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On some null-filiform algebras and solvable Leibniz algebras

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On some null-filiform algebras and solvable Leibniz algebras

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On some null-filiform algebras and solvable Leibniz algebras

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Santiago de Compostela, a 27 de abril de 2017.

Fdo.: Bakhrom Omirov　　　　Fdo.: Manuel Ladra González

On some null-filiform algebras and solvable Leibniz algebras

AUTORIZACIÓN DEL DIRECTOR/TUTOR DE LA TESIS

Dr. Bakhrom Omirov, Profesor del Institute of Mathematics, National University of Uzbekistan (Uzbekistán) y Dr. Manuel Ladra González, Profesor del Departamento de Matemáticas de la Universidad de Santiago de Compostela, como Directores de la Tesis Doctoral titulada **On some null-filiform algebras and solvable Leibniz algebras**, presentada por Dª. **Kamilyam Masutova**, alumna del programa de Doctorado *Matemáticas*,

AUTORIZAMOS la presentación de la Tesis Doctoral indicada para optar al grado de Doctor por la Universidad de Santiago de Compostela, considerando que reúne los requisitos exigidos en el artículo 34 del reglamento de Estudos de Doutoramento, y que como Director de la misma no incurre en causas de abstención establecidas en la ley 40/2015.

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Contents

Resumen de la Tesis Doctoral (in Spanish)

On some null-filiform algebras and solvable Leibniz algebras

Resumen abreviado:
Sobre algunas álgebras de Leibniz solubles y nulo-filiformes

En esta tesis se estudian algunas álgebras de Leibniz solubles y nulo-filiformes. Estudiamos la descripción de las álgebras solubles con un nilradical dado. En concreto, clasificamos las álgebras de Leibniz solubles con el nilradical filiforme. Además, establecemos que las álgebras de Leibniz solubles con nilradical Lie filiforme son álgebras de Lie. Tenemos que señalar que la descripción obtenida en la tesis de las álgebras de Lie solubles con nilradicales filiformes es también un resultado nuevo para álgebras de Lie.

Investigamos la variedad de las álgebras Leibniz de dimensión $n+1$, obteniendo resultados generales sobre algunas componentes irreducibles de la variedad de álgebras de Leibniz de dimensión finita, e indicamos representantes de las álgebras de Leibniz solubles, cuyas clausuras de las órbitas forman componentes irreducibles. También describimos el segundo grupo de cohomología de las álgebras de Leibniz solubles de dimensión $n+1$ con nilradical F_n^1, probamos que las álgebras R_1 y $R(F_n^1)$ son rígidas, y para otras álgebras encontramos una base del segundo grupo de cohomología.

Además, investigamos algunas clases especiales de álgebras de Leibniz nilpotentes. En concreto, consideramos álgebras de Leibniz nilpotentes naturalmente graduadas de dimensión n con sucesión característica igual a $(n-m, m)$ para el caso $m \geq 4$, y proporcionamos una clasificación de las álgebras de Leibniz descomponibles naturalmente graduadas con sucesión característica igual a $(n-m, m)$ en el caso $m \geq 4$.

En el último capítulo de esta tesis describimos álgebras con el índice máximo de nilpotencia que satisfacen algunas identidades generalizadas de una forma especial. En concreto, clasificamos las álgebras de Leibniz generalizadas y álgebras de Zinbiel generalizadas con el índice máximo de nilpotencia.

Las álgebras de Leibniz fueron introducidas a principios de los años 90 del siglo pasado por el matemático francés J.-L. Loday como un álgebra caracterizada por la identidad de Leibniz [45]. Cabe señalar que el álgebra que satisface la identidad de Leibniz fue considerada por primera vez por A. Bloh [10] en

1965 bajo el nombre de D-álgebra. Sin embargo, el estudio de las D-álgebras después de este artículo estuvo parado y solo después de los trabajos publicados por J.-L. Loday y T. Pirashvili [47, 48], las álgebras de Leibniz han sido estudiadas activamente, como lo demuestran numerosos trabajos dedicados a estas álgebras [5, 8, 9, 16, 50, 51, 58, 59].

Las álgebras de Leibniz generalizan álgebras de Lie de manera natural. La teoría de álgebras de Leibniz está siendo investigada activamente en las últimas dos décadas. Muchos resultados de la teoría de álgebras de Lie se extendieron a las álgebras de Leibniz. Por ejemplo, los resultados clásicos sobre las subálgebras de Cartan [58], la descomposición de Levi [9], las propiedades de las álgebras solubles con nilradical dado [19] y otros resultados de la teoría de álgebras de Lie también son verdaderos para álgebras de Leibniz [5,8,60,62,63].

En 1955, N. Jacobson [34] demostró que toda álgebra de Lie sobre un cuerpo de característica cero que admite una derivación no singular es nilpotente. El problema de si la recíproca de esta afirmación es correcta permaneció abierto hasta que J. Dixmier y W.G. Lister [22] construyeron un ejemplo de un álgebra de Lie nilpotente de dimensión 8 cuyas derivaciones son nilpotentes. Ellos llamaron a este tipo de álgebras, álgebras de Lie característicamente nilpotentes.

Si tenemos que todas las derivaciones de un álgebra son nilpotentes (las derivaciones interiores son también nilpotentes), entonces por el teorema de Engel concluimos que un álgebra de Lie característicamente nilpotente es nilpotente. La afirmación recíproca no es verdadera, porque existen álgebras de Lie nilpotentes que admiten derivaciones no nilpotentes. Por lo tanto, el subconjunto de las álgebras de Lie característicamente nilpotentes está estrictamente incluido en el conjunto de las álgebras de Lie nilpotentes.

Los trabajos [21,36,43], entre otros más, se dedican a la investigación de las álgebras de Lie característicamente nilpotentes. La clasificación de las álgebras de Lie nilpotentes hasta dimensión 8 revela que no hay álgebras de Lie característicamente nilpotentes en dimensiones menores que 7. Además, se probó que existen álgebras de Lie característicamente nilpotentes en cada dimensión, desde dimensión 7 hasta 13. Teniendo en cuenta que una suma directa de álgebras de Lie característicamente nilpotentes es característicamente nilpotente, entonces tenemos la existencia de álgebras de Lie característicamente nilpotentes en cada dimensión finita a partir de dimensión 7.

Recuérdese que, como en el caso de álgebras de Lie, se ha demostrado que existen álgebras de Leibniz nilpotentes, en las cuales todas las derivaciones son nilpotentes, y por lo tanto, no son singulares [57]. En particular, se demostró un análogo del teorema de Jacobson para álgebras de Leibniz [5]. Además, se comprueba que al igual que en el caso de álgebras de Lie, no se cumple el enunciado recíproco de Jacobson para álgebras de Leibniz. En [57], análogamente al caso de las álgebras de Lie, se definió la noción de álgebra de Leibniz característicamente nilpotente y se encontraron algunas familias de álgebras de Leibniz filiformes característicamente nilpotentes. Además, en dicho trabajo se presentó un criterio de característicamente nilpotencia de algunas álgebras de Leibniz filiformes. Debido a la existencia de un ejemplo de un álgebra de Leibniz característicamente nilpotente que no satisface la condición de [57], el criterio no es correcto.

El primer trabajo que se dedicó a la descripción de las álgebras de Lie solubles es el artículo [49]. De hecho, se demostró que el espacio complementado al nilradical forma una subálgebra abeliana, que consiste en los elementos semisimples de un álgebra. Sin embargo, la estructura del nilradical depende de esta subálgebra. Posteriormente, G.M. Mubarakzjanov propuso la descripción de las álgebras de Lie solubles con una estructura dada del nilradical [53] mediante las derivaciones exteriores del nilradical. Los trabajos [3, 4, 17, 54, 65, 67] se dedicaron a la aplicación del método de Mubarakzjanov en álgebras de Lie solubles con diferentes tipos de nilradicales. Algunos resultados de la teoría de álgebras de Lie extendidos a las álgebras de Leibniz en [5] nos permiten aplicar el método de Mubarakzjanov al caso de las álgebras de Leibniz. En esta dirección, los trabajos [19] y [20] se ocupan de la descripción de las álgebras de Leibniz solubles con nilradicales nulo-filiformes y filiformes naturalmente graduados, respectivamente. Nosotros continuamos la descripción de las álgebras solubles con un nilradical dado. En el Capítulo 1 se consideran y se clasifican las álgebras de Leibniz solubles con nilradical filiforme. Gracias a los trabajos [37] y [39], ya tenemos la clasificación de las álgebras de Leibniz filiformes no característicamente nilpotentes.

Además, establecemos que las álgebras de Leibniz solubles con el nilradical Lie filiforme son álgebras de Lie. Cabe señalar que la descripción obtenida en este capítulo de las álgebras de Lie solubles con nilradicales filiformes es también un resultado nuevo para álgebras de Lie.

Proposición 1.2.2. *No existe ninguna álgebra de Leibniz soluble de dimensión $n+2$ cuyo nilradical sea $F_1(0,0,\ldots,0,1)$.*

Proposición 1.2.3. *No existe ninguna álgebra de Leibniz soluble de dimensión $n+2$ cuyo nilradical sea $F_1^s(\alpha_3,\alpha_4,\ldots,\alpha_{n-1},\alpha_n,\alpha_n)$.*

Teorema 1.2.5. *Cualquier álgebra de Leibniz soluble de dimensión $n+2$ (en el caso de n impar) con nilradical $F_2(0,0,\dots,0,0,1)$ es isomorfa a la siguiente álgebra:*

$$L_1 : \begin{cases} [e_0,e_0]=e_2, & [x,e_1]=-\frac{n}{2}e_1, \\ [e_i,e_0]=e_{i+1}, \quad 2\leq i\leq n-1, & [e_0,x]=e_0, \\ [x,e_0]=-e_0, & [e_1,x]=\frac{n}{2}e_1, \\ [e_1,e_1]=e_n, & [e_i,x]=ie_i, \qquad 2\leq i\leq n. \end{cases}$$

Teorema 1.2.6. *Cualquier álgebra de Leibniz soluble de dimensión $n+2$ (en el caso de n par) con nilradical $F_2^1(0,0,\dots,0,\underbrace{\beta_{\frac{n+2}{2}}}_{\frac{n-2}{2}},0,\dots,0,0,1)$ es isomorfa a un álgebra de la siguiente familia de álgebras:*

$$L_2^{\beta_{\frac{n+2}{2}}} : \begin{cases} [e_0,e_0]=e_2, & [e_0,e_1]=\beta_{\frac{n+2}{2}}e_{\frac{n+2}{2}}, \\ [e_i,e_0]=e_{i+1}, \quad 2\leq i\leq n-1, & [e_1,e_1]=e_n, \\ [x,e_0]=-e_0, & [e_i,e_1]=\beta_{\frac{n+2}{2}}e_{\frac{n+2i}{2}}, \ 2\leq i\leq \frac{n}{2}, \\ [e_0,x]=e_0, & [x,e_1]=-\frac{n}{2}e_1-\beta_{\frac{n+2}{2}}e_{\frac{n}{2}}, \\ [e_1,x]=\frac{n}{2}e_1, & \\ [e_i,x]=ie_i, \quad 2\leq i\leq n. & \end{cases}$$

Teorema 1.2.7. *Cualquier álgebra de Leibniz soluble de dimensión $n+2$ con nilradical F_2^j, con $3\leq j\leq n$, es isomorfa a la siguiente álgebra:*

$$L_3^j : \begin{cases} [e_0,e_0]=e_2, & [e_0,e_1]=e_1, \\ [e_i,e_0]=e_{i+1}, \ 2\leq i\leq n-1, & [e_i,e_1]=e_{j+i-1}, \ 2\leq i\leq n-1-j, \\ [x,e_0]=-e_0, & [x,e_1]=-(j-1)e_1-e_{j-1}, \\ [e_0,x]=e_0, & [e_i,x]=ie_i, \ 2\leq i\leq n. \\ [e_1,x]=(j-1)e_1, & \end{cases}$$

Teorema 1.2.9. *No existe ninguna álgebra de Leibniz soluble cuyo nilradical sea un álgebra que no sea de Lie no característicamente filiforme de la familia $F_3(\theta_1,\theta_2,\theta_3)$.*

En el caso en que el nilradical sea un álgebra de Lie filiforme no característicamente nilpotente $A_{n+1}^r(\alpha_1,\dots,\alpha_t)$ o $B_{n+1}^r(\alpha_1,\dots,\alpha_t)$, obtenemos familias de álgebras de Lie. Para la clasificación de dichas familias de álgebras tenemos una conjetura sobre la transformación general de las bases. La veracidad de dicha conjetura se comprobó solo para dimensiones bajas fijas con el programa Mathematica. Sin embargo, debido al gran número de cálculos complejos necesarios no podríamos generalizar el cálculo para cualquier dimensión finita a partir de dimensiones bajas.

En muchos casos donde el álgebra de Leibniz involucrada puede depender de parámetros es útil conocer la estructura del conjunto de todas las álgebras de Leibniz de una dimensión dada. Cualquier ley del álgebra de Leibniz puede considerarse como un punto de una variedad algebraica afín definida por las ecuaciones polinómicas procedentes de la identidad de Leibniz para una base dada. Esta vía proporciona una descripción de las dificultades en los problemas de la clasificación relativas a las clases de las álgebras de Leibniz nilpotentes y solubles. Las órbitas relacionadas con la acción del grupo lineal general corresponden a las clases de isomorfismos de álgebras de Leibniz y así los problemas de clasificación (salvo isomorfismo) se pueden reducir a la clasificación de estas órbitas. Una variedad algebraica afín es una unión de un número finito de componentes irreducibles y las órbitas abiertas de Zariski proporcionan clases interesantes de álgebras de Leibniz para estudiar su clasificación. Las álgebras de Leibniz de estas clases se llaman rígidas.

Las variedades de las leyes de las álgebras de Lie sobre el cuerpo $\mathbb{C}$ de los números complejos han sido ampliamente estudiadas, estableciéndose varios resultados estructurales y propiedades importantes. Por el contrario, el problema para las variedades de álgebras de Leibniz no ha sido considerado en detalle. La investigación de las variedades de las leyes de las álgebras de Lie y de Leibniz se basa esencialmente en el estudio cohomológico de las álgebras de Leibniz y en la teoría de la deformación. Las deformaciones de anillos arbitrarios y álgebras asociativas, resultados sobre las álgebras rígidas y cuestiones de cohomología relacionadas, fueron investigadas por primera vez por Gerstenhaber [27] en 1964. Posteriormente, la noción de deformación fue aplicada en álgebras de Lie por Nijenhuis y Richardson [55], donde transforman el problema topológico relacionado con la rigidez en un problema cohomológico, demostrando que un álgebra de Lie $\mathfrak{g}$ es rígida si el segundo grupo $H^2(\mathfrak{g}, \mathfrak{g})$ de cohomología de Chevalley-Eilenberg se anula.

En el Capítulo 2 nos preocupamos de la estructura de la variedad $\mathfrak{Leib}_{n+1}$, la variedad de las álgebras Leibniz de dimensión $n+1$, en particular, dando respuestas a la siguiente cuestión: ¿Cuáles son las componentes irreducibles de $\mathfrak{Leib}_{n+1}$? Las respuestas a esta cuestión permitirían describir parcialmente las estructuras de algunas álgebras de Leibniz de dimensión $n+1$. Nosotros obtenemos resultados generales sobre algunas componentes irreducibles de la variedad de álgebras de Leibniz de dimensión finita e indicamos representantes de las álgebras de Leibniz solubles, cuyas clausuras de las órbitas forman componentes irreducibles. En el Capítulo 2, continuamos la investigación iniciada en el Capítulo 1, en concreto, consideramos las álgebras de Leibniz complejas solubles cuyo nilradical es un álgebra filiforme naturalmente graduada. A continuación damos una clasificación de estas álgebras.

Teorema 2.1.2. *Cualquier álgebra de Leibniz compleja filiforme naturalmente graduada de dimensión n es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$$F_n^1: \ [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1,$$
$$F_n^2: \ [e_i, e_1] = e_{i+1}, \ 1 \leq i \leq n-2,$$
$$F_n^3(\alpha): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1} e_n, & 2 \leq i \leq n-1, \end{cases}$$

donde $\alpha \in \{0, 1\}$ para n par, y, $\alpha = 0$ para n impar.

Los siguientes teoremas describen las álgebras de Leibniz solubles con nilradical F_n^1.

Teorema 2.1.3 ([20]). *Cualquier álgebra de Leibniz soluble de dimensión $n+1$ con nilradical F_n^1 es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

$R_1:$

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1, \\ [x, e_1] = -e_1 - e_2, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-1)e_i, \ 2 \leq i \leq n, \end{cases}$$

$R_2(\alpha):$

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-1+\alpha)e_i, \ 2 \leq i \leq n, \end{cases}$$

R_3 :

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-n)e_i, \ 2 \leq i \leq n, \\ [x, x] = e_n, \end{cases}$$

R_4 :

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1 + e_n, \\ [e_i, x] = (i+1-n)e_i, \ 2 \leq i \leq n, \\ [x, x] = -e_{n-1}, \end{cases}$$

$$R_5(\alpha_i) = R_5(\alpha_4, \dots, \alpha_n) : \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = e_2 + \sum_{i=4}^{n-1} \alpha_i \, e_i, & \\ [e_i, x] = e_i + \sum_{j=i+2}^{n} \alpha_{j-i+2} \, e_j, & 2 \leq i \leq n \, . \end{cases}$$

Además, el primer parámetro que no se anula $\{\alpha_4, \dots, \alpha_n\}$ en el álgebra $R_5(\alpha_4, \dots, \alpha_n)$ puede ajustar la escala a 1.

Teorema 2.1.4 ([20]). *No existe ninguna álgebra de Leibniz soluble de dimensión $(n+2)$ con nilradical F_n^1.*

Se describe en la Sección 2.2 el segundo grupo de cohomología de las álgebras de Leibniz solubles de dimensión $(n+1)$ con nilradical F_n^1. Con respecto a la dimensión $(n+2)$, en el trabajo [20] se afirmó que no existe tal álgebra de Leibniz soluble. Sin embargo, nosotros encontramos una descripción mejorada probando que existe una única álgebra de Leibniz soluble de dimensión $(n+2)$ con nilradical F_n^1.

Teorema 2.2.16. *Cualquier álgebra de Leibniz soluble de dimensión $(n+2)$ con nilradical F_n^1 es isomorfa al álgebra $R(F_n^1)$ con la siguiente tabla de multiplicación:*

$$\begin{array}{lll} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, & [e_1, x] = e_1, \\ [e_i, y] = e_i, & 2 \leq i \leq n, & [e_i, x] = (i-1)e_i, \quad 2 \leq i \leq n, \\ & & [x, e_1] = -e_1. \end{array}$$

Por otra parte, establecemos que el segundo grupo de cohomología de esta álgebra de Leibniz con coeficientes en sí misma es trivial, y, en consecuencia, es un álgebra rígida.

Probamos también que las álgebras R_1 y $R(F_n^1)$ son rígidas. Para otras álgebras encontramos una base del segundo grupo de cohomología.

Corolario 2.2.4. $\dim ZL^2(R_3, R_3) = (n+1)^2 - 2$ *y* $\dim HL^2(R_3, R_3) = 1$.

Proposición 2.2.5. *La clase de equivalencia* $\overline{\xi}$ *forma una base de* $HL^2(R_3, R_3)$*, donde*

$$\xi : \begin{cases} \xi(e_1, x) = e_1, \\ \xi(e_i, x) = (i-2)e_i, \quad 3 \le i \le n, \\ \xi(x, e_1) = -e_1. \end{cases}$$

Corolario 2.2.8. $\dim ZL^2(R_4, R_4) = (n+1)^2 - 2$*,* $\dim HL^2(R_4, R_4) = 1$*, y la clase de equivalencia* $\overline{\rho}$ *forma una base de* $HL^2(R_4, R_4)$*, donde*

$$\rho : \begin{cases} \rho(e_1, x) = e_1, \\ \rho(e_i, x) = (i-2)e_i, \ 3 \le i \le n, \\ \rho(x, e_1) = -e_1, \\ \rho(x, x) = -e_{n-1}. \end{cases}$$

Corolario 2.2.11.

$$\dim HL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} 2, & \alpha = 0;\ \pm 1; 1-n; 2-n, \\ 1, & \alpha \neq 0;\ \pm 1; 1-n; 2-n, \end{cases} \qquad \textit{para } n > 3;$$

$$\dim HL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} 4, & \alpha = -1, \\ 2, & \alpha = 0; 1; -2, \\ 1, & \alpha \neq 0;\ \pm 1; -2, \end{cases} \qquad \textit{para } n = 3.$$

En la siguiente proposición encontramos una base de $HL^2\big(R_2(\alpha), R_2(\alpha)\big)$.

Proposición 2.2.12. *La base de* $HL^2\big(R_2(\alpha), R_2(\alpha)\big)$ *consta de las siguientes clases de equivalencias*

$$\begin{cases} \overline{\rho}, \overline{\psi_1}, & \alpha = 0; 1; 1-n; 2-n, \\ \overline{\psi_1}, \overline{\psi_2}, & \alpha = -1, \\ \overline{\rho}, & \alpha \neq 0;\ \pm 1; 1-n; 2-n, \end{cases} \qquad \textit{para } n > 3;$$

$$\begin{cases} \overline{\rho}, \overline{\psi_1} & \alpha = 0; -1; -2, \\ \overline{\psi_1}, \overline{\psi_2}, \overline{\psi_3}, \overline{\psi_4}, & \alpha = -1, \qquad\qquad \textit{para } n = 3, \\ \overline{\rho}, & \alpha \neq 0;\ \pm 1; -2, \end{cases}$$

donde

$$\rho : \begin{cases} \rho(e_1, x) = e_1, \\ \rho(e_i, x) = (i-2)e_i, \quad 3 \leq i \leq n, \\ \rho(x, e_1) = -e_1, \end{cases}$$

$$\psi_1 : \psi_1(e_i, x) = e_i, \quad 2 \leq i \leq n, \quad \psi_2 : \psi_2(e_i, e_2) = e_i, \quad 2 \leq i \leq n,$$
$$\psi_3 : \begin{cases} \psi_3(e_1, x) = e_n, \\ \psi_3(x, x) = -e_{n-1}, \end{cases} \qquad \psi_4 : \begin{cases} \psi_4(e_1, e_2) = e_n, \\ \psi_4(x, e_2) = -e_{n-1}. \end{cases}$$

Corolario 2.2.14.

$$\dim ZL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = n^2 + 3n - 3,$$
$$\dim HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = \begin{cases} 2n-4, & \alpha_i = 0 \quad \textit{para todo } i, \\ 2n-5, & \alpha_i \neq 0 \quad \textit{para algún } i. \end{cases}$$

Proposición 2.2.15. *Las clases de equivalencias* $\overline{\rho}$, $\overline{\psi_k}$ $(4 \leq k \leq n)$ *y* $\overline{\varphi_{n,k}}$ $(2 \leq k \leq n-1)$ *forman una base de* $HL^2\big(R_5(0), R_5(0)\big)$*. La base de* $HL^2\big(R_5(\alpha_4, \ldots, \alpha_n), R_5(\alpha_4, \ldots, \alpha_n)\big)$ *con* $(\alpha_4, \ldots, \alpha_n) \neq (0, \ldots, 0)$ *es también la misma excepto un cociclo* $\overline{\psi_k}$ *con* $\alpha_k \neq 0$.

Teorema 2.2.16. *Cualquier álgebra de Leibniz soluble de dimensión* $(n+2)$ *con nilradical* F_n^1 *es isomorfa al álgebra* $R(F_n^1)$ *con la siguiente tabla de multiplicación:*

$$\begin{array}{lll} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, & [e_1, x] = e_1, \\ [e_i, y] = e_i, & 2 \leq i \leq n, & [e_i, x] = (i-1)e_i, \quad 2 \leq i \leq n, \\ & & [x, e_1] = -e_1. \end{array}$$

Corolario 2.2.19. $\dim ZL^2(R(F_n^1), R(F_n^1)) = \dim BL^2(R(F_n^1), R(F_n^1)) = (n+2)^2 - 3$ *y* $\dim HL^2(R(F_n^1), R(F_n^1)) = 0$.

Teorema 2.2.20. *El álgebra* $R(F_n^1)$ *es rígida.*

El Capítulo 3 de esta tesis se dedica a la investigación de algunas clases especiales de álgebras de Leibniz nilpotentes. En concreto, consideramos álgebras de Leibniz nilpotentes naturalmente graduadas de dimensión n con sucesión característica igual a $(n-m,m)$. Anteriormente, se había estudiado el problema de clasificar tales álgebras para $m<4$. El presente capítulo continúa esta investigación para el caso $m \geq 4$.

Recordemos que un álgebra de Lie nilpotente es un álgebra soluble de un tipo especial. Se sabe que las investigaciones de las álgebras de Lie de dimensión finita se condensan en el estudio de las álgebras nilpotentes. En este sentido, es natural aplicar los resultados y métodos de la teoría de álgebras de Lie en el estudio de las álgebras de Leibniz. Debido a que la descripción de las álgebras de Lie nilpotentes es un problema ilimitado, su estudio debe llevarse a cabo con restricciones adicionales como restricciones en el índice de nilpotencia del álgebra, en la sucesión característica, graduación, etc. Al estudiar las álgebras de Leibniz casi-filiformes naturalmente graduadas [15] se observó que, en contraste con el caso de Lie, las álgebras de Leibniz contienen una clase de álgebras de dimensión n, cuya sucesión característica es $(n-2,2)$. El estudio posterior de las álgebras naturalmente graduadas con sucesión característica igual a $(n-3,3)$ ([13]) revela, en este caso, que la clase de las álgebras de Leibniz que no son de Lie es suficientemente amplia. El presente capítulo se dedica a la descripción de las álgebras de Leibniz que no son de Lie con sucesión característica igual a $(n-m,m)$ para el caso $m \geq 4$.

El siguiente resultado proporciona una clasificación de las álgebras de Leibniz descomponibles naturalmente graduadas con sucesión característica igual a $(n-m,m)$ en el caso $m \geq 4$.

Teorema 3.2.2. *Sea L un álgebra de Leibniz con sucesión característica $C(L)=(n-m,m)$. El álgebra L es descomponible si y solo si M y N son álgebras de Leibniz nulo-filiformes con* $\dim M = n-m$ *y* $\dim N = m$.

Ahora consideraremos álgebras de Leibniz indescomponibles. La definición de la sucesión característica de un álgebra de Leibniz implica la existencia de una base tal que la matriz del operador de la multiplicación por la derecha R_{e_1} tiene una de las dos formas posibles. Dependiendo de estas formas tenemos álgebras de Leibniz de tipo I o II.

Teorema 3.2.4. *Sea L un álgebra de Leibniz con sucesión característica $C(L) = (n-m, m)$ de tipo I. Entonces existe una base $\{e_1, e_2, \ldots, e_{n-m}, f_1, \ldots, f_m\}$ del álgebra L en la cual la tabla de multiplicación tiene la siguiente forma:*

$$\begin{array}{ll}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-m-1, \\
[f_i, e_1] = f_{i+1}, & 1 \leq i \leq m-1, \\
[e_i, f_j] = A_{i,j}(\alpha)e_{i+j} + A_{i,j}(\beta)f_{i+j}, & 1 \leq i \leq m-j, \\
[f_i, f_j] = A_{i,j}(\gamma)e_{i+j} + A_{i,j}(\delta)f_{i+j}, & 1 \leq i \leq m-j, \\
[e_i, f_j] = A_{i,j}(\alpha)e_{i+j}, & m-j+1 \leq i \leq n-m-j, \\
[f_i, f_j] = B_{i,j}(\gamma)e_{i+j}, & m-j+1 \leq i \leq \min\{m, n-m-j\}
\end{array}$$

con $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Teorema 3.2.5. *Sea L un álgebra de Leibniz de tipo I. Sea $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ para algún $k \in \{1, 2, \ldots, m-1\}$. Entonces las siguientes relaciones se verifican:*

$$\begin{array}{lll}
\beta_1 = -1, & \beta_i = 0, & 2 \leq i \leq k-1, \\
\beta_{k+t} = C_{k+t-1}^{k-1}\beta + (-1)^k C_{k+t-2}^{k-1}, & & 1 \leq t \leq m-k-1,
\end{array}$$

donde $\beta = \beta_k$ es un número fijo.

Análogamente, obtenemos expresiones para los otros parámetros α_{k+t}, γ_{k+t} y δ_{k+t}.

Además, en la Sección 3.3 obtenemos expresiones para los cambios de los parámetros, mediante un isomorfismo, en la tabla de multiplicación de tales álgebras; estas expresiones se pueden utilizar para obtener una clasificación completa en una dimensión fija y un valor dado de m.

Como se mencionó anteriormente, el estudio de las álgebras nilpotentes es un problema ilimitado y solo debería tener lugar con la imposición de restricciones sobre el índice de nilpotencia. Cabe señalar que el nilíndice máximo del álgebra de Lie coincide con la dimensión del álgebra, y tal álgebra se llama filiforme, y para álgebras de Leibniz su nilíndice máximo es una dimensión mayor que su dimensión (en este caso el álgebra de Leibniz se llama nulo-filiforme). Sin embargo, en vista de la gran diversidad de clases de álgebras, las álgebras con el índice máximo de nilpotencia se describen por separado en cada caso especial. En el último capítulo de esta tesis describimos álgebras con el índice máximo de nilpotencia que satisfacen algunas identidades generalizadas de una forma especial. En concreto, en el Capítulo 4, clasificamos las álgebras de Leibniz generalizadas y álgebras de Zinbiel generalizadas con el índice máximo de nilpotencia.

Definición 4.1.1. *Un álgebra L sobre un cuerpo $\mathbb{F}$ se llama un álgebra de Leibniz generalizada si, para cualesquiera elementos $x, y, z \in L$, la siguiente identidad se verifica:*

$$a(bc) = A_1(ab)c + A_2(ac)b,$$

donde $A_1, A_2 \in \mathbb{F}$.

Definición 4.1.2. *Un álgebra L sobre un cuerpo $\mathbb{F}$ se llama un álgebra de Zinbiel generalizada si, para cualesquiera elementos $x, y, z \in L$, la siguiente identidad se verifica:*

$$(ab)c = A_1a(bc) + A_2a(cb),$$

donde $A_1, A_2 \in \mathbb{F}$.

Obsérvese que las álgebras de Leibniz son ejemplos de álgebras de Leibniz generalizadas para $A_1 = 1$ y $A_2 = -1$, y, en particular, la clasificación de las álgebras de Leibniz nulo-filiformes en [6] es un caso especial del Teorema 4.2.1 (para el caso $A_1 + A_2 = 0$). La estructura de las álgebras de Zinbiel (Koszul dual de las álgebras de Leibniz) de dimensión finita fue investigada en [1,23,56]. A su vez, observamos que las álgebras de Zinbiel son ejemplos de álgebras de Zinbiel generalizadas para $A_1 = A_2 = 1$. La clasificación de las álgebras de Zinbiel nulo-filiformes en [1] es un caso especial de los Teoremas 4.3.1 y 4.3.5 (en el caso $A_1 = A_2$).

Teorema 4.2.1. *Un álgebra de Leibniz generalizada nulo-filiforme A de dimensión n existe solo para $A_1 + A_2 = 0$ o $A_1 + A_2 = 1$. Además, en cada caso, existe una base $\{e_1, e_2, \ldots, e_n\}$ del álgebra A tal que la tabla de multiplicación tiene la siguiente forma:*

$$\text{caso I: } A_1 + A_2 = 0: \qquad e_ie_1 = e_{i+1}, \quad 1 \leq i \leq n-1;$$

$$\text{caso II: } A_1 + A_2 = 1: \qquad e_ie_j = e_{i+j}, \quad 2 \leq i+j \leq n.$$

Teorema 4.3.1. *Un álgebra de Zinbiel generalizada nulo-filiforme A de dimensión n existe solo para $A_1 = A_2$, o $A_1 = -A_2$, o $A_1 + A_2 = 1$. Además, en el álgebra A existe una base $\{e_1, e_2, \ldots, e_n\}$ tal que la tabla de multiplicación tiene la siguiente forma:*

$$e_i e_j = B_{i,j} e_{i+j},$$

donde $B_{1,j} = 1,\ 1 \leq j \leq n-1$ *y* $B_{i,j} = A_1 B_{i-1,j} + A_2 B_{j,i-1},\ 2 \leq i \leq n-j.$

Si consideramos las álgebras de Leibniz generalizadas para $A_1 = 1$ y $A_2 = 0$ (el caso $A_1 + A_2 = 1$) o las álgebras de Zinbiel generalizadas para $A_1 = 1$ y $A_2 = 0$ (el caso $A_1 + A_2 = 1$), entonces obtenemos las álgebras asociativas. Obsérvese que las álgebras asociativas nulo-filiformes de cualquier dimensión no fueron estudiadas anteriormente como álgebras con el índice máximo de nilpotencia. Los resultados obtenidos en el presente capítulo cubren este hueco. Por lo tanto, las clasificaciones obtenidas para las álgebras de Leibniz generalizadas nulo-filiformes y álgebras de Zinbiel generalizadas nulo-filiformes extienden los resultados bien conocidos para las álgebras de Leibniz, álgebras de Zinbiel y álgebras asociativas. Además, el desarrollo del álgebra abstracta moderna conduce a la aparición de nuevas variedades de álgebras y las clasificaciones obtenidas se pueden usar para estudiar las álgebras obtenidas como casos especiales de las álgebras de Leibniz generalizadas y álgebras de Zinbiel generalizadas.

Introduction

Leibniz algebras were introduced in the early 1990s of the last century by the French mathematician J.-L. Loday as an algebra characterized by the Leibniz identity [45]. It should be noted that an algebra satisfying the Leibniz identity was first considered by A. Bloh [10] in 1965 under the name of D-algebra. However, the study of D-algebras after this article on this topic stopped and only after the works of J.-L. Loday and T. Pirashvili [47, 48] Leibniz algebras have been actively studied, as they evidence the numerous works devoted to these algebras [5, 8, 9, 30, 46, 50, 51, 58, 59].

Leibniz algebras generalize Lie algebras in natural way. The theory of Leibniz algebras has been actively investigated in the last two decades. Many results of the theory of Lie algebras have been extended to Leibniz algebras. For instance, the classical results on Cartan subalgebras [58], Levi's decomposition [9], the properties of solvable algebras with given nilradical [19] and other from the theory of Lie algebras are also true for Leibniz algebras [5, 8, 60, 62, 63].

