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NEW METHODS FOR THE STUDY AND RESOLUTION OF EQUATIONS INVOLVING FRACTIONAL OPERATORS AND THEIR APPLICATIONS

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New methods for the study and resolution of equations involving fractional operators and their applications

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New methods for the study and resolution of equations involving fractional operators and their applications

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Polas persoas que as defenden até o final.
Polos proxectos que as poñen en práctica.

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Resumo

O obxectivo principal desta tese de doutoramento é, como o seu título indica, proporcionar novos métodos para a resolución de ecuacións diferenciais e integrais de orde fraccionaria, así como a aplicación dos mesmos a diversos modelos. A pesar disto e, como quedará resaltado no documento, realizáronse pequenas contribucións a outras cuestións relativas ao Cálculo Fraccionario, á Análise Matemática ou mesmo a outras áreas das Matemáticas.

O documento atópase dividido en seis capítulos, dos cales resumimos o seu contido a continuación.

Os dous primeiros capítulos están orientados a describir as nocións básicas necesarias para o desenvolvemento propio da tese. O primeiro deles posúe uns contidos posiblemente máis coñecidos para un matemático, mentres que o capítulo segundo contén resultados básicos xa relativos ao Cálculo Fraccionario. Estes capítulos beben, principalmente, de fontes bibliográficas de referencia [40, 55, 59, 69] de autores como Samko, Kilbas, Marichev, Sristava, Trujillo, Podlubny, Miller e Ross. Non obstante, xa podemos atopar contribucións orixinais na sección 1.3, ver [10], así como nas subseccións 2.1.5 e 2.2.3, ver [13] e [18], respectivamente. A sección 2.1, a pesar de conter resultados coñecidos, está fortemente reelaborada. O motivo disto é que, aínda que os resultados consultados na bibliografía resultaban correctos, algunhas das súas probas omitían pequenos detalles que semellaban relevantes. En consecuencia, parte das demostracións da devandita sección teñen sido reelaboradas, baseándose nos resultados de carácter fundamental do capítulo primeiro.

No capítulo segundo, debemos destacar o papel sobranceiro do Teorema 2.32, o cal non atopamos na literatura e resulta relativamente sinxelo de probar. Para entender a súa relevancia dentro da tese, debemos ter claro que, ao contrario do caso usual, os espazos de diferenciabilidade de orde α non están contidos uns noutros a medida que α medra. En consecuencia, a

intersección de varios espazos de diferenciabilidade de distintas ordes non é, necesariamente, o espazo de diferenciabilidade de maior orde. O rol do Teorema 2.32 é, precisamente, describir esta intersección de xeito explícito, o cal terá relevancia para utilizar os resultados obtidos no capítulo terceiro (relativos a ecuacións integrais fraccionarias) para deducir cuestións propias das ecuacións diferenciais fraccionarias no capítulo cuarto. Detallamos o contido destes capítulos a continuación:

- O contido do capítulo terceiro está, fundamentalmente, orientado ao estudo e resolución de ecuacións integrais fraccionarias lineais de coeficientes constantes. Mediante argumentos que involucran propiedades alxébricas dos polinomios, ver Teorema 3.5, somos quen de probar diversos resultados. En concreto, unha vez establecido o teorema alxébrico mencionado, onde reside a idea da construción, deducimos que os problemas integrais fraccionarios previamente descritos non son máis complicados ca o seu análogo enteiro se as ordes de partida son números racionais, ver Corolario 3.7. En particular, explicítanse dous modos para transformar un problema fraccionario nun problema enteiro equivalente e resolvelo a través do paso a unha ecuación diferencial ordinaria, se o termo fonte presenta suficiente regularidade. Como corolario, obténse que ambos problemas teñen sempre solución única, a pesar de que isto último xa podía ser deducido de xeito máis ou menos directo facendo uso de resultados clásicos propios da teoría de convolucións. Estes resultados recóllense en [15].

  Precisamente, a aplicación destes resultados do eido da teoría de convolucións permite estender, tras algo de traballo, o resultado de existencia e unicidade de solución aos casos onde as ordes de integración non son, necesariamente, racionais e o termo fonte non é, necesariamente, regular. En consecuencia, preguntámonos se o método de cálculo explícito de solución xa desenvolto pode ser xeneralizado a estes casos. A resposta, detallada en [16], é afirmativa:

  1. O termo fonte non ten por que ser suficientemente regular. Para probar isto, na subsección 3.5.1 desenvolvemos un algoritmo que permite expresar a solución dun problema arbitrario con termo fonte w en función da solución ao mesmo problema para un termo fonte estritamente máis regular ca w. Aplicando esta idea unha cantidade finita de veces, dedúcese un procedemento para resolver os problemas non regulares en función dos problemas regulares.

2. As ordes na ecuación poden ser irracionais, como amosamos na subsección 3.5.2. Para probar isto, tomamos inicialmente unha sucesión de problemas con ordes racionais que converxen a un problema con ordes non necesariamente racionais. Tras xustificar que o problema límite ten solución única, argumentamos que as solucións únicas dos problemas con orde racional deben converxer cara á solución única do problema límite.

O Corolario 3.26 resume toda a teoría desenvolta no capítulo terceiro, agás a parte algorítmica e construtiva para resolver as ecuacións.

- O capítulo cuarto, cuxo contido está desenvolto en [12], está enfocado a adaptar os resultados do capítulo previo ao marco das ecuacións diferenciais fraccionarias. En poucas palabras, amosamos como calquera ecuación diferencial lineal de coeficientes constantes de orde β está asociada a un espazo de dimensión $\lceil\beta\rceil$ de problemas integrais. O significado desta asociación é o seguinte: toda solución da ecuación diferencial é unha solución dun dos problemas integrais, mais a solución dun problema integral é solución do problema diferencial se e só se a devandita solución do problema integral posúe certa regularidade. Dado que os problemas integrais teñen solución única, a ecuación diferencial posúe un espazo de solucións de dimensión $\lceil\beta\rceil$ ao que chamamos "solucións febles". Como xa remarcamos, estas "solucións febles" non son, a priori, solucións da ecuación diferencial, posto que a aplicación do operador diferencial fraccionario á "solución feble" pode non estar ben definida por cuestións de regularidade. Despois desta análise, estudamos o subespazo de "solucións febles" que son, de feito, "fortes". Somos quen de describir este subespazo e calcular a súa dimensión, que resulta ser $\lceil\beta-\beta_*\rceil$, onde β_* é a orde máis alta na ecuación diferencial fraccionaria que non ten a mesma parte decimal ca β. Se dita orde non existe, o resultado segue a ser certo definindo $\beta_* = 0$. Finalmente, probamos que dito espazo de solucións "fortes" pode ser codificado a través de $\lceil\beta-\beta_*\rceil$ condicións iniciais de ordes $\beta-1, \beta-2, \dots, \beta-\lceil\beta-\beta_*\rceil$. Se a última orde descrita é un número negativo, a condición respectiva refírese á condición integral da orde oposta, concretamente, á condición integral de orde $\lceil\beta-\beta_*\rceil-\beta$.

No capítulo quinto, discutimos tres aplicacións do Cálculo Fraccionario e das ecuacións estudadas previamente.

A primeira delas consiste no estudo dunha familia de ecuacións diferenciais fraccionarias coa derivada de Caputo, a partir dos resultados previamente obtidos para o caso de Riemann-Liouville. Unha vez temos establecida a conexión entre estes dous tipos de ecuacións, estudamos o problema de Basset. Este problema consiste nunha ecuación diferencial fraccionaria con derivadas de Caputo. Exemplificamos como somos quen de resolver analíticamente este problema para uns datos concretos, a partir dos resultados desenvoltos nos capítulos anteriores.

A segunda das aplicacións queda inscrita no marco da Relatividade Especial. A sección comeza cunha introducción á devandita teoría, dende un punto de vista matemático e axiomático. Posteriormente, estudamos cando unha magnitude, que poida ser computada como unha integral temporal dende o punto de vista dun observador, pode ser medida como unha integral fraccionaria temporal dende a perspectiva doutro observador. Finalmente, formulamos un experimento mental no cal a resolución dunha ecuación integral fraccionaria é de interese para calcular a velocidade dun móbil con respecto a outro, a partir de certas observacións perturbadas por efectos relativistas.

O terceiro exemplo está adicado a unha aplicación no contexto da ecuación da viga. Supoñendo que o material da viga satisfai certas propiedades e que estamos nunha situación onde a ecuación aplica, podemos emular as ideas desenvoltas no artigo [76]. Estas ideas permiten concluír a existencia dunha dependencia fraccionaria entre a forza cortante e a deformación da viga, a pesar de que na ecuación en derivadas parciais de partida só figuran ordes enteiras.

Para finalizar, no capítulo sexto, resumimos os resultados obtidos máis relevantes e sinalamos algunhas liñas futuras, co gallo de ampliar a investigación xa realizada.

Resumen

El objetivo principal de esta tesis de doctorado es, como su título indica, proporcionar nuevos métodos para la resolución de ecuaciones diferenciales e integrales de orden fraccionaria, así como la aplicación de los mismos a diversos modelos. A pesar de esto, como quedará resaltado en el documento, se han realizado pequeñas contribuciones a otras cuestiones relativas al Cálculo Fraccionario, al Análisis Matemático o incluso a otras áreas de las Matemáticas.

El documento se encuetra dividido en seis capítulos, de los cuales resumimos su contenido a continuación.

Los dos primeros capítulos están orientados a describir las nociones bási-

cas necesarias para el desarrollo propio de la tesis. El primero de ellos posee unos contenidos posiblemente más conocidos para un matemático, mientras que el capítulo segundo contiene resultados básicos ya relativos al Cálculo Fraccionario. Estos capítulos beben, principalmente, de fuentes bibliográficas de referencia [40, 55, 59, 69] de autores como Samko, Kilbas, Marichev, Sristava, Trujillo, Podlubny, Miller e Ross. No obstante, ya podemos encontrar contribuciones originales en la sección 1.3, ver [10], así como en las subsecciones 2.1.5 y 2.2.3, ver [13] y [18], respectivamente. La sección 2.1, a pesar de contener resultados conocidos, está fuertemente reelaborada. El motivo de esto es que, a pesar de que los resultados consultados en la bibliografía resultaban correctos, algunas de sus pruebas omitían pequeños detalles que parecían relevantes. En consecuencia, parte de las demostraciones de dicha sección han sido reelaboradas, basándose en los resultados de carácter fundamental del capítulo primero.

En el capítulo segundo, debemos destacar el papel singular del Teorema 2.32, el cual no hemos hallado en la literatura y resulta relativamente fácil de probar. Para entender su relevancia dentro de la tesis, debemos tener claro que, al contrario que en el caso usual, los espacios de diferenciabilidad de orden α no están contenidos unos en otros a medida que α crece. En consecuencia, la intersección de varios espacios de diferenciabilidad de órdenes distintos no es, necesariamente, el espacio de diferenciabilidad de mayor orden. El rol del Teorema 2.32 es, precisamente, describir esta intersección de modo explícito, lo cual tendría relevancia para utilizar los resultados obtenidos en el capítulo terceiro (relativos a ecuaciones integrales fraccionarias) para deducir cuestiones propias de las ecuaciones diferenciales fraccionarias en el capítulo cuarto. Detallamos el contenido de estos capítulos a continuación:

- El contenido del tercer capítulo está, fundamentalmente, orientado al estudio y resolución de ecuaciones integrales fraccionarias lineales de coeficientes constantes. Mediante argumentos que involucran ciertas propiedades algebraicas de los polinomios, ver Teorema 3.5, somos capaces de probar diversos resultados. En concreto, una vez establecido el teorema algebraico mencionado, donde reside la idea de la construcción, deducimos que los problemas integrales fraccionarios previamente descritos no son más complicados que su análogo entero si los órdenes de partida son números racionales, ver Corolario 3.7. En particular, se explicitan dos modos para transformar un problema fraccionario en un problema entero equivalente y resolverlo a través del paso a una ecuación diferencial ordinaria, si el término fuente presenta suficiente regularidad. Como coralario, se obtiene que ambos problemas tienen siempre solución única, a pesar de que esto último ya podía ser deducido de modo más o menos directo haciendo uso de resultados clásicos propios de teoría de convoluciones. Estos resultados se recogen en [15].

Precisamente, la aplicación de estos resultados del campo de la teoría de convoluciones permite extender, tras algo de trabajo, el resultado de existencia y unicidad de solución a los casos donde los órdenes de integración no son, necesariamente, racionales y el término fuente no es, necesariamente, regular. En consecuencia, nos preguntamos si el método de cálculo explícito de solución previamente desarrollado puede ser generalizado a estos casos. La respuesta, detallada en [16], es afirmativa:

1. El término fuente no tiene por qué ser suficientemente regular. Para probar esto, en la subsección 3.5.1 desarrollamos un algoritmo que permite expresar la solución dun problema arbitrario con término fuente w en función de la solución al mismo problema para un término fuente estrictamente más regular que w. Aplicando esta idea una cantidad finita de veces, se deduce un procedimiento para resolver los problemas no regulares en función de los problemas regulares.
2. Los órdenes de la ecuación pueden ser irracionales, como mostramos en la subsección 3.5.2. Para probar esto, tomamos inicialmente una sucesión de problemas con órdenes racionales que convergen a un problema con órdenes no necesariamente racionales. Tras justificar que el problema límite tiene solución única, argumentamos que las soluciones únicas de los problemas con orden racional deben converger hacia la solución única del problema límite.

El Corolario 3.26 resume toda la teoría desarrollada en el capítulo tercero, salvo la parte algorítmica y constructiva para resolver las ecuaciones.

- El capítulo cuarto, cuyo contenido está desarrollado en [12], está enfocado a adaptar los resultados del capítulo previo al marco de las ecuaciones diferenciales fraccionarias. En pocas palabras, demostramos cómo cualquier ecuación diferencial lineal de coeficientes constantes de orden β está asociada a un espacio de dimensión $\lceil \beta \rceil$ de problemas integrales, pero la solución de un problema integral es solución del

> problema diferencial si y solo si dicha solución del problema integral posee cierta regularidad. Dado que los problemas integrales tienen solución única, la ecuación diferencial posee un espacio de soluciones de dimensión $\lceil \beta \rceil$ al que llamamos "soluciones débiles". Como ya remarcamos, estas "soluciones débiles" no son, a priori, soluciones de la ecuación diferencial, puesto que la aplicación del operador fraccionario a la "solución débil" puede no estar bien definida por cuestiones de regularidad. Tras este análisis, estudiamos el subespacio de "soluciones débiles" que son, de hecho, "fuertes". Somos capaces de describir este subespacio y calcular su dimensión, que resulta ser $\lceil \beta - \beta_* \rceil$, donde β_* es el orden más alto en la ecuación diferencial fraccionaria que no tiene la misma parte decimal que β. Si dicho orden no existe, el resultado sigue siendo cierto definiendo $\beta_* = 0$. Finalmente, probamos que dicho espacio de "soluciones fuertes" puede ser codificado a través de $\lceil \beta - \beta_* \rceil$ condiciones iniciales de órdenes $\beta - 1, \beta - 2, \ldots, \beta - \lceil \beta - \beta_* \rceil$. Si el último orden descrito es un número negativo, la condición respectiva se refiere a la condición integral del orden opuesto, concretamente, a la condición integral del orden $\lceil \beta - \beta_* \rceil - \beta$.

En el capítulo quinto, discutimos tres aplicaciones del Cálculo Fraccionario y de las ecuaciones estudiadas previamente.

La primera de ellas consiste en el estudio de una familia de ecuaciones diferenciales fraccionarias con la derivada de Caputo, a partir de los resultados previamente obtenidos para el caso de Riemann-Liouville. Una vez tenemos establecida la conexión entre estos dos tipos de ecuaciones, estudiamos el problema de Basset. Este problema consiste en una ecuación diferencial fraccionaria con derivadas de Caputo. Ejemplificamos cómo somos capaces de resolver analíticamente este problema para unos datos concretos, a partir de los resultados desarrollados en los capítulos anteriores.

