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An introduction to statistical methods for circular data

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Abstract

Angles, directions, events, occurrences along time... all of them can be viewed as data on a circle (circular data). The particular nature of this type of data requires specific and adapted inferential and modelling procedures. Although there are quite a few references on this topic, and despite circular data are quite common in many applied sciences, they are frequently overlooked. This brief introduction aims to give the reader just some basic ideas on circular data analysis (with some mentions to the general case of spherical or directional data), providing some relevant references and tools for their application in practice.

Keywords: Circular data, Kernel estimation, von Mises.
AMS Subject classifications: 62G07, 62G08, 62P10.

1. Introduction

A simple definition of circular data would state that a circular observation can be expressed as a point on a unit circle or as a unit vector, once an initial direction and a sense of rotation have been chosen. Specific examples of circular data are encountered in a variety of disciplines such as biology (animal orientation; Batschelet (1981) devotes a whole volume to the description and analysis of examples of circular data in biology), environmetrics and oceanography (wind and waves direction; Oliveira *et al.* (2014) and Jona–Lasinio *et al.* (2012)), medicine (bone fractures; Mann *et al.* (2003)) or geology (cross–beds; Mardia (1972)). In addition, circular data can be also generated by wrapping time around, when the goal is to describe patterns on a certain (daily, monthly, yearly) scale. Under an assumption of *stationarity* (in the sense that the events occurrence pattern remains stable along time), sample observations can be written and analyzed as circular data. For instance, times when frosting and thawing cycles occur along a day (Oliveira *et al.*, 2013), or also on a daily basis, circular methods have been applied for analyzing the chronography of domestic terrorism (Gill and Hangartner, 2010).

This idea of viewing temporal patterns as circular data is somehow reflected in the polar area diagram, introduced by Florence Nightingale (see Figure 1) when reporting the causes of mortality in the British army in the mid of the nineteenth century. But despite circular data appear quite frequently in many applied sciences (and they have been there for a long time...), they are frequently overlooked and analyzed without accounting for their particular nature.

In the statistical literature, there are also some (but not many) classical books dealing with the analysis of circular data, such as the ones by Fisher (1993), Jammalamadaka and SenGupta (2001) and more recently, Pewsey *et al.* (2013).

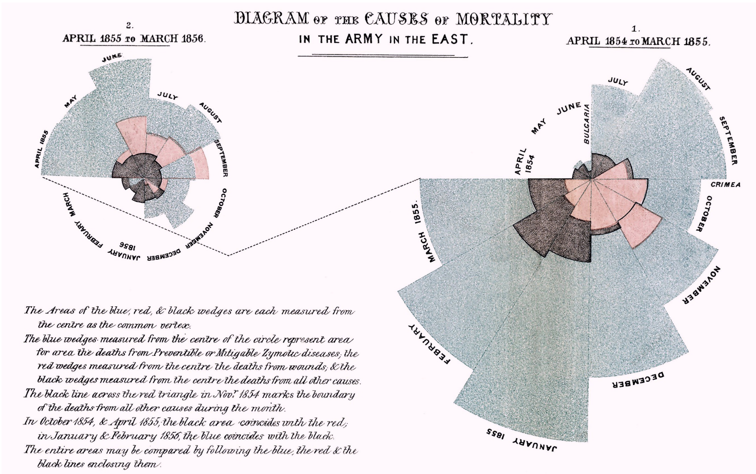

Figure 1: Polar area diagram, by Florence Nightingale (1858). Image from Wikimedia Commons.

Circular data can be viewed as a particular case of spherical or directional data (data whose support is a hypersphere of arbitrary dimension, Mardia (1972)). Hence, methods for spherical data analysis can be adapted to the unit circle. But sometimes this up–bottom focus does not take advantage of the simpler nature of a circle, where data points can be written in polar coordinates and are still easy to handle. In that respect, Fisher (1993) classifies circular data somewhere between linear and spherical. On the one hand, the connection between spherical and circular data has been already noticed. And on the other hand, there may be argued that there are not many differences between circular and linear analysis. If data are concentrated in a part of an arc, then a linearization argument may seem enough to directly apply linear methods, or even a periodicity argument could be used to extend periodically the data from $[0, 2\pi)$ to the real line. This work will be mainly focused on the specific case of circular data, without attempting to give a general view on the spherical case. For that purpose, the reader is referred to Mardia and Jupp (2000) and the forthcoming volume by Ley and Verdebout (2017), with un updated overview on modern directional statistics. So, it seems that finding a balance between both perspectives may be a good option: imitate methods from linear data, adapted and accounting for the circular nature but in a way that their extension may facilitate the construction of more sophisticated tools for spherical data.