In 1955, N. Jacobson [34] proved that every Lie algebra over a field of characteristic zero admitting a non-singular derivation is nilpotent. The problem whether the inverse of this statement is correct remained open until that an example of an 8-dimensional nilpotent Lie algebra whose all derivations are nilpotent was constructed by J. Dixmier and W.G. Lister [22]. They called such type of algebras characteristically nilpotent Lie algebras.

If all derivations of an algebra are nilpotent (the inner derivations are nilpotent, as well), then by Engel's theorem we conclude that a characteristically nilpotent Lie algebra is nilpotent. The inverse statement is not true, because there exist nilpotent Lie algebras admitting non-nilpotent derivations. Therefore, the subset of characteristically nilpotent Lie algebras is strictly embedded into the set of nilpotent Lie algebras.

The papers [21,36,43] and others are devoted to the investigation of characteristically nilpotent Lie algebras. The classification of nilpotent Lie algebras till dimension 8 shows that there is no characteristically nilpotent Lie algebras in dimensions less than 7. Moreover, it is shown that there exist characteristically nilpotent Lie algebras in each dimension from 7 till 13-dimensional. Taking into account that a direct sum of characteristically nilpotent Lie algebras is characteristically nilpotent, then we have the existence of characteristically nilpotent Lie algebras in each finite dimension starting from 7.

Recall that, as in the case of the Lie algebra, it has been shown that there are nilpotent Leibniz algebras, in which all derivations are nilpotent, and hence, are non-singular [57]. In particular, an analogue of Jacobson's theorem was proved for Leibniz algebras [5]. Moreover, it is shown that similarly to the case of Lie algebras for Leibniz algebras the inverse of Jacobson's statement does not hold. In [57], analogously as for Lie algebras, the notion of characteristically nilpotent Leibniz algebra was defined and some families of characteristically nilpotent filiform Leibniz algebras were found. Moreover, there was presented a criterion of characteristically nilpotency of some filiform Leibniz algebras. Due to the existence of a example of a characteristically nilpotent Leibniz algebra which does not satisfy the condition of [57], the criterion is not correct.

The first work which was devoted to the description of the solvable Lie algebras is the paper [49]. In fact, it was proved that the complemented space to the nilradical forms an abelian subalgebra, consisting of the semisimple elements of an algebra. However, the structure of the nilradical depends on this subalgebra. Later, G.M. Mubarakzjanov proposed the description of solvable Lie algebras with a given structure of the nilradical [53] by means of outer derivations of the nilradical. The papers [3, 4, 17, 54, 65, 67] were devoted to the application of Mubarakzjanov's method for solvable Lie algebras with different kinds of nilradicals. Some results of the Lie algebra theory generalized to Leibniz algebras in [5] allow us to apply the Mubarakzjanov's method to the case of Leibniz algebras. In this direction the papers [19] and [20] deal with the description of solvable Leibniz algebras with null-filiform and naturally graded filiform nilradicals, respectively. We continue the description of solvable algebras with a given nilradical. Solvable Leibniz algebras with filiform nilradical are considered and classified in Chapter 1. Thanks to the papers [37] and [39], we already have the classification of the non-characteristically nilpotent filiform Leibniz algebras. Moreover, we establish that solvable Leibniz algebras with filiform Lie nilradical are Lie algebras. It should be noted that the obtained description of solvable Lie algebras with filiform nilradicals in this chapter is a new result for Lie algebras, as well.

In the case when the nilradical is a non-characteristically nilpotent filiform Lie algebra $A^r_{n+1}(\alpha_1, \dots, \alpha_t)$ or $B^r_{n+1}(\alpha_1, \dots, \alpha_t)$, we get families of Lie algebras. For the classification of these families of algebras we have a conjecture about the general transformation of basis. The correctness of the conjecture was checked only for fixed low dimensions by the program Mathematica. However, because of the great number of complex calculations needed we could not generalize the calculation from low dimensions for any finite dimension.

In many cases where the Leibniz algebra involved may depend on the parameters it is useful to know the structure of the set of all Leibniz algebras of a given dimension. Any Leibniz algebra law can be considered as a point of an affine algebraic variety defined by the polynomial equations coming from the Leibniz identity for a given basis. This way provides a description of the difficulties in the classification problems referring to the classes of nilpotent and solvable Leibniz algebras. The orbits relative to the action of the general linear group correspond to the isomorphism classes of Leibniz algebras. Therefore, the classification problems (up to isomorphism) can be reduced to the classification of these orbits (see [2,12,32,64]). An affine algebraic variety is a union of a finite number of irreducible components and the Zariski open orbits provide interesting classes of Leibniz algebras to be classified. The Leibniz algebras of this class are called rigid.

The research of the varieties of Lie algebras laws over the field $\mathbb{C}$ of the complex numbers have been extensively studied, establishing various important structural results and properties. On the contrary, the problem for varieties of Leibniz algebras has not been considered in detail. The research of the varieties of Lie and Leibniz algebra laws is essentially based on the cohomological study of Leibniz algebras and on deformation theory (see [18, 24, 25, 38, 40, 41, 52]). Deformations of arbitrary rings and associative algebras, results about rigid algebras and related cohomology questions, were first investigated in 1964 by Gerstenhaber [27]. Later, the notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [55], where they transform the topological problem related to the rigidity into a cohomological problem, proving that a Lie algebra $\mathfrak{g}$ is rigid if the second group $H^2(\mathfrak{g}, \mathfrak{g})$ of the Chevalley-Eilenberg cohomology vanishes.

In Chapter 2 we are concerned with the structure of the variety $\mathfrak{Leib}_{n+1}$, the variety of the $(n+1)$-dimensional Leibniz algebras, in particular, with answers to the following question: What irreducible components do $\mathfrak{Leib}_{n+1}$ fall into? The answers to this question would allow to describe partially the structures of some Leibniz algebras of dimension $n+1$. We obtain general results on some irreducible components of the variety of finite-dimensional Leibniz algebras and indicate representatives of solvable Leibniz algebras, whose closures of orbits form irreducible components. In Chapter 2, we continue the research started in Chapter 1, namely, we consider complex solvable Leibniz algebras whose nilradical is a naturally graded filiform algebra. The second cohomology group of the $(n+1)$-dimensional solvable Leibniz algebras with nilradical F_n^1 is described in Section 2.2. With regard to the dimension $(n+2)$, in the work [20] it was asserted that there is no such solvable Leibniz algebra. However, we find in our ameliorated description that there exists a unique $(n+2)$-dimensional solvable algebra Leibniz with nilradical F_n^1. Moreover, we establish that the second group of cohomology of this Leibniz algebra with coefficients in itself is trivial, and, consequently, it is a rigid algebra.

Chapter 3 of this thesis is devoted to the investigation of some special classes of nilpotent Leibniz algebras. Namely, we consider n-dimensional naturally graded nilpotent Leibniz algebras with characteristic sequence equal to $(n-m, m)$. Earlier, the problem of classifying such algebras was studied for $m < 4$. The present chapter continues this investigation for the case $m \geq 4$.

Recall that a nilpotent Lie algebra is a solvable algebra of a special type. It is known that the investigation of finite-dimensional Lie algebras was reduced to the study of nilpotent algebras. In this connection, it is natural to apply results and methods from the Lie algebra theory to the study of Leibniz algebras. Due to the fact that the description of nilpotent Lie algebras is a boundless problem, their study should be carried out with additional restrictions such as constraints on the index of nilpotency of the algebra, on the characteristic sequence, grading, etc. In studying naturally graded quasi-filiform Leibniz algebras [15], it was noted that, in contrast to the Lie case, the Leibniz algebras contain a class of n-dimensional algebras whose characteristic sequence is $(n-2, 2)$. The subsequent study of naturally graded algebras with characteristic sequence equal to $(n-3, 3)$ ([13]) shows that the class of non-Lie Leibniz algebras in this case is sufficiently wide. The present chapter is devoted to the description of non-Lie Leibniz algebras with characteristic sequence equal to

$(n-m,m)$ for the case $m \geq 4$. Moreover, in Section 3.3 we obtain expressions for the changes of the parameters in the multiplication table of such algebras under an isomorphism; these expressions can be used to obtain a complete classification in fixed dimension and a given value of m.

As mentioned above, the study of nilpotent algebras is a boundless problem and only should take place with the imposition of restrictions on the nilpotency index. It should be noted that the maximum nilindex of Lie algebra coincides with the dimension of the algebra, and such algebra is called filiform (see [14,28, 29]), and for Leibniz algebras their maximum nilindex is one greater than the dimension of them (the so-called null-filiform Leibniz algebras). However, in view of the great diversity of classes of algebras, the algebras with the maximum nilpotency index are described separately in each special case. In the last chapter of this thesis, we describe algebras with the maximum nilpotency index satisfying some generalized identities of a special form. Namely, in Chapter 4, we classify generalized Leibniz algebras and generalized Zinbiel algebras with the maximum nilpotency index. Note that the Leibniz algebras are examples of generalized Leibniz algebras for $A_1 = 1$ and $A_2 = -1$, and, in particular, the classification of null-filiform Leibniz algebras in [6] is a special case of Theorem 4.2.1 (for the case $A_1 + A_2 = 0$). The structure of finite-dimensional Zinbiel algebras (Koszul dual to the Leibniz algebras) was investigated in [1, 23, 56]. In turn, we note that the Zinbiel algebras are examples of generalized Zinbiel algebras for $A_1 = A_2 = 1$. The classification of null-filiform Zinbiel algebras in [1] is a special case of Theorems 4.3.1 and 4.3.5 (in the case where $A_1 = A_2$). If we consider the generalized Leibniz algebras for $A_1 = 1$ and $A_2 = 0$ (the case $A_1 + A_2 = 1$) or generalized Zinbiel algebras for $A_1 = 1$ and $A_2 = 0$ (the case $A_1 + A_2 = 1$), then we get the associative algebras.

Note that the null-filiform associative algebras of any dimension were not studied earlier as algebras with the maximum nilpotency index. The results obtained in the present chapter fill this gap. Thus, the obtained classifications of null-filiform generalized Leibniz algebras and generalized Zinbiel algebras generalize well-known results for Leibniz algebras, Zinbiel algebras and associative algebras. In addition, the development of the modern abstract algebra leads to the appearance of new varieties of algebras, and the obtained classifications can be used to study the algebras obtained as special cases of generalized Leibniz algebras and generalized Zinbiel algebras.

Chapter 1

Solvable Leibniz algebras with filiform nilradical

In the first section of this chapter we present some known notations and results concerning filiform Lie and filiform Leibniz algebras, solvable algebras, outer derivations of nilradicals of Leibniz algebras. Solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra are known [20]. Here we extend the description to solvable Leibniz algebras whose nilradical is a filiform algebra. In Section 1.2 we investigate solvable Leibniz algebras whose nilradical is one from the list of Theorem 1.1.12. Moreover, in Section 1.3 we establish that solvable Leibniz algebras with filiform Lie nilradical are Lie algebras. It should be noted that the description of solvable Lie algebras with filiform nilradicals obtained in this work is a new result for Lie algebras, as well.

Throughout the chapter we consider finite-dimensional vector spaces and algebras over the field of the complex numbers. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noted, we shall consider non-nilpotent solvable algebras.

1.1 Introduction to solvable Leibniz algebras

In this section we give the necessary definitions and preliminary results concerning filiform Lie and filiform Leibniz algebras, solvable algebras, outer derivation, nilradical.

Before passing to preliminary results concerning Leibniz algebras, we will remind concepts of binomial coefficients and Catalan's numbers.

Here and elsewhere, C_n^m denotes the binomial coefficient $\binom{n}{m}$, i.e.

$$C_n^m = \frac{n!}{(n-m)!m!}.$$

We will also use the concept of the Catalan's number of degree p. Recall that the number of Catalan's number of degree p is defined as follows [33]:

$$Q_n^p = \frac{1}{(p-1)n+1} C_{pn}^n = \frac{1}{(p-1)n+1} \cdot \frac{(pn)!}{((p-1)n)!n!}.$$

Now we remind the notion of Lie algebra.

Definition 1.1.1 ([11,35]). *An algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called a Lie algebra if its multiplication (denoted by $[x,y]$) satisfies the identities:*

(1) $[x,x]=0$,

(2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$,

for all x,y,z in $\mathfrak{g}$.

The product $[x,y]$ is called the bracket of x and y. Identity (2) is called the Jacobi identity.

Definition 1.1.2. *An algebra $(L,[-,-])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x,y,z \in L$, the so-called Leibniz identity*

$$\big[[x,y],z\big] = \big[[x,z],y\big] + \big[x,[y,z]\big]$$

holds, where $[-,-]$ is the multiplication in L.

Any Lie algebra clearly is a Leibniz algebra but the converse is not true.

Definition 1.1.3. *The set $\mathrm{Ann}_r(L) = \{x \in L \mid [L,x]=0\}$ of a Leibniz algebra L is called the right annihilator of L.*

From the Leibniz identity we conclude that for any $x, y \in L$ the elements $[x, x]$ and $[x, y] + [y, x]$ lie in the right annihilator of the algebra L.

Definition 1.1.4. *A linear map $d : L \to L$ of a Leibniz algebra L is said to be a derivation if for any $x, y \in L$, the following condition holds:*

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element x of a Leibniz algebra L the operator of right multiplication $R_x : L \to L$, defined as $R_x(y) = [y, x]$ for $y \in L$, is a derivation. This kind of derivations are called *inner derivations.*

Definition 1.1.5 ([53]). *Let $d_1, d_2, \dots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \dots, d_n$ are said to be nil-independent if*

$$\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$$

is not nilpotent for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$.

In other words, if for any $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ there exists a natural number k such that $(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n)^k = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

For a Leibniz algebra L, the sequences of two-sided ideals defined recursively as follows:

$$\begin{aligned} L^1 &= L, & L^{k+1} &= [L^k, L^1], & k &\geq 1, \\ L^{[1]} &= L, & L^{[s+1]} &= [L^{[s]}, L^{[s]}], & s &\geq 1, \end{aligned}$$

are said to be *the lower central* and *the derived series of L*, respectively.

Definition 1.1.6. *A Leibniz algebra L is called nilpotent (respectively, solvable) if there exists $m \in \mathbb{N}$ ($t \in \mathbb{N}$) such that $L^m = 0$ (respectively, $L^{[t]} = 0$). The minimal number m (respectively, t) with such property is said to be the index of nilpotency (respectively, solvability) of the algebra L.*

Obviously, the index of nilpotency of an n-dimensional Leibniz algebra is not greater than $n + 1$.

Since the sum of two nilpotent ideals is a nilpotent ideal, we can consider the maximal nilpotent ideal of a Leibniz algebra.

Definition 1.1.7. *The (unique) maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.*

Definition 1.1.8. *An n-dimensional Leibniz algebra L is said to be filiform if* $\dim L^i = n - i$ *for* $2 \leq i \leq n$.

Let R be a solvable Leibniz algebra. Then it can be decomposed in the form $R = N \oplus Q$, where N is the nilradical and Q is the complementary vector space. Since the square of a solvable Leibniz algebra is contained into the nilradical [5], we get the nilpotency of the ideal R^2 and, consequently, $Q^2 \subseteq N$.

Theorem 1.1.9 ([19]). *Let R be a solvable Leibniz algebra and N its nilradical. Then the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N.*

A nilpotent Leibniz algebra is called *characteristically nilpotent* if all its derivations are nilpotent. If the nilradical N of a Leibniz algebra is characteristically nilpotent then, according to Theorem 1.1.9, a solvable Leibniz algebra is nilpotent. Therefore, we shall consider solvable Leibniz algebras with non-characteristically nilpotent filiform nilradical.

Theorem 1.1.10 ([37]). *Let $\mathfrak{g}$ be an $(n+1)$-dimensional non-characteristically nilpotent filiform Lie algebra. Then, it is isomorphic to one of the following non-isomorphic algebras:*

$$L_n : \left\{ \ [e_0, e_i] = e_{i+1}, \qquad 1 \leq i \leq n-1; \right.$$

$$Q_n,\ n\text{-}odd:$$

$$\left\{ \begin{array}{ll} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 1 \leq i \leq n-1; \end{array} \right.$$

$$A^r_{n+1}(\alpha_1, \ldots, \alpha_t), \quad 1 \leq r \leq n-3,\ t = \lfloor \tfrac{n-r-1}{2} \rfloor :$$

$$\left\{ \begin{array}{ll} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_j] = \left(\sum\limits_{k=i}^{t} (-1)^{k-i} C^{k-i}_{j-k-1} \alpha_k \right) e_{i+j+r}, & 1 \leq i < j \leq n-2, \\ & i + j + r \leq n; \end{array} \right.$$

$B_{n+1}^r(\alpha_1,\ldots,\alpha_t),\ 1\le r\le n-4,\ t=\lfloor\frac{n-r-2}{2}\rfloor,\ n\text{-}odd,\ n=2m+1:$

$$\begin{cases}
[e_0,e_i]=e_{i+1}, & 1\le i\le n-2,\\
[e_i,e_{n-i}]=(-1)^i e_n, & 1\le i\le m,\\
[e_i,e_j]=\left(\sum\limits_{k=i}^{t}(-1)^{k-i}C_{j-k-1}^{k-i}\alpha_k\right)e_{i+j+r}, & 1\le i,j\le n-1,\\
 & i+j+r\le n-1,
\end{cases}$$

where the parameters $(\alpha_1,\ldots,\alpha_t)$ satisfy the polynomial relations arising from the Jacobi identity and at least one parameter $\alpha_i\neq 0$.

In the following theorem all $(n+1)$-dimensional filiform Leibniz algebras decompose into three families of algebras.

Theorem 1.1.11 ([59]). *Any complex $(n+1)$-dimensional filiform Leibniz algebra admits a basis $\{e_0,e_1,\ldots,e_n\}$ such that the multiplication table of the algebra has one of the following forms:*

$F_1(\alpha_3,\alpha_4,\ldots,\alpha_n,\theta):$

$$\begin{cases}
[e_0,e_0]=e_2,\\
[e_0,e_1]=\sum\limits_{k=3}^{n-1}\alpha_k e_k+\theta e_n,\\
[e_i,e_0]=e_{i+1}, & 1\le i\le n-1,\\
[e_i,e_1]=\sum\limits_{k=i+2}^{n}\alpha_{k+1-i}e_k, & 1\le i\le n-2;
\end{cases}$$

$F_2(\beta_3,\beta_4,\ldots,\beta_n,\gamma):$

$$\begin{cases}
[e_0,e_0]=e_2,\\
[e_0,e_1]=\sum\limits_{k=3}^{n}\beta_k e_k,\\
[e_i,e_0]=e_{i+1}, & 2\le i\le n-1,\\
[e_1,e_1]=\gamma e_n,\\
[e_i,e_1]=\sum\limits_{k=i+2}^{n}\beta_{k+1-i}e_k, & 2\le i\le n-2;
\end{cases}$$

$F_3(\theta_1, \theta_2, \theta_3)$:

$$\begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = \theta_1 e_n, & \\ [e_0, e_1] = -e_2 + \theta_2 e_n, & \\ [e_1, e_1] = \theta_3 e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in \langle e_{i+j+1}, e_{i+j+2}, \dots, e_n \rangle, & \begin{cases} 1 \le i \le n-2, \\ 2 \le j \le n-i, \ i < j, \end{cases} \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i \alpha e_n, & 1 \le i \le n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for odd n and $\alpha = 0$ for even n. Moreover, the structure constants of an algebra from the family $F_3(\theta_1, \theta_2, \theta_3)$ should satisfy the Leibniz identity.

It is easy to see that algebras of the first and the second families are non-Lie algebras. Moreover, an algebra of the third family is a Lie algebra if and only if $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$.

From the list of Theorem 1.1.11 we only indicate the non-characteristically nilpotent filiform non-Lie Leibniz algebras.

Theorem 1.1.12 ([39]). *An arbitrary non-characteristically nilpotent filiform non-Lie Leibniz algebra is isomorphic to one of the following non-isomorphic algebras:*

- $F_1(0, 0, \dots, 0, 1)$ *and* $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$, $3 \le s \le n$, *where* $\alpha_k = \begin{cases} 0, & k \not\equiv s \mod (s-2), \\ (-1)^t Q_{t+1}^{s-1}, & k \equiv s \mod (s-2) \end{cases}$ *and* $t = \frac{k-s}{s-2}$, $3 \le k \le n$ *and* Q_n^p *is the p-th Catalan number;*
- ✓ *for odd n:*

$$F_2(0, 0, \dots, 0, 0, 1) \quad \text{and} \quad F_2^j(0, 0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0, 0, 0)$$

with $n \ge 4$ and $3 \le j \le n$;

✓ *for even n:*

$$F_2^1(0,0,\dots,0,\underbrace{\beta_{\frac{n+2}{2}}}_{\frac{n-2}{2}},0,\dots,0,0,1) \qquad \text{and}$$

$$F_2^j(0,0,\dots,0,\underbrace{1}_{j},0,\dots,0,0,0)$$

with $n \geq 4$ and $3 \leq j \leq n$,

- $F_3(1,0,0)$, $\quad F_3(0,1,0)$, $\quad F_3(0,0,1)$.

1.2 The classification of solvable Leibniz algebras with filiform non-Lie Leibniz nilradicals

In this section we describe solvable Leibniz algebras whose nilradical is one from the list of Theorem 1.1.12. In order to demonstrate the considering cases for the three families of Theorem 1.1.11, we divide this section into three subsections.

1.2.1 Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_1(\alpha_3,\alpha_4,\dots,\alpha_n,\theta)$

In the following proposition we present the matrix form of any derivation of an algebra of the family $F_1(\alpha_3,\alpha_4,\dots,\alpha_n,\theta)$.

Proposition 1.2.1 ([39]). *Any derivation of a filiform Leibniz algebra from the family $F_1(\alpha_3,\alpha_4,\dots,\alpha_n,\theta)$ has the following matrix form:*

$$\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\
0 & a_0+a_1 & a_2 & a_3 & \dots & b_{n-1} & b_n \\
0 & 0 & 2a_0+a_1 & a_2+a_1\alpha_3 & \dots & a_{n-2}+a_1\alpha_{n-1} & a_{n-1}+a_1\alpha_n \\
0 & 0 & 0 & 3a_0+a_1 & \dots & a_{n-3}+2a_1\alpha_{n-1} & a_{n-2}+2a_1\alpha_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & a_2+(n-3)a_1\alpha_3 & a_3+(n-3)a_1\alpha_4 \\
0 & 0 & 0 & 0 & \dots & (n-1)a_0+a_1 & a_2+(n-2)a_1\alpha_3 \\
0 & 0 & 0 & 0 & \dots & 0 & na_0+a_1
\end{pmatrix},$$

where

$$a_0(\theta - \alpha_n) = 0, \quad a_1(\alpha_n - \theta) = a_{n-1} - b_{n-1}, \quad \alpha_3(a_1 - a_0) = 0,$$

$$\alpha_k(a_1 - (k-2)a_0) = \frac{k}{2} a_1 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, \quad 4 \le k \le n.$$

Case $F_1(0, 0, \ldots, 0, 1)$**.**

From Proposition 1.2.1 we conclude that the number of nil-independent outer derivations of the algebra $F_1(0, 0, \ldots, 0, 1)$ is equal to one. Thus, if there is any solvable Leibniz algebra with nilradical $F_1(0, 0, \ldots, 0, 1)$, then it should be of dimension $n + 2$.

Proposition 1.2.2. *There is not any* $(n+2)$*-dimensional solvable Leibniz algebra with nilradical* $F_1(0, 0, \ldots, 0, 1)$*.*

Proof. Let L be a solvable Leibniz algebra with nilradical $F_1(0, 0, \ldots, 0, 1)$. We extend the basis $\{e_0, e_1, \ldots, e_n\}$ of the nilradical with a basis element x of Q.

From the multiplication table of $F_1(0, 0, \ldots, 0, 1)$ we conclude that $\langle e_2, e_3, \ldots, e_n \rangle \subseteq \mathrm{Ann}_r(L)$. Using Proposition 1.2.1, we derive the following products in the algebra L:

$$\begin{array}{ll}
[e_0, e_0] = e_2, & \\
[e_i, e_0] = e_{i+1}, \ 1 \le i \le n-1, & \\
[e_0, e_1] = e_n, & \\
[e_0, x] = \sum\limits_{i=1}^{n} a_i e_i, & [x, e_0] = \sum\limits_{i=0}^{n} \beta_i e_i, \\
[e_1, x] = \sum\limits_{i=1}^{n-2} a_i e_i + (a_{n-1} + a_1)e_{n-1} + b_n e_n, & [x, e_1] = \sum\limits_{i=0}^{n} \gamma_i e_i, \\
[e_i, x] = \sum\limits_{k=i}^{n} a_{k-i+1} e_k, \ 2 \le i \le n, & [x, x] = \sum\limits_{i=0}^{n} \delta_i e_i.
\end{array}$$

Taking the change as follows:

$$x' = x - \sum_{i=2}^{n-1} \beta_{i+1} e_i,$$

we can assume $[x, e_0] = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$.

The equalities

$$0 = [e_0, [e_0, x] + [x, e_0]] = [e_0, [x, x]] = [e_0, [e_1, x] + [x, e_1]]$$

imply

$$\beta_0 = 0, \quad \beta_1 = -a_1, \quad \delta_0 = \delta_1 = 0, \quad \gamma_0 = 0, \quad \gamma_1 = -a_1.$$

Considering

$$0 = [x, e_2] = -a_1 e_2 + \sum_{i=3}^{n} \gamma_i e_i,$$

we obtain $a_1 = 0$. Consequently, the restriction of the operator R_x to the nilradical $F_1(0, 0, \dots, 0, 1)$ is a nilpotent derivation. Therefore, we get a contradiction with the existence of an $(n+2)$-dimensional solvable Leibniz algebra. □

Case $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$**.**

Let us fix the first nonzero parameter $\alpha_s \neq 0$ of the algebra $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$. Then, from the relations of Proposition 1.2.1, we deduce $a_1 = (s-2)a_0$ and $b_{n-1} = a_{n-1}$. Therefore, the number of nil-independent outer derivations of the nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$ is equal to one.

Proposition 1.2.3. *There is not any $(n+2)$-dimensional solvable Leibniz algebra with nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$.*

Proof. Let L be a solvable Leibniz algebra with nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$. Since in the general form of a non-nilpotent derivation of the nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$ the parameter $a_0 \neq 0$ (otherwise, due to equality $a_1 = (s-2)a_0$ any derivation is nilpotent), without loss of generality, one can assume $a_0 = 1$.

Since for a basis element of the space Q the general form of the derivation R_x is presented in Proposition 1.2.1, we have the following multiplications:

$$
\begin{array}{ll}
[e_0,e_0]=e_2, & \\
[e_i,e_0]=e_{i+1}, & 1\le i\le n-1,\\
[e_0,e_1]=\sum\limits_{k=3}^{n}\alpha_k e_k, & \\
[e_i,e_1]=\sum\limits_{k=i+2}^{n}\alpha_{k+1-i}e_k, & 1\le i\le n-2,\\
[e_0,x]=e_0+(s-2)e_1+\sum\limits_{i=2}^{n}a_ie_i, & \\
[x,e_0]=\sum\limits_{i=0}^{n}\beta_ie_i, & \\
[e_1,x]=(s-1)e_1+\sum\limits_{i=2}^{n-1}a_ie_i+b_ne_n, & \\
[x,e_1]=\sum\limits_{i=0}^{n}\gamma_ie_i, & \\
[e_i,x]=(s-2+i)e_i+\sum\limits_{j=i+1}^{n}(a_{j+1-i}+(i-1)(s-2)\alpha_{j-i+2})e_j, & 2\le i\le n,\\
[x,x]=\sum\limits_{i=0}^{n}\delta_ie_i. &
\end{array}
$$

Obviously, $\langle e_2,e_3,\dots,e_n\rangle\subseteq \mathrm{Ann}_r(L)$, and consequently, $[x,e_i]=0$ with $2\le i\le n$.

The equality $[e_0,[x,x]]=0$ implies $\delta_0=\delta_1=0$.

Making the following change:

$$x'=x-\sum_{i=2}^{n-1}\beta_{i+1}e_i,$$

we obtain

$$
\begin{aligned}
&[e_0,x']=[e_0,x]=e_0+(s-2)e_1+\sum_{i=2}^{n}a_ie_i,\\
&[x',e_0]=\beta_0e_0+\beta_1e_1+\beta_2e_2,\\
&[e_1,x']=[e_1,x]=(s-1)e_1+\sum_{i=2}^{n-1}a_ie_i+b_ne_n,\\
&[x',e_1]=\sum_{i=0}^{n}\gamma'_ie_i,\\
&[x',x']=\sum_{i=0}^{n}\delta'_ie_i.
\end{aligned}
$$

From the equalities

$$[e_0,[e_0,x]+[x,e_0]]=[e_0,[e_1,x]+[x,e_1]]=0$$

we conclude

$$\beta_0=-1,\quad \beta_1=-s+2,\quad \gamma_0=0,\quad \gamma_1=-(s-1).$$

A contradiction obtained from the equality $0=[x,e_2]=[x,[e_1,e_0]]=-(s-1)e_2+\sum\limits_{k=3}^{n}\Delta_k e_k$ with $s\geq 3$, completes the proof of the proposition. □

1.2.2 Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_2(\beta_3,\beta_4,\dots,\beta_n,\gamma)$

In this subsection we consider the family of algebras $F_2(\beta_3,\beta_4,\dots,\beta_n,\gamma)$. Similar to the above subsection, firstly we describe the derivations of such algebras.

Proposition 1.2.4 ([39]). *Any derivation of a filiform Leibniz algebra of the family $F_2(\beta_3,\beta_4,\dots,\beta_n,\gamma)$ has the following matrix form:*

$$\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n\\
0 & b_1 & 0 & 0 & \dots & -a_1\gamma & b_n\\
0 & 0 & 2a_0 & a_2+a_1\beta_3 & \dots & a_{n-2}+a_1\beta_{n-1} & a_{n-1}+a_1\beta_n\\
0 & 0 & 0 & 3a_0 & \dots & a_{n-3}+2a_1\beta_{n-2} & a_{n-2}+2a_1\beta_{n-1}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & a_2+(n-3)a_1\beta_3 & a_3+(n-3)a_1\beta_4\\
0 & 0 & 0 & 0 & \dots & (n-1)a_0 & a_2+(n-2)a_1\beta_3\\
0 & 0 & 0 & 0 & \dots & 0 & na_0
\end{pmatrix},$$

where

$$\begin{aligned}
&\gamma(2b_1-na_0)=0, && \beta_3(b_1-2a_0)=0,\\
&\beta_k(b_1-(k-1)a_0) &&=\tfrac{k}{2}a_1\sum_{j=4}^{k}\beta_{j-1}\beta_{k-j+3}, \qquad 4\leq k\leq n-1,\\
&\beta_n(b_1-(n-1)a_0) &&=-a_1\gamma+\tfrac{n}{2}a_1\sum_{j=4}^{n}\beta_{j-1}\beta_{n-j+3}.
\end{aligned}$$

Case $F_2(0,0,\dots,0,0,1)$ and n-odd.

From the relations of Proposition 1.2.4, it follows $b_1 = \frac{n}{2}a_0$ and $a_1 = 0$. Therefore, the number of nil-independent outer derivations of the algebra $F_2(0,0,\dots,0,0,1)$ is equal to one. According to Theorem 1.1.9, we conclude that any solvable Leibniz algebra whose nilradical is $F_2(0,0,\dots,0,0,1)$ has dimension $n+2$.

Theorem 1.2.5. *Any $(n+2)$-dimensional (the case of odd n) solvable Leibniz algebra with nilradical $F_2(0,0,\dots,0,0,1)$ is isomorphic to the following algebra:*

$$L_1 : \left\{ \begin{array}{lll} [e_0,e_0] = e_2, & & [x,e_1] = -\frac{n}{2}e_1, \\ [e_i,e_0] = e_{i+1}, & 2 \le i \le n-1, & [e_0,x] = e_0, \\ [x,e_0] = -e_0, & & [e_1,x] = \frac{n}{2}e_1, \\ [e_1,e_1] = e_n, & & [e_i,x] = ie_i, \qquad 2 \le i \le n. \end{array} \right.$$

Proof. Using Proposition 1.2.4, we have the products:

$$\begin{array}{lll} [e_0,e_0] = e_2, & & [e_0,x] = a_0e_0 + \sum\limits_{i=2}^{n} a_ie_i, \\ [e_i,e_0] = e_{i+1}, & 2 \le i \le n-1, & [e_1,x] = \frac{n}{2}a_0e_1 + b_ne_n, \\ [x,e_0] = \sum\limits_{i=0}^{n} \mu_ie_i, & & [e_i,x] = ie_i + \sum\limits_{j=i+1}^{n} a_{j+1-i}e_j, \quad 2 \le i \le n, \\ [e_1,e_1] = e_n, & & [x,x] = \sum\limits_{i=0}^{n} \delta_ie_i. \\ [x,e_1] = \sum\limits_{i=0}^{n} \gamma_ie_i, & & \end{array}$$

Without loss of generality one can assume $a_0 = 1$. It is easy to see that $\langle e_2, e_3, \dots, e_n \rangle \subseteq \mathrm{Ann}_r(L)$. Hence, $[x,e_i] = 0$ for $2 \le i \le n$.

Let us take the following transformation of basis:

$$e_0' = e_0 + \sum_{i=2}^{n} A_ie_i, \quad e_1' = e_1, \quad e_i' = e_i + \sum_{k=i+1}^{n} A_{k-i+1}e_k, \ 2 \le i \le n, \ x' = x$$

with $A_2 = -a_2$, $A_i = \frac{1}{1-i}\left(a_i + \sum\limits_{j=2}^{i-1} A_ja_{i-j+1}\right)$, $3 \le i \le n$. Then we obtain

$$\begin{aligned}
[x', e_0'] &= [x, e_0] = \sum_{i=0}^{n} \mu_i e_i,\\
[e_0', x'] &= e_0 + \sum_{i=2}^{n} a_i e_i + \sum_{i=2}^{n} A_i \left(i e_i + \sum_{j=i+1}^{n} a_{j+1-i} e_j \right) =\\
&= e_0 + \sum_{i=2}^{n} a_i e_i + A_2(2e_2 + \sum_{j=3}^{n} a_{j-1} e_j) + A_3(3e_3 + \sum_{j=4}^{n} a_{j-2} e_j) + \cdots\\
&+ A_{n-2}((n-2)e_{n-2} + \sum_{j=n-1}^{n} a_{j-n+3} e_j) + A_{n-1}((n-1)e_{n-1} + a_2 e_n)\\
&+ A_n(n e_n) = e_0 + (a_2 + 2A_2)e_2 + \sum_{i=3}^{n} (iA_i + a_i + \sum_{j=2}^{i-1} A_j a_{i-j+1}) e_i\\
&= e_0 + \sum_{i=2}^{n} A_i e_i = e_0'.
\end{aligned}$$

Thus, we can assume $a_i = 0$ for $2 \le i \le n$.