La segunda de las aplicaciones queda inscrita en el marco de la Relatividad Especial. La sección comienza con una introducción de dicha teoría, desde un punto de vista matemático y axiomático. Posteriormente, estudiamos cuando una magnitud, que pueda ser computada como una integral temporal desde el punto de vista de un observador, puede ser medida como una integral fraccionaria temporal desde la perspectiva de otro observador. Finalmente, formulamos un experimento mental en el cual la resolución de una ecuación integral fraccionaria es de interés para calcular la velocidad de un móvil respecto a otro, a partir de ciertas observaciones perturbadas por efectos relativistas.

El tercer ejemplo está dedicado a una aplicación en el contexto de la ecuación de la viga. Suponiendo que el material de la viga satisface ciertas propiedades y que estamos en una situación donde la ecuación aplica, podemos emular las ideas desarrolladas en el artículo [76]. Estas ideas permiten concluir la existencia de una dependencia fraccionaria entre la fuerza cortante y la deformación de la viga, a pesar de que en la ecuación en derivadas parciales de partida solo figuran órdenes enteros.

Para finalizar, en el capítulo sexto, resumimos los resultados obtenidos más relevantes y señalamos algunas líneas futuras, con el objetivo de ampliar la investigación ya realizada.

Abstract

As the title indicates, the main scope of this PhD Thesis is to provide new methods to solve differential and integral equations of fractional order, and to apply those methods to study various models. Besides, as it can be seen in the dissertation, some other contributions to theory of Fractional Calculus, Mathematical Analysis and other areas of mathematics have also been made.

The Thesis is divided into six chapters. We summarize their content hereunder.

The first two chapters describe the basic notions underpinning the development of this research. The contents of first chapter are mostly well known, while the second chapter consists of basic results on Fractional Calculus. These chapters are based on some of the most relevant references [40, 55, 59, 69] from authors like Samko, Kilbas, Marichev, Sristava, Trujillo, Podlubny, Miller and Ross. However, we can find original contributions in the section 1.3, (see [10]) and in the subsections 2.1.5 and 2.2.3 (see [13] and [18]). Section 2.1, despite containing well known results, is strongly rewritten. Although the bibliographical results are correct, some of their proofs omit small details that seemed relevant. Consequently, some of these proofs have been remastered using the fundamental results from the first chapter.

In the second chapter, we highlight the remarkable role of Theorem 2.32 that we could not find in the literature and is relatively simple to prove. To understand its relevance in this Thesis, we must realize that, contrary to what happens classically, the space of differentiability of order α generally does not contain the space of differentiability of order β, even when $\alpha < \beta$. Consequently, the intersection of several spaces of different orders is not necessarily the space of differentiability of the highest order. The role of Theorem 2.32 is to describe this intersection explicitly. This is relevant, in order to use the results obtained in the third chapter (relative to fractional

integral equations) to deduce other questions involving fractional differential equations in the fourth chapter.

We briefly describe the content of these chapters below:

- The content of third chapter is fundamentally oriented to the study and resolution of linear fractional integral equations with constant coefficients. After using arguments that involve some algebraic properties of the polynomials, see Theorem 3.5, we prove several results. Specifically, once the algebraic theorem is established, we deduce that the previously described fractional integral problems are not more complicated that their integer analogues if the orders in the original equation are rational numbers, see Corollary 3.7. In particular, two different ideas to transform a fractional problem into an equivalent integer one are developed. Besides, we can solve this problem provided that the source term is smooth enough. As a corollary, we obtain that this type of problems always have a unique solution, although this last claim could already be deduced directly by using classical tools of convolution theory. These results are included in [15].

  The application of these results on convolution theory allows us to extend, after some work, the uniqueness and existence result to the cases where the integration orders are not necessarily rational, and the source term is not necessarily smooth. Therefore, we ask ourselves if the previously developed method for calculating the solution can be generalized for these cases. The answer, detailed in [16], is positive:

  1. The source term does not have to be smooth. To prove this, in the subsection 3.5.1, we develop an algorithm that allows us to express the problem with source term w in terms of an equation with a smoother source term. If we apply this idea a finite number of times, we deduce a process to solve the non-smooth problems using the properties of smooth problems.

  2. The orders in our equation can be irrational, as we show in the subsection 3.5.2. To solve this, we initially consider a sequence of problems with rational orders that converge to a problem with non-necessarily rational orders. After contending that the limit problem has a unique solution, we show that the unique solutions to the rational order problems must converge to the unique solution to the limit problem.

The Corollary 3.26 summarizes all the theory developed in this chapter except the algorithmic and constructive part to solve the equations.

- The fourth chapter, content developed in [12], is focused on adapting the results from previous chapters to the framework of fractional differential equations. We prove that every linear fractional differential equation of constant coefficients of order β is associated to a space of dimension β of integral problems. But the solution to the integral problem is a solution to the differential problem if and only if the solution to the integral problem is smooth enough. Since the integral problem has a unique solution, the differential equation has a space of solutions of dimension $\lceil\beta\rceil$ that are called "weak solutions". As we have already highlighted, these "weak solutions" are not a priori solutions to the differential equation since the application of the fractional operator to the "weak solution" can be not well defined due to smoothness issues. Through an appropriate analysis, we find out that there is always a subspace of the set of "weak solutions" that is, in fact, the set of "strong solutions". We describe this subspace and compute its dimension which is $\lceil\beta - \beta_*\rceil$. This β_* is the largest order in the fractional differential equation that does not share the decimal part with β. If such an order does not exist, the result is still true after defining $\beta_* = 0$. Finally, we prove that the space of "strong solutions" can be coded from $\lceil\beta-\beta_*\rceil$ initial values of orders $\beta-1, \beta-2, \ldots, \beta-\lceil\beta-\beta_*\rceil$. If the last described order is a negative number, the condition describes the integral condition of the opposite order, specifically, the integral condition of order $\lceil\beta - \beta_*\rceil - \beta$.

In the fifth chapter, we work on three applications of Fractional Calculus and the previously studied equations.

The first one consists of the study of a family of fractional differential equations with Caputo derivative using the previously obtained results for the Riemann-Liouville case. Once the connection between the two types of equations has been established, we study the Basset problem. This problem consists of a fractional differential equation with Caputo derivative. We show how we can solve this problem analytically for some specific data using the results developed in the previous chapters.

The second application deals with Special Relativity. The section begins with an introduction of this concept from a mathematical and axiomatic point of view. After this, we study how a magnitude, computed as a temporal integral from the point of view of one observer, can be measured as a temporal fractional integral from the point of view of the other observer. Finally, we elaborate a mental experiment and we obtain a fractional integral equation whose solution is relevant to calculate the relative speed of a mobile object using some relativistic perturbed observations.

The third example is devoted to an application in the context of the beam equation. Supposing that the material of the beam fulfils certain properties and that we are in a situation where the beam equation applies, we can mimic the ideas developed in the paper [76]. This allow us to conclude the existence of a fractional dependence between the shear force and the deflection of the beam, although the partial differential equation does only include integer orders.

Ending with the sixth chapter, we summarize the more relevant results obtained, and we point out some future lines of research with the aim of expanding the scope in future.

Chapter 1

Fundamental notions and notation

In this first chapter, we present the fundamental results underpinning the contributions in this Thesis. In particular, we establish the notation that will be used during the rest of the dissertation. It is important to highlight that we exclude from this initial description the concepts involving Fractional Calculus (FC), since they will be presented in the following chapter.

We begin stating several results and concepts that are useful for the development of the Thesis. The main interest of these statements is to have in mind what are the exact hypotheses that ensure that a certain result holds. In this sense, this manuscript aspires to be self-contained, in order to ease the readability and full understanding of the mathematical details. In any case, we will provide suitable references for the sake of completeness.

Namely, we present some basic notions concerning Banach spaces and integration theory. With respect to integration theory, we make special emphasis in some results involving absolutely continuous functions, convolution theory and special functions. Finally, we devote a subsection to a concrete topological result that will be helpful for the rest of our quest.

1.1 Banach spaces

The notion of Banach space has played a key role in the development of Modern Mathematical Analysis. In some sense, it is a very reasonable framework where it is possible to combine tools from Mathematical Analysis, Linear Algebra and Topology to study functional operators and their properties. However, before presenting some basic aspects of Banach spaces, we invoke the following result, due to Cauchy, whose proof can be found in [42] (page 128, Theorem 5.2.1).

Proposition 1.1. *If $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous map such that*

$$g(x) + g(y) = g(x + y) \quad \forall x, y \in \mathbb{R}^n,$$

then $g(\lambda\, x) = \lambda\, g(x)$ holds for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

We state hereunder the main notions and results that we need, concerning Banach spaces. All the results and definitions of the rest of this section can be found in [5].

Definition 1.2. We say that X is a (real) normed space if it is a vector space endowed with a map $\|\cdot\| : X \longrightarrow \mathbb{R}^+ \cup \{0\}$ such that:

- If $x \in X$ is such that $\|x\| = 0$, then $x = 0$.
- For any pair $x, y \in X$, we have that $\|x + y\| \leq \|x\| + \|y\|$.
- For any $\lambda \in \mathbb{R}$ and any $x \in X$, we have that $\|\lambda\, x\| = |\lambda| \cdot \|x\|$.

Definition 1.3. A normed space X is said to be a Banach space if the following conditions are equivalent for any sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$:

1. The sequence is convergent (to $x \in X$), that is: there exists $x \in X$ such that, for each $\varepsilon > 0$, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that, if $n \geq N_1(\varepsilon)$, we have $\|x_n - x\| \leq \varepsilon$.
2. The sequence is a Cauchy sequence, that is: for each $\varepsilon > 0$, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that, if $n, m \geq N_2(\varepsilon)$, we have $\|x_n - x_m\| \leq \varepsilon$.

Remark 1.4. In fact, the first condition always implies the second one, and this happens even in the much more general context of metric spaces. However, the second condition might not imply the first one. If it does, we say that our normed space is a complete normed space or, more commonly, a Banach space.

The main interest of Banach spaces occurs when they are infinite dimensional. Note that, indeed, most of the spaces of functions where we look for solutions to equations have non-finite dimension. In fact, the theory of vector spaces is somehow satisfactory in the finite dimensional case, providing a really nice and interesting framework. However, when translating these results to the infinite dimensional case, we observe that some of them do not hold. In short, the main problem is that a set which is simultaneously bounded and closed may not be compact. This implies that, for instance, a linear function defined on the closed unit ball may not be bounded. For this purpose, one has to add some extra hypotheses in the infinite dimensional case to achieve some partial results. Despite of doing this, some additional problems might appear and complicate the subject. For instance, one of the most classical ones is the non-compactness of the closed bounded sets. Another example is that injective (or surjective) endomorphisms in X are not necessarily automorphisms. To partially overcome these difficulties, particular sets of linear maps between Banach spaces are commonly studied. In our case, we pay particular attention to the notion of linear bounded map.

Proposition 1.5. *Given two Banach spaces X, Y and a linear map between them $F : X \longrightarrow Y$, the following three statements are equivalent:*

1. *There exists $c \in \mathbb{R}^+$ such that, for every $x \in X$, we have the inequality $\|F(x)\| \leq c\,\|x\|$.*
2. *There exists $c \in \mathbb{R}^+$ such that, for every $x \in X$ with $\|x\| = 1$, we have the inequality $\|F(x)\| \leq c$.*
3. *F is a continuous map, where we consider the topology induced by the norms in X and Y, respectively.*

If any of the previous conditions holds, we say that F is continuous or, equivalently, that F is bounded.

The terminology "F is bounded" is somehow misleading, but it obeys the following reason. At first, note that the previous proposition is trivial in the case where X is finite dimensional, since any linear operator F would fulfil automatically the three conditions. In fact, the vector space of linear maps from X to Y, denoted by $\mathrm{Mor}(X, Y)$, will be a Banach space, after inducing the norm on X in the following way

$$\|F\| := \sup\{\|F(x)\| \in \mathbb{R}^+ \cup \{0\} : x \in X,\ \|x\| = 1\}.$$

However, in the general case where X is infinite dimensional, one can not expect every linear map to be continuous and $\mathrm{Mor}(X, Y)$ is not a Banach space, since we can find discontinuous linear operators with "infinite" norm.

Therefore, the logical solution is to deal with $\mathrm{Mor}_B(X, Y)$, which consists in the previous morphisms that fulfil Proposition 1.5. These maps are the ones in $\mathrm{Mor}(X, Y)$ that have "finite norm", so that is why they receive the adjective "bounded". If we use the common notation $\mathrm{End}(X) := \mathrm{Mor}(X, X)$, it makes sense to define $\mathrm{End}_B(X)$ as

$$\mathrm{End}_B(X) := \{F \in \mathrm{End}(X) : \|F\| < \infty\}.$$

Of course, in the case where the linear operator F is bijective, we can consider the analogous sets $\mathrm{Aut}(X)$ and $\mathrm{Aut}_B(X)$. We note that they are not Banach spaces even if X is, but this already happened in the finite dimensional case, since the limit operator of a convergent sequence of automorphisms is not necessarily bijective. We give now one of the main classical results concerning the structure of $\mathrm{Aut}_B(X)$, guaranteeing that linear continuous invertible operators between Banach spaces have a continuous inverse. The proof can be consulted in [5], although we will use it in the particular case of the Banach space $L^1[a, b]$ that will be introduced later.

Theorem 1.6 (Bounded Inverse Theorem). *If $F \in \mathrm{Aut}_B(X)$, then we have that $F^{-1} \in \mathrm{Aut}_B(X)$.*

Finally, we will use the following result. It roughly states that the composition operator, between Banach spaces of linear operators, is continuous.

Lemma 1.7. *Consider three Banach spaces X, Y, Z and consider the Banach space of continuous linear maps from X to Y with the norm topology, denoted by $\mathrm{Mor}_B(X, Y)$. Then, the operator*

$$\mathrm{Comp} : \mathrm{Mor}_B(X, Y) \times \mathrm{Mor}_B(Y, Z) \longrightarrow \mathrm{Mor}_B(X, Z)$$

given by

$$\mathrm{Comp}(g, f) = f \circ g$$

is continuous.

Proof. From the triangle inequality, we deduce

$$\|\mathrm{Comp}(g_1, f_1) - \mathrm{Comp}(g_2, f_2)\| \leq \|f_1 \circ g_1 - f_1 \circ g_2\| + \|f_1 \circ g_2 - f_2 \circ g_2\|.$$

Moreover, the right hand side can be bounded from above by

$$\|f_1\| \cdot \|g_1 - g_2\| + \|f_1 - f_2\| \cdot \|g_2\|.$$

Finally, we observe that if (g_2, f_2) tends to (g_1, f_1) the previous bound goes to zero, since $\|g_1 - g_2\|$, $\|f_1 - f_2\|$ go to zero and $\|g_2\|$ tends to $\|g_1\|$ due to the continuity of the norm. $\square$

From the previous result, we have the following immediate corollary

Corollary 1.8. *Consider three Banach spaces X, Y, Z and the functions $g \in \mathrm{Mor}_B(X,Y)$, $f \in \mathrm{Mor}_B(Y,Z)$. Denote by*

$$\mathrm{Comp}_{(g,\cdot)} : \mathrm{Mor}_B(Y,Z) \longrightarrow \mathrm{Mor}_B(X,Z)$$

and

$$\mathrm{Comp}_{(\cdot,f)} : \mathrm{Mor}_B(X,Y) \longrightarrow \mathrm{Mor}_B(X,Z)$$

the maps defined by $\mathrm{Comp}_{(g,\cdot)}(f) = g \circ f$ and $\mathrm{Comp}_{(\cdot,f)}(g) = g \circ f$. We have that $\mathrm{Comp}_{(g,\cdot)}$ and $\mathrm{Comp}_{(\cdot,f)}$ are continuous.

1.2 Integration concepts

We will begin this section by stating two of the most relevant and classical theorems for integration in several variables. Later, we will also sketch some fundamental results concerning convolutions and special functions. For basic notions concerning Lebesgue measure and Lebesgue integral, we refer to [63], where it is possible to find an exhaustive description. Here, we provide the following short summary.

