Existence of solutions of nth-order nonlinear difference equations with general boundary conditions.

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Abstract

The aim of this paper is to prove the existence of one or multiple solutions of nonlinear difference equations coupled to a general set of boundary conditions.

Before to do this, we construct a discrete operator whose fixed points coincide with the solutions of the problem we are looking for. Moreover, we introduce a strong positiveness condition on the related Green's function that allows us to construct suitable cones where to apply adequate fixed point theorems.

Once we have the general existence result, we deduce, as a particular case, the existence of solutions of a second order difference equation with nonlocal perturbed Dirichlet conditions.

Keywords: difference equation, multiplicity of solutions, Green's function, positive solutions, parameter dependence.

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1 Introduction

It is very well known that the theory of difference equations appears in many different fields as, among others, computer science, economical models or population dinamics. We refer to the reader the classical books by Agarwal [1] and Kelly and Peterson [15] for a general overview on the basic theory of this type of equations, coupled with some interesting examples and mathematical models of the related topics.

More recently, several authors have focused their investigation in proving the existence and multiplicity of solutions of difference problems by using various methods from nonlinear analysis.

In particular, the method of upper and lower solutions coupled to Leray-Schauder degree theory and some different kinds of fixed point theorems in cones are very useful tools to obtain the existence of solutions of nonlinear boundary value problems. We make special mention of the paper [16], where Legget and Williams established a fixed point result, which has been extended during the next decades. Such results have been improved by the same authors in [17], where, as an application, it is proved the existence of multiple solutions of the following third order boundary value problem:

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \ t \in (0,1), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} k_i u(\xi_i), \ u'(0) = u'(1) = 0, \end{cases}$$

with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $k_i \in \mathbb{R}^+$ for $i = 1, \ldots, m-2$ and $0 < \sum_{i=1}^{m-2} k_i < 1$.

A result in the line of Legget-Williams fixed point theorem, that ensures the existence of at least a positive fixed point on different sets defined by means of suitable functionals, is obtained in [2].

In [8] the authors generalized the triple fixed point theorem of Legget-Williams, which allow them to prove the existence of three positive symmetric solutions of the discrete second order nonlinear conjugate boundary value problem

$$\begin{cases} \Delta^2 x(t-1) + f(x(t)) = 0, \ \text{for all } t \in [a+1, b+1], \\ x(a) = 0 = x(b+2), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $x \geq 0$.

Recently, in [5], the authors proved a new fixed point theorem that gives us a different existence result for problem studied in [2].

In [14] it is proved the existence of one or multiple solutions of a wide range of nonlinear ordinary differential equations, coupled to boundary value conditions, by imposing the following hypothesis on the kernel G :

(Pg_1) There exist Φ, k_1 and k_2 continuous functions on $[a,b]$ such that $\Phi(s) > 0$ for all $s \in (a,b)$, $0 < k_1(t) \leq k_2(t)$ for all $t \in (a,b)$ and

$$\Phi(s) k_1(t) \leq G(t,s) \leq \Phi(s) k_2(t), \ \forall (t,s) \in [a,b] \times [a,b].$$

This kind of conditions have been introduced in [9] and ensure the validity of monotone iterative techniques in a general framework. Moreover, under this condition, a characterization of the set of real parameters where the Green's function has constant sign is given. The extremes of the corresponding intervals are the first eigenvalues of the operator defined on related functional spaces, see [11, 12, 13] for details.

In this paper, we assume the discrete version of the above hypothesis and, moreover

(F) $f : I \times [0, \infty) \to [0, \infty)$ is a continuous function.

Here we denote $I \equiv \{a, \ldots, b\}$, with $b - a \geq 2$.

We obtain multiplicity results for a family of n-th order boundary value problems given by

$$u(k+n) + \sum_{i=0}^{n-1} p_i(k) u(k+i) = f(k, u(k)), \ k \in I, \tag{1}$$

$$L_i\, u = 0, \quad i = 1, \ldots, n. \tag{2}$$

Here

$$L_i : \mathbb{R}^{b-a+1} \to \mathbb{R}, \quad i = 1, \ldots, n,$$

are linear operators, for which the following condition for the related Green's function is fulfilled:

(G) There exist non negative functions on I, Φ, l_1 and l_2, such that $\Phi(s) > 0$ for all $s \in J \equiv \{a+1, \ldots, b-1\}$, $0 < l_1(k) \leq l_2(k)$ for all $k \in J$ and

$$\Phi(s) l_1(k) \leq G(k,s) \leq \Phi(s) l_2(k), \ \forall \ (k,s) \in I \times I. \tag{3}$$

As an application of these results, in Section 4 we continue the ones given in [10] for a second order problem. Moreover, we prove the existence of at least two or three solutions of the considered problem.