Now, making the change $x' = x - \sum_{i=2}^{n-1} \mu_{i+1} e_i$, we obtain the family:

$$\begin{array}{ll}
[e_0, e_0] = e_2, & [e_0, x] = e_0,\\
[e_i, e_0] = e_{i+1},\ 2 \le i \le n-1, & [e_1, x] = \frac{n}{2} e_1 + b_n e_n,\\
[x, e_0] = \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2, & [e_i, x] = i e_i,\ 2 \le i \le n,\\
[e_1, e_1] = e_n, & [x, x] = \sum\limits_{i=0}^{n} \delta_i e_i.\\
[x, e_1] = \sum\limits_{i=0}^{n} \gamma_i e_i, &
\end{array}$$

By setting $e_1' = e_1 - \frac{2}{n} b_n e_n$ we get $b_n' = 0$.

The equalities

$$0 = [e_0, [x, x]] = [e_1, [x, x]] = [e_0, [e_0, x] + [x, e_0]]$$
$$= [e_1, [e_0, x] + [x, e_0]] = [e_1, [e_1, x] + [x, e_1]]$$

derive

$$\delta_0 = \delta_1 = \mu_1 = \gamma_0 = 0, \quad \mu_0 = -1, \quad \gamma_1 = -\frac{n}{2}.$$

Applying the Leibniz identity to the triples $\{x, e_0, e_1\}$, $\{x, x, e_0\}$ and $\{x, x, e_1\}$, we conclude

$$\gamma_i = 0, \ \ 2 \le i \le n, \quad \delta_i = 0, \ \ 2 \le i \le n-1, \quad \mu_2 = 0.$$

Finally, putting $x' = x - \frac{\delta_n}{n} e_n$, we obtain the algebra L_1. $\square$

Case $F_2^1(\underbrace{0,0,\dots,0}_{\frac{n-2}{2}},\beta_{\frac{n+2}{2}},0,\dots,0,0,1)$ **and** n**-even.**

Similarly, we get $a_0 = 1,\ b_1 = \frac{n}{2},\ a_1 = 0$ and that any solvable Leibniz algebra whose nilradical is $F_2^1(0,\dots,0,\beta_{\frac{n+2}{2}},0,\dots,0,1)$ has dimension $(n+2)$.

Theorem 1.2.6. *Any $(n+2)$-dimensional (the case of even n) solvable Leibniz algebra with nilradical $F_2^1(\underbrace{0,0,\dots,0}_{\frac{n-2}{2}},\beta_{\frac{n+2}{2}},0,\dots,0,0,1)$ is isomorphic to an algebra of the following family of algebras:*

$$L_2^{\beta_{\frac{n+2}{2}}}: \left\{ \begin{array}{lll} [e_0,e_0] = e_2, & & [e_0,e_1] = \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}}, \\ [e_i,e_0] = e_{i+1}, & 2 \le i \le n-1, & [e_1,e_1] = e_n, \\ [x,e_0] = -e_0, & & [e_i,e_1] = \beta_{\frac{n+2}{2}} e_{\frac{n+2i}{2}},\ 2 \le i \le \frac{n}{2}, \\ [e_0,x] = e_0, & & [x,e_1] = -\frac{n}{2}e_1 - \beta_{\frac{n+2}{2}} e_{\frac{n}{2}}, \\ [e_1,x] = \frac{n}{2}e_1, & & \\ [e_i,x] = ie_i, & 2 \le i \le n. & \end{array} \right.$$

Proof. Using the previous arguments and Proposition 1.2.4, we obtain the multiplications:

$$\begin{array}{ll} [e_0,e_0] = e_2, & \\ [e_0,e_1] = \beta_{\frac{n+2}{2}}\ e_{\frac{n+2}{2}}, & \\ [e_i,e_0] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1,e_1] = e_n, & \\ [x,e_0] = \displaystyle\sum_{i=0}^{n} \mu_i e_i, & \\ [e_i,e_1] = \beta_{\frac{n+2}{2}}\ e_{\frac{n+2i}{2}}, & 2 \le i \le \dfrac{n}{2}, \end{array}$$

$$
\begin{aligned}
&[x, e_1] = \sum_{i=0}^{n} \gamma_i e_i, \\
&[e_0, x] = e_0 + \sum_{i=2}^{n} a_i e_i, \\
&[e_1, x] = \frac{n}{2} e_1 + b_n e_n, \\
&[e_i, x] = i e_i + \sum_{j=i+1}^{n} a_{j+1-i} e_j, \qquad 2 \leq i \leq n, \\
&[x, x] = \sum_{i=0}^{n} \delta_i e_i.
\end{aligned}
$$

Taking the following transformation of basis:

$$e_0' = e_0 + \sum_{i=2}^{n} A_i e_i, \quad e_1' = e_1, \quad e_i' = e_i + \sum_{k=i+1}^{n} A_{k-i+1} e_k, \; 2 \leq i \leq n, \quad x' = x$$

with

$$A_2 = -a_2, \quad A_i = \frac{1}{1-i} \left(a_i + \sum_{j=2}^{i-1} A_j a_{i-j+1} \right), \; 3 \leq i \leq n,$$

we can assume that $a_i = 0$ for $2 \leq i \leq n$.

By setting

$$x' = x - \sum_{i=2}^{n-1} \mu_{i+1} e_i, \quad e_1' = e_1 - \frac{2}{n} b_n e_n$$

we reduce the above multiplication to the following one:

$$\begin{array}{ll}
[e_0,e_0]=e_2, & [x,e_0]=\mu_0e_0+\mu_1e_1+\mu_2e_2,\\
[e_i,e_0]=e_{i+1},\ 2\le i\le n-1, & [x,e_1]=\sum\limits_{i=0}^{n}\gamma_ie_i,\\
[e_0,e_1]=\beta_{\frac{n+2}{2}}e_{\frac{n+2}{2}}, & [e_0,x]=e_0,\\
[e_1,e_1]=e_n, & [e_1,x]=\frac{n}{2}e_1,\\
[e_i,e_1]=\beta_{\frac{n+2}{2}}e_{\frac{n+2i}{2}},\ 2\le i\le \frac{n}{2}, & [e_i,x]=ie_i,\ 2\le i\le n,\\
 & [x,x]=\sum\limits_{i=0}^{n}\delta_ie_i.
\end{array}$$

From the equalities

$$[e_1,[x,x]]=[e_0,[x,x]]=[e_1,[e_0,x]+[x,e_0]]=[e_0,[e_0,x]+[x,e_0]]$$
$$=[e_1,[e_1,x]+[x,e_1]]=[e_0,[e_1,x]+[x,e_1]]=0$$

we derive

$$\delta_1=\delta_0=\mu_1=\gamma_0=0,\quad \mu_0=-1,\quad \gamma_1=-\frac{n}{2}.$$

Applying the Leibniz identity to the triples $\{x,e_0,x\}$, $\{x,e_0,e_1\}$ and $\{x,x,e_1\}$, we obtain

$$\mu_2=0,\ \delta_i=0,\ \gamma_j=0,\ \gamma_{\frac{n}{2}}=-\beta_{\frac{n+2}{2}},\ 2\le i,j\le n-1,\ j\ne\frac{n}{2}.$$

The following change: $x'=x-\frac{\delta_n}{n}e_n$ implies $\delta'_n=0$, and so we get the family of algebras $L_2^{\beta_{\frac{n+2}{2}}}$. $\square$

Case $F_2^j(0,0,\dots,0,\underbrace{1}_{j},0,\dots,0,0,0)$**,** $n\ge 4$ **and** $3\le j\le n$**.**

Similar to the above cases and using Proposition 1.2.4, we derive

$$b_1=(j-1)a_0,\quad \begin{cases} a_1=0, & \text{if } 3\le j\le \lfloor\frac{n+2}{2}\rfloor,\\ a_1, & \text{in other case.}\end{cases}$$

Similar to the previous cases, we can suppose $a_0=1$ and $b_1=j-1$. Therefore, the number of nil-independent outer derivations of the algebras

$F_2^j(0, 0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0, 0, 0)$ is equal to one. We shall use F_2^j to denote $F_2^j(0, 0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0, 0, 0)$.

Theorem 1.2.7. *Any $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_2^j, with $3 \leq j \leq n$, is isomorphic to the following algebra:*

$$L_3^j : \begin{cases} [e_0, e_0] = e_2, & [e_0, e_1] = e_1, \\ [e_i, e_0] = e_{i+1}, \ 2 \leq i \leq n-1, & [e_i, e_1] = e_{j+i-1}, \ 2 \leq i \leq n-1-j, \\ [x, e_0] = -e_0, & [x, e_1] = -(j-1)e_1 - e_{j-1}, \\ [e_0, x] = e_0, & [e_i, x] = ie_i, \ 2 \leq i \leq n. \\ [e_1, x] = (j-1)e_1, & \end{cases}$$

Proof. The proof is carried out by applying arguments used in Theorems 1.2.5 and 1.2.6. □

1.2.3 Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_3(\theta_1, \theta_2, \theta_3)$

Let L be a filiform Leibniz algebra from the family $F_3(\theta_1, \theta_2, \theta_3)$ and $\{e_0, e_1, \dots, e_n\}$ its basis. The following proposition describes the derivations of such algebras.

Proposition 1.2.8 ([39]). *A derivation d of a filiform Leibniz algebra of the family $F_3(\theta_1, \theta_2, \theta_3)$ has the following form:*

$$\begin{aligned} & d(e_0) = \sum_{i=0}^{n} a_i e_i, \quad d(e_1) = \sum_{i=1}^{n} b_i e_i, \\ & d(e_i) = ((i-1)a_0 + b_1)e_i + \sum_{j=i+1}^{n-1} b_{j-i+1} e_j + (b_{n-i+1} + (-1)^{i-1}\alpha a_{n-i+1})e_n, \\ & d(e_n) = ((n-1)a_0 + b_1 + \alpha a_1)e_n \end{aligned}$$

with the following restrictions:

$$\begin{aligned}
&\theta_1((n-3)a_0+b_1)=a_1\theta_2,\\
&2a_1\theta_3=(n-2)a_0\theta_2,\\
&\theta_3((n-1)a_0-b_1)=0.
\end{aligned}$$

Now we consider the case of solvable Leibniz algebras with non-Lie filiform nilradical of the family $F_3(\theta_1,\theta_2,\theta_3)$.

Theorem 1.2.9. *There is not any solvable Leibniz algebra whose nilradical is a non-characteristically filiform non-Lie algebra of the family $F_3(\theta_1,\theta_2,\theta_3)$.*

Proof. Let us consider firstly the **Case** $F_3^1(1,0,0)$**.** Proposition 1.2.8 leads to $b_1=(3-n)a_0$, and we can suppose $a_0=1$, making the change $x'=\frac{1}{a_0}x$. Thus, we have the following products:

$$\begin{array}{ll}
[e_i,e_0]=-[e_0,e_i]=e_{i+1}, & 1\leq i\leq n-1,\\
[e_0,e_0]=e_n, & \\
[e_i,e_{n-i}]=-[e_{n-i},e_i]=(-1)^i\alpha e_n, & 1\leq i\leq n-1,\\
[e_0,x]=e_0+\sum\limits_{i=1}^{n}a_ie_i, & \\
[e_1,x]=(3-n)e_1+\sum\limits_{i=2}^{n}b_ie_i, & \\
[e_i,x]=((i-1)+b_1)e_i+ & \\
\qquad +\sum\limits_{j=i+1}^{n-1}b_{j-i+1}e_j+(b_{n-i+1}+(-1)^{i-1}\alpha a_{n-i+1})e_n, & 2\leq i\leq n-1,\\
[e_n,x]=((n-1)+b_1+\alpha a_1)e_n, & \\
[x,e_0]=\sum\limits_{i=0}^{n}\beta_ie_i, & \\
[x,e_1]=\sum\limits_{i=0}^{n}\gamma_ie_i, & \\
[x,x]=\sum\limits_{i=0}^{n}\delta_ie_i. &
\end{array}$$

It is easy to check that by the change $x'=x+\sum\limits_{i=1}^{n-1}a_{i+1}e_i$ we can suppose $a_i=0$ with $2\leq i\leq n$.

Applying the Leibniz identity to the elements $\{e_2,e_{n-2},x\}$ and $\{e_0,e_0,x\}$, we get a contradiction with $\alpha=1$. Thus, $\alpha=0$.

Taking the following transformation of basis:

$$e_0' = e_0 + \frac{a_1}{n-2}e_1, \quad e_i' = e_i, \ 1 \le i \le n, \quad x' = x + \frac{a_1}{n-2}\sum_{i=1}^{n-1} b_{i+1}e_i,$$

we can suppose $a_1 = 0$.

Putting

$$e_0' = e_0, \quad e_i' = e_i + \sum_{j=i+1}^{n} A_{j-i+1}e_j, \ 1 \le i \le n,$$

with

$$A_2 = -b_2, \ A_i = \frac{1}{1-i}(b_i + \sum_{j=2}^{i-1} A_j b_{i-j+1}), \ 3 \le i \le n,$$

one can assume $b_i = 0$ for $2 \le i \le n$.

Let us resume the products of the family

$$\begin{array}{lll}
[e_0, e_0] = e_n, & & \\
[e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \le i \le n-1, & [x, e_0] = \sum_{i=0}^{n} \beta_i e_i, \\
[e_0, x] = e_0, & & [x, e_1] = \sum_{i=0}^{n} \gamma_i e_i, \\
[e_i, x] = (i-n+2)e_i, & 1 \le i \le n, & [x, x] = \sum_{i=0}^{n} \delta_i e_i.
\end{array}$$

Considering the Leibniz identity for the elements of the form

$$\{e_0, x, e_0\}, \quad \{e_0, x, e_1\}, \quad \{e_1, x, e_1\}, \quad \{e_1, x, e_0\},$$

we obtain that

$$\beta_0 = -1, \ \beta_i = 0, \ 1 \le i \le n-2, \qquad \gamma_0 = 0, \ \gamma_1 = n-3, \ \gamma_i = 0, \ 2 \le i \le n-1.$$

Using the induction method, we get $[x, e_i] = -(i-n+2)e_i$, with $2 \le i \le n$.

From the equality $0 = [x, [e_0, e_0]]$, we have $[x, e_n] = 0$, but this is a contradiction, because $[x, e_n] = -2e_n$. Therefore, there is not any solvable Leibniz algebra with nilradical $F_3^1(1, 0, 0)$.

Similar study of **Case** $F_3^2(0, 1, 0)$ and **Case** $F_3^2(0, 0, 1)$ leads to the non-existence of solvable Leibniz algebras with nilradicals $F_3^2(0, 1, 0)$ and $F_3^2(0, 0, 1)$. $\square$

1.3 The description of solvable Leibniz algebras with filiform Lie nilradical

In this section we study solvable Leibniz algebras whose nilradical is a filiform Lie algebra. Since we consider non-nilpotent solvable Leibniz algebras, it is sufficient to consider non-characteristically nilpotent filiform Lie nilradicals. In this section we restrict ourselves to the study of the families $A^r_{n+1}(\alpha_1,\dots,\alpha_t)$, $B^r_{n+1}(\alpha_1,\dots,\alpha_t)$ of Theorem 1.1.10, because the other two algebras of Theorem 1.1.10 have been already studied in [20].

Case $A^r_{n+1}(\alpha_1,\dots,\alpha_t)$ with $1\leq r\leq n-3$ and $t=\lfloor\frac{n-r-1}{2}\rfloor$.

Below we present some description of the derivations of the family of algebras $A^r_{n+1}(\alpha_1,\dots,\alpha_t)$.

Proposition 1.3.1. *Any derivation of a filiform Lie algebra of the family $A^r_{n+1}(\alpha_1,\dots,\alpha_t)$, with $1\leq r\leq n-3$ and $t=\lfloor\frac{n-r-1}{2}\rfloor$, has the following form:*

$$\begin{aligned}
d(e_0)&=\sum_{i=0}^{n}a_ie_i,\\
d(e_1)&=(1+r)a_0e_1+\sum_{i=2}^{n}b_ie_i,\\
d(e_i)&=(i+r)a_0e_i+\sum_{j=i+1}^{n}(*)e_j,\quad 2\leq i\leq n,
\end{aligned}$$

where $\{e_0,e_1,\dots,e_n\}$ is a basis of the family A^r_{n+1} and (*) are appropriate coefficients.

Proof. Let us denote

$$d(e_0)=\sum_{i=0}^{n}a_ie_i,\quad d(e_1)=\sum_{i=0}^{n}b_ie_i.$$

Using the induction method and the properties of a derivation, we establish

$$d(e_i)=((i-1)a_0+b_1)e_i+\sum_{j=i+1}^{n}(*)e_j,\quad 2\leq i\leq n.$$

From the equalities

$$0 = d([e_1, e_{n-1}]) = [d(e_1), e_{n-1}] + [e_1, d(e_{n-1})] = b_0 e_n$$

we derive $b_0 = 0$.

- If $\alpha_1 \neq 0$, then from the equality $d([e_1, e_2]) = d(\alpha_1 e_{r+3})$ we have $b_1 = (r+1)a_0$.
- If $\alpha_1 = 0$, then there exists i, $2 \leq i \leq t$, such that $\alpha_i \neq 0$. The equality $d([e_i, e_{i+1}]) = d(\alpha_i e_{2i+1+r})$ implies $b_1 = (r+1)a_0$.

Thus, we obtain the form of a derivation which is asserted in the proposition. $\square$

From Proposition 1.3.1, we conclude that the number of nil-independent derivations of the algebra $A^r_{n+1}(\alpha_1, \dots, \alpha_t)$ is equal to one. Consequently, according to Theorem 1.1.9, any solvable Leibniz algebra whose nilradical is $A^r_{n+1}(\alpha_1, \dots, \alpha_t)$ has dimension $n+2$.

In this section we use similar arguments to those from the previous section.

Theorem 1.3.2. *Any solvable Leibniz algebra with nilradical $A^r_{n+1}(\alpha_1, \dots, \alpha_t)$ is isomorphic to the following family of Lie algebras:*

$$\begin{aligned}
[e_0, e_i] &= e_{i+1}, \quad 1 \leq i \leq n-1, \\
[e_i, e_j] &= \Big(\sum_{k=i}^{t} (-1)^{k-i} C^{k-i}_{j-k-1} \alpha_k\Big) e_{i+j+r}, \ 1 \leq i < j \leq n-2, \ i+j+r \leq n, \\
[e_0, x] &= e_0 + a_1 e_1, \\
[e_1, x] &= (1+r)e_1 + \sum_{i=2}^{n} b_i e_i, \\
[e_2, x] &= (2+r)e_2 + \sum_{i=3}^{n} b_{i-1} e_i, \\
[e_i, x] &= (i+r)e_i + \sum_{j=i+1}^{i-1+r} b_{j-i+1} e_j \\
&\quad + \left(b_{1+r} + a_1 \left(\sum_{k=2}^{i-1} \left(\sum_{s=1}^{t} (-1)^{s-1} C^{s-1}_{k-s-1} \alpha_s\right)\right)\right) e_{i+r} \\
&\quad + \sum_{j=i+1+r}^{n} b_{j-i+1} e_j \quad \text{with } \ 3 \leq i \leq n-r, \\
[e_i, x] &= (i+r)e_i + \sum_{j=i+1}^{n} b_{j-i+1} e_j, \quad n-r+1 \leq i \leq n.
\end{aligned}$$

Proof. From Proposition 1.3.1 we have the following products:

$$\begin{array}{ll}
[e_0,e_i]=e_{i+1}, & 1\le i\le n-1,\\
[e_i,e_j]=\left(\sum\limits_{k=i}^{t}(-1)^{k-i}C_{j-k-1}^{k-i}\alpha_k\right)e_{i+j+r}, & 1\le i<j\le n-2, i+j+r\le n,\\
[e_0,x]=\sum\limits_{i=0}^{n}a_ie_i, & \\
[e_1,x]=(1+r)a_0e_1+\sum\limits_{i=2}^{n}b_ie_i, & \\
[e_i,x]=(i+r)a_0e_i+\sum\limits_{j=i+1}^{n}(*)e_j, & 2\le i\le n.
\end{array}$$

Since $a_0\neq 0$, then by scaling of basis element x, we can suppose $a_0=1$. The change $x'=x-\sum\limits_{i=1}^{n-1}a_{i+1}e_i$ allow us to suppose $a_i=0$, for $2\le i\le n$.

Thus, we have

$$\begin{array}{ll}
[e_0,e_i]=e_{i+1}, & 1\le i\le n-1,\\
[e_i,e_j]=\left(\sum\limits_{k=i}^{t}(-1)^{k-i}C_{j-k-1}^{k-i}\alpha_k\right)e_{i+j+r}, & 1\le i<j\le n-2, i+j+r\le n,\\
[e_0,x]=e_0+a_1e_1, & \\
[e_1,x]=(1+r)e_1+\sum\limits_{i=2}^{n}b_ie_i, &
\end{array}$$

Using the Jacobi identity and the induction method, we obtain the expressions of the products $[e_i,x]$, with $2\le i\le n$. Let us denote

$$[x,e_0]=\sum_{i=0}^{n}\beta_ie_i,\qquad [x,e_1]=\sum_{i=0}^{n}\gamma_ie_i,\qquad [x,x]=\sum_{i=0}^{n}\delta_ie_i.$$

Applying the Leibniz identity to the elements

$$\{e_0,x,x\},\quad \{e_1,x,x\},\quad \{e_0,x,e_0\},\quad \{e_1,x,e_0\}$$

we derive

$$\delta_i=0,\ 0\le i\le n-1,\quad \beta_i=0,\ 2\le i\le n-1,\quad \beta_1=-a_1,\quad \beta_0=-1.$$

By means of the Leibniz identity for the products and the induction method, we compute the following products:

$$\begin{aligned}
[x, e_2] &= -(1-\gamma_1)e_2 + \sum_{i=3}^{n} \gamma_{i-1} e_i,\\
[x, e_i] &= -(i-1-\gamma_1)e_i + \sum_{j=i+1}^{i-1+r} \gamma_{j-i+1} e_j\\
&\quad + \left(\gamma_{1+r} + a_1 \left(\sum_{k=2}^{i-1} \left(\sum_{s=1}^{t} (-1)^{s-1} C_{k-s-1}^{s-1} \alpha_s\right)\right)\right) e_{i+r}\\
&\quad + \sum_{j=i+1+r}^{n} \gamma_{j-i+1} e_j, \quad 3 \le i \le n-r,\\
[x, e_i] &= -(i-1-\gamma_1)e_i + \sum_{j=i+1}^{n} \gamma_{j-i+1} e_j, \quad n-r+1 \le i \le n.
\end{aligned}$$

Since $[e_1, x] + [x, e_1] \in \mathrm{Ann}_r(L)$, we get

$$[e_{n-1}, [e_1, x] + [x, e_1]] = [e_0, [e_1, x] + [x, e_1]] = [x, [e_1, x] + [x, e_1]] = 0,$$

from which we obtain

$$\gamma_0 = 0, \quad \gamma_1 = -1-r, \quad \gamma_i = -b_i, \qquad 2 \le i \le n.$$

Similarly, from $[x, [e_0, x] + [x, e_0]] = 0$ we conclude $\beta_n = 0$.

Considering the equality $0 = [x, [x, x]]$ we get $\delta_n = 0$. Thus, we obtain a Lie algebra. □

Case $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ with $1 \le r \le n-4$, $t = \lfloor \frac{n-r-2}{2} \rfloor$ and $n = 2m+1$.

The study of this case is similar to previous case.

Proposition 1.3.3. *Any derivation of a filiform Lie algebra of the family $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$, with $1 \le r \le n-4$, $t = \lfloor \frac{n-r-2}{2} \rfloor$ and $n = 2m+1$, has the following form:*

$$\begin{aligned}
d(e_0) &= a_0 e_0 + \sum_{i=2}^{n} a_i e_i,\\
d(e_1) &= (1+r)a_0 e_1 + \sum_{i=2}^{n} b_i e_i,\\
d(e_i) &= (i+r)a_0 e_i + \sum_{j=i+1}^{n} (*) e_j, \quad 2 \le i \le n-1,\\
d(e_n) &= (n+2r)a_0 e_n,
\end{aligned}$$

where $\{e_0, e_1, \ldots, e_n\}$ is a basis of the family B^r_{n+1} and (*) are appropriate coefficients.

Proof. In an analogous way to the proof of Proposition 1.3.1. □

From Proposition 1.3.3 and Theorem 1.1.9, we conclude that any solvable Leibniz algebra whose nilradical is $B^r_{n+1}(\alpha_1, \ldots, \alpha_t)$ has dimension $(n+2)$.

Theorem 1.3.4. *Any solvable Leibniz algebra with nilradical $B^r_{n+1}(\alpha_1, \ldots, \alpha_t)$ is isomorphic to an algebra of the following family of Lie algebras:*

$$
\begin{aligned}
&[e_0, e_i] = e_{i+1}, \ 1 \le i \le n-2,\\
&[e_i, e_{n-i}] = (-1)^i e_n, \ 1 \le i \le n-1,\\
&[e_i, e_j] = \Big(\sum_{k=i}^{t} (-1)^{k-i} C^{k-i}_{j-k-1} \alpha_k\Big) e_{i+j+r}, \ 1 \le i, j \le n-1, i+j+r \le n-1,\\
&[e_0, x] = e_0,\\
&[e_1, x] = (1+r)e_1 + \sum_{i=2}^{n-1} b_i e_i,\\
&[e_i, x] = (i+r)e_i + \sum_{j=i+1}^{n-1} b_{j-i+1} e_j, \ 2 \le i \le n-1,\\
&[e_n, x] = (n+2r)e_n.
\end{aligned}
$$

Proof. The proof is similar to that of Theorem 1.3.2. □

From Theorems 1.3.2 and 1.3.4, we resume that any solvable Leibniz algebra whose nilradical is either $A^r_{n+1}(\alpha_1, \ldots, \alpha_t)$ or $B^r_{n+1}(\alpha_1, \ldots, \alpha_t)$ is a Lie algebra.

For the classification of the received families of algebras it is necessary to consider the general transformation of basis.

Conjecture.

(i) By the following transformation of basis of an algebra of the family from Theorem 1.3.2:

$$
e'_0 = e_0, \quad e'_1 = e_1 + \sum_{i=2}^{n-1} A_i e_i, \quad e'_i = e_i + \sum_{j=i+1}^{n} A_{j-i+1} e_j, \quad 2 \le i \le n, \tag{TB}
$$

with

$$A_2 = -b_2, \quad A_i = \frac{1}{1-i}(b_i + \sum_{j=2}^{i-1} A_j b_{i-j+1}), \quad 3 \leq i \leq n,$$

we can eliminate the parameters $b_i = 0$ with $2 \leq i \leq n$;

(ii) By the transformation of basis (TB), in an algebra of the family from Theorem 1.3.4, with

$$A_2 = -b_2, \qquad A_{2k} = \frac{1}{1-2k}(b_{2k} + \sum_{j=2}^{k} A_{2j-1} b_{2k-2j+2}), \quad 2 \leq k \leq \frac{n-1}{2},$$

$$A_3 = \frac{b_2^2}{2}, \qquad A_{2k+1} = -\frac{1}{2k}(\sum_{j=1}^{k} A_{2j} b_{2k-2j+2}), \qquad 2 \leq k \leq \frac{n-3}{2},$$

we can eliminate the parameters $b_i = 0$ with $2 \leq i \leq n$.

The correctness of the conjecture was checked for fixed low dimensions by program Mathematica. However, we could not generalize the calculation from low dimensions for any finite dimension because of the great number of complex calculations needed.

Chapter 2

Second cohomology group of the solvable Leibniz algebras with F_n^1 nilradical

This chapter is devoted to establish some results from a geometrical viewpoint in the study of variety of Leibniz algebras. In Section 2.1 we remind preliminary results concerning naturally graded filiform Leibniz algebras, cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras. Representatives of irreducible components of the variety $\mathfrak{Leib}_{n+1}$ of the $(n+1)$-dimensional Leibniz algebras and descriptions of the second cohomology group of the solvable Leibniz algebras whose nilradical is a filiform algebra of type F_n^1 are given in Section 2.2. In the case where the dimension of the solvable Leibniz algebra with nilradical F_n^1 is equal to $(n+2)$, it was asserted in [20] that there is no such algebra. In Section 2.2 we find in our ameliorated description that there exits a unique $(n+2)$-dimensional solvable algebra Leibniz with nilradical F_n^1. Moreover, we establish that the second group of cohomology of this Leibniz algebra with coefficients in itself is trivial, and, consequently, it is a rigid algebra.

2.1 Preliminary definitions and results

In this section we give necessary definitions on the natural graduation for a Leibniz algebra, cohomology and known results.

Let us define the notion of the naturally graded algebra.

Let L be a finite-dimensional nilpotent Leibniz algebra. Set

$$\mathrm{gr}(L)_i := L^i/L^{i+1}, \qquad 1 \leq i \leq s-1,$$

where s is the nilindex of the algebra L, and denote

$$\mathrm{gr}\, L = \mathrm{gr}(L)_1 \oplus \mathrm{gr}(L)_2 \oplus \cdots \oplus \mathrm{gr}(L)_{s-1}.$$

Since $[\mathrm{gr}(L)_i, \mathrm{gr}(L)_j] \subseteq \mathrm{gr}(L)_{i+j}$, we obtain the graded algebra $\mathrm{gr}\, L$. The grading constructed above will be called the *natural graduation*.

Definition 2.1.1. *If a Leibniz algebra G is isomorphic to the algebra $\mathrm{gr}\, L$, then G is called a naturally graded Leibniz algebra.*

Due to [6] and [66] it is known that there are three naturally graded filiform Leibniz algebras. In fact, the third type encloses the class of naturally graded filiform Lie algebras.

Theorem 2.1.2. *Any complex n-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_n^1: \ [e_i, e_1] = e_{i+1}, \ 2 \leq i \leq n-1,$$
$$F_n^2: \ [e_i, e_1] = e_{i+1}, \ 1 \leq i \leq n-2,$$
$$F_n^3(\alpha): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1} e_n, & 2 \leq i \leq n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

Due to Chapter 1 we conclude that there is no solvable Leibniz algebra whose nilradical is an algebra from the family $F_1(\alpha_4, \ldots, \alpha_n, \theta)$ except the algebra F_n^1. Below, we present the description of solvable Leibniz algebras whose nilradical is isomorphic to the algebra F_n^1.

Theorem 2.1.3 ([20]). *An arbitrary $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to one of the following pairwise non-isomorphic algebras:*

R_1 :

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1, \\ [x, e_1] = -e_1 - e_2, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-1)e_i, \ 2 \le i \le n, \end{cases}$$

$R_2(\alpha)$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-1+\alpha)e_i, \ 2 \le i \le n, \end{cases}$$

R_3 :

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1, \\ [e_i, x] = (i-n)e_i, \ 2 \le i \le n, \\ [x, x] = e_n, \end{cases}$$

R_4 :

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1, \\ [x, e_1] = -e_1, \\ [e_1, x] = e_1 + e_n, \\ [e_i, x] = (i+1-n)e_i, \ 2 \le i \le n, \\ [x, x] = -e_{n-1}, \end{cases}$$

$$R_5(\alpha_i) = R_5(\alpha_4, \ldots, \alpha_n) : \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, x] = e_2 + \sum\limits_{i=4}^{n-1} \alpha_i \, e_i, & \\ [e_i, x] = e_i + \sum\limits_{j=i+2}^{n} \alpha_{j-i+2} \, e_j, & 2 \le i \le n\,. \end{cases}$$

Moreover, the first non-vanishing parameter $\{\alpha_4, \ldots, \alpha_n\}$ in the algebra $R_5(\alpha_4, \ldots, \alpha_n)$ can be scaled to 1.

Theorem 2.1.4 ([20]). *There does not exist any $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^1.*

Nevertheless, the last theorem is not true. We will find the unique $(n+2)$-dimensional solvable algebra Leibniz with nilradical F_n^1 in Subsection 2.2.2.

Let $GL_n(\mathbb{F})$ be the group of nonsingular $n \times n$ matrices with entries in the field $\mathbb{F}$.

The bilinear maps $V \times V \to V$ form a vector space $\mathrm{Hom}(V \otimes V, V)$ of dimension $\dim(V)^3$, which can be viewed with its natural structure of an affine algebraic variety over the field $\mathbb{F}$. An n-dimensional Leibniz algebra L of the variety $\mathfrak{Leib}_n$ may be considered as an element $\lambda(L)$ via the bilinear mapping $\lambda\colon L \otimes L \to L$ satisfying Leibniz identity.

The group $GL_n(\mathbb{F})$ naturally acts on $\mathfrak{Leib}_n$ via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\Big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\Big), \quad g \in GL_n(\mathbb{F}),\ \lambda \in \mathfrak{Leib}_n.$$

The orbits $\mathrm{Orb}(-)$ under this action are the isomorphism classes of algebras.

Note that solvable Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

Recall that the algebra L is called *rigid* if $\mathrm{Orb}(L)$ is an open set in the Zariski topology.

The concept of cohomology group of Leibniz algebras was introduced by J.-L. Loday and T. Pirashvili [47, 61]. For acquaintance with the definition of cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras (similar to Lie algebras case) we refer the reader to the papers [7, 27, 31, 45, 47, 55].

Now we give the definition of cohomology groups for Leibniz algebras.

A *representation* M of a Leibniz algebra L is introduced in [44].

Definition 2.1.5. *A vector space M is called a representation or bimodule over a Leibniz algebra L if there are two bilinear maps:*

$$[-,-]\colon L \times M \to M \qquad \textit{and} \qquad [-,-]\colon M \times L \to M$$

satisfying the following three axioms

$$[m,[x,y]] = [[m,x],y] - [[m,y],x],$$
$$[x,[m,y]] = [[x,m],y] - [[x,y],m],$$
$$[x,[y,m]] = [[x,y],m] - [[x,m],y],$$

for any $m \in M$, $x, y \in L$.