Summary of Lebesgue integration

Consider the sets in $\mathbb{R}$ that can be described as a countable disjoint union of open intervals. Call $\mathcal{F}$ the family consisting in the previous sets. Of course, any $X \in \mathcal{F}$ is associated to a length in the obvious way (sum of the lengths of the intervals), taking into account that this length can be infinite. For any $A \subset \mathbb{R}$ we can consider the sets $X \in \mathcal{F}$ such that $A \subset X$ and we can define the infimum of the lengths among all the possible choices for X. Note again that this infimum could be $+\infty$ if there is no finite length covering of A. The previous infimum is called the outer measure of A and we should highlight that every set $A \subset \mathbb{R}$ is, by construction, outer measurable.

The main point is that, to successfully develop the Lebesgue Measure Theory, the notion of what is a measurable set has to be restricted. In other words, not every outer measurable subset is going to be Lebesgue measurable. We say that $E \subset \mathbb{R}$ is Lebesgue measurable if, for every $A \subset \mathbb{R}$, the sum of the outer measure of $A \cap E$ and the outer measure of $A \cap (\mathbb{R} \setminus E)$ equals the outer measure of A. Thus, measurable sets are the ones that allow to break any set into two pieces, via intersection with the measurable set and its complementary, in such a way that the original outer measure is the sum of the outer measures of the two obtained pieces. The Lebesgue measure is defined as the outer measure restricted to the measurable sets.

Lebesgue measure has several properties, but we emphasize that any zero outer measure set is, indeed, Lebesgue measurable. Moreover, when a property holds at every point of a set except for, at most, a zero Lebesgue measure set, it is said to hold "almost everywhere".

After a technical process, one may construct the Lebesgue integral for the so-called Lebesgue integrable functions: a function is Lebesgue integrable if it is Lebesgue measurable (the preimage of any open set is Lebesgue measurable) and if the Lebesgue integral of its absolute value is finite (which is defined after a technical process of approximation by the so-called "simple functions"). It is a consequence of the construction of the Lebesgue integral that functions which are equal almost everywhere have the same Lebesgue integral. Moreover, the complement of a zero Lebesgue measure set is dense and, hence, two continuous functions that are equal almost everywhere are equal everywhere. Furthermore, on compact intervals, Riemann integrable functions in the proper sense are Lebesgue integrable and the value of both integrals do coincide.

Thus, the main role of Lebesgue integration is not to provide an additional method for computing more primitives, although "more" functions are Lebesgue integrable than Riemann integrable, but to define integration under a more robust framework. Indeed, this allows a very nice interplay between Lebesgue integration and Functional and Real Analysis, providing very interesting theoretical results.

We end this summary concerning Lebesgue integration recalling that the set of all Lebesgue integrable functions defined on the interval $[a, b]$ is denoted by $\mathscr{L}^1[a, b]$. If we consider the quotient set of $\mathscr{L}^1[a, b]$ via the equivalence relation "two functions are related if they are equal almost everywhere" we obtain $L^1[a, b]$. Passing to this quotient is not relevant in terms of integration and, as usual, we will refer to $L^1[a, b]$ as the set of integrable functions on $[a, b]$. Analogously, we will call Lebesgue measurable functions just "measurable functions".

Of course, this construction can be adapted for higher dimensions: we only need to replace the role of the intervals, in the construction of the outer measure, for the same idea with hyperrectangles.

Remark 1.9. We define the norm of an element $f \in L^1[a, b]$ as the Lebesgue integral $\int_a^b |f|$. We observe that this norm is well defined, since it is independent of the chosen representative. Besides, the space $L^1[a, b]$ endowed with the previous norm is a Banach space.

1.2.1 Absolutely continuous functions

We also need to recall the notion of absolutely continuous function. This concept plays a key role in FC, since the set of absolutely continuous functions is made up of the functions that, essentially, are antiderivatives of some function in L^1. We will see later, how this notion is highly relevant to construct the spaces where fractional derivatives are well defined.

Definition 1.10. A real function f of real variable is absolutely continuous on $[a,b] \subset \mathbb{R}$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for every family of subintervals $\{[a_1, b_1], ..., [a_n, b_n]\}$ with disjoint interiors, we have

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Longrightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

We denote the set of these functions by $AC[a,b]$.

Remark 1.11. It follows trivially from the definition that any absolutely continuous function is uniformly continuous and, hence, continuous.

This definition appears to carry little information. Furthermore, it seems a bit complicated to check the absolute continuity of a given function if its expression is not very manageable. However, the following result characterizes the absolutely continuous functions in a simple way.

Theorem 1.12 (Fundamental Theorem of Calculus)**.** *Consider a real function f defined on an interval $[a,b] \subset \mathbb{R}$. Then, $f \in AC[a,b]$ if and only if there exists $\varphi \in L^1[a,b]$ such that*

$$f(t) = f(a) + \int_a^t \varphi(s)\, ds. \tag{1.1}$$

Remark 1.13. This result establishes that, essentially, absolutely continuous functions defined on $[a,b]$ are the primitives of the functions in $L^1[a,b]$, that is, antiderivatives of measurable functions whose absolute value has finite integral.

Remark 1.14. In general, we can use indistinctly the notations $L^1(a,b)$ and $L^1[a,b]$, since the considered functions are defined almost everywhere.

This last result allows us to define the derivative of an absolutely continuous function on $[a,b]$ as a certain function in $L^1[a,b]$.

Definition 1.15. If $f \in AC[a,b]$, we define its derivative f' as the unique function (up to a measure zero set) $\varphi \in L^1[a,b]$ such that (1.1) holds.

Remark 1.16. It is relevant to have in mind that the previous definition makes sense because, once fixed $f(a)$, the antiderivative operator is injective when defined on $L^1[a,b]$.

1.2.2 Fubini and Tonelli theorems

As we announced before, there are several versions for Fubini and Tonelli theorems, which are slightly different. We will use a specific one for each result, which will be enough for our purposes. The statement for Tonelli theorem can be found at [3] (page 118, Theorem 10.9) and the one for Fubini theorem can be found as a particular case of the statement in [63] (page 164, Theorem 8.8).

Theorem 1.17 (Tonelli theorem)**.** *Consider two closed hyperrectangles $\Omega_1 \subset \mathbb{R}^{m_1}$, $\Omega_2 \subset \mathbb{R}^{m_2}$ (bounded or unbounded) and let $f(x,y)$ be a non-negative real measurable function defined on $\Omega_1 \times \Omega_2$. If we denote the section functions by $f^x(y)$ and $f^y(x)$, we have that f^x is measurable for almost every $x \in \Omega_1$ and f^y is measurable for almost every $y \in \Omega_2$. Furthermore, the functions*

$$\varphi(x) = \int_{\Omega_2} f^x(y)\,dy \quad \text{and} \quad \psi(y) = \int_{\Omega_1} f^y(x)\,dx$$

are measurable and the following identity holds

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,dy\ dx = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,dx\ dy = \int_{\Omega_1\times\Omega_2} f(x,y)\ dx \times dy,$$

where the possible values of the integrals are in $[0,+\infty]$.

Theorem 1.18 (Fubini theorem)**.** *Consider two closed hyperrectangles $\Omega_1 \subset \mathbb{R}^{m_1}$, $\Omega_2 \subset \mathbb{R}^{m_2}$ (bounded or unbounded) and let $f(x,y)$ be a real measurable function defined on $\Omega_1 \times \Omega_2$ such that*

$$\int_{\Omega_1\times\Omega_2} |f(x,y)|\ dx \times dy$$

is finite. If we denote the section functions by $f^x(y)$ and $f^y(x)$, we have that f^x is measurable for almost every $x \in \Omega_1$ and f^y is measurable for almost every $y \in \Omega_2$. Furthermore, the functions

$$\varphi(x) = \int_{\Omega_2} f^x(y)\,dy \quad \text{and} \quad \psi(y) = \int_{\Omega_1} f^y(x)\,dx$$

are measurable and their absolute value is integrable. Moreover, the following identity holds

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,dy\ dx = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,dx\ dy = \int_{\Omega_1\times\Omega_2} f(x,y)\ dx\times dy,$$

where the three integrals are finite.

It is quite frequent to find mathematical texts intending to prove that a function is in L^1 by showing that it is absolutely integrable, without checking the measurability condition. Moreover, in the case of $\Omega_1, \Omega_2 \subset \mathbb{R}$, it is also common to find "arguments" that pretend to show that a function $f(x,y)$ is in $L^1(\Omega_1 \times \Omega_2)$ if one of the iterated integrals of $|f(x,y)|$ exists. This is correct, provided that $f(x,y)$ is measurable, although the underlying reason is due neither to Tonelli nor Fubini theorems. Nevertheless, there is a very interesting interplay between the two previous results that allows to overcome these situations in a very standard and clear way. This is the so-called Fubini-Tonelli theorem.

Theorem 1.19 (Fubini-Tonelli)**.** *Consider two closed hyperrectangles denoted by $\Omega_1 \subset \mathbb{R}^{m_1}$, $\Omega_2 \subset \mathbb{R}^{m_2}$ (bounded or unbounded) and let $f(x,y)$ be a real measurable function defined on $\Omega_1 \times \Omega_2$ such that one of the three following integrals*

$$\int_{\Omega_1}\int_{\Omega_2} |f(x,y)|\,dy\ dx, \quad \int_{\Omega_2}\int_{\Omega_1} |f(x,y)|\,dx\ dy, \quad \int_{\Omega_1\times\Omega_2} |f(x,y)|\ dx\times dy$$

is finite. Then, all of them exist and coincide (Tonelli's thesis).

Moreover, if we denote the section functions by $f^x(y)$ and $f^y(x)$, we have that f^x is measurable for almost every $x \in \Omega_1$ and f^y is measurable for almost every $y \in \Omega_2$. Furthermore, the functions

$$\varphi(x) = \int_{\Omega_2} f^x(y)\,dy \quad \text{and} \quad \psi(y) = \int_{\Omega_1} f^y(x)\,dx$$

are measurable and their absolute value is integrable. Moreover, the three integrals

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,dy\ dx, \quad \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,dx\ dy, \quad \int_{\Omega_1\times\Omega_2} f(x,y)\ dx\times dy$$

are finite and coincide (Fubini's thesis).

Proof. We know that $f(x,y)$ is measurable on the product space $\Omega_1 \times \Omega_2$. Hence, $|f(x,y)|$ is measurable on the product space $\Omega_1 \times \Omega_2$ and the Tonelli's thesis is an immediate consequence of the application of Tonelli theorem to the non-negative function $|f(x,y)|$. Observe that $|f(x,y)|$ is measurable due to the continuity of $|\cdot|$, since the preimage of an open set is still Lebesgue measurable. Consequently, $f(x,y)$ is absolutely integrable in $\Omega_1 \times \Omega_2$ and we can apply Fubini theorem to derive the Fubini thesis. □

The previous theorems, as formulated, only apply for products of two hyperrectangles. As one can expect, they can be applied to much more general sets and, even, to measure spaces providing that additional technical conditions are imposed. We will not discuss about these topics, since they are not a matter of interest for us. However, we would like to have an analogous statement for cartesian products involving more than two factors. In fact, it is trivial to develop an inductive argument to adapt Theorem 1.19 to the case of a product of n-hyperrectangles. In particular, we will need the three factors case. This is enough for our purposes and, moreover, it suffices to imagine how the statement would be in the case of having more factors. We have the following result.

Theorem 1.20 (Fubini-Tonelli, three factors)**.** *Consider three closed hyperrectangles $\Omega_1 \subset \mathbb{R}^{m_1}$, $\Omega_2 \subset \mathbb{R}^{m_2}$, $\Omega_3 \subset \mathbb{R}^{m_3}$ (bounded or unbounded) and let $f(x_1,x_2,x_3)$ be a real measurable function defined on $\Omega_1 \times \Omega_2 \times \Omega_3$ such that one of the, a priori distinct, thirteen following integrals*

$$\begin{aligned}
\int_{\Omega_i}\int_{\Omega_j}\int_{\Omega_k} |f(x_1,x_2,x_3)|\, dx_k\ dx_j\ dx_i \quad &\text{(six choices)},\\
\int_{\Omega_i \times \Omega_j}\int_{\Omega_k} |f(x_1,x_2,x_3)|\, dx_k\ dx_i \times dx_j \quad &\text{(three choices)},\\
\int_{\Omega_k}\int_{\Omega_i \times \Omega_j} |f(x_1,x_2,x_3)|\, dx_i \times dx_j\ dx_k \quad &\text{(three choices)},\\
\int_{\Omega_i \times \Omega_j \times \Omega_k} |f(x_1,x_2,x_3)|\, dx_i \times dx_j \times dx_k \quad &\text{(one choice)},
\end{aligned}$$

is finite, where $i,j,k \in \{1,2,3\}$ are all different. Then, all of them exist and coincide (Tonelli's thesis).

Moreover, if we denote the section functions by $f^{x_i}(x_j,x_k)$ and $f^{(x_i,x_j)}(x_k)$, we have that f^{x_i} is measurable for almost every $x_i \in \Omega_i$ and $f^{(x_i,x_j)}$ is mea-

surable for almost every $(x_i, x_j) \in \Omega_i \times \Omega_j$. Furthermore, the functions

$$\varphi(x_i) = \int_{\Omega_j \times \Omega_k} f^{x_i}(x_j, x_k)\, dx_j \times dx_k,$$
$$\psi(x_i, x_j) = \int_{\Omega_k} f^{(x_i,x_j)}(x_k)\, dx_k$$

are measurable and their absolute value is integrable. Moreover, the thirteen following integrals

$$\int_{\Omega_i} \int_{\Omega_j} \int_{\Omega_k} f(x_1, x_2, x_3)\, dx_k\ dx_j\ dx_i \quad \text{(six choices)},$$
$$\int_{\Omega_i \times \Omega_j} \int_{\Omega_k} f(x_1, x_2, x_3)\, dx_k\ dx_i \times dx_j \quad \text{(three choices)},$$
$$\int_{\Omega_k} \int_{\Omega_i \times \Omega_j} f(x_1, x_2, x_3)\, dx_i \times dx_j\ dx_k \quad \text{(three choices)},$$
$$\int_{\Omega_i \times \Omega_j \times \Omega_k} f(x_1, x_2, x_3)\, dx_i \times dx_j \times dx_k \quad \text{(one choice)},$$

are finite and coincide (Fubini's thesis).

Proof. If we describe Ω as the product $(\Omega_i \times \Omega_j) \times \Omega_k$ and we apply the standard version of Fubini-Tonelli theorem (Theorem 1.19), we conclude immediately the result if we omit the theses involving the integrals

$$\int_{\Omega_i} \int_{\Omega_j} \int_{\Omega_k} |f(x_1, x_2, x_3)|\, dx_k\ dx_j\ dx_i,$$
$$\int_{\Omega_i} \int_{\Omega_j} \int_{\Omega_k} f(x_1, x_2, x_3)\, dx_k\ dx_j\ dx_i.$$

Thus, we have proved the result for the last seven possible choices and we need to prove it for the first six ones. Now, if we consider the section function $f^{x_k}(x_i, x_j)$, we know that it is measurable for almost every $x_k \in \Omega_k$. Moreover, we can apply Fubini-Tonelli theorem (Theorem 1.19) to $f^{x_k}(x_i, x_j)$ for almost every $x_k \in \Omega_k$. If we integrate with respect to x_k we conclude the equivalence between the existence of the following two integrals

$$\int_{\Omega_k} \int_{\Omega_j} \int_{\Omega_i} |f(x_1, x_2, x_3)|\, dx_i\ dx_j\ dx_k,$$
$$\int_{\Omega_k} \int_{\Omega_i \times \Omega_j} |f(x_1, x_2, x_3)|\, dx_i \times dx_j\ dx_k,$$

and, also, between the pair

$$\int_{\Omega_k}\int_{\Omega_j}\int_{\Omega_i} f(x_1,x_2,x_3)\,dx_i\ dx_j\ dx_k,$$
$$\int_{\Omega_k}\int_{\Omega_i\times\Omega_j} f(x_1,x_2,x_3)\,dx_i\times dx_j\ dx_k.$$

However, in both cases, the existence of the last integral was equivalent to the existence of one of the last seven choices. Hence, the existence of the thirteen choices is equivalent and we are done. □

Obviously, it is interesting to have similar results for the calculation of integrals in non-rectangular regions. In this sense, there are several versions of the Theorem of Change of Variables that allow to perform the integration in circular, cylindrical or spherical enclosures, for instance. A much simpler case, which will be enormously useful for our purposes, is the triangular version of Fubini-Tonelli theorem. This result, often known as Dirichlet formula, states under pretty general assumptions that the "vertical" integration of a function defined on a triangle gives the same result that its "horizontal" analogue.

