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Response Matrix/Discrete Ordinates Solution of the 1D Fokker-Planck Equation

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ABSTRACT

The Fokker-Planck equation (FPE) is one of the quintessential equations of particle transport theory. Representing small angle scattering characteristics of electron and photon transport by differential scattering indeed is a mathematical/numerical challenge. Here, we address the challenge with the method of response matrix applied to the S_N approximation to arrive at a nearly six place precision benchmark. Our approach aligns with the response matrix solution of the radiative transfer equation for forward peaked scattering previously published. We conclude with the application of the benchmark to a finite difference approximation.

I. INTRODUCTION

Appropriate characterization of particle scattering is crucial in any transport application. For neutrons, there is the scattering phase function, which generally is expressible as a Legendre polynomial expansion. For the FPE, say for electrons, the scattering is differential, but, like neutrons, has a matrix representation in the discrete ordinates sense. The matrix formulations for both neutrons and electrons enables application of the response matrix method of solution RM/DOM[1] previously applied for neutrons and photons. In extreme forward scattering however, such as required for detection of tumors by light or for screen-Rutherford scattering of electrons in cancer treatment, Fokker-Planck scattering could be an effective scattering model. Thus, our focus in this work is the application of response matrix theory for application of the 1D FPE, which is a new and promising application.

We begin by specifying the discrete ordinates and their ordering, determined by the S_N balance equation incorporating the discrete version of the directional FP operator. The formal exponential solution, leads to the evaluation of the matrix exponential through diagonalization, which we show to be equivalent to a spectral eigenfunction expansion the numerical evaluation of which is unstable. To stabilize, we introduce a stabilizing factor. A straightforward numerical evaluation of the vector equation, identical to the solution found in Ref[2], follows achieving nearly six-place precision for reflected and transmitted angular intensities for the slab.

II. THEORY

A. The 1D Fokker-Planck SN Balance Equation

The 1D Fokker-Plank equation (FPE) without a volume source is

$$\left[\mu\frac{\partial}{\partial z}+\alpha\right]\psi(z,\mu)=\sigma\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi(z,\mu)}{\partial\mu}\right], \tag{1a}$$

including half-range boundary conditions

$$\begin{aligned}\psi(0,\mu)&=f(\mu),\mu\in[0,1]\\ \psi(a,\mu)&=g(\mu),\mu\in[-1,0]\end{aligned} \tag{1b}$$

at boundaries of a slab $z\in[0,a]$. The momentum transfer,

$$\sigma=\frac{\sigma_{tr}}{2}=(1-g)\frac{\sigma_s}{2}, \tag{1c}$$

also known as the transport cross section, is characteristic of the FPE, where g is the average scattering cosine.

The solution via response matrix requires discretization in direction μ, and spectral expansion in the spatial variable z. By following the matrix- exponential formalism of radiative transfer [3,4], we arrive at several solution representations, which are obtained through numerical linear algebra.

1. Discretization in Direction

We first consider the directional (or angular) discretization (in μ) on the half-range intervals [1,0] and [0,–1] denoted directions + and – respectively. The orientation is

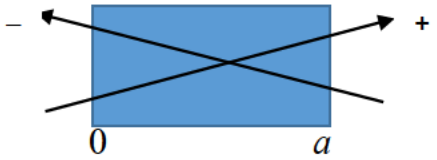

Fig. 1. Directions of motion.

positive directions from left to right and negative directions from right to left as shown in Fig. 1. The choice of how one discretizes is essentially arbitrary but we have found some choices may give better numerical results than others may. For example, for the integrated intensity or current density

$$\begin{aligned} \psi(z) &\equiv \int_{-1}^{1} d\mu \psi(z,\mu) \simeq \sum_{m=1}^{2N} \omega_m \psi_m(z) \\ J(z) &\equiv \int_{-1}^{1} d\mu \mu \psi(z,\mu) \simeq \sum_{m=1}^{2N} \omega_m \mu_m \psi_m(z) \end{aligned} \tag{2a,b}$$

the discrete ordinate (μ_j) should be chosen for precise integration over the entire range [-1,1]. Thus, the most common choices are half- range (HR) and full range (FR) Legendre Gauss quadrature (LGQ) and Clenshaw-Curtis quadrature (CCQ). Once the discrete ordinates are fixed, the derivation of the solution is independent of the ordinates. After experimentation, our choice will be FRLGQ. Apparently, HRLGQ as used for the neutron transport equation[1] performs poorly for the FPE.

The discrete ordinates for FRLGQ come from the zeros of the Legendre Polynomials of degree $2N$

$$P_{2N}(\mu_j) = 0; j = 1,2,...,2N, \tag{3}$$

where μ_j algebraically descends over the interval [1,-1]. The ordinates are formed into two sets algebraically descending in each half range

$$\begin{aligned} \tilde{\boldsymbol{\mu}}^{+} &\equiv \left[\mu_j; j = 1,...,N\right] \underset{\text{Renamed}}{=} \left[\tilde{\mu}_j; j = N,...,1\right] \\ \tilde{\boldsymbol{\mu}}^{-} &\equiv \left[\mu_j; j = N+1,...,2N\right] \underset{\text{Renamed}}{=} \left[-\tilde{\mu}_j; j = 1,...,N\right]. \end{aligned} \tag{4a,b}$$

In vector notation, the two disjoint sets form a single set to define $\boldsymbol{\mu}$ as

$$\boldsymbol{\mu} \equiv \begin{bmatrix} \mu_1 & \mu_2 & \ldots & \mu_{2N} \end{bmatrix}^T = \begin{bmatrix} \tilde{\boldsymbol{\mu}}^{+} & \tilde{\boldsymbol{\mu}}^{-} \end{bmatrix}^T, \tag{4c}$$

with corresponding quadrature weights

$$\tilde{\boldsymbol{\omega}}^{+} \equiv \left[\tilde{\boldsymbol{\omega}}_j ; j = N,\ldots,1 \right]$$
$$\tilde{\boldsymbol{\omega}}^{-} \equiv \left[\tilde{\boldsymbol{\omega}}_j ; j = 1,\ldots,N \right] \qquad \text{(5a,b)}$$

$$\boldsymbol{\omega} \equiv \left[\tilde{\boldsymbol{\omega}}^{+} \;\; \tilde{\boldsymbol{\omega}}^{-} \right]^T . \qquad \text{(5c)}$$

It is important to note the complementary ordering of each directional set-- the reason becomes clear in the Appendix, where the angular Laplacian operator $\nabla_\mu^2 \psi(z,\mu)$ is finite differenced according to the Morel scheme[5] to preserve the zeroth and first moments.

Once we have specified the vectors of Eqs(4a,b) and (5b), any symmetric quadrature set is permissible, such as for finite difference found in Ref [6], as long as the numbering convention is obeyed. In general, however, the zero and first moments of the intensity may not be preserved.