A *cohomology* of a Leibniz algebra L with coefficients in a representation M is defined in [47] as follows.

Define the space of *n-cochains* $CL^n(L,M) = \mathrm{Hom}_{\mathbb{F}}(L^{\otimes n}, M)$, for $n \geq 0$, and an $\mathbb{F}$-homomorphism $d^n\colon CL^n(L,M) \to CL^{n+1}(L,M)$ by

$$\begin{aligned}(d^n f)(x_1,\dots,x_{n+1}) &:= [x_1, f(x_2,\dots,x_{n+1})]\\ &+ \sum_{i=2}^{n+1}(-1)^i[f(x_1,\dots,\widehat{x}_i,\dots,x_{n+1}),x_i]\\ &+ \sum_{1\leq i<j\leq n+1}(-1)^{j+1} f(x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\widehat{x}_j,\dots,x_{n+1}).\end{aligned}$$

This $(CL^*(L,M),d)$ is a cochain complex. Its n-th cohomology group is well defined by $HL^n(L,M) := ZL^n(L,M)/BL^n(L,M)$, where the elements $ZL^n(L,M) := \ker d^n$ and $BL^n(L,M) := \operatorname{im} d^{n-1}$ are called *n-cocycles* and *n-coboundaries*, respectively.

Here we will just see the second cohomology group of a Leibniz algebra L with coefficients in a corepresentation M is the quotient space

$$HL^2(L,M) = ZL^2(L,M)/BL^2(L,M),$$

where the 2-cocycles $\varphi \in ZL^2(L,M)$ and the 2-coboundaries $f \in BL^2(L,M)$ are defined as follows

$$\begin{aligned}(d^2\varphi)(a,b,c) = [a,\varphi(b,c)] - [\varphi(a,b),c] + [\varphi(a,c),b]\\ + \varphi(a,[b,c]) - \varphi([a,b],c) + \varphi([a,c],b) = 0\end{aligned} \tag{2.1.1}$$

and

$$f(a,b) = [d(a),b] + [a,d(b)] - d([a,b]) \text{ for some linear map } d. \tag{2.1.2}$$

The following proposition summarizes results from the work [42] regarding derivations and the second group cohomology for the algebra R_1 from Theorem 2.1.3.

Proposition 2.1.6.

$$\begin{aligned}
&\dim \mathrm{Der}(R_1) = 2,\\
&\dim BL^2(R_1, R_1) = (n+1)^2 - 2,\\
&\dim ZL^2(R_1, R_1) = (n+1)^2 - 2,\\
&\dim HL^2(R_1, R_1) = 0.
\end{aligned}$$

Thus, according to the results of the paper [7], we have the following theorem.

Theorem 2.1.7. *The algebra R_1 is rigid.*

2.2 Description of the second cohomology group of solvable Leibniz algebras with F_n^1 nilradical

In the following section we describe the second cohomology group of the $(n+1)$-dimensional solvable Leibniz algebras whose nilradical is the naturally graded filiform Leibniz algebra F_n^1. In the work [20], for dimension $(n+2)$, it was asserted that there is no such solvable Leibniz algebra. However, here we find that there exists a unique $(n+2)$-dimensional solvable algebra Leibniz with nilradical F_n^1. Moreover, we establish the triviality of the second group of cohomology of this Leibniz algebra and, consequently, the rigidity of the algebra.

2.2.1 Second cohomology group of $(n+1)$-dimensional solvable Leibniz algebras with F_n^1 nilradical

In this subsection we present some irreducible components of the variety $\mathfrak{Leib}_{n+1}$ in terms of closures of the orbits of some Leibniz algebras. The following equality $\dim \mathrm{Orb}(L) = (n+1)^2 - \dim \mathrm{Der}(L)$ gives us the dimensional relation between orbits and derivations of an algebra L.

Let L be an $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is the filiform algebra F_n^1. The proposition below describes the derivations of the algebras from $R_2(\alpha) - R_5(\alpha_i)$ in the list of Theorem 2.1.3.

Proposition 2.2.1. *The derivations of the algebras $R_2(\alpha) - R_5(\alpha_i)$ have the following form:*

$$\mathrm{Der}(R_2(\alpha)) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i-2)e_i, & 3 \le i \le n, \\ d_2(e_i) = e_i, & 2 \le i \le n, \\ d_3(x) = -e_1,\ d_3(e_i) = e_{i+1}, & 2 \le i \le n-1; \end{cases}$$

with $\alpha \neq 2-n;\ 1-n,$

$$\mathrm{Der}(R_2(2-n)) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i-2)e_i, & 3 \le i \le n, \\ d_2(e_i) = e_i, & 2 \le i \le n, \\ d_3(x) = -e_1,\ d_3(e_i) = e_{i+1}, & 2 \le i \le n-1, \\ d_4(x) = -e_{n-1},\ d_4(e_1) = e_n; \end{cases}$$

$$\mathrm{Der}(R_2(1-n)) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i-2)e_i, & 3 \le i \le n, \\ d_2(e_i) = e_i, & 2 \le i \le n, \\ d_3(x) = -e_1,\ d_3(e_i) = e_{i+1}, & 2 \le i \le n-1, \\ d_4(x) = e_n; \end{cases}$$

$$\mathrm{Der}(R_3) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i-n)e_i, & 2 \le i \le n-1, \\ d_2(x) = e_1,\ d_2(e_i) = -e_{i+1}, & 2 \le i \le n-1, \\ d_3(x) = e_n; \end{cases}$$

$$\mathrm{Der}(R_4) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i+1-n)e_i, & 2 \le i \le n, \\ d_2(x) = -e_1,\ d_2(e_i) = e_{i+1}, & 2 \le i \le n-1, \\ d_3(e_1) = e_n,\ d_3(x) = -e_{n-1}; \end{cases}$$

$$\mathrm{Der}(R_5(0)) : \begin{cases} d_1(e_1) = e_1,\ d_1(e_i) = (i-1)e_i, & 2 \le i \le n, \\ d_2(e_1) = e_2,\ d_2(e_i) = e_i, & 2 \le i \le n, \\ d_j(e_1) = e_j,\ d_j(e_i) = e_{i+j-2}, & 3 \le j \le n, \\ & 2 \le i \le n-j+2; \end{cases}$$

$$\mathrm{Der}(R_5(\alpha_i)) : \begin{cases} d_1(e_1) = e_2,\ d_1(e_2) = e_2 + \alpha_n e_n,\ d_1(e_i) = e_i, & 3 \le i \le n, \\ d_j(e_1) = e_{j+1},\ d_j(e_i) = e_{i+j-1}, & 2 \le j \le n-1, \\ & 2 \le i \le n-j+1, \end{cases}$$

where in the case of $R_5(\alpha_i)$ *one of the parameters* α_i *is non-zero.*

Proof. The proof of the proposition is carrying out by direct computations. □

Due to equality (2.1.2) defining the space BL^2 we have

Corollary 2.2.2.

$$\dim BL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} (n+1)^2 - 4, & \alpha = 2-n \ \textit{or}\ 1-n, \\ (n+1)^2 - 3, & \alpha \neq 2-n, 1-n; \end{cases}$$

$$\dim BL^2(R_3, R_3) = (n+1)^2 - 3;$$

$$\dim BL^2(R_4, R_4) = (n+1)^2 - 3;$$

$$\dim BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = \begin{cases} n^2 + n + 1, & \alpha_i = 0 \ \textit{for all}\ i, \\ n^2 + n + 2, & \alpha_i \neq 0 \ \textit{for some}\ i. \end{cases}$$

Now we give descriptions of the second cohomology group of the algebras $R_2 - R_5(\alpha_i)$ by presenting their bases. In fact, we find the bases of the spaces BL^2 and ZL^2 for these algebras.

In the next theorem we present the general form of 2-cocycles for the algebra R_3.

Theorem 2.2.3. *An arbitrary* $\varphi \in ZL^2(R_3, R_3)$ *has the following form:*

$$\varphi(e_1, e_1) = \sum_{i=3}^{n} a_{1,i} e_i,$$

$$\varphi(e_i, e_1) = \sum_{s=1}^{n} a_{i,s} e_s + a_{i,0} x, \quad 2 \leq i \leq n-1,$$

$$\varphi(e_n, e_1) = a_{n-1,0} e_1 - \sum_{i=3}^{n-1} \left(\sum_{t=2}^{i-1} a_{n+t-i,t} \right) e_i + a_{n,n} e_n,$$

$$\varphi(e_1, e_2) = b_{1,1} e_1,$$

$$\varphi(e_i, e_2) = (i-n) b_{1,1} e_i + b_{2,3} e_{i+1}, \quad 2 \leq i \leq n,$$

$$\varphi(e_1, e_i) = -a_{i-1,0} e_1, \ 3 \leq i \leq n,$$

$$\varphi(e_i, e_3) = (n-i) a_{2,0} e_i - (a_{2,1} + b_{1,1}) e_{i+1}, \quad 2 \leq i \leq n-1,$$

$$\varphi(e_i, e_j) = (n-i) a_{j-1,0} e_i + (a_{j-2,0} - a_{j-1,1}) e_{i+1}, \ 2 \leq i \leq n-1, \ 4 \leq j \leq n,$$

$$\varphi(e_1, x) = c_{1,1} e_1 + \sum_{i=2}^{n-1} (i-n-1) a_{1,i+1} e_i + c_{1,n} e_n + \frac{2}{(n-2)(n-1)} \sum_{t=2}^{n} a_{t,t} x,$$

$$\varphi(e_2, x) = (n-1) b_{2,3} e_1 + \sum_{i=2}^{n} c_{2,i} e_i + (n-2) b_{1,1} x,$$

$$
\begin{aligned}
\varphi(e_3,x) &= (2-n)(a_{2,1}+b_{1,1})e_1 + \left(a_{2,2} - \frac{2}{n-1}\sum_{t=2}^{n} a_{t,t}\right)e_2 \\
&\quad + (c_{2,2}+c_{1,1})e_3 + \sum_{i=4}^{n-1}(c_{2,i-1}+(3-i)a_{2,i})e_i \\
&\quad + (c_{2,n-1}+(3-n)a_{2,n}-a_{2,0})e_n + (3-n)a_{2,0}x, \\
\varphi(e_i,x) &= (n-i+1)(a_{i-2,0}-a_{i-1,1})e_1 + \sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s} a_{i+t-s-1,t}e_s \\
&\quad + \left(\sum_{t=2}^{i-1} a_{t,t} + \frac{(i+1-2n)(i-2)}{(n-2)(n-1)}\sum_{t=2}^{n} a_{t,t}\right)e_{i-1} \\
&\quad + (c_{2,2}+(i-2)c_{1,1})e_i \\
&\quad + \sum_{s=i+1}^{n-1}\left(c_{2,s-i+2}+(i-s)\sum_{t=2}^{i-1} a_{t,s-i+t+1}\right)e_s \\
&\quad + \left(c_{2,n-i+2}-a_{i-1,0}+(i-n)\sum_{t=2}^{i-1} a_{t,n-i+t+1}\right)e_n + (i-n)a_{i-1,0}x, \\
&\qquad\qquad 4 \le i \le n, \\
\varphi(x,e_1) &= -c_{1,1}e_1 + a_{1,3}e_2 + \sum_{i=3}^{n} d_{1,i}e_i - \frac{2}{(n-2)(n-1)}\sum_{t=2}^{n} a_{t,t}x, \\
\varphi(x,e_2) &= -b_{2,3}e_1 + b_{1,1}e_n, \\
\varphi(x,e_3) &= (a_{2,1}+b_{1,1})e_1 - a_{2,0}e_n, \\
\varphi(x,e_i) &= (a_{i-1,1}-a_{i-2,0})e_1 - a_{i-1,0}e_n, \quad 4 \le i \le n, \\
\varphi(x,x) &= (a_{n-1,1}-a_{n-2,0})e_1 - (2c_{1,0}+d_{1,n}+c_{1,n}+a_{n,n})e_{n-1} \\
&\quad + \sum_{i=2}^{n-2}\left((n-i)(a_{1,i+2}-d_{1,i+1}) + \sum_{s=2}^{i} a_{n-i+s-1,s}\right)e_i \\
&\quad + \beta_n e_n + a_{n-1,0}x.
\end{aligned}
$$

Proof. We set

$$
\varphi(e_i,e_1) = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \qquad 1 \le i \le n,
$$

$$\varphi(e_1, x) = c_{1,0}x + \sum_{i=1}^{n} c_{1,i}e_i,$$

$$\varphi(e_1, e_2) = b_{1,0}x + \sum_{i=1}^{n} b_{1,i}e_i,$$

$$\varphi(e_2, x) = c_{2,0}x + \sum_{i=1}^{n} c_{2,i}e_i,$$

$$\varphi(e_2, e_2) = b_{2,0}x + \sum_{i=1}^{n} b_{2,i}e_i,$$

$$\varphi(x, e_1) = d_{1,0}x + \sum_{i=1}^{n} d_{1,i}e_i,$$

$$\varphi(x, x) = \beta_0 x \sum_{i=1}^{n} \beta_i e_i,$$

$$\varphi(x, e_2) = d_{2,0}x + \sum_{i=1}^{n} d_{2,i}e_i.$$

Applying equality (2.1.1) to the triple (e_i, e_1, e_1), we obtain $[e_i, \varphi(e_1, e_1)] = 0$. Hence, $\varphi(e_1, e_1) = \sum\limits_{i=2}^{n} a_{1,i}e_i$. Similarly, the equation $(d^2\varphi)(e_i, e_j, e_k) = 0$, for $2 \leq j, k \leq n$, leads to $[e_i, \varphi(e_j, e_k)] = 0$. Consequently, we have $\varphi(e_j, e_k) \in \operatorname{span}\langle e_2, e_3, \ldots, e_n\rangle$.

Considering $(d^2\varphi)(e_1, e_1, e_2) = 0$, we derive

$$\varphi(e_1, e_2) = b_{1,0}x + b_{1,1}e_1 + b_{1,n}e_n.$$

Moreover, from $(d^2\varphi)(e_i, e_1, e_2) = 0$, with $2 \leq i \leq n-1$, we deduce

$$\varphi(e_{i+1}, e_j) = [e_i, \varphi(e_1, e_2)] + [\varphi(e_i, e_2), e_1],$$

which we get inductively for $2 \leq i \leq n$,

$$\varphi(e_i, e_2) = \frac{(i-2)(i+1-2n)}{2} b_{1,0}e_{i-1} + \big((i-2)b_{1,1} + b_{2,2}\big)e_i + \sum_{s=i+1}^{n} b_{2,s-i+2}e_s.$$

Similarly, we have

$$\begin{array}{lll} (d^2\varphi)(e_n,e_1,e_2)=0 & \Rightarrow & b_{1,0}=0, \\ (d^2\varphi)(e_1,e_2,x)=0 & \Rightarrow & b_{1,n}=0,\ c_{2,0}=(n-2)b_{1,1}, \\ (d^2\varphi)(e_2,e_2,x)=0 & \Rightarrow & \left\{\begin{array}{l} b_{2,2}=(2-n)b_{1,1},\ c_{2,1}=(n-1)b_{1,3}, \\ b_{2,i}=0,\ 4\leq i\leq n. \end{array}\right. \end{array}$$

Now we consider $(d^2\varphi)(e_i,e_j,e_1)=0$ with $2\leq j\leq n-1$. Then we have

$$\varphi(e_i,e_{j+1})=[\varphi(e_i,e_j),e_1]-[e_i,\varphi(e_j,e_1)]-\varphi([e_i,e_1],e_j).$$

Applying the induction on j for any i and the equality above we obtain the following:

$$\begin{array}{ll} \varphi(e_1,e_j)=-a_{j-1,0}e_1, & 3\leq j\leq n, \\ \varphi(e_i,e_3)=(n-i)a_{2,0}e_i-(a_{2,1}+b_{1,1})e_{i+1}, & 2\leq i\leq n-1, \\ \varphi(e_i,e_j)=(n-i)a_{j-1,0}e_i+(a_{j-2,0}-a_{j-1,1})e_{i+1}, & 2\leq i\leq n-1, \\ & \ 4\leq j\leq n. \end{array}$$

On the other hand, the condition $(d^2\varphi)(e_i,e_n,e_1)=0$ implies

$$a_{n,0}=0, \quad a_{n,1}=a_{n-1,0}.$$

We consider equality (2.1.1) for the triple (e_1,x,e_1), then we get

$$d_{1,0}=-c_{1,0}, \quad a_{1,2}=0, \quad c_{1,i}=(i-1-n)a_{1,i+1},\ 2\leq i\leq n-1.$$

Thus, we have

$$\varphi(e_1,x)=c_{1,0}x+c_{1,1}e_1+\sum_{i=2}^{n-1}(i-n-1)a_{1,i+1}e_i+c_{1,n}e_n,$$

$$\varphi(e_2,x)=(n-2)b_{1,1}x+(n-1)b_{2,3}e_1+\sum_{i=2}^{n}c_{2,i}e_i$$

and $[e_1,\varphi(e_i,x)]=(i-n)a_{i-1,0}e_1$ for $3\leq i\leq n$.

From the equality $(d^2\varphi)(e_i,e_1,x)=0,\ 2\leq i\leq n-1$, we deduce

$$\begin{aligned} \varphi(e_{i+1},x)&=[\varphi(e_i,x),e_1]+[e_i,\varphi(e_1,x)]-[\varphi(e_i,e_1),x]+\varphi(e_i,[e_1,x]) \\ &\quad+\varphi([e_i,x],e_1)=[\varphi(e_i,x),e_1]+(i+1-n)x+c_{1,0}e_i \\ &\quad+c_{1,1}e_{i+1}+(i-n)a_{i,1}e_1+\sum_{s=2}^{n}(i+1-s)a_{i,s}e_s. \end{aligned}$$

Hence, we derive inductively that

$$\begin{aligned}
\varphi(e_3,x) &= (3-n)a_{2,0}x + (2-n)(a_{2,1}+b_{1,1})e_1 + (a_{2,2}+(2-n)c_{1,0})e_2 \\
&+(c_{2,2}+c_{1,1})e_3 + \sum_{i=4}^{n-1}(c_{2,i-1}+(3-i)a_{2,i})e_i \\
&+(c_{2,n-1}+(3-n)a_{2,n}-a_{2,0})e_n, \\
\varphi(e_i,x) &= (i-n)a_{i-1,0}x + (n-i+1)(a_{i-2,0}-a_{i-1,1})e_1 \\
&+\sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s}a_{i+t-s-1,t}e_s + \left(\tfrac{(i+1-2n)(i-2)}{2}c_{1,0}+\sum_{t=2}^{i-1}a_{t,t}\right)e_{i-1} \\
&+(c_{2,2}+(i-2)c_{1,1})e_i + \sum_{s=i+1}^{n-1}\left(c_{2,s-i+2}+(i-s)\sum_{t=2}^{i-1}a_{t,s-i+t+1}\right)e_s \\
&+\left(c_{2,n-i+2}-a_{i-1,0}+(i-n)\sum_{t=2}^{i-1}a_{t,n-i+t+1}\right)e_n,
\end{aligned}$$

where $4 \le i \le n$.

Moreover, the condition $(d^2\varphi)(e_n,e_1,x)=0$ implies

$$[\varphi(e_n,x),e_1] - [\varphi(e_n,e_1),x] + \varphi(e_n,[e_1,x]) = 0,$$

which derives

$$(n-1)a_{n,2}e_2 + \sum_{s=3}^{n-1}(n-s+1)\left(a_{n,s}+\sum_{t=2}^{s-1}a_{n+t-s,t}\right)e_s$$

$$+\left(\sum_{t=2}^{n}a_{t,t} - \frac{(n-1)(n-2)}{2}c_{1,0}\right)e_n = 0.$$

Thus, we get

$$a_{n,2}=0,\ a_{n,s} = -\sum_{t=2}^{s-1}a_{n+t-s,t},\ 3\le s\le n-1,$$

$$c_{1,0} = \frac{2}{(n-1)(n-2)}\sum_{t=2}^{n}a_{t,t}.$$

Considering equality (2.1.1) for the following triples (e_2,x,e_1), (e_1,x,e_2), (x,e_1,e_2), (e_2,x,e_2) we obtain:

$$d_{1,1}=-c_{1,1},\quad d_{2,0}=0,\quad d_{2,s}=0,\ 2\le s\le n-1,\quad d_{2,1}=-b_{2,3}.$$

From $(d^2\varphi)(x, e_i, e_1) = 0$ with $2 \leq i \leq n-1$ we have

$$\varphi(x, e_{i+1}) = [\varphi(x, e_i), e_1] - [x, \varphi(e_i, e_1)] + \varphi(e_1, e_i),$$

which inductively implies

$$\begin{aligned} \varphi(x, e_3) &= (a_{2,1} + b_{1,1})e_1 - a_{2,0}e_n, \\ \varphi(x, e_i) &= (a_{i-1,1} - a_{i-2,0})e_1 - a_{i-1,0}e_n, \qquad 4 \leq i \leq n. \end{aligned}$$

Finally, the equalities $(d^2\varphi)(e_1, x, x) = (d^2\varphi)(x, e_1, x) = (d^2\varphi)(x, e_2, x) = 0$ imply

$$\begin{aligned} d_{1,2} &= a_{1,3}, \qquad\qquad d_{2,n} = b_{1,1}, \\ \beta_0 &= a_{n-1,0}, \qquad\qquad \beta_1 = a_{n-1,1} - a_{n-2,0}, \\ \beta_{n-1} &= -d_{1,n} - c_{1,n} - a_{n,n} - \frac{4}{(n-1)(n-2)} \sum_{t=2}^{n} a_{t,t}, \\ \beta_i &= (n-i)(a_{1,i+2} - d_{1,i+i}) + \sum_{s=2}^{i} a_{n-i+s-1}, \quad 2 \leq i \leq n-2, \end{aligned}$$

which complete the proof of the theorem. $\square$

Corollary 2.2.4. $\dim ZL^2(R_3, R_3) = (n+1)^2 - 2$ *and* $\dim HL^2(R_3, R_3) = 1$.

Now we present a basis of $HL^2(R_3, R_3)$.

Proposition 2.2.5. *The equivalence class $\overline{\xi}$ forms a basis of $HL^2(R_3, R_3)$, where*

$$\xi : \begin{cases} \xi(e_1, x) = e_1, \\ \xi(e_i, x) = (i-2)e_i, \quad 3 \leq i \leq n, \\ \xi(x, e_1) = -e_1. \end{cases}$$

Proof. In order to find a basis of $HL^2(R_3, R_3)$ we need to describe linearly independent elements which lie in $ZL^2(R_3, R_3)$ and do not lie in $BL^2(R_3, R_3)$. For this purpose we will find a basis of 2-cocycles and 2-coboundaries.

Since an arbitrary element of $ZL^2(R_3, R_3)$ has the form of Theorem 2.2.3 we shall use this description.

Note that there are parameters $\{a_{i,j}, b_{1,1}, b_{2,3}, c_{1,1}, c_{1,n}, \beta, c_{2,k}, d_{1,s}\}$ in the general form of elements $ZL^2(R_3, R_3)$. One of the natural basis of the space

ZL^2 is a basis whose basis elements are obtained by the instrumentality of these parameters. For the fixed pair (i,j) we denote by $\varphi(a_{i,j})$ a cocycle which has $a_{i,j}=1$ and all other parameters are equal to zero. Define such type of notation for other parameters.

Set

$$\begin{array}{llll} \varphi_{i,j}=\varphi(a_{i,j}), & \psi_1=\varphi(b_{1,1}), & \psi_2=\varphi(b_{2,3}), & \psi_3=\varphi(c_{1,1}), \\ \psi_4=\varphi(c_{1,n}), & \psi_5=\varphi(\beta), & \eta_k=\varphi(c_{2,k}), & \rho_s=\varphi(d_{1,s}), \end{array}$$

where $1\leq i\leq n$, $0\leq j\leq n$, $2\leq k\leq n$, $3\leq s\leq n$ and

$$(i,j)\notin\{(1,0),(1,1),(1,2),(n,0),(n,1),\dots,(n,n-1)\}.$$

In order to find a basis of $BL^2(R_3,R_3)$ we consider the endomorphisms $f_{j,k}\colon R_3\to R_3$ defined as follows

$$\begin{array}{rl} f_{i,j}(e_i)=e_j, & 1\leq i,j\leq n, \\ f_{i,n+1}(e_i)=x, & 1\leq i\leq n, \\ f_{n+1,j}(x)=e_j, & 1\leq j\leq n, \\ f_{n+1,n+1}(x)=x, & \end{array}$$

where in the expansion of endomorphisms the omitted values are assumed to be zero.

Consider

$$g_{i,j}(x,y)=[f_{i,j}(x),y]+[x,f_{i,j}(y)]-f_{i,j}([x,y]).$$

Note that $g_{i,j}\in BL^2(R_3,R_3)$ and now we separate a basis from these elements. Since the dimension of the space $\mathrm{Der}(R_3)$ is equal to 3, then to take a basis we should exclude three elements $g_{i,j}$. The description of $\mathrm{Der}(R_3)$ allow us to release the elements $g_{1,1}, g_{2,3}$ and $g_{n+1,n}$.

By direct computation we obtain

$$g_{j,k}:\begin{cases} g_{1,2}=-\varphi_{1,3}, & \\ g_{1,i}=-\rho_i-\varphi_{1,i+1}, & 3\leq i\leq n-1,\\ g_{1,n}=\psi_4-\rho_n, & \\ g_{1,n+1}=\sum\limits_{k=2}^{n-1}(n-k)\varphi_{k,k}-\psi_4-\rho_n, & \\ g_{2,1}=-\psi_2, & \\ g_{2,i}=-\varphi_{2,i+1}+(2-i)\eta_i, & 2\leq i\leq n-1,\ i\neq 3,\\ g_{2,n}=(2-n)\eta_n, & \\ g_{2,n+1}=\varphi_{2,1}-\psi_1-\eta_n, & \\ g_{i,k}=\varphi_{i-1,k}-\varphi_{i,k+1}, & 3\leq i\leq n-1,\ 1\leq k\leq n-1,\\ g_{i,n}=\varphi_{i-1,n}, & 3\leq i\leq n-1,\\ g_{i,n+1}=\varphi_{i-1,0}+\varphi_{i,1}, & 3\leq i\leq n-1,\\ g_{n,1}=\varphi_{n-1,1}, & \\ g_{n,i}=\varphi_{n-1,i}, & 2\leq i\leq n-2,\\ g_{n,n-1}=\varphi_{n-1,n-1}-\varphi_{n,n}, & \\ g_{n,n}=\varphi_{n-1,n}+\psi_5, & \\ g_{n,n+1}=\varphi_{n-1,0}, & \\ g_{n+1,1}=-\eta_3, & \\ g_{n+1,i}=-\rho_{i+1}, & 2\leq i\leq n-1,\\ g_{n+1,n+1}=\psi_3-2\psi_5+(n-2)\eta_2. & \end{cases}$$

From these equalities it is not difficult to check that ψ_3 and η_2 do not belong to $BL^2(R_3,R_3)$, but $\psi_3+(n-2)\eta_2\in BL^2(R_3,R_3)$. Thus, we can take the equivalence class of ψ_3 as a basis of $ZL^2(R_3,R_3)$. □

Similar to the algebra R_3, we will study the algebra R_4. In the beginning we will present the general form of 2-cocycles for this algebra.

Theorem 2.2.6. *An arbitrary $\varphi\in ZL^2(R_4,R_4)$ has the following form:*

$$\varphi(e_1,e_1)=\sum_{i=3}^{n}a_{1,i}e_i,$$

$$\begin{aligned}
\varphi(e_i,e_1) &= \sum_{s=1}^{n} a_{i,s}e_s + a_{i,0}x,\ \ 2\leq i\leq n-1,\\
\varphi(e_n,e_1) &= a_{n-1,0}e_1 - \sum_{i=3}^{n-1}\Big(\sum_{t=2}^{i-1} a_{n+t-i,t}\Big)e_i\\
&\quad -\Big(\sum_{t=2}^{n-1} a_{t,t} + \frac{(n-4)(n-1)}{2}d_{1,0}\Big)e_n,\\
\varphi(e_1,e_2) &= b_{1,1}e_1 + b_{1,1}e_n,\\
\varphi(e_i,e_2) &= (i-n+1)b_{1,1}e_i + b_{2,3}e_{i+1}, \quad 2\leq i\leq n,\\
\varphi(e_1,e_i) &= -a_{i-1,0}e_1 - a_{i-1,0}e_n, \quad 3\leq i\leq n,\\
\varphi(e_i,e_3) &= (n-i-1)a_{2,0}e_i - (a_{2,1}+b_{1,1})e_{i+1}, \quad 2\leq i\leq n-1,\\
\varphi(e_n,e_j) &= -a_{j-1,0}e_n, \quad 3\leq j\leq n,\\
\varphi(e_i,e_j) &= (n-i-1)a_{j-1,0}e_i + (a_{j-2,0}-a_{j-1,1})e_{i+1},\ 2\leq i\leq n-1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4\leq j\leq n,\\
\varphi(e_1,x) &= (a_{n-1,0}-d_{1,0})x + (a_{n-1,1}-a_{n-2,0}-d_{1,1})e_1\\
&\quad + \sum_{i=2}^{n-2}\Big(\sum_{t=2}^{i} a_{n+t-i-1,t} - (n-i)a_{1,i+1}\Big)e_i\\
&\quad + \Big(\sum_{t=2}^{n-1} a_{t,t} - a_{1,n} + \frac{(n-2)(n-3)}{2}d_{1,0}\Big)e_{n-1} + c_{1,n}e_n,\\
\varphi(e_2,x) &= (n-2)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i + (n-3)b_{1,1}x,\\
\varphi(e_3,x) &= (4-n)a_{2,0}x + (3-n)(a_{2,1}+b_{1,1})e_1 + (a_{2,2}+(n-3)d_{1,0})e_2\\
&\quad + (c_{2,2}-d_{1,1})e_3 + \sum_{i=4}^{n-2}(c_{2,i-1}+(3-i)a_{2,i})e_i\\
&\quad + (c_{2,n-2}+(4-n)a_{2,n-1}+a_{2,0})e_{n-1}\\
&\quad + (c_{2,n-1}+(3-n)a_{2,n}-a_{2,1})e_n,\\
\varphi(e_i,x) &= (i-n+1)a_{i-1,0}x + (n-i)(a_{i-2,0}-a_{i-1,1})e_1\\
&\quad + \sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s} a_{i+t-s-1,t}e_s
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{t=2}^{i-1} a_{t,t} + \frac{(2n-i-3)(i-2)}{2} d_{1,0}\Big) e_{i-1} \\
&+ (c_{2,2} + (2-i)d_{1,1})e_i \\
&+ \sum_{s=i+1}^{n-2} \Big(c_{2,s-i+2} + (i-s)\sum_{t=2}^{i-1} a_{t,s-i+t+1}\Big) e_s \\
&+ \Big(c_{2,n-i+1} + a_{i-1,0} + (i-n+1)\sum_{t=2}^{i-1} a_{t,n-i+t}\Big) e_{n-1} \\
&+ \Big(c_{2,n-i+2} + a_{i-2,0} - a_{i-1,1} + (i-n)\sum_{t=2}^{i-1} a_{t,n-i+t+1}\Big) e_n, \\
&\hspace{6cm} 4 \le i \le n-2,
\end{aligned}
$$

$$
\begin{aligned}
\varphi(e_{n-1},x) &= (a_{n-3,0} - a_{n-2,1})e_1 + \sum_{s=2}^{n-3}(n-1-s)\sum_{t=2}^{s} a_{n+t-s-2,t} e_s \\
&+ \Big(\sum_{t=2}^{n-2} a_{t,t} + \frac{(n-2)(n-3)}{2} d_{1,0}\Big) e_{n-2} \\
&+ (c_{2,2} + a_{n-2,0} - (n-3)d_{1,1})e_{n-1} \\
&+ \Big(c_{2,3} + a_{n-3,0} - a_{n-2,1} - \sum_{t=2}^{n-2} a_{t,t+2}\Big) e_n, \\
\varphi(e_n,x) &= \sum_{s=2}^{n-2}(n-s)\sum_{t=2}^{s} a_{n+t-s-1,t} e_s \\
&+ \Big(\sum_{t=2}^{n-1} a_{t,t} + \frac{(n-2)(n-3)}{2} d_{1,0} + a_{n-1,0}\Big) e_{n-1} \\
&+ (c_{2,2} + a_{n-2,0} - a_{n-1,1} - (n-2)d_{1,1})e_n + a_{n-1,0}x, \\
\varphi(x,e_1) &= d_{1,0}x + d_{1,1}e_1 + a_{1,3}e_2 + \sum_{i=3}^{n} d_{1,i}e_i, \\
\varphi(x,e_2) &= -b_{2,3}e_1 - b_{1,1}e_{n-1}, \\
\varphi(x,e_3) &= (a_{2,1} + b_{1,1})e_1 + a_{2,0}e_{n-1}, \\
\varphi(x,e_i) &= (a_{i-1,1} - a_{i-2,0})e_1 + a_{i-1,0}e_{n-1}, \quad 4 \le i \le n,
\end{aligned}
$$

$$\begin{aligned}
\varphi(x,x) &= (a_{n-3,0} - a_{n-2,1})e_1 \\
&+ \sum_{i=2}^{n-3}\Big((i-n+1)(d_{1,i+1} - a_{1,i+2}) - \sum_{t=2}^{i} a_{n-i+t-2,t}\Big)e_i \\
&+ \big(a_{1,n} - d_{1,n-1} - \frac{n^2-5n+10}{2}d_{1,0} - \sum_{t=2}^{n-2} a_{t,t}\big)e_{n-2} \\
&+ (a_{n-1,n} + d_{1,1} - c_{1,n})e_{n-1} + \beta_n e_n - a_{n-2,0}x.
\end{aligned}$$

Proof. The proof of this theorem is carrying out by applying similar arguments as in the proof of Theorem 2.2.3. $\square$

Remark 2.2.7. *It should be noted that in the case $n = 3$ an arbitrary 2-cocycle for the algebra R_4 is different from the description of Theorem 2.2.6 and it has the form:*

$$\begin{aligned}
\varphi(e_1,e_1) &= a_{1,3}e_3, \\
\varphi(e_2,e_1) &= \sum_{s=1}^{3} a_{i,s}e_s + a_{2,0}x, \\
\varphi(e_3,e_1) &= a_{2,0}e_1 - (a_{2,2} - d_{1,0})e_3, \\
\varphi(e_1,e_2) &= b_{1,1}e_1 + b_{1,1}e_3, \\
\varphi(e_2,e_2) &= b_{2,3}e_3, \\
\varphi(e_3,e_2) &= b_{1,1}e_3, \\
\varphi(e_1,e_3) &= -a_{2,0}e_1 - a_{2,0}e_3, \\
\varphi(e_2,e_3) &= -(a_{2,1} + b_{1,1})e_3, \\
\varphi(e_3,e_3) &= -a_{2,0}e_3, \\
\varphi(e_1,x) &= (a_{2,1} + b_{1,1} - d_{1,1})e_1 + (a_{2,2} - a_{1,3})e_2 - c_{1,3}e_3 + (a_{2,0} - d_{1,0})x, \\
\varphi(e_2,x) &= b_{2,3}e_1 + c_{2,2}e_2 + c_{2,3}e_3, \\
\varphi(e_3,x) &= (a_{2,2} + a_{2,0})e_2 - (a_{2,1} + c_{2,2} - d_{1,1})e_3 + a_{2,0}x, \\
\varphi(x,e_1) &= d_{1,1}e_1 + (a_{1,3} - 2d_{1,0})e_2 + d_{1,3}e_3 + d_{1,0}x, \\
\varphi(x,e_2) &= b_{2,3}e_1 - b_{1,1}e_2, \\
\varphi(x,e_3) &= (a_{2,1} + b_{1,1})e_1 + a_{2,0}e_2, \\
\varphi(x,x) &= b_{2,3}e_1 + (a_{2,3} - c_{1,3} + d_{1,1})e_2 + \beta_3 e_3 + b_{1,1}x.
\end{aligned}$$

The following corollary is proved by applying similar arguments as in the case of the algebra R_3.