Corollary 1.21 (Dirichlet formula)**.** *Consider the triangle T with vertices $\{(a,a),(a,b),(b,b)\}\subset\mathbb{R}^2$, where $a<b$. Note that T is exactly the set of points (x,y) such that $a\le x\le y\le b$.*

If f is a measurable function on T and one of the following two integrals

$$\int_a^b\int_x^b |f(x,y)|\,dy\,dx,\qquad \int_a^b\int_a^y |f(x,y)|\,dx\,dy$$

is finite, then both exist and coincide (Tonelli's thesis).

Moreover, if we denote the section functions by $f^x(y)$ (defined on $[x,b]$) and $f^y(x)$ (defined on $[a,y]$), we have that f^x is measurable for almost every $x\in[a,b]$ and f^y is measurable for almost every $y\in[a,b]$. Furthermore, the functions

$$\varphi(x)=\int_x^b f^x(y)\,dy\quad\text{and}\quad \psi(y)=\int_a^y f^y(x)\,dx$$

are measurable and their absolute value is integrable. Moreover, the three integrals

$$\int_a^b\int_x^b f(x,y)\,dy\,dx,\qquad \int_a^b\int_a^y f(x,y)\,dx\,dy,\qquad \int_T f(x,y)\ dx\times dy$$

are finite and coincide (Fubini's thesis).

Proof. We define $\bar{f}$ as the extension of the function f to the whole square $[a,b] \times [a,b]$ with value 0 out of T. Clearly, the measurability of f implies the measurability of $\bar{f}$. The result follows from Fubini-Tonelli theorem by considering the zone where $\bar{f}$ vanishes to reconsider the endpoints of integration. □

Now, we state and prove an analogue of Dirichlet formula for the tridimensional case. We will not state it in its whole generality, since later we will only need to use it in a very particular situation.

Corollary 1.22 (Dirichlet formula, tetrahedron). *Consider the simplex S whose set of vertices is $\{(a,a,a),(a,a,b),(a,b,b),(b,b,b)\} \subset \mathbb{R}^3$, where $a<b$. Note that S is exactly the set of points (x,y,z) such that $a \leq x \leq y \leq z \leq b$.*

If f is a measurable function on S and one of the following two integrals

$$\int_a^b \int_x^b \int_y^b |f(x,y,z)| \, dz \, dy \, dx, \qquad \int_a^b \int_a^z \int_a^y |f(x,y,z)| \, dx \, dy \, dz$$

is finite, then both exist and coincide (Tonelli's thesis).

Moreover, the three integrals

$$\int_a^b \int_x^b \int_y^b f(x,y,z) \, dz \, dy \, dx, \qquad \int_a^b \int_a^z \int_a^y f(x,y,z) \, dx \, dy \, dz,$$

$$\int_S f(x,y,z) \, dx \times dy \times dz$$

are finite and coincide (Fubini's thesis).

Proof. We define $\bar{f}$ as the extension of the function f to the whole cube $[a,b] \times [a,b] \times [a,b]$ with value 0 out of S. Clearly, the measurability of f implies the measurability of $\bar{f}$. The result follows from the Fubini-Tonelli theorem for three factors (Theorem 1.20) by considering the zone where $\bar{f}$ vanishes to reconsider the endpoints of integration. □

1.2.3 Convolution theory

As we have announced, we will also need some standard results concerning convolution theory. Apart from the definition and the classical fact that the convolution of two functions in $L^1[0,b]$ is again in $L^1[a,b]$, we will state two results. One gives necessary conditions on the factors for their convolution to be identically null. The other one gives suitable conditions ensuring that identity perturbed versions of convolution operators are surjective.

We recall that a function $f \in L^1[a,b]$ induces, in a natural way, a convolution operator via the map $C_a : L^1[a,b] \longrightarrow \mathrm{End}_B(L^1[a,b])$.

Definition 1.23. Given $f \in L^1[a,b]$, we defined its associated convolution operator as $C_a(f) : L^1[a,b] \longrightarrow L^1[a,b]$ defined as

$$(f * g)(t) := (C_a(f)\, g)(t) := \int_a^t f(t-s+a) \cdot g(s)\, ds$$

for $g \in L^1[a,b]$. Under the previous notation, we say that f is the kernel of the convolution operator $C_a(f)$.

There are some well known properties about convolution operators that we summarize below. For details, see cite [63].

Remark 1.24. The convolution of two functions is commutative, that is, $f * g = g * f$.

Moreover, we have already commented that the convolution is well defined, as an operation in $L^1[a,b]$, in the following sense

Proposition 1.25. *If $f, g \in L^1[a,b]$, then $f * g \in L^1[a,b]$.*

Proof. To show that $f * g$ is measurable, we can apply Dirichlet formula in Corollary 1.21. We define $F(t,s) = f(t) \cdot g(t-s+a)$, which is a measurable function on its triangular domain $a \le s \le t \le b$, since it is a product of measurable functions.

To show that the convolution of two integrable functions is, indeed, integrable, it suffices to prove that

$$\int_a^b \left| \int_a^t f(t-s+a) \cdot g(s)\, ds \right| dt$$

is finite. Of course, it is enough to see that the following upper bound

$$\int_a^b \int_a^t |f(t-s+a) \cdot g(s)|\, ds\, dt$$

is finite. However, in virtue of Dirichlet formula, we just prove that the other iterated integral is finite. This can be done easily, since

$$\int_a^b \int_s^b |f(t-s+a) \cdot g(s)|\, dt\, ds = \int_a^b |g(s)| \int_s^b |f(t-s+a)|\, dt\, ds$$

and the right hand side is less than or equal to

$$\int_a^b |g(s)|\, ds \cdot \int_a^b |f(t)|\, dt,$$

whose factors are finite by hypothesis. $\square$

Remark 1.26. We observe that this proof, indeed, shows that the linear operator $C_a(f) : L^1[a,b] \longrightarrow L^1[a,b]$ is continuous, since we have checked that $\|C_a(f)\| \leq \|f\|$.

We also present a similar result involving the convolution between an integrable and a continuous function.

Proposition 1.27. *If $f \in \mathcal{C}[a,b]$ and $g \in L^1[a,b]$, then $f * g \in \mathcal{C}[a,b]$.*

Proof. To show that $f * g$ is continuous, it is clear that it suffices to prove that $|(f*g)(t+d) - (f*g)(t)| \to 0$, for each fixed $t \in [a,b]$, when $d \to 0$ and $t+d \in [a,b]$. We simply observe that the difference

$$\left| \int_a^{t+d} f(t+d-s+a) \cdot g(s)\, ds - \int_a^t f(t-s+a) \cdot g(s)\, ds \right|$$

can be bounded from above by

$$\int_a^t |(f(t+d-s+a) - f(t-s+a)) \cdot g(s)|\; ds \\ + \int_t^{t+d} |f(t+d-s+a) \cdot g(s)|\; ds,$$

but it is possible to provide the following upper estimate

$$\max_{t,\, t+d \in [a,b]} |f(t+d) - f(t)| \cdot \int_a^b |g(s)|\, ds + \max_{t \in [a,b]} |f(t)| \cdot \int_t^{t+d} |g(s)|\, ds.$$

The left addend goes to zero, since f is uniformly continuous (continuous on the compact interval $[a,b]$) and $g \in L^1[0,b]$. The right addend also tends to zero when $d \to 0$, since f is bounded on the compact interval $[a,b]$ and the right factor tends to zero when $d \to 0$ because $g \in L^1[0,b]$. □

Proposition 1.28. *Given $f, g, h \in L^1[a,b]$, we have $(f*g)*h = f*(g*h)$.*

Proof. The key is to realise that the expression

$$((f*g)*h)(t) = \int_a^t \left(\int_a^r f(r-s+a) \cdot g(s)\, ds \right) h(t-r+a)\, dr,$$

where $a \leq s \leq r \leq t$, can be turned into

$$\int_a^t \left(\int_s^t f(r-s+a) \cdot h(t-r+a)\, dr \right) g(s)\, ds$$

via the Dirichlet formula (Corollary 1.21). The hypotheses that ensure that Dirichlet formula can be applied may be checked as in the previous proposition. After the change of variables given by $\rho = r - s + a$, we see that the previous expression equals

$$\int_a^t \left(\int_a^{t-s+a} f(\rho) \cdot h((t-s+a) - \rho + a)\, d\rho \right) g(s)\, ds = ((f * h) * g)(t).$$

Now, we just combine the previous idea with the commutativity of the convolution to obtain the desired result

$$(f * g) * h = (f * h) * g = (h * f) * g = (h * g) * f = (g * h) * f = f * (g * h).$$

□

Finally, we will also need a result concerning the annulation of a convolution. In a few words, we need to know what are the possibilities for the factors of a convolution, provided that the obtained result is the zero function. Roughly speaking, the classical result in this direction, known as Titchmarsh theorem, states that this happens only when the integrand of the convolution from 0 to t is always zero, independently of t.

Theorem 1.29 (Titchmarsh theorem). *Suppose that $f, g \in L^1[a,b]$ are such that $f * g \equiv 0$. Then, there exist $\lambda, \mu \in \mathbb{R}^+$ such that the following three conditions hold:*

i) $f \equiv 0$ *on the interval* $[a, a+\lambda]$*,*

ii) $g \equiv 0$ *on the interval* $[a, a+\mu]$*,*

iii) $\lambda + \mu \geq b - a$*.*

We will not provide the proof of this result, since it is a bit technical and it is not interesting from the point of view of the work that we are going to develop. The proof can be consulted in [75].

Remark 1.30. In particular, Titchmarsh theorem states that the operator $C_a(f) : L^1[a,b] \longrightarrow L^1[a,b]$ is injective, provided that $f \in L^1[a,b]$ and that f is not identically null at any interval $[a, a+\lambda]$ for $\lambda > 0$.

We will also use the following result, concerning Volterra integral equations of the second kind. This result essentially states that some family of integral equations do always have a continuous solution, provided that the source term is continuous.

Theorem 1.31 (Rust theorem)**.** *Given $k \in L^1[a,b]$, the Volterra integral equation*

$$[C_a(k)\, v]\,(t) + v(t) := \int_a^t k(t-s) \cdot v(s)\, ds + v(t) = w(t)$$

has exactly one continuous solution $v \in \mathcal{C}[a,b]$, provided that $w \in \mathcal{C}[a,b]$ and that the following two conditions hold:

i) If $h \in \mathcal{C}[a,b]$, then $C_a(k)\, h \in \mathcal{C}[a,b]$,

ii) If $n \in \mathbb{Z}^+$ is large enough, then $(C_a(k))^n = C_a(\widetilde{k})$ for some $\widetilde{k} \in \mathcal{C}[a,b]$.

We will not provide the proof, analogously to what we previously did with Titchmarsh theorem. The result can be found in a more general context in [66], since in this dissertation we only state it for the particular case of convolution kernels.

Remark 1.32. In particular, after suitable minor modifications, we will use Rust theorem to ensure that $C_a(k) : L^1[a,b] \longrightarrow L^1[a,b]$ is a surjective operator, once a particular choice for the kernel $k \in L^1[a,b]$ has been made.

As one can imagine, Remarks 1.30 and 1.32 will be used to ensure that some integral operators of our interest are bijective, implying the existence and uniqueness of solution for the associated problems.

1.2.4 Laplace transforms

In this subsection, we present the fundamental notions and results concerning Laplace Transform (LaT). However, before discussing those topics, we will set a suitable functional space where this transform is defined. Further information about LaT or the proofs that are ommited can be found in [25]. For results showing connections between LaT and FC, that will be explained in the next chapter, see [59, 69].

The classical functional space where LaT is defined is

$$D_{\mathcal{L}} := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : u(t) = 0 \;\forall\, t < 0,\, \exists\, c \in \mathbb{R} : \int_0^\infty e^{-ct} \cdot u(t)\, dt < \infty \right\}.$$

This implies that the functions $u(t)$ to be transformed are zero when $t < 0$. If a function u is asymptotically bounded by some exponential term in absolute value, that is, if there is $M > 0$ such that $|u(t)| < e^{ct}$ for some $c > 0$ and any $t > M$, then $u \in D_{\mathcal{L}}$. If we define

$$\lambda(u) := \inf\{c \in \mathbb{R} : e^{-ct} u(t) \in L^1(\mathbb{R})\},$$

for each $u \in D_{\mathcal{L}}$, then we can build the following set, related to the region of convergence of the LaT,

$$\mathbb{R}_{\lambda(u)} := \{s \in \mathbb{R} : s > \lambda(u)\}.$$

Of course, from the definition of $D_{\mathcal{L}}$, we have that $\lambda(u) < +\infty$ for every $u \in D_{\mathcal{L}}$, which means that $\mathbb{R}_{\lambda(u)} \neq \emptyset$ for every $u \in D_{\mathcal{L}}$. Furthermore, it is known that, if $u \in D_{\mathcal{L}}$, then any primitive of u lies in $D_{\mathcal{L}}$.

Definition 1.33. Given $u \in D_{\mathcal{L}}$, we define the LaT as

$$U(s) := \mathcal{L}[u](s) := \int_{\mathbb{R}} e^{-st} \cdot u(t)\, dt = \int_0^\infty e^{-st} \cdot u(t)\, dt,$$

where the LaT, denoted by $U(s)$, is well defined for every $s \in \mathbb{R}_{\lambda(u)}$.

One powerful application of the LaT is its ability to simplify the task of finding the solution to some PDE. As one can see, the LaT only depends on the value of the function in the positive real half-line. Therefore, the LaT fits naturally with functions that depend on a positive variable. Probably, the main example is the time variable when we are studying some evolution process: heat transfer, diffusion and several others. The previously mentioned usefulness of LaT is summarized in the following well known result, [25] (page 15, Theorem 2.3).

Proposition 1.34. *Consider a function u and an integer $n \geq 1$ such that $u^{n-1)}$ is an absolutely continuous function on $(0, +\infty)$. If $u^{n)} \in D_{\mathcal{L}}$ and we can ensure the existence of the limits $u(0^+), \ldots, u^{n-1)}(0^+)$, then*

$$\mathcal{L}\left[u^{n)}\right](s) = s^n \cdot U(s) - \sum_{j=0}^{n-1} s^{n-j-1} u^{j)}(0^+).$$

In particular, provided that $u(0^+) = 0, \ldots, u^{n-1)}(0^+) = 0$, we get

$$\mathcal{L}\left[u^{n)}\right](s) = s^n \cdot U(s).$$

It is also important to take into account the following classical and relevant result about LaT. It is common that, when we solve the transformed problem, we get a solution that is the product of some "elementary" functions. The Convolution theorem states that, provided that all the involved expressions make sense, the LaT of a convolution is the product of the corresponding LaT.

Theorem 1.35 (Convolution)**.** *If $u_1, u_2 \in D_{\mathcal{L}}$, then we have that*

$$\mathcal{L}[u_1 * u_2](s) = U_1(s) \cdot U_2(s) \in D_{\mathcal{L}}.$$

1.2.5 Special functions

When introducing the notions of fractional integral or derivative, it will be necessary to extend functions defined on a discrete domain, like $\mathbb{N}$ or $\mathbb{Z}$, to functions of a real variable. In this sense, we will use the Euler Gamma function (Γ), that extends the factorial function.

