COMPONENTWISE LOCALIZATION OF SOLUTIONS TO SYSTEMS OF OPERATOR INCLUSIONS VIA HARNACK TYPE INEQUALITIES

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Abstract. We establish compression-expansion type fixed point theorems for systems of operator inclusions with decomposable multivalued maps. The approach is vectorial allowing to localize individually the components of solutions and to obtain multiple solutions with multiplicity not necessarily concerned with all components of the solution. A general scheme of applicability of the theory is elaborated based on Harnack type inequalities and illustrated on systems of differential inclusions with one-dimentional ϕ-Laplacian.

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1. Introduction

The main goal of this paper is to discuss the existence, componentwise localization and multiplicity of positive solutions for systems of differential inclusions with one-dimensional ϕ-Laplacian, namely

$$\begin{cases} -\left(\phi_1(u_1')\right)' \in G_1(t,u_1,u_2) \\ -\left(\phi_2(u_2')\right)' \in G_2(t,u_1,u_2), \end{cases} \tag{1.1}$$

for $t\in(0,1)$, subject to the boundary conditions

$$u_1'(0)=u_2'(0)=u_1(1)=u_2(1)=0,$$

where for each $i\in\{1,2\}$, $\phi_i:(-a_i,a_i)\to(-b_i,b_i)$ $(0<a_i,\ b_i\leq+\infty)$ is an increasing homeomorphism with $\phi_i(0)=0$ and G_i is a multivalued map. In particular, the homeomorphisms ϕ_1,ϕ_2 can be one of the following

$$\phi:\mathbb{R}\to\mathbb{R},\quad \phi(x)=|x|^{p-2}x;$$

$$\phi:\mathbb{R}\to(-1,1),\quad \phi(x)=\frac{x}{\sqrt{1+x^2}};$$

$$\phi:(-1,1)\to\mathbb{R},\quad \phi(x)=\frac{x}{\sqrt{1-x^2}},$$

corresponding to the one-dimensional p-Laplace operator, the mean curvature operator in the Euclidian and Minkowski space, respectively. Such type of equations involving the ϕ-Laplacian has been investigated in a large number of papers using fixed point methods, degree theory, upper and lower solution techniques and variational methods. We refer the reader to the papers [1–8, 10–12, 14, 15, 17, 18, 24], to the survey work [16], and the bibliographies therein.

It is worth noting that in our work on system (1.1), the two homeomorphisms ϕ_1,ϕ_2 can be of different types. Also, for a solution $u=(u_1,u_2)$, different localizations of the components

u_1 and u_2 are possible, allowing to obtain multiple solution results where the multiplicity concerns either only one of the components or both of them.

More general, we shall discuss systems of operator inclusions in Banach spaces, of the form

$$\begin{cases} L_1(u_1) \in F_1(u_1, u_2) \\ L_2(u_2) \in F_2(u_1, u_2), \end{cases} \tag{1.2}$$

where, in this abstract setting, the 'boundary conditions' are simulated by the belonging of u_i to the domain $D(L_i)$ of the not necessarily linear operator L_i $(i = 1, 2)$, while the 'positivity' of solutions means their belonging to a given cone.

If the operators L_i $(i = 1, 2)$ are invertible, as is the case of the boundary value problem related to system (1.1), then (1.2) is equivalent to the fixed point problem

$$\begin{cases} u_1 \in L_1^{-1}F_1(u_1, u_2) \\ u_2 \in L_2^{-1}F_2(u_1, u_2), \end{cases} \tag{1.3}$$

with decomposable multivalued maps $L_1^{-1}F_1$ and $L_2^{-1}F_2$.

Thus, it is natural to develop a theory for abstract systems of inclusions, of the form

$$\begin{cases} u_1 \in N_1(u_1, u_2) \\ u_2 \in N_2(u_1, u_2), \end{cases} \tag{1.4}$$

in a Banach space X, where $N = (N_1, N_2) : A \subset X^2 \to 2^{X^2}$ is a multivalued operator such that each of its components is a decomposable map, that is, the composition of two multivalued maps

$$N_i(u) = \Psi_i(u, \Phi_i(u)), \quad u = (u_1, u_2) \in A \qquad (i = 1, 2).$$

Our theoretical strategy is to make the reverse path, from abstract to concrete. Thus, first, in Section 2, we develop a fixed point theory with localization on components for the general system (1.4); next, in Section 3, we apply this theory to systems of type (1.2); and, finally, in Section 4, we particularize even more to obtain results for the considered boundary value problem related to system (1.1). Of course, one may use the abstract theory to obtain similar results for system (1.1) subject to some other boundary conditions. Also the results can be immediately extended to systems of more than two inclusions. Our restriction to systems of two inclusions is only to simplify the presentation.

The common approach to systems is to look at them as generalizations of equations. On the contrary, in our vectorial approch, a system is seen as a particular equation that has the splitting property. Consequently, a larger diversity of results may be expected in the case of systems.

This paper extends for inclusions the theory established in [9], [20], [21] and [22] for equations, and in [13] for inclusions with convex-valued maps. The extension is not trivial since, as explain in [19], several difficulties arise when treating compositions of multivalued maps. One of them consists in guaranteeing continuity properties for the maps, another one concerns the geometric properties of their values. For example, even if one of the maps is single-valued (but nonlinear) and the values of the other one are convex, the values of the composed map can be non-convex, contrary to the hypotheses of basic fixed point theorems for multivalued maps.

