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System of fractional boundary value problem with p-Laplacian and advanced arguments

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Abstract

In this paper, we discuss the existence and multiplicity of positive solutions for a system of fractional differential equations with boundary condition and advanced arguments. The existence result is proved via Leray–Schauder's fixed point theorem type in a vector Banach space. Further, by using a new fixed point theorem in order Banach space, we study the multiplicity of positive solutions. Finally, some examples are given to illustrate our results.

MSC: 34A08; 34B15; 34B18

Keywords: Fractional differential equations; p-Laplacian operator; Cone; Fixed point theorem; Positive solutions; Multiplicity of solutions

1 Introduction

Fractional calculus and differential equations have now proved to be important tools modeling many real world phenomena like chemistry and physics [11, 22, 23, 25]). For the description of hereditary properties of fractional calculus, see [20, 24, 32, 37] and the references therein.

Recently, there have been some papers dealing with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis, see [2–7, 9, 33, 35, 38].

For example, Chai obtained in [10] the existence of at least one nonnegative solution and two positive solutions by using fixed point theorem on cone for the following problem:

$$\begin{cases} (\varphi_p(u'(t))' + a(t)f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = 0. \end{cases} \tag{1.1}$$

Su *et al.* [31] studied the existence of one and two positive solutions by using the fixed point index theory of the following boundary values problems:

$$\begin{cases} (\varphi_p(u'(t))' + a(t)f(u(t)) = 0, \quad 0 < t < 1, \\ \alpha\varphi_p(u(0)) - \beta\varphi_p(u'(\xi)) = 0, \qquad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta)) = 0. \end{cases} \tag{1.2}$$

Tang *et al.* [34] studied the existence of positive solutions of fractional differential equation with p-Laplacian of the following type (1.3) by using the coincidence degree theory.

$$\begin{cases} D^{\alpha}_{0^+}(\phi(D^{\beta}_{0^+}u(t)))(t) = f(t,u(t),D^{\beta}_{0^+}u(t)), & 0<t<1, \\ D^{\alpha}_{0^+}u(0)=0, \qquad D^{\beta}_{0^+}u(1)=\gamma D^{\beta}_{0^+}u(1). \end{cases} \tag{1.3}$$

In this work, we study the existence and multiplicity of positive solutions of the following problem:

$$\begin{cases} (\varphi_p(D^{\alpha}_{0^+}u(t)))' + a_1(t)f(u(\theta_1(t)),v(\theta_2(t))) = 0, & 0<t<1, \\ (\varphi_{\tilde{p}}(D^{\alpha}_{0^+}v(t)))' + a_2(t)g(u(\theta_1(t)),v(\theta_2(t))) = 0, & 0<t<1, \\ D^{\alpha}_{0^+}u(0) = u(0) = u'(0) = 0, \qquad D^{\beta}_{0^+}u(1) = \gamma D^{\beta}_{0^+}u(\eta), \\ D^{\alpha}_{0^+}v(0) = v(0) = v'(0) = 0, \qquad D^{\beta}_{0^+}v(1) = \gamma D^{\beta}_{0^+}v(\eta), \end{cases} \tag{1.4}$$

where $\eta\in(0,1), \gamma\in(0,\frac{1}{\eta^{\alpha-\beta-1}}), D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ are the standard Riemann–Liouville fractional derivatives with $\alpha\in(2,3), \beta\in(1,2)$ such that $\alpha\geq\beta+1$, p-Laplacian operator is defined as $\varphi_p(s)=|s|^{p-2}s, p>1$, and the functions $f,g\in C(\mathbb{R}^2,\mathbb{R})$.

In recent years, many authors studied the existence of solutions for systems of difference and differential equations with and without fractional derivative by using the vector version of the fixed point theorem (see [1, 8, 13, 15–19, 21, 26–28], the monograph of Graef *et al.* [12], and the references therein).

For establishing the existence and multiplicity of positive solutions of problem (1.4), let us list the following assumptions:

(H_1) $a_i\in L^1[0,1]$ is nonnegative and $a_i(t)\not\equiv 0$ on any subinterval of $[0,1]$ for $i=1,2$.

(H_2) The advanced argument $\theta\in C((0,1),(0,1])$ and $0\leq\theta(t)\leq 1, \forall t\in(0,1)$.

This work is organized as follows: In Sect. 2, we introduce all the background material used in this paper such as fractional calculus analysis and some results from fixed point theory. In Sects. 3, 4, the existence and multiplicity results of solutions for a system of fractional p-Laplace differential equations (1.4) are discussed by using the fixed point theorems in the generalized Banach space. We end the paper with two examples to illustrate our main results.

2 Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper.

Definition 2.1 ([14]) Let X be a real Banach space. A nonempty closed convex set $P\subset X$ is called cone if

(1) $x\in P, \lambda\geq 0$, then $\lambda x\in P$;

(2) $x\in P, -x\in P$, then $x=0$.

If $x,y\in\mathbb{R}^n$ with $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, then by $x\leq y$ we mean $x_i\leq y_i$ for all $i=1,\ldots,n$. Also we set $|x|=(|x_1|,\ldots,|x_n|)$, $\max(x,y)=(\max(x_1,y_1),\ldots,\max(x_n,y_n))$, and $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x_i>0\}$. If $c\in\mathbb{R}$, then $x\leq c$ means $x_i\leq c$ for each $i=1,\ldots,n$.

Definition 2.2 Let X be a nonempty set, and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d: X\times X\to\mathbb{R}^m_+$, which satisfies all the usual axioms of the metric, is called a generalized metric in Perov's sense and (X,d) is called a generalized metric space.

Let (X,d) be a generalized metric space in Perov's sense. For $r := (r_1,\dots,r_m) \in \mathbb{R}^m_+$, we will denote by

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\}$$

the open ball centered in x_0 with radius r, and by

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \leq r\}$$

the closed ball centered at x_0 with radius r.

Theorem 2.1 ([12, 36]) *Let X be a generalized Banach space, and let $N : X \to X$ be a completely continuous operator. Then either*
(i) *the equation $N(x) = x$ has at least one solution, or*
(ii) *the set $\mathcal{M} = \{x \in X | \mu N(x) = x, \mu \in (0,1)\}$ is unbounded.*

Theorem 2.2 ([30]) *Let $(X, \|\cdot\|)$ be a normed space, $P_1,P_2 \subset X$ be two cones; $P := P_1 \times P_2; r,R \in \mathbb{R}^2_+, P_{r,R} := \{u \in P_i : r_i \leq \|u_i\| \leq R_i\}$ with $0 < r < R$; and let $N : P_{r,R} \to P, N = (N_1,N_2)$ be a compact map. Assume that, for each $i \in \{1,2\}$, one of the following conditions is satisfied in $P_{r,R}$:*
(1) *$N_i(u_i) \nprec u_i$ if $\|u_i\| = r_i$, and $N_i(u_i) \nsucc u_i$ if $\|u_i\| = R_i$;*
(2) *$N_i(u_i) \nsucc u_i$ if $\|u_i\| = r_i$, and $N_i(u_i) \nprec u_i$ if $\|u_i\| = R_i$.*
Then N has a fixed point u in P with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1,2\}$, where $\preceq$, namely $u \preceq v$ if and only if $v - u \in P$. We shall say that $u \prec v$ if $v - u \in P \setminus \{0\}$.

