$. But the Cauchy problem has the unique solution and therefore

$$y(t) = C_0 y_0(t) + C_1 y_1(t) + \cdots + C_{m-1} y_{m-1}(t) \equiv 0.$$

Since not all C_s are zero, the latter implies linear dependent of the system $y_s(t)$. Thus we have a contradiction, which proves the proposition. □

Next we show that the maximal number of linearly independent solutions of equation (1) is m.

Proposition 2. *Let $x(t)$, $t \geq 0$, be any solution of equation (1). Then $x(t)$ is a linear combination of solutions $y_s(t)$, $s = 0, ..., m-1$.*

Proof. Let $x(t)$ be a solution of (1) and $x^{(n)}(0) = x_n$, $n = 0, 1, ..., m-1$. Obviously $y(t) = (Y^{-1}(0)\mathbf{x}_0, \mathbf{y}_F(t))$ is a solution of equation (1), where $\mathbf{x}_0 = (x_0, ..., x_{m-1})^T$ satisfies the same initial conditions. Since the Cauchy problem has a unique solution, then $x(t) = (Y^{-1}(0)\mathbf{x}_0, \mathbf{y}_F(t))$. □

Thus, formula (12) gives us the expression of the solution of the Cauchy problem (1), (2). Further we find the explicit form of the matrix $Y^{-1}(0)$.

Let $p \geq 0$ be any real number. Consider the functions

$$y_p(t) = \sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_{m-1} D^{\alpha-(m-1)} + a_m)^k \frac{t^{\alpha k + p}}{\Gamma(\alpha k + p + 1)}, \; t \geq 0. \tag{13}$$

Obviously, if $p = 0, 1, ..., m-1$, then $y_p(t)$ is one of the fundamental solutions of equation (1). From here on it is convenient to denote $y^{(\beta)}(t) = D^\beta y(t)$ for any positive real number β.

Theorem 3. *Let $p \geq 0$, n be integer and $n < \alpha$. Then*

$$y_p^{(n)}(t) = \begin{cases} y_{p-n}(t), & p \geq n, \\ a_1 y_p^{(n-1)}(t) + ... + a_{m-1} y_{p+m-2}^{(n-1)}(t) + a_m y_{p+\alpha-1}^{(n-1)}(t), & p+1 \leq n. \end{cases}$$

Proof. If $n \leq p$, then

$$y_p^{(n)}(t) = \sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_{m-1} D^{\alpha-(m-1)} + a_m)^k \frac{t^{\alpha k + p - n}}{\Gamma(\alpha k + p + 1 - n)},$$

i.e. $y_p^{(n)}(t) = y_{p-n}(t)$ and first part of the theorem is proved.

If $p + 1 \leq n < \alpha$, then making use of the equality above one has

$$\begin{aligned} y_p^{(n)}(t) &= y_0^{(n-p)}(t) = \Big[\sum_{k=1}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \frac{t^{\alpha(k-1)+\alpha-1}}{\Gamma(\alpha\, k)} \Big]^{(n-p-1)} \\ &= (a_1 D^{\alpha-1} + \cdots + a_m) \left[\sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \frac{t^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + (\alpha - 1) + 1)} \right]^{(n-p-1)} \\ &= (a_1 D^{\alpha-1} + \cdots + a_m) y_{\alpha-1}^{(n-p-1)}(t) = (a_1 D^{\alpha-1} + \cdots + a_m) y_{\alpha-1+p}^{(n-1)}(t) \\ &= \left[\sum_{k=0}^{\infty} (a_1 D^{\alpha-1} + \cdots + a_m)^k \frac{(a_1 D^{\alpha-1} + \cdots + a_m) t^{\alpha k + \alpha - 1 + p}}{\Gamma(\alpha k + (\alpha - 1) + p + 1)} \right]^{(n-1)} \\ &= a_1 y_p^{(n-1)}(t) + a_2 y_{p+1}^{(n-1)}(t) + \cdots + a_{m-1} y_{p+m-2}^{(n-1)}(t) + a_m y_{p+\alpha-1}^{(n-1)}(t). \end{aligned}$$