2. Existence results for operator systems with decomposable multivalued maps

Let X be a Banach space and consider the fixed point problem

$$\begin{cases} u_1 \in N_1(u_1, u_2) \\ u_2 \in N_2(u_1, u_2), \end{cases}$$

where $N = (N_1, N_2) : A \subset X^2 \to 2^A$ is a multivalued operator such that each of its components is a decomposable map, that is, the composition of two upper semicontinuous (usc, for short) multivalued maps:

$$N_i(u) = \Psi_i(u, \Phi_i(u)), \quad u = (u_1, u_2) \in A \qquad (i = 1, 2).$$

2.1. **A Schauder type fixed point theorem.** First, we present a generalization of Kakutani's fixed point theorem for a system of two decomposable maps. For the case of a single equation, the reader may find a similar result in [19].

Theorem 2.1. *Let A be a nonempty, convex and compact subset of X^2, Y_1, Y_2 Banach spaces and $N = (N_1, N_2) : A \subset X^2 \to 2^A$, $(N_1, N_2)(u) = (\Psi_1(u, \Phi_1(u)), \Psi_2(u, \Phi_2(u)))$, a multivalued operator such that for each $i \in \{1, 2\}$,*

(i) *$\Phi_i : A \to 2^{Y_i}$ is usc with compact convex values;*
(ii) *$\Psi_i : A \times Y_i \to 2^X$ is usc with closed convex values.*

Then N has a fixed point in A.

Proof. Consider the set $B = A \times \overline{\text{co}}\Phi_1(A) \times \overline{\text{co}}\Phi_2(A)$. Since A is compact and Φ_1 is usc with compact values, $\overline{\text{co}}\Phi_1(A)$ is compact (similarly, $\overline{\text{co}}\Phi_2(A)$ is also compact). Hence, B is the Cartesian product of compact convex sets, so it is a compact convex subset of $X^2 \times Y^2$.

Now, we associate to N the map $\Pi : B \to 2^B$ given by

$$\Pi(u_1, u_2, v_1, v_2) = \Psi_1(u_1, u_2, v_1) \times \Psi_2(u_1, u_2, v_2) \times \Phi_1(u_1, u_2) \times \Phi_2(u_1, u_2).$$

Note that Π is usc with closed convex values and thus the Bohnenblust–Karlin fixed point theorem ensures that Π has a fixed point $(u_1, u_2, v_1, v_2) \in B$, that is,

$$u_1 \in \Psi_1(u_1, u_2, v_1), \quad u_2 \in \Psi_2(u_1, u_2, v_2), \quad v_1 \in \Phi_1(u_1, u_2), \quad v_2 \in \Phi_2(u_1, u_2).$$

Therefore, the pair $(u_1, u_2) \in A$ is a fixed point of N. □

Corollary 2.2. *Let C be a nonempty, convex and closed subset of X^2, Y_1, Y_2 Banach spaces and $N = (N_1, N_2) : C \subset X^2 \to 2^C$, $(N_1, N_2)(u) = (\Psi_1(u, \Phi_1(u)), \Psi_2(u, \Phi_2(u)))$, a multivalued operator such that $N(C)$ is relatively compact and for each $i \in \{1, 2\}$,*

(i) *$\Phi_i : C \to 2^{Y_i}$ is usc with compact convex values;*
(ii) *$\Psi_i : C \times Y_i \to 2^X$ is usc with closed convex values.*

Then N has a fixed point in C.

Proof. It suffices to take the compact and convex set $A = \overline{\text{co}}N(C) \subset C$ and to apply Theorem 2.1. □

2.2. **A vector version of Krasnosel'skiĭ's fixed point theorem in cones.** Now we present the vector version of Krasnosel'skiĭ's fixed point theorem in cones for the class of decomposable maps. Note that this type of componentwise localization of positive fixed points for systems was initiated in [20, 21] for single-valued compact maps and continued in [13] by a similar result for usc multivalued maps. Also, a Krasnosel'skiĭ type fixed point theorem in cones for a single decomposable map was proved in [5] (see also [23] for a fixed point index theory for decomposable maps).

Recall that a closed convex subset K of a Banach space X is a cone if it satisfies that $K \cap (-K) = \{0\}$ and $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geq 0$. Any cone K induces a partial order relation in X, denoted by $\preceq$, namely $x \preceq y$ if $y - x \in K$. One says that $x \prec y$ if $y - x \in K \setminus \{0\}$.

In the sequel, let K_1 and K_2 be two cones of the Banach space X and so $K := K_1 \times K_2$ is a cone of X^2. For $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we denote

$$\begin{aligned} (K_i)_{r_i,R_i} &:= \{u \in K_i \,:\, r_i \leq \|u\| \leq R_i\} \quad (i = 1, 2), \\ K_{r,R} &:= \{u = (u_1, u_2) \in K \,:\, r_i \leq \|u_i\| \leq R_i \quad \text{for } i = 1, 2\}. \end{aligned}$$

We look for fixed points of an operator $N = (N_1, N_2) : K \longrightarrow 2^K$, whose components N_1 and N_2 are of the form

$$N_i(u) = \Psi_i(u, \Phi_i(u)), \quad u \in K \quad (i = 1, 2).$$

Here we assume that for a Banach space Y and two closed convex sets $K_{Y_1}, K_{Y_2} \subset Y$, the following conditions on Ψ_i and Φ_i are satisfied for each $i \in \{1, 2\}$:

(H_Φ) $\Phi_i : K \to 2^{K_{Y_i}}$ is usc with compact convex values;
(H_Ψ) $\Psi_i : K \times K_{Y_i} \to 2^{K_i}$ is usc with closed convex values.