Remark 2.1 ([30]) In Theorem (2.2) four cases are possible for $u \in p_{r,R}$:
(c_1) $N_1(u) \nprec u_1$ if $\|u_1\| = r_1$, and $N_1(u) \nsucc u_1$ if $\|u_1\| = R_1$, $N_2(u) \nprec u_2$ if $\|u_2\| = r_2$, and $N_2(u) \nsucc u_2$ if $\|u_2\| = R_2$;
(c_2) $N_1(u) \nprec u_1$ if $\|u_1\| = r_1$, and $N_1(u) \nsucc u_1$ if $\|u_1\| = R_1$, $N_2(u) \nsucc u_2$ if $\|u_2\| = r_2$, and $N_2(u) \nprec u_2$ if $\|u_2\| = R_2$;
(c_3) $N_1(u) \nsucc u_1$ if $\|u_1\| = r_1$, and $N_1(u) \nprec u_1$ if $\|u_1\| = R_1$, $N_2(u) \nprec u_2$ if $\|u_2\| = r_2$, and $N_2(u) \nsucc u_2$ if $\|u_2\| = R_2$;
(c_4) $N_1(u) \nsucc u_1$ if $\|u_1\| = r_1$, and $N_1(u) \nprec u_1$ if $\|u_1\| = R_1$, $N_2(u) \nsucc u_2$ if $\|u_2\| = r_2$, and $N_2(u) \nprec u_2$ if $\|u_2\| = R_2$.

Theorem 2.3 ([29]) *Let $(X, \|\cdot\|)$ be a Banach space, $P_1,P_2 \subset X$ be two cones, and $P := P_1 \times P_2$ be the corresponding cone of $X^2 = X \times X$, and let $\alpha_i, \beta_i > 0$. We denote*

$$U_{\alpha_i} = \{u \in P_i : \|u\| < \alpha_i\} \quad \textit{and} \quad V_{\beta_i} = \{u \in P_i : \|u\| < \beta_i\},$$

with $\alpha_i \neq \beta_i, r_i = \min\{\alpha_i,\beta_i\}$ and $R_i = \max\{\alpha_i,\beta_i\}$ for $i = 1,2$. Assume that $N : \overline{W_1 \times W_2} \to P, N = (N_1,N_2)$ is a compact map (where $W_i = U_{\alpha_i} \cup V_{\beta_i}$ for $i = 1,2$) and there exist $h_i \in P_i \setminus \{0\}, i = 1,2$, such that for each $i \in \{1,2\}$ the following condition is satisfied in $\overline{W_1 \times W_2}$:

$$\lambda u_i \neq N_i u \quad \textit{for } \|u_i\| = \alpha_i \textit{ and } \lambda \geq 1; \tag{2.1}$$

$$u_i \neq N_i u + \mu h_i \quad \text{for } \|u_i\| = \beta_i \text{ and } \mu \geq 0. \tag{2.2}$$

Then

(1) *N has at least one fixed point* $u = (u_1, u_2)$ *in P such that* $u_i \in U_{\alpha_i} \setminus \overline{V_{\beta_i}}$ *for* $i = 1, 2$ *if* $\alpha_i > \beta_i$ *for* $i = 1, 2$;

(2) *N has at least two fixed points located in* $(U_{\alpha_1} \setminus \overline{V_{\beta_1}}) \times U_{\alpha_2}$ *and* $(U_{\alpha_1} \setminus \overline{V_{\beta_1}}) \times (V_{\beta_2} \setminus U_{\alpha_2})$ *if* $\beta_1 < \alpha_1$ *and* $\beta_2 > \alpha_2$;

(3) *N has at least two fixed points located in* $U_{\alpha_1} \times (U_{\alpha_2} \setminus \overline{V_{\beta_2}})$ *and* $(V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times (U_{\alpha_2} \setminus \overline{V_{\beta_2}})$ *if* $\beta_1 > \alpha_1$ *and* $\beta_2 < \alpha_2$;

(4) *N has at least four (three nontrivial) fixed points in* $U_{\alpha_1} \times U_{\alpha_2}, U_{\alpha_1} \times (V_{\beta_2} \setminus \overline{U_{\alpha_2}}), (V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times U_{\alpha_2}$, *and* $(V_{\beta_1} \setminus \overline{U_{\alpha_1}}) \times (V_{\beta_2} \setminus \overline{U_{\alpha_2}})$ *if* $\alpha_i < \beta_i$ *for* $i = 1, 2$.

Remark 2.2 ([29]) Our previous results can be easily generalized to systems of n operator equations.

Definition 2.3 ([7]) The fractional integral of Riemann–Liouville of the function $h \in L^1((0, \infty), \mathbb{R})$ of order $\alpha > 0$ is defined by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where $\Gamma(\alpha)$ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt.$$

Definition 2.4 For a function $h \in AC^n(J)$, the Riemann–Liouville fractional order derivative of order $\alpha > 0$ of h is defined by

$$D_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Remark 2.3 ([7])

(1) If $\lambda > -1$

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1} t^{\lambda-\alpha},$$

and $D_{0^+}^{\alpha} t^{\alpha-m} = 0, m = 1, 2, \dots, n$, where $n = [\alpha] + 1$.

(2) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$ for all $u \in C(0,1) \cap L^1(0,1)$.

(3) If $u \in L^1(0,1), \alpha > \beta > 0$, then

$$D_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t).$$

Lemma 2.1 ([7]) *If we assume that* $u \in C(0,1) \cap L^1(0,1)$, *then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0, \quad \alpha > 0$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$, $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *as a unique solution, where* $n = [\alpha] + 1$.

Lemma 2.2 ([7]) *Suppose that* $u \in C(0,1) \cap L^1(0,1)$ *is such that* $D_{0^+}^{\alpha} u \in C(0,1) \cap L^1(0,1)$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *where* $n = [\alpha] + 1$.

Lemma 2.3 ([10]) *If* $x, y \geq 0$, $\gamma > 0$, *then*

$$(x+y)^{\gamma} \leq \max\{2^{\gamma-1}, 1\}(x^{\gamma} + y^{\gamma}).$$

Lemma 2.4 ([10]) *Let* $c > 0$, $\gamma > 0$. *For any* $x, y \in [0, c]$, *we have that*

(1) *if* $\gamma > 1$, *then* $|x^{\gamma} - y^{\gamma}| \leq \gamma c^{\gamma-1} |x - y|$;
(2) *if* $0 < \gamma \leq 1$, *then* $|x^{\gamma} - y^{\gamma}| \leq |x - y|^{\gamma}$.

3 Existence result

Denote by $C([0,1])$ the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \max\{|u(t)| : t \in [0,1]\}.$$

Define the cone P in $C([0,1]^2)$ as $P = \{u \in C([0,1]) : u(t) \geq 0, t \in [0,1]\}$. Let $q > 1$ and $\tilde{q} > 1$ satisfy the relation $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$, where $p, \tilde{p}$ are given by (1.4).

To prove the existence of solutions to (1.4), we need the following auxiliary lemma.