Definition 1.36. We define the Euler Gamma function Γ as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad \Re(z) > 0.$$

Remark 1.37. It is possible to give an analytical extension of Γ that allows a definition on the whole complex plane $\mathbb{C}$ except for the non-positive integers $0, -1, -2, \ldots$ However, due to the scope of this Thesis, we can think of the real valued function obtained as the restriction of Γ to positive real arguments in $\mathbb{R}^+$.

Remark 1.38. The recurrence formula $\Gamma(z+1) = z\,\Gamma(z)$, for $\Re(z) > 0$ is easily derived after the application of integration by parts. In particular, after observing that $\Gamma(1) = 1$, it trivially follows that $\Gamma(n+1) = n!$ for every natural number n. Consequently, Γ extends the factorial function.

Moreover, in certain situations, it will be useful to handle the Beta function B. It is well known its linkage with the Γ function and, as we shall see, it allows to work more easily with some identities involving both of them.

Definition 1.39. We define the Beta function B as

$$B(z,w) = \int_0^1 x^{z-1}(1-x)^{w-1} dx, \quad \Re(z) > 0, \quad \Re(w) > 0. \tag{1.2}$$

Lemma 1.40. *We have the identity*

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}, \quad \Re(z) > 0, \quad \Re(w) > 0. \tag{1.3}$$

As we have already mentioned, our interest in these special functions is motivated because of their relevance in the calculation of many integrals. Some of these computations will appear so often during this manuscript that we will perform them only once. We develop them now, and later we will refer to them when necessary.

Lemma 1.41. *If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, the following identity holds*

$$\int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\, dt = B(\beta,\alpha)\,(x-s)^{\alpha+\beta-1}. \tag{1.4}$$

Proof. If we apply the change of variables $t = s + \tau(x - s)$, we obtain

$$\int_0^1 (1-\tau)^{\alpha-1}(x-s)^{\alpha-1}\tau^{\beta-1}(x-s)^{\beta-1}(x-s)\,d\tau,$$

which can be simplified into

$$\int_0^1 \tau^{\beta-1}(1-\tau)^{\alpha-1}(x-s)^{\alpha+\beta-1}\,d\tau.$$

Then, if we use the expression (1.2), we get

$$\int_0^1 \tau^{\beta-1}(1-\tau)^{\alpha-1}(x-s)^{\alpha+\beta-1}\,d\tau = B(\beta,\alpha)\,(x-s)^{\alpha+\beta-1}.$$

□

1.3 Some questions about Topology

Homeomorphisms are one the main objects of study in topology and, hence, it is interesting to provide conditions to ensure that a map between topological spaces is a homeomorphism. Some examples of this kind of results are the Invariance of Domain theorem (IDT) or the well known lemma stating that a continuous bijection from a compact topological space to a Hausdorff topological space is a homeomorphism. In this section, a similar result is provided.

Most specifically, we show that any continuous bijection from a path-connected topological space to a totally ordered set endowed with the order topology is a homeomorphism. In particular, we show how this assertion generalizes the easiest case of the IDT for dimension $n = 1$. To the best of our knowledge, although the statement is reasonably general and the proof given is not extremely complicated, this result does not appear anywhere else or, at least, it has not been widely described in the extant literature, except in [10]. We begin recalling some basic facts about general topology to motivate our result. Later, we will provide the statement and proof of the theorem, together with several immediate corollaries and comments concerning the applicability of our result. Any required definition of topological character or previous result can be found in [56].

As it is well known, one of the main tasks in Topology is to study homeomorphisms and the properties that are preserved by them, which are called "topological properties". A homeomorphism is no more than a bijective continuous map between two topological spaces whose inverse is also continuous.

The part of the definition “whose inverse is also continuous” can be dropped under certain conditions on the topological spaces, but not in general. In this sense, any student who has attended a Topology course knows that it is possible to construct a continuous bijective map from a real interval $[0, 1)$ to the circle $\mathbb{S}^1$. This map is generally described via the correspondence $x \in [0, 1) \to e^{2\pi x i} \in \mathbb{S}^1$, where i is the imaginary unit. The student also knows that the inverse map is not continuous and, thus, we have the very first example of a continuous bijective map with discontinuous inverse.

The previous example is sometimes confused with a well known theorem in Real Analysis that states that the inverse of any continuous bijection between real intervals is continuous too, see Theorem 5.6.5 in [3]. The key, of course, is that here the domain and codomain are real intervals and not just general topological spaces. Therefore, the very natural question arises of what can we require from the topological spaces to ensure that any continuous bijection between them is automatically a homeomorphism.

It so happens that the previous theorem of Real Analysis is a particular case, for dimension $n = 1$, of the IDT stated below. The proof, for $n = 2$, can be found in [56] as Theorem 62.3.

Theorem 1.42 (Invariance of Domain theorem)**.** *Assume that $U \subset \mathbb{R}^n$ is an open set and consider an arbitrary continuous injective map $f : U \longrightarrow \mathbb{R}^n$. Then the set $f(U)$ is open and f induces a homeomorphism between U and $f(U)$.*

The proof of the IDT in dimensions $n > 1$, due to Brouwer in 1911 [6], requires techniques that involve algebraic topology but, as we mentioned, the case $n = 1$ is covered on any course in Real Analysis since the proof in that case is much simpler. In short, the key argument when $n = 1$ is to use the well known fact that a continuous map between intervals is injective if and only if it is monotone. As we shall see in our generalization, the properties that play a key role here are that intervals are path-connected and their topology is given by the order topology.

If we are not in the Euclidean case, there are still results that ensure that a continuous bijection is a homeomorphism under different assumptions. Perhaps one of the most famous of such results is the following one, see Theorem 26.6 in [56].

Lemma 1.43. *If $f : X \longrightarrow Y$ is a continuous bijection, where X is compact and Y is Hausdorff, then f is a homeomorphism.*

We sketch the basic ideas that appear in the proof, noting that each bullet point is a well known point-set topology result:

- A continuous bijective map $f : X \longrightarrow Y$ is a homeomorphism if and only if it is a closed map, i.e., if it maps any closed set in X to a closed set in Y.

- Since X is compact, any closed set in X is compact.

- Since f is continuous, the image of a compact set, via f, is compact.

- Since Y is Hausdorff, any compact set in Y is closed.

This theorem does not apply when $X = [0, 1)$ and $Y = \mathbb{S}^1$, because $[0, 1)$ is not compact. However, one could consider the following argument. First, we describe $[0, 1)$ as an increasing union of compact sets, for instance, $X = \bigcup_{n=2}^{\infty} X_n$, where $X_n = \left[0, 1 - \frac{1}{n}\right]$ for $n \geq 2$. It is a direct consequence of Lemma 1.43 that f induces a homeomorphism between X_n and $f(X_n)$ for $n \geq 2$. Second, we use the following result, whose proof is a standard exercise in point-set topology.

Lemma 1.44. *If $f : X \longrightarrow Y$ is a bijection between two topological spaces such that for every $x \in X$ there exist open sets $U_x \subset X, V_x \subset Y$ fulfilling:*

i) $x \in U_x$, $f(x) \in V_x$,

ii) *f induces an homeomorphism between U_x and V_x,*

then f is a homeomorphism.

Then we see that the problem involving the compactness has disappeared but, of course, there is no global homeomorphism at all. The problem is that the hypotheses of Lemma 1.44 do not hold for $x = 0$, since there is no open set in $\mathbb{S}^1$ containing $f(0)$ and, simultaneously, contained in some $f(X_n)$. Intuitively, the obstruction is that the topology of $\mathbb{S}^1$ is more complicated than the topology of any $f(X_n)$. It seems reasonable that one could avoid the problem if the codomain had a simpler topology. In fact, we will prove that it is enough for X to be path-connected and for Y to have the order topology to ensure that a continuous bijection is a homeomorphism, deriving two general corollaries later.

Remark 1.45. The condition required in Lemma 1.44 for every $x \in X$ is often referred as "f is a local homeomorphism at x." In consequence, Lemma 1.44 can be stated as follows: A bijection that is a local homeomorphism at every point is a homeomorphism."

Finally, we make a small remainder concerning the order topology. This topology can be induced in any totally ordered set and, indeed, it coincides with the usual one for the common choices of totally ordered spaces: Euclidean topology in $\mathbb{R}$ or discrete topology in $\mathbb{N}$ and $\mathbb{Z}$. The specific construction is the following one.

Definition 1.46. Given a totally ordered set X, we define the order topology in X as the topology generated by the following sets, which are called "open intervals",

$$\begin{aligned}(x_1, x_2) &:= \{x \in X : x_1 <_X x <_X x_2\},\\ (-\infty, x_1) &:= \{x \in X : x <_X x_1\},\\ (x_1, +\infty) &:= \{x \in X : x_1 <_X x\},\end{aligned}$$

where $<_X$ denotes the strict order relation in X. Of course, it is possible to define closed intervals after including the endpoints in the previous sets. Moreover, we observe that the existence of a minimum element $m_1 \in X$ implies that $[m_1, x) = (-\infty, x)$. Analogously, if there is a maximum element $m_2 \in X$, we have that $(x, m_2] = (x, +\infty)$. This observation allows us to establish the following remark.

Remark 1.47. Consider a totally ordered set X, with $\#X \geq 2$, endowed with the order topology and an arbitrary $x \in X$. We can always find an open interval containing x of one of the following three kinds:

i) If x is not the minimum nor the maximum of X, there are $x_1, x_2 \in X$ such that $x \in (x_1, x_2)$.

ii) If x is the minimum of X, there is $x_2 \in X$ such that $x \in [x, x_2) = (-\infty, x_2)$.

iii) If x is the maximum of X, there is $x_1 \in X$ such that $x \in (x_1, x] = (x_1, +\infty)$.

We also recall the following result, which will be used in the proof of the main theorem.

Lemma 1.48. *If X is a topological space endowed with the order topology, then X is Hausdorff.*

Proof. Let $x, y \in X$. We assume, without loss of generality, that $x <_X y$ and distinguish two cases:

- If there is $z \in X$ such that $x <_X z <_X y$, we observe that $x \in (-\infty, z)$, $y \in (z, +\infty)$ and both sets are disjoint.
- In other case, we observe that $x \in (-\infty, y)$, $y \in (x, +\infty)$ and both sets are disjoint, since there is no element strictly greater than x and strictly lower than y simultaneously.

□

Now, we prove the main result, to derive two almost immediate corollaries that will relax the assumptions on the space X later.

Theorem 1.49. *Assume that X is a path-connected topological space and Y is a totally ordered set, endowed with the order topology. Then any continuous bijection $f : X \longrightarrow Y$ is a homeomorphism.*

The idea of the proof is to show that f is a homeomorphism as a consequence of Lemma 1.44. First, we induce a definition of interval in X via f and check some basic properties. Second, we show that f is a homeomorphism between intervals in X and Y, which is the most technical part. Finally, we argue that any $x \in X$ lies in one of these open intervals in X and we conclude due to Lemma 1.44.

Proof of Theorem 1.47. It is possible to define an order on X by pulling back the one on Y. Given $x_1, x_2 \in X$, we define $x_1 \leq_X x_2$ if and only if $f(x_1) \leq_Y f(x_2)$. Since f is bijective, it is clear that X is a totally ordered set and we will use the previously described interval notation.

It is direct to see, from the definition of the intervals in X and using that f is bijective, that $f^{-1}[f(x_1), f(x_2)] = [x_1, x_2]$ and $f^{-1}(f(x_1), f(x_2)) = (x_1, x_2)$ for any $x_1, x_2 \in X$, where $x_1 <_X x_2$. Furthermore, since f is continuous and $(f(x_1), f(x_2))$ is an open set in Y, the interval (x_1, x_2) is open in X. Analogously, the interval $[x_1, x_2]$ is closed in X. In other words, the topology induced from f is coarser than the original topology of X.

Now, we want to conclude that $[x_1, x_2]$ is compact in X, in order to apply Lemma 1.43. This follows from the fact that X is path-connected, and hence connected. To see this, consider a path from x_1 to x_2, i.e., a continuous map

$$p : [0, 1] \longrightarrow X,$$

such that $p(0) = x_1$ and $p(1) = x_2$. In fact $[x_1, x_2] \subset p[0, 1]$, because, if there were some $c \in [x_1, x_2]$ with $c \notin p[0, 1]$, the pair of sets $(-\infty, c)$ and $[c, +\infty)$ would induce a nontrivial separation of the connected set $p[0, 1]$. Moreover,

$p[0,1]$ is compact, so $[x_1, x_2]$ is a closed set inside a compact set and, thus, also compact.

Next, a direct application of Lemma 1.43 shows that the map

$$g : [x_1, x_2] \longrightarrow [f(x_1), f(x_2)]$$

defined as $g(x) = f(x)$ for all $x \in [x_1, x_2]$ is a homeomorphism because it has compact domain and Hausdorff codomain since the order topology is Hausdorff, recall Lemma 1.48. If we remove the endpoints, it is a standard exercise in point-set topology to show that any interval $(x_1, x_2]$, $[x_1, x_2)$, or (x_1, x_2) is homeomorphic via f to $(f(x_1), f(x_2)]$, $[f(x_1), f(x_2))$, or $(f(x_1), f(x_2))$, respectively.

Moreover, without loss of generality, we can assume that $\#X \geq 2$, since the case $X = \{p\}$ is trivial. In virtue of Remark 1.47, any $f(x)$ will belong to an open set V of the form $(f(x_1), f(x)]$, $[f(x), f(x_2))$, or $(f(x_1), f(x_2))$, depending on whether $f(x)$ is the maximum in Y, the minimum in Y or none of them. Now, the previous paragraph implies that f induces a homeomorphism between the open sets V and $f^{-1}(V)$, since x is the maximum or minimum in X if and only if $f(x)$ is the maximum or minimum in Y, respectively. Finally, we recall that x is arbitrary and invoke Lemma 1.44 to conclude that f is a homeomorphism. $\square$

Corollary 1.50. *Assume that X is a topological space whose path-connected components are open and that Y is a totally ordered set endowed with the order topology. Then any continuous bijection $f : X \longrightarrow Y$ that maps the path-connected components of X to open sets, is automatically a homeomorphism.*

Proof. The proof is straightforward since f is bijective and a local homeomorphism. To show this last claim, given any $x \in X$, consider its associated path-connected component C_x, which is open. Analogously, $f(x)$ belongs to the open set $f(C_x)$. Then we apply Theorem 1.49 to the restriction of f with domain C_x and codomain $f(C_x)$. Hence, this restriction is a homeomorphism and, since C_x and $f(C_x)$ are open sets, f is a local homeomorphism at x. $\square$

It is well known that the path-connected components of X are open, provided that X is locally path-connected in the sense of [56], see Theorem 25.4. Moreover, if we define locally path-connectedness as "every point has a path-connected neighbourhood", it is still true that, if X is locally path-connected, then the path-connected components of X are open sets. Therefore, under any of the two previous definitions of "locally path-connected set", we can state the following corollary.

Corollary 1.51. *Assume that X is a locally path-connected topological space and that Y is a totally ordered set endowed with the order topology. Then any continuous bijection $f : X \longrightarrow Y$ that maps the path-connected components of X into open sets is a homeomorphism.*

Remark 1.52. In Corollaries 1.50 and 1.51, it is possible to change "bijection" for "injection". In this case, since X is the union of its path-connected components and f maps them into open sets, we have that $f(X)$ is open in Y. Moreover, since the relative topology in $f(X)$ is the order topology, Theorem 1.49 ensures that f induces a homeomorphism between X and $f(X)$.

Remark 1.53. In particular, Corollaries 1.50, 1.51 and Remark 1.52 generalize the IDT in the case $n = 1$. Note that our results do not require X to be an open subset of $\mathbb{R}$. Moreover, Y is not necessarily contained in $\mathbb{R}$, since it can be any set with the order topology. We show now how to deduce the IDT for $n = 1$ from the previous results, using the notation in Theorem 1.42. Note that the IDT, for $n = 1$, is a direct consequence of Remark 1.52, provided that we ensure that the image of an arbitrary path-connected component is open.