Now we state and prove the main result of this section.

Theorem 2.3. *Let $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : K_{r,R} \to 2^K$,*

$$N(u) = (\Psi_1(u, \Phi_1(u)), \Psi_2(u, \Phi_2(u))),$$

$N(K_{r,R})$ is relatively compact and for each $i \in \{1, 2\}$, Φ_i and Ψ_i satisfy (H_Φ) and (H_Ψ), respectively. In addition assume that the following conditions

$$\begin{aligned} u_i \notin N_i(u_1, u_2) + \mu h_i \quad &\text{for any } (u_1, u_2) \in K_{r,R} \text{ with } \|u_i\| = \alpha_i \text{ and } \mu > 0; \\ \lambda u_i \notin N_i(u_1, u_2) \quad &\text{for any } (u_1, u_2) \in K_{r,R} \text{ with } \|u_i\| = \beta_i \text{ and } \lambda > 1 \end{aligned}$$

hold for some $h_i \in K_i \setminus \{0\}$ $(i = 1, 2)$. Then N has a fixed point $u = (u_1, u_2) \in K$ with $r_i \leq \|u_i\| \leq R_i$ for $i = 1, 2$.

Proof. We consider four cases which cover all the possible combinations of compression–expansion conditions for N_1 and N_2.

Case 1: Assume first that $\alpha_i < \beta_i$ for both $i = 1, 2$ (compression for both N_1 and N_2). Then $r_i = \alpha_i$ and $R_i = \beta_i$ for $i = 1, 2$. Define the map $\tilde{N} = (\tilde{N}_1, \tilde{N}_2) : K \to 2^K$ by

$$\tilde{N}_i(u_1, u_2) = \mu(u_1, u_2) N_i \left(\delta_1(u_1) \frac{u_1}{\|u_1\|}, \delta_2(u_2) \frac{u_2}{\|u_2\|} \right) + (1 - \mu(u_1, u_2))\, h_i,$$

where $\mu(u_1, u_2) = \min\{\|u_1\|/r_1, \|u_2\|/r_2, 1\}$ and $\delta_i(u_i) = \max\{\min\{u_i, R_i\}, r_i\}$ for $i = 1, 2$. The map $\tilde{N}$ is a decomposable map, namely, $\tilde{N}_i(u_1, u_2) = \tilde{\Psi}_i(u_1, u_2, \tilde{\Phi}_i(u_1, u_2))$ $(i = 1, 2)$, with

$$\tilde{\Phi}_i(u_1,u_2) = \Phi_i\left(\delta_1(u_1)\frac{u_1}{\|u_1\|}, \delta_2(u_2)\frac{u_2}{\|u_2\|}\right),$$

$$\tilde{\Psi}_i(u_1,u_2,v) = \mu(u_1,u_2)\Psi_i\left(\delta_1(u_1)\frac{u_1}{\|u_1\|}, \delta_2(u_2)\frac{u_2}{\|u_2\|}, v\right) + (1-\mu(u_1,u_2))\,h_i.$$

Observe that $\tilde{\Phi}_i$ and $\tilde{\Psi}_i$ satisfy conditions (H_Φ) and (H_Ψ), respectively. Moreover, note that $\tilde{N}(K)$ is relatively compact since its values belong to the compact set

$$C = \overline{\mathrm{co}}\,(N(K_{r,R}) \cup \{h\}),$$

where $h = (h_1, h_2)$. Therefore, Corollary 2.2 applies and guarantees the existence of a fixed point of $\tilde{N}$ in $C \subset K$. It can be shown in a similar way to the proof of [13, Theorem 2] (or [20, 21]) that the fixed point u belongs to $K_{r,R}$. Hence, u is also a fixed point of the operator N which is located in $K_{r,R}$.

Case 2: Assume that $\alpha_1 < \beta_1$ (compression for N_1) and $\beta_2 < \alpha_2$ (expansion for N_2). Note that this case can be reduced to the previous one by considering the operator $N^* = (N_1^*, N_2^*) : K \to 2^K$ given by

$$N_1^*(u_1,u_2) = N_1\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2\right),$$

$$N_2^*(u_1,u_2) = \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)^{-1} N_2\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2\right).$$

Each component of N^* can be written as the composition $N_i^*(u) = \Psi_i^*(u, \Phi_i^*(u))$, where

$$\Phi_i^*(u_1,u_2) = \Phi_i\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2\right) \qquad (i = 1,2),$$

$$\Psi_1^*(u_1,u_2,v) = \Psi_1\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2, v\right),$$

$$\Psi_2^*(u_1,u_2,v) = \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)^{-1} \Psi_2\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2, v\right).$$

Notice that the map N^* is in case 1 and thus it has a fixed point $v \in K_{r,R}$. Then the point $u = (u_1, u_2)$ defined as $u_1 = v_1$ and $u_2 = \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right) v_2$ is a fixed point of the operator N.

Case 3: Assume that $\beta_1 < \alpha_1$ (expansion for N_1) and $\alpha_2 < \beta_2$ (compression for N_2). This case is completely analogous to the previous one.