2. Angular Discretization of the FPE

The partitioned intensity vector formed by vectors in forward and backward directions is

$$\boldsymbol{\psi}(z) \equiv \begin{bmatrix} \boldsymbol{\psi}^{+}(z) \\ \boldsymbol{\psi}^{-}(z) \end{bmatrix}, \qquad \text{(6a)}$$

where

$$\boldsymbol{\psi}^{+}(z) \equiv \left[\psi_N\left(\tilde{\mu}_N^{+}\right) \;\; \psi_{N-1}\left(\tilde{\mu}_{N-1}^{+}\right) \;\; \ldots. \;\; \psi_1\left(\tilde{\mu}_1^{+}\right) \right]^T$$
$$\boldsymbol{\psi}^{-}(z) \equiv \left[\psi_{N+1}\left(\tilde{\mu}_1^{-}\right) \;\; \psi_{N+2}\left(\tilde{\mu}_2^{-}\right) \;\; \ldots. \;\; \psi_{2N}\left(\tilde{\mu}_N^{-}\right) \right]^T \qquad \text{(6b)}$$

As discretized, the FP balance equation becomes

$$\frac{d\boldsymbol{\psi}(z)}{dz} = \boldsymbol{A}\boldsymbol{\psi}(z)$$
$$\boldsymbol{\psi}^{+}(0) = \boldsymbol{f}, \;\; \boldsymbol{\psi}^{-}(a) = \boldsymbol{g} \qquad \text{(7a,b)}$$

where

$$\boldsymbol{A} \equiv \boldsymbol{M}^{-1}\left[\sigma \boldsymbol{W}^{-1}\boldsymbol{L} - \alpha \boldsymbol{I}\right] \tag{7c}$$

$$\boldsymbol{M} \equiv \begin{bmatrix} \tilde{\boldsymbol{\mu}}^{+} & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{\mu}}^{-} \end{bmatrix} \qquad \text{(7d,e,f)}$$

$$\boldsymbol{W} \equiv \begin{bmatrix} \tilde{\boldsymbol{\varpi}}^{+} & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{\varpi}}^{-} \end{bmatrix}.$$

L is the discretization of the angular Laplacian given in the appendix. In its derivation, a recurrence relation appears requiring that the ordinates be either monotonically increasing or decreasing. Hence, the reason for ordering the ordinates to be decreasing as shown above.

B. Evaluation of the Matrix Exponential

The formal solution to Eqs(7) for a constant ***A*** matrix is

$$\boldsymbol{\psi}(z) \equiv e^{\boldsymbol{A}z}\boldsymbol{\psi}(0). \tag{8}$$

Thus, the spatial treatment is analytical and centers on the evaluation of the exponential matrix, which is challenging [7]. For a reasonable *N*, say under 500, a convenient evaluation is via diagonalization, which is permitted if the matrix has independent eigenvectors as is true here. Diagonalization is more direct than the alternative of specifying eigenvectors and assuming a complete set and expanding as found in Ref [2]. As shown, spectral expansion becomes a consequence of the diagonalization.

The following diagonalization gives

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{\lambda}\boldsymbol{T}^{-1}, \tag{9a}$$

where the eigenvalues are real symmetric sets of negative and positive values ordered as

$$\begin{aligned} \tilde{\boldsymbol{\lambda}}^{-} &\equiv \left[-\lambda_j; j = 1,\ldots,N\right] \\ \tilde{\boldsymbol{\lambda}}^{+} &\equiv \left[\lambda_j; j = N+1,\ldots,2N\right] \end{aligned} \tag{9b}$$

Ordering of the eigenvalues can be arbitrary as long as the positives and negatives are grouped together. Then combining the sets into a diagonal matrix gives

$$\boldsymbol{\lambda} \equiv \begin{bmatrix} \tilde{\boldsymbol{\lambda}}^- & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{\lambda}}^+ \end{bmatrix}. \tag{9c}$$

If $\boldsymbol{T}$ is the eigenvector matrix, where each column corresponds to an eigenvalue of the diagonal column, then

$$\boldsymbol{A}\boldsymbol{T}_k^{\pm} = \pm\lambda_k \boldsymbol{T}_k^{\pm};\ k = 1,\ldots,N. \tag{10a}$$

The eigenvectors, by column, inherit the ordering of the eigenvalues, thus

$$\boldsymbol{T} \equiv \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 & \ldots & \boldsymbol{T}_{2N} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}^- & \boldsymbol{T}^+ \end{bmatrix} \tag{10b}$$

with ordering

$$\begin{aligned} \boldsymbol{T}^- &\equiv \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 & \ldots & \boldsymbol{T}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_N^- & \boldsymbol{T}_{N-1}^- & \ldots & \boldsymbol{T}_1^- \end{bmatrix} \\ \boldsymbol{T}^+ &\equiv \begin{bmatrix} \boldsymbol{T}_{N+1} & \boldsymbol{T}_{N+2} & \ldots & \boldsymbol{T}_{2N} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1^+ & \boldsymbol{T}_2^+ & \ldots & \boldsymbol{T}_N^+ \end{bmatrix}. \end{aligned} \tag{10c}$$

With Eqs(9a and c), $\boldsymbol{A}$ becomes

$$\boldsymbol{A} = \boldsymbol{T} \begin{bmatrix} \tilde{\boldsymbol{\lambda}}^- & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{\lambda}}^+ \end{bmatrix} \boldsymbol{T}^{-1} \tag{11a}$$

giving the matrix exponential

$$e^{\boldsymbol{A}z} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{\Gamma}^-(z) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}^+(z) \end{bmatrix} \boldsymbol{T}^{-1}, \tag{11b}$$

where for $k = 1,\ldots,N$

$$\begin{aligned}\boldsymbol{\Gamma}^{-}(z) &\equiv diag\left\{e^{-\lambda_{N-k+1}z}\right\}\\ \boldsymbol{\Gamma}^{+}(z) &\equiv diag\left\{e^{\lambda_k z}\right\}.\end{aligned} \tag{11c,d}$$

The solution therefore becomes

$$\boldsymbol{\psi}(z) = \boldsymbol{T}\begin{bmatrix}\boldsymbol{\Gamma}^{-}(z) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Gamma}^{+}(z)\end{bmatrix}\boldsymbol{T}^{-1}\boldsymbol{\psi}(0). \tag{12}$$

Continuing, if one partitions $\boldsymbol{T}$ into four square matrices each of size N

$$\boldsymbol{T} \equiv \begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix} \tag{13a}$$

and similarly partitions

$$\boldsymbol{T}^{-1} \equiv \begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}^{-1} = \begin{bmatrix}\boldsymbol{T}_1^{*} & \boldsymbol{T}_2^{*}\\ \boldsymbol{T}_3^{*} & \boldsymbol{T}_4^{*}\end{bmatrix}, \tag{13b}$$

the exponential matrix becomes

$$e^{\boldsymbol{A}z} = \begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}\begin{bmatrix}\boldsymbol{\Gamma}^{-}(z) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Gamma}^{+}(z)\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_1^{*} & \boldsymbol{T}_2^{*}\\ \boldsymbol{T}_3^{*} & \boldsymbol{T}_4^{*}\end{bmatrix}. \tag{14}$$

An alternative representation is possible by considering the transpose $\boldsymbol{A}^T$ satisfying

$$\boldsymbol{A}^T\boldsymbol{X}_k = \lambda_k\boldsymbol{X}_k;\ k = 1,\ldots,N, \tag{15a}$$

where the eigenvalues are identical to those of matrix $\boldsymbol{A}$. Assuming $\boldsymbol{X}_k$ and $\boldsymbol{T}_k$ are normalized such that

$$\boldsymbol{X}_k^T\boldsymbol{T}_k = 1, \tag{15b}$$

then

$$\boldsymbol{T}^{-1} = \boldsymbol{X}^{T}; \tag{15c}$$

and Eq(14) becomes

$$e^{\boldsymbol{A}z} = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^{-}(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}^{+}(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \\ \boldsymbol{X}_3 & \boldsymbol{X}_4 \end{bmatrix}^{T}, \tag{16a}$$

with

$$\boldsymbol{X} \equiv \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 \\ \boldsymbol{X}_3 & \boldsymbol{X}_4 \end{bmatrix} \tag{16b}$$

implying

$$\boldsymbol{X}^{T} \equiv \begin{bmatrix} \boldsymbol{X}_1^{T} & \boldsymbol{X}_3^{T} \\ \boldsymbol{X}_2^{T} & \boldsymbol{X}_4^{T} \end{bmatrix} \tag{16c}$$

and

$$e^{\boldsymbol{A}z} = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^{-}(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}^{+}(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1^{T} & \boldsymbol{X}_3^{T} \\ \boldsymbol{X}_2^{T} & \boldsymbol{X}_4^{T} \end{bmatrix}. \tag{17}$$