Lemma 3.1 *Given* $h_1, h_2 \in C[0,1]$, $\eta \in (0,1)$, $\gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}})$, *and* $\alpha \geq \beta + 1$, *the unique solution of C boundary value problem for a coupled system*

$$\left(\varphi_p\left(D_{0^+}^{\alpha} u(t)\right)\right)' + h_1(t) = 0, \quad 0 < t < 1, \tag{3.1}$$

$$\left(\varphi_{\tilde{p}}\left(D_{0^+}^{\alpha} v(t)\right)\right)' + h_2(t) = 0, \quad 0 < t < 1, \tag{3.2}$$

$$D_{0^+}^{\alpha} u(0) = u(0) = u'(0) = 0, \qquad D_{0^+}^{\beta} u(1) = \gamma D_{0^+}^{\beta} u(\eta), \tag{3.3}$$

$$D_{0^+}^{\alpha} v(0) = v(0) = v'(0) = 0, \qquad D_{0^+}^{\beta} v(1) = \gamma D_{0^+}^{\beta} v(\eta), \tag{3.4}$$

is

$$\begin{aligned} u(t) = {} & \int_0^1 G_1(t,s) \varphi_q\left(\int_0^s h_1(\tau)\, d\tau\right) ds \\ & + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \varphi_q\left(\int_0^s h_1(\tau)\, d\tau\right) ds \end{aligned} \tag{3.5}$$

and

$$v(t) = \int_0^1 G_1(t,s)\varphi_{\tilde{q}}\left(\int_0^s h_2(\tau)\,d\tau\right)ds + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_{\tilde{q}}\left(\int_0^s h_2(\tau)\,d\tau\right)ds, \tag{3.6}$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(\eta,s) = \begin{cases} \frac{[\eta(1-s)]^{\alpha-\beta-1}-(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le s \le \eta \le 1, \\ \frac{[\eta(1-s)]^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le \eta \le s \le 1. \end{cases}$$

Proof Integrating equation (3.1) from 0 to t, we have

$$\varphi_p\bigl(D_{0^+}^{\alpha}u(t)\bigr) - \varphi_p\bigl(D_{0^+}^{\alpha}u(0)\bigr) = \int_0^t h_1(s)\,ds$$

and so,

$$D_{0^+}^{\alpha}u(t) = -\varphi_q\left(\int_0^t h_1(s)\,ds\right).$$

From Lemma 2.2,

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha}\varphi_q\left(\int_0^t h_1(s)\,ds\right) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3}. \end{aligned}$$

From (3.3), $B = C = 0$, and so

$$u(t) = -I_{0^+}^{\alpha}\varphi_q\left(\int_0^t h_1(s)\,ds\right) + At^{\alpha-1}. \tag{3.7}$$

Now, from Remark 2.3

$$\begin{aligned} D_{0^+}^{\beta}u(t) &= -I_{0^+}^{\alpha-\beta}\varphi_q\left(\int_0^t h_1(s)\,ds\right) + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} \\ &= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}. \end{aligned}$$

Therefore

$$D_{0^+}^{\beta}u(1) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)},$$

$$\gamma D_{0^+}^{\beta}u(\eta) = -\frac{\gamma}{\Gamma(\alpha-\beta)}\int_0^{\eta} (\eta-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds + A\frac{\Gamma(\alpha)\gamma}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}\eta^{\alpha-\beta-1}.$$

By boundary condition (3.3), we have

$$
\begin{aligned}
A &= \frac{1}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad - \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds,
\end{aligned}
$$

and replacing in (3.7), we obtain

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \frac{t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds.
\end{aligned}
$$

Splitting the second integral in two parts of the form

$$
t^{\alpha-1} + \frac{k}{1-\gamma\eta^{\alpha-\beta-1}} = \frac{t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}},
$$

we have $k = \gamma\eta^{\alpha-\beta-1}t^{\alpha-1}$, and thus

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \frac{\gamma\eta^{\alpha-\beta-1}t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad - \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{[\eta(1-s)]^{\alpha-\beta-1}-(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\times\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^\eta \frac{[\eta(1-s)]^{\alpha-\beta-1}}{\Gamma(\alpha)}\times\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds \\
&= \int_0^1 G_1(t,s)\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \\
&\quad \times \int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s h_1(\tau)\,d\tau\right)ds.
\end{aligned}
$$

This completes the proof. □

Lemma 3.2 ([35]) *Let $\rho \in (0,1)$ be fixed. The kernel $G_1(t,s)$ satisfies the following properties*:

(1) $G_1(t,s) \in C([0,1]\times[0,1])$ *and* $G_1(t,s) > 0$ *for all* $s,t \in (0,1)$;
(2) $G_1(t,s) \leq G_1(1,s)$ *for all* $s \in (0,1)$;
(3) $\min_{\rho\leq t\leq 1} G_1(t,s) \geq \rho^{\alpha-1}G_1(1,s)$ *for all* $s \in [0,1]$.

We are now ready to present our main result. In this section we give an existence result based on the nonlinear alternative of Leray–Schauder type.

Theorem 3.1 *Assume* (H_1)–(H_2) *and that the following condition holds*:

(H_3) *There exist functions* $p, q, h, \breve{p}, \breve{q}$, *and* $\bar{h} \in L^1([0,1],\mathbb{R}_+)$ *and constants* $\alpha_1, \alpha_2, \alpha_3$, *and* $\alpha_4 \in [0,1)$ *such that*

$$\bigl|f(u,v)\bigr| \leq p(t)|u|^{\alpha_1} + q(t)|v|^{\alpha_2} + h(t) \quad \textit{for each } t \in [0,1] \textit{ and } u,v \in \mathbb{R}$$

and

$$\bigl|g(u,v)\bigr| \leq \breve{p}(t)|u|^{\alpha_3} + \breve{q}(t)|v|^{\alpha_4} + \breve{h}(t) \quad \textit{for each } t \in [0,1] \textit{ and } u,v \in \mathbb{R}.$$

If $\alpha_1 p, \alpha_2 p, \alpha_3 q$, *and* $\alpha_4 q \in [0,1)$, *then system* (1.4) *has at least one solution.*

Proof Let N be the operator

$$N : C(0,1)\times C(0,1) \to C(0,1)\times C(0,1)$$

defined by

$$N(u,v) = \bigl(N_1(u,v), N_2(u,v)\bigr),$$

where

$$\begin{aligned} &N_1(u,v)(t) \\ &\quad= \int_0^1 G_1(t,s)\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\ &\qquad+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} N_2(u,v) &= \int_0^1 G_1(t,s)\varphi_{\tilde{q}}\biggl(\int_0^s a_2(\tau)g\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\ &\quad+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_{\tilde{q}}\biggl(\int_0^s a_2(\tau)g\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds. \end{aligned} \tag{3.9}$$

We shall use the Leray–Schauder fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1. To show that N is continuous, let (u_n, v_n) be a sequence such that $(u_n, v_n) \to (u, v) \in C[0,1] \times C[0,1]$ as $n \to \infty$. Then we have

$$\begin{aligned}
&\bigl|N_1(u_n, v_n)(t) - N_1(u, v)(t)\bigr| \\
&\quad= \biggl|\int_0^1 G_1(t,s)\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u_n\bigl(\theta_1(\tau)\bigr), v_n\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u_n\bigl(\theta_1(\tau)\bigr), v_n\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad- \biggl[\int_0^1 G_1(t,s)\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr), v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad+ \int_0^1 \frac{\gamma t^{\alpha-1}G_2(\eta,s)}{1-\gamma\eta^{\alpha-\beta-1}}\varphi_q\biggl(\int_0^s a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr), v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds\biggr]\biggr|.
\end{aligned}$$

By Lemma 3.2 and $t \in [0,1]$,

$$\begin{aligned}
&\bigl|N_1(u_n, v_n)(t) - N_1(u, v)(t)\bigr| \\
&\quad\le \int_0^1 G_1(1,s)\biggl(\int_0^s \bigl|a_1(\tau)f\bigl(u_n\bigl(\theta_1(\tau)\bigr), v_n\bigl(\theta_2(\tau)\bigr)\bigr)\bigr|^{q-1} \\
&\qquad- \bigl|a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr), v\bigl(\theta_2(\tau)\bigr)\bigr)\bigr|^{q-1}\,d\tau\biggr)\,ds \\
&\qquad+ \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\biggl(\int_0^s \bigl|a_1(\tau)f\bigl(u_n\bigl(\theta_1(\tau)\bigr), v_n\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\bigr|^{q-1} \\
&\qquad- \bigl|a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr), v\bigl(\theta_2(\tau)\bigr)\bigr)\bigr|^{q-1}\,d\tau\biggr)\,ds.
\end{aligned}$$

On the other hand, since f is a continuous function combined with the fact that

$$\|u_n - u\| \to 0, \quad \text{as } n \to \infty,$$

then there exists $N \ge 1$ such that for all $\tau \in [0,1]$ the following estimate

$$\bigl|f\bigl(u_n\bigl(\theta_1(\tau)\bigr), v_n\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau) - f\bigl(u\bigl(\theta_1(\tau)\bigr), v\bigl(\theta_2(\tau)\bigr)\bigr)\bigr| < \epsilon,$$

holds for $n \ge N$. By the Lebesgue dominated convergence theorem, we have

$$\bigl\|N_1(u_n, v_n) - N_1(u, v)\bigr\| \to 0, \quad \text{as } n \to \infty.$$

Similarly,

$$\bigl\|N_2(u_n, v_n) - N_2(u, v)\bigr\| \to 0, \quad \text{as } n \to \infty.$$

Consequently, N is continuous.