Corollary 2.2.8. $\dim ZL^2(R_4, R_4) = (n+1)^2 - 2$, $\dim HL^2(R_4, R_4) = 1$ *and the equivalence class* $\overline{\rho}$ *forms a basis of* $HL^2(R_4, R_4)$, *where*

$$\rho : \begin{cases} \rho(e_1, x) = e_1, \\ \rho(e_i, x) = (i-2)e_i, \ 3 \leq i \leq n, \\ \rho(x, e_1) = -e_1, \\ \rho(x, x) = -e_{n-1}. \end{cases}$$

Now we will pass to studying of algebras with parameters from the Theorem 2.1.3.

Theorem 2.2.9. *An arbitrary* $\varphi \in ZL^2\big(R_2(\alpha), R_2(\alpha)\big)$ *has the following form:*

$$\begin{aligned}
\varphi(e_1, e_1) &= \sum_{i=2}^{n} a_{1,i} e_i, \\
\varphi(e_i, e_1) &= \sum_{s=1}^{n} a_{i,s} e_s + a_{i,0} x, \ 2 \leq i \leq n-1, \\
\varphi(e_i, e_1) &= \sum_{s=1}^{n} a_{i,s} e_s + a_{i,0} x, \ 2 \leq i \leq n-1, \\
\varphi(e_n, e_1) &= a_{n-1,0} e_1 - \sum_{i=3}^{n-1} \Big(\sum_{t=2}^{i-1} a_{n+t-i,t} \Big) e_i \\
&\quad + \Big(\frac{(n-1)(2\alpha+n)}{2} d_{1,0} - \sum_{t=2}^{n-1} a_{t,t} \Big) e_n, \\
\varphi(e_1, e_2) &= b_{1,1} e_1 + b_{1,n} e_n, \\
\varphi(e_i, e_2) &= \big((i-2) b_{1,1} + b_{2,2}\big) e_i + b_{2,3} e_{i+1}, \qquad 2 \leq i \leq n, \\
\varphi(e_1, e_i) &= -a_{i-1,0} e_1, \qquad 3 \leq i \leq n, \\
\varphi(e_i, e_3) &= -(\alpha+i-1) a_{2,0} e_i - (a_{2,1} + b_{1,1}) e_{i+1}, \qquad 2 \leq i \leq n,
\end{aligned}$$

$$\varphi(e_i,e_j) = -(\alpha+i-1)a_{j-1,0}e_i + (a_{j-2,0}-a_{j-1,1})e_{i+1}, \quad 2\leq i\leq n, \; 4\leq j\leq n,$$

$$\varphi(e_1,x) = -d_{1,1}e_1 + \sum_{i=2}^{n-1}(\alpha+i-2)a_{1,i+1}e_i + c_{1,n}e_n - d_{1,0}x,$$

$$\varphi(e_2,x) = -\alpha b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i - (\alpha+1)b_{1,1}x,$$

$$\varphi(e_3,x) = (\alpha+1)(a_{2,1}+b_{1,1})e_1 + (a_{2,2}-(\alpha+1)d_{1,0})e_2 + (c_{2,2}-d_{1,1})e_3 + \sum_{i=4}^{n}(c_{2,i-1}+(3-i)a_{2,i})e_i + (\alpha+2)a_{2,0}x,$$

$$\begin{aligned}\varphi(e_i,x) = {} & (\alpha+i-2)(a_{i-1,1}-a_{i-2,0})e_1 + \sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s} a_{i+t-s-1,t}e_s \\ & + \Big(\sum_{t=2}^{i-1} a_{t,t} - \frac{(2\alpha+i-1)(i-2)}{2}d_{1,0}\Big)e_{i-1} + (c_{2,2}-(i-2)d_{1,1})e_i \\ & + \sum_{s=i+1}^{n}\Big(c_{2,s-i+2} + (i-s)\sum_{t=2}^{i-1} a_{t,s-i+t+1}\Big)e_s + (\alpha+i-1)a_{i-1,0}x, \\ & \qquad 4\leq i\leq n,\end{aligned}$$

$$\varphi(x,e_1) = \sum_{i=1}^{n} d_{1,i}e_i + d_{1,0}x,$$

$$\varphi(x,e_2) = -b_{2,3}e_1 - b_{1,n}e_{n-1},$$

$$\varphi(x,e_3) = (a_{2,1}+b_{1,1})e_1,$$

$$\varphi(x,e_i) = (a_{i-1,1}-a_{i-2,0})e_1, \quad 4\leq i\leq n,$$

$$\varphi(x,x) = \sum_{i=2}^{n-2}(\alpha+i-1)(d_{1,i+1}-a_{1,i+2})e_i + \big((\alpha+n-2)d_{1,n}-c_{1,n}\big)e_{n-1} + \beta_n e_n$$

with restrictions

$$\left\{\begin{array}{l}(\alpha-1)a_{1,2}=0, \quad (\alpha+1)b_{1,n}=0, \quad (\alpha+1)(b_{2,2}-(\alpha+1)b_{1,1})=0,\\ (n-3)b_{1,n}=0, \quad \alpha(d_{1,2}-a_{1,3})=0.\end{array}\right. \tag{2.2.1}$$

Proof. The proof of this theorem is carrying out by applying similar arguments as in the proof of Theorem 2.2.3. □

From the equalities (2.2.1), we get that if $n \neq 3$, then $b_{1,n} = 0$. Thus, we distinguish the cases $n = 3$ and $n > 3$. Moreover, the general form of infinitesimal deformations also depends on the value of α. Therefore, we have

Corollary 2.2.10.

$$\dim ZL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} (n+1)^2 - 1, & \alpha = 0; \pm 1, \\ (n+1)^2 - 2, & \alpha \neq 0; \pm 1, \end{cases} \qquad for\ n > 3;$$

$$\dim ZL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} 15, & \alpha = 0; 1, \\ 16, & \alpha = -1, \\ 14, & \alpha \neq 0; \pm 1, \end{cases} \qquad for\ n = 3.$$

Corollary 2.2.11.

$$\dim HL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} 2, & \alpha = 0; \pm 1; 1-n; 2-n, \\ 1, & \alpha \neq 0; \pm 1; 1-n; 2-n, \end{cases} \qquad for\ n > 3;$$

$$\dim HL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} 4, & \alpha = -1, \\ 2, & \alpha = 0; 1; -2, \\ 1, & \alpha \neq 0; \pm 1; -2, \end{cases} \qquad for\ n = 3.$$

In the following proposition similarly to the Proposition 2.2.5 we find a basis of $HL^2\big(R_2(\alpha), R_2(\alpha)\big)$.

Proposition 2.2.12. *The basis of $HL^2\big(R_2(\alpha), R_2(\alpha)\big)$ consists of the following equivalence classes*

$$\begin{cases} \overline{\rho}, \overline{\psi_1}, & \alpha = 0; 1; 1-n; 2-n, \\ \overline{\psi_1}, \overline{\psi_2}, & \alpha = -1, \\ \overline{\rho}, & \alpha \neq 0; \pm 1; 1-n; 2-n, \end{cases} \qquad for\ n > 3;$$

$$\begin{cases} \overline{\rho}, \overline{\psi_1} & \alpha = 0; -1; -2, \\ \overline{\psi_1}, \overline{\psi_2}, \overline{\psi_3}, \overline{\psi_4}, & \alpha = -1, \\ \overline{\rho}, & \alpha \neq 0; \pm 1; -2, \end{cases} \qquad for\ n = 3,$$

where

$$\rho:\begin{cases}\rho(e_1,x)=e_1,\\ \rho(e_i,x)=(i-2)e_i, \quad 3\le i\le n,\\ \rho(x,e_1)=-e_1,\end{cases}$$

$$\psi_1:\psi_1(e_i,x)=e_i, \quad 2\le i\le n, \quad \psi_2:\psi_2(e_i,e_2)=e_i, \quad 2\le i\le n,$$

$$\psi_3:\begin{cases}\psi_3(e_1,x)=e_n,\\ \psi_3(x,x)=-e_{n-1},\end{cases} \qquad \psi_4:\begin{cases}\psi_4(e_1,e_2)=e_n,\\ \psi_4(x,e_2)=-e_{n-1}.\end{cases}$$

We need to investigate only the algebra $R_5(\alpha_i)$ from the list of Theorem 2.1.3 and to give description of the second cohomology group of this algebra.

Theorem 2.2.13. *An arbitrary 2-cocycle of $ZL^2\big(R_5(\alpha_i),R_5(\alpha_i)\big)$ has the following form:*

$$\varphi(e_1,e_1)=a_{2,1}e_1+(a_{2,2}-\alpha_n a_{n,2})e_2+\sum_{i=3}^{n}a_{1,i}e_i+a_{2,0}x,$$

$$\varphi(e_i,e_1)=\sum_{s=1}^{n}a_{i,s}e_s, \quad 2\le i\le n-1+a_{i,0}x,$$

$$\varphi(e_n,e_1)=\sum_{i=2}^{n}a_{n,i}e_i,$$

$$\begin{aligned}\varphi(e_1,e_2)=&\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,0}-c_0\Big)e_2+\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,1}-c_1\Big)e_3\\&+\sum_{i=4}^{n-1}\alpha_i\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,0}-c_0\Big)e_i,\end{aligned}$$

$$\begin{aligned}\varphi(e_i,e_2)=&\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,0}-c_0\Big)e_i+\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,1}-c_1\Big)e_{i+1}\\&+\sum_{k=i+2}^{n}\alpha_i\Big(\sum_{j=4}^{n}\alpha_j a_{j-1,0}-c_0\Big)e_k, \quad 2\le i\le n,\end{aligned}$$

$$\varphi(e_1,e_j) = -a_{j-1,0}e_2 - a_{j-1,1}e_3 - a_{j-1,0}\sum_{k=4}^{n-1}\alpha_k e_k, \quad 3 \le j \le n,$$

$$\varphi(e_i,e_j) = -a_{j-1,0}e_i - a_{j-1,1}e_{i+1} - a_{j-1,0}\sum_{k=i+2}^{n}\alpha_{k-i+2}e_k, \ 2 \le i \le n, \qquad 3 \le j \le n,$$

$$\begin{aligned}
\varphi(x,e_1) &= -d_2e_1 + \sum_{i=2}^{n} d_ie_i,\\
\varphi(e_1,x) &= (c_1 + d_2 - \alpha_n a_{n-1,1})e_1 + (c_2\\
&+ a_{1,3} - a_{2,3} - d_2 + \alpha_n a_{n-1,1} + \alpha_n a_{n,3})e_2\\
&+ \Big(c_3 + a_{1,4} - a_{2,4} + \alpha_n(a_{n,3} - \alpha_4 a_{n,2})\Big)e_3\\
&+ \sum_{i=4}^{n-1}\Big[c_i + a_{1,i+1} - a_{2,i+1} + \alpha_n(a_{n,i} - \alpha_{i+1}a_{n,2})\\
&+ \sum_{j=4}^{i}\alpha_j(a_{1,i-j+3} - a_{2,i-j+3})\Big]e_i + c_{n+1}e_n + (c_0 - \alpha_n a_{n-1,0})x,\\
\varphi(e_2,x) &= \sum_{i=1}^{n} c_ie_i + c_0x,\\
\varphi(e_3,x) &= \Big(a_{2,1} + \sum_{j=4}^{n-1}\alpha_j a_{j,1}\Big)e_1 + \Big(\sum_{j=4}^{n}\alpha_j a_{j,2} - a_{2,1}\Big)e_2\\
&+ \Big(c_1 + c_2 + \sum_{j=4}^{n}\alpha_j a_{j,3}\Big)e_3\\
&+ \sum_{k=4}^{n}\Big(c_{k-1} - a_{2,1}\alpha_k - \sum_{j=2}^{k-2}\alpha_{k-j+2}a_{2,j} + \sum_{j=4}^{n}\alpha_j a_{j,k}\Big)e_k\\
&+ a_{2,1}\alpha_n e_n + \Big(a_{2,0} + \sum_{j=4}^{n}\alpha_j a_{j,0}\Big)x,\\
\varphi(e_i,x) &= \Big(a_{i-1,1} + \sum_{j=i+1}^{n-1}\alpha_{j-i+3}a_{j,1}\Big)e_1 + \Big(\sum_{j=i+1}^{n}\alpha_{j-i+3}a_{j,2} - a_{i-1,1}\Big)e_2
\end{aligned}$$

$$+\sum_{k=3}^{i-2}\Big(\sum_{j=i}^{n-1}\alpha_{j-i+k+1}a_{j,1}+\sum_{s=2}^{k-1}\sum_{j=i}^{n}\alpha_{j+k-i-s+3}a_{j,s}$$

$$+\sum_{j=i+1}^{n}\alpha_{j-i+3}a_{j,k}\Big)e_k$$

$$+\Big(\sum_{j=i}^{n-1}\alpha_j a_{j,1}+\sum_{s=2}^{i-2}\sum_{j=i}^{n}\alpha_{j-s+2}a_{j,s}+\sum_{j=i+1}^{n}\alpha_{j-i+3}a_{j,i-1}\Big)e_{i-1}$$

$$+\Big[c_1+c_2-(i-3)d_2-\sum_{j=3}^{i-1}\alpha_{j+1}(a_{j,1}+a_{j,2})+\sum_{s=3}^{i-1}\sum_{j=i}^{n}\alpha_{j-s+3}a_{j,s}$$

$$+\sum_{j=i+1}^{n}\alpha_{j-i+3}a_{j,i}\Big]e_i+\sum_{k=i+1}^{n-1}\Big(c_{k+2-i}+\sum_{j=2}^{i-1}\alpha_{k-i+j+1}a_{j,1}$$

$$-\sum_{s=2}^{i-1}\sum_{j=2}^{k-s}\alpha_{k+4-j-s}a_{i-s+1,j}+\sum_{s=2}^{i-1}\sum_{j=s+2}^{n}\alpha_{j-s+2}a_{j,k-i+s+1}\Big)e_k$$

$$+\alpha_n a_{i-1,1}e_n+\Big(a_{i-1,0}+\sum_{j=i+1}^{n}\alpha_{j-i+3}a_{j,0}\Big)x,\ 4\le i\le n,$$

$$\varphi(x,x)=d_3e_2+d_4e_3+\sum_{i=4}^{n-2}\Big(d_{i+1}+\sum_{j=4}^{i}\alpha_j d_{i+3-j}\Big)e_i$$

$$+\Big(d_n+\sum_{j=4}^{n}\alpha_j d_{n+2-j}\Big)e_{n-1}+\beta e_n.$$

Proof. The proof of this proposition is carrying out by applying similar arguments as in the proof of Theorem 2.2.3. $\square$

Corollary 2.2.14.

$$\dim ZL^2\big(R_5(\alpha_i),R_5(\alpha_i)\big)=n^2+3n-3,$$
$$\dim HL^2\big(R_5(\alpha_i),R_5(\alpha_i)\big)=\begin{cases}2n-4, & \alpha_i=0 \quad \textit{for all } i,\\ 2n-5, & \alpha_i\neq 0 \quad \textit{for some } i.\end{cases}$$

Let us introduce the notations

$$\rho : \begin{cases} \rho(e_1, x) = -e_1 + e_2, \\ \rho(e_i, x) = (i-3)e_i, \quad 4 \le i \le n, \\ \rho(x, e_1) = e_1 - e_2, \\ \rho(x, x) = -\alpha_n e_{n-1}, \end{cases}$$

$$\psi_k \, (4 \le k \le n-1) : \begin{cases} \psi_k(e_1, x) = e_k, \\ \psi_k(e_i, x) = e_{k+i-2}, \quad 2 \le i \le n-k+2, \end{cases}$$

$$\psi_n : \{ \; \psi_n(e_2, x) = e_n,$$

$$\varphi_{n,2} : \begin{cases} \varphi_{n,2}(e_n, e_1) = e_2, \\ \varphi_{n,2}(e_1, x) = -\alpha_n \sum\limits_{j=3}^{n-1} \alpha_{j+1} e_j, \\ \varphi_{n,2}(e_i, x) = \sum\limits_{j=2}^{i-1} \alpha_{n+j+1-i} e_j, \quad 3 \le i \le n-1, \\ \varphi_{n,2}(e_n, x) = \sum\limits_{j=3}^{n-1} \alpha_{j+1} e_j, \end{cases}$$

$$\varphi_{n,k} \, (3 \le k \le n-1) : \begin{cases} \varphi_{n,k}(e_n, e_1) = e_k, \\ \varphi_{n,k}(e_1, x) = -\alpha_n e_k, \\ \varphi_{n,k}(e_i, x) = \sum\limits_{j=k}^{i+k-3} \alpha_{n+j+3-i-k} e_j, \\ \qquad\qquad 3 \le i \le n+2-k, \\ \varphi_{n,k}(e_i, x) = \sum\limits_{j=k}^{n} \alpha_{n+j+3-i-k} e_j, \\ \qquad\qquad n+3-k \le i \le n-1, \\ \varphi_{n,k}(e_n, x) = \sum\limits_{j=k+1}^{n} \alpha_{j+3-k} e_j. \end{cases}$$

Proposition 2.2.15. *The equivalence classes* $\overline{\rho}$, $\overline{\psi_k}$ $(4 \le k \le n)$ *and* $\overline{\varphi_{n,k}}$ $(2 \le k \le n-1)$ *form a basis of* $HL^2\big(R_5(0), R_5(0)\big)$. *The basis of* $HL^2\big(R_5(\alpha_4, \dots, \alpha_n), R_5(\alpha_4, \dots, \alpha_n)\big)$ *with* $(\alpha_4, \dots, \alpha_n) \neq (0, \dots, 0)$ *is also the same except one cocycle* $\overline{\psi_k}$ *with* $\alpha_k \neq 0$.

Proof. Since there are parameters $(a_{i,j}, c_k, \beta, d_s)$ in the general form of 2-cocycles for the algebra $R_5(\alpha_4, \dots, \alpha_n)$, we consider the natural basis of the space ZL^2 whose basis elements are obtained by the instrumentality of these parameters.

Similarly as in the proof of Proposition 2.2.5, we denote by $\varphi(a_{i,j})$ the cocycle which satisfies $a_{i,j} = 1$ and all other parameters are equal to zero. We define such type of notation for other parameters by notations

$$\varphi_{i,j} = \varphi(a_{i,j}), \quad \psi_k = \varphi(c_k), \quad \rho_s = \varphi(d_{1,s}), \quad \eta = \varphi(\beta),$$

where $1 \leq i \leq n$, $0 \leq j \leq n$, $0 \leq k \leq n+1$, $2 \leq s \leq n$ and

$$(i,j) \notin \{(1,0),(1,1),(1,2)\}.$$

To define the basis of $BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ we consider the endomorphisms $f_{j,k} : R_5(\alpha_i) \to R_5(\alpha_i)$ defined as follows

$$\begin{aligned} f_{i,j}(e_i) &= e_j, && 1 \leq i,j \leq n,\\ f_{i,n+1}(e_i) &= x, && 1 \leq i \leq n,\\ f_{n+1,j}(x) &= e_j, && 1 \leq j \leq n,\\ f_{n+1,n+1}(x) &= x, \end{aligned}$$

where in the expansion of the endomorphisms the omitted values are assumed to be zero.

Consider

$$g_{i,j}(x,y) = [f_{i,j}(x), y] + [x, f_{i,j}(y)] - f_{i,j}([x,y]).$$

Note that $g_{i,j} \in BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ and by direct computation we express $g_{i,j}$ via the elements $\varphi_{i,j}, \psi_k, \rho_s$ and η.

In the case of $\alpha_4 = \alpha_5 = \cdots = \alpha_n = 0$ we obtain that any linear combination of elements ρ_2, ψ_k, $4 \leq k \leq n$ and $\varphi_{n,j}$, $2 \leq j \leq n-1$ does not belong to $BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$. Therefore, the equivalence classes of these elements form a basis of $HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$.

However, in the case of $(\alpha_4, \alpha_5, \ldots, \alpha_n) \neq (0,0,\ldots 0)$ we obtain that

$$2\alpha_4\psi_4 + 3\alpha_5\psi_5 + \cdots + (n-2)\alpha_n\psi_n \in BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big).$$

Hence, in this case we get that the basis of $HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ also consists from $\overline{\rho_2}$, $\overline{\psi_k}$ $(4 \leq k \leq n)$ and $\overline{\varphi_{n,k}}$ $(2 \leq k \leq n-1)$, except one cocycle $\overline{\psi_k}$ with $\alpha_k \neq 0$. $\square$

2.2.2 $(n+2)$-dimensional solvable Leibniz algebras with F_n^1 nilradical and its rigidity

As mentioned above, in the work [20] it was made a mistake. Therefore, in the beginning we consider a solvable Leibniz algebra whose dimension is equal to $(n+2)$ and the nilradical is the filiform Leibniz algebra F_n^1. We give a complete proof of the following theorem and specify the exact place of the error in Theorem 2.1.4.

Theorem 2.2.16. *An arbitrary $(n+2)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to the algebra $R(F_n^1)$ with the multiplication table:*

$$\begin{array}{lll} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, & [e_1, x] = e_1, \\ [e_i, y] = e_i, & 2 \leq i \leq n, & [e_i, x] = (i-1)e_i, \quad 2 \leq i \leq n, \\ & & [x, e_1] = -e_1. \end{array}$$

Proof. From the conditions of the theorem we have the existence of a basis, with $\{e_1, e_2, \dots, e_n\}$ a basis of the nilradical F_n^1, and $\{x, y\}$ a basis of a complementary vector space such that the multiplication table of F_n^1 remains the same. The non-nilpotent derivations of F_n^1, denoted by D_x and D_y, are of the form given in [20, Proposition 4.1], with the set of entries $\{\alpha_i, \gamma\}$ and $\{\beta_i, \delta\}$, respectively, where $[e_i, x] = D_x(e_i)$ and $[e_i, y] = D_y(e_i)$.

By taking the following change of basis:

$$x' = \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}x - \frac{\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}y, \quad y' = -\frac{\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}x + \frac{\alpha_1}{\alpha_1\beta_2 - \alpha_2\beta_1}y,$$

we may assume that $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$.

Therefore, we have the products

$$\begin{array}{ll} [e_1, x] = e_1 + \sum\limits_{i=3}^{n} \alpha_i e_i, & [e_2, x] = e_2 + \sum\limits_{i=3}^{n-1} \alpha_i e_i + \gamma e_n, \\ [e_i, x] = (i-1)e_i + \sum\limits_{j=i+1}^{n} \alpha_{j-i+2} e_j, & 3 \leq i \leq n, \\ [e_1, y] = e_2 + \sum\limits_{i=3}^{n} \beta_i e_i, & [e_2, y] = e_2 + \sum\limits_{i=3}^{n-1} \beta_i e_i + \delta e_n, \\ [e_i, y] = e_i + \sum\limits_{j=i+1}^{n} \beta_{j-i+2} e_j, & 3 \leq i \leq n. \end{array}$$

Let us introduce the notations

$$[x, e_1] = \sum_{i=1}^{n} \lambda_i e_i, \qquad [x, e_2] = \sum_{i=1}^{n} \delta_i e_i, \qquad [x, x] = \sum_{i=1}^{n} \mu_i e_i.$$

From the Leibniz identity we get $[x, e_i] = 0,\ 3 \leq i \leq n$.

By applying similar arguments as in Case 1 of the proof of Theorem 2.1.3 (see [20, Theorem 4.2]) and taking into account that the products $[e_i, y]$, $1 \leq i \leq n$, will not be changed under the transformations of bases that were used there, we obtain two cases of products of elements:

$$\text{Case } \lambda_2 \neq 1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1)e_i, & 2 \leq i \leq n, \\ [x, e_1] = -e_1 - e_2, & \\ [e_1, y] = \beta e_n, & \\ [e_i, y] = e_i + \sum\limits_{j=i+1}^{n} \beta_{j-i+2} e_j, & 2 \leq i \leq n; \end{cases}$$

$$\text{Case } \lambda_2 = 1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_1, x] = e_1, & \\ [e_i, x] = (i-1)e_i, & 2 \leq i \leq n, \\ [x, e_1] = -e_1, & \\ [e_1, y] = \delta e_n, & \\ [e_i, y] = e_i + \sum\limits_{j=i+1}^{n} \beta_{j-i+2} e_j, & 2 \leq i \leq n. \end{cases}$$

The omission of Case $\lambda_2 = 1$ in the proof of Theorem 2.1.4 is an error in the work [20]. We fill this gap in the proof.

We consider **Case** $\lambda_2 = 1$**.**

Let us introduce notations

$$[y, e_1] = \sum_{i=1}^{n} \eta_i e_i, \quad [y, e_2] = \sum_{i=1}^{n} \theta_i e_i, \quad [y, y] = \sum_{i=1}^{n} \tau_i e_i,$$

$$[x, y] = \sum_{i=1}^{n} \sigma_i e_i, \quad [y, x] = \sum_{i=1}^{n} \rho_i e_i.$$

By taking the following change of basis elements $y' = y + \sigma_1 e_1$, we derive $[x, y] = \sum\limits_{i=2}^{n} \sigma_i e_i$.

From the Leibniz identity $0 = [e_2, [x, y]] = [[e_2, x], y] - [[e_2, y], x]$, we deduce $\beta_i = 0, \ 3 \leq i \leq n$.

By making the change $y' = y + \sum\limits_{i=2}^{n-1} \eta_{i+1} e_i$, we obtain $[y, e_1] = \eta_1 e_1 + \eta_2 e_2$ and

$$
\begin{array}{llll}
[e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1, & [e_i, y] = e_i, & 2 \leq i \leq n, \\
[e_1, y] = \delta e_n, & & [e_1, x] = e_1, & \\
[y, e_1] = \eta_1 e_1 + \eta_2 e_2, & & [y, x] = \sum\limits_{i=1}^{n} \rho_i e_i, & \\
[e_i, x] = (i-1) e_i, & 2 \leq i \leq n, & [y, e_2] = \sum\limits_{i=1}^{n} \theta_i e_i, & \\
[y, y] = \sum\limits_{i=1}^{n} \tau_i e_i, & & [x, e_1] = -e_1, & \\
[y, e_i] = \sum\limits_{j=1}^{n} \mu_{i,j} e_j, & 3 \leq i \leq n, & [x, y] = \sum\limits_{i=2}^{n} \sigma_i e_i. &
\end{array}
$$

By applying the Leibniz identity to the products above we derive

$$
\begin{array}{c}
\delta = \eta_1 = \eta_2 = \theta_i = \mu_{i,j} = \sigma_i = 0 \qquad (1 \leq i, j \leq n), \\
\rho_i = \tau_i = 0 \quad (1 \leq i \leq n-1), \qquad \rho_n = (n-1)\tau_n.
\end{array}
$$

Finally, by taking the change of basis elements $y' = y - \tau_n e_n$, we obtain the algebra $R(F_n^1)$. $\square$

In order to describe the second group of cohomology of the algebra $R(F_n^1)$ we need the description of its derivations. The general matrix form of the derivations of $R(F_n^1)$ is given in the following proposition.

Proposition 2.2.17. *Any derivation of the algebra $R(F_n^1)$ has the following matrix form:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \beta_2 & \beta_3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1+\beta_2 & \beta_3 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha_1+\beta_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & (n-2)\alpha_1+\beta_2 & 0 & 0 \\ -\beta_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Proof. The proof is carried out by straightforward calculations of the derivation property. □

From Proposition 2.2.17 we conclude that $\dim BL^2(R(F_n^1), R(F_n^1)) = n^2 + 4n + 1$. The general form of an element of the space $ZL^2(R(F_n^1), R(F_n^1))$ is presented below.

Proposition 2.2.18. *An arbitrary element φ of the space $ZL^2\big(R(F_n^1), R(F_n^1)\big)$ has the following form:*

$$\begin{aligned}
\varphi(e_1, e_1) &= \sum_{k=3}^{n} a_{1,1}^k e_k, \\
\varphi(e_i, e_1) &= \sum_{k=1}^{n} a_{i,1}^k e_k + a_{i,1}^{n+1} x + a_{i,1}^{n+2} y, \ 2 \le i \le n-1, \\
\varphi(e_n, e_1) &= a_{n-1,1}^{n+1} e_1 - \sum_{k=3}^{n-1} \Big(\sum_{j=1}^{k-2} a_{n-j,1}^{k-j} \Big) e_k \\
&\quad + \Big(\frac{n(n-1)}{2} a_{n+1,1}^{n+1} + (n-1) a_{n+1,1}^{n+2} - \sum_{k=2}^{n-1} a_{k,1}^k \Big) e_n, \\
\varphi(e_1, e_2) &= a_{1,2}^1 e_1, \\
\varphi(e_i, e_2) &= ((i-2)a_{1,2}^1 + a_{2,2}^2) e_i + a_{2,2}^3 e_{i+1}, \ 2 \le i \le n, \\
\varphi(e_1, e_i) &= -a_{i-1,1}^{n+1} e_1, \ 3 \le i \le n, \\
\varphi(e_i, e_3) &= -((i-1)a_{2,1}^{n+1} + a_{2,1}^{n+2}) e_i - (a_{1,2}^1 + a_{2,1}^1) e_{i+1}, \ 2 \le i \le n,
\end{aligned}$$

$$
\begin{aligned}
\varphi(e_i,e_j) &= -((i-1)a_{j-1,1}^{n+1}+a_{j-1,1}^{n+2})e_i+(a_{j-2,1}^{n+1}-a_{j-1,1}^{1})e_{i+1}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4\leq j\leq n,\ 2\leq i\leq n,\\
\varphi(x,e_1) &= a_{n+1,1}^{1}e_1+a_{1,1}^{3}e_2+\sum_{k=3}^{n}a_{n+1,1}^{k}e_k+a_{n+1,1}^{n+1}x+a_{n+1,1}^{n+2}y,\\
\varphi(y,e_1) &= a_{n+2,1}^{1}e_1+\sum_{k=3}^{n}a_{n+2,1}^{k}e_k,\\
\varphi(x,e_2) &= -a_{2,2}^{3}e_1,\\
\varphi(x,e_3) &= (a_{1,2}^{1}+a_{2,1}^{1})e_1,\\
\varphi(x,e_i) &= (a_{i-1,1}^{1}-a_{i-2,1}^{n+1})e_1,\ 4\leq i\leq n,\\
\varphi(e_1,x) &= -a_{n+1,1}^{1}e_1+\sum_{k=3}^{n-1}(k-2)a_{1,1}^{k+1}e_k+(n-2)a_{1,n+2}^{n}e_n\\
&\quad -a_{n+1,1}^{n+1}x-a_{n+1,1}^{n+2}y,\\
\varphi(e_2,x) &= \sum_{k=2}^{n}a_{2,n+1}^{k}e_k-a_{1,2}^{1}x+(a_{1,2}^{1}-a_{2,2}^{2})y,\\
\varphi(e_3,x) &= (a_{1,2}^{1}+a_{2,1}^{1})e_1+(a_{2,1}^{2}-a_{n+1,1}^{n+1}-a_{n+1,1}^{n+2})e_2+(a_{2,n+1}^{2}-a_{n+1,1}^{1})e_3\\
&\quad +\sum_{k=4}^{n}\Big(a_{2,n+1}^{k-1}-(k-3)a_{2,1}^{k}\Big)e_k+2a_{2,1}^{n+1}x+2a_{2,1}^{n+2}y,\\
\varphi(e_i,x) &= (i-2)(a_{i-1,1}^{1}-a_{i-2,1}^{n+1})e_1+\sum_{k=2}^{i-2}(i-k)\sum_{j=1}^{k-1}a_{i-j,1}^{k-j+1}e_k\\
&\quad +\Big(\sum_{k=2}^{i-1}a_{k,1}^{k}-\frac{(i-1)(i-2)}{2}a_{n+1,1}^{n+1}-(i-2)a_{n+1,1}^{n+2}\Big)e_{i-1}\\
&\quad +\Big(a_{2,n+1}^{2}-(i-2)a_{n+1,1}^{1}\Big)e_i\\
&\quad +\sum_{k=i+1}^{n}\Big(a_{2,n+1}^{k-i+2}-(k-i)\sum_{j=2}^{i-1}a_{j,1}^{k-i+j+1}\Big)e_k\\
&\quad +(i-1)a_{i-1,1}^{n+1}x+(i-1)a_{i-1,1}^{n+2}y, \qquad 4\leq i\leq n,\\
\varphi(e_1,y) &= -a_{n+2,1}^{1}e_1+\sum_{k=2}^{n-1}a_{1,1}^{k+1}e_k+a_{1,n+2}^{n}e_n,
\end{aligned}
$$

$$
\begin{aligned}
\varphi(e_2,y) &= -a^3_{2,2}e_1 + a^2_{2,n+2}e_2 + a^3_{2,n+2}e_3 - a^1_{1,2}x + (a^1_{1,2} - a^2_{2,2})y,\\
\varphi(e_3,y) &= (a^1_{1,2} + a^1_{2,1})e_1 + (a^2_{2,n+2} - a^1_{n+2,1})e_3 + a^3_{2,n+2}e_4 + a^{n+1}_{2,1}x + a^{n+2}_{2,1}y,\\
\varphi(e_i,y) &= (a^1_{i-1,1} - a^{n+1}_{i-2,1})e_1 + (a^2_{2,n+2} - (i-2)a^1_{n+2,1})e_i + a^3_{2,n+2}e_{i+1}\\
&\quad + a^{n+1}_{i-1,1}x + a^{n+2}_{i-1,1}y,\ 4 \le i \le n,\\
\varphi(x,x) &= \sum_{k=2}^{n-2}(k-1)(a^{k+1}_{n+1,1} - a^{k+2}_{1,1})e_k + (n-2)(a^n_{n+1,1} - a^n_{1,n+2})e_{n-1}\\
&\quad + (n-1)a^n_{n+1,n+2}e_n,\\
\varphi(y,x) &= a^3_{2,n+2}e_1 + \sum_{k=2}^{n-1}(k-1)a^{k+1}_{n+2,1}e_k + (n-1)a^n_{n+2,n+2}e_n,\\
\varphi(x,y) &= -a^3_{2,n+2}e_1 + \sum_{k=2}^{n-2}(a^{k+1}_{n+1,1} - a^{k+2}_{1,1})e_k\\
&\quad + (a^n_{n+1,1} - a^n_{1,n+2})e_{n-1} + a^n_{n+1,n+2}e_n,\\
\varphi(y,y) &= \sum_{k=2}^{n-1}a^{k+1}_{n+2,1}e_k + a^n_{n+2,n+2}e_n.
\end{aligned}
$$

Proof. Let $\varphi \in ZL^2(R(F^1_n), R(F^1_n))$. We set $x_1 := x,\ x_2 := y$ and

$$
\begin{aligned}
\varphi(e_i,e_j) &= \sum_{k=1}^{n} a^k_{i,j}e_k + a^{n+1}_{i,j}x_1 + a^{n+2}_{i,j}x_2, && 1 \le i,j \le n,\\
\varphi(e_i,x_j) &= \sum_{k=1}^{n} a^k_{i,n+j}e_k + a^{n+1}_{i,n+j}x_1 + a^{n+2}_{i,n+j}x_2, && 1 \le i \le n,\ 1 \le j \le 2,\\
\varphi(x_j,e_i) &= \sum_{k=1}^{n} a^k_{n+j,i}e_k + a^{n+1}_{n+j,i}x_1 + a^{n+2}_{n+j,i}x_2, && 1 \le i \le n,\ 1 \le j \le 2,\\
\varphi(x_i,x_j) &= \sum_{k=1}^{n} a^k_{n+i,n+j}e_k + a^{n+1}_{n+i,n+j}x_1 + a^{n+2}_{n+i,n+j}x_2, && 1 \le i,j \le 2.
\end{aligned}
$$

For $\varphi \in ZL^2(R(F^1_n), R(F^1_n))$ we shall verify equation (2.1.1).