C. Superposition of Eigenfunctions

1. Without Scaling

Equation (17), while disguised, represents the matrix of eigenfunctions for the solution expansion. To see this and to reproduce the solution corresponding to that of Ref [2], we first do the matrix multiplication explicitly in Eq(17)

$$e^{\boldsymbol{A}z} = \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^{-}(z)\boldsymbol{X}_1^{T} + \boldsymbol{T}_2\boldsymbol{\Gamma}^{+}(z)\boldsymbol{X}_2^{T} & \boldsymbol{T}_1\boldsymbol{\Gamma}^{-}(z)\boldsymbol{X}_3^{T} + \boldsymbol{T}_2\boldsymbol{\Gamma}^{+}(z)\boldsymbol{X}_4^{T} \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^{-}(z)\boldsymbol{X}_1^{T} + \boldsymbol{T}_4\boldsymbol{\Gamma}^{+}(z)\boldsymbol{X}_2^{T} & \boldsymbol{T}_3\boldsymbol{\Gamma}^{-}(z)\boldsymbol{X}_3^{T} + \boldsymbol{T}_4\boldsymbol{\Gamma}^{+}(z)\boldsymbol{X}_4^{T} \end{bmatrix} \tag{18a}$$

and then decompose into a sum of the matrices

$$\boldsymbol{W}_{-}\left(z\right)=\begin{bmatrix}\boldsymbol{T}_{1}\boldsymbol{\Gamma}^{-}\left(z\right)\boldsymbol{X}_{1}^{T} & \boldsymbol{T}_{1}\boldsymbol{\Gamma}^{-}\left(z\right)\boldsymbol{X}_{3}^{T}\\ \boldsymbol{T}_{3}\boldsymbol{\Gamma}^{-}\left(z\right)\boldsymbol{X}_{1}^{T} & \boldsymbol{T}_{3}\boldsymbol{\Gamma}^{-}\left(z\right)\boldsymbol{X}_{3}^{T}\end{bmatrix}$$

$$\boldsymbol{W}_{+}\left(z\right)=\begin{bmatrix}\boldsymbol{T}_{2}\boldsymbol{\Gamma}^{+}\left(z\right)\boldsymbol{X}_{2}^{T} & \boldsymbol{T}_{2}\boldsymbol{\Gamma}^{+}\left(z\right)\boldsymbol{X}_{4}^{T}\\ \boldsymbol{T}_{4}\boldsymbol{\Gamma}^{+}\left(z\right)\boldsymbol{X}_{2}^{T} & \boldsymbol{T}_{4}\boldsymbol{\Gamma}^{+}\left(z\right)\boldsymbol{X}_{4}^{T}\end{bmatrix} \qquad (18b,c)$$

to give

$$e^{\boldsymbol{A}z}=\boldsymbol{W}_{-}\left(z\right)+\boldsymbol{W}_{+}\left(z\right). \qquad (18d)$$

The explicit matrix multiplications give the following matrix elements for indices (r,s) = 1,…,4 and (i,j)=1,…,N

$$\left(\boldsymbol{T}_{r}\boldsymbol{\Gamma}^{-}\left(z\right)\boldsymbol{X}_{s}^{T}\right)_{ij}=\sum_{k=1}^{N}e^{-\left|\lambda_{N-k+1}\right|z}T_{r,ik}X_{s,kj}^{T}$$

$$\left(\boldsymbol{T}_{r}\boldsymbol{\Gamma}^{+}\left(z\right)\boldsymbol{X}_{s}^{T}\right)_{ij}=\sum_{k=1}^{N}e^{\left|\lambda_{k}\right|z}T_{r,ik}X_{s,kj}^{T}. \qquad (19)$$

On substitution

$$\boldsymbol{W}_{-}\left(z\right)=\begin{bmatrix}\boldsymbol{W}_{1}^{-} & \boldsymbol{W}_{2}^{-}\\ \boldsymbol{W}_{3}^{-} & \boldsymbol{W}_{4}^{-}\end{bmatrix}$$

$$\boldsymbol{W}_{+}\left(z\right)=\begin{bmatrix}\boldsymbol{W}_{1}^{+} & \boldsymbol{W}_{2}^{+}\\ \boldsymbol{W}_{3}^{+} & \boldsymbol{W}_{4}^{+}\end{bmatrix}, \qquad (20a)$$

into Eqs(18b,c), the matrix exponential becomes

$$e^{\boldsymbol{A}z}=\begin{bmatrix}\sum_{k=1}^{N}e^{-\lambda_{N-k+1}z}\boldsymbol{T}_{1,k}\boldsymbol{X}_{1,k}^{T} & \sum_{k=1}^{N}e^{-\lambda_{N-k+1}z}\boldsymbol{T}_{1,k}\boldsymbol{X}_{3,k}^{T}\\ \sum_{k=1}^{N}e^{-\lambda_{N-k+1}z}\boldsymbol{T}_{3,k}\boldsymbol{X}_{1,k}^{T} & \sum_{k=1}^{N}e^{-\lambda_{N-k+1}z}\boldsymbol{T}_{3,k}\boldsymbol{X}_{3,k}^{T}\end{bmatrix}+\begin{bmatrix}\sum_{k=1}^{N}e^{\lambda_{k}z}\boldsymbol{T}_{2,k}\boldsymbol{X}_{2,k}^{T} & \sum_{k=1}^{N}e^{\lambda_{k}z}\boldsymbol{T}_{2,k}\boldsymbol{X}_{4,k}^{T}\\ \sum_{k=1}^{N}e^{\lambda_{k}z}\boldsymbol{T}_{4,k}\boldsymbol{X}_{2,k}^{T} & \sum_{k=1}^{N}e^{\lambda_{k}z}\boldsymbol{T}_{4,k}\boldsymbol{X}_{4,k}^{T}\end{bmatrix}. \qquad (20b)$$

By extracting the common summation from each element, there results

$$e^{\boldsymbol{A}z}=\sum_{k=1}^{N}\left\{e^{-\lambda_{N-k+1}z}\begin{bmatrix}\boldsymbol{T}_{1,k}\boldsymbol{X}_{1,k}^{T} & \boldsymbol{T}_{1,k}\boldsymbol{X}_{3,k}^{T}\\ \boldsymbol{T}_{3,k}\boldsymbol{X}_{1,k}^{T} & \boldsymbol{T}_{3,k}\boldsymbol{X}_{3,k}^{T}\end{bmatrix}+e^{\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{2,k}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{T}_{2,k}\boldsymbol{X}_{4,k}^{T}\\ \boldsymbol{T}_{4,k}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{T}_{4,k}\boldsymbol{X}_{4,k}^{T}\end{bmatrix}\right\}, \quad (21a)$$

re-expressing the inner matrices and re-indexing the first sum, one finds

$$e^{\boldsymbol{A}z}=\sum_{k=1}^{N}\left\{e^{-\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{1,N-k+1}\\ \boldsymbol{T}_{3,N-k+1}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_{1,N-k+1}^{T} & \boldsymbol{X}_{3,N-k+1}^{T}\end{bmatrix}+e^{\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{2,k}\\ \boldsymbol{T}_{4,k}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{X}_{4,k}^{T}\end{bmatrix}\right\}. \quad (21b)$$

Introducing Eq(21b) into Eq(8) results in the intensity

$$\boldsymbol{\psi}(z)\equiv\sum_{k=1}^{N}\left\{e^{-\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{1,N-k+1}\\ \boldsymbol{T}_{3,N-k+1}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_{1,N-k+1}^{T} & \boldsymbol{X}_{3,N-k+1}^{T}\end{bmatrix}+e^{\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{2,k}\\ \boldsymbol{T}_{4,k}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{X}_{4,k}^{T}\end{bmatrix}\right\}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix} \quad (22a)$$

or

$$\boldsymbol{\psi}(z)\equiv\sum_{k=1}^{N}\left\{\begin{array}{r}e^{-\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{1,N-k+1}\\ \boldsymbol{T}_{3,N-k+1}\end{bmatrix}\left\{\begin{bmatrix}\boldsymbol{X}_{1,N-k+1}^{T} & \boldsymbol{X}_{3,N-k+1}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix}\right\}+\\ +e^{\lambda_k z}\begin{bmatrix}\boldsymbol{T}_{2,k}\\ \boldsymbol{T}_{4,k}\end{bmatrix}\left\{\begin{bmatrix}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{X}_{4,k}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix}\right\}\end{array}\right\}. \quad (22b)$$

