Existence results for singular φ – laplacian problems in presence of lower and upper solutions

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Abstract

This paper is devoted to the study of the existence of solutions to singular φ – laplacian problems, coupled with nonlinear functional boundary value conditions, in presence of a pair of well ordered lower and upper solutions. The results follow from a general existence result for a functional problem that improves a previous one due to Bereanu and Mawhin related

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to non homogeneous Dirichlet equations. The arguments are in the line of the showed in some previous papers devoted to regular φ – laplacian operators.

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1 Introduction

The study of N – dimensional hypersurfaces of a given mean curvature $H : \Omega \times \mathbb{R} \to \mathbb{R}$, with Ω a bounded domain in $\mathbb{R}^N$, delivers us to a differential equation of the type

$$\operatorname{div}\left(\frac{\nabla v(x)}{\sqrt{1 \pm |\nabla v(x)|^2}}\right) = H(x, v(x)), \quad x \in \Omega. \tag{1.1}$$

The positive sign is given in differential geometry, by using the Euclidian space, and the negative one appears on relativity theory, when one is working in the Minkowski space (see [1] for details).

When $0 \notin \bar{\Omega}$, the study or radial solutions of problem (1.1) is translated to the study of the ordinary differential equation [2, 3]

$$(\varphi_{\pm}(u'(r)))' = f(r, u(r), u'(r)), \quad r \in I, \qquad \text{with } \varphi_{\pm}(x) \equiv \frac{x}{\sqrt{1 \pm x^2}}.$$

In this case $\varphi_+ : \mathbb{R} \to (-1, 1)$ and $\varphi_- : (-1, 1) \to \mathbb{R}$ are two homeomorphisms that satisfy $\varphi_{\pm}(0) = 0$. Using the terminology employed in [2, 3, 4, 5, 6] we will say that operator φ_+ is bounded and φ_- is singular.

The behavior of these two operators differs substantially from the so called p – laplacian operator, defined, for all $x \in \mathbb{R}$, as $\varphi_p(x) = |x|^{p-2}\, x, \quad p > 1$. This operator is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, and appears in the Rheological law [19] when describing the behavior of the shear of a Non Newtonian fluid, in which we arrive at an equation of the type

$$\frac{d}{dt}\varphi_p(u'(t)) = f(t, u(t), u'(t)),\ t \in I.$$

We note that the particular case $p = 2$ coincides with the identity function and, as consequence the p - laplacian equation is the general second order differential equation for the Newtonian mechanics.

These kind of equations have been studied under the framework of the φ - laplacian operator (φ is an increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ that vanishes at 0) by several authors in the last few years. The nonlinear part have been considered continuous [12, 13], Carathéodory [9, 15, 16, 17], and even discontinuous and with functional dependence [10, 11]. The existence of solutions is mainly related to the lower and upper solutions method. We point out that in [9, 10, 11] the case of the bounded φ – laplacian ($\varphi(\mathbb{R})$ bounded) is also considered. In [9], by an extension argument, the authors adapt the bounded case to the regular one without any additional assumption.

Recently, Bereanu, Jebelean and Mawhin have developed the method of lower and upper solutions for the singular φ - laplacian problem, $\varphi : (-a, a) \rightarrow \mathbb{R}$, coupled with linear two – point boundary value conditions as Dirichlet, Neumann and periodic. So, in [2] the authors prove the validity of such a method for the Neumann case with a singularity at $r = 0$:

$$(r^{N-1}\,\varphi(u'(r)))' = r^{N-1} f(r, u(r), u'(r)),\ r \in I,$$

$$u'(R_1) = u'(R_2) = 0,\ 0 \leq R_1 < R_2.$$

In [6] Bereanu and Mawhin show that the result is valid for the periodic equation

$$(\varphi(u'(r)))' = f(r, u(r), u'(r)),\ r \in I, \quad u(0) - u(T) = u'(0) - u'(T) = 0.$$

In [4, 5] are obtained some "universal" existence results for the non homogeneous Dirichlet problem

$$(\varphi(u'(r)))' = f(r, u(r), u'(r)),\ r \in I, \quad u(0) = A, u(T) = B.$$

They show that if f is continuous then this problem is solvable if and only if $|B - A| < a\,T$. Such result generalizes the one given in [6] for the homogeneous case and on its proof it is not necessary to impose any growth condition in the behavior of function f with respect its first derivative (Nagumo's condition).

In this paper we develop the validity of the method of lower and upper solutions for a functional equation with a singular φ - laplacian operator and functional nonlinear boundary conditions. As far as we know, this is the first time in which such kind of problems have been considered under this point of view. The arguments combine some arguments presented by Bereanu and Mawhin for non homogeneous Dirichlet problems and the techniques used in [9, 10, 11] for regular φ - laplacian equations.