We consider $c = b$ in this equation, then we get $[a, \varphi(b,b)] = 0$ for all $a \in R(F^1_n)$. From the multiplication table of the algebra $R(F^1_n)$ it is easy to see that $\varphi(b,b) \in I$ for all $b \in R(F^1_n)$, where $I = \mathrm{span}\langle e_2, \ldots, e_n\rangle$.

If $b, c \in I$, then we obtain $[a, \varphi(b,c)] = 0$ for all $a \in R(F_n^1)$, and consequently, $\varphi(b,c) \in I$, i.e. $\varphi(I, I) \subseteq I$.

We consider the triples (a, b, c), with $a, b, c \in R(F_n^1)$ and apply the equation (2.1.1) for them.

If $(a, b, c) = (x_2, x_1, e_i)$, $i \neq 1$, then we derive $\varphi(x_2, I) \subseteq Q$, where Q is the complementary vector space of the nilradical F_n^1 to the algebra $R(F_n^1)$. The triple (e_i, x_2, e_j), $j \neq 1$, implies $[e_i, \varphi(x_2, e_j)] = 0$, from which we get $\varphi(x_2, I) = 0$.

If we take (e_i, x_1, e_j), $j \neq 1$, then we deduce $\varphi(x_1, e_j) \in \langle e_1, x_2 \rangle$.

By considering the triples (x_1, e_i, e_1) and (x_1, e_i, x_1), $i \neq 1$, we obtain $\varphi(x_1, I) \subseteq \langle e_1 \rangle$.

The triples (e_1, x_1, e_i), $i \neq 1$, imply $\varphi(e_1, I) \subseteq \langle e_1, e_2 \rangle$.

By considering the equation (2.1.1) for the triples (e_1, x_2, e_i), $i \neq 1$, we get $\varphi(e_1, I) \subseteq \langle e_1 \rangle$.

Similarly, we obtain

$$\begin{aligned}
(e_n, x_1, e_1) &\Rightarrow \ \varphi(e_n, e_1) \in F_n^1;\\
(x_1, x_2, e_1) &\Rightarrow \ \varphi(e_1, x_2) \in F_n^1;\\
(x_2, x_i, e_1),\ i = 1, 2 &\Rightarrow \ \varphi(x_2, e_1) \in F_n^1 \setminus \{e_2\};\\
(x_1, e_i, x_1),\ i \neq 1 &\Rightarrow \ \varphi(e_2, x_1) \in R(F_n^1) \setminus \{e_1\};\\
(e_1, e_1, x_2) &\Rightarrow \ \varphi(e_1, e_1) \in I \setminus \{e_2\}.
\end{aligned}$$

Given the restrictions above, by applying the multiplication of the algebra $R(F_n^1)$ and the property of cocycle for $(d^2\varphi)(e_1, e_i, e_1) = 0$, we obtain

$$\varphi(e_1, e_2) = a_{1,2}^1 e_1, \qquad \varphi(e_1, e_i) = -a_{i-1,1}^{n+1} e_1, \quad 3 \leq i \leq n.$$

In an analogous way, from $(d^2\varphi)(e_i, e_1, e_j) = 0$ and $(d^2\varphi)(e_i, e_j, e_1) = 0$, $1 \leq i, j \leq n$, we conclude

$$\varphi(e_1,e_1)=\sum_{k=3}^{n}a_{1,1}^{k}e_k,$$

$$\varphi(e_n,e_1)=a_{n-1,1}^{n+1}e_1+\sum_{k=2}^{n}a_{n,1}^{k}e_k,$$

$$\varphi(e_i,e_2)=((i-2)a_{1,2}^{1}+a_{2,2}^{2})e_i+\sum_{k=i+1}^{n}a_{2,2}^{k-i+2}e_k, \qquad 2\leq i\leq n,$$

$$\varphi(e_i,e_3)=-\big((i-1)a_{2,1}^{n+1}+a_{2,1}^{n+2}\big)e_i-(a_{1,2}^{1}+a_{2,1}^{1})e_{i+1}, \qquad 2\leq i\leq n,$$

$$\varphi(e_i,e_j)=-\big((i-1)a_{j-1,1}^{n+1}+a_{j-1,1}^{n+2}\big)e_i+(a_{j-2,1}^{n+1}-a_{j-1,1}^{1})e_{i+1}, \quad 2\leq i\leq n, \quad 4\leq j\leq n.$$

The equations $(d^2\varphi)(e_i,e_j,x_1)=0,\ 1\leq i,j\leq n$, imply $a_{2,2}^{k}=0$, for $4\leq k\leq n$, and

$$\varphi(e_1,x_1)=a_{1,n+1}^{1}e_1+\sum_{k=3}^{n-1}(k-2)a_{1,1}^{k+1}e_k+a_{1,n+1}^{n}e_n+a_{1,n+1}^{n+1}x_1+a_{1,n+1}^{n+2}x_2,$$

$$\varphi(e_2,x_1)=\sum_{k=2}^{n}a_{2,n+1}^{k}e_k-a_{1,2}^{1}x_1+(a_{1,2}^{1}-a_{2,2}^{2})x_2,$$

$$\varphi(e_3,x_1)=(a_{1,2}^{1}+a_{2,1}^{1})e_1+(a_{2,1}^{2}+a_{1,n+1}^{n+1}+a_{1,n+1}^{n+2})e_2+(a_{2,n+1}^{2}+a_{1,n+1}^{1})e_3$$
$$+\sum_{k=4}^{n}\big(a_{2,n+1}^{k-1}-(k-3)a_{2,1}^{k}\big)e_k+2a_{2,1}^{n+1}x_1+2a_{2,1}^{n+2}x_2,$$

$$\begin{aligned}
\varphi(e_i, x_1) &= (i-2)(a_{i-1,1}^{1} - a_{i-2,1}^{n+1})e_1 + \sum_{k=2}^{i-2}(i-k)\sum_{j=1}^{k-1} a_{i-j,1}^{k-j+1} e_k \\
&+ \Big(\sum_{k=2}^{i-1} a_{k,1}^{k} + \frac{(i-1)(i-2)}{2} a_{1,n+1}^{n+1} + (i-2)a_{1,n+1}^{n+2}\Big)e_{i-1} \\
&+ (a_{2,n+1}^{2} + (i-2)a_{1,n+1}^{1})e_i \\
&+ \sum_{k=i+1}^{n} \Big(a_{2,n+1}^{k-i+2} - (k-i)\sum_{j=2}^{i-1} a_{j,1}^{k-i+j+1}\Big)e_k \\
&+ (i-1)a_{i-1,1}^{n+1}x_1 + (i-1)a_{i-1,1}^{n+2}x_2, \qquad 4 \le i \le n, \\
\varphi(e_n, e_1) &= a_{n-1,1}^{n+1}e_1 - \sum_{k=3}^{n-1}\Big(\sum_{j=2}^{k-2} a_{n-j,1}^{k-j}\Big)e_k \\
&- \Big(\frac{n(n-1)}{2} a_{1,n+1}^{n+1} + (n-1)a_{1,n+1}^{n+2} + \sum_{k=2}^{n-1} a_{k,1}^{k}\Big)e_n.
\end{aligned}$$

The equations $(d^2\varphi)(e_1, x_1, e_1) = 0$ and $(d^2\varphi)(e_2, x_1, e_1) = 0$ imply

$$a_{1,n+1}^{i} = -a_{n+1,1}^{i}, \qquad i = 1, n+1, n+2.$$

From the conditions $(d^2\varphi)(e_i, x_1, e_j) = 0$ for $2 \le i, j \le n$, we conclude

$$\begin{aligned}
\varphi(x_1, e_2) &= -a_{2,2}^{3}e_1, \\
\varphi(x_1, e_3) &= (a_{1,2}^{1} + a_{2,1}^{1})e_1, \\
\varphi(x_1, e_i) &= (a_{i-1,1}^{1} - a_{i-2,1}^{n+1})e_1, \qquad 4 \le i \le n.
\end{aligned}$$

The equalities $(d^2\varphi)(e_i, e_j, x_2) = 0,\ 1 \le i, j \le n$, determine

$$\varphi(e_1, x_2) = a^1_{1,n+2}e_1 + \sum_{k=2}^{n-1} a^{k+1}_{1,1}e_k + a^n_{1,n+2}e_n,$$

$$\varphi(e_2, x_2) = -a^3_{2,2}e_1 + \sum_{k=2}^{n} a^k_{2,n+2}e_k - a^1_{1,2}x_1 + (a^1_{1,2} - a^2_{2,2})x_2,$$

$$\varphi(e_3, x_2) = (a^1_{1,2} + a^1_{2,1})e_1 + (a^2_{2,n+2} + a^1_{1,n+2})e_3 + \sum_{k=4}^{n} a^{k-1}_{2,n+2}e_k + a^{n+1}_{2,1}x_1 + a^{n+2}_{2,1}x_2,$$

$$\varphi(e_i, x_2) = (a^1_{i-1,1} - a^{n+1}_{i-2,1})e_1 + (a^2_{2,n+2} + (i-2)a^1_{1,n+2})e_i + \sum_{k=i+1}^{n} a^{k-i+2}_{2,n+2}e_k + a^{n+1}_{i-1,1}x_1 + a^{n+2}_{i-1,1}x_2, \qquad 4 \le i \le n.$$

From the equality $(d^2\varphi)(e_2, x_2, e_1) = 0$ we get $a^1_{1,n+2} = -a^1_{n+2,1}$.

Given the restrictions above, by using the property of cocycle for $(d^2\varphi)(x_i, x_j, e_1) = 0$ for $1 \le i, j \le 2$, we obtain

$$\varphi(x_1, e_1) = a^1_{n+1,1}e_1 + a^3_{1,1}e_2 + \sum_{k=3}^{n} a^k_{n+1,1}e_k + a^{n+1}_{n+1,1}x_1 + a^{n+2}_{n+1,1}x_2,$$

$$\varphi(x_1, x_1) = \sum_{k=2}^{n-2}(k-1)(a^{k+1}_{n+1,1} - a^{k+2}_{1,1})e_k + ((n-2)a^n_{n+1,1} - a^n_{1,n+1})e_{n-1} + a^n_{n+1,n+1}e_n,$$

$$\varphi(x_2, x_1) = a^1_{n+2,n+1}e_1 + \sum_{k=2}^{n-1}(k-1)a^{k+1}_{n+2,1}e_k + a^n_{n+2,n+1}e_n + a^{n+2}_{n+2,n+1}x_2,$$

$$\varphi(x_1, x_2) = a^1_{n+1,n+2}e_1 + \sum_{k=2}^{n-2}(a^{k+1}_{n+1,1} - a^{k+2}_{1,1})e_k + (a^n_{n+1,1} - a^n_{1,n+2})e_{n-1} + a^n_{n+1,n+2}e_n + a^{n+2}_{n+1,n+2}x_2,$$

$$\varphi(x_2, x_2) = \sum_{k=2}^{n-1} a^{k+1}_{n+2,1}e_k + a^n_{n+2,n+2}e_n.$$

By checking the general condition of cocycle for the other basis elements, we get the following

$$\begin{aligned}
(d^2\varphi)(e_1,x_1,x_2) = 0 \Rightarrow\ & a^n_{1,n+1} = (n-2)a^n_{1,n+2};\\
(d^2\varphi)(e_2,x_1,x_2) = 0 \Rightarrow\ & a^1_{n+1,n+2} = -a^3_{2,n+2},\\
& a^{n+2}_{n+1,n+2} = a^k_{2,n+2} = 0, \quad 4 \le k \le n;\\
(d^2\varphi)(e_2,x_2,x_1) = 0 \Rightarrow\ & a^1_{n+2,n+1} = a^3_{2,n+2}, \quad a^{n+2}_{n+2,n+1} = 0.
\end{aligned}$$

Considering the equations $(d^2\varphi)(x_i,x_j,x_k) = 0$ for $1 \le i,j,k \le 2$, we have

$$x_i\varphi(x_j,x_k) - \varphi(x_i,x_j)x_k + \varphi(x_i,x_k)x_j = 0,$$

from which we obtain

$$a^n_{n+1,n+1} = (n-1)a^n_{n+1,n+2}, \qquad a^n_{n+2,n+1} = (n-1)a^n_{n+2,n+2}.$$

Thus, we have a general form of the 2-cocycle φ. $\square$

Proposition 2.2.18 implies the following corollary.

Corollary 2.2.19. $\dim ZL^2(R(F_n^1),R(F_n^1)) = (n+2)^2 - 3$ *and* $\dim HL^2(R(F_n^1),R(F_n^1)) = 0$.

Thus, according to the results of the paper [7], we have the following theorem.

Theorem 2.2.20. *The algebra* $R(F_n^1)$ *is rigid.*

Chapter 3

Naturally graded Leibniz algebras with characteristic sequence $(n - m, m)$

It is known that the investigation of the finite-dimensional Lie algebras was reduced to the study of the nilpotent algebras. In this connection, it is natural to apply results and methods from the Lie algebra theory to the study of Leibniz algebras. Since the description of nilpotent Lie algebras is itself a boundless problem, the study of nilpotent Leibniz algebras must be accompanied by imposing additional conditions such as constraints on the index of nilpotency of the algebra, on the characteristic sequence, grading, etc. In studying naturally graded quasi-filiform Leibniz algebras [15], it was noted that, in contrast to the Lie case, the Leibniz algebras contain a class of n-dimensional algebras whose characteristic sequence is $(n-2, 2)$. The subsequent study of naturally graded algebras with characteristic sequence equal to $(n-3, 3)$ shows that the class of non-Lie Leibniz algebras in this case is sufficiently wide. The present chapter is devoted to non-Lie Leibniz algebras with characteristic sequence equal to $(n - m, m)$ for the case $m \geq 4$. We provide a description of such algebras in Section 3.2. Moreover, in Section 3.3 we obtain expressions for the changes of the parameters in the multiplication table of such algebras under an isomorphism; these expressions can be used to obtain a complete classification in fixed dimension and a given value of m.

3.1 Preliminary definitions and results

In this section, we present some necessary definitions and results.

Let L be a nilpotent Leibniz algebra of dimension n.

Definition 3.1.1. *The algebra L is said to be null-filiform if* $\dim L^i = (n+1) - i,\ 1 \leq i \leq n+1$.

It is readily seen from the definition that an algebra L being null-filiform is equivalent to that the algebra has the maximal index of nilpotency.

The following theorem asserts that, in each dimension, up to isomorphism, there exists a unique null-filiform Leibniz algebra.

Theorem 3.1.2 ([6]). *In any n-dimensional null-filiform Leibniz algebra L, there exists a basis $\{e_1, e_2, \ldots, e_n\}$ such that the multiplication in the algebra L has the following form:*

$$[e_i, e_1] = e_{i+1}, \qquad 1 \leq i \leq n-1$$

(the omitted products vanish).

Let L be an n-dimensional nilpotent Leibniz algebra, and let x be an arbitrary element of the set $L \setminus L^2$. For the nilpotent right multiplication operator R_x, we define the decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$ consisting of the dimensions of the Jordan blocks of the operator R_x. On the set of such sequences, we define the lexicographic order.

Definition 3.1.3. *The sequence $C(L) = \max_{x \in L \setminus L^2} C(x)$ is called the characteristic sequence of the algebra L.*

Example 3.1.4. *Let L be an n-dimensional Leibniz algebra. L is abelian if and only if $C(L) = (1, 1, \ldots, 1)$.*

Example 3.1.5. *An n-dimensional Leibniz algebra L is null-filiform if and only if $C(L) = (n)$.*

Definition 3.1.6. *An algebra L is said to be decomposable if there exist subalgebras M and N of the algebra L such that $L = M \oplus N$ and $[M, N] = [N, M] = 0$.*

We also need the following auxiliary lemmas.

Lemma 3.1.7 ([26]). *For an arbitrary polynomial P of degree less than n, the following equality holds:*

$$\sum_{i=0}^{n}(-1)^i C_n^i P(i) = 0.$$

Lemma 3.1.8 ([26]). *For arbitrary $a, n \in \mathbb{N}$, the following identity holds:*

$$\sum_{i=0}^{n}(-1)^i C_a^i C_{a+(n-i)-1}^{n-i} = 0.$$

3.2 Description of naturally graded Leibniz algebras with characteristic sequence $(n-m, m)$, $m \geq 4$

Taking into account results from [13,15], in what follows, we shall consider n-dimensional naturally graded Leibniz algebras with characteristic sequence $C(L) = (n-m, m)$, for $m \geq 4$.

The definition of the characteristic sequence of a Leibniz algebra implies the existence of a basis $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ such that the matrix of the right multiplication operator R_{e_1} has one of the following two forms:

$$\mathrm{I}: \begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix}, \qquad \mathrm{II}: \begin{pmatrix} J_m & 0 \\ 0 & J_{n-m} \end{pmatrix},$$

where $n - m \geq m$.

Definition 3.2.1. *A Leibniz algebra L is called an algebra of type I if there exists an element $e_1 \in L \setminus L^2$ such that the right multiplication operator R_{e_1} has a matrix of the form*

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix};$$

if R_{e_1} has a matrix of the second form, then L is called an algebra of type II.

Suppose that M and N are null-filiform Leibniz algebras with $\dim M = n-m$ and $\dim N = m$, respectively. Therefore, $C(M) = (n-m)$ and $C(N) = (m)$. It is readily verified that the decomposable algebra $L = M \oplus N$ has the characteristic sequence $C(L) = (n-m, m)$. In the following theorem, it is asserted that the decomposable Leibniz algebras whose characteristic sequence is equal to $C(L) = (n-m, m)$ consist only of the direct sum of two null-filiform Leibniz algebras.

Theorem 3.2.2. *Let L be a Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$. The algebra L is decomposable if and only if M and N are null-filiform Leibniz algebras with* $\dim M = n-m$, $\dim N = m$.

Proof. Necessity. Let L be a decomposable Leibniz algebra whose characteristic sequence is $C(L) = (n-m, m)$, i.e., there exist subalgebras M and N of the algebra L such that $L = M \oplus N$ and $[M, N] = [N, M] = 0$. Then there exists an element $a \in L$ such that $a = x + y$, $x \in M$, $y \in N$, and the matrix of the right multiplication operator R_a has the following form:

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix}.$$

Therefore, there exist bases $\{e_1, e_2, \dots, e_{n-m}\} \subseteq M$ and $\{f_1, \dots, f_m\} \subseteq N$ such that

$$[a, e_i] = e_{i+1}, \ \ 1 \le i \le n-m-1, \ \ [a, f_i] = f_{i+1}, \ \ 1 \le i \le m-1.$$

Hence we have

$$[x, e_i] = e_{i+1}, \ \ 1 \le i \le n-m-1, \ \ [y, f_i] = f_{i+1}, \ \ 1 \le i \le m-1.$$

Then the matrix of the restriction of the right multiplication operator to M (respectively, to N) has the form (J_{n-m}) (respectively, (J_m)). Thus, it follows from Example 3.1.5 that M and N are null-filiform algebras.

Sufficiency. Let M and N be null-filiform Leibniz algebras with $\dim M = n-m$ and $\dim N = m$. Then there exist bases $\{e_1, e_2, \dots, e_{n-m}\}$ and $\{f_1, \dots, f_m\}$ in M and N, respectively, such that

$$\begin{aligned} &[e_i, e_1] = e_{i+1}, && 1 \le i \le n-m-1, \\ &[f_i, f_1] = f_{i+1}, && 1 \le i \le m-1. \end{aligned}$$

Take an element $a = e_1 + f_1 \in L = M \oplus N$. Consider

$$R_a(x) = [x, a] = [x, e_1 + f_1] = [x, e_1] + [x, f_1] = R_{e_1}(x) + R_{f_1}(x).$$

Obviously,

$$R_a(e_i) = e_{i+1},\ 1 \leq i \leq n - m - 1 \text{ and } R_a(f_i) = f_{i+1},\ 1 \leq i \leq m - 1.$$

Then the matrix of the operator R_a has the following form:

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix};$$

and so, $C(a) = (n - m, m)$.

Suppose that there exists an element $y \in L \backslash L^2 : C(y) > C(a) = (n-m, m)$. Then $C(y) = (k_1, \ldots, k_s)$ satisfies $k_1 > n - m$. Therefore, there exists an element $z \in L$:

$$[\ldots[[z, \underbrace{y], y], \ldots, y]}_{k_1 \text{ times}} \neq 0.$$

Since the nilindex of L is $n-m$, i.e., $L^{n-m} = 0$, it follows that $k_1 \leq n-m$; a contradiction. But if

$$C(y) = (n - m, k_2, \ldots, k_s), \text{ where } \sum_{p=2}^{s} k_p = m,$$

then $k_2 > m$; a contradiction. Therefore, $C(L) = (n - m, m)$. $\square$

Theorem 3.2.2 provides a classification of the naturally graded decomposable Leibniz algebras with characteristic sequence equal to $(n-m, m),\ m \geq 4$. Further, we shall consider indecomposable Leibniz algebras.

Let L be an n-dimensional indecomposable naturally graded Leibniz algebra over a field $\mathbb{F}$. Suppose that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$.

Let us introduce the maps $A_{i,j}, B_{i,j} : \mathbb{F}^n \to \mathbb{F}$ as follows:

$$A_{i,j}(x) = \sum_{l=0}^{j-1} (-1)^l C_{j-1}^l x_{l+i}, \quad 2 \leq i + j \leq n,$$
$$B_{i,j}(x) = \sum_{l=0}^{m-i} (-1)^l C_{j-1}^l x_{l+i}, \quad 2 \leq i + j \leq n.$$

For these maps the following lemma is true.

Lemma 3.2.3. *For arbitrary $i, j \in \mathbb{N}$, the following equalities are valid:*

$$A_{i,j}(x) - A_{i+1,j}(x) = A_{i,j+1}(x),$$

$$B_{i,j}(x) - B_{i+1,j}(x) = B_{i,j+1}(x).$$

Proof. The proof is carried out by induction, making use of the equality $C_{j-1}^{l} + C_{j-1}^{l-1} = C_j^l$. $\square$

In the following theorem, the multiplication table of the Leibniz algebra of type I with characteristic sequence equal to $(n-m, m)$, $m \geq 4$, is provided.

Theorem 3.2.4. *Let L be a Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$ of type I. Then there exists a basis $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ of the algebra L in which the multiplication table has the following form:*

$$\begin{array}{ll}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-m-1, \\
[f_i, e_1] = f_{i+1}, & 1 \leq i \leq m-1, \\
[e_i, f_j] = A_{i,j}(\alpha)e_{i+j} + A_{i,j}(\beta)f_{i+j}, & 1 \leq i \leq m-j, \\
[e_i, f_j] = A_{i,j}(\alpha)e_{i+j}, & m-j+1 \leq i \leq n-m-j, \\
[f_i, f_j] = A_{i,j}(\gamma)e_{i+j} + A_{i,j}(\delta)f_{i+j}, & 1 \leq i \leq m-j, \\
[f_i, f_j] = B_{i,j}(\gamma)e_{i+j}, & m-j+1 \leq i \leq \min\{m, n-m-j\}
\end{array} \tag{3.2.1}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $1 \leq j \leq m$.

Proof. The condition

$$R_{e_1} = \begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix},$$

where $n-m \geq m$, implies that there exists a basis $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ such that

$$\begin{array}{lll}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-m-1, & [e_{n-m}, e_1] = 0, \\
[f_i, e_1] = f_{i+1}, & 1 \leq i \leq m-1, & [f_m, e_1] = 0.
\end{array}$$

It is readily verified that

$$\begin{array}{lll}
L_i = \langle e_i, f_i \rangle & \text{for} & 1 \leq i \leq m, \\
L_i = \langle e_i \rangle & \text{for} & m+1 \leq i \leq n-m,
\end{array}$$

$$\langle e_2, \dots, e_{n-m}\rangle \in \mathrm{Ann}_r(L).$$

Consider the multiplication in L by the element f_1 on the right. Let

$$\begin{array}{lll}
[e_i, f_1] = \alpha_i e_{i+1} + \beta_i f_{i+1}, & 1 \le i \le m-1, & \\
[e_i, f_1] = \alpha_i e_{i+1}, & m \le i \le n-m-1, & [e_{n-m}, f_1] = 0, \\
[f_i, f_1] = \gamma_i e_{i+1} + \delta_i f_{i+1}, & 1 \le i \le m-1, & [f_m, f_1] = \gamma_m e_{m+1}.
\end{array}$$

Applying induction on j for any value of i and the Leibniz identity for the products

$$[e_i, f_j],\ 1 \le i \le n-m \ \text{ and } \ [f_i, f_j],\ 1 \le i \le m,$$

we obtain the multiplication on the right side by the element $f_j, 2 \le j \le m$.

For $j = 2$, we consider all possible values of i and obtain the following products:

- $1 \le i \le m-2$,

$$\begin{aligned}
[e_i, f_2] = [e_i, [f_1, e_1]] &= [[e_i, f_1], e_1] - [e_{i+1}, f_1] \\
&= \alpha_i e_{i+2} + \beta_i f_{i+2} - (\alpha_{i+1} e_{i+2} + \beta_{i+1} f_{i+2}) \\
&= (\alpha_i - \alpha_{i+1}) e_{i+2} + (\beta_i - \beta_{i+1}) f_{i+2};
\end{aligned}$$

- $i = m-1$,

$$[e_{m-1}, f_2] = [e_{m-1}, [f_1, e_1]] = [[e_{m-1}, f_1], e_1] - [e_m, f_1]$$

$$= (\alpha_{m-1} e_m + \beta_{m-1} f_m) e_1 - \alpha_m e_{m+1} = (\alpha_{m-1} - \alpha_m) e_{m+1};$$

- $m \le i \le n-m-2$,

$$[e_i, f_2] = [e_i, [f_1, e_1]] = [[e_i, f_1], e_1] - [e_{i+1}, f_1]$$

$$= \alpha_i e_{i+2} - \alpha_{i+1} e_{i+2} = (\alpha_i - \alpha_{i+1}) e_{i+2};$$

- $i = n-m-1$ or $i = n-m$,

$$[e_i, f_2] = 0;$$

- $1 \le i \le m-2$,

$$[f_i, f_2] = [f_i, [f_1, e_1]] = [[f_i, f_1], e_1] - [f_{i+1}, f_1] = \gamma_i e_{i+2} + \delta_i f_{i+2}$$

$$-(\gamma_{i+1} e_{i+2} + \delta_{i+1} f_{i+2}) = (\gamma_i - \gamma_{i+1}) e_{i+2} + (\delta_i - \delta_{i+1}) f_{i+2};$$

- $i = m-1$,

$$[f_{m-1}, f_2] = [f_{m-1}, [f_1, e_1]] = [[f_{m-1}, f_1], e_1] - [f_m, f_1]$$

$$= (\gamma_{m-1}e_m + \delta_{m-1}f_m)e_1 - \gamma_m e_{m+1} = (\gamma_{m-1} - \gamma_m)e_{m+1};$$

- $i = m$,

$$[f_m, f_2] = [f_m, [f_1, e_1]] = [[f_m, f_1], e_1] - [[f_m, e_1], f_1] = \gamma_m e_{m+2}.$$

Thus, products (3.2.1) for $j = 2$ are obtained.

Suppose that products (3.2.1) are true for $j = q$, and let us prove them for $j = q+1$.

It follows from the Leibniz identity that

$$[e_i, f_{q+1}] = [e_i, [f_q, e_1]] = [[e_i, f_q], e_1] - [e_{i+1}, f_q],$$

$$[f_i, f_{q+1}] = [f_i, [f_q, e_1]] = [[f_i, f_q], e_1] - [f_{i+1}, f_q].$$

Using Lemma 3.2.3, for all i we obtain

- $1 \leq i \leq m-q-1$,

$$[e_i, f_{q+1}] = A_{i,q+1}(\alpha)e_{i+q+1} + A_{i,q+1}(\beta)f_{i+q+1};$$

- $i = m-q$,

$$[e_{m-q}, f_{q+1}] = A_{m-q,q+1}(\alpha)e_{m+1};$$

- $m-q+1 \leq i \leq n-m-q-1$,

$$[e_i, f_{q+1}] = A_{i,q+1}(\alpha)e_{i+q+1};$$

- $i = n-m-q$,

$$[e_{n-m-q}, f_{q+1}] = A_{n-m-q,q}(\alpha)e_{n-m}e_1 = 0;$$

- $n-m-q+1 \leq i \leq n-m$,

$$[e_i, f_{q+1}] = 0.$$

Similarly, for the products of the form $[f_i, f_j]$, we obtain

- $1 \leq i \leq m-q-1$,
$$[f_i, f_{q+1}] = A_{i,q+1}(\gamma)e_{i+q+1} + A_{i,q+1}(\delta)f_{i+q+1};$$
- $i = m-q$,
$$[f_{m-q}, f_{q+1}] = B_{m-q,q+1}(\gamma)e_{m+1};$$
- $m-q+1 \leq i \leq \min\{n-m-q-1, m\}$,
$$[f_i, f_{q+1}] = B_{i,q+1}(\gamma)e_{i+q+1}.$$

$\square$

Thus, from Theorem 3.2.4, we obtain the following set of parameters in the structure constants defining the multiplication table of the algebra:

$$\begin{array}{llllllllll}
\alpha_1, & \dots, & \alpha_{k-1}, & \alpha_k, & \alpha_{k+1}, & \dots, & \alpha_{m-1}, & \alpha_m, & \alpha_{m+1}, & \dots, & \alpha_{n-m-1}, \\
\beta_1, & \dots, & \beta_{k-1}, & \beta_k, & \beta_{k+1}, & \dots, & \beta_{m-1}, & & & & \\
\gamma_1, & \dots, & \gamma_{k-1}, & \gamma_k, & \gamma_{k+1}, & \dots, & \gamma_{m-1}, & \gamma_m, & & & \\
\delta_1, & \dots, & \delta_{k-1}, & \delta_k, & \delta_{k+1}, & \dots, & \delta_{m-1}. & & & &
\end{array}$$

For finding of values of these parameters let $f_k \notin \mathrm{Ann}_r(L)$ and $f_{k+1} \in \mathrm{Ann}_r(L)$ for some fixed number k, $1 \leq k \leq m-1$. Then, for different k, we obtain non-intersecting classes. Indeed, the dimensions of the right annihilators of algebras from these classes will differ.

In the following theorem we will find values of β_i, $1 \leq i \leq k-1$, and the expressions for the structure constants $\beta_{k+t}, 1 \leq t \leq m-k-1$, in terms of β_k.

Theorem 3.2.5. *Let L be a Leibniz algebra of type I. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some $k \in \{1, 2, \dots, m-1\}$. Then the following relations hold:*

$$\beta_1 = -1, \;\; \beta_i = 0, \;\; 2 \leq i \leq k-1,$$

$$\beta_{k+t} = C_{k+t-1}^{k-1}\beta + (-1)^k C_{k+t-2}^{k-1}, \;\; 1 \leq t \leq m-k-1, \tag{3.2.2}$$

where $\beta := \beta_k$.

Proof. For some $k \in \{1, 2, \dots, m-1\}$ let $f_k \notin \mathrm{Ann}_r(L)$ and $f_{k+1} \in \mathrm{Ann}_r(L)$. Obviously,

$$[f_i, e_1] + [e_1, f_i] \in \mathrm{Ann}_r(L), \quad 1 \leq i \leq k-1.$$

Since

$$[f_i, e_1] + [e_1, f_i] = f_{i+1} + A_{1,i}(\alpha) e_{i+1} + A_{1,i}(\beta) f_{i+1}$$

and $\langle e_2, \dots, e_{n-m} \rangle \in \mathrm{Ann}_r(L)$, we obtain $1 + A_{1,i}(\beta) = 0$.