The goal of this paper is to introduce in a simple way some methods for circular data analysis, without aiming to be exhaustive, and showing the potential use of R (R Core Team, 2016) packages to perform such analysis. Pewsey *et al.* (2013) is indeed an invaluable contribution for this purpose. The authors present a thorough an detailed review on circular statistics with R, focusing on the use of package `circular`. There are also other packages such `isocirc` (Barragán *et al.*, 2013), `CircNNTS` (Fernández-Durán and Gregorio-Domínguez, 2016) or `movMF`. The recently released package `Directional` also includes some specific functions for circular data. A special attention will be placed in `NPCirc` (Oliveira *et al.*, 2014), focused on the use of smoothing methods for circular density and regression.

This paper is organized as follows. Section 2 is focused on circular densities, presenting some parametric models and density estimation procedures. Circular variables are analyzed in a different role in Section 3, as explanatory and/or response variables in a regression model. Other modelling and inference problems are briefly mentioned in Section 4, jointly with some comments on available software for analyzing circular data.

2. Circular densities

This section will be devoted to present some classical parametric models for circular data, progressing from simple parametric formulations to more flexible approaches. Along this section, denote by Θ a circular random variable, measured in radians in $[0, 2\pi)$. Jammalamadaka and SenGupta (2001, Section 2.1) specify the conditions for a function f to be a circular density ($f(\theta) \geq 0$, $\int_0^{2\pi} f(\theta)d\theta = 1$ and $f(\theta) = f(\theta + 2\pi k)$, for $\theta \in [0, 2\pi)$ and $k \in \mathbb{Z}$).

2.1. Parametric models

The most widely used parametric circular density is the von Mises, a symmetric unimodal function, satisfying that the maximum likelihood estimator of the location parameter is the sample mean (it is also known as the circular normal). The von Mises density is given by:

$$f_{\mu,\kappa}(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left(\kappa \cos(\theta - \mu)\right), \quad 0 \leq \theta < 2\pi, \tag{2.1}$$

where I_0 denotes the modified Bessel function of the first kind and order zero (which is used as a normalizing constant in the density expression). As it can be seen in Figure 2 (left plot, solid line), the von Mises is a symmetric model with mean/mode at μ and concentration controlled by κ. Taking $\kappa = 0$ gives the uniform density on the circle, $f(\theta) = (2\pi)^{-1}$. There are other popular circular models, such as the cardiod (which also contains the uniform as a particular case), but a general way of constructing circular densities is by using a wrapping procedure. That is, if X is a random variable on $\mathbb{R}$ with density g, then $\theta = X(\mathrm{mod}(2\pi))$ has density $f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2\pi k)$. Some examples are the wrapped–normal and wrapped–Cauchy, two of them depicted in Figure 2, left plot. All of these models are symmetric densities (apart from the wrapped skew–normal, they can be seen as particular cases of the Jones–Pewsey family; Jones and Pewsey (2005)). This wrapping argument allows to obtain non symmetric models, for instance, by wrapping skew–normal densities (Pewsey, 2000), shown in Figure 2, right plot, for different degrees of skewness. Pewsey *et al.* (2013) makes in Sec. 4.3.14 a nice summary of the main features of these and other densities.

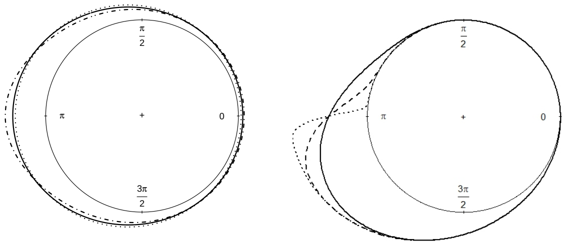

Figure 2: Some circular density examples. Left plot: von Mises (solid line), wrapped–normal (dotted line) and wrapped–Cauchy (dash–dotted line), with mean at π radians. Right plot: wrapped skew–normal densities, with different asymmetry parameters.