2 A general existence result

Let $T > 0$ be given, and define $I = [0, T]$. Along the paper, for any u, $v \in \mathcal{C}(I)$ such that $u(t) \leq v(t)$ for all $t \in I$, we shall write

$$[u, v] = \{z \in \mathcal{C}(I) \,:\, u(t) \leq z(t) \leq v(t) \quad \text{for all } t \in I\}\,.$$

Let us consider the functional problem

$$(P) \quad \begin{cases} \dfrac{d}{dt}\varphi(u'(t)) = f(t, u, u(t), u'(t)) \text{ for a.e. } t \in I, \\[2em] u(0) = \mathcal{A}(u), \qquad\quad u(T) = \mathcal{B}(u), \end{cases}$$

under the following set of assumptions:

(H_1) $\varphi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\varphi(0) = 0$, being $a > 0$ a given constant.

(H_2) $f : I \times \mathcal{C}(I) \times \mathbb{R}^2 \to \mathbb{R}$ is such that for each $u \in \mathcal{C}^1(I)$ the composition $f(\cdot, u, u(\cdot), u'(\cdot))$ is measurable in I and

$$\lim_{n\to\infty} f(t, u_n, u_n(t), u_n'(t)) = f(t, u, u(t), u'(t)) \text{ for a.e. } t \in I$$

whenever $u_n \to u$ in $\mathcal{C}^1(I)$. Moreover, there exists $\psi \in L^1(I)$ and a null measure set $N \subset I$ such that

$$|f(t, \xi, x, y)| \leq \psi(t) \text{ for all } (t, \xi, x, y) \in (I \setminus N) \times \mathcal{C}(I) \times \mathbb{R}^2.$$

(H_3) $\mathcal{A}$, $\mathcal{B} \in \mathcal{C}(\mathcal{C}^1(I), \mathbb{R})$ are nonlinear operators that satisfy the following property:

$$|\mathcal{A}(u) - \mathcal{B}(u)| < a\,T, \quad \text{for all } u \in \mathcal{C}^1(I). \tag{2.1}$$

We say that ***u* is a solution of problem (*P*)** if $u \in \mathcal{C}^1(I)$, $\varphi(u'(\cdot)) \in AC(I)$ and u satisfies both the differential equation (a.e. in I) and the functional conditions (here $AC(I)$ denotes the space of absolutely continuous functions in I). We remark that a necessary condition for a $\mathcal{C}^1$ function to be a solution of (P) is that $\|u'\|_\infty < a$.

Remark 2.1. *We note that condition (H_2) allows functional dependence of f on its second variable. Moreover, as we will point out after, function f can be discontinuous at some of its variables.*

Before considering the nonlinear problem (P), we show the following technical result, that follows by direct integration.

LEMMA 2.1. *Let the function $w \in \mathcal{C}^1([0,T])$ satisfying $\|w'\|_\infty < a$ be fixed. Given constants u_l, u_r such that $|u_l - u_r| < a\,T$ there exists a unique function $u \in \mathcal{C}^1([0,T])$ satisfying $\|u'\|_\infty < a$, $u(0) = u_l$, $u(T) = u_r$ and $\varphi\left(u'\right) - \varphi\left(w'\right) = c \in \mathbb{R}$. Moreover, u is in the form*

$$u(t) = u_l + \int_0^t \varphi^{-1}\left(\varphi(w'(s)) + c\right) \mathrm{d}s, \tag{2.2}$$

where the constant c is determined uniquely by the equation

$$\int_0^T \varphi^{-1}\left(\varphi(w'(t)) + c\right) \mathrm{d}t = u_r - u_l. \tag{2.3}$$

To construct the fixed point operator associated to problem (P), we follow the approach given in [5] (for $\mathcal{A}$ and $\mathcal{B}$ two real constants) and in [10, 11] (for regular φ – laplacian), and study the solvability of the quasilinear equation

$$\frac{d}{dt}\varphi(u'(t)) = e(t) \text{ for a.e. } t \in I, \quad u(0) = \mathcal{A}(u), \quad u(T) = \mathcal{B}(u), \tag{2.4}$$

with φ satisfying (H_1), $e \in L^1(I)$, and $\mathcal{A}$ and $\mathcal{B}$ fulfilling (H_3).

Moreover, to ensure the existence and uniqueness of solutions for this problem, we introduce the following two cases about the operators $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathcal{C}^1([0,T]); \mathbb{R})$:

Case 1.

$(H1_a)$ There exists an $\epsilon > 0$ such that for any $u, v \in \mathcal{C}^1([0,T])$ satisfying $\|u'\|_\infty < a$, $\|v'\|_\infty < a$ and $\varphi(u') - \varphi(v') =$ constant, the inequality

$$\begin{aligned}(\mathcal{A}(u) - \mathcal{A}(v))(u(0) - v(0)) + (\mathcal{B}(u) - \mathcal{B}(v))(u(T) - v(T)) \\ \leq (1-\epsilon)\left((u(0) - v(0))^2 + (u(T) - v(T))^2\right)\end{aligned} \tag{2.5}$$

holds.

$(H1_b)$ For any real number C, the two following inequalities hold:

$$\begin{pmatrix} \mathcal{A}(-at + C) \\ \mathcal{B}(-at + C) \end{pmatrix} \neq \begin{pmatrix} C \\ -aT + C \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{2.6}$$

$$\begin{pmatrix} \mathcal{A}(at + C) \\ \mathcal{B}(at + C) \end{pmatrix} \neq \begin{pmatrix} C \\ aT + C \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{2.7}$$

for any $\lambda \geq 0$.