If U is an open set in $\mathbb{R}$, then U is locally path-connected in any of the senses previously described. Obviously, the topology in $\mathbb{R}$ is the order topology. Now, consider an arbitrary path-connected component C of U. Since U is open, C is an open interval and, by connectedness, $f(C)$ is an interval, too. Moreover, C and $f(C)$ are homeomorphic because of Theorem 1.49. To show that the interval $f(C)$ is open, it is enough to prove that it does not contain an endpoint. If $f(p) \in f(C)$ were an endpoint, then $C \setminus \{p\}$ and $f(C) \setminus \{f(p)\}$ would be still homeomorphic, but $C \setminus \{p\}$ would not be connected, in contrast to $f(C) \setminus \{f(p)\}$. Therefore, $f(C)$ is open for any path-connected component $C \subset X$. Hence, from Remark 1.52, we conclude the IDT for $n = 1$, as it is stated in Theorem 1.42.

Remark 1.54. In general, Theorem 1.49 allows us to apply the following philosophy: Imagine that X is a topological space that is indexed bijectively by a totally ordered set Y endowed with the order topology. Assume that X is path-connected and that the map $f : X \longrightarrow Y$, which is the inverse of the indexing map, is continuous. In some sense, the continuity of f says that, if two elements of X are close, then the associated indices have to be close, too. Theorem 1.49 ensures that, under the previous hypotheses, the indexing map f^{-1} is continuous. Therefore, if two indices are close to each other, the associated indexed elements are also close. This situation seems to be of special interest and to appear naturally when X is a set of indexed maps or operators.

Chapter 2

An introduction to Fractional Calculus

The main purpose of this chapter is to introduce the basic notions concerning fractional integration and fractional differentiation. The first step, that which already been taken, consisted in giving technical results to support the theoretical development that we are going to make. We begin by introducing the FC in the sense of Riemann and Liouville, which was born as a generalization of the Cauchy formula for repeated integration. Our first task is to present the notions of the Riemann-Liouville fractional integral and derivative, together with some elementary results involving such concepts. These descriptions can be consulted in [69]. After that, we will present an axiomatic characterization of the Riemann-Liouville fractional integral, that justifies its relevance. This is a not very well known result, despite being available at [19], which we have generalized for the Stieltjes case in [13]. We also give a proof, developed in [18], stating the impossibility of the existence of an "Index Law" for fractional derivatives.

2.1 The Riemann-Liouville fractional integral

In this section, we present the basic ideas and results concerning fractional integration. Unless otherwise indicated, these notions can be found at [69]. We will begin with the fundamental idea (the Cauchy formula for repeated integration), which allows to generalize integer order integration to a positive real order. After this exposition, which permits to understand fractional integration as a natural extension of usual integration, we describe the main properties of the fractional integral operator. Finally, we reproduce a result ensuring that this interpolation is the unique one satisfying some intuitive properties.

2.1.1 The Cauchy formula for repeated integration

In 1832, Bernhard Riemann and Joseph Liouville put the first stone in the formalization of the theory of FC by defining the a fractional integral. The novelty consists in generalizing the Cauchy formula for repeated integration. In a few words, this result gives an identity that equals n consecutive integrals of a continuous function to a convolution of such function with a certain polynomial kernel, that involves the factorial expression $(n-1)!$. The generalization of integer order powers to complex exponents and the construction of the Γ function as an interpolation of the factorial allow us to give a definition for the integral of complex order. However, because of technical reasons, the real part of the order is required to be positive and, in this Thesis, we will only consider the case of positive real orders. In this subsection, we provide the proof for the Cauchy formula for repeated integration, which can be consulted in [69].

Theorem 2.1 (Cauchy formula for repeated integration)**.** *Given a function $f \in L^1[a,b]$ and a positive integer $n \in \mathbb{Z}^+$, we have the following identity:*

$$\begin{aligned}\left(I^n_{a^+} f\right)(t) :&= \int_a^t \int_a^{s_1} \cdots \int_a^{s_{n-1}} f(s_n)\, ds_n \cdots ds_2\, ds_1 \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds,\end{aligned}$$

where $t \in [a,b]$.

Proof. We apply an inductive argument concerning the positive integer n.

The case $n = 1$ is evident, since the identity would read

$$\int_a^t f(s_1)\, ds_1 = \int_a^t f(s)\, ds.$$

We suppose that the identity holds for a certain n and we prove it for $n+1$. By definition, we know that

$$\left(I^{n+1}_{a^+} f\right)(t) := \int_a^t \int_a^{s_1} \cdots \int_a^{s_n} f(s_{n+1})\, ds_{n+1} \cdots ds_2\, ds_1.$$

We apply the inductive hypothesis to the n most internal integrals. It follows that

$$\left(I^{n+1}_{a^+} f\right)(t) = \frac{1}{(n-1)!} \int_a^t \int_a^{s_1} (s_1 - s)^{n-1} f(s)\, ds\, ds_1.$$

If we could interchange the order of the integrals, via Corollary 1.21, we would get

$$\left(I_{a^+}^{n+1} f\right)(t) = \frac{1}{(n-1)!} \int_a^t \int_s^t (s_1 - s)^{n-1} f(s)\, ds_1\, ds,$$

and we could derive the desired identity

$$\left(I_{a^+}^{n+1} f\right)(t) = \frac{1}{n!} \int_a^t (t-s)^n f(s)\, ds.$$

To ensure the applicability of Corollary 1.21, consider the function

$$F(s, s_1) := \frac{1}{(n-1)!} (s_1 - s)^{n-1} f(s), \text{ where } a \le s \le s_1 \le t.$$

This function is clearly Lebesgue measurable and we have to see that one iterated integral is finite. If we repeat the previous calculations, we get

$$\int_a^t \int_s^t |F(s, s_1)|\, ds_1\, ds = \frac{1}{n!} \int_a^t (t-s)^n |f(s)|\, ds \le \frac{(t-a)^n}{n!} \int_a^t |f(s)|\, ds,$$

and the hypotheses for applying Corollary 1.21 are fulfilled. □

Remark 2.2. The result is valid for any choice of the base point $a \in \mathbb{R}$. It is possible to state an analogous result, where the right limit of integration is fixed and the left limit of integration is the variable.

Although we will give another motivation to introduce the FC, we have begun with the most standard one. From this point of view, the definition for the Riemann-Liouville fractional integral arises as a natural generalization of the main identity in Theorem 2.1. This is easily seen, just by taking into account Remark 1.38, which stated $\Gamma(z+1) = z!$ for $z \in \mathbb{N}$. Later, it will become clear why this is the most reasonable extension for the usual integration, instead of using another different interpolation for the factorial function or even more complicated formulae.

Definition 2.3. We define the left Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1[a,b]$ with base point a as

$$I_{a^+}^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

for almost every $t \in [a,b]$. In the case that $\alpha = 0$, we just define $I_{a^+}^{\alpha}$ as

$$I_{a^+}^{0} f(t) = \operatorname{Id} f(t) = f(t).$$

Remark 2.4. Note that the left Riemann-Liouville fractional integral of a certain function is built as a new function defined at almost every point, contrary to what happens when $\alpha \in \mathbb{Z}^+$. The main difference is that $I_{a^+}^{\alpha}$ is, in principle, a map from $L^1[a,b]$ to itself but, when $\alpha \geq 1$, we will show that its image is contained in $\mathcal{C}[a,b]$.

Remark 2.5. Although it is possible to make a definition for complex orders with positive real part, we observe that we have already restricted our attention to the case of a real order of integration.

We can also make the analogous definition with endpoint b but, as we shall see in the next subsection, this consideration does not carry more generality.

Definition 2.6. We define the right Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1[a,b]$ with endpoint b as

$$I_{b^-}^{\alpha} f(t) = \int_t^b \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

for almost every $t \in [a,b]$. In the case that $\alpha = 0$, we just define

$$I_{b^-}^{0} f(t) = \operatorname{Id} f(t) = f(t).$$

2.1.2 Simplifications on the Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral operator has several properties that are relevant to be inspected. Before describing them, we make three observations that will allow us to make some simplifications in this study. More specifically, we see how the Riemann-Liouville fractional integral operator behaves with respect to translations, reflections and dilatations/contractions.

Lemma 2.7. *If Q is the reflection operator given by $(Q\varphi)(t) = \varphi(a+b-t)$, mapping $L^1[a,b]$ into itself, we have the identities*

$$QI_{a^+}^{\alpha} = I_{b^-}^{\alpha} Q, \quad QI_{b^-}^{\alpha} = I_{a^+}^{\alpha} Q.$$

Proof. It is obvious that $Q \circ Q = \operatorname{Id}$. We only have to check the first identity, since the second one follows after applying Q at both sides.

On the one hand,

$$(QI_{a^+}^{\alpha} f)(t) = (I_{a^+}^{\alpha} f)(a+b-t) = \frac{1}{\Gamma(\alpha)} \int_a^{a+b-t} (a+b-t-s)^{\alpha-1} f(s)\, ds.$$

We make the change of variables $s = a+b-r$ or, equivalently, $r = a+b-s$, implying the formal substitution $ds = -dr$. The last expression equals

$$\frac{-1}{\Gamma(\alpha)} \int_b^t (r-t)^{\alpha-1} f(a+b-r)\, dr,$$

where we should note the change in the endpoints of integration because of the change of variables. If we invert the order of the endpoints, cancelling the minus sign, we obtain

$$\frac{1}{\Gamma(\alpha)} \int_t^b (r-t)^{\alpha-1} f(a+b-r)\, dr.$$

On the other hand,

$$(I^{\alpha}_{b^-} Qf)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} Q\, f(s)\, ds$$

and it is enough to take into account the identity $Q\, f(s) = f(a+b-s)$ to equal the previous expression to

$$\frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(a+b-s)\, ds. \qquad \square$$

Lemma 2.8. *If $\widehat{Q}$ is the translation operator given by $(\widehat{Q}\varphi)(t) = \varphi(a+t)$, mapping $L^1[a, a+b]$ into $L^1[0,b]$, we have the identity*

$$\widehat{Q}\, I^{\alpha}_{a^+} = I^{\alpha}_{0^+}\, \widehat{Q}.$$

Proof. We know that

$$(\widehat{Q}\, I^{\alpha}_{a^+} f)(t) = (I^{\alpha}_{a^+} f)(a+t) = \frac{1}{\Gamma(\alpha)} \int_a^{a+t} (a+t-s)^{\alpha-1} f(s)\, ds.$$

We make the change of variables $s = a+r$ or, equivalently, $r = s-a$, implying the formal substitution $ds = dr$. The last expression equals

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(a+r)\, dr = (I^{\alpha}_{0^+} \widehat{Q}\, f)(t). \qquad \square$$

The two previous Lemmas 2.7 and 2.8 imply that, without loss of generality, we can restrict the study to the left Riemann-Liouville fractional integral with zero base point. From now on, we shall use the nomenclature "Riemann-Liouville fractional integral of order α" to denote the operator $I^{\alpha}_{0^+}$.

Moreover, we can see how the Riemann-Liouville fractional integral behaves with respect to dilatations or contractions of the domain interval, in the following sense.

Lemma 2.9. *If $\widetilde{Q}$ is the contraction/dilatation operator given by the expression $(\widetilde{Q}\varphi)(t) = \varphi(b \cdot t)$, mapping $L^1[0,b]$ into $L^1[0,1]$, we have the identity*

$$\widetilde{Q}\, I^{\alpha}_{0+} = b^{\alpha} \cdot I^{\alpha}_{0+}\, \widetilde{Q}.$$

Proof. The left hand side equals, for each $t \in [0,1]$,

$$\int_0^{b\,t} \frac{(b\,t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

Now, to apply the change of variable, we rescale the integrand linearly as a function of $r \in [0,t]$, instead of $s \in [0, b\,t]$. The identity $s = b\,r$ and the symbolic change $ds = b\,dr$ imply that the previous expression equals

$$\int_0^{t} \frac{b^{\alpha-1}\,(t - r)^{\alpha-1}}{\Gamma(\alpha)}\, b\,(\widetilde{Q}f)(r)\, dr,$$

which coincides with the right hand side. □

We will not assume that $b = 1$ in general, and we will keep working on $L^1[0,b]$. However, the previous result will be used to simplify some technical proofs that, in principle, would distinguish the cases $b < 1$ and $b > 1$.

2.1.3 Basic properties of the fractional integral

In this subsection, we will discuss the main properties of the Riemann-Liouville fractional integral. We will prove that it is a continuous well defined operator from $L^1[0,b]$ to itself, and satisfying the Index Law, together with other several properties. As usual, these results can be consulted in [69].

The first step is to show that I^{α}_{0+} preserves the Lebesgue integrability.

Proposition 2.10. *If $f \in L^1[0,b]$, then $I^{\alpha}_{0+} f \in L^1[0,b]$ for $\alpha \geq 0$.*

Proof. If $\alpha = 0$ the claim is trivial. In other case, the proof will be a direct consequence of Dirichlet formula in Corollary 1.21, provided that we make a suitable choice for the integrand. Consider the function

$$F(s,t) := \frac{1}{\Gamma(\alpha)}\,(t-s)^{\alpha-1}\, f(s), \text{ where } 0 \leq s \leq t \leq b.$$

On the one hand, the function F is clearly measurable on its triangular domain. At first, we note that the function is a product of Lebesgue measurable functions of (s,t) and, thus, Lebesgue measurable.

On the other hand, we shall see that the integral of its absolute value is bounded in $[0,b]\times[0,b]$. We invoke Dirichlet formula and we know that it is enough to check the finiteness of one of the iterated integrals (Corollary 1.21). We need to see that

$$\int_0^b \int_s^b |F(s,t)|\, dt\, ds$$

is finite. To check this, we observe that

$$\int_0^b \int_s^b |F(s,t)|\, dt\, ds = \frac{1}{\Gamma(\alpha)} \int_0^b |f(s)| \left(\int_s^b (t-s)^{\alpha-1}\, dt \right) ds.$$

When we compute the inner integral, we get

$$\frac{1}{\Gamma(\alpha+1)} \int_0^b |f(s)|\, (b-s)^{\alpha} ds \le \frac{b^{\alpha}}{\Gamma(\alpha+1)} \int_0^b |f(s)| ds.$$

Hence, since $f \in L^1[0,b]$, we have checked the hypotheses of Dirichlet formula.

In particular, one of the theses of Dirichlet formula states that

$$\left(I_{0+}^{\alpha} f\right)(t) = \int_0^t F(s,t)\, ds = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

lies in $L^1[0,b]$. □

Furthermore, the Riemann-Liouville fractional integral fulfils a property which is called Index Law. It states that the composition of two fractional integrals is, indeed, another fractional integral whose order is the addition of the two previous ones.

Proposition 2.11. *The Riemann-Liouville fractional integral fulfils the following Index Law for any function* $f \in L^1[0,b]$

$$\left(I_{0+}^{\alpha}\, I_{0+}^{\beta} f\right)(t) = \left(I_{0+}^{\alpha+\beta} f\right)(t), \quad \alpha,\beta \ge 0. \tag{2.1}$$

Proof. The case where α or β equals zero is trivial. In the general case, we have

$$\left(I_{0+}^{\alpha} I_{0+}^{\beta} f\right)(t) = \frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \int_0^s (s-r)^{\beta-1} f(r)\, dr\, ds.$$

We consider the function

$$F(r,s,t) = \frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)}\,(t-s)^{\alpha-1}(s-r)^{\beta-1}f(r), \text{ where } 0 \le r \le s \le t \le b.$$

The function F is measurable, since it is a product of measurable functions. As we did before, we shall see that one iterated integral of the absolute value is finite to apply Dirichlet formula in its tridimensional version, Corollary 1.22. Thus, it is enough to check that

$$\frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)}\int_0^b |f(r)|\int_r^b (s-r)^{\beta-1}\int_s^b (t-s)^{\alpha-1}\,dt\,ds\,dr$$

is finite. After computing the inner integral we arrive to

$$\frac{1}{\alpha\cdot\Gamma(\alpha)\cdot\Gamma(\beta)}\int_0^b |f(r)|\int_r^b (s-r)^{\beta-1}(b-s)^{\alpha}\,ds\,dr.$$

Now, we can recognize an integral involving the B function and, by Remark 1.38 and Lemma 1.41, the previous expression equals to

$$\frac{B(\beta,\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^b (b-r)^{\alpha+\beta}|f(r)|\,dr \le \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\int_0^b |f(r)|\,dr,$$

where, in the last inequality, we have used Lemma 1.40.