For $i = 1$, we have $1 + A_{1,1}(\beta) = 1 + \beta_1 = 0$ and, therefore, $\beta_1 = -1$.

For $i > 1$, using the equality

$$1 + A_{1,i}(\beta) = 1 + \beta_1 + \sum_{l=1}^{i-1} (-1)^l C_{i-1}^l \beta_{l+1} = 0$$

we obtain

$$\sum_{l=1}^{i-1} (-1)^l C_{i-1}^l \beta_{l+1} = 0, \;\; i > 2.$$

This yields

$$\beta_i = (-1)^i \sum_{l=1}^{i-2} (-1)^l C_{i-1}^l \beta_{l+1} = 0, \;\; i > 2.$$

Using the resulting relation, we shall show by induction on i that $\beta_i = 0$ for all $2 \leq i \leq k-1$. Indeed, for $i = 2$ we have $\beta_2 = 0$, and for $i = 3$ the equality $-2\beta_2 + \beta_3 = 0$ implies $\beta_3 = 0$.

Suppose that $\beta_i = 0$ for $i = q < k-1$. Then, using

$$\sum_{l=1}^{q} (-1)^l C_q^l \beta_{l+1} = \sum_{l=1}^{q-1} (-1)^l C_q^l \beta_{l+1} + (-1)^q \beta_{q+1} = 0,$$

we obtain $\beta_{q+1} = 0$.

Thus, we have proved $\beta_1 = -1$, $\beta_2 = 0, \dots, \beta_{k-1} = 0$.

Now consider the products $[e_i, f_{k+1}]$, $1 \leq i \leq m-k-1$.

The condition $f_{k+1} \in \mathrm{Ann}_r(L)$ implies that

$$[e_1, f_{k+1}] = 0 \;\text{ and }\; A_{1,k+1}(\alpha) e_{k+2} + A_{1,k+1}(\beta) f_{k+2} = 0.$$

Therefore,

$$\begin{cases} A_{1,k+1}(\alpha) = 0, \\ A_{1,k+1}(\beta) = 0. \end{cases}$$

Since

$$A_{1,k+1}(\beta) = \sum_{l=0}^{k}(-1)^l C_k^l \beta_{l+1} = \beta_1 + 0 + \cdots + 0 + (-1)^{k-1}k\beta_k + (-1)^k\beta_{k+1},$$

we have $\beta_{k+1} = k\beta + (-1)^k$, where $\beta := \beta_k$.

Similarly, considering the products of the form $[e_i, f_{k+1}],\ 2 \leq i \leq m-k-1$, we conclude $A_{i,k+1}(\beta) = 0$. Let us prove the following dependence by induction:

$$\beta_{k+t} = C_{k+t-1}^{k-1}\beta + (-1)^k C_{k+t-2}^{k-1},\ 1 \leq t \leq m-k-1.$$

The base case of the induction was obtained before. It follows from the condition

$$A_{i,k+1}(\beta) = \sum_{l=0}^{k}(-1)^l C_k^l \beta_{l+i}$$

that

$$\beta_{k+i} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_k^l \beta_{l+i},\ \ 2 \leq i \leq m-k-1.$$

Suppose that the equality

$$\beta_{k+i} = C_{k+i-1}^{k-1}\beta + (-1)^k C_{k+i-2}^{k-1}$$

is valid for all $1 \leq i \leq q-1,\ 2 \leq q \leq m-k-1$. Let us prove its validity for $i = q$. Using the equality $A_{i,k+1}(\beta) = 0$, we obtain

$$\sum_{l=0}^{k}(-1)^l C_k^l \beta_{l+q} = (-1)^{k-q} C_k^{k-q}\beta \\ + \sum_{l=1-q}^{-1}(-1)^{l+k} C_k^{k+l}\beta_{k+(l+q)} + (-1)^k\beta_{k+q} = 0.$$

Using Lemma 3.1.8 in the following equalities:

$$\beta_{k+q} = (-1)^{k+1} \cdot \Big[(-1)^{k-q} C_k^{k-q} \beta$$

$$+ (-1)^k \sum_{l=1-q}^{-1} (-1)^l C_k^{l+k} \Big(C_{k+l+q-1}^{k-1} \beta + (-1)^k C_{k+l+q-2}^{k-1} \Big) \Big]$$

$$= (-1)^{1-q} C_k^{k-q} \beta - \sum_{l=1-q}^{-1} (-1)^l \left(C_k^{k+l} C_{k+l+q-1}^{k-1} \beta + (-1)^k C_k^{k+l} C_{k+l+q-2}^{k-1} \right)$$

$$= (-1)^{1-q} C_k^{k-q} \beta - \sum_{l=1-q}^{-1} (-1)^l C_k^{k+l} C_{k+l+q-1}^{k-1} \beta$$

$$- (-1)^k \sum_{l=1-q}^{-1} (-1)^l C_k^{k+l} C_{k+l+q-2}^{k-1}$$

$$= -\beta \sum_{l=-q}^{-1} (-1)^l C_k^{k+l} C_{k+l+q-1}^{k-1} - (-1)^k \sum_{l=1-q}^{-1} (-1)^l C_k^{k+l} C_{k+l+q-2}^{k-1}$$

$$= -\beta \sum_{l=1}^{q} (-1)^l C_k^{k-l} C_{k-l+q-1}^{k-1} - (-1)^k \sum_{l=1}^{q-1} (-1)^l C_k^{k-l} C_{k-l+q-2}^{k-1}$$

$$= \beta\, C_{k+q-1}^{k-1} - \beta \sum_{l=0}^{q} (-1)^l C_k^l C_{k-l+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1}$$

$$- (-1)^k \sum_{l=0}^{q-1} (-1)^l C_k^l C_{k-l+q-2}^{k-1} = \beta C_{k+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1},$$

we find that

$$\beta_{k+q} = \beta C_{k+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1}.$$

Therefore, for $i = q$, Formula (3.2.2) is also valid. □

Theorem 3.2.5 shows that the constants $\beta_{k+1}, \ldots, \beta_{m-1}$ are expressed in terms of β_k. In the subsequent theorems, we show that the constants $\alpha_i, \gamma_i, \delta_i,\ i \geq k+1$, can also be linearly expressed in terms of $\alpha_i, \gamma_i, \delta_i,\ 1 \leq i \leq k$.

Let us present the following lemma.

Lemma 3.2.6. *For arbitrary $i, k, l \in \mathbb{N}$, the following equality holds:*

$$C_{k+i}^{l} - \sum_{p=0}^{i-1} (-1)^p C_{k+i}^{k+p} C_{k+p}^{l} C_{k-l+p-1}^{k-l-1} = (-1)^i C_{k+i}^{l} C_{k-l+i-1}^{k-l-1}. \qquad (3.2.3)$$

Proof. The proof is carried out by induction, making use of Lemma 3.1.8. □

Theorem 3.2.7. *Let L be a Leibniz algebra of type I. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some k $(1 \le k \le m-1)$. Then*

$$\alpha_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \alpha_{l+1}, \ 1 \le t \le m-k-1. \qquad (3.2.4)$$

Proof. From the assumptions of the theorem it follows that $[e_1, f_{k+t}] = 0$ for $1 \le t \le m-k-1$. Then

$$A_{1,k+t}(\alpha) = \sum_{l=0}^{k+t-1} (-1)^l C_{k+t-1}^{l} \alpha_{l+1} = 0,$$

from where

$$\alpha_{k+t} = (-1)^{k+t} \sum_{l=0}^{k+t-2} (-1)^l C_{k+t-1}^{l} \alpha_{l+1}, \ 1 \le t \le m-k-1.$$

Using the resulting relations, let us prove equalities (3.2.4) by induction.
For $t = 1$, we obviously have

$$\alpha_{k+1} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k}^{l} \alpha_{l+1}.$$

For $t \le q < m-k-1$, we suppose that relations (3.2.4) hold. Let us prove them for $t = q+1$.

In view of the relations

$$\alpha_{k+t} = (-1)^{k+t} \sum_{l=0}^{k+t-2} (-1)^l C_{k+t-1}^{l} \alpha_{l+1},$$

$$\alpha_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^l C_{k-l+t-2}^{k-l-1}\alpha_{l+1}, \ \ t \le q,$$

and equality (3.2.3) from the chain of equalities

$$\begin{aligned}
\alpha_{k+q+1} &= (-1)^{k+q+1}\sum_{l=0}^{k+q-1}(-1)^l C_{k+q}^l\alpha_{l+1}\\
&= (-1)^{k+q+1}\Big(\sum_{l=0}^{k-1}(-1)^l C_{k+q}^l\alpha_{l+1} + \sum_{l=k}^{k+q-1}(-1)^l C_{k+q}^l\alpha_{l+1}\Big)\\
&= (-1)^{k+q+1}\Big(\sum_{l=0}^{k-1}(-1)^l C_{k+q}^l\alpha_{l+1} + \sum_{l=0}^{q-1}(-1)^{l+k} C_{k+q}^{l+k}\alpha_{l+k+1}\Big)\\
&= (-1)^{k+q+1}\sum_{l=0}^{k-1}(-1)^l C_{k+q}^l\alpha_{l+1}\\
&+ (-1)^{k+q+1}\sum_{l=0}^{q-1}(-1)^{l+k}C_{k+q}^{k+l}\Big((-1)^{k+1}\sum_{p=0}^{k-1}(-1)^p C_{k+l}^p C_{k-p+l-1}^{k-p-1}\alpha_{p+1}\Big)\\
&= (-1)^{k+q+1}\Big[\sum_{l=0}^{k-1}(-1)^l C_{k+q}^l\alpha_{l+1}\\
&- \sum_{p=0}^{q-1}(-1)^p C_{k+q}^{k+p}\sum_{l=0}^{k-1}(-1)^l C_{k+p}^l C_{k-l+p-1}^{k-l-1}\alpha_{l+1}\Big]\\
&= (-1)^{k+q+1}\sum_{l=0}^{k-1}(-1)^l\Big[C_{k+q}^l - \sum_{p=0}^{q-1}(-1)^p C_{k+q}^{k+p}C_{k+p}^l C_{k-l+p-1}^{k-l-1}\Big]\alpha_{l+1}\\
&= (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+q}^l C_{k-l+q-1}^{k-l-1}\alpha_{l+1}
\end{aligned}$$

we establish that for $t = q+1$ relations (3.2.4) hold; thus, the theorem is proved. □

Remark 3.2.8. *Using Theorem 3.2.7, we have obtained the expression for the structure constants* $\alpha_{k+1}, \dots, \alpha_{m-1}$ *in terms of* $\alpha_1, \dots, \alpha_k$*. In the case* $n - m > m$*, the parameters* $\alpha_m, \dots, \alpha_{n-m-1}$ *are found from the equalities*

$[e_i, f_{k+t}] = 0$ *for* $1 \leq i \leq n-2m,\ 1 \leq t \leq m-k$. *Thus, relations* (3.2.4) *are extended to the case* $m-k \leq t \leq n-m-k-1$, *i.e.,*

$$\alpha_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C^l_{k+t-1}C^{k-l-1}_{k-l+t-2}\alpha_{l+1},\ 1 \leq t \leq n-m-k-1.$$

Considering the identities $[f_1, f_{k+t}] = 0,\ 1 \leq t \leq m-k$, analogously to Theorem 3.2.7 the following theorem is proved.

Theorem 3.2.9. *Let* L *be a Leibniz algebra of type I. Let* $f_k \notin \mathrm{Ann}_r(L),\ f_{k+1} \in \mathrm{Ann}_r(L)$ *for some* k $(1 \leq k \leq m-1)$. *Then*

$$\gamma_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C^l_{k+t-1}C^{k-l-1}_{k-l+t-2}\gamma_{l+1},\ 1 \leq t \leq m-k,$$

$$\delta_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C^l_{k+t-1}C^{k-l-1}_{k-l+t-2}\delta_{l+1},\ 1 \leq t \leq m-k-1.$$

Remark 3.2.10. *Using Theorem 3.2.9, we obtain the expression for* δ_{k+t} *in terms of the constants* $\delta_1, \delta_2, \ldots, \delta_k$. *Taking into account the condition* $[f_i, f_1] + [f_1, f_i] \in \mathrm{Ann}_r(L),\ 1 \leq i \leq m$, *it is easy to see that all the odd* δ_i *are linearly expressed in terms of the even ones, i.e., the number of free parameters decreases twofold.*

Let L be a naturally graded Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$ of type II.

Theorem 3.2.11. *Let* L *be a Leibniz algebra with characteristic sequence* $C(L) = (n-m, m)$ *of type II. Then there exists a basis* $\{e_1, e_2, \ldots, e_m, f_1, \ldots, f_{n-m}\}$ *of* L *such that the table of multiplications in the algebra has the following form:*

$$\begin{array}{ll}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq m-1,\\
[f_i, e_1] = f_{i+1}, & 1 \leq i \leq n-m-1,\\
[e_i, f_j] = A_{i,j}(\alpha)e_{i+j} + A_{i,j}(\beta)f_{i+j}, & 1 \leq i \leq m-j,\\
[e_i, f_j] = B_{i,j}(\beta)f_{i+j}, & m-j+1 \leq i \leq \min\{m, n-m-j\},\\
[f_i, f_j] = A_{i,j}(\gamma)e_{i+j} + A_{i,j}(\delta)f_{i+j}, & 1 \leq i \leq m-j,\\
[f_i, f_j] = A_{i,j}(\delta)f_{i+j}, & m-j+1 \leq i \leq n-m-j
\end{array}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *and* $1 \leq j \leq n-m$.

Proof. The proof is carrying out by applying similar arguments as in the proof Theorem 3.2.4. □

Thus, we have obtained the following collection of parameters in the structure constants defining the algebra:

$$\begin{array}{llllllllll}
\alpha_1, & \dots, & \alpha_{k-1}, & \alpha_k, & \alpha_{k+1}, & \dots, & \alpha_{m-1}, & & & \\
\beta_1, & \dots, & \beta_{k-1}, & \beta_k, & \beta_{k+1}, & \dots, & \beta_{m-1}, & \beta_m, & & \\
\gamma_1, & \dots, & \gamma_{k-1}, & \gamma_k, & \gamma_{k+1}, & \dots, & \gamma_{m-1}, & & & \\
\delta_1, & \dots, & \delta_{k-1}, & \delta_k, & \delta_{k+1}, & \dots, & \delta_{m-1}, & \delta_m, & \delta_{m+1}, & \dots, \ \delta_{n-m-1}.
\end{array}$$

Further we specify the relations among these parameters.

Theorem 3.2.12. *Let L be a Leibniz algebra of type II. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some k, $1 \le k \le m-1$. Then*

$$\beta_1 = -1, \quad \beta_i = 0, \ 2 \le i \le k-1,$$
$$\beta_{k+t} = C_{k+t-1}^{k-1}\beta + (-1)^k C_{k+t-2}^{k-1}, \ 1 \le t \le m-k,$$

where $\beta := \beta_k$.

Proof. The proof is similar to the demonstration of Theorem 3.2.5. □

Applying the same arguments as in Theorem 3.2.7, we obtain the following theorem.

Theorem 3.2.13. *Let L be a Leibniz algebra of type II. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some k, $1 \le k \le m-1$. Then*

$$\alpha_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1}\alpha_{l+1}, \quad 1 \le t \le m-k-1,$$
$$\gamma_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1}\gamma_{l+1}, \quad 1 \le t \le m-k-1,$$
$$\delta_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1}\delta_{l+1}, \quad 1 \le t \le m-k.$$

Remark 3.2.14. *If a number k is such that $m \le k \le n-m-1$, then all the fixed constants α_i, γ_i, $1 \le i \le m-1$, and β_i, $1 \le i \le m$, remain as* *free parameters. Therefore, it suffices to find the dependence only for δ_i, $k+1 \leq i \leq n-m-1$, in terms of δ_i, $1 \leq i \leq k$. In Theorem 3.2.12, the constants δ_{k+t}, $1 \leq t \leq m-k$, are linearly expressed in terms of $\delta_1, \dots, \delta_k$ for $1 \leq k \leq m-1$. We can extend the relation for δ_{k+t} from Theorem 3.2.12 in the interval $1 \leq k \leq n-m-1$. Then we obtain*

$$\delta_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \delta_{l+1} \; 1 \leq t \leq n-m-k-1.$$

Moreover, similar to Remark 3.2.10, we conclude that the odd constants δ_i are expressed in terms of even ones.

3.3 On transformations of basis of naturally graded Leibniz algebras with characteristic sequence $(n-m, m)$, $m \geq 4$

Let L be an n-dimensional naturally graded Leibniz algebra whose characteristic sequence is $C(L) = (n-m, m)$, $m \geq 4$, of type I, and let $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ be a basis of L. Then from Theorem 3.2.4 we have that the multiplication in L is defined by the equalities (3.2.1). Therefore, the classification problem can be reduced to the problem of finding the structure constants $\alpha_i, \beta_i, \gamma_i$ and δ_i for $1 \leq i \leq k$.

Proposition 3.3.1. *Let L be a Leibniz algebra of type I. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some k, $1 \leq k \leq m-1$. Then the general transformation of basis can be reduced to the following form: $e'_i = A_i e_i + B_i f_i$, $f'_i = C_i e_i + D_i f_i$, $1 \leq i \leq k$, where*

$$A_2 = A_1^2 + A_1 B_1 \alpha_1 + B_1^2 \gamma_1, \;\; B_2 = B_1^2 \delta_1,$$

$$C_2 = A_1 C_1 + B_1 C_1 \alpha_1 + B_1 D_1 \gamma_1, \;\; D_2 = A_1 D_1 - B_1 C_1 + B_1 D_1 \delta_1,$$

$$\begin{aligned}
A_i &= A_{i-1}(A_1 + B_1\alpha_{i-1}) + B_{i-1}B_1\gamma_{i-1},\\
B_i &= B_{i-1}(A_1 + B_1\delta_{i-1}) = B_1\prod_{l=1}^{i-2}(A_1 + B_1\delta_{l+1}),\\
C_i &= C_{i-1}(A_1 + B_1\alpha_{i-1}) + D_{i-1}B_1\gamma_{i-1},\\
D_i &= D_{i-1}(A_1 + B_1\delta_{i-1}) = D_1\prod_{l=1}^{i-2}(A_1 + B_1\delta_{l+1}).
\end{aligned} \tag{3.3.1}$$

Proof. Let us consider the general transformation of the basic elements. It is well known that for naturally graded Leibniz algebras it suffices to consider the transformation

$$e'_1 = A_1e_1 + B_1f_1, \quad f'_1 = C_1e_1 + D_1f_1.$$

The proof of the statement is concluded by using the products $[e'_i, e'_1] = e'_{i+1}$ and $[f'_i, e'_1] = f'_{i+1}$. □

In the general transformation of the basis elements, the new parameters $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i$ are expressed via the initial constants $\alpha_i, \beta_i, \gamma_i, \delta_i$ in the following theorem.

Theorem 3.3.2. *Let L be a Leibniz algebra of type I. Let $f_k \notin \mathrm{Ann}_r(L)$, $f_{k+1} \in \mathrm{Ann}_r(L)$ for some k, $1 \le k \le m-1$. Then, by applying the general transformation of basis, the parameters $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i$, $1 \le i \le k$, have the form*

$$\alpha'_i = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))\,D_{i+1} - B_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ 1 \le i \le k-1,$$

$$\beta'_i = \frac{A_{i+1}B_i(C_1 + \delta_iD_1) - (A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))\,B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ 1 \le i \le k-1,$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))\,D_{i+1} - D_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ 1 \le i \le k-1,$$

$$\delta'_i = \frac{A_{i+1}D_i(C_1 + \delta_iD_1) - (C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))\,B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, 1 \le i \le k-1,$$

$$\alpha'_k = \frac{(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))\,(D_{k+1} + \beta_kB_1C_k) - (B_kC_1 + D_1(\beta_kA_k + \delta_kB_k))\,C_{k+1}}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_kB_1(A_{k+1}C_k - A_kC_{k+1})},$$

$$\beta'_k = \frac{A_{k+1}\,(C_1B_k + D_1(\beta_kA_k + \delta_kB_k)) - (B_{k+1} + \beta_kA_kB_1)\,(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_kB_1(A_{k+1}C_k - A_kC_{k+1})},$$

$$\gamma_k' = \frac{(C_1C_k + D_1(\alpha_kC_k + \gamma_kD_k))\,(D_{k+1} + \beta_kB_1C_k) - (C_1D_k + D_1(\beta_kC_k + \delta_kD_k))\,C_{k+1}}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_kB_1(A_{k+1}C_k - A_kC_{k+1})},$$

$$\delta_k' = \frac{A_{k+1}\,(C_1D_k + D_1(\beta_kC_k + \delta_kD_k)) - (B_{k+1} + \beta_kA_kB_1)\,(C_kC_1 + D_1(\alpha_kC_k + \gamma_kD_k))}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_kB_1(A_{k+1}C_k - A_kC_{k+1})},$$

where A_i, B_i, C_i, D_i *satisfy relations* (3.3.1).

Proof. From Proposition 3.3.1 it follows that the general transformation of basis can be replaced with an equivalent transformation of basis of the form

$$e_i' = A_ie_i + B_if_i,\ 1 \le i \le k,$$

$$f_i' = C_ie_i + D_if_i,\ 1 \le i \le k,$$

where the A_i, B_i, C_i, D_i are defined by identities (3.3.1).

For $1 \le i \le k-1$ we have

$$[e_i', f_1'] = (A_ie_i + B_if_i)(C_1e_1 + D_1f_1)$$

$$= (A_iC_1 + \alpha_iA_iD_1 + \gamma_iB_iD_1)e_{i+1} + (B_iC_1 + \beta_iA_iD_1 + \delta_iB_iD_1)f_{i+1}.$$

On the other hand,

$$[e_i', f_1'] = \alpha_i'e_{i+1}' + \beta_i'f_{i+1}' = \alpha_i'(A_{i+1}e_{i+1} + B_{i+1}f_{i+1}) + \beta_i'(C_{i+1}e_{i+1} + D_{i+1}f_{i+1}).$$

Thus, we obtain the system of equalities

$$\begin{cases} A_{i+1}\alpha_i' + C_{i+1}\beta_i' = A_iC_1 + \alpha_iA_iD_1 + \gamma_iB_iD_1, \\ B_{i+1}\alpha_i' + D_{i+1}\beta_i' = B_iC_1 + \beta_iA_iD_1 + \delta_iB_iD_1; \end{cases}$$

that yield

$$\alpha_i' = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))\,D_{i+1} - (B_iC_1 + D_1(\beta_iA_i + \delta_iB_i))\,C_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}},$$

$$\beta_i' = \frac{(B_iC_1 + D_1(\beta_iA_i + \delta_iB_i))\,A_{i+1} - (A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))\,B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}.$$

Similarly, from the equalities

$$[f_i', f_1'] = (C_ie_i + D_if_i)(C_1e_1 + D_1f_1)$$

$$= (C_iC_1 + \alpha_iC_iD_1 + \gamma_iD_iD_1)e_{i+1} + (D_iC_1 + \beta_iC_iD_1 + \delta_iD_iD_1)f_{i+1},$$

$$[f'_i, f'_1] = \gamma'_ie'_{i+1} + \delta'_if'_{i+1} = \gamma'_i(A_{i+1}e_{i+1} + B_{i+1}f_{i+1}) + \delta'_i(C_{i+1}e_{i+1} + D_{i+1}f_{i+1}),$$

we obtain

$$\begin{cases} A_{i+1}\gamma'_i + C_{i+1}\delta'_i = C_iC_1 + \alpha_iC_iD_1 + \gamma_iD_iD_1, \\ B_{i+1}\gamma'_i + D_{i+1}\delta'_i = D_iC_1 + \beta_iC_iD_1 + \delta_iD_iD_1, \end{cases}$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))\,D_{i+1} - (D_iC_1 + D_1(\beta_iC_i + \delta_iD_i))\,C_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}},$$

$$\delta'_i = \frac{(D_iC_1 + D_1(\beta_iC_i + \delta_iD_i))\,A_{i+1} - (C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))\,B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}.$$

For $i = k$, the following relations are also valid:

$$\begin{aligned} e'_k &= A_ke_k + B_kf_k, \\ f'_k &= C_ke_k + D_kf_k. \end{aligned}$$

We have

$$e'_{k+1} = [e'_k, e'_1] = A_{k+1}e_{k+1} + (B_{k+1} + \beta_kA_kB_1)f_{k+1},$$

$$f'_{k+1} = [f'_k, e'_1] = C_{k+1}e_{k+1} + (D_{k+1} + \beta_kB_1C_k)f_{k+1}.$$

Consider

$$[e'_k, f'_1] = (A_kC_1 + \alpha_kA_kD_1 + \gamma_kB_kD_1)e_{k+1} + (B_kC_1 + \beta_kA_kD_1 + \delta_kB_kD_1)f_{k+1}.$$

On the other hand,

$$\begin{aligned} [e'_k, f'_1] &= \alpha'_ke'_{k+1} + \beta'_kf'_{k+1} \\ &= \alpha'_k(A_{k+1}e_{k+1} + (B_{k+1} + \beta_kA_kB_1)f_{k+1}) \\ &\quad + \beta'_k(C_{k+1}e_{k+1} + (D_{k+1} + \beta_kB_1C_k)f_{k+1}). \end{aligned}$$

Therefore,

$$\begin{cases} A_{k+1}\alpha'_k + C_{k+1}\beta'_k = A_kC_1 + \alpha_kA_kD_1 + \gamma_kB_kD_1, \\ (B_{k+1} + \beta_kA_kB_1)\alpha'_k + (D_{k+1} + \beta_kB_1C_k)\beta'_k = B_kC_1 + \beta_kA_kD_1 + \delta_kB_kD_1, \end{cases}$$

from where we find the constants α'_k, β'_k:

$$\alpha'_k = \frac{(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))\,(D_{k+1} + \beta_kB_1C_k) - (B_kC_1 + D_1(\beta_kA_k + \delta_kB_k))\,C_{k+1}}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)},$$

$$\beta'_k = \frac{A_{k+1}\,(B_kC_1 + D_1(\beta_kA_k + \delta_kB_k)) - (B_{k+1} + \beta_kA_kB_1)\,(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)}.$$

Similarly, from the product $[f'_k, f'_1]$, we derive the following identities:

$$\gamma'_k = \frac{(C_1C_k + D_1(\alpha_kC_k + \gamma_kD_k))\,(D_{k+1} + \beta_kB_1C_k) - (C_1D_k + D_1(\beta_kC_k + \delta_kD_k))\,C_{k+1}}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)},$$

$$\delta'_k = \frac{A_{k+1}\,(C_1D_k + D_1(\beta_kC_k + \delta_kD_k)) - (B_{k+1} + \beta_kA_kB_1)\,(C_kC_1 + D_1(\alpha_kC_k + \gamma_kD_k))}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)}.$$

□

Proposition 3.3.1 and Theorem 3.3.2 are also valid for Leibniz algebras of type II for the case $k,\ 1 \le k \le m-1$. We need to consider the case of Leibniz algebras of type II for $k,\ m \le k \le n-m-1$. Using the same arguments for Leibniz algebras of type II as for type I, we obtain the following results.

Proposition 3.3.3. *Let L be a Leibniz algebra of type II. Let $f_k \notin \mathrm{Ann}_r(L),\ f_{k+1} \in \mathrm{Ann}_r(L)$ for some $k,\ m \le k \le n-m-1$. Then, the general transformation of basis can be reduced to the following transformation of basis:*

$$\begin{aligned}
e'_i &= A_ie_i + B_if_i, && 1 \le i \le m,\\
f'_i &= C_ie_i + D_if_i, && 1 \le i \le m,\\
f'_{m+1} &= (\beta_mB_1C_m + D_{m+1})f_{m+1}, &&\\
f'_i &= \left(\beta_mB_1C_m \prod_{l=m+1}^{i-1}(A_1 + B_1\delta_l) + D_i\right) f_i, && m+2 \le i \le n-m,
\end{aligned}$$

where A_i, B_i, C_i, D_i are defined by equalities (3.3.1).

Similarly, we have the following theorem.

Theorem 3.3.4. *Let L be a Leibniz algebra of type II. Let $f_k \notin \mathrm{Ann}_r(L),\ f_{k+1} \in \mathrm{Ann}_r(L)$ for some $k,\ m \le k \le n-m-1$. Then we have*

for $k = m$:

$$\alpha'_i = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))\,D_{i+1} - B_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}},\ \ 1 \le i \le m-1,$$

$$\beta'_i = \frac{A_{i+1}B_i(C_1+\delta_i D_1) - (A_iC_1 + D_1(\alpha_i A_i + \gamma_i B_i))\, B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))\, D_{i+1} - D_iC_{i+1}(C_1+\delta_i D_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\delta'_i = \frac{A_{i+1}D_i(C_1+\delta_i D_1) - (C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))\, B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\beta'_m = \frac{\beta A_m D_1 + B_m(C_1+\delta_m D_1)}{\beta B_1 C_m + D_{m+1}}, \quad \delta'_m = \frac{\beta C_m D_1 + D_m(C_1+\delta_m D_1)}{\beta B_1 C_m + D_{m+1}};$$

for $m+1 \leq k \leq n-m-1$:

$$\alpha'_i = \frac{(A_iC_1 + D_1(\alpha_i A_i + \gamma_i B_i))\, D_{i+1} - B_iC_{i+1}(C_1+\delta_i D_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\beta'_i = \frac{A_{i+1}B_i(C_1+\delta_i D_1) - (A_iC_1 + D_1(\alpha_i A_i + \gamma_i B_i))\, B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))\, D_{i+1} - D_iC_{i+1}(C_1+\delta_i D_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\delta'_i = \frac{A_{i+1}D_i(C_1+\delta_i D_1) - (C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))\, B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \ \ 1 \leq i \leq m-1,$$

$$\beta'_m = \frac{B_m(C_1+\delta_m D_1)}{D_{m+1}}, \ \delta'_m = \frac{D_m(C_1+\delta_m D_1)}{D_{m+1}},$$

$$\delta'_i = \frac{D_i(C_1+D_1\delta_i)}{D_{i+1}}, \ \ m+1 \leq i \leq n-m-1.$$

It should be noted that the solution of the classification problem consists of the following stages: the description of the algebras satisfying the given conditions, i.e., the determination of the multiplication table of the algebras with the least number of parameters; the determination of the relations determining the change of the parameters in the new basis (in the general transformation of basis); the study of the given relations for the parameters and the determination of pairwise of non-isomorphic algebras with given conditions. The results presented in this chapter carry out the first two stages of the classification of naturally graded Leibniz algebras with characteristic sequence equal to $(n-m, m)$. The available classifications for $m=2$ and $m=3$ ([13, 15])

show that, in the general case (i.e., for any values of m), the classification is boundless. However, for fixed values of n and m, it can be obtained by using the results of the present chapter.

Chapter 4

On some null-filiform algebras

In the study of nilpotent algebras, we encounter the following natural problem: To describe the algebras with the maximum nilpotency index. However, in view of the great diversity of classes of algebras, the algebras with the maximum nilpotency index are described separately in each special case. In the present chapter, we describe the algebras with the maximum nilpotency index satisfying certain special identities. These identities and algebras satisfying them are introduced in Section 4.1. Generalized Leibniz algebras and generalized Zinbiel algebras with the maximum nilindex are classified in Section 4.2 and 4.3, respectively. If we consider the generalized Leibniz algebras or generalized Zinbiel algebras for $A_1 = 1$ and $A_2 = 0$ (the case $A_1 + A_2 = 1$), then we get the associative algebras. Note that the null-filiform associative algebras of any dimension have not been considered earlier as algebras with the maximum nilpotency index. The results obtained in the present chapter fill this gap.

4.1 Preliminary definitions

Now we introduce some identities and algebras satisfying these identities.

Definition 4.1.1. *An algebra L over a field $\mathbb{F}$ is called a generalized Leibniz algebra if, for any elements $x, y, z \in L$, the following identity is true:*

$$a(bc) = A_1(ab)c + A_2(ac)b, \tag{4.1.1}$$

where $A_1, A_2 \in \mathbb{F}$.

Definition 4.1.2. *An algebra L over a field $\mathbb{F}$ is called a generalized Zinbiel algebra if, for any elements $x, y, z \in L$, the following identity is true:*

$$(ab)c = A_1 a(bc) + A_2 a(cb), \tag{4.1.2}$$

where $A_1, A_2 \in \mathbb{F}$.

Note that the Leibniz algebras are examples of generalized Leibniz algebras for $A_1 = 1$ and $A_2 = -1$. In turn, we note that the Zinbiel algebras are examples of generalized Zinbiel algebras for $A_1 = A_2 = 1$.

If we consider the generalized Leibniz algebras for $A_1 = 1$ and $A_2 = 0$ or generalized Zinbiel algebras for $A_1 = 1$ and $A_2 = 0$, then we get the associative algebras.

For an algebra A of an arbitrary variety, we consider the series

$$A^1 = A, \qquad A^{k+1} = \sum_{i=1}^{k} A^i A^{k+1-i}, \quad k \geq 1. \tag{4.1.3}$$

Definition 4.1.3. *An n-dimensional algebra A is called null-filiform if* $\dim A^i = (n+1) - i,\ 1 \leq i \leq n+1.$

It is easy to see that an algebra has a maximum nilpotency index if and only if it is null-filiform. For a nilpotent algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated. Moreover, we consider algebras over a field of characteristic zero.

4.2 Generalized Leibniz algebras

Let A be an arbitrary n-dimensional generalized Leibniz algebra. It is easy to see that the series (4.1.3) for the algebra A is reduced to the sequence

$$A^1 = A, \qquad A^{k+1} = A^k A^1, \quad k \geq 1. \tag{4.2.1}$$

Therefore, in what follows, we use sequence this (4.2.1) instead of the series.

The following theorem gives a classification of null-filiform generalized Leibniz algebras.