Case 2.

$(H2_a)$ There exists an $\epsilon > 0$ such that for any $u, v \in \mathcal{C}^1([0,T])$ satisfying $\|u'\|_\infty < a$, $\|v'\|_\infty < a$ and $\varphi(u') - \varphi(v') =$ constant, the inequality

$$\begin{aligned}(\mathcal{A}(u) - \mathcal{A}(v))(u(0) - v(0)) + (\mathcal{B}(u) - \mathcal{B}(v))(u(T) - v(T)) \\ \geq (1+\epsilon)\left((u(0) - v(0))^2 + (u(T) - v(T))^2\right)\end{aligned} \tag{2.8}$$

holds.

$(H2_b)$ For any real number C, the two following inequalities are fulfilled:

$$\begin{pmatrix} \mathcal{A}(-at+C) \\ \\ \mathcal{B}(-at+C) \end{pmatrix} \neq \begin{pmatrix} C \\ \\ -aT+C \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ \\ 1 \end{pmatrix}, \tag{2.9}$$

$$\begin{pmatrix} \mathcal{A}(at+C) \\ \\ \mathcal{B}(at+C) \end{pmatrix} \neq \begin{pmatrix} C \\ \\ aT+C \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \\ -1 \end{pmatrix}, \tag{2.10}$$

for any $\lambda \geq 0$.

Remark 2.2. *Note that the constant operators $\mathcal{A}(u) = A$, $\mathcal{B}(u) = B$ satisfy assumptions $(H1_a)$ and the condition $|B - A| < aT$ is equivalent to the assumptions $(H1_b)$.*

EXAMPLE 2.1. *Define $\mathcal{A}(u) = \alpha \sin u(0) + \gamma$ and $\mathcal{B}(u) = \beta \cos u(T) + \delta$, with $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$.*

It is clear that condition $(H1_b)$ holds for all $\alpha,\ \beta \in (-1, 1)$.

Condition (2.6) in $(H1_b)$ is rewriting as

If $\alpha \sin(C_0) + \gamma = C_0 + \lambda_0$ for some $C_0 \in \mathbb{R}$ and $\lambda_0 \geq 0$ then $\beta \cos(-aT + C_0) + \delta + a\,T \neq C_0 - \lambda_0$, and vice versa.

Now, (2.7) is writing as

If $\alpha \sin(C_0) + \gamma = C_0 - \lambda_0$ for some $C_0 \in \mathbb{R}$ and $\lambda_0 \geq 0$ then $\beta \cos(aT + C_0) + \delta - a\,T \neq C_0 + \lambda_0$, and vice versa.

Choosing the particular case $\alpha = 1/2$, $\beta = 1/3$ and $\gamma = \delta = 0$, we have that if

$$\frac{\sin(C_0)}{2} - C_0 = \lambda_0, \quad \textit{for some } \lambda_0 \geq 0, \textit{ then } C_0 \leq 0.$$

Thus, we have that

$$\frac{\cos\left(-aT+C_0\right)}{3}-C_0=-a\,T-\lambda_0=-a\,T-\frac{\sin\left(C_0\right)}{2}+C_0$$

if and only if

$$2\,C_0=\frac{\cos\left(-aT+C_0\right)}{3}+\frac{\sin\left(C_0\right)}{2}+a\,T.$$

On the other hand

$$0\geq 2\,C_0=\frac{\cos\left(-aT+C_0\right)}{3}+\frac{\sin\left(C_0\right)}{2}+a\,T\geq-\frac{1}{3}-\frac{1}{2}+a\,T,$$

which is not possible when $a\,T>5/6$.

Suppose now that

$$\frac{\cos\left(-aT+C_0\right)}{3}-C_0=-a\,T-\lambda_0\quad \textit{for some } \lambda_0\geq 0, \textit{ then } C_0\geq a\,T-1/3,$$

then

$$\frac{\sin\left(C_0\right)}{2}-C_0=\lambda_0=-\frac{\cos\left(-aT+C_0\right)}{3}+C_0-a\,T.$$

Which implies that

$$2\,a\,T-\frac{2}{3}\leq 2\,C_0=\frac{\cos\left(-aT+C_0\right)}{3}+\frac{\sin\left(C_0\right)}{2}+a\,T\leq\frac{5}{6}+a\,T,$$

which is not possible when $a\,T>9/6$.

As a conclusion the condition (2.6) holds when $a\,T>9/6$.

Analogously we can conclude that if $a\,T>9/6$ *then condition (2.7) is fulfilled.*

So, we are in a position to prove the following existence and uniqueness result.

Proposition 2.1. *Suppose that either conditions* $(H1_a)$ *and* $(H1_b)$ *or* $(H2_a)$ *and* $(H2_b)$ *hold and let* $e \in L^1(0,T)$ *be fixed. Then problem (2.4) has a unique solution* $u \in AC(I)$ *such that* $\|u\|_\infty < a$.