However, in practice, the aforementioned models may not be flexible enough to capture the underlying structure of some observed samples. Just for illustration, consider a classical example from Batschelet (1981) reporting the orientation of 214 dragonflies with respect to the sun's azimuth. In Figure 3 (left plot), a circular representation of the data (in radians, points on the circumference) is provided. It is clear from the plot that there are two preferred directions for dragonflies, so it seems obvious that a unimodal density is not appropriate for modelling such a dataset. A rose diagram (a circular equivalent to the histogram) is also plotted.

Further comments on this figure will be given later on. The generalized von Mises distribution (Gatto and Jammalamadaka, 2007), or the models with diametrically opposed modes by Abe and Pewsey (2011) are parametric options for fitting a circular density to this dataset.

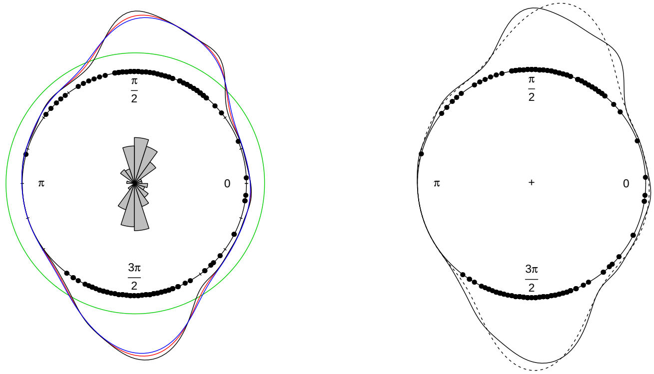

Figure 3: Orientation of dragonflies from Batschelet (1981). Left plot: data observations (points), rose diagram (grey bars) and kernel density estimators with different bandwidths (black: plug–in rule; red: cross–validation; green: rule–of–thumb; blue: bootstrap bandwidth). Right plot: data observations (points), kernel density estimator with plug–in bandwidth (solid line) and von Mises mixture with 4 components.

Another possibility is to construct a mixture model such as:

$$f(\theta) = \sum_{m=1}^{M} p_m f_m(\theta), \quad 0 \leq \theta < 2\pi, \tag{2.2}$$

$p_m > 0$ and $\sum_m p_m = 1$, with f_m circular densities. Specifically, mixture of von Mises densities can be built just taking $f_m = f_{\mu_m,\kappa_m}$

as in (2.1). In Figure 4, some examples are shown, as an illustration of the flexibility of this approach. A simple case of a mixture of two von Mises with antipodal modes (in 0 and π), same concentration and same weights ($p_1 = p_2 = 0.5$) is shown in the top–left part. Another mixture of two von Mises, this time with modes at 0 and $\pi/\sqrt{3}$, same concentrations but different weights ($p_1 = 0.25$), is shown in the top–right part. In the bottom row, there is a mixture of three von Mises, with same concentration and weights and modes at $\pi, 3$, π and $5\pi/3$. Finally, in the bottom–right plot, a mixture of five von Mises is shown. It should be noted that the number of components in the mixture does not necessarily reflect the number of modes, as in this case. More examples can be seen in Oliveira *et al.* (2012) and reproduced with `NPCirc` package. For the dragonfly data, an estimated mixture with 4 components is depicted in Figure 3. Solid line is a kernel density estimator, with plug–in bandwidth, which will be introduced in the next section.

For parametric models, also for mixtures, maximum likelihood approaches are feasible. See Jammalamadaka and SenGupta (2001) for classical models and note that practical implementation is available in `circular`. Parameter estimation for mixtures of von Mises has been also studied by Banerjee *et al.* (2005), who present an adaptation of the EM algorithm for this case, available in `movMF`. In this setting, the number of components of the mixture must be fixed previously to the fitting process, so in practice, the estimation should be accompanied by a model selection criterion in order to select the number of components. For the dragonflies example, mixtures with 2 and 3 components have been also fitted, but the mixture with 4 components gives a better AIC, despite the appearance of the different estimators is quite similar.

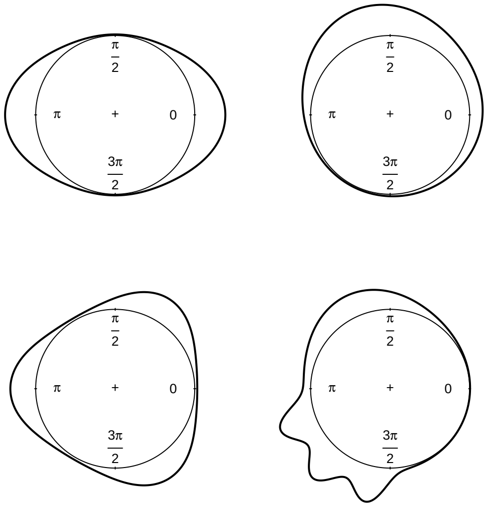

Figure 4: Some examples of mixtures of von Mises. Top–left: mixture of two von Mises with antipodal modes. Top–right: mixture of two von Mises. Bottom–left: mixture of two von Mises. Bottom–right: mixture of five von Mises.

