

Research Article

Existence of Solutions for Nonlocal Boundary Value Problems of Higher-Order Nonlinear Fractional Differential Equations

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We study some existence results in a Banach space for a nonlocal boundary value problem involving a nonlinear differential equation of fractional order q given by ${}^c D^q x(t) = f(t, x(t))$, $0 < t < 1$, $q \in (m-1, m]$, $m \in \mathbb{N}$, $m \geq 2$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 0, \dots, x^{(m-2)}(0) = 0$, $x(1) = \alpha x(\eta)$. Our results are based on the contraction mapping principle and Krasnoselskii's fixed point theorem.

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1. Introduction

Fractional differential equations involve derivatives of fractional order. They arise in many engineering and scientific disciplines such as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, and polymer rheology. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1–17] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

The subject of multipoint nonlocal boundary value problems, initiated by Ilin and Moiseev [18, 19], has been addressed by many authors, for instance, [20–26]. The multipoint boundary conditions appear in certain problems of thermodynamics, elasticity, and wave propagation, see [27] and the references therein. The multipoint boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to sensors located at intermediate positions.

For $m \in \mathbb{N}$, $m \geq 2$, and $q \in (m-1, m]$, we consider the following nonlinear fractional differential equation of order q with nonlocal boundary conditions:

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \quad x(1) &= \alpha x(\eta), \\ 0 < \eta < 1, \quad \alpha \eta^{m-1} &\neq 1, \quad \alpha \in \mathbb{R}, \end{aligned} \quad (1.1)$$

where ${}^c D$ is the Caputo fractional derivative and $f : [0, 1] \times X \rightarrow X$ is continuous. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathcal{C} = C([0, 1], X)$ denotes the Banach space of all continuous functions from $[0, 1] \rightarrow X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

By a solution of (1.1), we mean a function $x \in \mathcal{C}$ of class $C^m[0, 1]$ which satisfies the nonlocal fractional boundary value problem (1.1).

2. Preliminaries

Let us recall some basic definitions [12, 15, 17] on fractional calculus.

Definition 2.1. For a function $g : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1, \quad (2.1)$$

where $[q]$ denotes the integer part of the real number q .

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \quad (2.2)$$

provided that the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1, \quad (2.3)$$

provided the right hand side is pointwise defined on $(0, \infty)$.

We remark that the Caputo derivative becomes the conventional n th derivative of the function as $q \rightarrow n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer-order derivatives. On the other hand, the Riemann-Liouville fractional derivative could hardly produce the physical interpretation of the initial conditions required for the initial value problems

involving fractional differential equations (the same applies to the boundary value problems of fractional differential equations). Moreover, the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [17].

Lemma 2.4 (see [28]). *For $q > 0$, the general solution of the fractional differential equation ${}^c D^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n - 1$ ($n = [q] + 1$).

In view of Lemma 2.4, it follows that

$$I^{qc} D^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n - 1$ ($n = [q] + 1$).

Now, we state a known result due to Krasnoselskii [29] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.5. *Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be the operators such that*

- (i) $Ax + By \in M$ whenever $x, y \in M$,
- (ii) A is compact and continuous,
- (iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

To study the nonlinear problem (1.1), we first consider the associated linear problem and obtain its solution.

Lemma 2.6. *For a given $\sigma \in C[0, 1]$, the unique solution of the boundary value problem,*

$$\begin{aligned} & {}^c D^q x(t) = \sigma(t), \quad 0 < t < 1, \quad q \in (m - 1, m], \quad m \in \mathbb{N}, \quad m \geq 2, \\ & x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \quad x(1) = \alpha x(\eta), \\ & 0 < \eta < 1, \quad \alpha \eta^{m-1} \neq 1, \quad \alpha \in \mathbb{R}, \end{aligned} \tag{2.6}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \frac{t^{m-1}}{(1-\alpha \eta^{m-1})} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds \right]. \tag{2.7}$$

Proof. Using (2.5), we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - c_0 - c_1 t - c_2 t^2 - \dots - c_{m-1} t^{m-1}, \quad (2.8)$$

where $c_0, c_1, c_2, \dots, c_{m-1} \in \mathbb{R}$ are arbitrary constants. In view of the relations ${}^c D^q I^q x(t) = x(t)$ and $I^q I^p x(t) = I^{q+p} x(t)$ for $q, p > 0, x \in L(0, 1)$, we obtain

$$\begin{aligned} x'(t) &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds - c_1 - 2c_2 t - \dots - (m-1)c_{m-1} t^{m-2}, \\ x''(t) &= \int_0^t \frac{(t-s)^{q-3}}{\Gamma(q-2)} \sigma(s) ds - 2c_2 - \dots - (m-1)(m-2)c_{m-1} t^{m-3}, \dots \end{aligned} \quad (2.9)$$