Step 2. N maps bounded sets into bounded sets in $C[0,1] \times C[0,1]$, it suffices to show that for any $r > 0$ there exists a positive constant vector $l = (l_1, l_2)$ such that, for each $(u, v) \in$

$B_r = \{(u,v) \in C[0,1] \times C[0,1] : \|u\| \leq r, \|v\| \leq r\}$, we have

$$\big\|N(u,v)\big\| \leq l.$$

For each $t \in [0,1]$, we have

$$\begin{aligned}
&\big|N_1(u,v)(t)\big| \\
&\quad \leq \int_0^1 \big|G_1(t,s)\big|\varphi_q\biggl(\int_0^s \big|a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\big|\biggr)\,ds \\
&\qquad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 \big|G_2(\eta,s)\big|\varphi_q\biggl(\int_0^s \big|a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\big|\biggr)\,ds \\
&\quad \leq \max\bigl\{2^{q-1},1\bigr\}\int_0^1 G_1(1,s)\int_0^s \big|a_1(\tau)\big|^{q-1}\big|p(\tau)\big|^{q-1}\big|u\bigl(\theta_1(\tau)\bigr)\big|^{\alpha_1(q-1)} \\
&\qquad + \big|a_1(\tau)\big|^{q-1}\big|q(\tau)\big|^{q-1}\big|v\bigl(\theta_2(\tau)\bigr)\big|^{\alpha_2(q-1)} + \big|a_1(\tau)\big|^{q-1}\big|h(\tau)\big|^{q-1}\,d\tau\,ds \\
&\qquad + \max\bigl\{2^{q-1},1\bigr\}\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 \big|G_2(\eta,s)\big| \int_0^s \big|a_1(\tau)\big|^{q-1}\big|p(\tau)\big|^{q-1}\big|u\bigl(\theta_1(\tau)\bigr)\big|^{\alpha_1(q-1)} \\
&\qquad + \big|a_1(\tau)\big|^{q-1}\big|q(\tau)\big|^{q-1}\big|v\bigl(\theta_2(\tau)\bigr)\big|^{\alpha_2(q-1)} + \big|a_1(\tau)\big|^{q-1}\big|h(\tau)\big|^{q-1}\,d\tau\,ds \\
&\quad \leq \max\bigl\{2^{q-1},1\bigr\}\bigl(\|u\|^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + \|v\|^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\qquad + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\bigl(\|u\|^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} \\
&\qquad + \|v\|^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr)\int_0^1 \frac{\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \\
&\quad \leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr) \\
&\qquad + \frac{\max\{2^{q-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\qquad + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr).
\end{aligned}$$

Hence

$$\begin{aligned}
&\big\|N_1(u,v)\big\| \\
&\quad \leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr) \\
&\qquad + \frac{\max\{2^{q-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} + r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&\qquad + \|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr) := l_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\big\|N_2(u,v)\big\| \\
&\quad \leq \frac{\max\{2^{\tilde{q}-1},1\}}{(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1} + r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1} + \|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1}\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\max\{2^{\tilde{q}-1},1\}\eta^{\alpha-\beta-1}\gamma}{(1-\gamma\eta^{\alpha-\beta-1})(\alpha-\beta)\Gamma(\alpha)}\bigl(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1}+r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1} \\
&+\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1}\bigr):=l_2.
\end{aligned}
$$

Step 3. N maps bounded sets into equicontinuous. Let $u\in B_r$ be a bounded set as in Step 2, $t_1,t_2\in[0,1]$ with $t_1<t_2$, from (3.5) and Lemma 2.3, we have

$$
\begin{aligned}
&\bigl|N_1(u,v)(t_2)-N_1(u,v)(t_1)\bigr| \\
&\le\int_0^1\bigl|G_1(t_2,s)-G_1(t_1,s)\bigr|\varphi_q\biggl(\int_0^s|a_1(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)|\,ds \\
&+\frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1\bigl|G_2(\eta,s)\bigr| \\
&\times\varphi_q\biggl(\int_0^s|a_1|(\tau)f\bigl(u\bigl(\theta_1(\tau)\bigr),v\bigl(\theta_2(\tau)\bigr)\bigr)\,d\tau\biggr)|\,ds \\
&\le\int_0^1\bigl|G_1(t_2,s)-G_1(t_1,s)\bigr|\int_0^s|a_1(\tau)\bigl[p(\tau)\bigl|u(\theta_1\bigr|^{\alpha_1}+q(\tau)\bigl|v\bigl(\theta_2(\tau)\bigr)\bigr|^{\alpha_2} \\
&+h(\tau)\,d\tau\,ds\bigr]|^{q-1}+\frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1\bigl|G_2(\eta,s)\bigr| \\
&\times\int_0^s\bigl|a_1(\tau)\bigl[p(\tau)\bigl|u(\theta_1\bigr|^{\alpha_1}+q(\tau)\bigl|v\bigl(\theta_2(\tau)\bigr)\bigr|^{\alpha_2}+h(\tau)\,d\tau\,ds\bigr]\bigr|^{q-1} \\
&\le\max\bigl\{2^{q-1},1\bigr\}\int_0^1\bigl|G_1(t_2,s)-G_1(t_1,s)\bigr|\int_0^s\bigl|a_1(\tau)\bigr|^{q-1}\bigl|p(\tau)\bigr|^{q-1} \\
&\times\bigl|u\bigl(\theta_1(\tau)\bigr)\bigr|^{\alpha_1(q-1)}+\bigl|a_1(\tau)\bigr|^{q-1}\bigl|q(\tau)\bigr|^{q-1}\bigl|v\bigl(\theta_2(\tau)\bigr)\bigr|^{\alpha_2(q-1)}+\bigl|a_1(\tau)\bigr|^{q-1} \\
&\times\bigl|h(\tau)\bigr|^{q-1}\,d\tau\,ds+\max\bigl\{2^{q-1},1\bigr\}\frac{\gamma|t_2-t_1|^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1\bigl|G_2(\eta,s)\bigr| \\
&\times\int_0^s\bigl|a_1(\tau)\bigr|^{q-1}\bigl|p(\tau)\bigr|^{q-1}u\bigl(\theta_1(\tau)\bigr)\bigr)|^{\alpha_1(q-1)} \\
&+\bigl|a_1(\tau)\bigr|^{q-1}\bigl|q(\tau)\bigr|^{q-1}\bigl|v\bigl(\theta_2(\tau)\bigr)\bigr|^{\alpha_2(q-1)}+\bigl|a_1(\tau)\bigr|^{q-1}\bigl|h(\tau)\bigr|^{q-1}\,d\tau\,ds.
\end{aligned}
$$