Case 4: Assume that $\beta_i < \alpha_i$ for $i = 1,2$ (expansion for both N_1 and N_2). This situation reduces to case 1 by taking the decomposable map $N^{**} = (N_1^{**}, N_2^{**})$ defined as

$$N_1^{**}(u_1,u_2) = \left(\frac{R_1}{\|u_1\|} + \frac{r_1}{\|u_1\|} - 1\right)^{-1} N_1\left(\left(\frac{R_1}{\|u_1\|} + \frac{r_1}{\|u_1\|} - 1\right)u_1, u_2\right),$$

$$N_2^{**}(u_1,u_2) = \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)^{-1} N_2\left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1\right)u_2\right).$$

Thus, the proof is finished. □

In particular, if $\Psi(u,v) = \Psi(v)$, then from Theorem 2.3 we obtain the following compression–expansion fixed point theorem for inclusion systems involving compositions of the form $\Psi \circ \Phi$.

Theorem 2.4. *Let $\alpha_i,\beta_i>0$ with $\alpha_i\neq\beta_i$, $r_i=\min\{\alpha_i,\beta_i\}$ and $R_i=\max\{\alpha_i,\beta_i\}$ for $i=1,2$. Assume that $N:K_{r,R}\to 2^K$,*

$$N=(\Psi_1\Phi_1,\Psi_2\Phi_2),$$

$N(K_{r,R})$ is relatively compact and for each $i\in\{1,2\}$, Φ_i and Ψ_i satisfy (H$_\Phi$) *and* (H$_\Psi$)*, respectively. In addition assume that the following conditions*

$$\begin{aligned} u_i &\notin N_i(u_1,u_2)+\mu h_i \quad \text{for any } (u_1,u_2)\in K_{r,R} \text{ with } \|u_i\|=\alpha_i \text{ and } \mu>0;\\ \lambda u_i &\notin N_i(u_1,u_2) \quad \text{for any } (u_1,u_2)\in K_{r,R} \text{ with } \|u_i\|=\beta_i \text{ and } \lambda>1 \end{aligned}$$

hold for some $h_i\in K_i\setminus\{0\}$ $(i=1,2)$. Then N has a fixed point $u=(u_1,u_2)\in K$ with $r_i\leq\|u_i\|\leq R_i$ for $i=1,2$.

As a straightforward consequence we obtain the following result.

Corollary 2.5. *Let $\alpha_i,\beta_i>0$ with $\alpha_i\neq\beta_i$, $r_i=\min\{\alpha_i,\beta_i\}$ and $R_i=\max\{\alpha_i,\beta_i\}$ for $i=1,2$. Assume that $N:K_{r,R}\to 2^K$,*

$$N=(\Psi_1\Phi_1,\Psi_2\Phi_2),$$

$N(K_{r,R})$ is relatively compact and for each $i\in\{1,2\}$, Φ_i and Ψ_i satisfy (H$_\Phi$) *and* (H$_\Psi$)*, respectively. In addition assume that the following conditions hold:*

$$\begin{aligned} y &\not\prec u_i \quad \text{for all } y\in N_i(u_1,u_2),\ (u_1,u_2)\in K_{r,R} \text{ with } \|u_i\|=\alpha_i;\\ y &\not\succ u_i \quad \text{for all } y\in N_i(u_1,u_2),\ (u_1,u_2)\in K_{r,R} \text{ with } \|u_i\|=\beta_i. \end{aligned}$$

Then N has a fixed point $u=(u_1,u_2)\in K$ with $r_i\leq\|u_i\|\leq R_i$ for $i=1,2$.

3. Positive solutions for systems of operator inclusions

In this section, we apply the general fixed point theorems established from above to the following system of operator inclusions

$$\begin{cases} L_i(u_i)\in F_i(u_1,u_2)\\ u_i\in D(L_i) \quad (i=1,2), \end{cases} \tag{3.1}$$

where for each i, $L_i:D(L_i)\subset X\to Y$ is a not necessarily linear map which is invertible, $F_i:X\times X\to 2^Y$ is a set-valued map, and X, Y are Banach spaces with continuous embedding $X\subset Y$.

Equivalently, we deal with the following inclusion system of type (1.4),

$$u_i\in L_i^{-1}F_i(u_1,u_2), \quad u_i\in X \quad (i=1,2). \tag{3.2}$$

We look for *positive solutions* for (3.1), that is, solutions $u=(u_1,u_2)$ with $u_i\in K_0^i\cap X$, where $K_0^i\subset Y$ is a cone for every $i=1,2$. We use the same symbol $\preceq$ to denote the ordering in Y induced either by K_0^1 or K_0^2. Moreover, we assume that $L_i^{-1}(K_0^i)\subset K_0^i$ for every $i\in\{1,2\}$.

Let P be a cone in X and for each $i\in\{1,2\}$, consider the following basic assumptions:

(H$_1$) $L_i^{-1}:K_0^i\to D(L_i)$ can be written as the composition $L_i^{-1}=T_i\circ S_i$, where
- (a) $S_i:K_0^i\to P$ is a continuous linear operator which maps bounded sets into relatively compact sets and $S_i\circ F_i$ has closed and convex values;
- (b) $T_i:P\to D(L_i)$ is a continuous map.