Proof. Define

$$w(t) = \int_0^t \varphi^{-1}\left(\int_0^s e(r)\,dr\right) ds.$$

It is obvious that $w \in \mathcal{C}^1([0,T])$ and $\|w'\|_\infty < a$. As consequence, from Lemma 2.1, we know that for any pair of real constants u_l, u_r with $|u_l - u_r| < a\,T$, there exists a unique function $u \in \mathcal{C}^1([0,T])$ satisfying $\|u'\|_\infty < a$, $u(0) = u_l$, $u(T) = u_r$ and $\varphi(u') - \varphi(w') = c \in \mathbb{R}$. Moreover, the expressions of u and c are given by formulas (2.2) and (2.3).

The constant c depends continuously on the difference $u_r - u_l$, hence u is a continuous function of u_l and u_r. Then, we can define the (single-valued) operator $\mathcal{M}_w : \mathcal{D}(\mathcal{M}_w) \subset \mathbb{R}^2 \to \mathbb{R}^2$ with the strip domain

$$\mathcal{D}(\mathcal{M}_w) := \left\{(u_l, u_r)^T : |u_l - u_r| < aT\right\}$$

in the following manner:

$$\mathcal{M}_w \begin{pmatrix} u_l \\ u_r \end{pmatrix} := \begin{pmatrix} \mathcal{A}(u) \\ \mathcal{B}(u) \end{pmatrix},$$

where u is the function defined in (2.2) and (2.3).

From the definition of this operator and Lemma 2.1, we deduce that problem (2.4) has a unique solution if and only if operator $\mathcal{M}_w$ has a unique fixed point in $\mathbb{R}^2$.

Note that $\mathcal{M}_w$ is continuous. Moreover, the operator $(1-\epsilon)\mathrm{id}-\mathcal{M}_w$ (respectively $\mathcal{M}_w-(1+\epsilon)\mathrm{id}$) is monotone in Case 1 defined on the real Hilbert space $\mathbb{R}^2$ with the usual inner product $(x,y)=x^T\cdot y$. Respectively, $\mathcal{M}_w-(1+\epsilon)\mathrm{id}$ is monotone in Case 2.

Next, assume Case 1 and define the following (multi-valued) extension of the operator $\mathcal{M}_w^\epsilon := (1-\epsilon)\mathrm{id}-\mathcal{M}_w$:

$$\widetilde{\mathcal{M}}_w^\epsilon(x) := \begin{cases} \mathcal{M}_w^\epsilon(x), & \text{if} \quad x\in\mathcal{D}(\mathcal{M}_w), \\ \{y : (y-\mathcal{M}_w^\epsilon(z), x-z)\geq 0, \forall z\in\mathcal{D}(\mathcal{M}_w)\}, & \text{if} \quad x\in\partial\mathcal{D}(\mathcal{M}_w). \end{cases}$$

Note that the values of the operator $\widetilde{\mathcal{M}}_w^\epsilon$ on the boundary

$$\partial\mathcal{D}(\mathcal{M}_w) = \left\{\begin{pmatrix} C \\ C-aT \end{pmatrix} : C\in\mathbb{R}\right\}\bigcup\left\{\begin{pmatrix} C \\ C+aT \end{pmatrix} : C\in\mathbb{R}\right\}$$

are in fact in the following form:

$$\widetilde{\mathcal{M}}_w^\epsilon\begin{pmatrix} C \\ C-aT \end{pmatrix} = \left\{\begin{pmatrix} (1-\epsilon)C-\mathcal{A}(C-at) \\ (1-\epsilon)(C-aT)-\mathcal{B}(C-at) \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix} : \lambda\geq 0\right\},$$

and

$$\widetilde{\mathcal{M}}_w^\epsilon\begin{pmatrix} C \\ C+aT \end{pmatrix} = \left\{\begin{pmatrix} (1-\epsilon)C-\mathcal{A}(C+at) \\ (1-\epsilon)(C+aT)-\mathcal{B}(C+at) \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix} : \lambda\geq 0\right\}.$$

Indeed, let $z\in\mathcal{D}(\mathcal{M}_w)$ and $C\in\mathbb{R}$ be arbitrary. Denote $\xi=(C,C-aT)^T$, $m=(\mathcal{A}(C-at),\mathcal{B}(C-at))^T$, $\eta=(1,-1)^T$(an outer normal vector to the boundary at ξ) and take $\mu\in(0,1]$. Obviously, $(\xi-z,\eta)>0$.

Then the monotonicity of $\mathcal{M}_w^\epsilon$ implies that $(\mathcal{M}_w^\epsilon(\mu z+(1-\mu)\xi),\xi-z)$ is non-increasing function of μ.