2.2. Nonparametric estimation

Initially introduced by Bai *et al.* (1988) and Hall *et al.* (1987) (in the general spherical case) and Fisher (1989), nonparametric density estimation for circular data can be viewed as an alternative and flexible approach. Consider a random sample of angles $\{\Theta_1, \ldots, \Theta_n\} \in [0, 2\pi)$ from a circular random variable Θ with unknow density f. The kernel density estimator can be defined as:

$$\widehat{f}_\nu(\theta) = \frac{1}{n}\sum_{i=1}^{n} K_\nu(\theta - \Theta_i), \quad 0 \leq \theta < 2\pi, \tag{2.3}$$

where K_ν is a circular kernel function with concentration parameter $\nu > 0$, which controls the degree of smoothness of the estimate. If a von Mises kernel is considered, then the estimator (2.3) can be interpreted as a mixture of n von Mises densities, centered at the sample observations with common concentration parameter ν. If ν is taken too large (highly concentrated kernel), the resulting estimator will be very wiggly. On the contrary, if ν is taken too small (kernels with low concentration), then the estimator becomes smoother and in the limit case with $\nu = 0$, it gives a uniform density on the circle. So, as it is in the linear case, the selection of the concentration parameter is a key problem for circular density estimation.

There have been several proposals for choosing the smoothing parameter, trying to minimize some error criterion. For instance, Hall *et al.* (1987) propose the use of cross–validation (likelihood and least squares) selectors, for spherical densities. Taylor (2008) extends the basic ideas from linear data and designed a *rule–of–thumb* selector. Di Marzio *et al.* (2011) present a bootstrap selector and Oliveira *et al.* (2012) design a plug–in rule, and provide a thorough simulation study comparing the different proposals. In Figure 3, some kernel density estimators (2.3) with von Mises kernel have been computed, trying different bandwidths. Except for the rule-of-thumb, which provides a too small concentration value (yielding an almost uniform estimator), the other proposals give similar results. This is not the case in general, and as Oliveira *et al.* (2012) mention, the bootstrap bandwidth may lead to uniform estimates in small samples (the reader is referred to the previous work for further details).

Kernel density estimators can be used obviously for inferential purposes, as estimators of the density curve, but a (non expert) straightforward use of these methods can be found in exploratory tasks (as in Figure 3). Instead of plotting a rose diagram for data visualization, the kernel density estimator can be a good replacement given its continuous nature. Nevertheless, the appearance of the estimator heavily depends on the value of the concentration parameter, being its selection a crucial task as already noticed. However, the concentration (bandwidth) selection problem can be overcome by a basic exploration tool, the CircSiZer. (Oliveira *et al.*, 2014). The SiZer map was originally introduced by Chaudhuri and Marron (1999), and it is a visualization method which allows to represent which features on a smooth estimate are not due to sampling-noise. The CircSiZer is an adaptation of this tool for circular data. For constructing the CircSiZer map (see Figure 5), the kernel density estimator (2.3) is evaluated on a range of concentrations ν, represented along the circle radius in the figure (on a log–scale, so concentrations closer to the center will give rougher curves). Each ring of the circle corresponds to a concentration value. Along the ring, significantly increasing/decreasing patterns are marked in colour (blue if significantly increasing; red if significantly decreasing); purple areas indicate no significantly increasing nor decreasing and grey zones do not have enough data to extract conclusions. The significantly increasing/decreasing patterns are identified by means of confidence intervals for the derivative of the smoothed curve. In Figure 5, a CircSiZer map is shown for a simulated sample of size $n = 250$ from the bottom–left model in Figure 4, and the three modes are detected. From Figure 5, following the anticlockwise sense of rotation, the two modes are identified for the whole range of concentrations indicating that these ones are genuine features of the underlying structure.