If we let the expansion coefficients be

$$\begin{aligned}\boldsymbol{\alpha}_k &\equiv\begin{bmatrix}\boldsymbol{X}_{1,N-k+1}^{T} & \boldsymbol{X}_{3,N-k+1}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix}\\ \boldsymbol{\beta}_k &\equiv\begin{bmatrix}\boldsymbol{X}_{2,k}^{T} & \boldsymbol{X}_{4,k}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix},\end{aligned} \quad (23a)$$

then

$$\boldsymbol{\psi}(z) \equiv \sum_{k=1}^{N} \left\{ e^{-\lambda_k z} \begin{bmatrix} \boldsymbol{T}_{1,N-k+1} \\ \boldsymbol{T}_{3,N-k+1k} \end{bmatrix} \alpha_k + e^{\lambda_k z} \begin{bmatrix} \boldsymbol{T}_{2,k} \\ \boldsymbol{T}_{4,k} \end{bmatrix} \beta_k \right\} \tag{23b}$$

in terms of eigenfunctions

$$e^{-\lambda_k z} \begin{bmatrix} \boldsymbol{T}_{1,N-k+1} \\ \boldsymbol{T}_{3,N-k+1} \end{bmatrix},\ e^{\lambda_k z} \begin{bmatrix} \boldsymbol{T}_{2,k} \\ \boldsymbol{T}_{4,k} \end{bmatrix}. \tag{23c}$$

The solution, Eq(23b), is similar to Eq(32) in Ref [2], with one major difference. One of the chosen eigenfunctions is a growing exponential and therefore is unstable for large eigenvalues. Hence, we must introduce a stability factor to obtain our final result. Before doing so however, let us complete the solution by finding the expansion coefficients from the initial conditions.

To find the expansion coefficients, rather than use Eqs(23), it is more convenient to use Eq(17) in Eq(8)

$$\boldsymbol{\psi}(z) \equiv \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^-(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}^+(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1^T & \boldsymbol{X}_3^T \\ \boldsymbol{X}_2^T & \boldsymbol{X}_4^T \end{bmatrix} \boldsymbol{\psi}(0). \tag{24a}$$

If, we express

$$\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{X}_1^T & \boldsymbol{X}_3^T \\ \boldsymbol{X}_2^T & \boldsymbol{X}_4^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}^+(0) \\ \boldsymbol{\psi}^-(0) \end{bmatrix}, \tag{24b}$$

then

$$\boldsymbol{\psi}(z) \equiv \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^-(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}^+(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}. \tag{24c}$$

Equation(24c) splits into the following two vector equations for forward and backward directions:

$$\begin{aligned}\boldsymbol{\psi}^{+}(z)&=\boldsymbol{T}_1\boldsymbol{\Gamma}^{-}(z)\boldsymbol{\alpha}+\boldsymbol{T}_2\boldsymbol{\Gamma}^{+}(z)\boldsymbol{\beta}\\ \boldsymbol{\psi}^{-}(z)&=\boldsymbol{T}_3\boldsymbol{\Gamma}^{-}(z)\boldsymbol{\alpha}+\boldsymbol{T}_4\boldsymbol{\Gamma}^{+}(z)\boldsymbol{\beta}\end{aligned}\tag{25a,b}$$

If we let $z = 0$ and a in Eqs(25a,b) respectively and introduce the known boundary conditions, Eqs(25) become

$$\begin{aligned}\boldsymbol{\psi}^{+}(0)&=f=\boldsymbol{T}_1\boldsymbol{\alpha}+\boldsymbol{T}_2\boldsymbol{\beta}\\ \boldsymbol{\psi}^{-}(a)&=g=\boldsymbol{T}_3\boldsymbol{\Gamma}^{-}(a)\boldsymbol{\alpha}+\boldsymbol{T}_4\boldsymbol{\Gamma}^{+}(a)\boldsymbol{\beta}.\end{aligned}\tag{26a,b}$$

Solving for the expansion coefficients gives

$$\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}=\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3\boldsymbol{\Gamma}^{-}(a) & \boldsymbol{T}_4\boldsymbol{\Gamma}^{+}(a)\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\end{bmatrix}.\tag{27}$$

The exiting intensities come from Eq(25a,b) with $z = a$ and 0 respectively

$$\begin{aligned}\boldsymbol{\psi}^{+}(a)&=\boldsymbol{T}_1\boldsymbol{\Gamma}^{-}(a)\boldsymbol{\alpha}+\boldsymbol{T}_2\boldsymbol{\Gamma}^{+}(a)\boldsymbol{\beta}\\ \boldsymbol{\psi}^{-}(0)&=\boldsymbol{T}_3\boldsymbol{\alpha}+\boldsymbol{T}_4\boldsymbol{\beta}.\end{aligned}\tag{28a,b}$$

In vector notation

$$\begin{bmatrix}\boldsymbol{\psi}^{+}(a)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix}=\begin{bmatrix}\boldsymbol{T}_1\boldsymbol{\Gamma}^{-}(a) & \boldsymbol{T}_2\boldsymbol{\Gamma}^{+}(a)\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}\tag{29a}$$

and substituting Eq(27) gives

$$\begin{bmatrix}\boldsymbol{\psi}^{+}(a)\\ \boldsymbol{\psi}^{-}(0)\end{bmatrix}=\boldsymbol{R}\begin{bmatrix}\boldsymbol{\psi}^{+}(0)\\ \boldsymbol{\psi}^{-}(a)\end{bmatrix},\tag{29b}$$

where the response matrix is

$$\boldsymbol{R} \equiv \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^-(a) & \boldsymbol{T}_2\boldsymbol{\Gamma}^+(a) \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a) & \boldsymbol{T}_4\boldsymbol{\Gamma}^+(a) \end{bmatrix}^{-1}. \tag{29c}$$

The response matrix response directly relates the ingoing to the outgoing intensities at the boundaries.

For the slab interior, the final solution without scaling, from Eq(24a,b) cast in vector form, is

$$\boldsymbol{\psi}(z) \equiv \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^-(z) & \boldsymbol{T}_2\boldsymbol{\Gamma}^+(z) \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(z) & \boldsymbol{T}_4\boldsymbol{\Gamma}^+(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}. \tag{30a}$$

or expanded

$$\boldsymbol{\psi}(z) \equiv \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^-(z) & \boldsymbol{T}_2\boldsymbol{\Gamma}^+(z) \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(z) & \boldsymbol{T}_4\boldsymbol{\Gamma}^+(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a) & \boldsymbol{T}_4\boldsymbol{\Gamma}^+(a) \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\psi}^+(0) \\ \boldsymbol{\psi}^-(a) \end{bmatrix}. \tag{30b}$$

The obvious numerical shortcoming of the solution is the exponential with the positive exponent in $\boldsymbol{\Gamma}^+(z)$. We now address this with a scaling factor.