This is true, because if A is monotone, then we have that $(Ax - Ay, x - y) \geq 0$. And now if we take $\mu_1 < \mu_2$, from the monotonicity of $\mathcal{M}_w^\epsilon$, we will have that

$$(\mathcal{M}_w^\epsilon(\mu_1 z + (1-\mu_1)\xi) - \mathcal{M}_w^\epsilon(\mu_2 z + (1-\mu_2)\xi),$$

$$\mu_1 z + (1-\mu_1)\xi - \mu_2 z - (1-\mu_2)\xi) \geq 0.$$

But $\mu_1 z + (1-\mu_1)\xi - \mu_2 z - (1-\mu_2)\xi = (\mu_2 - \mu_1)\xi - (\mu_2 - \mu_1)z$, so we can write the previous one as

$$(\mathcal{M}_w^\epsilon(\mu_1 z + (1-\mu_1)\xi) - \mathcal{M}_w^\epsilon(\mu_2 z + (1-\mu_2)\xi),$$

$$(\mu_2 - \mu_1)(\xi - z)) \geq 0.$$

Since $\mu_2 - \mu_1 > 0$, we can write that

$$(M_w^\varepsilon(\mu_1 z + (1-\mu_1)\xi), \xi - z) \geq (M_w^\varepsilon(\mu_2 z + (1-\mu_2)\xi), \xi - z).$$

So, $(\mathcal{M}_w^\epsilon(\mu z + (1-\mu)\xi), \xi - z)$ is a non-increasing function of μ.

The fact that $\mathcal{M}_w^\epsilon(\mu z + (1-\mu)\xi) \to (1-\epsilon)\xi - m$ as $\mu \to 0$ (since $\mu z + (1-\mu)\xi \to \xi$ and then $c \to -\infty$ in (2.3) thus $u \to -at + C$ in C^1) yields that $(1-\epsilon)\xi - m + \lambda\eta \in \widetilde{\mathcal{M}}_w^\epsilon(\xi)$ for any $\lambda \geq 0$.

Next, if $y \in \widetilde{\mathcal{M}}_w^\epsilon(\xi)$ we know that $0 \leq (y - \mathcal{M}_w^\epsilon(\mu z + (1-\mu)\xi), \xi - (\mu z + (1-\mu)\xi)) = \mu\,(y - \mathcal{M}_w^\epsilon(\mu z + (1-\mu)\xi), (\xi - z))$, which implies that $(y - ((1-\epsilon)\xi - m), \xi - z) \geq 0$ for any $z \in \mathcal{D}(\mathcal{M}_w)$.

Hence $y - ((1-\epsilon)\xi - m) = \lambda\eta$ for some non-negative constant λ.

The next step is to prove that the operator $\widetilde{\mathcal{M}}_w^\epsilon$ is maximal monotone [7, 8]. Assume by contradiction that there is $[x^*, y^*]$ which is not in the graph $\widetilde{\mathcal{M}}_w^\epsilon$ and $(y^* - y, x^* - x) \geq 0$ for any $[x, y]$ such that $y \in \widetilde{\mathcal{M}}_w^\epsilon(x)$.

Let's see that $x^* \notin \mathcal{D}(\widetilde{\mathcal{M}}_w^\epsilon) = \overline{\mathcal{D}(\mathcal{M}_w)}$:

We have to proof that $y^* = \widetilde{\mathcal{M}}_w^\epsilon(x^*)$, i.e., that $[x^*, y^*]$ is in the graph. We have two cases.

1) if $x^* \in \mathcal{D}(\mathcal{M}_w)$ We have that

$$(\widetilde{\mathcal{M}}_w^\epsilon(x) - y^*, x - x^*) \geq 0$$

and

$$(\widetilde{\mathcal{M}}_w^\epsilon(x) - \widetilde{\mathcal{M}}_w^\epsilon(x^*), x - x^*) \geq 0.$$

Let $v \in \mathbb{R}^2$ and $\lambda > 0$ be given, and define $x_\lambda = x^* + \lambda v$. Then $\lambda(\widetilde{\mathcal{M}}_w^\epsilon(x_\lambda) - y^*, v) \geq 0$. When $\lambda \to 0$ we have that $\widetilde{\mathcal{M}}_w^\epsilon(x_\lambda) \to \widetilde{\mathcal{M}}_w^\epsilon(x^*)$ and, as consequence, Since $(\widetilde{\mathcal{M}}_w^\epsilon(x^*) - y^*, v) \geq 0$.

By redefining $x_\lambda = x^* - \lambda v$, analogously we deduce that $(\widetilde{\mathcal{M}}_w^\epsilon(x^*) - y^*, -v) \geq 0$.

As a consequence of the last two inequalities we obtain that $(\widetilde{\mathcal{M}}_w^\epsilon(x^*) - y^*, v) = 0$, i.e. $y^* = \widetilde{\mathcal{M}}_w^\epsilon(x^*)$.

2) the other case is when $x^* \in \partial\mathcal{D}(\mathcal{M}_w)$. From $(y^* - y, x^* - x) \geq 0$ for every $[x, y], y \in \widetilde{\mathcal{M}}_w^\epsilon(x)$ we arrive to the same result as before.

Next, take the closest point ξ from the boundary of $\mathcal{D}(\widetilde{\mathcal{M}}_w^\epsilon)$ and denote by η the outer normal vector, i.e. $(x^* - x, \eta) > 0$. Now, we have that if $y \in \widetilde{\mathcal{M}}_w^\epsilon(\xi)$ then $y + \lambda\eta \in \widetilde{\mathcal{M}}_w^\epsilon(\xi)$ as well for any arbitrarily large λ, implying a contradiction with $(y^* - (y + \lambda\eta), x^* - x) \geq 0$.