(H$_2$) S_i and T_i are positive and increasing, i.e.,

$$0 \preceq u \preceq v \quad \text{implies} \quad 0 \preceq S_i(u) \preceq S_i(v) \quad \text{and} \quad 0 \preceq T_i(u) \preceq T_i(v).$$

In addition, $0 \in D(L_i)$ and $L_i(0) = 0$.

(H$_3$) There exists $\psi_i \in K_0^i \setminus \{0\}$ such that for every $u \in K_0^i \cap X$, one has

$$u \preceq \|u\| \, \psi_i.$$

(H$_4$) There exists $\varphi_i \in K_0^i \setminus \{0\}$ such that for every $u \in K_0^i \cap X$ with $L_i u \in K_0^i$, one has

$$\|u\| \, \varphi_i \preceq u \quad \text{(abstract Harnack inequality)}.$$

(H$_5$) $F_i : X^2 \to 2^Y$ is usc, maps bounded sets into bounded sets and

$$F_i((K_0^1 \times K_0^2) \cap X^2) \subset K_0^i.$$

(H$_6$) For each vector $\lambda = (\lambda_1, \lambda_2) \in (0, +\infty)^2$, there exist elements $\underline{F_i}(\lambda), \overline{F_i}(\lambda) \in K_0^i$ such that

$$\underline{F_i}(\lambda) \preceq y \preceq \overline{F_i}(\lambda) \ \text{ for every } y \in F_i(u_1, u_2)$$

and every (u_1, u_2) such that $u_j \in K_0^j \cap X$ and $\lambda_j \, \varphi_j \preceq u_j \preceq \lambda_j \, \psi_j$, $j = 1, 2$.

For each $i \in \{1, 2\}$, we consider the cone in X

$$K_i = \{u \in K_0^i \cap X \; : \; \|u\| \, \varphi_i \preceq u\}$$

and the product cone in X^2, $K = K_1 \times K_2$.

Let $N_i : X^2 \to 2^X$ be the operators

$$N_i(u_1, u_2) = L_i^{-1} F_i(u_1, u_2), \quad i = 1, 2. \tag{3.3}$$

Note that we can write $N = (N_1, N_2) = (\Psi_1 \Phi_1, \Psi_2 \Phi_2)$, where

$$\Psi_i = T_i, \quad \Phi_i = S_i \, F_i \quad \text{for } i = 1, 2. \tag{3.4}$$

For $r, R \in (0, +\infty)^2$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, the notation $K_{r,R}$ stands for

$$K_{r,R} = \{u \in K \; : \; r_i \leq \|u_i\| \leq R_i \ \text{ for } i = 1, 2\},$$

and from (H$_3$) and (H$_4$), for every $(u_1, u_2) \in K_{r,R}$ one has

$$r_j \, \varphi_j \preceq \|u_j\| \, \varphi_j \preceq u_j \preceq \|u_j\| \, \psi_j \preceq R_j \psi_j, \quad j = 1, 2.$$

Observe that clearly for $i = 1, 2$, the map Ψ_i is usc with convex and compact values since it is a single-valued continuous map. Moreover, Φ_i is usc (as the composition of a continuous single-valued map and an usc multivalued map) and $\Phi_i(K_{r,R})$ is relatively compact since $F_i(K_{r,R})$ is bounded and S_i maps bounded sets into relatively compact sets.

Lemma 3.1. *Assume that conditions* (H$_1$)-(H$_5$) *hold. Then for* $N = (N_1, N_2)$, *one has*

$$N(K) \subset K.$$

Proof. Let us see that $N_i(K) \subset K_i$ for $i = 1, 2$. Indeed, if $u \in K$ and $v \in N_i(u) = L_i^{-1} F_i(u)$, then $u_1 \in K_0^1 \cap X$, $u_2 \in K_0^2 \cap X$ and thus, by (H$_5$), $F_i(u) \subset K_0^i$. Next, by (H$_2$), we obtain that $v \in L_i^{-1} F_i(u) \subset K_0^i \cap D(L_i) \subset K_0^i \cap X$. Finally, $L_i v \in F_i(u)$ and $F_i(u) \subset K_0^i$ imply that $\|v\| \, \varphi_i \preceq v$, as a consequence of the abstract Harnack inequality in (H$_4$). Therefore, $v \in K_i$, as wished. $\square$

Now we present the following general existence and localization result.

Theorem 3.2. *Assume that for each* $i \in \{1,2\}$, *conditions* (H_1)-(H_6) *hold and there exist* $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$ *such that*

$$L_i^{-1}\underline{F_i}(\|u_1\|,\|u_2\|) \not\succeq u_i \quad \text{for all } u \in K_{r,R} \text{ with } \|u_i\| = \alpha_i, \tag{3.5}$$

$$u_i \not\succeq L_i^{-1}\overline{F_i}(\|u_1\|,\|u_2\|) \quad \text{for all } u \in K_{r,R} \text{ with } \|u_i\| = \beta_i, \tag{3.6}$$

where $r_i = \min\{\alpha_i, \beta_i\}$ *and* $R_i = \max\{\alpha_i, \beta_i\}$.

Then problem (3.1) *has at least one positive solution* $u \in K$ *with*

$$r_i \leq \|u_i\| \leq R_i \qquad (i = 1,2).$$

Proof. Let us apply Corollary 2.5 to the operator N defined in (3.3) and the cone K considered above. Note that N is a decomposable map with Φ_i and Ψ_i as in (3.4).