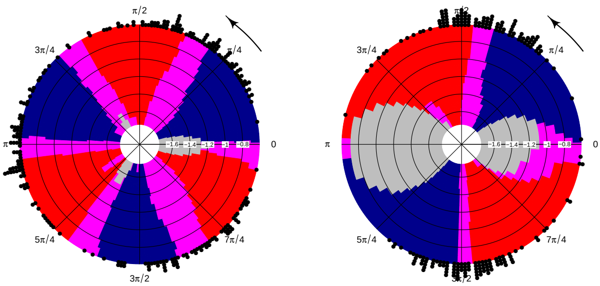

Figure 5: CircSiZer maps. Left plot: simulated sample of size $n = 250$ from bottom–left model in Figure 4. Right plot: orientations of dragonflies.

2.3. Some extensions

It was already remarked in the Introduction that circular data were a particular case of spherical data (data on a hypersphere of general dimension). For this broader context, a similar section could be rewritten, with Mardia and Jupp (2000) as a reference. Just some quick notes for the reader: parametric models can be similarly constructed for spherical data (for instance, the von Mises density can be extended to spheres) and nonparametric density estimation, as already noted, was introduced by Bai *et al.* (1988) and Hall *et al.* (1987). From a practical view, it is worth mentioning the package

`Directional`, recently available for R, which includes a variety of functions for directional data analysis.

Circular data may also be accompanied by other circular or linear data, giving rise to toroidal and cylindrical data, respectively. Semiparametric models have been introduced by Johnson and Wehrly (1978) and Wehrly and Johnson (1980). See also García–Portugués *et al.* (2013) for a practical application using copulas, and Jones *et al.* (2015) for the introduction of circular.

From a methodological perspective, the two alternatives presented in this section are restricted to parametric modelling (by means of maximum likelihood methods) and nonparametric estimation (via kernel smoothers). Recently, Di Marzio *et al.* (2016) introduced the circular local likelihood approach, as an adaptation of a local polynomial method for density estimation, which could be extended to other contexts such as spherical or toroidal data.

3. Regression models

Circular observations may not come alone in practical applications, and their measurements may be accompanied by other circular or linear samples. The analysis of the joint behaviour of such pairs of variables can be approached from a density estimation perspective (see previous section and Di Marzio *et al.* (2016) for conditional density estimation), but also from a regression approach.

Just for motivation, consider the dataset presented in Oliveira *et al.* (2014), regarding wind speed and wind direction measurements on a buoy in the north–west coast of Spain. A graphical representation of these dataset is displayed in Figure 6. From a regression formulation, consider Θ (the circular variable) as the wind direction recorded and as Y (the linear response), the wind speed. In the circular representation, values of the wind speed increase from the center of the of the circle to the border. Data observations are

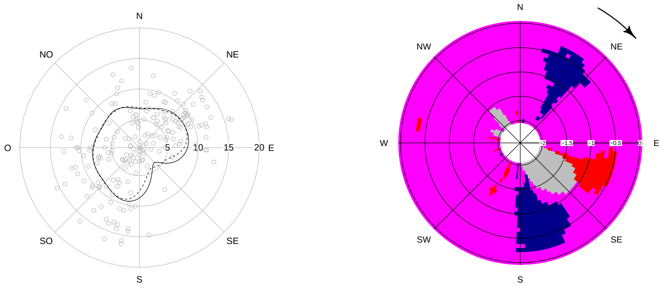

Figure 6: Wind direction and wind speed data. Left plot: data observations (grey points) with smooth regression estimates: Nadaraya–Watson (solid line) and local–linear (dashed line). Right plot: CircSiZer.

depicted as grey points. Then, a regression model for explaining the wind speed in terms of the wind direction could be written as:

$$Y_i = f(\Theta_i) + \sigma(\Theta_i)\varepsilon_i, \tag{3.1}$$

being (Θ_i, Y_i), with $i = 1, \ldots, n$, the random sample and ε_i the error term. As a previous step to regression, a circular–linear correlation coefficient could be computed (see Pewsey *et al.* (2013), Sec. 8.2 and Sec. 8.3 for circular–linear and circular–circular coefficients, or also Fisher (1993), Sec. 6.2; available also in package `Directional`, which includes some tools for computing correlations with spherical variables).