Thus $\widetilde{\mathcal{M}}_w^\epsilon$ is a maximal monotone operator and then by Minty's theorem there exists a unique $x \in \mathcal{D}(\widetilde{\mathcal{M}}_w^\epsilon)$ such that $0 \in \epsilon x + \mathcal{D}(\widetilde{\mathcal{M}}_w^\epsilon)(x)$. On the other hand, assumptions $(H1_b)$ imply that $x \notin \partial\mathcal{D}(\widetilde{\mathcal{M}}^\epsilon_w)$, i.e. x is from the interior of $\mathcal{D}(\widetilde{\mathcal{M}}^\epsilon_w)$ and then $\mathcal{M}_w(x) = x$.

The same result can be derived in Case 2. The monotone operator to be considered in that case is $\mathcal{M}_w - (1+\epsilon)\mathrm{id}$. □

As an immediate consequence, we attain at the following result

Corollary 2.1. *Assume that assumptions $(H_1) - (H_3)$ are fulfilled. Then u is a solution of problem (P) if and only if u is a fixed point of the operator $\mathcal{T} : \mathcal{C}^1(I) \to \mathcal{C}^1(I)$, defined by*

$$\mathcal{T}u\,(t) = \mathcal{A}(u) + \int_0^t \varphi^{-1}\left(c(u) + \int_0^s f\,(r, u, u(r), u'(r))\;dr\right) ds, \qquad (2.11)$$

being $c(u)$ the unique solution of the expression

$$\int_0^T \varphi^{-1}\left(c(u) + \int_0^s f\,(r, u, u(r), u'(r))\;dr\right) ds = \mathcal{B}(u) - \mathcal{A}(u). \qquad (2.12)$$

Now, we arrive at the main result of this section, in which we ensure the solvability of problem (P).

Theorem 2.1. *Suppose that $(H_1) - (H_3)$ hold. If, in addition, operator $\mathcal{A}$ is bounded in $\mathcal{C}^1(I)$, then problem (P) has at least one solution.*

Proof. From Corollary 2.1, it suffices to prove that the operator $\mathcal{T}$, defined in (2.11) – (2.12), has a fixed point. To see this, we use that

$$\begin{aligned} |\mathcal{T}u(t)| &\leq |\mathcal{A}(u)| + \int_0^t \left|\varphi^{-1}\left(c(u) + \int_0^s f\,(r, u, u(r), u'(r))\,dr\right)\right| ds \\ &\leq K + \int_0^T \left|\varphi^{-1}\left(c(u) + \int_0^s f\,(r, u, u(r), u'(r))\,dr\right)\right| ds \end{aligned}$$

In consequence

$$\|\mathcal{T}u\|_\infty \quad \leq \quad K + a\,T.$$

On the other hand, since

$$(\mathcal{T}u)'(t) = \varphi^{-1}\left(c(u) + \int_0^t f\left(r, u, u(r), u'(r)\right)\, dr\right),$$

we conclude that

$$\|(\mathcal{T}u)'\|_\infty \quad \leq \quad a\,T.$$

These two last inequalities, together with the fact that operator $\mathcal{T}$ is completely continuous on $\mathcal{C}^1(I)$ (see [10, Theorem 2.2] for details), imply, by the Schauder's fixed point theorem [14], that operator $\mathcal{T}$ has at least one fixed point.

□

3 Upper and lower solutions

In this section, by assuming the existence of a pair of well ordered lower and upper solutions, we provide an existence result for the following nonlinear functional problem with nonlinear functional boundary conditions:

$$(P^*)\quad \begin{cases} \dfrac{d}{dt}\varphi(u'(t)) = f(t,u,u(t),u'(t)) \text{ for a.e. } t \in I, \\ L_1(u(a),u'(a),u) = 0, \\ L_2(u(b),u'(b),u) = 0. \end{cases}$$

In this case φ satisfies condition (H_1) and f and L_i, $i = 1, 2$, satisfy the following conditions:

(H_2^*) f is a locally L^1-bounded Carathéodory function, in the following standard sense:

$f(t, \cdot, \cdot, \cdot)$ is continuous in $\mathcal{C}(I) \times \mathbb{R}^2$ for a. e. $t \in I$; $f(\cdot, \xi, x, y)$ is measurable for all $(\xi, x, y) \in \mathcal{C}(I) \times \mathbb{R}^2$; and for every $R > 0$ there exists $\psi \in L^1(I)$ and a null measure set $N \subset I$ such that $|f(t, \xi, x, y)| \leq \psi(t)$ for all $(t, \xi, x, y) \in (I \setminus N) \times \mathcal{C}(I) \times \mathbb{R}^2$ with $\|(\xi, x, y)\|_\infty \leq R$.

(H_3^*) $L_i \in \mathcal{C}(\mathbb{R}^2 \times \mathcal{C}(I), \mathbb{R})$, $i = 1, 2$. Furthermore, for every $(x, u) \in \mathbb{R} \times \mathcal{C}(I)$, $L_1(x, \cdot, u)$ is nondecreasing and $L_2(x, \cdot, u)$ is nonincreasing.