First, we prove that

$$L_i^{-1}y \not\succeq u_i \quad \text{for all } y \in F_i(u) \text{ and all } u \in K_{r,R} \text{ with } \|u_i\| = \alpha_i. \tag{3.7}$$

By condition (H_6), $\underline{F_i}(\|u_1\|,\|u_2\|) \preceq y$ for all $y \in F_i(u_1,u_2)$ and all $u \in K_{r,R}$. Hence, by (H_2), $L_i^{-1}\underline{F_i}(\|u_1\|,\|u_2\|) \preceq L_i^{-1}y$ for all $y \in F_i(u_1,u_2)$ and all $u \in K_{r,R}$. Now condition (3.5) implies (3.7).

Secondly, we show that

$$u_i \not\succeq L_i^{-1}y \quad \text{for all } y \in F_i(u) \text{ and all } u \in K_{r,R} \text{ with } \|u_i\| = \beta_i. \tag{3.8}$$

By conditions (H_2) and (H_6), $L_i^{-1}y \preceq L_i^{-1}\overline{F_i}(\|u_1\|,\|u_2\|)$ for all $y \in F_i(u_1,u_2)$, $u \in K_{r,R}$. Then condition (3.6) implies (3.8).

Therefore, Corollary 2.5 ensures the existence of at least one fixed point for N located in $K_{r,R}$, which is a positive solution of the system (3.1). □

It is said that the norm $\|\cdot\|$ of the Banach space X is monotone with respect to the ordering given by the cones K_0^i $(i = 1,2)$ if $0 \preceq x \preceq y$ implies $\|x\| \leq \|y\|$. In that case, conditions (3.5) and (3.6) hold provided that (3.9) and (3.10) below are satisfied.

Corollary 3.3. *Assume that, for each* $i \in \{1,2\}$, *conditions* (H_1)-(H_6) *hold, the norm* $\|\cdot\|$ *is monotone with respect to the ordering and there exist* $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$ *such that*

$$\left\|L_i^{-1}\underline{F_i}(\|u_1\|,\|u_2\|)\right\| > \alpha_i \quad \text{for all } u \in K_{r,R} \text{ with } \|u_i\| = \alpha_i, \tag{3.9}$$

$$\left\|L_i^{-1}\overline{F_i}(\|u_1\|,\|u_2\|)\right\| < \beta_i \quad \text{for all } u \in K_{r,R} \text{ with } \|u_i\| = \beta_i, \tag{3.10}$$

where $r_i = \min\{\alpha_i, \beta_i\}$ *and* $R_i = \max\{\alpha_i, \beta_i\}$.

Then problem (3.1) *has at least one positive solution* $u \in K$ *with*

$$r_i \leq \|u_i\| \leq R_i \qquad (i = 1,2).$$

A number of particular examples of differential problems which illustrate the applicability of the previous theory can be seen in [9] (see also [22]). Roughly speaking, the abstract theory works provided that a Harnack type inequality can be obtained for the problem.

4. Application to ϕ-Laplacian systems of inclusions

The aim of this section is to derive sufficient conditions for the existence and localization of positive solutions for systems of the form

$$\begin{cases} -(\phi_1(u_1'))' \in G_1(t,u_1,u_2) & \text{in } (0,1) \\ -(\phi_2(u_2'))' \in G_2(t,u_1,u_2) & \text{in } (0,1) \\ u_1'(0) = u_1(1) = 0 = u_2'(0) = u_2(1), \end{cases} \tag{4.1}$$

where for each $i \in \{1,2\}$, $\phi_i : (-a_i, a_i) \to (-b_i, b_i)$ $(0 < a_i, b_i \leq +\infty)$ is an increasing homeomorphism such that $\phi_i(0) = 0$ and $G_i : (0,1) \times \mathbb{R}_+^2 \to 2^{\mathbb{R}_+}$ is an usc multivalued map with closed convex values, which maps bounded sets into bounded sets.

Problem (4.1) can be studied by means of the abstract scheme described in Section 3. Here $X = C[0,1]$, $Y = L^\infty(0,1)$, $\|\cdot\| = \|\cdot\|_\infty$ the sup-norm in X, P is the positive cone of $C[0,1]$, K_0^i is the positive cone of $L^\infty(0,1)$, $\psi_i \equiv 1$ and

$$L_i(w)(t) = -\left(\phi_i(w'(t))\right)', \quad F_i(u_1,u_2) = G_i(\cdot, u_1(\cdot), u_2(\cdot)) \qquad (i = 1,2).$$

Note that, for each $w \in L^\infty(0,1)$ and each $i \in \{1,2\}$, one has

$$L_i^{-1}(w)(t) = -\int_t^1 \phi_i^{-1}\left(-\int_0^s w(\tau)\,d\tau\right) ds \tag{4.2}$$

and $L_i^{-1} = T_i \circ S_i$, with

$$T_i(w)(t) = -\int_t^1 \phi_i^{-1}\left(-w(s)\right) ds, \qquad S_i(w)(t) = \int_0^t w(\tau)\,d\tau.$$

In the sequel, we assume that G_i is such that $S_i \circ F_i$ is an usc multivalued map with closed and convex values. Clearly, conditions (H_1), (H_2), (H_3) and (H_5) hold.

On the other hand, condition (H_4) holds here thanks to the following Harnack type inequality proved in [10] for the differential operator $Lu := -(\phi(u'))'$ (where $\phi : (-a,a) \to (-b,b)$ is an increasing homeomorphism with $\phi(0) = 0$) and the boundary conditions $u'(0) = u(1) = 0$.