A basic parametric model, considering the circular variable as explanatory and the linear one as the response, could be formulated just by appropriately transforming the circular values through a cosine function (taking $f(\theta) = \beta_0 + \beta_1 \cos(\theta - \theta_0)$ in (3.1)), which can be extended to include more terms or variants (see Pewsey *et al.* (2013), Sec. 8.4). The role of the variables (explanatory and response) can be also reversed and consider the circular one as the response. Linear models are feasible just by the introduction of a proper link function (for instance, Fisher (1993) suggests the use of $g(u) = 2\tan^{-1}(u)$, which is indeed the most widely used in this context). Classical regression estimation methods can be applied for these models.

As it happened for density estimation, the search for more flexible models which could explain the relation between variables, justifies the introduction of a nonparametric perspective. Di Marzio *et al.* (2009) and Di Marzio *et al.* (2012) introduced the local–linear estimators for circular–linear, circular–circular and linear–circular regressions (these procedures are implemented in `NPCirc`). Specifically, in Figure 6, a local–linear estimator and a local–constant one (Nadaraya–Watson estimator) are computed for the wind data. For bandwidth selection, a direct approach is to compute a cross–validation bandwidth. Just for illustration, two other examples of smooth regression are displayed in Figure 7, on the cylinder and the torus, respectively (see `NPCirc` package for details on the data). The selection of the smoothing parameter is also a crucial problem for nonparametric regression estimation, but which can be again avoided (with an exploratory aim), by the construction of a CircSiZer, as the one shown in Figure 6 (right plot). See Oliveira *et al.* (2014) for details, but note that the plot can be read as the one obtained for the density case.

The consideration of spherical variables as response and/or explanatory variables in a regression model, approaching the estimation problem from a nonparametric perspective, is the goal of Di Marzio *et al.* (2014). This problem is also tackled by

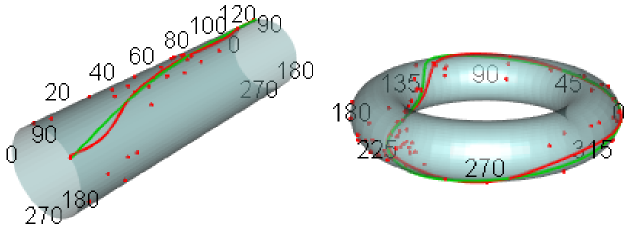

Figure 7: Two examples of regression on the cylinder (circular response and linear explanatory variable) and the torus (circular response and explanatory variables). Left plot: an example of distance and direction of periwinkles movements. Right plot: two wind directions, observed with a time lag.

García–Portugués *et al.* (2016). In both cases, local polynomial fits are proposed, but as discussed by García–Portugués *et al.* (2016), this second proposal presents some technical and practical advantages.

4. Other modeling and inference strategies

In this concise introduction to circular data analysis, there are at least two relevant issues that have been ignored along the text. The first one hast to do with hypothesis testing on this context, and the second one which must be mentioned is related to Bayesian modelling.

With respect to hypothesis testing for circular data, there are some classical contributions which answer the first questions that may appear when analyzing a circular data sample: is the distribution uniform? is it unimodal? is the von Mises an appropriate model? Different tests have been proposed for assessing these simplifying hypotheses (see Fisher, 1993), most of them available in `circular`. A general goodness–of–fit test for a parametric density model (in the general setting of directional–linear data) has been proposed by García–Portugués *et al.* (2015), considering the nonparametric density estimator as a pilot. With this same tool, an independence test between directional and linear variables can be constructed (see García–Portugués *et al.* (2014) for an application). For the regression case, Deschepper *et al.* (2008) present a test for circular–linear regression, and based on nonparametric estimators García–Portugués *et al.* (2016) propose a goodness–of–fit test for parametric models with directional covariate and linear response.

Bayesian analysis of circular data has been approached by Ravindran and Ghosh (2011) considering wrapping probability distributions and a data augmentation method for overcoming the computational difficulties that may arise in a EM algorithm. A special mentioned should be made to the use of wrapped and projected Gaussian processes for complex modelling problems accounting for spatial and spatio–temporal dependencies. Mastrantonio (2016), Wang and Gelfand (2014) and Jona–Lasinio *et al.* (2012) are interesting contributions.

4.1. Available software

Practical application of statistical methods for circular data is nowadays straightforward, thanks to the available software. Package `circular`, thoroughly described by Pewsey *et al.* (2013) as a companion to different inferential procedures (one sample and two sample problems) includes parametric methods and some nonparametric functions. Density estimation by mixtures of von Mises is available in `movMF`. Specifically focused on nonparametric methods is `NPCirc`, which includes functions for density and regression estimation, different bandwidths selectors and the CircSizer construction.