Theorem 4.2.1. *An n-dimensional null-filiform generalized Leibniz algebra A exists only for $A_1 + A_2 = 0$ or $A_1 + A_2 = 1$. Moreover, in each case, there exists a basis $\{e_1, e_2, \dots, e_n\}$ of the algebra A such that the multiplication table has the following form:*

$$\text{case I: } A_1 + A_2 = 0 : \qquad e_i e_1 = e_{i+1}, \quad 1 \le i \le n-1;$$

$$\text{case II: } A_1 + A_2 = 1 : \qquad e_i e_j = e_{i+j}, \quad 2 \le i + j \le n.$$

Proof. Let A be null-filiform generalized Leibniz algebra of dimension n and let $\{e_1, e_2, \dots, e_n\}$ be a basis of the algebra A such that $e_i \in A^i \setminus A^{i+1},\ 1 \le i \le n$. Thus, we get

$$e_2 = \Big(\sum_{i=1}^{n} \gamma_i e_i\Big)\Big(\sum_{i=1}^{n} \delta_i e_i\Big) = (\gamma_1 \delta_1)(e_1 e_1) + f,$$

where $f \in A^3$. Note that $\gamma_1 \delta_1 \neq 0$ and $e_1 e_1 \in A^2 \setminus A^3$; otherwise, $e_2 \in A^3$. Denoting

$$e_2' = \frac{1}{\gamma_1 \delta_1}(e_2 - f),$$

one can assume that $e_2 = e_1 e_1$.

Let $e_2 e_1 = \alpha_1 e_3$ and $e_1 e_2 = \beta_1 e_3$, where $(\alpha_1, \beta_1) \neq (0,0)$. If $\alpha_1 = 0$, then $e_2 e_1 = 0$. Hence,

$$e_1 e_2 = e_1(e_1 e_1) = A_1(e_1 e_1)e_1 + A_2(e_1 e_1)e_1 = (A_1 + A_2)e_2 e_1 = 0,$$

i.e., $\beta_1 = 0$. Then the algebra is not null-filiform.

Thus, $\alpha_1 \neq 0$. We perform the following change of a basis element: $e_3' := \alpha_1 e_3$. Without loss of generality we can suppose that $\alpha_1 = 1$. Hence, $e_2 e_1 = e_3$ and $e_1 e_2 = \beta_1 e_3$, where $\beta_1 = A_1 + A_2$.

Denote $e_i e_1 = \alpha_{i-1} e_{i+1},\ 2 \le i \le n-1$. Consider the equalities

$$e_i e_2 = e_i(e_1 e_1) = A_1(e_i e_1)e_1 + A_2(e_i e_1)e_1 = \beta_1(e_i e_1)e_1$$

$$= \beta_1 \alpha_{i-1} e_{i+1} e_1 = \beta_1 \alpha_{i-1} \alpha_i e_{i+2}.$$

By induction on j, for any i, we prove the equality

$$e_i e_j = \alpha_{i-1} \alpha_i \dots \alpha_{i+j-2} \beta_1^{j-1} e_{i+j}, \quad 2 \le i + j \le n.$$

If there exists j_0 $(2 \leq j_0 \leq n-1)$ such that $\alpha_{j_0-1} = 0$, then $e_{j_0-i+1}e_i = 0$ for any $2 \leq i \leq j_0$. Hence, we arrive at a contradiction. Thus, $\alpha_{j_0-1} \neq 0$ for any $2 \leq j_0 \leq n-1$. By changing the basis elements $e'_{j_0+1} := \alpha_{j_0-1}e_{j_0+1}$, we can assume that $\alpha_{j_0-1} = 1$ for any $2 \leq j_0 \leq n-1$. We have

$$e_ie_1 = e_{i+1}, \ 1 \leq i \leq n-1, \ e_ie_j = \beta_1^{j-1}e_{i+j} \quad 2 \leq i+j \leq n.$$

In view of identity (4.1.1) for elements $\{e_1, e_1, e_2\}$, we find $\beta_1 e_1e_3 = \beta_1^2 e_4$, i.e., $\beta_1 \in \{0, 1\}$. Hence, we get the following algebras:

$$\text{case I}: \quad e_ie_1 = e_{i+1}, \ 1 \leq i \leq n-1 \quad \text{for} \quad \beta_1 = 0;$$

$$\text{case II}: \quad e_ie_j = e_{i+j}, \ 2 \leq i+j \leq n \quad \text{for} \quad \beta_1 = 1.$$

Since the dimension of the right annihilator of the algebra I is equal to $n-1$ and the dimension of the right annihilator of the algebra II is equal to 1, the algebras I and II are not isomorphic. □

Note that, in particular, the classification of null-filiform Leibniz algebras (Theorem 3.1.2) is a special case of Theorem 4.2.1 (for the case $A_1 + A_2 = 0$). The case $A_1 + A_2 = 1$ in the Theorem 4.2.1 give us the classification of null-filiform associative algebras of any dimension.

4.3 Generalized Zinbiel algebras

Let A be a given generalized Zinbiel algebra. Then it follows from identity (4.1.2) that in the algebra A it suffices to consider the sequence

$$A^1 = A, \qquad A^{k+1} = AA^k, \qquad k \geq 1,$$

instead of the series (4.1.3).

The following theorem describes null-filiform generalized Zinbiel algebras.

Theorem 4.3.1. *An n-dimensional null-filiform generalized Zinbiel algebra A exists only for $A_1 = A_2$, or $A_1 = -A_2$, or $A_1 + A_2 = 1$. Moreover, in the algebra A there exists a basis $\{e_1, e_2, \ldots, e_n\}$ such that the multiplication table has the following form:*

$$e_ie_j = B_{i,j}e_{i+j},$$

where $B_{1,j} = 1, \ 1 \leq j \leq n-1$ and $B_{i,j} = A_1B_{i-1,j} + A_2B_{j,i-1}, \ 2 \leq i \leq n-j$.

Proof. Similarly as in the proof of Theorem 4.2.1, there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of the algebra A such that $e_i \in A^i \setminus A^{i+1}$, $1 \le i \le n$, and $e_1e_1 = e_2$. We set $e_1e_2 = \alpha_1 e_3$ and $e_2e_1 = \beta_1 e_3$, where $(\alpha_1, \beta_1) \ne (0,0)$. If $\alpha_1 = 0$, then $e_1e_2 = 0$. It follows from the equality

$$e_2e_1 = (e_1e_1)e_1 = A_1e_1(e_1e_1) + A_2e_1(e_1e_1) = (A_1 + A_2)e_1e_2 = 0,$$

that $\beta_1 = 0$. We arrive at a contradiction.

Thus, $\alpha_1 \ne 0$. By changing a basis element $e_3' := \alpha_1 e_3$, we can assume that $\alpha_1 = 1$, i.e., $e_1e_2 = e_3$ and $e_2e_1 = \beta_1 e_3$, where $\beta_1 = A_1 + A_2$.

Let $e_1e_3 = \alpha_2 e_4$. By applying identity (4.1.2) to the products

$$(e_1e_1)e_2,\ (e_1e_2)e_1,\ (e_2e_1)e_1,$$

we obtain

$$e_2e_2 = (A_1 + A_2\beta_1)e_1e_3 \quad \text{and} \quad e_3e_1 = (A_1\beta_1 + A_2)e_1e_3.$$

If $\alpha_2 = 0$, then $e_2e_2 = e_3e_1 = 0$, which is a contradiction. Hence, $\alpha_2 \ne 0$. By the change of variables $e_4' := \alpha_2 e_4$, we can assume that $\alpha_2 = 1$, $e_2e_2 = (A_1 + A_2\beta_1)e_4$ and $e_3e_1 = (A_1\beta_1 + A_2)e_4$.

In view of identity (4.1.2) for the basis elements $\{e_2, e_1, e_1\}$, we obtain three possible cases for the parameters A_1 and A_2:

$$\begin{array}{ll} \text{case I:} & A_1 = A_2 \ \ (\beta_1 = 2 \cdot A_1 \ne 0); \\ \text{case II:} & A_1 = -A_2 \ \ (\beta_1 = 0); \\ \text{case III:} & A_1 + A_2 = 1 \ \ (\beta_1 = 1. \end{array}$$

Indeed,

$$\beta_1 e_3e_1 = (e_2e_1)e_1 = A_1e_2e_2 + A_2e_2e_2 = \beta_1 e_2e_2,$$

which yields either $\beta_1 = 0$ or $e_3e_1 = e_2e_2$. Thanks to the equality $e_3e_1 = e_2e_2$, we obtain $A_1 = A_2$ or $\beta_1 = 1$.

Denote $e_1e_{i+j-1} = \alpha_{i+j-2}e_{i+j}$. We show that the equality $e_ie_j = B_{i,j}e_1e_{i+j-1}$, $2 \le i+j \le s+1$, where $B_{i,j}$ satisfy the conditions of the theorem, holds for some fixed s, $1 \le s \le n-1$.

Assume that, for $s = s_0$, this dependence is true. Then, under the condition $e_1e_{i+j-1} = 0$, we get $e_ie_j = 0$ for any $2 \le i+j \le s_0+1$, which contradicts the existence of an algebra. Thus, $e_1e_{i+j-1} \neq 0$ (i.e., $\alpha_{i+j-2} \neq 0$). By the change of variables $e'_{i+j} := \alpha_{i+j-2}e_{i+j}$, we can assume that $\alpha_{i+j-2} = 1$. Hence, we obtain

$$e_1e_{i+j-1} = e_{i+j}, \qquad 2 \leq i+j \leq s_0+1,$$

$$e_ie_j = B_{i,j}e_1e_{i+j-1} = B_{i,j}e_{i+j}, \quad 2 \leq i+j \leq s_0+1.$$

By induction, we show that $e_ie_j = B_{i,j}e_1e_{i+j-1}$ for $i+j = s_0+2$.
Let $e_1e_{s_0+1} = \alpha_{s_0}e_{s_0+2}$. For $i = 2$, we get

$$e_2e_{s_0} = (e_1e_1)e_{s_0} = A_1e_1(e_1e_{s_0}) + A_2e_1(e_{s_0}e_1) = (A_1 + A_2B_{s_0,1})e_1e_{s_0+1}.$$

We set $A_1 + A_2B_{s_0,1} = B_{2,s_0}$. Then

$$e_2e_{s_0} = B_{2,s_0}e_1e_{s_0+1}.$$

Assume that the relation

$$e_{i_0}e_{s_0+2-i_0} = B_{i_0,s_0+2-i_0}e_1e_{s_0+1},$$

where B_{i_0,s_0+2-i_0} satisfy the conditions of the theorem, is true for $i = i_0,\ 1 \leq i_0 \leq s_0$. Consider the equalities

$$e_{i_0+1}e_{s_0+1-i_0} = (e_1e_{i_0})e_{s_0+1-i_0} = A_1e_1(e_{i_0}e_{s_0+1-i_0}) + A_2e_1(e_{s_0+1-i_0}e_{i_0})$$

$$= (A_1B_{i_0,s_0+1-i_0} + A_2B_{s_0+1-i_0,i_0})e_1e_{s_0+1} = B_{i_0+1,s_0+1-i_0}e_1e_{s_0+1}.$$

Thus,

$$e_ie_j = B_{i,j}e_1e_{i+j-1}$$

for $2 \leq i+j \leq s_0+2$. Similarly, one can assume that $\alpha_{s_0} = 1$, i.e., $e_1e_{s_0+1} = e_{s_0+2}$. Then

$$e_ie_j = B_{i,j}e_{i+j},$$

where $B_{1,j} = 1,\ 1 \leq j \leq n-1$ and $B_{i,j} = A_1B_{i-1,j} + A_2B_{j,i-1},\ 2 \leq i+j \leq n$. $\square$

In the previous theorem we have described null-filiform generalized Zinbiel algebras. To classify them completely, we shall need the following auxiliary lemmas.

Lemma 4.3.2. *For any natural numbers $a, i, s \in \mathbb{N}$, $i \geq 2$, $s \geq 3$, the following equalities are true:*

$$\sum_{l=2}^{i} C_{l+a-s}^{a-1} = C_{i+a-(s-1)}^{a}, \qquad \sum_{l=2}^{i} C_{l-2}^{j-1} = C_{i-1}^{j}.$$

Proof. The lemma is proved by induction. $\square$

Lemma 4.3.3. *For any $i,\ j \in \mathbb{N}$ and any $x \in \mathbb{F}$, the following equalities are true:*

$$\sum_{l=2}^{i} \sum_{k=1}^{j-3} (C_{k+l-2}^{k} + C_{k+l-2}^{j-1}) x^{k+i} = \sum_{k=2}^{j-2} (C_{k+i-2}^{k} + C_{k+i-2}^{j}) x^{k+i-1},$$

$$\sum_{l=2}^{i} \sum_{k=1}^{l-j} C_{k+j-2}^{j-1} x^{k+j+i-l} = \sum_{k=1}^{i-j} C_{k+j-1}^{j} x^{k+j}.$$

Proof. The first equality follows from the following chain of equalities:

$$\begin{aligned}
&\sum_{l=2}^{i} \sum_{k=1}^{j-3} (C_{k+l-2}^{k} + C_{k+l-2}^{j-1}) x^{k+i} \\
&= \sum_{l=2}^{i} \sum_{k=2}^{j-2} (C_{k+l-3}^{k-1} + C_{k+l-3}^{j-1}) x^{k+i-1} \\
&= \sum_{k=2}^{j-2} \sum_{l=2}^{i} (C_{k+l-3}^{k-1} + C_{k+l-3}^{j-1}) x^{k+i-1} \\
&= \sum_{k=2}^{j-2} x^{k+i-1} \Big(\sum_{l=2}^{i} C_{k+l-3}^{k-1} + \sum_{l=2}^{i} C_{k+l-3}^{j-1} \Big) \\
&\overset{\text{by Lemma 4.3.2}}{=} \sum_{k=2}^{j-2} (C_{k+i-2}^{k} + C_{k+i-2}^{j}) x^{k+i-1}.
\end{aligned}$$

The second equality follows from the equalities

$$
\begin{aligned}
\sum_{l=2}^{i}\sum_{k=1}^{l-j} C^{j-1}_{k+j-2}x^{k+j+i-l} &= \sum_{l=2}^{i}\Big[\sum_{k=1}^{l-j-1} C^{j-1}_{k+j-2}x^{k+j+i-l} + C^{j-1}_{l-2}x^{i}\Big]\\
&\overset{\text{by Lemma 4.3.2}}{=} C^{j}_{i-1}x^{i} + x^{i+j}\sum_{l=2}^{i}\sum_{k=1}^{l-j-1} C^{j-1}_{k+j-2}x^{k-l}\\
&= C^{j}_{i-1}x^{i} + x^{i+j}\Big[\sum_{k=1}^{1-j} C^{j-1}_{k+j-2}x^{k-2} + \sum_{k=1}^{2-j} C^{j-1}_{k+j-2}x^{k-3} + \cdots\\
&\qquad + \sum_{k=1}^{i-j-1} C^{j-1}_{k+j-2}x^{k-i}\Big]\\
&= C^{j}_{i-1}x^{i} + x^{i+j}\Big[\sum_{k=1}^{1-j} C^{j-1}_{k+j-2}(x^{k-2} + x^{k-3} + \cdots + x^{k-(i-1)} + x^{k-i})\\
&\quad + C^{j-1}_{0}x^{-j-1} + (C^{j-1}_{0}x^{-j-2} + C^{j-1}_{1}x^{-j-1})\\
&\quad + (C^{j-1}_{0}x^{-j-3} + C^{j-1}_{1}x^{-j-2} + C^{j-1}_{2}x^{-j-1}) + \sum_{k=0}^{3} C^{j-1}_{k}x^{-j-4+k} + \cdots\\
&\quad + \sum_{k=0}^{i-5} C^{j-1}_{k}x^{-j-(i-4)+k} + \sum_{k=0}^{i-4} C^{j-1}_{k}x^{-j-(i-3)+k} + \sum_{k=0}^{i-3} C^{j-1}_{k}x^{-j-(i-2)+k}\Big]\\
&= C^{j}_{i-1}x^{i} + x^{i+j}\Big[x^{-j-1}\sum_{k=1}^{i-3} C^{j-1}_{k} + x^{-j-2}\sum_{k=1}^{i-4} C^{j-1}_{k} + \cdots + \underbrace{x^{1-i}\sum_{k=1}^{j-1} C^{j-1}_{k}}_{(i-j-1)\text{th place}}\\
&\quad + \underbrace{x^{-i}\sum_{k=1}^{j-2} C^{j-1}_{k}}_{(i-j)\text{th place}} + \cdots + \underbrace{x^{-j-(i-4)}\sum_{k=1}^{2} C^{j-1}_{k}}_{(i-4)\text{th place}} + \underbrace{x^{-j-(i-3)}C^{j-1}_{1}}_{(i-3)\text{th place}}\Big]\\
&\overset{\text{by Lemma 4.3.2}}{=} C^{j}_{i-1}x^{i} + x^{i+j}\Big[C^{j}_{i-2}x^{-j-1} + C^{j}_{i-3}x^{-j-2} + \cdots\\
&\qquad + C^{j}_{j+1}x^{2-i} + C^{j}_{j}x^{1-i}\Big]\\
&= C^{j}_{i-1}x^{i} + \sum_{k=1}^{i-j-1} C^{j}_{k+j-1}x^{k+j} = \sum_{k=1}^{i-j} C^{j}_{k+j-1}x^{k+j}.
\end{aligned}
$$

□

Lemma 4.3.4. *For any $i, j \in \mathbb{N}$ and any $x \in \mathbb{F}$, the following equality is true:*

$$\sum_{l=2}^{i}\sum_{k=1}^{l-4} C_{k+j-2}^{j-1} x^{k+j+i-l} = \sum_{k=1}^{i-4} C_{k+j-1}^{j} x^{k+j}.$$

Proof. The proof of the lemma is based on the following chain of equalities:

$$\sum_{l=2}^{i}\sum_{k=1}^{l-4} C_{k+j-2}^{j-1} x^{k+j+i-l} = \sum_{l=5}^{i}\sum_{k=1}^{l-4} C_{k+j-2}^{j-1} x^{k+j+i-l} = \underbrace{C_{j-1}^{j-1} x^{i+j-4}}_{\text{at } l=5}$$

$$+ \underbrace{\sum_{k=0}^{1} C_{k+j-1}^{j-1} x^{k+i+j-5}}_{\text{at } l=6} + \cdots + \underbrace{\sum_{k=0}^{i-6} C_{k+j-1}^{j-1} x^{k+j+2}}_{\text{at } l=i-1} + \underbrace{\sum_{k=0}^{i-5} C_{k+j-1}^{j-1} x^{k+j+1}}_{\text{at } l=i}$$

$$= (C_{j-1}^{j-1} + C_{j}^{j-1} + \cdots + C_{i+j-6}^{j-1}) x^{i+j-4}$$

$$+ (C_{j-1}^{j-1} + C_{j}^{j-1} + \cdots + C_{i+j-7}^{j-1}) x^{i+j-5}$$

$$+ (C_{j-1}^{j-1} + C_{j}^{j-1} + \cdots + C_{i+j-8}^{j-1}) x^{i+j-6} + \cdots + (C_{j-1}^{j-1} + C_{j}^{j-1} + C_{j+1}^{j-1}) x^{j+3}$$

$$+ (C_{j-1}^{j-1} + C_{j}^{j-1}) x^{j+2} + C_{j-1}^{j-1} x^{j+1}$$

$$= \sum_{l=1}^{i-4} C_{l+j-2}^{j-1} x^{i+j-4} + \sum_{l=1}^{i-5} C_{l+j-2}^{j-1} x^{i+j-5} + \cdots + \sum_{l=1}^{3} C_{l+j-2}^{j-1} x^{j+3}$$

$$+ \sum_{l=1}^{2} C_{l+j-2}^{j-1} x^{j+2} + x^{j+1}$$

$$\overset{\text{by Lemma 4.3.2}}{=} C_{i+j-5}^{j} x^{i+j-4} + C_{i+j-6}^{j} x^{i+j-5} + \cdots + C_{j+1}^{j} x^{j+2} + C_{j}^{j} x^{j+1}$$

$$= \sum_{k=1}^{i-4} C_{k+j-1}^{j} x^{k+j}.$$

□

Now we will directly start the classification of the null-filiform generalized Zinbiel algebras. For this purpose we will find a general view of the parameters.

The following theorem describes the parameters $B_{i,j}$ of Theorem 4.3.1 in the case when $A_1 = A_2 = x$.

Theorem 4.3.5. *For any n-dimensional null-filiform generalized Zinbiel algebra A with $A_1 = A_2 = x$ (case I), the following equalities are true for any $i > 1$:*

$$B_{i,1} = 2x^{i-1} + \sum_{k=1}^{i-2} x^k; \quad B_{i,2} = C_{i-1}^1 x^{i-1}(2x+1) + \sum_{k=1}^{i-3} C_k^1 x^{k+1},$$

$$B_{i,j} = C_{i+j-3}^{j-1} x^{i+j-3}(2x+1) + x^{i-1} \tag{4.3.1}$$

$$+ \sum_{k=1}^{i-j} C_{k+j-2}^{j-1} x^{k+j-1} + \sum_{k=1}^{j-3} (C_{k+i-2}^k + C_{k+i-2}^{j-1}) x^{k+i-1}, \ j \geq 3.$$

Proof. It follows from Theorem 4.3.1 that $e_i e_j = B_{i,j} e_{i+j}$, where

$$B_{1,j} = 1, \ 1 \leq j \leq n-1, \quad B_{i,j} = x(B_{i-1,j} + B_{j,i-1}), \qquad 2 \leq i \leq n-j.$$

Since $B_{j+1,i-1} = x(B_{j,i-1} + B_{i-1,j})$, we get $B_{i,j} = B_{j+1,i-1}, \ i \geq 2$. Hence, for $j > 1$, we obtain

$$B_{i,j} = x(B_{i-1,j} + B_{i,j-1}), \qquad 2 \leq i \leq n-j. \tag{4.3.2}$$

For $j = 1$, we arrive at the recurrence relation

$$B_{i,1} = x(B_{i-1,1} + B_{1,i-1}) = xB_{i-1,1} + x, \quad i > 1.$$

This yields

$$B_{i,1} = 2x^{i-1} + \sum_{k=1}^{i-2} x^k.$$

In view of equality (4.3.2) for $j = 2$ and the relation obtained for $B_{i,1}$, by induction, we prove the relation

$$B_{i,2} = C_{i-1}^1 x^{i-1}(2x+1) + \sum_{k=1}^{i-3} C_k^1 x^{k+1}. \tag{4.3.3}$$

We prove equality (4.3.1) for $j \geq 3$ by induction on j for any value of i.

Setting $j = 3$ in equality (4.3.2), we obtain $B_{i,3} = x(B_{i-1,3} + B_{i,2})$. By using relation (4.3.3), we now easily prove equality (4.3.1) for $j = 3$.

Assume that relation (4.3.1) is true for fixed $j > 3$. We now consider $B_{i,j+1}$ for any i:

$$B_{i,j+1} = \underbrace{x(x(x\dots(x}_{k \ \text{times}}(B_{i-k,j+1} + \underbrace{B_{i-k+1,j}) + B_{i-k+2,j}) + \dots + B_{i-1,j}) + B_{i,j})}_{k \ \text{times}}$$

$$= \underbrace{x(x(x\dots(x}_{k \ \text{times}}(x(B_{i-k-1,j+1} + B_{i-k,j})$$

$$+ \underbrace{B_{i-k+1,j}) + B_{i-k+2,j}) + \dots + B_{i-1,j}) + B_{i,j})}_{k \ \text{times}}$$

$$= \underbrace{x(x(x\dots(x}_{(k+1) \ \text{times}}(B_{i-k-1,j+1} + \underbrace{B_{i-k,j}) + B_{i-k+1,j}) + \dots + B_{i-1,j}) + B_{i,j})}_{(k+1) \ \text{times}}.$$

Thus, on the $(i-1)$th step, we can arrive at $B_{1,j+1}$ equal to 1, i.e.,

$$B_{i,j+1} = \underbrace{x(x(x\dots(x}_{(i-1) \ \text{times}}(1 + \underbrace{B_{2,j}) + B_{3,j}) + \dots + B_{i-1,j}) + B_{i,j})}_{(i-1) \ \text{times}}$$

$$= x^{i-1} + x^{i-1}B_{2,j} + x^{i-2}B_{3,j} + \dots + x^2 B_{i-1,j} + xB_{i,j}$$

$$= x^{i-1} + \sum_{l=2}^{i} x^{i+1-l} B_{l,j} = x^{i-1} + \sum_{l=2}^{i} x^{i+1-l}\Big[C_{l+j-3}^{j-1} x^{l+j-3}(2x+1) + x^{l-1}$$

$$+ \sum_{k=1}^{l-j} C_{k+j-2}^{j-1} x^{k+j-1} + \sum_{k=1}^{j-3} (C_{k+l-2}^{k} + C_{k+l-2}^{j-1}) x^{k+l-1}\Big]$$

$$= x^{i-1} + \sum_{l=2}^{i}\Big[C_{l+j-3}^{j-1} x^{i+j-2}(2x+1) + x^{i} + \sum_{k=1}^{l-j} C_{k+j-2}^{j-1} x^{k+j+i-l}$$

$$+ \sum_{k=1}^{j-3} (C_{k+l-2}^{k} + C_{k+l-2}^{j-1}) x^{k+i}\Big]$$

$$= x^{i-1} + x^{i+j-2}(2x+1)\sum_{l=2}^{i} C_{l+j-3}^{j-1} + (i-1)x^{i}$$

$$+ \sum_{l=2}^{i}\sum_{k=1}^{l-j} C_{k+j-2}^{j-1} x^{k+j+i-l} + \sum_{l=2}^{i}\sum_{k=1}^{j-3}(C_{k+l-2}^{k} + C_{k+l-2}^{j-1})x^{k+i}$$

$$\overset{\text{by Lemma 4.3.2}}{=} \quad x^{i-1} + C_{i+j-2}^{j} x^{i+j-2}(2x+1) + (i-1)x^{i}$$

$$+ \sum_{l=2}^{i}\sum_{k=1}^{l-j} C_{k+j-2}^{j-1} x^{k+j+i-l} + \sum_{l=2}^{i}\sum_{k=1}^{j-3}(C_{k+l-2}^{k} + C_{k+l-2}^{j-1})x^{k+i}.$$

By using Lemma 4.3.3, we obtain

$$\begin{aligned} B_{i,j+1} &= C_{i+j-2}^{j} x^{i+j-2}(2x+1) + x^{i-1} + \sum_{k=1}^{i-j-1} C_{k+j-1}^{j} x^{k+j} \\ &\quad + (i-1)x^{i} + C_{i-1}^{j} x^{i} + \sum_{k=2}^{j-2}(C_{k+i-2}^{k} + C_{k+i-2}^{j})x^{k+i-1} \\ &= C_{i+j-2}^{j} x^{i+j-2}(2x+1) + x^{i-1} + \sum_{k=1}^{i-j-1} C_{k+j-1}^{j} x^{k+j} \\ &\quad + \sum_{k=1}^{j-2}(C_{k+i-2}^{k} + C_{k+i-2}^{j})x^{k+i-1}. \end{aligned}$$

□

We now present a description of the parameters $B_{i,j}$ in the case II, i.e. $A_1 = -A_2$.

Theorem 4.3.6. *For any n-dimensional null-filiform generalized Zinbiel algebra A with $A_1 = -A_2 = x$ (case II), the following equalities are true for any $i > 1$:*

$$B_{i,1} = -\sum_{k=1}^{i-2} x^{k}, \quad B_{2,2} = x, \quad B_{i,2} = -C_{i-3}^{1} x^{i-2}(x+1) - \sum_{k=1}^{i-4} C_{k}^{1} x^{k+1},$$

$$\begin{aligned} B_{i,j} &= -(C_{i+j-5}^{j-1} - C_{i+j-5}^{j-3})x^{i+j-4}(x+1) \\ &\quad + \sum_{k=1}^{j-3} C_{i+k-3}^{k-1} x^{i+k-2} - \sum_{k=1}^{i-4} C_{k+j-2}^{j-1} x^{k+j-1}, \qquad j \geq 3. \end{aligned} \tag{4.3.4}$$

Proof. By using Theorem 4.3.1, we obtain $e_i e_j = B_{i,j} e_{i+j}$, where

$$B_{1,j} = 1,\ 1 \le j \le n-1, \quad B_{i,j} = x(B_{i-1,j} - B_{j,i-1}),\ 2 \le i \le n-j.$$

It is easy to see that $B_{j+1,i-1} = x(B_{j,i-1} - B_{i-1,j})$. Hence,

$$B_{i,j} = -B_{j+1,i-1}, \quad i \ge 2.$$

For $j = 1, 2, 3$, we get the recurrence relations

$$\begin{aligned} B_{i,1} &= x(B_{i-1,1} - B_{1,i-1}) = x(B_{i-1,1} - 1), && i > 1, \\ B_{i,2} &= x(B_{i-1,2} - B_{2,i-1}) = x(B_{i-1,2} + B_{i,1}), && i > 1, \\ B_{i,3} &= x(B_{i-1,3} - B_{3,i-1}) = x(B_{i-1,3} + B_{i,2}), && i > 1, \end{aligned}$$

respectively. By using these relations, we show that the first two equalities in (4.3.4) and the third equality in (4.3.4), for $j = 3$, are true.

We prove the third equality in (4.3.4), for $j > 3$, by induction.

Let relations (4.3.4) be true for any i and fixed $j > 3$. For any i, we consider $B_{i,j+1}$. Then

$$B_{i,j+1} = x(B_{i-1,j+1} + B_{i,j}).$$

Reasoning as in the proof of Theorem 4.3.5, we can show that

$$B_{i,j+1} = \underbrace{x(x(x \dots (x(x}_{(i-1)\ \text{times}}(B_{1,j+1} + \underbrace{B_{2,j}) + B_{3,j}) + B_{4,j}) + \dots + B_{i-1,j}) + B_{i,j})}_{(i-1)\ \text{times}}.$$

Thus,

$$B_{i,j+1} = \underbrace{x(x(x\ldots(x}_{(i-1)\ \text{times}}(1+\underbrace{B_{2,j})+B_{3,j})+\cdots+B_{i-1,j})+B_{i,j})}_{(i-1)\ \text{times}}$$

$$= x^{i-1} + x^{i-1}B_{2,j} + x^{i-2}B_{3,j} + x^{i-3}B_{4,j} + \cdots + x^2 B_{i-1,j} + xB_{i,j}$$

$$= x^{i-1} + \sum_{l=2}^{i} x^{i+1-l} B_{l,j}$$

$$= x^{i-1} + \sum_{l=2}^{i} x^{i+1-l}\Big[-(C_{l+j-5}^{j-1} - C_{l+j-5}^{j-3})x^{l+j-4}(x+1)$$

$$+ \sum_{k=1}^{j-3} C_{l+k-3}^{k-1} x^{l+k-2} - \sum_{k=1}^{l-4} C_{k+j-2}^{j-1} x^{k+j-1}\Big]$$

$$= x^{i-1} - \sum_{l=2}^{i}(C_{l+j-5}^{j-1} - C_{l+j-5}^{j-3})x^{i+j-3}(x+1) + \sum_{l=2}^{i}\sum_{k=1}^{j-3} C_{l+k-3}^{k-1} x^{i+k-1}$$

$$- \sum_{l=2}^{i}\sum_{k=1}^{l-4} C_{k+j-2}^{j-1} x^{i+k+j-l}$$

$$\overset{\text{by Lemma 4.3.4}}{=} \quad x^{i-1} - x^{i+j-3}(x+1)\Big[\sum_{l=2}^{i} C_{l+j-5}^{j-1} - \sum_{l=2}^{i} C_{l+j-5}^{j-3}\Big]$$

$$+ \sum_{k=1}^{j-3} x^{i+k-1} \sum_{l=2}^{i} C_{l+k-3}^{k-1} - \sum_{k=1}^{i-4} C_{k+j-1}^{j} x^{k+j}$$

$$\overset{\text{by Lemma 4.3.2}}{=} \quad x^{i-1} - (C_{i+j-4}^{j} - C_{i+j-4}^{j-2})x^{i+j-3}(x+1)$$

$$+ \sum_{k=1}^{j-3} C_{i+k-2}^{k} x^{i+k-1} - \sum_{k=1}^{i-4} C_{k+j-1}^{j} x^{k+j}$$

$$= (C_{i+j-4}^{j-2} - C_{i+j-4}^{j})x^{i+j-3}(x+1) + \sum_{k=1}^{j-2} C_{i+k-3}^{k-1} x^{i+k-2} - \sum_{k=1}^{i-4} C_{k+j-1}^{j} x^{k+j}.$$

□

The theorem presented below completes the description of the null-filiform generalized Zinbiel algebras.

Theorem 4.3.7. *For any n-dimensional null-filiform generalized Zinbiel algebra A with $A_1 + A_2 = 1$ (case III), the following relation is true:*

$$B_{i,j} = 1 \text{ for any } 1 \leq i, j \leq n-1.$$

Proof. Let $A_1 = x$. Then $A_2 = 1 - x$ and

$$B_{i,j} = xB_{i-1,j} + (1-x)B_{j,i-1} = x(B_{i-1,j} - B_{j,i-1}) + B_{j,i-1}.$$

We prove the theorem by induction on i for any j.

For $i = 1$, we have $B_{1,j} = 1,\ 1 \leq j \leq n-1$.

Consider $B_{i+1,j} = xB_{i,j} + (1-x)B_{j,i} = x + (1-x)B_{j,i}$. Since

$$B_{j,i} = xB_{j-1,i} + (1-x)B_{i,j-1} = xB_{j-1,i} + (1-x),$$

$$B_{j-1,i} = xB_{j-2,i} + (1-x)B_{i,j-2} = xB_{j-2,i} + (1-x),$$

we get

$$B_{j,i} = xB_{j-1,i} + (1-x) = x(xB_{j-2,i} + (1-x)) + (1-x).$$

Similarly, we can show that

$$B_{j,i} = \underbrace{x(x(x\dots(x(x}_{(j-1)\text{ times}} B_{1,i} + (1-x)\underbrace{) + (1-x)) + \dots + (1-x)) + (1-x))}_{(j-1)\text{ times}}.$$

Since $B_{1,j} = 1,\ 1 \leq j \leq n-1$, we find $B_{j,i} = 1$. Thus, $B_{i+1,j} = x + (1-x) \cdot 1 = 1$. Therefore, $B_{i,j} = 1$ for any $1 \leq i, j \leq n-1$. $\square$

In particular, the classification of null-filiform Zinbiel algebras in the work [1] is a special case of Theorems 4.3.1 and 4.3.5 (the case $A_1 = A_2$). Theorems 4.3.1 and 4.3.7 (the case $A_1 + A_2 = 1$) give us the classification of the null-filiform associative algebras of any dimension.

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