Proposition 4.1. *For each $c \in (0,1)$ and any $u \in C^1[0,1]$ with $u'(0) = u(1) = 0$, $u'(t) \in (-a,a)$ for every $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ on $[0,1]$, the following inequality holds:*

$$u(t) \geq (1-c)\,\|u\|_\infty \quad \text{for all } t \in [0,c].$$

Hence, condition (H_4) holds by taking the function φ_i given by

$$\varphi_i(t) = \begin{cases} 1 - c_i, & \text{for } t \in [0, c_i] \\ 0, & \text{for } t \in (c_i, 1], \end{cases}$$

where $0 < c_i < 1$ is fixed.

Assume that for each $i \in \{1,2\}$, the multivalued map G_i satisfies the following condition:

(C) There exist continuous functions $f_i, h_i : [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ such that
  (i) f_i and h_i are nondecreasing in the second and third variables on $\mathbb{R}_+$;
  (ii) for every $(t,x,y) \in [0,1] \times \mathbb{R}_+^2$,

$$f_i(t,x,y) \leq z \leq h_i(t,x,y) < b_i \quad \text{for all } z \in G_i(t,x,y).$$

Finally, hypothesis (H_6) is satisfied for

$$\underline{F_i}(\lambda_1,\lambda_2)(t) = f_i(t, \varphi_1(t)\lambda_1, \varphi_2(t)\lambda_2),$$
$$\overline{F_i}(\lambda_1,\lambda_2)(t) = h_i(t, \lambda_1, \lambda_2).$$

Therefore, as a consequence of Corollary 3.3, we obtain the following result concerning the existence of positive solutions for (4.1).

Theorem 4.2. *Assume that, for each $i \in \{1,2\}$, condition (C) holds and there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ such that*

$$-\int_{c_1}^{1} \phi_1^{-1}\left(-\int_0^{c_1} f_1(\tau,(1-c_1)\alpha_1,\varphi_2(\tau)r_2)\,d\tau\right)ds > \alpha_1, \tag{4.3}$$

$$-\int_{c_2}^{1} \phi_2^{-1}\left(-\int_0^{c_2} f_2(\tau,\varphi_1(\tau)r_1,(1-c_2)\alpha_2)\,d\tau\right)ds > \alpha_2, \tag{4.4}$$

$$-\int_0^{1} \phi_1^{-1}\left(-\int_0^{s} h_1(\tau,\beta_1,R_2)\,d\tau\right)ds < \beta_1, \tag{4.5}$$

$$-\int_0^{1} \phi_2^{-1}\left(-\int_0^{s} h_2(\tau,R_1,\beta_2)\,d\tau\right)ds < \beta_2, \tag{4.6}$$

where $r_i = \min\{\alpha_i,\beta_i\}$ and $R_i = \max\{\alpha_i,\beta_i\}$.

Then problem (4.1) has at least one positive solution $u=(u_1,u_2)\in K$ with

$$r_i \leq \|u_i\|_\infty \leq R_i \qquad (i=1,2).$$

Proof. Let us apply Corollary 3.3 with the operators L_i^{-1} as defined in (4.2).

First, let us show that condition (3.9) is satisfied for $i=1$ (the case $i=2$ is analogous). Since for each nonnegative function $w \in L^\infty(0,1)$,

$$\left\|L_1^{-1}(w)\right\|_\infty = -\int_0^1 \phi_1^{-1}\left(-\int_0^s w(\tau)\,d\tau\right)ds,$$

by the monotonicity assumptions on f_1, we have that for each $u \in K_{r,R}$ with $\|u_1\|_\infty = \alpha_1$,

$$\begin{aligned}\left\|L_1^{-1}\underline{F_1}(\|u_1\|_\infty,\|u_2\|_\infty)\right\|_\infty &\geq -\int_0^1 \phi_1^{-1}\left(-\int_0^s f_1(\tau,\varphi_1(\tau)\alpha_1,\varphi_2(\tau)r_2)\,d\tau\right)ds\\ &\geq -\int_{c_1}^1 \phi_1^{-1}\left(-\int_0^s f_1(\tau,\varphi_1(\tau)\alpha_1,\varphi_2(\tau)r_2)\,d\tau\right)ds\\ &\geq -\int_{c_1}^1 \phi_1^{-1}\left(-\int_0^{c_1} f_1(\tau,(1-c_1)\alpha_1,\varphi_2(\tau)r_2)\,d\tau\right)ds.\end{aligned}$$

Hence, applying (4.3), we deduce that condition (3.9) in Corollary 3.3 holds.