Two other packages dealing with circular data must be mentioned. First, methods for constrained inference on the circle, with applications in cell biology, are available in `isocir` (Barragán *et al.*, 2013). Functions for constructing circular (also multivariate and spherical) distributions based on nonnegative trigonometric sums, estimating parameters and plotting the constructed densities, are included in `CircNNTSR`, by Fernández–Durán and Gregorio–Domínguez (2016). Finally, the reader is also referred to package `Directional`, which also includes some specific functions for the circular case.

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References

[1] Abe, T., and Pewsey, A. (2011). Sine–skewed circular distributions, *Statistical Papers*, 52(3), 683-707.

[2] Bai, Z. D., Rao, C. R. and Zhao, L. C. (1988). Kernel estimators of density function of directional data. *Journal of Multivariate Analysis*, 27(1), 24-39.

[3] Banerjee, A., Dhillon, I.S., Ghosh, J. and Sra, S. (2005). Clustering on the unit hypersphere using von Mises-Fisher

distributions, *Journal of Machine Learning Research*, 6, 1345–1382.

[4] Barragán, S., Fernández, M. A., Rueda, C. and Peddada, S. D. (2013). isocir: An R package for constrained inference using isotonic regression for circular data, with an application to cell biology, *Journal of Statistical Software*, 54(4), 1-17.

[5] Batschelet, E. (1981). *Circular Statistics in Biology*, Academic Press, New York.

[6] Chaudhuri, P. and Marron, J. S. (1999). SiZer for exploration of structures in curves, *Journal of the American Statistical Association*, 94(447), 807-823.

[7] Deschepper, E., Thas, O. and Ottoy, J. P. (2008). Tests and diagnostic plots for detecting lack-of-fit for circular-linear regression models, *Biometrics*, 64(3), 912-920.

[8] Di Marzio, M., Fensore, S., Panzera, A. and C. Taylor, C. (2016). Practical performance of local likelihood for circular density estimation, *Journal of Statistical Computation and Simulation*, 86(13), 2560-2572.

[9] Di Marzio, M., Fensore, S., Panzera, A. and C. Taylor, C. (2016). A note on nonparametric estimation of circular conditional densities, *Journal of Statistical Computation and Simulation*, 86(13), 2573-2582.

[10] Di Marzio, M., Panzera, A. and Taylor, C. C. (2009). Local polynomial regression for circular predictors. *Statistics and Probability Letters*, 79(19), 2066-2075.

[11] Di Marzio, M., Panzera, A. and Taylor, C. C. (2011). Kernel density estimation on the torus, *Journal of Statistical Planning and Inference*, 141(6), 2156-2173.

[12] Di Marzio, M., Panzera, A. and Taylor, C. C. (2012). Non–parametric regression for circular responses, *Scandinavian Journal of Statistics*, 40(2), 238-255.

[13] Di Marzio, M., Panzera, A. and Taylor, C. C. (2014). Nonparametric regression for spherical data, *Journal of the American Statistical Association*, 109(506), 748-763.

[14] Fernández-Durán, J. J. and Gregorio-Domínguez, M. M. (2016). CircNNTSR: An R package for the statistical analysis of circular, multivariate circular, and spherical data using nonnegative trigonometric sums, *Journal of Statistical Software*, 70(1), 1-19.

[15] Fisher, N. I. (1989). Smoothing a sample of circular data, *Journal of Structural Geology*, 11(6), 775-778.

[16] Fisher, N.I. (1993). *Statistical Analysis of Circular Data*, Cambridge University Press.

[17] García-Portugués, E., Van Keilegom, I., Crujeiras, R. M. and González-Manteiga, W. (2016). Testing parametric models in linear-directional regression, *Scandinavian Journal of Statistics*, 43(4) 1178-1191.

[18] García-Portugués, E., Crujeiras, R.M. and González-Manteiga, W. (2015). Central limit theorems for directional and linear random variables with applications. *Statistica Sinica*, 25, 1207-1229.

[19] García-Portugués, E., Barros, A.M.G., Crujeiras, R.M., González-Manteiga, W. and Pereira, J.M. (2014). A test for assessing independence between wildfire orientation and size, *Stochastic Environmental Research and Risk Assessment*, 28, 1261-1275.

[20] García-Portugués, E., Crujeiras, R.M. and González-Manteiga, W. (2013). Kernel density estimation for directional-linear data, *Journal of Multivariate Analysis*, 121, 152-275.