It is well known that, provided problem (1)-(2) has $u \equiv 0$ as its unique solution when $f \equiv 0$, the solutions of problem (1)-(2) are given as the fixed points of the difference operator

$$Tu(k) = \sum_{s=a}^{b} G(k,s) f(s, u(s)), \tag{4}$$

where $G(k,s)$ is its associated Green's function.

Thus, in order to find the fixed points of operator T, we previously study in Section 3, the existence of at least two or three fixed points of the difference operator.

2 Description of the problem and some previous fixed point existence results

In this section, in order to study the existence of some fixed points of the difference operator, defined in (4) in an appropriate cone, we give some basic definitions and we recall some previous results.

First of all, we recall some definitions.

Definition 2.1. *Let B be a real Banach space. A nonempty closed convex set $P \subset B$ is called a cone if it satisfies the following two conditions:*
1) $\lambda x \in P$ for all $x \in P$ and $\lambda \geq 0$.
2) If $x \in P$ and $-x \in P$, then $x = 0$.

Then, consider a subinterval $I_1 = \{a_1, \ldots, b_1\} \subset I$ such that $l_1(k) > 0$ for all $k \in I_1$ and denote:

$$L_1 = \max_{k \in I} l_1(k) > 0, \ m = \min_{k \in I_1} l_1(k) > 0, \ L_2 = \max_{k \in I} l_2(k) > 0. \tag{5}$$

Finally, let us consider the cone

$$P = \left\{ u : I \to [0, \infty), \ u(k) \geq \frac{l_1(k)}{L_2} \left\| u \right\|_\infty, \ k \in I \right\},$$

where

$$\left\| u \right\|_\infty := \max_{k \in I} \left| u(k) \right|.$$

Now, we give definitions of concave and convex functional on a cone.

Definition 2.2. *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space B if $\alpha : P \to [0, +\infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \ \forall x, \ y \in P, \ t \in [0,1].$$

Similarly, a map β is said to be a nonnegative continuous convex functional on a cone P of a real Banach space B if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y), \ \forall x, \ y \in P, \ t \in [0,1].$$

Let β, γ and θ, be nonnegative continuous convex functionals on the cone P, and α and ψ, nonnegative concave functionals on P. Thus, for nonnegative real numbers d, p and q, we define the following subspaces of the cone P :

$$\begin{aligned}
P(\gamma, r) &= \{u \in P \mid \gamma(u) < r\}, \\
P(\gamma, \alpha, p, r) &= \{u \in P \mid p \leq \alpha(u), \ \gamma(u) \leq r\}, \\
Q(\gamma, \beta, d, r) &= \{u \in P \mid \beta(u) \leq d, \ \gamma(u) \leq r\}, \\
P(\gamma, \theta, \alpha, p, q, r) &= \{u \in P \mid p \leq \alpha(u), \ \theta(u) \leq q, \ \gamma(u) \leq r\}.
\end{aligned}$$

Recall a result, proved in [6], which ensures the existence of two fixed points on the cone P.

Theorem 2.3. *Let P be a cone in a real Banach space B. Let α and γ be increasing and nonnegative continuous functionals on P. Let θ be a nonnegative continuous functional on P with $\theta(0) = 0$, such that, for some positive constants r and M,*

$$\alpha(u) \leq \theta(u) \leq \gamma(u) \ \text{and} \ \|u\| \leq M\alpha(u), \ \text{for all} \ u \in \overline{P(\alpha, r)}.$$

Assume that there exist two positive numbers p and q with $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \ \text{for all} \ 0 \leq \lambda \leq 1 \ \text{and} \ u \in \partial P(\theta, q).$$

Suppose that $L : \overline{P(\alpha, r)} \to P$ is a completely continuous operator satisfying
i) $\alpha(Lu) > r$ for all $u \in \partial P(\alpha, r)$,
ii) $\theta(Lu) < q$ for all $u \in \partial P(\theta, q)$,
iii) $P(\gamma, p) \neq \emptyset$ and $\gamma(Lu) > p$ for all $u \in \partial P(\gamma, p)$.
Then, L has at least two fixed points u_1 and u_2 such that

$$p < \gamma(u_1), \ \text{with} \ \theta(u_1) < q,$$

and

$$q < \theta(u_2), \ \text{with} \ \alpha(u_2) < r.$$

Finally, we introduce a result, see[4], that ensures the existence of three fixed points of L on the cone P.