By Lemma 2.4 we obtain

$$
\begin{aligned}
&\le\max\bigl\{2^{q-1},1\bigr\}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1}+r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1} \\
&+\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr)\int_0^1\bigl|G_1(t_1,s)-G_1(t_2,s)\bigr|\,ds \\
&+\frac{\max\{2^{q-1},1\}\gamma(\alpha-1)|t_2-t_1|}{(1-\gamma\eta^{\alpha-\beta-1})}\bigl(r^{\alpha_1(q-1)}\|a_1\|_{L_1}^{q-1}\|p\|_{L_1}^{q-1} \\
&+r^{\alpha_2(q-1)}\|a_1\|_{L_1}^{q-1}\|q\|_{L_1}^{q-1}+\|a_1\|_{L_1}^{q-1}\|h\|_{L_1}^{q-1}\bigr)\int_0^1\bigl|G_2(\eta,s)\bigr|\,ds.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\bigl|N_2(u,v)(t_1)-N_2(u,v)(t_2)\bigr| \\
&\le\max\bigl\{2^{\tilde{q}-1},1\bigr\}\bigl(r^{\alpha_3(q-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1}+r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|a_2\|_{L_1}^{\tilde{q}-1} \|\breve{h}\|_{L_1}^{\tilde{q}-1}\Big) \int_0^1 \big|G_1(t_1,s) - G_1(t_2,s)\big|\, ds \\
&+ \frac{\max\{2^{\tilde{q}-1},1\}\gamma(\alpha-1)|t_2-t_1|}{(1-\gamma\eta^{\alpha-\beta-1})}\big(r^{\alpha_3(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{p}\|_{L_1}^{\tilde{q}-1} \\
&+ r^{\alpha_4(\tilde{q}-1)}\|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{q}\|_{L_1}^{\tilde{q}-1} + \|a_2\|_{L_1}^{\tilde{q}-1}\|\breve{h}\|_{L_1}^{\tilde{q}-1}\big) \int_0^1 \big|G_2(\eta,s)\big|\, ds.
\end{aligned}
$$

The continuity of G_1 implies that the right-hand side of the above inequality tends to zero if $t_2 \to t_1$. Therefore, by Arzela–Ascoli N is completely continuous.

Step 4. A priori bounds. Now it remains to show that the set

$$
\mathcal{M} = \big\{(u,v)C\big([0,1],\mathbb{R}\big) \times C\big([0,1],\mathbb{R}\big) : (u,v) = \lambda N(u,v) < \lambda < 1\big\}
$$

is bounded. Let $(u,v) \in \mathcal{M}$, then there exists $0 < \lambda < 1$ such that $u = \lambda N_1(u,v)$ and $v = \lambda N_2(u,v)$. Thus, for $t \in [0,1]$, we have

$$
\begin{aligned}
\big|u(t)\big| &\le \int_0^1 \big|G_1(t,s)\big|\varphi_q\bigg(\int_0^s |a_1(\tau)f\big(u(\theta_1(\tau)),v(\theta_2(\tau))\big)\, d\tau\bigg)|\, ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 \big|G_2(\eta,s)\big|\varphi_q\bigg(\int_0^s |a_1(\tau)f\big(u(\theta_1(\tau)),v(\theta_2(\tau))\big)\, d\tau\bigg)|\, ds \\
&\le \max\big\{2^{q-1},1\big\}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&\quad + \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big] \\
&\quad \times \int_0^1 G_1(1,s)\, ds + \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} \\
&\quad + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&\quad + \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big] \int_0^1 \big|G_2(\eta,s)\big|\, ds, \\
\|u\| &\le \max\big\{2^{q-1},1\big\}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&\quad + \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big]\bigg[\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\frac{\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\, ds\bigg], \\
\|u\| &\le \max\big\{2^{q-1},1\big\}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&\quad + \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big]\frac{1}{(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}\, ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|u\| \le{}& \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&+ \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big].
\end{aligned}
$$

Similarly, we obtain

$$
\|v\| \le \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big[\|\breve{p}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|u\|^{\alpha_3(\tilde{q}-1)} + \|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|v\|^{\alpha_4(\tilde{q}-1)}
$$

$$+ \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}\big].$$

Notice that if $\epsilon \leq \delta$ and $\|u\| > 1$, then $\|u\|^{\epsilon} \leq \|u\|^{\delta}$ Thus, $\|u\|^{\epsilon} \leq 1 + \|u\|^{\delta}$ for all u. We then have

$$\begin{aligned}
&\|u\| + \|v\| \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big[\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|u\|^{\alpha_1(q-1)} + \|q\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1}\|v\|^{\alpha_2(q-1)} \\
&\qquad + \|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}\big] + \frac{\max\{2^{q-1},1\}\varphi_q(\int_0^1 a_1(\tau)\,d\tau)}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big[\|\breve{p}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|u\|^{\alpha_3(\tilde{q}-1)} \\
&\qquad + \|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1}\|v\|^{\alpha_4(\tilde{q}-1)} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}\big] \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big(\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1}\big) \\
&\qquad \times \big(\|u\|^{\alpha_1(q-1)} + \|v\|^{\alpha_4(\tilde{q}-1)}\big) \\
&\qquad + \big(\|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big)\big(\|u\|^{\alpha_3(\tilde{q}-1)} + \|v\|^{\alpha_2(q-1)}\big) \\
&\qquad + \big(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}\big) \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} \\
&\qquad + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big)\big(\|u\|^{\alpha_\star} + \|v\|^{\alpha_\star}\big) + \big(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}\big) \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)} \\
&\qquad \times \big(\|p\|_{L^1}^{q-1}\|a_1\|_{L^1}^{q-1} + \|\breve{q}\|_{L^1}^{\tilde{q}-1}\|a_2\|_{L^1}^{\tilde{q}-1} + \|\breve{p}\|_{L^1}^{\tilde{q}-1} + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big) \\
&\qquad \times \big(\|u\| + \|v\|\big)^{\alpha_\star} + \big(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1}\big),
\end{aligned}$$

where

$$\alpha_\star = \max\big\{\alpha_1(q-1), \alpha_2(q-1), \alpha_3(\tilde{q}-1), \alpha_4(\tilde{q}-1)\big\}.$$

If $\|u\| + \|v\| > 1$, then

$$\begin{aligned}
&\frac{\|u\| + \|v\|}{(\|u\| + \|v\|)^{\alpha_\star}} \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1} \\
&\qquad + \|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big) + \frac{(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{h}\|_{L^1}^{\tilde{q}-1})}{(\|u\| + \|v\|)^{\alpha_\star}}
\end{aligned}$$

or

$$\begin{aligned}
&\big(\|u\| + \|v\|\big)^{1-\alpha_\star} \\
&\quad\leq \frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)}\big(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{q}\|_{L^1}^{\tilde{q}-1} + \|a_2\|_{L^1}^{\tilde{q}-1}\|\breve{p}\|_{L^1}^{\tilde{q}-1}
\end{aligned}$$

$$+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big)+\big(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{h}\|_{L^1}^{\tilde{q}-1}\big).$$