In the sequel, we introduce the concept of lower and upper solutions for problem (P^*) as follows:

Definition 3.1. *Two functions α, $\beta : I \to \mathbb{R}$ such that $\alpha \leq \beta$ on I are said to be a coupled lower and upper solution of problem (P^*) if the following conditions are satisfied:*

(i) $\alpha \in \mathcal{C}^1(I)$, $\|\alpha'\| < a$, $\varphi(\alpha'(\cdot)) \in AC(I)$ *and*

$$\frac{d}{dt}\varphi(\alpha'(t)) \geq f(t, \xi, \alpha(t), \alpha'(t)) \text{ for a.e. } t \in I \text{ and all } \xi \in [\alpha, \beta].$$

(ii) $L_1(\alpha(a), \alpha'(a), \xi) \geq 0$, *and* $L_2(\alpha(b), \alpha'(b), \xi) \geq 0$, *for all* $\xi \in [\alpha, \beta]$.

(iii) $\beta \in \mathcal{C}^1(I)$, $\|\beta'\| < a$, $\varphi(\beta'(\cdot)) \in AC(I)$ *and*

$$\frac{d}{dt}\varphi(\beta'(t)) \leq f(t, \xi, \beta(t), \beta'(t)) \text{ for a.e. } t \in I \text{ and all } \xi \in [\alpha, \beta].$$

(iv) $L_1(\beta(a), \beta'(a), \xi) \le 0$, *and* $L_2(\beta(b), \beta'(b), \xi) \le 0$, *for all* $\xi \in [\alpha, \beta]$.

Remark 3.1. *Note that in the definition of a lower solution it appears the concept of upper solution an vice-versa. This is why we use the concept of "coupled" lower and upper solutions.*

It is immediate to verify that this overlapping disappears on conditions (i) *and* (iii) *when* $f(t, \cdot, x, y)$ *is nondecreasing in* $\mathcal{C}(I)$ *for all* $(t, x, y) \in (I \setminus N') \times \mathbb{R}^2$, N' *being a null measure set. In particular, it holds for the nonfunctional case, i.e.,* $f(t, \xi, x, y) \equiv f(t, x, y)$.

Conditions (ii) *and* (iv) *remain independent when the nonlinear operators* L_i, $i = 1, 2$, *are nondecreasing with respect to the last variable, i.e., if* $u_1, u_2 \in \mathcal{C}(I)$ *are such that* $u_1(t) \le u_2(t)$ *for all* $t \in I$ *then*

$$L_i(x, y, u_1) \le L_i(x, y, u_2) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

As far as we know, this is the first time in which this kind of monotonicity assumption is avoided.

On the contrary to the regular case, in which one of the main difficulty consists on giving some a priori bounds of the first derivative of all the possible solutions. In this case, the definition of the singular φ operator, implies that $\|u'\|_\infty < a$ is a necessary condition for all the solutions of the studied problem. As we have seen, this important fact has been fundamental to prove the general existence result for problem (P). This property has been pointed out, at the first time, by Bereanu and Mawhin in [6], when they study the homogeneous Dirichlet problem, and in [5] for the nonhomogeneous boundary conditions $u(0) = A$,

$u(T) = B$, with A and B two given constants.

However, they show that this general existence result is not true for nonhomogeneous Neumann boundary conditions. In this direction, they prove the existence of solutions for the Neumann – Steklov boundary conditions $\varphi(u'(0)) = g_0(u(0))$, $\varphi(u'(T)) = g_0(u(T))$, with g_0 and g_T to real continuous functions, by means of the method of lower and upper solutions.

In the sequel we prove the existence of at least one solution of problem (P^*) lying between a pair of well ordered lower and upper solutions. We do not need to impose any growth condition on the function f with respect to the dependence on u' (Nagumo - type conditions), but we will assume the additional condition

$$(H_4^*) \qquad \max\{\beta(T) - \alpha(T), \beta(0) - \alpha(0)\} < aT.$$

Before proving the main result of this paper, we define

$$p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\} \text{ for all } (t,x) \in I \times \mathbb{R},$$

and present the following result given in [18]

LEMMA 3.1. *Given $v, v_n \in C^1(I)$ such that $v_n \to v$ in $C^1(I)$, then*

(i) $\dfrac{d}{dt}p(t, v(t))$ *exists for a.e.* $t \in I$;

(ii) $\dfrac{d}{dt}p(t, v_n(t)) \to \dfrac{d}{dt}p(t, v(t))$ *for a.e.* $t \in I$.

Now, we are in a position to prove the following existence result, in which we develop the classical theory of lower and upper solutions.

THEOREM 3.1. *Let φ, f and L_i, $i = 1, 2$, satisfy (H_1), (H_2^*) and (H_3^*). Assume that α and β are coupled lower and upper solutions for problem (P^*) that satisfy (H_4^*). Then problem (P^*) has at least one solution $u \in [\alpha, \beta]$.*

Proof. First, we define $\delta_a(y) = \max\{-a, \min\{y, a\}\}$ for all $y \in \mathbb{R}$ and consider the following modified problem

$$(P_M^*) \quad \begin{cases} \dfrac{d}{dt}\varphi(u'(t)) = f\left(t, p(\cdot, u(\cdot)), p(t, u(t)), \delta_a\left(\dfrac{d}{dt}p(t, u(t))\right)\right), \\ \\ u(0) = \mathcal{A}(u), \quad u(T) = \mathcal{B}(u), \end{cases}$$

with

$$\mathcal{A}(v) = p(0, v(0) + L_1(v(0), v'(0), v))$$

and

$$\mathcal{B}(v) = p(T, v(T) + L_2(v(T), v'(T), v))$$

for all $v \in \mathcal{C}^1(I)$.