Theorem 2.4. *Let P be a cone in a real Banach space B, and let r and M be positive numbers. Assume that α and ψ are nonnegative, continuous and concave functionals on P, and γ, β and θ are nonnegative, continuous and convex functional on P with*

$$\alpha(u) \leq \beta(u) \ \text{and} \ \|u\| \leq M\gamma(u), \ \text{for all} \ u \in \overline{P(\gamma, r)}.$$

Suppose that $L : \overline{P(\gamma, r)} \to \overline{P(\gamma, r)}$ *is a completely continuous operator and there exist nonnegative numbers* h, d, p, q *with* $0 < d < p$*, such that*

a) $\{u \in P(\gamma, \theta, \alpha, p, q, r) \mid \alpha(u) > p\} \neq \emptyset$ *and* $\alpha(Lu) > p$ *for* $u \in P(\gamma, \theta, \alpha, p, q, r)$,

b) $\{u \in P(\gamma, \beta, \psi, h, d, r) \mid \beta(u) < d\} \neq \emptyset$ *and* $\beta(Lu) < d$ *for* $u \in P(\gamma, \beta, \psi, h, d, r)$,

c) $\alpha(Lu) > p$ *for all* $u \in P(\gamma, \alpha, p, r)$ *with* $\theta(Lu) > q$,

d) $\beta(Lu) < d$ *for all* $u \in Q(\gamma, \beta, d, r)$ *with* $\psi(Lu) < h$.

Then, L *has at least three fixed points* u_1, u_2 *and* $u_3 \in \overline{P(\gamma, r)}$ *such that*

$$\beta(u_1) < d, \;\; p < \alpha(u_2) \text{ and } d < \beta(u_3) \text{ with } \alpha(u_3) < p.$$

3 Existence of multiple fixed points

This section is devoted to prove the existence of multiple solutions of problem (1) – (2). To this end, we previously obtain some useful properties of operator T defined in (4).

Let $u \in P$ be arbitrarily chosen. Clearly, from condition (G), we have that $Tu \geq 0$ on I and, moreover, we deduce that the following inequalities are fulfilled for all $k \in I$:

$$\begin{aligned} Tu(k) &= \sum_{s=a}^{b} G(k,s) f(s,u(s)) \geq l_1(k) \sum_{s=a}^{b} \Phi(s) f(s,u(s)) \\ &= \frac{l_1(k)}{L_2} \sum_{s=a}^{b} \max_{k \in I} l_2(k) \Phi(s) f(s,u(s)) \geq \frac{l_1(k)}{L_2} \sum_{s=a}^{b} \max_{k \in I} \{G(k,s)\} f(s,u(s)) \\ &= \frac{l_1(k)}{L_2} \sum_{s=a}^{b} \max_{k \in I} \{G(k,s) f(s,u(s))\} \geq \frac{l_1(k)}{L_2} \|Tu\|_\infty . \end{aligned}$$

In other words, $T : P \to P$.

Moreover, due to the continuity of function f, it is clear that T is a completely continuous operator.

Now, from Theorems 2.3 and 2.4, we deduce the existence of two or three fixed points, respectively, of operator T defined in (4). We follow the steps given in [3, 14].

Theorem 3.1. *Suppose that there exist positive integers* p, q *and* r *such that* $p < q < r$*, and assume that function* f *satisfies the following conditions:*

(i) $f(k,u) \geq \frac{u}{m \sum_{s=a_1}^{b_1} \Phi(s)}$ *for all* $k \in I_1$ *and* $u \in \left[r, \frac{L_2}{m} r\right]$*, being the inequality strict at* $u = r$,

(ii) $f(k,u) \leq \frac{q}{L_2 \sum_{s=a}^{b} \Phi(s)}$ *for all* $k \in I$ *and* $u \in \left[0, \frac{L_2}{m} q\right]$*, being the inequality strict at* $u = q$,

(iii) $f(k,u) > \frac{u L_2}{L_1 \sum_{s=a_1}^{b_1} l_1(s) \Phi(s)}$ *for all* $k \in I_1$ *and* $u \in \left[\frac{m}{L_2} p, p\right]$.