2. With Scaling

Here, we begin with the formal solution representation of Eq(8) and Eq(12)

$$\boldsymbol{\psi}(z) = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^-(z)_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}^+(z)_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix}^{-1} \boldsymbol{\psi}(0). \tag{31a}$$

Note that the exponential diagonals are further identified by their quadrant, either 1 or 4. This is because of the ordering of the diagonals, which gives

$$\boldsymbol{\Gamma}^-(z)_1 \neq \boldsymbol{\Gamma}^-(z)_4, \tag{31b}$$

where for $k = 1,...,N$ from Eq(11c)

$$\begin{aligned}\mathbf{\Gamma}^{-}\left(z\right)_1 &\equiv diag\left\{e^{-\lambda_{N-k+1}z}\right\}\\ \mathbf{\Gamma}^{-}\left(z\right)_4 &\equiv diag\left\{e^{-\lambda_k z}\right\}.\end{aligned} \tag{31c,d}$$

To include a scaling factor, we insert the identity

$$\begin{bmatrix}\mathbf{\Gamma}^{-}\left(z\right)_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{+}\left(z\right)_4\end{bmatrix}=\begin{bmatrix}\mathbf{\Gamma}^{-}\left(z\right)_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{+}\left(z\right)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}^{-1}, \tag{32a}$$

which is also

$$\begin{bmatrix}\mathbf{\Gamma}^{-}\left(z\right)_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{+}\left(z\right)_4\end{bmatrix}=\begin{bmatrix}\mathbf{\Gamma}^{-}\left(z\right)_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a-z\right)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}^{-1}. \tag{32b}$$

and noting that

$$\mathbf{\Gamma}^{+}\left(z\right)_4\mathbf{\Gamma}^{-}\left(-a\right)_4=\mathbf{\Gamma}^{-}\left(a-z\right)_4. \tag{32c}$$

Thus, Eq(31a) becomes

$$\boldsymbol{\psi}\left(z\right)=\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}\begin{bmatrix}\mathbf{\Gamma}^{-}\left(z\right)_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a-z\right)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}^{-1}\boldsymbol{\psi}\left(0\right) \tag{33a}$$

and combining inverses

$$\boldsymbol{\psi}\left(z\right)=\begin{bmatrix}\boldsymbol{T}_1\mathbf{\Gamma}^{-}\left(z\right)_1 & \boldsymbol{T}_2\mathbf{\Gamma}^{-}\left(a-z\right)_4\\ \boldsymbol{T}_3\mathbf{\Gamma}^{-}\left(z\right)_1 & \boldsymbol{T}_4\mathbf{\Gamma}^{-}\left(a-z\right)_4\end{bmatrix}\left[\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}\right]^{-1}\boldsymbol{\psi}\left(0\right) \tag{33b}$$

and multiplying through

$$\boldsymbol{\psi}\left(z\right)=\begin{bmatrix}\boldsymbol{T}_1\mathbf{\Gamma}^{-}\left(z\right)_1 & \boldsymbol{T}_2\mathbf{\Gamma}^{-}\left(a-z\right)_4\\ \boldsymbol{T}_3\mathbf{\Gamma}^{-}\left(z\right)_1 & \boldsymbol{T}_4\mathbf{\Gamma}^{-}\left(a-z\right)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\mathbf{\Gamma}^{-}\left(a\right)_4\\ \boldsymbol{T}_3 & \boldsymbol{T}_4\mathbf{\Gamma}^{-}\left(a\right)_4\end{bmatrix}^{-1}\boldsymbol{\psi}\left(0\right). \tag{33c}$$

If

$$\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \boldsymbol{\Gamma}^- \left(a\right)_4 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4 \boldsymbol{\Gamma}^- \left(a\right)_4 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\psi}^+ \left(0\right) \\ \boldsymbol{\psi}^- \left(0\right) \end{bmatrix}, \tag{34a}$$

then

$$\boldsymbol{\psi}\left(z\right) = \begin{bmatrix} \boldsymbol{T}_1 \boldsymbol{\Gamma}^- \left(z\right)_1 & \boldsymbol{T}_2 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \\ \boldsymbol{T}_3 \boldsymbol{\Gamma}^- \left(z\right)_1 & \boldsymbol{T}_4 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \tag{34b}$$

or

$$\boldsymbol{\psi}\left(z\right) = \begin{bmatrix} \boldsymbol{T}_1 \boldsymbol{\Gamma}^- \left(z\right)_1 \boldsymbol{\alpha} + \boldsymbol{T}_2 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \boldsymbol{\beta} \\ \boldsymbol{T}_3 \boldsymbol{\Gamma}^- \left(z\right)_1 \boldsymbol{\alpha} + \boldsymbol{T}_4 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \boldsymbol{\beta} \end{bmatrix}. \tag{34c}$$

By explicitly expanding the matrix multiplications as done for Eq(23b), we find

$$\boldsymbol{\psi}\left(z\right) \equiv \sum_{k=1}^{N} \left\{ e^{-\lambda_k z} \begin{bmatrix} \boldsymbol{T}_{1,N-k+1} \\ \boldsymbol{T}_{3,N-k+1} \end{bmatrix} \alpha_k + e^{-\lambda_k (a-z)} \begin{bmatrix} \boldsymbol{T}_{2,k} \\ \boldsymbol{T}_{4,k} \end{bmatrix} \beta_k \right\}, \tag{35}$$

which is essentially identical to Eq(32) of Ref [2].

To find the expansion coefficients as above, rather than use Eq(35), it is best to note that Eq(34c) is actually two equations

$$\begin{aligned} \boldsymbol{\psi}^+ \left(z\right) &= \boldsymbol{T}_1 \boldsymbol{\Gamma}^- \left(z\right)_1 \boldsymbol{\alpha} + \boldsymbol{T}_2 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \boldsymbol{\beta} \\ \boldsymbol{\psi}^- \left(z\right) &= \boldsymbol{T}_3 \boldsymbol{\Gamma}^- \left(z\right)_1 \boldsymbol{\alpha} + \boldsymbol{T}_4 \boldsymbol{\Gamma}^- \left(a-z\right)_4 \boldsymbol{\beta}, \end{aligned} \tag{36a}$$

which for *z* = 0, and *a* respectively give

$$\begin{aligned} \boldsymbol{f} &= \boldsymbol{T}_1 \boldsymbol{\alpha} + \boldsymbol{T}_2 \boldsymbol{\Gamma}^- \left(a\right)_4 \boldsymbol{\beta} \\ \boldsymbol{g} &= \boldsymbol{T}_3 \boldsymbol{\Gamma}^- \left(a\right)_1 \boldsymbol{\alpha} + \boldsymbol{T}_4 \boldsymbol{\beta} \end{aligned} \tag{36b}$$

or

$$\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2\boldsymbol{\Gamma}^-(a)_4 \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}. \tag{36c}$$

Solving for the coefficients yields

$$\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2\boldsymbol{\Gamma}^-(a)_4 \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_4 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}. \tag{37}$$

Next, evaluating, Eqs(36a) at $z = a$ and 0 respectively

$$\begin{aligned} \boldsymbol{\psi}^+(a) &= \boldsymbol{T}_1\boldsymbol{\Gamma}^-(a)_1\,\boldsymbol{\alpha} + \boldsymbol{T}_2\boldsymbol{\beta} \\ \boldsymbol{\psi}^-(0) &= \boldsymbol{T}_3\boldsymbol{\alpha} + \boldsymbol{T}_4\boldsymbol{\Gamma}^-(a)_4\,\boldsymbol{\beta} \end{aligned} \tag{38a}$$

or

$$\begin{bmatrix} \boldsymbol{\psi}^+(a) \\ \boldsymbol{\psi}^-(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4\boldsymbol{\Gamma}^-(a)_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \tag{38b}$$

and substituting Eq(37), we find

$$\begin{bmatrix} \boldsymbol{\psi}^+(a) \\ \boldsymbol{\psi}^-(0) \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}, \tag{39a}$$

where the response matrix without unbounded exponentials is

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4\boldsymbol{\Gamma}^-(a)_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2\boldsymbol{\Gamma}^-(a)_4 \\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_4 \end{bmatrix}^{-1}. \tag{39b}$$

Finally, Eq(34b) with Eq(36d) gives the interior intensity

$$\boldsymbol{\psi}(z)=\begin{bmatrix}\boldsymbol{T}_1\boldsymbol{\Gamma}^-(z)_1 & \boldsymbol{T}_2\boldsymbol{\Gamma}^-(a-z)_4\\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(z)_1 & \boldsymbol{T}_4\boldsymbol{\Gamma}^-(a-z)_4\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2\boldsymbol{\Gamma}^-(a)_4\\ \boldsymbol{T}_3\boldsymbol{\Gamma}^-(a)_1 & \boldsymbol{T}_4\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\end{bmatrix}. \quad (40)$$

We are now ready for numerical demonstrations.