This implies that

$$\begin{aligned}\|u\|+\|v\| &\le \big[A\big(\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{q}\|_{L^1}^{\tilde{q}-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{p}\|_{L^1}^{\tilde{q}-1}+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\big)\\ &\quad+\big(\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{h}\|_{L^1}^{\tilde{q}-1}\big)\big]^{1-\alpha_\star},\end{aligned}$$

then

$$\|u\|+\|v\|\le[AB+C]^{1-\alpha_\star}:=M_2,$$

where

$$\begin{aligned}A&=\frac{\max\{2^{q-1},1\}}{(\alpha-\beta)(1-\gamma\eta^{\alpha-\beta-1})\Gamma(\alpha)},\\ B&=\|a_1\|_{L^1}^{q-1}\|p\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{q}\|_{L^1}^{\tilde{q}-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{p}\|_{L^1}^{\tilde{q}-1}+\|a_1\|_{L^1}^{q-1}\|q\|_{L^1}^{q-1}\end{aligned}$$

and

$$C=\|a_1\|_{L^1}^{q-1}\|h\|_{L^1}^{q-1}+\|a_2\|_{L^1}^{\tilde{q}-1}\|\check{h}\|_{L^1}^{\tilde{q}-1}.$$

As a consequence of Theorem 2.1, the operator N has a fixed point that is a solution of system (1.4). This completes the proof of the theorem. □

4 Multiplicity of positive solutions

In this section, our goal is to establish positive solutions and multiplicity of solutions for the problem to system (1.4). To this end, first in this section we assume the functions $f,g\in C(\mathbb{R}^2,\mathbb{R}_+)$ and define the operator on P as $N:P^2\to P^2$ to be the completely continuous map $N=(N_1,N_2)$ given in the proof of Theorem 3.1. Then (3.5) and (3.6) are equivalent to the fixed point problem

$$u=N(u),\qquad u\in P^2.$$

If $v\in P$ and

$$\begin{aligned}u_i(t)&=\int_0^1 G_1(t,s)\varphi_q\bigg(\int_0^s a_1(\tau)v(\tau)\,d\tau\bigg)\,ds\\ &\quad+\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\bigg(\int_0^s a_1(\tau)v(\tau)\,d\tau\bigg)\,ds,\quad i=1,2,\end{aligned}$$

and $u_i(t_i)=\|u_i\|$, by Lemma 3.2 this implies that, for any $t\in[\rho,1]$,

$$\begin{aligned}u_i(t)&=\int_0^1 G_1(t,s)\varphi_q\bigg(\int_0^s a_1(\tau)v(\tau)\,d\tau\bigg)\,ds\\ &\quad+\frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\bigg(\int_0^s a_1(\tau)v(\tau)\,d\tau\bigg)\,ds\end{aligned}$$

$$
\begin{aligned}
u_i(t) \geq{}& \int_0^1 \min G_1(t,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds \\
&+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds \\
\geq{}& \int_0^1 \rho^{\alpha-1}G_1(1,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds \\
&+ \frac{\gamma\rho^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds \\
\geq{}& \rho^{\alpha-1}\left[\int_0^1 G_1(1,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds\right] \\
\geq{}& \rho^{\alpha-1}\left[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds\right. \\
&\left.+ \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}}\int_0^1 G_2(\eta,s)\varphi_q\left(\int_0^s a_1(\tau)v(\tau)\,d\tau\right)ds\right].
\end{aligned}
$$

Hence

$$u_i(t) \geq \rho^{\alpha-1}\|u_i\|, \quad i = 1,2.$$

Define the cone P_i for $i = 1,2$ in P by

$$P_i = \left\{u_i \in P : u_i(t) \geq \rho^{\alpha-1}\|u_i\| \text{ for all } t \in [\rho,1]\right\},$$

and the product cone $P = P_1 \times P_2$ in P^2, then $N(P) \subset P$. Before we state our main result, we introduce the following notations: $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, we let $r_i = \min\{\alpha_i,\beta_i\}$, $R_i = \max\{\alpha_i,\beta_i\} i = 1,2$.

$$
\begin{aligned}
\gamma_1 &= \min\left\{f\left(u_1\left(\theta_1(t)\right),u_2\left(\theta_1(t)\right)\right) : \rho \leq t \leq 1, \rho^{\alpha-1}\beta_1 \leq u_1 \leq \beta_1, \rho^{\alpha-1}r_2 \leq u_2 \leq R_2\right\}, \\
\gamma_2 &= \min\left\{g\left(u_1\left(\theta_1(t)\right),u_2\left(\theta_1(t)\right)\right) : \rho \leq t \leq 1, \rho^{\alpha-1}r_1 \leq u_1 \leq R_1, \rho^{\alpha-1}\beta_2 \leq u_2 \leq \beta_2\right\}, \\
\Gamma_1 &= \max\left\{f\left(u_1\left(\theta_1(t)\right),u_2\left(\theta_1(t)\right)\right) : \rho \leq t \leq 1, \rho^{\alpha-1}\alpha_1 \leq u_1 \leq \alpha_1, \rho^{\alpha-1}r_2 \leq u_2 \leq R_2\right\}, \\
\Gamma_2 &= \max\left\{g\left(u_1\left(\theta_1(t)\right),u_2\left(\theta_1(t)\right)\right) : \rho \leq t \leq 1, \rho^{\alpha-1}r_1 \leq u_1 \leq R_1, \rho^{\alpha-1}\alpha_2 \leq u_2 \leq \alpha_2\right\}.
\end{aligned}
$$

Also, let

$$A = \min\left\{G_1(t,s) : \rho \leq t \leq 1, 0 \leq s \leq 1\right\}$$

and

$$B = \max\left\{G_1(t,s) : \rho \leq t \leq 1, 0 \leq s \leq 1\right\}.$$

Theorem 4.1 *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i, i = 1, 2$, such that*

$$\begin{aligned} &B\Gamma_1^{q-1} \leq \alpha_1, \qquad A\gamma_1^{q-1} \geq \beta_1, \\ &B\Gamma_2^{q-1} \leq \alpha_2, \qquad A\gamma_2^{q-1} \geq \beta_2. \end{aligned} \tag{4.1}$$

Then (1.4) *has a positive solution $u = (u_1, u_2)$ with $r_i \leq \|u_i\| \leq R_i, i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the corresponding orbit of u is included in the rectangle $[\rho r_1, R_1] \times [\rho r_2, R_2]$.*

Proof First note that if $u \in P_{r,R}$, then $r_1 \leq \||u_1\| \leq R_1$ and $r_2 \leq \|u_2\| \leq R_2$, and by the definition of P,

$$\{\rho^{\alpha-1} r_1 \leq u_1(t) \leq R_1 \text{ and } \rho^{\alpha-1} r_2 \leq u_2(t) \leq R_2\}$$

for all t, showing that the orbit of u for $t \in [\rho, 1]$ is included in the rectangle $[\rho r_1, R_1] \times [\rho r_2, R_2]$.

Also, if we know for example that $\|u_1\| = \alpha_1$, then

$$\rho^{\alpha-1}\alpha_1 \leq u_1(t) \leq \alpha_1.$$

We now prove that, for every $u \in P_{r,R}$ and $i \in \{1, 2\}$, the following properties hold:

$$\begin{aligned} &\|u_i\| = \alpha_i \quad \text{implies } u_i \nprec N_i(u), \\ &\|u_i\| = \beta_i \quad \text{implies } u_i \nsucc N_i(u), \end{aligned} \tag{4.2}$$

guaranteeing the applicability of Theorem 2.2. Indeed, if $\|u_1\| = \alpha_1$ and we would have $u_1 \prec N_1(u)$, then

$$\begin{aligned} u_1(t) &< N_1(u)(t) \\ &\leq \int_0^1 \max G_1(t,s) \max\bigl|a_1(t) f\bigl(u\bigl(\theta_1(t)\bigr), v\bigl(\theta_2(t)\bigr)\bigr)\bigr|^{q-1}\,dt \\ &\quad + \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \max\bigl|a_1(t) f\bigl(u\bigl(\theta_1(t)\bigr), v\bigl(\theta_2(t)\bigr)\bigr)\bigr|^{q-1}\,dt \\ &\leq B\Gamma_1^{q-1} + \Gamma_1^{q-1} \frac{\gamma}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \\ &\leq B\Gamma_1^{q-1} \\ &\leq \alpha_1 \end{aligned}$$

for all t. This yields the contradiction $\alpha_1 < \alpha_1$.