Then if $G(k,s)$ *satisfies condition (G), then operator* T *has at least two fixed points,* u_1 *and* u_2*, such that*

$$p < \|u_1\|_\infty , \;\; \max_{k \in I_1} u_1(k) < q < \max_{k \in I_1} u_2(k), \;\; \min_{k \in I_1} u_2(k) < r.$$

Proof. Let us denote:

$$\alpha(u) := \min_{k\in I_1} u(k), \tag{6}$$

$$\theta(u) := \max_{k\in I_1} u(k), \tag{7}$$

and

$$\gamma(u) := \|u\|_\infty . \tag{8}$$

For all $u \in P$ we have that $\alpha(u) \le \theta(u) \le \gamma(u)$. The fact that $u \in P$ gives us that

$$\alpha(u) = \min_{k\in I_1} u(k) \ge \min_{k\in I_1} \frac{l_1(k)}{L_2} \|u\|_\infty = \frac{m}{L_2}\gamma(u).$$

Thus $\gamma(u) \le \frac{L_2}{m}\alpha(u)$ for all $u \in P$.

Hence, for all $\lambda \ge 0$ and $u \in P$, we verify that:

$$\theta(\lambda u) = \max_{k\in I_1} \{\lambda u(k)\} = \lambda \max_{k\in I_1} u(k) = \lambda\,\theta(u).$$

If $u \in \partial P(\alpha, r)$, i.e. $\min_{k\in I_1} u(k) = r$, then $\alpha(u) = r \ge \frac{m}{L_2} \|u\|_\infty$.

Using (i) and (G), we deduce the following inequalities:

$$\begin{aligned}\alpha(Tu) &= \min_{k\in I_1} \sum_{s=a}^{b} G(k,s)f(s,u(s)) \ge \min_{k\in I_1} \sum_{s=a}^{b} l_1(k)\Phi(s)f(s,u(s)) \\ &\ge \min_{k\in I_1} l_1(k) \sum_{s=a_1}^{b_1} \Phi(s)f(s,u(s)) \ge \sum_{s=a_1}^{b_1} \Phi(s)\frac{u(s)}{\sum\limits_{s=a_1}^{b_1} \Phi(s)}.\end{aligned}$$

The fact that $\alpha(u) = r$ gives us that there exists $k_1 \in I_1$ with $u(k_1) = r$. According to (i) we have $f(k_1, u(k_1)) > \frac{u(k_1)}{m \sum\limits_{s=a_1}^{b_1} \Phi(s)}$. Since $\Phi > 0$ on I_1, the inequality for α is strict too, and it follows that

$$\alpha(Tu) > \sum_{s=a_1}^{b_1} \Phi(s)\frac{u(s)}{\sum\limits_{s=a_1}^{b_1} \Phi(s)} \ge r \text{ for all } u \in \partial P(\alpha, r).$$

Next, if $u \in \partial P(\theta, q)$, i.e. $\max_{k\in I_1} u(k) = q$, we deduce that

$$\gamma(u) \ge \theta(u) = q \ge \alpha(u) \ge \frac{m}{L_2}\gamma(u).$$

The last one gives us that $q \le \|u\|_\infty \le \frac{L_2}{m} q$ and from (ii) and (G), we obtain:

$$\begin{aligned}\theta(Tu) &= \max_{k\in I_1} \sum_{s=a}^{b} G(k,s)f(s,u(s)) \le \max_{k\in I_1} \sum_{s=a}^{b} l_2(k)\Phi(s)f(s,u(s)) \\ &\le L_2 \sum_{s=a}^{b} \Phi(s)f(s,u(s)) \le \sum_{s=a}^{b} \Phi(s)\frac{q}{\sum\limits_{s=a}^{b} \Phi(s)}.\end{aligned}$$

Following the previous arguments, $\theta(u) = q$ gives us that there exists $k_2 \in I_1$ with $u(k_2) = q$. Using (ii) and the fact that $\Phi > 0$ on J we arrive at

$$\theta(Tu) < \sum_{s=a}^{b} \Phi(s) \frac{q}{\sum\limits_{s=a}^{b} \Phi(s)} = q \text{ for all } u \in \partial P(\theta, q).$$

Now, since $P(\gamma, p) = \{u \in P \mid \|u\|_\infty < p\} \neq \emptyset$, we deduce that $\|u\|_\infty = p$ and $\alpha(u) \geq \frac{m}{L_2} p$ for all $u \in \partial P(\gamma, p)$.