Applying the boundary conditions for (2.6), we find that $c_0 = 0, c_1 = 0, \dots, c_{m-2} = 0$, and

$$c_{m-1} = \frac{1}{(1-\alpha\eta^{m-1})} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds \right]. \quad (2.10)$$

Substituting the values of c_0, c_1, \dots, c_{m-1} in (2.8), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds \\ &\quad - \frac{t^{m-1}}{(1-\alpha\eta^{m-1})} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds \right]. \end{aligned} \quad (2.11)$$

This completes the proof. \square

3. Main Results

For the forthcoming analysis, we need the following assumptions:

- (A₁) $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$, for all $t \in [0, 1], x, y \in X$;
- (A₂) $\|f(t, x)\| \leq \mu(t)$, for all $(t, x) \in [0, 1] \times X, \mu \in L^1([0, 1], R^+)$.

In relation to the nonlocal problem (1.1), we define the constants:

$$\Lambda = \frac{L}{\Gamma(q+1)} + \lambda, \quad \lambda = \frac{L(1+|\alpha|\eta^q)}{\Gamma(q+1)|1-\alpha\eta^{m-1}|}. \quad (3.1)$$

Theorem 3.1. *Assume that $f : [0, 1] \times X \rightarrow X$ is a jointly continuous function and satisfies the assumption (A₁). Then the boundary value problem (1.1) has a unique solution provided $\Lambda < 1$, where Λ is given by (3.1).*

Proof. Define $F : \mathcal{C} \rightarrow \mathcal{C}$ by

$$(Fx)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \frac{t^{m-1}}{(1-\alpha\eta^{m-1})} \times \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \right], \quad t \in [0, 1]. \tag{3.2}$$

Let us set $\sup_{t \in [0,1]} \|f(t, 0)\| = M$, and choose

$$r \geq \frac{M}{(1-\beta)\Gamma(q+1)} \left(1 + \frac{1+|\alpha|\eta^q}{|1-\alpha\eta^{m-1}|} \right), \tag{3.3}$$

where β is such that $\Lambda \leq \beta < 1$. Now we show that $FB_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned} \|(Fx)(t)\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, x(s))\| ds \\ &\quad + \frac{t^{m-1}}{|1-\alpha\eta^{m-1}|} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f(s, x(s))\| ds + |\alpha| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \|f(s, x(s))\| ds \right] \\ &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|) ds \\ &\quad + \frac{t^{m-1}}{|1-\alpha\eta^{m-1}|} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|) ds \right. \\ &\quad \left. + |\alpha| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|) ds \right] \\ &\leq (Lr + M) \left(\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{t^{m-1}}{|1-\alpha\eta^{m-1}|} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds + |\alpha| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} ds \right] \right) \\ &\leq \left(\frac{L}{\Gamma(q+1)} + \frac{L(1+|\alpha|\eta^q)}{\Gamma(q+1)|1-\alpha\eta^{m-1}|} \right) r + \frac{M}{\Gamma(q+1)} \left(1 + \frac{1+|\alpha|\eta^q}{|1-\alpha\eta^{m-1}|} \right) \\ &= \Lambda r + \frac{M}{\Gamma(q+1)} \left(1 + \frac{1+|\alpha|\eta^q}{|1-\alpha\eta^{m-1}|} \right) \\ &\leq (\Lambda + 1 - \beta)r \leq r. \end{aligned} \tag{3.4}$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
& \| (F x)(t) - (F y)(t) \| \\
& \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \| f(s, x(s)) - f(s, y(s)) \| ds \\
& \quad + \frac{t^{m-1}}{|1 - \alpha \eta^{m-1}|} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \| f(s, x(s)) - f(s, y(s)) \| ds \right. \\
& \quad \quad \left. + |\alpha| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \| f(s, x(s)) - f(s, y(s)) \| ds \right] \\
& \leq L \| x - y \| \left(\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{t^{m-1}}{|1 - \alpha \eta^{m-1}|} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds + |\alpha| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} ds \right] \right) \\
& = L \| x - y \| \left(\frac{t^q}{\Gamma(q+1)} + \frac{t^{m-1}}{|1 - \alpha \eta^{m-1}|} \left(\frac{1 + |\alpha| \eta^q}{\Gamma(q+1)} \right) \right) \\
& \leq L \| x - y \| \frac{1}{\Gamma(q+1)} \left(1 + \frac{1 + |\alpha| \eta^q}{|1 - \alpha \eta^{m-1}|} \right) \\
& = \Lambda \| x - y \|.
\end{aligned} \tag{3.5}$$

Clearly Λ depends on the parameters (q, m, α, η, L) involved in the problem. As $\Lambda < 1$, therefore, F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. \square