III. Numerical Implementation

A. First Benchmark Demonstration

A MATLAB program is available to find the exiting intensities and the spatial variation of the intensity from Eqs(39) and (40) for a 1*cm* thick slab with the following properties:

$$\alpha = 0.02cm^{-1},\ \sigma = 0.01cm^{-1},\ g = 0.99,\ a = 1cm. \qquad (41a)$$

with boundary conditions

$$\boldsymbol{f} = \mathbf{1}, \boldsymbol{g} = \mathbf{2}. \qquad (41b)$$

Table 1 shows the progression of convergence of the angular reflectance $\psi^-(0)$ and transmittance $\psi^+(a)$ with quadrature order 2*N*. We find 11 angular fluxes for each quadrature approximation. The last two columns give the relative error between consecutive quadrature. The discrepant digits with respect to the last tabular entry (2*N* = 1600), which is expected to be precise to its last digit, are marked in bold in each 2*N* approximation. As observed, 5-place precision is maintained after 2*N* = 600. For N = 1400 and 1600, all but two entries are off by one digit in the sixth place.

Table I. Reflectance and Transmittance

2*N*= 400

μ	***Ref***	***Trn***	***RelErr/Ref***	***RelErr/Trn***
1.00e+00	1.000000e+00	9.79997**1**e-01	0.00e+00	2.76e-08
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	1.10e-07
6.00e-01	1.000000e+00	9.6786**15**e-01	0.00e+00	1.00e-05
4.00e-01	1.000000e+00	9.9373**84**e-01	0.00e+00	3.01e-06
2.00e-01	1.000000e+00	1.166696e+00	0.00e+00	3.66e-05
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.54649**5**e+00	2.000000e+00	1.82e-05	0.00e+00
-4.00e-01	1.81781**1**e+00	2.000000e+00	1.72e-05	0.00e+00
-6.00e-01	1.92580**4**e+00	2.000000e+00	7.13e-06	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	4.96e-07	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	2.76e-08	0.00e+00

$2N$= 600

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	5.11e-09
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	2.01e-08
6.00e-01	1.000000e+00	9.67859**7**e-01	0.00e+00	1.87e-06
4.00e-01	1.000000e+00	9.93739**0**e-01	0.00e+00	5.42e-07
2.00e-01	1.000000e+00	1.16670**4**e+00	0.00e+00	6.87e-06
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.54649**0**e+00	2.000000e+00	3.38e-06	0.00e+00
-4.00e-01	1.81780**5**e+00	2.000000e+00	3.18e-06	0.00e+00
-6.00e-01	1.92580**6**e+00	2.000000e+00	1.33e-06	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	9.19e-08	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	5.12e-09	0.00e+00

$2N$= 800

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	1.79e-09
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	7.01e-09
6.00e-01	1.000000e+00	9.67859**0**e-01	0.00e+00	6.56e-07
4.00e-01	1.000000e+00	9.93739**1**e-01	0.00e+00	1.88e-07
2.00e-01	1.000000e+00	1.16670**7**e+00	0.00e+00	2.42e-06
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.54648**8**e+00	2.000000e+00	1.18e-06	0.00e+00
-4.00e-01	1.81780**3**e+00	2.000000e+00	1.11e-06	0.00e+00
-6.00e-01	1.92580**7**e+00	2.000000e+00	4.67e-07	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	3.22e-08	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	1.79e-09	0.00e+00

$2N$=1000

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	8.67e-10
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	3.27e-09
6.00e-01	1.000000e+00	9.67858**7**e-01	0.00e+00	3.04e-07
4.00e-01	1.000000e+00	9.93739**2**e-01	0.00e+00	8.63e-08
2.00e-01	1.000000e+00	1.16670**8**e+00	0.00e+00	1.12e-06
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.54648**7**e+00	2.000000e+00	5.48e-07	0.00e+00
-4.00e-01	1.81780**2**e+00	2.000000e+00	5.13e-07	0.00e+00
-6.00e-01	1.925808e+00	2.000000e+00	2.16e-07	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	1.49e-08	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	8.31e-10	0.00e+00

$2N=1200$

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	4.59e-10
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	1.77e-09
6.00e-01	1.000000e+00	9.67858**6**e-01	0.00e+00	1.65e-07
4.00e-01	1.000000e+00	9.937393e-01	0.00e+00	4.67e-08
2.00e-01	1.000000e+00	1.16670**9**e+00	0.00e+00	6.11e-07
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.546487e+00	2.000000e+00	2.98e-07	0.00e+00
-4.00e-01	1.81780**2**e+00	2.000000e+00	2.78e-07	0.00e+00
-6.00e-01	1.925808e+00	2.000000e+00	1.18e-07	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	8.09e-09	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	4.49e-10	0.00e+00

$2N=1400$

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	3.03e-10
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	9.44e-10
6.00e-01	1.000000e+00	9.67858**5**e-01	0.00e+00	9.96e-08
4.00e-01	1.000000e+00	9.937393e-01	0.00e+00	2.80e-08
2.00e-01	1.000000e+00	1.16670**9**e+00	0.00e+00	3.69e-07
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.546486e+00	2.000000e+00	1.80e-07	0.00e+00
-4.00e-01	1.817801e+00	2.000000e+00	1.68e-07	0.00e+00
-6.00e-01	1.925808e+00	2.000000e+00	7.09e-08	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	4.89e-09	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	2.74e-10	0.00e+00

$2N=1600$

1.00e+00	1.000000e+00	9.799972e-01	0.00e+00	6.02e-11
8.00e-01	1.000000e+00	9.747933e-01	0.00e+00	9.19e-10
6.00e-01	1.000000e+00	9.678584e-01	0.00e+00	6.49e-08
4.00e-01	1.000000e+00	9.937393e-01	0.00e+00	1.81e-08
2.00e-01	1.000000e+00	1.166710e+00	0.00e+00	2.40e-07
0.00e+00	1.000000e+00	2.000000e+00	0.00e+00	0.00e+00
-2.00e-01	1.546486e+00	2.000000e+00	1.17e-07	0.00e+00
-4.00e-01	1.817801e+00	2.000000e+00	1.09e-07	0.00e+00
-6.00e-01	1.925808e+00	2.000000e+00	4.61e-08	0.00e+00
-8.00e-01	1.949563e+00	2.000000e+00	3.17e-09	0.00e+00
-1.00e+00	1.959994e+00	2.000000e+00	1.75e-10	0.00e+00

The table required about 5 minutes of computing time on a Dell/Precision 2.4GHz PC.

Figures 2 and 3 show the reflectance, transmission and spatial distributions across the medium. They are visually in complete agreement with References 2 and 6.

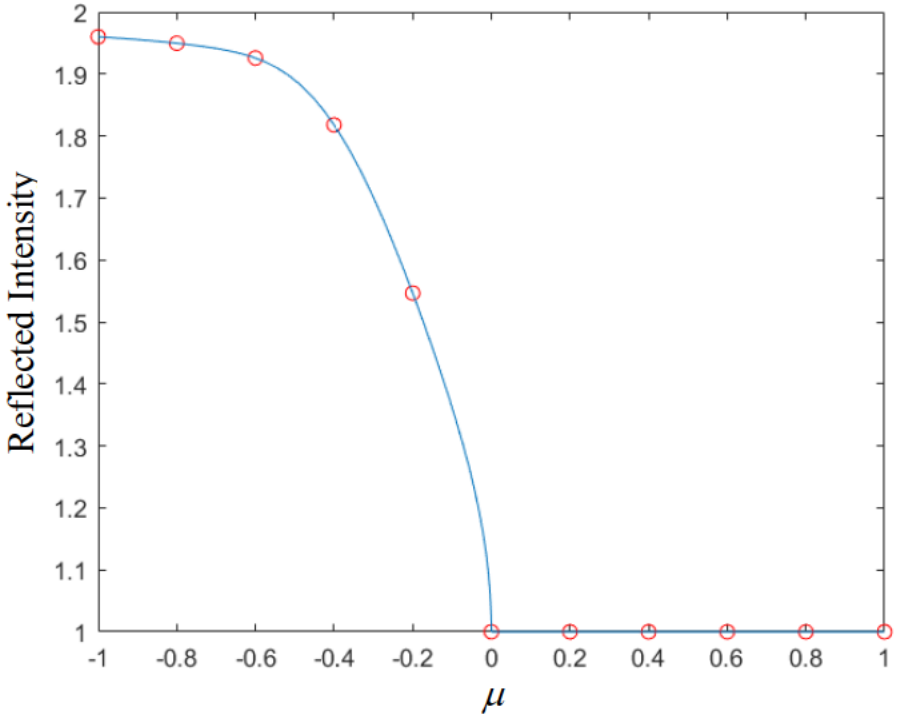

Fig. 2. Reflectance intensity distribution.