Finally, using (iii) and (G), one can check that:

$$\begin{aligned}
\gamma(Tu) &= \max_{k \in I} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \geq \max_{k \in I} \sum_{s=a}^{b} l_1(k) \Phi(s) f(s,u(s)) \\
&\geq L_1 \sum_{s=a_1}^{b_1} \Phi(s) f(s,u(s)) > L_1 \sum_{s=a_1}^{b_1} \Phi(s) \frac{L_2 u(s)}{L_1 \sum_{s=a_1}^{b_1} l_1(s) \Phi(s)} \\
&\geq \sum_{s=a_1}^{b_1} \Phi(s) \frac{l_1(s) \|u\|_\infty}{\sum\limits_{s=a_1}^{b_1} l_1(s) \Phi(s)} = p.
\end{aligned}$$

Thus, $\gamma(Tu) > p$ for all $u \in \partial P(\gamma, p)$ and all the assumptions of Theorem 2.3 are verified. Hence, T has at least two fixed points on P, u_1 and u_2, such that $p < \gamma(u_1) = \|u_1\|_\infty$ and $q > \theta(u_1) = \max\limits_{k \in I_1} u_1(k)$. Moreover, $q < \theta(u_2) = \max\limits_{k \in I_1} u_2(k)$ and $r > \alpha(u_2) = \min\limits_{k \in I_1} u_2(k)$. □

Remark 3.2. *We point out that due to the properties that the fixed points u_1 and u_2 satisfy, both of them are not trivial.*

As an application of Theorem 2.4, we formulate the next result that gives us the existence of at least three fixed points of operator T.

Theorem 3.3. *Let p, q and r be positive integers such that:*

$$p < q < \frac{L_2}{m} q \leq r.$$

Suppose that the function f satisfies the assumptions below:

(a) $f(k,u) \leq \frac{r}{L_2 \sum\limits_{s=a}^{b} \Phi(s)}$ for all $k \in I$ and $u \in [0, r]$,

(b) $f(k,u) < \frac{p}{L_2 \sum\limits_{s=a}^{b} \Phi(s)}$ for all $k \in I$ and $u \in [0, p]$,

(c) $f(k,u) \geq \frac{u}{m \sum\limits_{s=a_1}^{b_1} \Phi(s)}$ for all $k \in I_1$ and $u \in \left[q, \frac{L_2}{m} q\right]$, being the inequality strict for $u = q$.

Then, operator T has at least three fixed points $u_1, u_2, u_3 \in \{u \in P \mid \|u\|_\infty \leq r\}$ such that $\max\limits_{k \in I_1} u_1(k) < p$, $q < \min\limits_{k \in I_1} u_2(k)$ and $p < \max\limits_{k \in I_1} u_3(k)$ with $\max\limits_{k \in I_1} u_3(k) < q$.

Proof. Let α, θ, γ are defined as in (6)-(8), $\Psi(u) = \alpha(u)$ and $\beta(u) = \theta(u)$. It is easy to check α and Ψ are concave and nonnegative functionals in P, while β, θ and γ are convex and nonnegative functionals in P.

We already proved that $T(P) \subset P$. Now, let us show that $T\left(\overline{P(\gamma, r)}\right) \subset \overline{P(\gamma, r)}$. Indeed, if $u \in \overline{P(\gamma, r)}$, i.e. $\|u\|_\infty \le r$, then using (a) it follows that:

$$\begin{aligned} \|Tu\|_\infty &= \max_{k \in I} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \le \max_{k \in I} \sum_{s=a}^{b} l_2(k)\Phi(s) f(s,u(s)) \\ &= L_2 \sum_{s=a}^{b} \Phi(s) f(s,u(s)) \le \sum_{s=a}^{b} \Phi(s) \frac{r}{\sum\limits_{s=a}^{b} \Phi(s)} = r. \end{aligned}$$

Hence, $Tu \in \overline{P(\gamma, r)}$ and $T\left(\overline{P(\gamma, r)}\right) \subset \overline{P(\gamma, r)}$.

Clearly, $\alpha(u) \le \beta(u)$ and $\gamma(u) = \|u\|_\infty$.