Now, if $\|u_1\| = \beta_1$ and $u_1 \succ N_1(u)$, then for $t \in [\rho, 1]$ we obtain

$$\begin{aligned} u_1(t) &> N_1(u)(t) \\ &\geq \int_0^1 \min G_1(t,s) \min\bigl|a_1(t) f\bigl(u\bigl(\theta_1(t)\bigr), v\bigl(\theta_2(t)\bigr)\bigr)\bigr|^{q-1}\,dt \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\gamma t^{\alpha-1}}{1-\gamma\eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) \min\left|a_1(t) f\left(u\left(\theta_1(t)\right), v\left(\theta_2(t)\right)\right)\right|^{q-1} dt \\
&\geq A\gamma_1^{q-1} \\
&\geq \beta_1.
\end{aligned}
$$

Then we deduce that $\beta_1 > \beta_1$, which is a contradiction. Hence (4.2) holds for $i = 1$. Similarly, (4.2) is true for $i = 2$. By Theorem 2.2, we see that N has at least one fixed point in P. Therefore, system (1.4) has at least one positive solution. □

Now we study the existence of multiple positive solutions for the systems of fractional boundary value problem with p-Laplacian boundary conditions.

(H_4) f, g are positive and increasing, i.e.,

$$0 \leq u \leq x, 0 \leq v \leq y \quad \text{imply } 0 \leq f(u,v) \leq f(x,y), 0 \leq g(u,v) \leq g(x,y).$$

We present the following general existence, multiplicity, and localization result.

Theorem 4.2 *Let conditions $(H_1)-(H_2)-(H_4)$ hold, and assume that the norm $\|\cdot\|$ is monotone with respect to each cone $P_i (i = 1,2)$. Moreover, suppose that there exist $\alpha_i, \beta_i > 0$, with $\alpha_i \neq \beta_i$, $i = 1,2$, such that*

$$\left\|N_1\left(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1}\right)\right\| < \alpha_1, \qquad \left\|N_2\left(R_1\rho^{\alpha-1}, \alpha_2\rho^{\alpha-1}\right)\right\| < \alpha_2, \tag{4.3}$$

$$\left\|N_1\left(\beta_1\rho^{\alpha-1}, 0\right)\right\| > \beta_1, \qquad \left\|N_2\left(0, \beta_2\rho^{\alpha-1}\right)\right\| < \beta_2, \tag{4.4}$$

where $R_i = \max\{\alpha_i, \beta_i\} (i = 1,2)$.

Then problem (1.4) has at least

1. *one solution $u = (u_1, u_2)$ such that $\beta_i < \|u_i\| < \alpha_i$ for $i = 1,2$, if $\alpha_i > \beta_i$ for $i = 1,2$;*
2. *two solutions (u_1, u_2) and (v_1, v_2) such that $\beta_1 < \|u_1\| < \alpha_1$, $\beta_2 < \|u_2\| < \alpha_2$, $\beta_1 < \|v_1\| < \alpha_1$, and $\|v_2\| < \alpha_2$ if $\alpha_1 > \beta_1$ and $\alpha_2 < \beta_2$;*
3. *two solutions (u_1, u_2) and (v_1, v_2) such that $\alpha_1 < \|u_1\| < \beta_1$, $\alpha_2 < \|u_2\| < \beta_2$, $\|v_1\| < \alpha_1$, and $\beta_2 < \|v_2\| < \alpha_2$ if $\alpha_1 < \beta_1$ and $\alpha_2 > \beta_2$;*
4. *four solutions $(u_1, u_2), (v_1, v_2), (w_1, w_2)$, and (z_1, z_2) such that $\beta_i < \|u_i\| < \alpha_i, \alpha_1 < \|v_1\| < \beta_1$, and $\|v_2\| < \alpha_2, \|w_1\| < \alpha_1, \alpha_2 < \|w_1\| < \beta_2$, and $\|z_i\| < \alpha_i$, if $\alpha_i < \beta_i$ for $i = 1,2$.*

Proof We shall apply Theorem 2.3 to the operator $N = (N_1, N_2)$ defined as in (3.8) and (3.9). Let us see that it satisfies conditions (2.1)(2.2).

First we prove that

$$\lambda u_1 \neq N_1(u) \quad \text{for every } u \in k \text{ with } \|u_1\| = \alpha_1, \|u_2\| \leq R_2 \text{ and all } \lambda \geq 1. \tag{4.5}$$

Indeed, if not,

$$\lambda\|u_1\| = \lambda\alpha_1 = \left\|N_1(u)\right\|.$$

From $0 \le u_1 \le \alpha_1\rho^{\alpha-1}$ and $0 \le u_2 \le R_2\rho^{\alpha-1}$, by $(H_1), (H_4)$ it follows that

$$0 \le f(u_1,u_2) \le f(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1}),$$
$$0 \le \varphi_q\left(\int_0^s a_1(\tau)f(u_1,u_2)\,d\tau\right) \le \varphi_q\left(\int_0^s a_1(\tau)f(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1})\,d\tau\,ds\right).$$

By Lemma (3.2) we obtain

$$0 \le N_1(u_1,u_2) \le N_1(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1}),$$

and the norm of X being monotone,

$$\|N_1(u_1,u_2)\| \le \|N_1(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1})\|.$$

By assumption (4.3),

$$\|N_1(\alpha_1\rho^{\alpha-1}, R_2\rho^{\alpha-1})\| < \alpha_1,$$

so we obtain the contradiction

$$\lambda\alpha_1 < \alpha_1 \quad \text{for some } \lambda \ge 1.$$

Hence (4.5) holds.

Now, we prove that $u_1 \neq N_1(u) + \mu\rho^{\alpha-1}$ for every $u \in P$ with $\|u_1\| = \beta_1, \|u_2\| \le R_2$ and all $\mu \ge 0$.

Assume the contrary, i.e., $u_1 = N_1(u) + \mu\rho^{\alpha-1}$ for some $u \in P$ with $\|u_1\| = \beta_1, \|u_2\| \le R_2$ and some $\mu \ge 0$. Then $u_1 - N_1(u) \in P_1$, so $0 \le N_1(u) \le u_1$, and the norm of X being monotone

$$\|N_1(u)\| \le \|u_1\| = \beta_1. \tag{4.6}$$

Also, from condition (H_4), $0 \le \beta_1\rho^{\alpha-1} \le u_1$ and $0 \le u_2$, so we obtain

$$0 \le f_1(\beta_1\rho^{\alpha-1}, 0) \le f(u_1,u_2),$$

then by (H_1) we obtain

$$0 \le \varphi_q\left(\int_0^s a_1(\tau)f_1(\beta_1\rho^{\alpha-1},0)\,d\tau\right) \le \varphi_q\left(\int_0^s a_1(\tau)f(u_1,u_2)\,d\tau\right),$$

and by Lemma 3.2 we conclude $0 \le N_1(\beta_1\rho^{\alpha-1}, 0) \le N_1(u_1,u_2)$. Hence, by monotonicity of the norm,

$$\|N_1(\beta_1\rho^{\alpha-1},0)\| \le \|N_1(u_1,u_2)\|.$$

Now, from (4.6) we have

$$\|N_1(\beta_1\rho^{\alpha-1},0)\| \le \beta_1,$$

which contradicts assumption (4.4). Therefore, conditions (2.1)–(2.2) hold for i = 1. Similarly, they can be verified for $i = 2$. □