Finally, we check that condition (3.10) is also satisfied for $i=2$ (the case $i=1$ is similar). We have that for each $u \in K_{r,R}$ with $\|u_2\|_\infty = \beta_2$,

$$\left\|L_2^{-1}\overline{F_2}(\|u_1\|_\infty,\|u_2\|_\infty)\right\| \leq -\int_0^1 \phi_2^{-1}\left(-\int_0^s h_2(\tau,R_1,\beta_2)\,d\tau\right)ds,$$

and so the conclusion follows from (4.6). □

Note that, in particular, if $c_1 = c_2 =: c$ and ϕ_1 and ϕ_2 are odd homeomorphisms, then conditions (4.3)-(4.6) hold if the following inequalities are satisfied:

$$\begin{aligned}(1-c)\phi_1^{-1}\left(c\min_{\tau\in[0,c]} f_1(\tau,(1-c)\alpha_1,(1-c)r_2)\right) &> \alpha_1,\\ (1-c)\phi_2^{-1}\left(c\min_{\tau\in[0,c]} f_2(\tau,(1-c)r_1,(1-c)\alpha_2)\right) &> \alpha_2,\\ \phi_1^{-1}\left(\max_{\tau\in[0,1]} h_1(\tau,\beta_1,R_2)\right) &< \beta_1,\\ \phi_2^{-1}\left(\max_{\tau\in[0,1]} h_2(\tau,R_1,\beta_2)\right) &< \beta_2.\end{aligned}$$

Next we give an example of application of Theorem 4.2, where the operators associated to the two equations of the system have different behaviors: compression for one of them and expansion for the other one.

Example 4.3. *Consider the system*

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' \in G_1(u,v) \quad \text{in } (0,1)\\ -v'' \in G_2(u,v) \quad \text{in } (0,1)\\ u'(0)=u(1)=0=v'(0)=v(1),\end{cases} \tag{4.7}$$

where the usc multivalued maps G_1 and G_2 are given by

$$G_1(u,v) = \begin{cases} \frac{1}{2}\left([\sqrt[3]{u},\sqrt[4]{u}]e^{-u}+\cos^2 v\right) & \text{if } u\in[0,1]\\ \frac{1}{2}\left(e^{-1}+\cos^2 v\right) & \text{if } u>1\end{cases} \tag{4.8}$$

and

$$G_2(u,v) = [1+\sin^2 u, 2]v^2.$$

One may easily verify that condition (C) *holds for the functions f_i and h_i $(i=1,2)$ defined as*

$$f_1(u) = \begin{cases} \min\left\{\frac{1}{2}\sqrt[3]{u}e^{-u}, \frac{1}{2e}\right\} & \text{if } u\in[0,1],\\ \frac{1}{2e} & \text{if } u>1,\end{cases}$$

$$h_1(u) = 4/5, \quad f_2(v) = v^2, \quad h_2(v) = 2v^2.$$

Moreover, choosing $c = 1/2$, straightforward computations show that we can take $\alpha_1 = 1/50$, $\beta_1 = 2$, $\alpha_2 = 18$ and $\beta_2 = 1/3$. Therefore, according to Theorem 4.2*, problem* (4.7) *has at least one positive solution (u,v) such that*

$$\frac{1}{50} \le \|u\|_\infty \le 2 \quad \text{and} \quad \frac{1}{3} \le \|v\|_\infty \le 18.$$

Remark 4.1 (Asymptotic conditions)**.** *As shown in Example* 4.3*, it is meaningful the simple case where condition* (C) *is given by functions of the form $f_i(t,u_1,u_2) = f_i(u_i)$ and $h_i(t,u_1,u_2) = h_i(u_i)$ for $i=1,2$.*

In this case, the existence of the numbers α_i is guaranteed by the following asymptotic behavior at zero or infinity:

$$\limsup_{\lambda\to 0^+}\frac{(1-c)\phi_i^{-1}\left(cf_i((1-c)\lambda)\right)}{\lambda} > 1 \quad \text{or} \quad \limsup_{\lambda\to+\infty}\frac{(1-c)\phi_i^{-1}\left(cf_i((1-c)\lambda)\right)}{\lambda} > 1.$$

Similarly, the existence of the numbers β_i can be obtained from the following asymptotic behavior at zero or infinity:

$$\liminf_{\lambda\to 0^+}\frac{\phi_i^{-1}\left(h_i(\lambda)\right)}{\lambda}<1 \quad or \quad \liminf_{\lambda\to +\infty}\frac{\phi_i^{-1}\left(h_i(\lambda)\right)}{\lambda}<1.$$

Remark 4.2 (Multiple solutions)**.** *Multiplicity results can be immediately established if several pairs of numbers (α_1,β_1) or (α_2,β_2) as in* (4.3)-(4.6) *exist. Note that we may obtain multiple solutions with multiplicity not necessarily concerned with all components of the solution, as shown in the following example.*

Example 4.4. *Consider the system* (4.7) *with G_1 as defined in* (4.8) *and*

$$G_2(u,v)=[1+\sin^2 u,2]v^2+\frac{1}{9}\sqrt[3]{v}.$$

To check condition (C)*, take f_1 and h_1 as in Example* 4.3*,*

$$f_2(v)=v^2+\frac{1}{9}\sqrt[3]{v} \quad and \quad h_2(v)=2v^2+\frac{1}{9}\sqrt[3]{v}.$$

Again, with $c=1/2$, one may easily verify that conditions (4.3)-(4.6) *hold by taking $\alpha_1=1/50$, $\beta_1=2$ and as pair (α_2,β_2), any one of the following two pairs $(1/500,1/4)$, $(20,1/3)$.*

Thus Theorem 4.2 *applied twice ensures the existence of at least two positive solutions (u_1,v_1) and (u_2,v_2) such that*

$$\frac{1}{50}\le\|u_1\|_\infty,\|u_2\|_\infty\le 2,\quad \frac{1}{500}\le\|v_1\|_\infty\le\frac{1}{4}\quad and \quad \frac{1}{3}\le\|v_2\|_\infty\le 20.$$

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