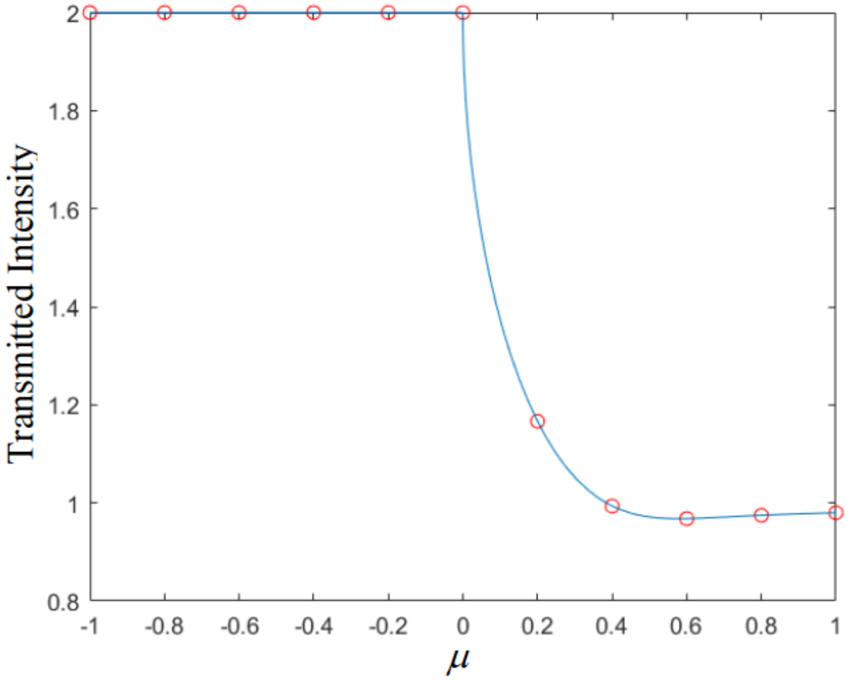

Fig. 3. Transmittance intensity distribution.

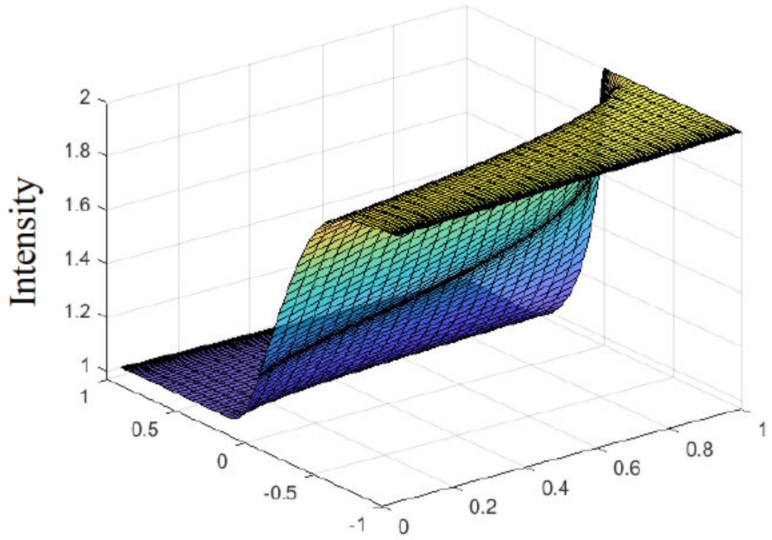

Fig. 4. Spatial intensity distribution

B. Second Benchmark Demonstration

For a second demonstration, we compare the eigenfunction and a finite difference benchmark solution[2]. This benchmark can be considered a standard, where the spatial and angular variables are simply discretized. The numerical scheme is verified by the method of manufactured solutions (MMS), where a solution is assumed yielding a source that will give the solution exactly. A numerical error then follows from the application of the numerical scheme to the MMS.

Table 2. Oscar's Finite Difference Solution
($I = 801$ and $N = 20000$):

μ	$z = 0$ (Ref)	$z = 1$ (Trn)
1.00e+00	1.00000e+00	9.799972e-01
8.00e-01	1.00000e+00	9.747933e-01
6.00e-01	1.00000e+00	9.678584e-01
4.00e-01	1.00000e+00	9.937384e-01
2.00e-01	1.00000e+00	1.166714e+00
0.00e+00	1.00000e+00	2.00000e+00
−2.00e-01	1.546487e+00	2.00000e+00
−4.00e-01	1.817804e+00	2.00000e+00
−6.00e-01	1.925808e+00	2.00000e+00
−8.00e-01	1.949563e+00	2.00000e+00
−1.00e+00	1.959994e+00	2.00000e+00

The comparison shows excellent agreement between the two benchmarks, which are only off one digit in the fifth place and three digits in the sixth place.

CONCLUSION

Another approach to the solution of the Fokker-Planck equation with constant coefficients has been presented. Our method features diagonalization of the matrix exponential representing the analytical solution. The result is the solution found in Ref 2, but by straightforward linear algebra that defines the spectral eigenfunction expansion directly. The solution expressible in terms of a response matrix that is independent of boundary conditions depending only on slab properties and thickness. We then provide a solid 5-place benchmark for a 1600 quadrature to take its place in the literature.

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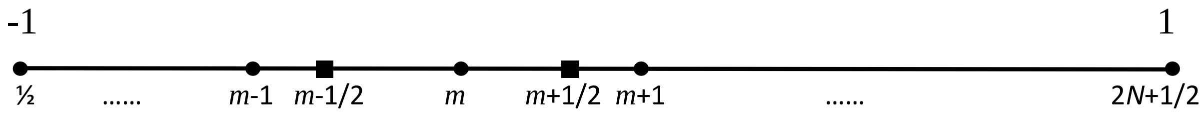
APPENDIX: Discretization of the FP Laplacian Operator

Define ∇_m^2 as the discretized approximation to the FP operator

$$\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi\left(z,\mu\right)}{\partial\mu}\right]_{\mu_m} \simeq \nabla_m^2\psi_m\left(z\right),. \tag{A1a}$$

specified at a Gauss abscissa μ_m. In between, each abscissae is an intermediate



abscissae, $\mu_{m+1/2}$

$$\mu_m < \mu_{m+1/2} < \mu_{m+1}, \tag{A1b}$$

not necessarily at the cell center. We approximate the angular derivative on the intermediate abscissae as

$$\left.\frac{\partial\psi\left(z,\mu\right)}{\partial\mu}\right|_{m+1/2} \simeq \frac{\psi_{m+1}\left(z\right)-\psi_m\left(z\right)}{\mu_{m+1}-\mu_m} \equiv \tilde{\psi}_{m+1/2}. \tag{A2}$$

On the abscissae, the approximate FP operator then becomes

$$\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi\left(z,\mu\right)}{\partial\mu}\right]_{\mu_m} \simeq \nabla_m^2\psi_m\left(z\right) \equiv \frac{\left(1-\mu_{m+1/2}^2\right)\tilde{\psi}_{m+1/2}-\left(1-\mu_{m-1/2}^2\right)\tilde{\psi}_{m-1/2}}{\mu_{m+1/2}-\mu_{m-1/2}}. \tag{A3}$$

To ensure the conservation of the zeroth and first moments, we replace $1-\mu_{m+1/2}^2$ by $\gamma_{m+1/2}$ and $\mu_{m+1/2}-\mu_{m-1/2}$ by ω_m

$$\nabla_m^2\psi_m\left(z\right) \equiv \frac{1}{\omega_m}\left[\gamma_{m+1/2}\tilde{\psi}_{m+1/2}-\gamma_{m-1/2}\tilde{\psi}_{m-1/2}\right], \tag{A4}$$

and force the following two moments to be satisfied.