5 Applications

Example 5.1 Consider the fractional differential equation with advanced argument for p-Laplacian:

$$\begin{cases} \varphi_{3/2}(D_{0^+}^{5/2}u(t))' + \frac{t^{-1/2}}{4}\frac{t}{1+t}(|u(\theta(t))|^{\frac{1}{4}} + |v(\theta(t))|^{\frac{1}{5}}) = 0, \quad 0 < t < 1, \\ \varphi_{3/2}(D_{0^+}^{5/2}v(t))' + \frac{7t^{-1/2}}{2}\frac{t^2}{1+t^2}(13 + |v(\theta(t))|^{1/4} + |u(\theta(t))|^{\frac{1}{6}}) = 0, \quad 0 < t < 1, \\ D_{0^+}^{5/2}u(0) = u(0) = u'(0) = 0, \qquad D_{0^+}^{7/6}u(1) = \frac{7}{10}D_{0^+}^{7/6}u(\frac{1}{2}), \\ D_{0^+}^{5/2}v(0) = v(0) = v'(0) = 0, \qquad D_{0^+}^{7/6}v(1) = \frac{7}{10}D_{0^+}^{7/6}v(\frac{1}{2}), \end{cases} \tag{5.1}$$

where $\alpha = \frac{5}{2}, \beta = \frac{7}{6}, \eta = \frac{7}{10}, p = \tilde{p} = \frac{3}{2}, q = \tilde{q} = 3, a_1(t) = \frac{t^{-1/2}}{4}, a_2(t) = \frac{7t^{-1/2}}{2}, \varphi_3(\int_0^1 a_1(t)\,dt) = \frac{1}{4}, \varphi_3(\int_0^1 a_2(t)\,dt) = \frac{\sqrt{7}}{2}, \alpha_1 p = \alpha_3\tilde{p} = \frac{3}{8} \in (0,1), \alpha_2 p = \frac{3}{10} \in (0,1), \alpha_4\tilde{p} = \frac{3}{12} \in (0,1)$

$$f\bigl(u(\theta(t)), v(\theta(t))\bigr) = \frac{t}{1+t}\bigl(\bigl|u(\theta(t))\bigr|^{\frac{1}{4}} + \bigl|v(\theta(t))\bigr|^{\frac{1}{5}}\bigr), \quad \theta(t) = t^{\gamma}, \gamma \in (0,1),$$

and

$$g\bigl(u(\theta(t)), v(\theta(t))\bigr) = \frac{t^2}{1+t^2}\bigl(13 + \bigl|v(\theta(t))\bigr|^{1/4} + \bigl|u(\theta(t))\bigr|^{\frac{1}{6}}\bigr), \quad \theta(t) = t^{\gamma}, \gamma \in (0,1).$$

It is clear that, for all $(t,u,v) \in [0,1] \times \mathbb{R}^2$,

$$\begin{cases} |f(u,v)| \le t(|u|^{\frac{1}{4}} + |v|^{\frac{1}{5}}), \\ |g(u,v)| \le t^2(13 + |v|^{1/4} + |u|^{\frac{1}{6}}). \end{cases}$$

Hence all the conditions of Theorem 3.1 hold, this implies that problem (5.1) has at least one solution.

Example 5.2 Consider the fractional differential equation with advanced argument for p-Laplacian:

$$\begin{cases} \varphi_{3/2}(D_{0^+}^{5/2}u(t))' + \frac{t^{-1/2}}{4}f(u(\theta(t)), v(\theta(t))) = 0, \qquad 0 < t < 1, \\ \varphi_{3/2}(D_{0^+}^{5/2}v(t))' + \frac{7t^{-1/2}}{2}g(u(\theta(t)), v(\theta(t))) = 0, \qquad 0 < t < 1, \\ D_{0^+}^{5/2}u(0) = u(0) = u'(0) = 0, \qquad D_{0^+}^{7/6}u(1) = \frac{7}{10}D_{0^+}^{7/6}u(\frac{1}{2}), \\ D_{0^+}^{5/2}v(0) = v(0) = v'(0) = 0, \qquad D_{0^+}^{7/6}v(1) = \frac{7}{10}D_{0^+}^{7/6}v(\frac{1}{2}), \end{cases} \tag{5.2}$$

where $f, g \in C(\mathbb{R}^2, \mathbb{R}_+)$ are nondecreasing in u and v, $\theta(t) = t^{\gamma}, \gamma \in (0,1)$. Assume that

$$\lim_{z\to\infty}\frac{f(z,z)}{z} = \lim_{z\to\infty}\frac{g(z,z)}{z} = 0 \tag{5.3}$$

and

$$\lim_{z\to 0}\frac{f(z,z)}{z} = \lim_{z\to\infty}\frac{g(z,z)}{z} = \infty. \tag{5.4}$$

From conditions (5.3) and (5.4), we can prove that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0, \alpha_1 < \beta_1, \alpha_2 = \beta_1$, and $\beta_2 = \alpha_1$ such that

$$\begin{aligned} &\frac{f(\rho^{\alpha-1}\beta_1, \rho^{\alpha-1}\alpha_1)}{\rho^{\alpha-1}\beta_1} \geq \frac{1}{\rho^{\alpha-1}A}, \\ &\frac{g(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}\alpha_1)}{\rho^{\alpha-1}\beta_2} \geq \frac{1}{\rho^{\alpha-1}A} \end{aligned} \tag{5.5}$$

and

$$\frac{f(\alpha_1, \beta_1)}{\alpha_1} \leq \frac{1}{B}, \qquad \frac{g(\beta_1, \beta_1)}{\alpha_2} \leq \frac{1}{B}. \tag{5.6}$$

Then we set

$$\begin{aligned} &r_i = \alpha_1, \qquad R_i = \beta_1 \quad \text{for } i \in \{1, 2\}, \\ &\Gamma_1 = f(\alpha_1, \beta_1), \qquad \Gamma_2 = g(\beta_1, \beta_1), \end{aligned}$$

and

$$\gamma_1 = f\bigl(\rho^{\alpha-1}\beta_1, \rho^{\alpha-1}\alpha_1\bigr), \qquad \gamma_2 = g\bigl(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}\alpha_1\bigr).$$

We concluded that (5.5) and (5.6) guarantee (4.1). Hence, by Theorem 4.2, problem (5.2) has a positive solution.

Since f, g are positive and increasing, we can easily show that

$$\begin{aligned} &\frac{f(\rho^{\alpha-1}\beta_1, 0)}{\beta_1} \geq \frac{1}{A}, \qquad \frac{f(\rho^{\alpha-1}\alpha_1, \rho^{\alpha-1}R_2)}{\alpha_1} < \frac{1}{B}, \\ &\frac{g(, 0, \rho^{\alpha-1}\beta_2)}{\beta_2} \leq \frac{1}{A}, \qquad \frac{g(\rho^{\alpha-1}R_1, \alpha_2\rho^{\alpha-1})}{\alpha_2} < \frac{1}{B}. \end{aligned}$$

Thus conditions (4.3) and (4.4) hold. Then, by Theorem 4.2, problem (5.2) has multiplicity of solutions.

6 Conclusions

In this present work, we discussed some existence multiplicity results for system of fractional differential equations, under various assumptions on the right-hand side nonlinearity. The main assumptions on the nonlinearity are the continuity and some Nagumo–Bernstein type growth conditions. We have used fixed point theory in vector metric spaces with general properties from functional analysis. Also the positivity result for a fractional system of differential equations was considered. We hope that this paper can provide contributions to the questions of existence, positivity, and multiplicity of solutions for fractional differential equations on bounded domains.

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