The previous argumentation allows us to consider the measurable function

$$F^t(r,s) = \frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)}\,(t-s)^{\alpha-1}(s-r)^{\beta-1}f(r), \text{ where } 0 \le r \le s \le t.$$

We know that it is absolutely integrable for almost every $t \in [0,b]$ because of Dirichlet formula applied to $F(r,s,t)$ and its section $F^t(r,s)$. Hence, for almost every $t \in [0,b]$, both iterated integrals of $F^t(r,s)$ exist and coincide.

On the one hand, one iterated integral is

$$\left(I^{\alpha}_{0^+}\circ I^{\beta}_{0^+}\right)f(t) = \frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1}\int_0^s (s-r)^{\beta-1}f(r)\,dr\,ds.$$

On the other hand, the other iterated integral is

$$\frac{1}{\Gamma(\alpha)\cdot\Gamma(\beta)}\int_0^t f(r)\int_r^t (t-s)^{\alpha-1}(s-r)^{\beta-1}\,ds\,dr.$$

Since $\alpha, \beta > 0$, we can apply the identity in Lemma 1.41, concerning the B function, to equal the previous expression to

$$\frac{B(\beta,\alpha)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^t (t-r)^{\alpha+\beta-1} f(r)\,dr = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-r)^{\alpha+\beta-1} f(r)\,dr,$$

where the right hand side equals $I_{0+}^{\alpha+\beta} f(t)$ in virtue of Definition 2.3. □

Proposition 2.12. *Let $n \in \mathbb{Z}^+$ and $f \in L^1[0,b]$. Then $I_{0+}^{\alpha} f \in C^{n-1}[0,b]$ provided that $\alpha \geq n$. Moreover, the derivative of order $n-1$ of $I_{0+}^{\alpha} f$ is absolutely continuous.*

Proof. The proof is straightforward from the two previous results. We just observe the factorization $I_{0+}^{\alpha} = I_{0+}^{n} \circ I_{0+}^{\alpha-n}$, where the first operator maps $L^1[0,b]$ into $L^1[0,b]$ and the second one maps $L^1[0,b]$ into $\mathcal{C}^{n-1}[a,b]$. Moreover, the derivative of order $n-1$ of $I_{0+}^{\alpha} f = \left(I_{0+}^{n-1} \circ I_{0+}^{\alpha-n+1}\right) f$ is clearly $I_{0+}^{\alpha-n+1} f = I_{0+}^{1}\left(I_{0+}^{\alpha-n} f\right)$ which is absolutely continuous, due to Theorem 1.12. □

Remark 2.13. It is relevant to note that fractional integration does not preserve every type of regularity. For instance, the fractional integral of a $\mathcal{C}^{\infty}$ function is not necessarily a $\mathcal{C}^{\infty}$ function, as we shall see at the end of this subsection in a particular example.

We now deduce another relevant property involving fractional integrals: their continuity with respect to the order of integration.

Proposition 2.14. *The map $\mathrm{Ind}_a : \mathbb{R}^+ \to \mathrm{End}_B(L^1[0,b])$ given by $\alpha \to I_{0+}^{\alpha}$ is continuous.*

Proof. It is clear that we have to compute an appropriate estimate for the expression

$$\left\|\left(I_{0+}^{\alpha} - I_{0+}^{\beta}\right) f\right\|_{L^1[0,b]} = \int_0^b \left|\int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\right) f(s)\,ds\right| dt.$$

The assumption $b = 1$ does not imply a loss of generality, since in other case we would only add a factor $\max\{b^{\alpha}, b^{\beta}\}$ to the estimate, recall Lemma 2.9. For simplicity of the computations we will assume the hypothesis $b = 1$. Clearly, it will be enough to deduce a suitable upper bound for

$$\int_0^1 \int_0^t \left|\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\right) f(s)\right| ds\,dt.$$

In virtue of the triangle inequality, it is enough to find an upper bound for the integral

$$\int_0^1 \int_0^t \left(\left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} \right| + \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right| \right) |f(s)|\, ds\, dt.$$

However, due to Dirichlet formula (Corollary 1.21), since the integrand is measurable, it is enough to estimate from above

$$\begin{aligned} &\int_0^1 |f(s)| \int_s^1 \left| \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} \right) \right| dt\, ds \\ +&\int_0^1 |f(s)| \int_s^1 \left| \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \right) \right| dt\, ds. \end{aligned}$$

The key point is that we can compute the inner integral for both addends, despite of the absolute value, and the result will be the absolute value of the integral. This happens because both differences have constant sign as functions of $t-s$. This claim involving the constant sign is obvious for the first integral, but we have to highlight the importance of $b=1$ in the case of the second one, since it ensures that $(t-s)^{\alpha-1}-(t-s)^{\beta-1}$ does not change sign and we can extract the absolute value outside the integral.

After computing the inner integrals, we obtain the following expression

$$\int_0^1 \frac{|f(s)|}{\alpha} \left| \frac{(1-s)^{\alpha}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha}}{\Gamma(\beta)} \right| ds + \int_0^1 \frac{|f(s)|}{\Gamma(\beta)} \left| \frac{(1-s)^{\alpha}}{\alpha} - \frac{(1-s)^{\beta}}{\beta} \right| ds.$$

The first addend can be bounded, in a trivial way, by

$$\frac{1}{\alpha} \left| \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\beta)} \right| \int_0^1 |f(s)|\, ds. \tag{2.2}$$

An upper estimate for the second addend can be written as the into two new terms

$$\int_0^1 \frac{|f(s)|}{\Gamma(\beta)} \left| \frac{(1-s)^{\alpha}}{\alpha} - \frac{(1-s)^{\alpha}}{\beta} \right| ds + \int_0^1 \frac{|f(s)|}{\Gamma(\beta)} \left| \frac{(1-s)^{\alpha}}{\beta} - \frac{(1-s)^{\beta}}{\beta} \right| ds.$$

The first new term can be trivially bounded by

$$\frac{1}{\Gamma(\beta)} \left| \frac{1}{\alpha} - \frac{1}{\beta} \right| \int_0^1 |f(s)|\, ds. \tag{2.3}$$

For any $\delta \in (0,1)$, the second new term can be split into

$$\int_0^{1-\delta} \frac{|f(s)|}{\Gamma(\beta)} \left| \frac{(1-s)^{\alpha}}{\beta} - \frac{(1-s)^{\beta}}{\beta} \right| ds + \int_{1-\delta}^1 \frac{|f(s)|}{\Gamma(\beta)} \left| \frac{(1-s)^{\alpha}}{\beta} - \frac{(1-s)^{\beta}}{\beta} \right| ds,$$

giving two final addends.

The first final addend can be bounded, in virtue of the Mean Value Theorem applied to $g(x) = (1-s)^x$, by

$$\frac{|\alpha-\beta| \cdot |\ln(\delta)|}{\Gamma(\beta+1)} \int_0^1 |f(s)|\, ds. \tag{2.4}$$

Moreover, it is straightforward to give the following upper bound for the second addend

$$\frac{\delta^\alpha + \delta^\beta}{\Gamma(\beta+1)} \int_0^1 |f(s)|\, ds. \tag{2.5}$$

Consider now a fixed value of β. It will be enough to show that, for any given $\varepsilon > 0$, there exists some d that makes each expression (2.2), (2.3), (2.4), (2.5) less than or equal to $\frac{\varepsilon}{4}\|f\|_{L^1[0,1]}$, whenever $|\alpha-\beta| < d$. We should note that we can select δ at our best convenience.

Clearly, since the Γ function is continuous and not vanishing on $\mathbb{R}^+$, there is a d_1 such that $|\alpha - \beta| < d_1$ implies that (2.2) and (2.3) is less than or equal to $\frac{\varepsilon}{4}\|f\|_{L^1[0,1]}$.

Moreover, we can consider a small enough value for $\delta > 0$ such that (2.5) is less than or equal to $\frac{\varepsilon}{4}\|f\|_{L^1[0,1]}$, whenever $\alpha \in (\beta - d_1, \beta + d_1)$.

After fixing this δ, we can find $d < d_1$ such that $|\alpha - \beta| < d$ implies that (2.4) is less than or equal to $\frac{\varepsilon}{4}\|f\|_{L^1[0,1]}$. □

Remark 2.15. Note that this result, in principle, does not hold for the case $\alpha \in \mathbb{R}^+ \cup \{0\}$, see [69] (page 48, Theorem 2.6).

Proposition 2.16. *For any $\alpha \geq 0$, the operator $I^\alpha_{0+} : L^1[0,b] \longrightarrow L^1[0,b]$ is injective.*

Proof. The operator I^1_{0+} is injective, which is an immediate consequence of Theorem 1.29. It trivially follows, by iterated composition, that I^n_{0+} is injective for any positive integer n. Therefore, if we consider a value for n such that $n > \alpha$, we have that $I^n_{0+} = I^{n-\alpha}_{0+} \circ I^\alpha_{0+}$. Since the composition is injective, the first factor I^α_{0+} in the composition has to be injective and we are done. □

We end this subsection by presenting an example of explicit computation of a fractional integral which is well known [55, 59, 69]. In general, particular calculations of fractional integrals are strongly linked with special functions and identities that hold between them.

Example 2.17. *More specifically, we are interested in the value of* $I_{0^+}^{\alpha} t^{\beta}$ *for* $\alpha > 0$ *and* $\beta > -1$*, where the restriction on* β *ensures that the integrand lies in* $L^1[0,b]$*. In virtue of Lemma 1.41,*

$$I_{0^+}^{\alpha} t^{\beta} = \frac{1}{\Gamma(\alpha)} \int_0^t s^{\beta}(t-s)^{\alpha-1}\, ds = \frac{B(\beta+1,\alpha)}{\Gamma(\alpha)}\, t^{\alpha+\beta}.$$

After using the relation between the B *and* Γ *functions, presented in Lemma 1.40, we arrive to*

$$I_{0^+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, t^{\alpha+\beta},$$

generalizing the classical result for an integer order of integration α*.*

Since the fractional integral is a linear operator, it is trivial to compute the Riemann-Liouville fractional integral of any "generalized polynomial" whose exponents are greater than -1*. Indeed, observe that* $I_{0^+}^{\alpha} t^{\beta} \in I_{0^+}^{\gamma} L^1[0,b]$ *if and only if* $\alpha + \beta > \gamma - 1$*.*

2.1.4 Axiomatic characterization for the Riemann-Liouville fractional integral

Before introducing a suitable definition for fractional derivative, we will invest some pages to convince the reader that the previous definition for the fractional integral is, in some sense, the most natural one. On the one hand, it is true that the Riemann-Liouville fractional integral extends the Cauchy formula for repeated integration. However, on the other hand, there could be different extensions for the Cauchy formula, implying several reasonable definitions for fractional integration, whatever does "reasonable" mean.

This idea of providing some axiomatic characterization for the Riemann-Liouville fractional integral was formulated by J. Lew during the "International Conference on Fractional Calculus and its Applications" held at the University of New Haven, in June 1974. He conjectured that Riemann-Liouville fractional integral was the unique interpolation for the usual integral satisfying some properties. This conjecture was confirmed to be true in the document [19], written by D. Cartwright and J. McMullen. We reproduce this result with minor modifications, since the original work is formulated for complex-valued functions and we are dealing with the real case.

Theorem 2.18 (Cartwright-McMullen)**.** *Given a fixed* $a \in \mathbb{R}$*, there is only one family of operators* $\left(J_{a^+}^{\alpha}\right)_{\alpha>0}$ *on* $L^1[a,b]$ *satisfying the following conditions:*

1. *The operator of order* 1 *is the usual integral with base point* a. *That is,* $J_{a^+}^1 = I_{a^+}^1$. *(Interpolation property)*

2. *The Index Law holds. That is,* $J_{a^+}^{\alpha} \circ J_{a^+}^{\beta} = J_{a^+}^{\alpha+\beta}$ *for all* $\alpha, \beta > 0$. *(Index Law)*

3. *The family is continuous with respect to the parameter. That is, the following map* $\mathrm{Ind}_a : \mathbb{R}^+ \longrightarrow \mathrm{End}_B\left(L^1[a,b]\right)$ *given by* $\mathrm{Ind}_a(\alpha) = J_{a^+}^{\alpha}$ *is continuous, where the codomain has the norm topology. (Continuity)*

This family is precisely given by the Riemann-Liouville fractional integrals and, hence, we have $J_{a^+}^{\alpha} = I_{a^+}^{\alpha}$ *for* $\alpha > 0$.

As we have already mentioned, we reproduce the proof presented in [19] with some slight modifications. The main idea is to split the proof into two steps.

The first step is to show that the result holds when $J_{a^+}^{\alpha}$ is assumed to be a convolution operator. More specifically, the idea is to show that it is enough to study $J_{a^+}^{\frac{1}{m}}$ for $m \in \mathbb{Z}^+$. The main reason is that the Index Law and the continuity with respect to the index allow us to describe $J_{a^+}^{\alpha}$ for $\alpha > 0$, provided that we know $J_{a^+}^{\frac{1}{m}}$. When $J_{a^+}^{\frac{1}{m}}$ is assumed to be a convolution operator, we use appropriate tools from convolution theory, namely Titchmarsh theorem, to conclude the uniqueness.

The second step is to show that the result holds when $J_{a^+}^{\frac{1}{m}}$ is not necessarily a convolution operator. At first, one uses the Index Law and the interpolation property in order to prove that $J_{a^+}^{\frac{1}{m}}$ commutes with any convolution operator with polynomial kernel. Indeed, the continuity of the convolution operator C_a and the density of polynomials in $L^1[a,b]$ imply that it commutes with any convolution operator. The previous fact is crucial to deduce that $I_{a^+}^1 \circ J_{a^+}^{\frac{1}{m}}$ is always a convolution operator, although $J_{a^+}^{\frac{1}{m}}$ is not. Finally, one mimics the discussion developed during the first step, now for the convolution operator $I_{a^+}^1 \circ J_{a^+}^{\frac{1}{m}}$, and uses the injectivity of $I_{a^+}^1$ to derive the desired result.

We provide the proof in detail hereunder.

Proof. It is well known that the family of Riemann-Liouville fractional integrals satisfy the three properties, recall Propositions 2.11 and 2.14. We shall see that, in fact, this is the unique possibility for the family $\left(J_{a^+}^{\alpha}\right)_{\alpha>0}$.