One can check that $u_q(k) = \frac{L_2}{m} q$ belongs to the set

$$\left\{ u \in P\left(\gamma, \theta, \alpha, q, \frac{L_2}{m} q, r\right) \mid \alpha(u) > q \right\}.$$

Thus

$$\left\{ u \in P \mid q < \min_{k \in I_1} u(k),\ \max_{k \in I_1} u(k) \le \frac{L_2}{m} q,\ \|u\|_\infty \le r \right\} \ne \emptyset.$$

Let $u \in P\left(\gamma, \theta, \alpha, q, \frac{L_2}{m} q, r\right)$. From (c), it follows that:

$$\begin{aligned} \alpha(Tu) &= \min_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \ge \min_{k \in I_1} \sum_{s=a}^{b} l_1(k)\Phi(s) f(s,u(s)) \\ &\ge m \sum_{s=a_1}^{b_1} \Phi(s) f(s,u(s)) \ge \sum_{s=a_1}^{b_1} \Phi(s) \frac{u(s)}{\sum\limits_{s=a_1}^{b_1} \Phi(s)}. \end{aligned}$$

If there exists $s_1 \in I_1$ such that $u(s_1) > q$, then from the last inequality, we have $\alpha(Tu) > q$.

Otherwise, if $u(s) = q$ for all $s \in I_1$, then by using (c), we obtain:

$$\alpha(Tu) \ge m \sum_{s=a_1}^{b_1} \Phi(s) f(s,q) > \sum_{s=a_1}^{b_1} \Phi(s) \frac{u(s)}{\sum\limits_{s=a_1}^{b_1} \Phi(s)} = q.$$

Similarly as above, function $u_p(k) = \frac{m}{L_2} p$ belongs to the set

$$\left\{ u \in P\left(\gamma, \beta, \Psi, \frac{m}{L_2} p, p, r\right) \mid \beta(u) < p \right\},$$

so

$$\left\{ u \in P \mid \frac{m}{L_2} p \le \min_{k \in I_1} u(k),\ \max_{k \in I_1} u(k) < p,\ \|u\|_\infty \le r \right\} \ne \emptyset.$$

If $u \in P\left(\gamma, \beta, \Psi, \frac{m}{L_2}p, p, r\right)$, then using (b), one can check that:

$$
\begin{aligned}
\beta(Tu) &= \max_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \leq \max_{k \in I_1} \sum_{s=a}^{b} L_2 \Phi(s) f(s,u(s)) \\
&< L_2 \sum_{s=a}^{b} \Phi(s) \frac{p}{L_2 \sum\limits_{s=a}^{b} \Phi(s)} = p.
\end{aligned}
$$

Hence, $\beta(Tu) < p$ for all $u \in P\left(\gamma, \beta, \Psi, \frac{m}{L_2}p, p, r\right)$.

Suppose that $u \in P(\gamma, \alpha, q, r)$ and $\theta(Tu) > \frac{L_2}{m}q$. One can verify that

$$
\begin{aligned}
\alpha(Tu) &= \min_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \geq \min_{k \in I_1} \sum_{s=a}^{b} l_1(k) \Phi(s) f(s,u(s)) \\
&= \frac{m}{L_2} \sum_{s=a}^{b} L_2\, \Phi(s) f(s,u(s)) \geq \frac{m}{L_2} \sum_{s=a}^{b} \max_{k \in I_1} G(k,s) f(s,u(s)) \\
&\geq \frac{m}{L_2} \max_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) = \frac{m}{L_2} \theta(Tu) > \frac{m}{L_2} \frac{L_2}{m} q = q.
\end{aligned}
$$

If $u \in Q(\gamma, \beta, p, r)$ with $\Psi(Tu) < \frac{m}{L_2}p$, it follows that

$$
\begin{aligned}
\beta(Tu) &= \max_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) \leq \sum_{s=a}^{b} L_2 \Phi(s) f(s,u(s)) \\
&= \frac{L_2}{m} \sum_{s=a}^{b} m \Phi(s) f(s,u(s)) \leq \frac{L_2}{m} \sum_{s=a}^{b} \min_{k \in I_1} G(k,s) f(s,u(s)) \\
&\leq \frac{L_2}{m} \min_{k \in I_1} \sum_{s=a}^{b} G(k,s) f(s,u(s)) = \frac{L_2}{m} \Psi(Tu) < \frac{L_2}{m} \frac{m}{L_2} p = p.
\end{aligned}
$$

Thus, all the assumptions of Theorem 2.4 are verified, which ensures us the existence of at least three critical points such that $p > \beta(u_1) = \max\limits_{k \in I_1} u_1(k)$, $q < \alpha(u_2) = \min\limits_{k \in I_1} u_2(k)$ and $p < \beta(u_3) = \max\limits_{k \in I_1} u_3(k)$ with $q > \alpha(u_3) = \min\limits_{k \in J} u_3(k)$. $\square$

Remark 3.4. *We point out that the fixed points u_1 and u_2 obtained in previous result are not trivial. However, without additional assumptions on the data of operator T, we cannot ensure such property for u_3.*

4 An application to a second order problem

In this section, in order to maintain a similar notation to the one used in [10], we redefine $I = \{0, \ldots, N\}$ and $I_1 = J = \{1, \ldots, N-1\}$, i. e., $a = 0$, $b = N$, $a_1 = 1$ and $b_1 = N - 1$.