We note that problem (P_M^*) is of the form (P), with the right hand side defined as

$$\begin{cases} f\left(t, p(\cdot, \xi), \beta(t), \beta'(t)\right), & \text{if} \quad x > \beta(t), \\ f\left(t, p(\cdot, \xi), x, \delta_a(y)\right), & \text{if} \quad \alpha(t) \le x \le \beta(t), \\ f\left(t, p(\cdot, \xi), \alpha(t), \alpha'(t)\right), & \text{if} \quad x < \alpha(t). \end{cases}$$

It is obvious that this function is discontinuous in the last variable. However, from condition (H_2^*) and Lemma 3.1, one can easily check that condition (H_2) holds in this case.

From the definition of function p it is clear that condition (H_3) is also fulfilled and that operator $\mathcal{A}$ is bounded in $C^1(I)$. In consequence Theorem 2.1 implies that problem (P_M^*) has at least one solution.

Now it suffices to prove that every solution of (P_M^*) is, actually, a solution of (P^*).

To this end, we prove that every solution of (P_M^*) belongs to the sector $[\alpha, \beta]$. This property follows from the fact that $u(0) \in [\alpha(0), \beta(0)]$, $u(T) \in [\alpha(T), \beta(T)]$, and that, for all $(a, b) \subset (0, T)$ such that $u(t) > \beta(t)$ for a.e. $t \in (a, b)$, it is satisfied that

$$\frac{d}{dt}\varphi(u'(t)) = f\left(t, p(\cdot, u(\cdot)), \beta(t), \beta'(t)\right) \geq \frac{d}{dt}\varphi(\beta'(t)), \quad \text{for a.e. } t \in (a, b).$$

But this implies that, if such interval exists, then $\varphi(u') - \varphi(\beta')$ is a nondecreasing function on (a, b), and we conclude that $u(T) > \beta(T)$, a contradiction.

To deduce that every solution of (P_M^*) satisfies the boundary conditions we must take into account that if

$$u(T) + L_2(u(T), u'(T), u) < \alpha(T)$$

the definition of $\mathcal{B}$ gives us that $u(T) = \alpha(T)$. Since $u \in [\alpha, \beta]$ we also have $u'(b) \leq \alpha'(b)$. Now using condition (H_3^*) and the definition of lower solution we conclude

$$\alpha(T) > \alpha(T) + L_2(\alpha(T), \alpha'(T), u) \geq \alpha(T).$$

The rest of the proof follows similar steps as in [10], and so the result is established. □

We finish the paper by presenting the following example in which we point out the kind of problems we are able to consider.

Example 3.1. *Consider the problem*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{u'(t)}{\sqrt{4-(u'(t))^2}}\right) &= 1-e^{-t}+9\,u^3(t)+2\,u^5(\cos(\pi\,t/2)) \\ & \quad -\displaystyle\int_0^t u^2(s)\,ds+\sinh(u'(t)), \quad t\in I=[0,1], \\ 5\,u(0) &= 1/2-\displaystyle\min_{t\in[1/3,1]} u(t)+u(1/2)+e^{u'(0)}. \\ u^3(1) &= -\displaystyle\int_0^1 u^5(s)\,ds-(u'(1))^7. \end{cases}$$

This problem is a particular case of (P)*, with*

$$\varphi(x)=\frac{x}{\sqrt{4-x^2}} \quad \text{for } |x|<2=a,$$

$$f(t,\xi,x,y)=1-e^{-t}+9\,x^3+2\,\xi^5(\cos(\pi\,t/2))-\int_0^t \xi^2(s)\,ds+\sinh y,$$

for $(t,\xi,x,y)\in I\times\mathcal{C}(I)\times\mathbb{R}^2$*,*

$$L_1(x,y,\xi)=1/2-5\,x-\min_{t\in[1/3,1]}\xi(t)+\xi(1/2)+e^y, \quad \text{for } (x,y,\xi)\in\mathbb{R}^2\times\mathcal{C}(I)$$

and

$$L_2(x,y,\xi)=-x^3-\int_0^1 \xi^5(s)\,ds-y^7, \quad \text{for } (x,y,\xi)\in\mathbb{R}^2\times\mathcal{C}(I).$$

It is clear that $\alpha(t)=-1/2$ *and* $\beta(t)=1/2$*,* $t\in I$*, are a pair of coupled lower and upper solutions of this problem. Since* (H_1)*,* (H_2^*)*,* (H_3^*) *and* (H_4^*) *are fulfilled, by Theorem 3.1 we have that it has at least one solution such that* $\|u\|_\infty\le 1/2$ *and* $\|u'\|_\infty<2$*.*

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