□

Corollary 1. *Let n be integer and $1 \leq n \leq m-1$. Then*

$$y_0^{(n)}(t) - a_1 y_0^{(n-1)}(t) - a_2 y_1^{(n-1)}(t) - ... - a_{m-1} y_{m-2}^{(n-1)}(t) - a_m y_{\alpha-1}^{(n-1)}(t) = 0.$$

Proof. If $p = 0$ in Theorem 3, then one has the above equality. □

Corollary 2. *Let n be integer and $1 \leq n \leq m-1$. Then*

$$y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_0^{(n-2)}(0) - ... - a_{n-1} y_0^{(1)}(0) - a_n = 0.$$

Proof. Obviously if $p > 0$, then $y_p(0) = 0$ and $y_0(0) = 1$. Therefore, if $p \geq n$, then from Theorem 3 we obtain $y_p^{(n)}(0) = \delta_{p,n}$–Kronecker delta. Using this and first part of Theorem 3 we have from Corollary 1

$$0 = y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_1^{(n-1)}(0) - ... - a_{m-1} y_{m-2}^{(n-1)}(0) - a_m y_{\alpha-1}^{(n-1)}(0) =$$

$$y_0^{(n)}(0) - a_1 y_0^{(n-1)}(0) - a_2 y_0^{(n-2)}(0) - a_3 y_0^{(n-3)}(0) - \cdots - a_{n-1} y_0^{(1)}(0) - a_n.$$

□

Theorem 4. *The inverse of the fundamental matrix at $t = 0$ is given by the following expression*

$$Y^{-1}(0) = A := \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ -a_1 & 1 & 0 & ... & 0 \\ -a_2 & -a_1 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ -a_{m-1} & -a_{m-2} & -a_{m-3} & ... & 1 \end{pmatrix}.$$

Proof. As it was stated in Theorem 3, all above the diagonal elements of matrix $Y(0)$ are zero, i.e. $y_s^{(n)}(0) = 0$ if $s > n$. Moreover it has the following form with the diagonal elements $y_s^{(s)}(0) = 1$:

$$Y(0) = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ y_0^{(1)}(0) & 1 & 0 & ... & 0 \\ y_0^{(2)}(0) & y_0^{(1)}(0) & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ y_0^{(n-1)}(0) & y_0^{(n-2)}(0) & y_0^{(m-3)}(0) & ... & 1 \end{pmatrix}.$$

Let us denote by Y_i the row of matrix $Y(0)$ with number i and by A_j the column of matrix A with number j, i.e.

$$Y_i = (y_0^{(i-1)}(0), ..., y_0^{(1)}(0), 1, 0, ..., 0),$$

$$A_j = (0, ..., 0, 1, a_1, ..., a_{m-j})^T.$$

Then $Y(0) \cdot A = (Y_i \cdot A_j)_{i,j=\overline{1,m}}$.

Note that the last $m - i$ elements of Y_i are zero, and the first $j - 1$ elements of A_j are zero. Therefore, if $i = j$, then $Y_i \cdot A_j = 1$ and if $i < j$, then $Y_i \cdot A_j = 0$. Finally, if $i > j$, then

$$Y_i \cdot A_j = 1 \cdot y_0^{(i-j)}(0) - a_1 \cdot y_0^{(i-j-1)}(0) - ... - 1 \cdot a_{i-j},$$

and if we use Corollary 2 with $n = i - j$, then we obtain $Y_i \cdot A_j = 0$. Thus $Y_i \cdot A_j = \delta_{i,j}$, which implies that $Y(0) \cdot A$ is the identity matrix. □