Our goal in this section is to extend the results given in [10] concerning the following second order problem with perturbed Dirichlet conditions:

$$-\Delta^2 u(k-1) = f(k,u(k)), \ k \in J, \ N \geq 2, \quad (9)$$

$$u(0) = 0, \ u(N) = \mu \sum_{k=c}^{d} u(k), \ \mu > 0, \ 1 \leq c \leq d \leq N-1. \quad (10)$$

In that case, existence of one or two nontrivial solutions are deduced by means of the Krasnoselskiĭ's fixed point theorem. In this section, as a direct application of the previous fixed point theorems, we deduce the existence of two or three solutions of problem (9) – (10). To this end, we assume the following property

$(H1)$ $\mu \in \left(0, \frac{2N}{(c+d)(d-c+1)}\right)$.

So, by denoting $J_1 = \{1, \ldots, N\}$, we have the following result

Theorem 4.1. *[10, Theorem 2.1] If μ satisfies hypothesis $(H1)$ then there is G the Green's function related to the linear part of problem (9) – (10). Moreover $G(k,s) > 0$ for all $k \in J_1$ and $s \in J$, and there are two positive constants $0 < m_1 < M_1$ for which*

$$m_1 \, G(N,s) \leq G(k,s) \leq M_1 \, G(N,s) \quad (11)$$

for all $k \in J_1$ and $s \in J$.

Since $G(0,s) = 0$ for all $s \in J$, it is clear that condition (G) is fulfilled in this situation.

Remark 4.2. *On [10] some explicit estimations of the constants m_1 and M_1 are obtained. Such expressions are very complicated and depends on the relative positions of s and a and b.*

In particular, we have that $\Phi(s) = G(N,s)$, $l_1(0) = l_2(0) = 0$, and

$$l_1(k) = m_1 \quad \text{and} \quad l_2(k) = M_1, \quad \text{for all } k \in J_1.$$

As consequence, the constants defined in (5) satisfy, in this case:

$$m = L_1 = m_1 \quad \text{and} \quad M_1 = L_2.$$

Using these properties, we deduce, as in Theorems 3.1 and 3.3, the existence of two or three solutions (with at least two of them non trivial on J) respectively, of problem (9) – (10).

Theorem 4.3. *Suppose that there exist positive integers p, q and r such that $p < q < r$, and assume that function f satisfies the following conditions:*

(i) $f(k,u) \geq \frac{u}{m_1 \sum_{s=1}^{N-1} G(N,s)}$ for all $k \in J$ and $u \in \left[r, \frac{M_1}{m_1} r\right]$, being the inequality strict at $u = r$,

(ii) $f(k,u) \leq \frac{q}{M_1 \sum_{s=1}^{N-1} G(N,s)}$ for all $k \in J$ and $u \in \left[0, \frac{M_1}{m_1} q\right]$, being the inequality strict at $u = q$,

(iii) $f(k,u) > \frac{u\,M_1}{m_1^2 \sum\limits_{s=1}^{N-1} G(N,s)}$ *for all* $k \in J$ *and* $u \in \left[\frac{m_1}{M_1}p, p\right]$.

Then problem (9) – (10) *has at least two nontrivial solutions,* u_1 *and* u_2*, such that*

$$p < \|u_1\|, \;\; \max_{k\in J} u_1(k) < q < \max_{k\in J} u_2(k), \;\; \min_{k\in J} u_2(k) < r.$$

Theorem 4.4. *Let* p, q *and* r *be positive numbers such that:*

$$p < q < \frac{M_1}{m} q \le r.$$

Assume, moreover, that the function f *satisfies the following conditions:*

(a) $f(k,u) \le \frac{r}{M_1 \sum\limits_{s=1}^{N-1} G(N,s)}$ *for all* $k \in I$ *and* $u \in [0,r]$,

(b) $f(k,u) < \frac{p}{M_1 \sum\limits_{s=1}^{N-1} G(N,s)}$ *for all* $k \in I$ *and* $u \in [0,p]$,

(c) $f(k,u) \ge \frac{u}{m_1 \sum\limits_{s=1}^{N-1} G(N,s)}$ *for all* $k \in J$ *and* $u \in \left[q, \frac{M_1}{m_1} q\right]$, *being the inequality strict for* $u = q$.