[21] García-Portugués, E., Crujeiras, R.M. and González-Manteiga, W. (2013). Exploring wind direction and SO2 concentration by circular–linear density estimation. *Stochastic Environmental Research and Risk Assessment*, 27 1055-1067.

[22] Gatto, R. and Jammalamadaka, S. R. (2007). The generalized von Mises distribution. *Statistical Methodology*, 4(3), 341-353.

[23] Gill, J. and Hangartner, D. (2010). Circular data in political science and how to handle it, *Political Analysis*, 18(3), 316-336.

[24] Hall, P., Watson, G. S. and Cabrera, J. (1987). Kernel density estimation with spherical data, *Biometrika*, 74(4), 751-762.

[25] Jammalamadaka, S.R. and SenGupta, A. (2001). *Topics in Circular Statistics*, World Scientific, Singapore.

[26] Johnson, R. A. and Wehrly, T. E. (1978). Some angular-linear distributions and related regression models, *Journal of the American Statistical Association*, 73(363), 602-606.

[27] Jona–Lasinio, G., Gelfand, A. and Jona-Lasinio, M. (2012). Spatial analysis of wave direction data using wrapped gaussian processes, *Annals of Applied Statistics*, 1478-1498.

[28] Jones, M. C., and Pewsey, A. (2005). A family of symmetric distributions on the circle, *Journal of the American Statistical Association*, 100(472), 1422-1428.

[29] Jones, M. C., Pewsey, A., and Kato, S. (2015). On a class of circulas: copulas for circular distributions, *Annals of the Institute of Statistical Mathematics*, 67(5), 843-862.

[30] Ley, C. and Verdebout, T. (2017). *Modern Directional Statistics*, Chapman and Hall.

[31] Mann, K. A., Gupta, S., Race, A., Miller, M. A. and Cleary, R. J. (2003). Application of circular statistics in the study of crack distribution around cemented femoral components, *Journal of Biomechanics*, 36(8), 1231-1234.

[32] Mardia, K. (1972). *Statistics for Directional Data*, Academic Press, New York.

[33] Mardia, K. and Jupp, P. (2000). *Directional Statistics*, Wiley, New York.

[34] Mastrantonio, G., Gelfand, A. E. and Lasinio, G. J. (2016). Spatio-temporal circular models with non-separable covariance structure. *Test*, 25(2), 331-350.

[35] Oliveira, M., Crujeiras, R. M. and Rodríguez-Casal, A. (2012). A plug-in rule for bandwidth selection in circular density estimation, *Computational Statistics and Data Analysis*, 56(12), 3898-3908.

[36] Oliveira, M., Crujeiras, R. M. and Rodríguez-Casal, A. (2013). Nonparametric circular methods for exploring environmental data, *Journal of Environmental and Ecological Statistics*, 20(1), 1-17.

[37] Oliveira, M., Crujeiras, R. M. and Rodríguez-Casal, A. (2014). CircSiZer: an exploratory tool for circular data, *Journal of Environmental and Ecological Statistics*, 21(1), 143-159.

[38] Oliveira, M., Crujeiras, R. M. and Rodríguez-Casal, A. (2014). NPCirc: an R package for nonparametric circular methods, *Journal of Statistical Software*, 61(9), 1-26.

[39] Pewsey, A. (2000). The wrapped skew-normal distribution on the circle, *Communications in Statistics-Theory and Methods*, 29(11), 2459-2472.

[40] Pewsey, A., Neuhäuser, M. and Ruxton, G.D. (2013). *Circular Statistics in R*, Oxford University Press.

[41] R Core Team (2016). *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna, Austria. URL http://www.R-project.org/.

[42] Ravindran, P. and Ghosh, S. K. (2011). Bayesian analysis of circular data using wrapped distributions. *Journal of Statistical Theory and Practice*, 5(4), 547-561.

[43] Taylor, C.C. (2008) Automatic bandwidth selection for circular density estimation, *Computational Statistics and Data Analysis*, 52(7), 3493-3500.

[44] Wang, F. and Gelfand, A. E. (2014). Modeling space and space-time directional data using projected Gaussian processes. *Journal of the American Statistical Association*, 109(508), 1565-1580.

[45] Wehrly, T. E. and Johnson, R. A. (1980). Bivariate models for dependence of angular observations and a related Markov process, *Biometrika*, 67(1), 255-256.

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