For the zeroth moment

$$\int_{-1}^{1} d\mu \frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi\left(z,\mu\right)}{\partial\mu}\right]=0 \qquad \text{(A5a)}$$

or

$$\sum_{m=1}^{2N}\omega_m\nabla_m^2\tilde{\psi}=0 \qquad \text{(A5b)}$$

which gives from Eq(A4)

$$\sum_{m=1}^{2N}\omega_m\nabla_m^2\psi_m\left(\mathrm{z}\right)\equiv\sum_{m=1}^{2N}\left[\gamma_{m+1/2}\tilde{\psi}_{m+1/2}-\gamma_{m-1/2}\tilde{\psi}_{m-1/2}\right]=0\,. \qquad \text{(A5c)}$$

On cancellation

$$\sum_{m=1}^{2N}\left[\gamma_{m+1/2}\tilde{\psi}_{m+1/2}-\gamma_{m-1/2}\tilde{\psi}_{m-1/2}\right]=\gamma_{2N+1/2}\tilde{\psi}_{2N+1/2}-\gamma_{1/2}\tilde{\psi}_{1/2}=0 \qquad \text{(A6a)}$$

satisfied if

$$\begin{aligned}\gamma_{2N+1/2}&=0\\ \gamma_{1/2}&=0,\end{aligned} \qquad \text{(A6b,c)}$$

and the zeroth moment holds.

For the first moment, from two integrations by parts

$$\int_{-1}^{1} d\mu\mu\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi\left(\mathrm{z},\mu\right)}{\partial\mu}\right]=-2\int_{-1}^{1} d\mu\mu\psi\left(\mathrm{z},\mu\right) \qquad \text{(A7a)}$$

and Gauss quadrature

$$\sum_{m=1}^{2N}\omega_m\mu_m\nabla_m^2\psi_m\left(z\right)=-2\sum_{m=1}^{2N}\omega_m\mu_m\psi_m\left(z\right). \qquad \text{(A7b)}$$

On substitution of Eq(A4)

$$\sum_{m=1}^{2N}\left\{\mu_m\left[\gamma_{m+1/2}\tilde{\psi}_{m+1/2}-\gamma_{m-1/2}\tilde{\psi}_{m-1/2}\right]+2\omega_m\mu_m\psi_m\left(z\right)\right\}=0 \qquad \text{(A8a)}$$

and from the definition of Eq(A2)

$$\sum_{m=1}^{2N}\mu_m\left\{\begin{array}{r}\left[\gamma_{m+1/2}\left[\dfrac{\psi_{m+1}\left(z\right)-\psi_m\left(z\right)}{\mu_{m+1}-\mu_m}\right]-\gamma_{m-1/2}\left[\dfrac{\psi_m\left(z\right)-\psi_{m-1}\left(z\right)}{\mu_m-\mu_{m-1}}\right]\right] \\ +2\omega_m\psi_m\left(z\right)\end{array}\right\}=0. \qquad \text{(A8b)}$$

Gathering terms

$$\sum_{m=1}^{2N}\mu_m\left\{\left[\begin{array}{l}\left[\gamma_{m+1/2}\dfrac{\psi_{m+1}\left(z\right)}{\mu_{m+1}-\mu_m}+\gamma_{m-1/2}\dfrac{\psi_{m-1}\left(z\right)}{\mu_m-\mu_{m-1}}\right]- \\ \quad -\left[\dfrac{\gamma_{m+1/2}}{\mu_{m+1}-\mu_m}+\dfrac{\gamma_{m-1/2}}{\mu_m-\mu_{m-1}}\right]\psi_{m+1}\left(z\right)\end{array}\right]+2\omega_m\psi_m\left(z\right)\right\}=0 \qquad \text{(A9)}$$

and re-indexing the first term in square brackets, we find

$$\sum_{m=1}^{2N}\left[\gamma_{m-1/2}\frac{\mu_{m-1}}{\mu_m-\mu_{m-1}}+\gamma_{m+1/2}\frac{\mu_{m+1}}{\mu_{m+1}-\mu_m}\right]\psi_m\left(z\right) \qquad \text{(A10a)}$$

to give for Eq(A9)

$$\sum_{m=1}^{2N}\left\{\begin{array}{l}\gamma_{m-1/2}\left[\dfrac{\mu_{m-1}}{\mu_m-\mu_{m-1}}-\dfrac{\mu_m}{\mu_m-\mu_{m-1}}\right]\psi_m\left(z\right)- \\ -\gamma_{m+1/2}\left[\dfrac{\mu_m}{\mu_{m+1}-\mu_m}-\dfrac{\mu_{m+1}}{\mu_{m+1}-\mu_m}\right]\psi_m\left(z\right)+2\mu_m\omega_m\psi_m\left(z\right)\end{array}\right\}=0 \qquad \text{(A10b)}$$

or

$$\sum_{m=1}^{2N}\left\{\left[\gamma_{m+1/2}-\gamma_{m+1/2}\right]+2\mu_m\omega_m\right\}\psi_m\left(z\right)=0, \tag{A10c}$$

satisfied by

$$\gamma_{m+1/2}=\gamma_{m+1/2}-2\mu_m\omega_m. \tag{A10c}$$

Thus, with Eq(A6c), one conserves the first moment by introducing $\gamma_{m+1/2}$.

The final form of the approximation of the angular Laplacian is

$$\nabla_m^2\psi_m\left(z\right)\equiv\frac{1}{\omega_m}\left[\beta_{m+1/2}\left(\psi_{m+1}\left(z\right)-\psi_m\left(z\right)\right)-\beta_{m-1/2}\left(\psi_m\left(z\right)-\psi_{m-1}\left(z\right)\right)\right] \tag{A11a}$$

with

$$\begin{gathered}\beta_{m+1/2}\equiv\frac{\gamma_{m+1/2}}{\mu_{m+1}-\mu_m}\\ \gamma_{m+1/2}=\gamma_{m-1/2}-2\mu_m\omega_m,\ \gamma_{1/2}\equiv0.\end{gathered} \tag{A11b}$$

If one defines the ***L*** matrix as

$$\boldsymbol{L}=\begin{bmatrix} -\nu_1 & \beta_{3/2} & 0 & 0 & \ldots & \ldots & & 0 \\ \beta_{3/2} & -\nu_2 & \beta_{5/2} & 0 & \ldots & & & \ldots \\ \ldots & \ldots & \ldots & \ldots. & & & & \\ 0 & \ldots & \beta_{m-1/2} & -\nu_m & \beta_{m+1/2} & 0 & \ldots & 0 \\ \ldots & & \ldots & \ldots & \ldots & & & \\ & & & \ldots & \ldots & \ldots & & \ldots \\ \ldots & & & & \ldots & \beta_{2N-3/2} & -\nu_{2N-1} & \beta_{2N-1/2} \\ 0 & \ldots & & & \ldots & 0 & \beta_{2N-1/2} & -\nu_{2N} \end{bmatrix} \tag{A11c}$$

and

$$\nu_m\equiv\beta_{m+1/2}+\beta_{m-1/2}, \tag{A11d}$$

then the approximation to the angular Laplacian is

$$\nabla_m^2 \psi_m(z) \equiv \sum_{j=1}^{2N} \boldsymbol{L}_{m,j} \psi_j(z). \tag{A.11e}$$

Cover Letter:

The solution to the FPE represents a new solution. This work was presented at ICTT27 and should be included in the NSE special issue.