Example 2. *Let $3 < \alpha \le 4$. Consider the Cauchy problem*

$$D^\alpha y(t) - a_1 D^{\alpha-1} y(t) - a_2 D^{\alpha-2} y(t) - a_3 D^{\alpha-3} y(t) - a_4 y(t) = 0,$$

$$y^{(j)}(0) = b_j, j = 0, 1, 2, 3.$$

According to (12) the solution of this problem has the form

$$y(t) = (Y^{-1}(0) \cdot b, y_F(t)), \tag{14}$$

where

$$Y^{-1}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 \\ -a_2 & -a_1 & 1 & 0 \\ -a_3 & -a_2 & -a_1 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, y_F(t) = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix},$$

and

$$y_s(t) = \sum_{k=0}^{\infty}(a_1D^{\alpha-1}+a_2D^{\alpha-2}+a_3D^{\alpha-3}+a_4)^k \frac{t^{\alpha k+s}}{\Gamma(\alpha k+s+1)}, s = 0,1,2,3.$$

The initial conditions give

$$y(t) = b_0y_0(t)+(b_1-a_1b_0)y_1(t)+(b_2-a_2b_0-a_1b_1)y_2(t)+(b_3-a_3b_0-a_2b_1-a_1b_2)y_3(t).$$

In particular, if $a_1 = a_2 = a_3 = 0, a_4 \neq 0$ and $b_0 = 1, b_1 = b_2 = b_3 = 0$, then the solution is

$$y(t) = b_0\, y_0(t) = b_0 \sum_{k=0}^{\infty} a_4^k \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} = b_0\, E_{\alpha,1}(a_4 t^\alpha).$$

Next consider the case when $a_1 \neq 0, a_2 = a_3 = a_4 = 0$, and $b_3 = 1, b_0 = b_1 = b_2 = 0$.

In this case one has

$$y(t) = (b_3-a_3b_0-a_2b_1-a_1b_2)y_3(t) = y_3(t) = \sum_{k=0}^{\infty} a_1^k D^{(\alpha-1)k} \frac{t^{\alpha k+3}}{\Gamma(\alpha k+4)} =$$

$$= \sum_{k=0}^{\infty} a_1^k \frac{t^{\alpha k+3-(\alpha-1)k}}{\Gamma(\alpha k+4-(\alpha-1)k)} = \sum_{k=0}^{\infty} a_1^k \frac{t^{k+3}}{(k+3)!} = a_1^{-3} \sum_{k=3}^{\infty} a_1^k \frac{t^k}{k!} =$$

$$= a_1^{-3}[e^{a_1 t} - 1 - a_1 t - a_1^2 \frac{t^2}{2}].$$

Thus the solution of the Cauchy problem

$$D^\alpha y(t) - a_1 D^{\alpha-1} y(t) = 0,$$
$$y^{(j)}(0) = 0, j = 0,1,2,\ y^{(3)}(0) = 1,$$

has the form

$$y(t) = a_1^{-3}[e^{a_1 t} - 1 - a_1 t - a_1^2 \frac{t^2}{2}].$$

Example 3. *Let m be a positive integer and $m-1<\alpha\leq m$. Consider the Cauchy problem*

$$D^{\alpha}y(t)-\lambda y(t)=0,$$

$$y^{(j)}(0)=b_j,\ j=0,1,\cdots,m-1.$$

Obviously, for this equation $Y^{-1}(0)=E$, i.e. the identity matrix and the fundamental system of solutions was found in Example 1. Therefore according to (12) the solution of the Cauchy problem has the form

$$y(t)=\sum_{s=0}^{m-1}b_s t^s E_{\alpha,s+1}(\lambda t^{\alpha}),$$

i.e. we have the known result from [5].

4. Non-homogeneous equations

Let $f(t)$ be an arbitrary continuous function in the domain $[0,T)$. In the present section we consider a non-homogeneous equation

$$D^{\alpha}y(t)-a_1D^{\alpha-1}y(t)-...-a_{m-1}D^{\alpha-(m-1)}y(t)-a_my(t)=f(t),$$

$$t\in(0,T), \tag{15}$$

and the Cauchy problem (2),(15). Again, as in the homogenous case, if $m=1$, i.e. $0<\alpha\leq 1$ then we have the equation $D^{\alpha}y(t)-a_1y(t)=f(t)$.