Theorem 3.2. *Let $f : [0, 1] \times X \rightarrow X$ be a jointly continuous function mapping bounded subsets of $[0, 1] \times X$ into relatively compact subsets of X . Further, the assumptions $(A_1) - (A_2)$ hold with $\lambda < 1$, where λ is given by (3.1). Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.*

Proof. Let us fix

$$r \geq \frac{\|\mu\|_{L^1}}{\Gamma(q)} \left(1 + \frac{1 + |\alpha| \eta^{q-1}}{|1 - \alpha \eta^{m-1}|} \right), \tag{3.6}$$

and consider $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. We define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \\
(\Psi x)(t) &= -\frac{t^{m-1}}{(1 - \alpha \eta^{m-1})} \left[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \right].
\end{aligned} \tag{3.7}$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Psi y\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q)} \left(1 + \frac{1 + |\alpha|\eta^{q-1}}{|1 - \alpha\eta^{m-1}|} \right) \leq r. \tag{3.8}$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_1) that Ψ is a contraction mapping for $\lambda < 1$. Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q)}. \tag{3.9}$$

To show that the operator Φ is compact, we use the classical Arzela-Ascoli theorem. Let \mathcal{A} be a bounded subset of \mathcal{C} . We have to show that $\Phi(\mathcal{A})$ is equicontinuous and for each t , the set $\Phi(\mathcal{A})(t)$ is relatively compact in X . In view of $(A_1), (A_2)$, we define $\sup_{(t,x) \in [0,1] \times B_r} \|f(t, x)\| = f_{\max}$, and consequently we have

$$\begin{aligned} & \|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\ &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) f(s, x(s)) ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds \right\| \\ &\leq \frac{f_{\max}}{\Gamma(q+1)} \left| 2(t_2 - t_1)^q + t_1^q - t_2^q \right|, \end{aligned} \tag{3.10}$$

which is independent of x . Thus, Φ is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $\Phi(\mathcal{A})(t)$ is relatively compact in X for every t . Therefore, Φ is relatively compact on B_r . Hence, By Arzela-Ascoli theorem, Φ is compact on B_r . Thus all the assumptions of Theorem 2.5 are satisfied and the conclusion of Theorem 2.5 implies that the boundary value problem (1.1) has at least one solution on $[0, 1]$. \square

Example 3.3. Consider the following boundary value problem:

$$\begin{aligned} {}^c D^q x(t) &= \frac{1}{(t+7)^2} \frac{\|x\|}{1 + \|x\|}, \quad 2 < q \leq 3, \quad t \in [0, 1], \\ x(0) &= 0, \quad x'(0) = 0, \quad x(1) = x\left(\frac{1}{2}\right). \end{aligned} \tag{3.11}$$

Here, $m = 3, f(t, x(t)) = (1/(t+7)^2)(\|x\|/(1 + \|x\|)), \alpha = 1, \eta = 1/2$. As $\|f(t, x) - f(t, y)\| \leq (1/49)\|x - y\|$, therefore, (A_1) is satisfied with $L = 1/49$. Further,

$$\Lambda = \frac{L}{\Gamma(q+1)} \left(1 + \frac{1 + |\alpha|\eta^q}{|1 - \eta^{m-1}|} \right) = \frac{1}{49\Gamma(q+1)} \left(1 + \frac{4}{3} \left(1 + \left(\frac{1}{2}\right)^q \right) \right) < 1, \quad 2 < q \leq 3. \tag{3.12}$$

Thus, by Theorem 3.1, the boundary value problem (3.11) has a unique solution on $[0, 1]$.

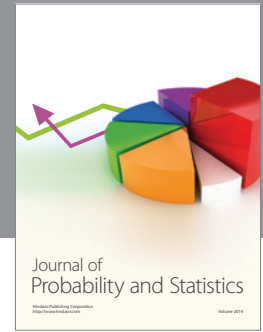
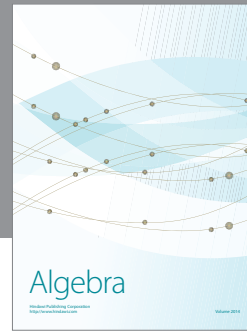
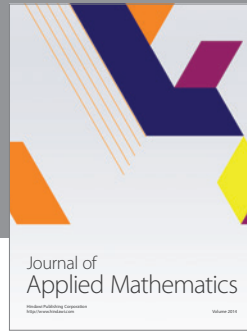
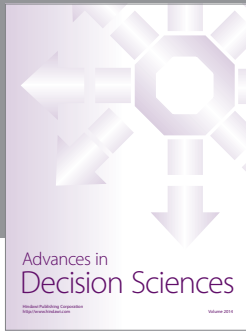
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