Then problem (9) – (10) *has at least three solutions* $u_1, u_2, u_3 \in \{u \in P \mid \|u\| \le r\}$ *such that* $\max\limits_{k\in J} u_1(k) < p$, $q < \min\limits_{k\in J} u_2(k)$ *and* $p < \max\limits_{k\in J} u_3(k)$ *with* $\max\limits_{k\in J} u_3(k) < q$.

In the sequel, we consider a particular case of problem (9) – (10). We fix the values of $c = 1, d = N-1$ and $\mu = \frac{1}{N-1}$. It is easy to check that condition (H1) holds. In this case the Green's function is given by the expression

$$G(k,s) = \begin{cases} \frac{k\left(N^2-N+s-s^2\right)}{N(N-1)}, & k \le s, \\ \frac{s\left(N^2-N+k-ks\right)}{N(N-1)}, & k \ge s. \end{cases}$$

Moreover, from (11) we have that

1) if $k \le s$ then $m_1 \le \frac{k(N+s-1)}{Ns}$ and

2) if $k \ge s$ then $m_1 \le \frac{N^2-N+k-ks}{N(N-s)}$.

Since $\frac{N+s-1}{s}$ is decreasing on s and $\frac{N^2-N+k-ks}{N-s}$ is increasing on s we obtain that $m_1 = 2/N$.

Using similar arguments we deduce that $M_1 = 2$.

By direct calculations we obtain that

$$\sum_{s=1}^{N-1} G(N,s) = \sum_{s=1}^{N-1} \frac{s(N-s)}{N-1} = \frac{1}{N-1}\left(N \sum_{s=1}^{N-1} s - \sum_{s=1}^{N-1} s^2\right) = \frac{N\,(N+1)}{6}.$$

Finally, as a direct consequence of Theorems 4.3 and 4.4, we obtain the following results

Theorem 4.5. *Suppose that there exist positive integers* p, q *and* r *such that* $p < q < r$, *and assume that function* f *satisfies the following conditions:*

(i) $f(k,u) \ge \frac{3\,u}{N+1}$ *for all* $k \in J$ *and* $u \in [r, Nr]$, *being the inequality strict at* $u = r$,

(ii) $f(k,u) \leq \frac{3\,q}{N(N+1)}$ *for all* $k \in J$ *and* $u \in [0, Nq]$*, being the inequality strict at* $u = q$*,*

(iii) $f(k,u) > \frac{3\,N\,u}{N+1}$ *for all* $k \in J$ *and* $u \in \left[\frac{p}{N}, p\right]$.

Then problem (9) – (10) *with* $c = 1, d = N - 1$ *and* $\mu = \frac{1}{N-1}$ *has at least two nontrivial solutions,* u_1 *and* u_2*, such that*

$$p < \|u_1\|\,, \quad \max_{k \in J} u_1(k) < q < \max_{k \in J} u_2(k), \quad \min_{k \in J} u_2(k) < r.$$

Theorem 4.6. *Let* p, q *and* r *be positive numbers such that:*

$$p < q < Nq \leq r.$$

Assume, moreover, that the function f *satisfies the following conditions:*

(a) $f(k,u) \leq \frac{3r}{N(N+1)}$ *for all* $k \in I$ *and* $u \in [0, r]$*,*

(b) $f(k,u) < \frac{3p}{N(N+1)}$ *for all* $k \in I$ *and* $u \in [0, p]$*,*

(c) $f(k,u) \geq \frac{3u}{N+1}$ *for all* $k \in J$ *and* $u \in [q, Nq]$*, being the inequality strict for* $u = q$*.*

Then problem (9) – (10) *with* $c = 1, d = N - 1$ *and* $\mu = \frac{1}{N-1}$ *has at least three solutions* $u_1, u_2, u_3 \in \{u \in P \mid \|u\| \leq r\}$ *such that* $\max_{k \in J} u_1(k) < p$*,* $q < \min_{k \in J} u_2(k)$ *and* $p < \max_{k \in J} u_3(k)$ *with* $\max_{k \in J} u_3(k) < q$*.*

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