If we consider the initial data

$$y^{(n)}(0)=0,\ \ n=0,...,m-1, \tag{16}$$

with equation (15), then, as it was noted above, this Cauchy problem has a unique solution. We denote this solution by $y_f(t)$. Let $\tilde{y}(t)$ be the unique solution of the problem (1), (2), which has the form (12). Then, because of the linearity, the function $y_f(t)+\tilde{y}(t)$ will be the unique solution of the problem (15), (2). Thus, to solve the Cauchy problem (15), (2) it is sufficient to find $y_f(t)$.

Let $y_{\alpha-1}(t)$ be the function defined in (13) and $L_0:=\frac{d}{dt}D^{\alpha-1}$. We first study some properties of $y_{\alpha-1}(t)$.

Lemma 3. *Function $y_{\alpha-1}(t)$ is the solution of the Cauchy problem:*

$$(L_0-L_2)y(t)=0,\ t>0,$$

$$y^{(j)}(0)=0, j=0,1,...,m-2,\ \textit{and}\ y^{(\alpha-1)}(0)=1.$$

Proof. Consider the system of functions

$$f_{\alpha-1,k}(t)=\frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)},\ \ k=0,1,2,...$$

By direct calculation we can verify that $D^{\alpha-1}f_{\alpha-1,0}(t)=1$ and therefore $L_0f_{\alpha-1,0}(t)=0$. Hence in the same way as in Section 2, one can show that this system is 0-normed with respect to L_0 and satisfies conditions (i) and (ii) from Section 1. Therefore

$$y_{\alpha-1}(t)=\sum_{k=0}^{\infty}L_2^k f_{\alpha-1,k}(t)$$

is a solution of the equation $(L_0-L_2)y(t)=0, t>0$. Now it is not hard to show that $y_{\alpha-1}(t)$ satisfies the Cauchy conditions. □

Theorem 5. *The unique solution of the Cauchy problem (15), (16) has the form*

$$y_f(t)=\int_0^t f(\tau)y_{\alpha-1}(t-\tau)d\tau. \tag{17}$$

Proof. Since $f(t)$ is a continuous function in the domain $[0,T)$, using the Cauchy conditions for $y_{\alpha-1}(t)$ one obtains

$$\frac{d^j}{dt^j}y_f(t)=\int_0^t f(\tau)\frac{d^j}{dt^j}y_{\alpha-1}(t-\tau)d\tau,\ \ j=1,...,m-1,\ \ t\in[0,T).$$

Therefore $y_f(t)$ satisfies the Cauchy conditions (16). On the other hand (see the proof of Theorem 1)

$$D^{\alpha-j}y_f(t)=\int_0^t f(\tau)(D^{\alpha-j}y_{\alpha-1})(t-\tau)d\tau,\ \ j=1,...,m-1,\ \ t\in[0,T). \tag{18}$$

The function $F(t):=\frac{d^{m-1}}{dt^{m-1}}y_f(t)$ is absolutely continuous in $[0,T)$ and $F(0)=0$. Therefore (see [10], p. 40)

$$I^{m-\alpha}\frac{d}{dt}F(t)=\frac{d}{dt}I^{m-\alpha}F(t).$$

Making use of this equality we apply the operator $\frac{d}{dt}$ to (18) with $j=1$. If we note that $D^{\alpha-1}y_{\alpha-1}(0)=1$, then

$$L_1y_f(t)=D^{\alpha}y_f(t)=f(t)+\int_0^t f(\tau)(L_0y_{\alpha-1})(t-\tau)d\tau.$$

Hence, due to Lemma 3,

$$(L_1-L_2)y_f(t)=f(t)+\int_0^t f(\tau)((L_0-L_2)y_{\alpha-1})(t-\tau)d\tau=f(t).$$

□

If $f(t)$ is a real analytic function in $(0,T)$, i.e.

$$f(t)=\sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!},\ t\in(0,T), \tag{19}$$

then the function $y_f(t)$ has a particularly simple form.

Lemma 4. *If $f(t)=t^n/n!$, n is non-negative integer, then $y_f(t)=y_{\alpha+n}(t)$.*

Proof. Consider the system of functions

$$f_{\alpha+n,k}(t)=\frac{t^{\alpha k+\alpha+n}}{\Gamma(\alpha k+\alpha+n+1)},\ k=0,1,2,...$$

By direct calculation we can verify that $L_1 f_{\alpha+n,0}(t)=f(t)$. Therefore in the same way as in Section 2, one can show that this system is f-normed with respect to L_1 and, from Section 1, it satisfies conditions (i) and (ii). Hence

$$y_{\alpha+n}(t)=\sum_{k=0}^{\infty} L_2^k f_{\alpha+n,k}(t)$$

is a solution of the equation $(L_1-L_2)y(t)=f(t), t\in(0,T)$.

Obviously $y_{\alpha+n}(t)$ satisfies the Cauchy conditions (16). □

Example 4. *Let m be a positive integer and $m-1<\alpha\le m$. Consider the Cauchy problem*

$$D^\alpha y(t)-y(t)=\frac{t^n}{n!},$$
$$y^{(j)}(0)=0, j=0,1,...,m-1.$$

According to Lemma 4 the solution of this problem has the form

$$y_f(t)=y_{\alpha+n}(t)=\sum_{k=0}^{\infty}\frac{t^{\alpha k+\alpha+n}}{\Gamma(\alpha k+\alpha+n)}=t^{\alpha+n}E_{\alpha,\alpha+n}(t^\alpha).$$

The following statement is an easy corollary of Lemma 4.

Theorem 6. *Let $f(t)$ be a real analytic function in $(0,T)$, i.e. $f(t)$ has the form (19). Then*

$$y_f(t)=\sum_{n=0}^{\infty} f^{(n)}(0)y_{\alpha+n}(t).$$

Remark. *Since*

$$y_{\alpha+n}(t) = \int_0^t \frac{(t-\tau)^n}{n!} y_{\alpha-1}(\tau) d\tau,$$

then one has

$$\begin{aligned} y_f(t) &= \sum_{n=0}^{\infty} f^{(n)}(0) y_{\alpha+n}(t) \\ &= \int_0^t \sum_{n=0}^{\infty} f^{(n)}(0) \frac{(t-\tau)^n}{n!} y_{\alpha-1}(\tau) d\tau \\ &= \int_0^t f(\tau) y_{\alpha-1}(t-\tau) d\tau, \end{aligned}$$

i.e. $y_f(t)$ *has the form (17).*

5. Conclusion

30 years ago a mathematician from Uzbekistan B.A. Bondarenko introduced the Operator Algorithms method to solve partial differential equations (see [1]). Recently, in 2012 V.V.Karachik [4] adopted this method to solve the ordinary differential equations and in [11] the authors used the same method for solving some fractional differential equations. The main purpose of this paper is to show that by use of the Bondarenko method one can construct the fundamental solutions of more general fractional differential equations (1) (in fact, we may apply this method for the general linear differential equation with constant coefficients and the Caputo derivatives considered in [7]). As it was shown in Introduction, this method is very simple, a solution of the equation has the form (3), and to use this method, unlike to other methods, we do not need to introduce and investigate many new notions (for example, in the modified Mikusinski method (see [7] and [12]) we introduce a new spaces C_α and with the operations of the Laplace convolution we obtain a commutative ring, then extend this ring to the quotient field). We also note that in the Bondorenko method the solution of the Cauchy problem has a particularly simple form (see formula (12) and Theorem 4).

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