Consider a family of operators $\left(J^{\alpha}_{a^+}\right)_{\alpha>0}$ fulfilling the three hypotheses in the statement. Then, for any positive integer m, we have

$$\left(J^{\frac{1}{m}}_{a^+}\right)^m = J^1_{a^+} = I^1_{a^+} = \left(I^{\frac{1}{m}}_{a^+}\right)^m = C_a\left(g_{a,\frac{1}{m}}\right)^m,$$

where we use the notation

$$g_{a,\alpha}(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}$$

and we recall that the operator C_a was presented at Definition 1.23.
Step 1: Assume initially that, for any given $m \in \mathbb{Z}^+$, there exists a function $f \in L^1[a,b]$ such that $J^{\frac{1}{m}} = C_a(f)$. Therefore, it is immediate to see that

$$0 \equiv C_a(f)^m - C_a\left(g_{a,\frac{1}{m}}\right)^m = C_a(h),$$

where $h = \left(f * \overset{(m)}{\cdots} * f\right) - \left(g_{a,\frac{1}{m}} * \overset{(m)}{\cdots} * g_{a,\frac{1}{m}}\right)$ and each convolution has been done m times. Thus, due to Titchmarsh theorem 1.29, $h \equiv 0$. Moreover, since convolution is associative, we can write

$$h = \left(f - g_{a,\frac{1}{m}}\right) * \left(f - \xi^1 g_{a,\frac{1}{m}}\right) * \cdots * \left(f - \xi^{m-1} g_{a,\frac{1}{m}}\right) \equiv 0,$$

where ξ is a primitive m-th root of unity. Note that, if $f - \xi^j g_{a,\frac{1}{m}} \equiv 0$ on some interval $[a, a+\lambda]$ for an integer $j \in \{0,1,\ldots,m-1\}$, the same can not happen for another integer j', since it would imply that $g_{a,\frac{1}{m}}$ vanishes on $[a, a+\lambda]$. As a consequence of Titchmarsh theorem, we deduce that

$$f - \xi^j g_{a,\frac{1}{m}} \equiv 0 \text{ on } [a,b] \text{ for some } j \in \{0,1,\ldots,m-1\}.$$

But, since f and $g_{a,\frac{1}{m}}$ are real valued, it follows that $J^{\frac{1}{m}}_{a^+} = \pm I^{\frac{1}{m}}_{a^+}$ for each $m \in \mathbb{Z}^+$. In fact, the Index Law implies that

$$J^{\alpha}_{a^+} = \pm I^{\alpha}_{a^+}, \text{ where } \alpha \in \mathbb{Q}^+. \tag{2.6}$$

Since the map $\alpha \to J^{\alpha}_{a^+}$ is continuous and $J^{\alpha}_{a^+} \not\equiv 0$ for any $\alpha > 0$ (in other case, we would have $J^{\beta}_{a^+} \equiv 0$ for every $\beta \geq \alpha$ due to the Index Law), the sign choice in equation (2.6) has to be independent of α and the identity can be extended for $\alpha \in \mathbb{R}$. Finally, noting that $J^1_{a^+} = I^1_{a^+}$, we must have $J^{\alpha}_{a^+} = I^{\alpha}_{a^+}$.

Step 2: Now, we face the general case, where we do not assume $J_{a^+}^{\frac{1}{m}}$ to be a convolution operator. At first, observe that $J_{a^+}^{\frac{1}{m}}$ commutes with $I_{a^+}^1 = C_a\left(g_{a,1}\right)$ because

$$J_{a^+}^{\frac{1}{m}} \circ I_{a^+}^1 = J_{a^+}^{\frac{m+1}{m}} = I_{a^+}^1 \circ J_{a^+}^{\frac{1}{m}},$$

due to the Index Law and the interpolation property. Besides, it commutes with the operator $I_{a^+}^n = C_a(g_{a,n})$ for any $n \in \mathbb{Z}^+$, whose kernel is a monomial of degree $n-1$. Then, by linearity, $J_{a^+}^{\frac{1}{m}}$ commutes with any convolution operator with polynomial kernel. Since polynomials are dense in $L^1[a,b]$ and the operator C_a is continuous, recall Remark 1.26, we conclude that $J_{a^+}^{\frac{1}{m}}$ commutes with any convolution operator $C_a(f)$ with kernel $f \in L^1[a,b]$.

The previous remark allows to prove that $S_a := I_{a^+}^1 J_{a^+}^{\frac{1}{m}} = J_{a^+}^{\frac{1}{m}} I_{a^+}^1$ is, indeed, a convolution operator. More specifically, we deduce

$$S_a\, f = J_{a^+}^{\frac{1}{m}} C_a\left(g_{a,1}\right)\, f = J_{a^+}^{\frac{1}{m}} C_a(f)\, g_{a,1} = C_a(f) J_{a^+}^{\frac{1}{m}}\, g_{a,1} = C_a\left(J_{a^+}^{\frac{1}{m}}\, g_{a,1}\right)\, f.$$

Moreover, if we denote $\gamma_{a,m} = J_{a^+}^{\frac{1}{m}}\, g_{a,1}$ and we observe that $S_a^m = I_{a^+}^{m+1}$, we obtain

$$\left(C_a\left(g_{a,\,\frac{m+1}{m}}\right)\right)^m = \left(I_{a^+}^{\frac{m+1}{m}}\right)^m = I_{a^+}^{m+1} = S_a^m = \left(C_a(\gamma_{a,m})\right)^m,$$

which implies $\gamma_{a,m} = \xi^j\, g_{a,\,\frac{m+1}{m}}$, as we already proved in Step 1. Thus,

$$I_{a^+}^1 J_{a^+}^{\frac{1}{m}} = S_a = C_a(\gamma_{a,m}) = \xi^j\, C_a\left(g_{a,\,\frac{m+1}{m}}\right) = \xi^j\, I_{a^+}^{\frac{m+1}{m}} = I_{a^+}^1 \xi^j I_{a^+}^{\frac{1}{m}}.$$

Since $I_{a^+}^1$ is injective, this implies that $J_{a^+}^{\frac{1}{m}} = \xi^j I_{a^+}^{\frac{1}{m}}$. As we already checked in Step 1, the conclusion $\left(J_{a^+}^\alpha\right)_{\alpha>0} = \left(I_{a^+}^\alpha\right)_{\alpha>0}$ follows. □

2.1.5 Axiomatic characterization for the Stieltjes case

We are also interested in the "Riemann-Liouville fractional integral with respect to a function h", which is also defined in [69]. Analogously to what is done in [69], in the rest of the subsection we will assume that $h \in \mathcal{C}^1[a,b]$ with $h'(t) > 0$ for every $t \in [a,b]$.

Given $\alpha \in \mathbb{R}^+$ and $a \in \mathbb{R}$ we define the Riemann-Liouville fractional integral of order α and base point a of a function $f \in L^1[a,b]$ with respect to h as

$$\left(I_{h,a^+}^\alpha f\right)(t) := \int_a^t \frac{(h(t)-h(s))^{\alpha-1}}{\Gamma(\alpha)}\, f(s)\, h'(s)\, ds.$$

There are several properties of this operator that are derived in [69]. We highlight that, for $\alpha = 1$, the operator $I^1_{h,\,a^+}$ is the Stieltjes integral operator with integrator h and that $I^\alpha_{h,\,a^+} \in \operatorname{End}_B\left(L^1[a,b]\right)$. Moreover, we observe that the choice $h(t) = t$ recovers the Riemann-Liouville definition.

It is a reasonable question whether we can give, for the case of the Stieltjes integral operator, a similar result to the one in the previous subsection. We would like to ensure that there is only one continuous interpolation for the Stielthes integral operator such that the Index Law holds. The answer is positive when the integrator is given by a function $h \in \mathcal{C}^1[a,b]$ such that $h'(t) > 0$ for any $t \in [a,b]$. Furthermore, we can give an explicit construction of the interpolation in this case, which is the "Riemann-Liouville fractional integral with respect to the function h". Instead of developing a technical proof for this result, we will deduce it as a corollary of the Cartwright-McMullen theorem after some suitable remarks. This result is a new contribution appearing in [13].

Theorem 2.19. *There is only one family of operators $(J^\alpha_{h,\,a^+})_{\alpha>0}$ on $L^1[a,b]$ satisfying the following conditions:*

1. *The operator of order 1 is the usual Stieltjes integral, $J^1_{h,\,a^+} = I^1_{h,\,a^+}$. (Interpolation property)*
2. *The Index Law holds. That is, $J^\alpha_{h,\,a^+} \circ J^\beta_{h,\,a^+} = J^{\alpha+\beta}_{h,\,a^+}$ for all $\alpha, \beta > 0$. (Index Law)*
3. *The family is continuous with respect to the parameter. That is, the following map $\operatorname{Ind}_{h,a} : \mathbb{R}^+ \longrightarrow \operatorname{End}_B\left(L^1[a,b]\right)$ given by $\operatorname{Ind}_{h,\,a}(\alpha) = J^\alpha_{h,\,a^+}$ is continuous, where the codomain has the norm topology. (Continuity)*

The family is precisely given by the "Riemann-Liouville fractional integral with respect to the function h".

Proof. Consider the operator $R_h : L^1[h(a), h(b)] \longrightarrow L^1[a,b]$ given by the expression $R_h(f) = f \circ h$. Since h is continuously differentiable and $h'(t) > 0$ for $t \in [a,b]$, it is a consequence of the Change of Variables Theorem that R_h is well-defined, meaning that $f \circ h \in L^1[a,b]$ for $f \in L^1[h(a), h(b)]$. Although h is not necessarily linear, it is straightforward to check that R_h is an invertible linear operator, where $R_h^{-1} = R_{h^{-1}}$. To see that the operator R_h is continuous, we recall that, for a function $f \in L^1[h(a), h(b)]$, we have

$$\|f\|_{L^1[h(a),h(b)]} = \int_{h(a)}^{h(b)} |f(t)|\, dt = \int_a^b |f(h(t))| \cdot \left|h'(t)\right| dt \geq m \cdot \int_a^b |f(h(t))|\, dt,$$

where $m = \min\{|h'(t)| \in \mathbb{R}^+ : t \in [a,b]\} > 0$ exists since $|h'|$ is continuous on the compact interval $[a,b]$. Thus,

$$\|R_h(f)\| = \int_a^b |f(h(t))|\, dt \leq \frac{1}{m} \|f\|_{L^1[h(a),h(b)]},$$

and we have proved that R_h is continuous.

The previous properties concerning R_h are of our interest because

$$I^1_{h,\,a^+} = R_h \circ I^1_{h(a)^+} \circ R_h^{-1}. \tag{2.7}$$

This claim follows by direct calculation, since

$$\left(R_h \circ I^1_{h(a)^+} \circ R_h^{-1} f\right)(t) = \left(R_h \circ I^1_{h(a)^+}(f \circ h^{-1})\right)(t),$$

and the right hand side in the previous expression equals, by definition,

$$R_h \left(\int_{h(a)}^t f\left(h^{-1}(s)\right)\, ds \right) = \int_{h(a)}^{h(t)} f\left(h^{-1}(s)\right)\, ds.$$

Finally, the Change of Variables theorem allows us to rewrite the last term as

$$\int_a^t f(s) \cdot h'(s)\, ds = \left(I^1_{h,\,a^+} f\right)(t).$$

In fact, it is immediate to show that $I^n_{h,\,a^+} = R_h \circ I^n_{h(a)^+} \circ R_h^{-1}$, for every positive integer n, after iterated composition. Our intuition tells us that, if $J^\alpha_{h(a)^+}$ is a "nice interpolation" for $I^n_{h(a)^+}$, then the definition $R_h \circ J^\alpha_{h(a)^+} \circ R_h^{-1}$ should be a "nice interpolation" for $I^n_{h,\,a^+}$. Conversely, a choice for $J^\alpha_{h,\,a^+}$ fulfilling the hypotheses in Theorem 2.19 should imply that

$$K^\alpha_{h(a)^+} := R_h^{-1} \circ J^\alpha_{h,\,a^+} \circ R_h. \tag{2.8}$$

is under the hypotheses of the Cartwright-McMullen theorem (Theorem 2.18). This intuition can be confirmed after making the following three remarks. Before doing this, note that the continuity of $J^\alpha_{h,\,a^+}$, R_h and R_h^{-1} implies directly that $K^\alpha_{h(a)^+} \in \mathrm{End}_B\left(L^1[h(a),h(b)]\right)$. Therefore, we have:

1. If $J^1_{h,\,a^+} = I^1_{h,\,a^+}$, then $K^1_{h(a)^+} = I^1_{h(a)^+}$. This is a consequence of Equations (2.7) and (2.8).

2. If $J^\alpha_{h,\,a^+} \circ J^\beta_{h,\,a^+} = J^{\alpha+\beta}_{h,\,a^+}$, then $K^\alpha_{h(a)^+} \circ K^\beta_{h(a)^+} = K^{\alpha+\beta}_{h(a)^+}$. This is a consequence of Equation (2.8).

3. If the map $\operatorname{Ind}_{h,\,a}(\alpha) = J^{\alpha}_{h,\,a^+}$ is continuous, then $\operatorname{Ind}_{h(a)}(\alpha) = K^{\alpha}_{h(a)^+}$ is continuous. This is a consequence of Equation (2.8), together with the continuity of the composition operator described in Corollary 1.8. If we recall the notation in Corollary 1.8, it is straightforward express $\operatorname{Ind}_{h(a)}$ as the composition of continuous maps

$$\operatorname{Ind}_{h(a)} = \operatorname{Comp}_{(R_h^{-1},\cdot)} \circ \operatorname{Comp}_{(\cdot, R_h)} \circ \operatorname{Ind}_{h,\,a}.$$

Therefore, since R_h and R_h^{-1} are bijective, two different choices for $J^{\alpha}_{h,\,a^+}$ would induce two different possibilities for $J^{\alpha}_{h(a)^+}$ in the Cartwright-McMullen theorem. Since this is not possible, there is also a unique choice for $J^{\alpha}_{h,\,a^+}$. Moreover, this unique extension will be given by

$$I^{\alpha}_{h,\,a^+} = R_h \circ I^{\alpha}_{h(a)^+} \circ R_h^{-1}.$$

□

2.2 The fractional derivative

In this section, we present the basic ideas and results concerning fractional differentiation. Unless otherwise indicated, these notions can be found in [69]. We will begin with the definition of Riemann-Liouville fractional derivative and we will discuss about the suitable space where it can be defined. We will study its basic properties, highlighting the ones that do not have an analogue in the integer order case. Probably, some of the most notorious properties is that being differentiable of a certain order does not imply being differentiable for lower lower order, or that fractional derivatives do not fulfil the Index Law, which will be proved in this section. We will also emulate this discussion for the Caputo derivative. Finally, we will prove that it is not possible to define a fractional derivative, fulfilling certain reasonable properties, in such a way that the Index Law holds.

2.2.1 The Riemann-Liouville fractional derivative

We will define the Riemann-Liouville fractional derivative as the left inverse operator of the fractional integral. After that, an easy analytical expression for its computation will follow. One of the most relevant remarks at this point is that the fractional derivative will be no more a local operator, meaning that two functions that coincide in a neighbourhood of a point do not need to have the same fractional derivative at that point.

Definition 2.20. Consider $\alpha \geq 0$. We define the Riemann-Liouville fractional derivative of order α and base point 0 as the left inverse of the corresponding Riemann-Liouville fractional integral, meaning

$$D_{0+}^{\alpha} I_{0+}^{\alpha} f = f,$$

for every $f \in L^1[a, b]$.

We note that the Riemann-Liouville fractional derivative is well defined, due to the injectivity of the fractional integral (recall Proposition 2.16). Moreover, it will be a surjective operator from $I_{0+}^{\alpha} L^1[0, b]$ to $L^1[0, b]$.

Remark 2.21. There is an equivalent explicit version of Definition 2.20, that is defined over $I_{0+}^{\alpha} L^1[0, b]$, since we can see that the left inverse for the fractional integral is

$$D_{0+}^{\alpha} = D^{\lceil \alpha \rceil} I_{0+}^{\lceil \alpha \rceil - \alpha}.$$

However, it is clear that we need to complete the definition if we pretend that D_{0+}^{α} matches perfectly the usual derivative when α is an integer. In particular, note that a constant function can be differentiated an arbitrary number of times, giving always zero. But constant functions do not lie in $I_{0+}^{n} L^1[0, b]$ for $n \in \mathbb{Z}^+$ and hence, under the previous definition for D_{0+}^{α}, its derivative is not defined for $\alpha \geq 1$.

The solution to this problem is to consider a suitable collection of spaces, depending on α, to correctly perform fractional differentiation. The key idea to develop a successful construction is to recall the role of absolutely continuous functions, described in Theorem 1.12, and their expression.

Following [69], for $n \geq 1$, we will denote by $AC^n[0, b]$ the functions $f \in L^1[0, b]$ which are $n - 1$ times differentiable and $D^{n-1} f$ is absolutely continuous. So, essentially, we are imposing that our function has at least n derivatives, but the last one might only be found in the previous weak sense. In this direction, Proposition 2.12 states that $I_{0+}^{\alpha} f$ lies in $AC^{\lfloor \alpha \rfloor}[0, b]$ for $\alpha \geq 1$.

Lemma 2.22. *We have that*

$$AC^n[0, b] = \left\langle \left\{ 1, \ldots, t^{n-2}, t^{n-1} \right\} \right\rangle \oplus I_{0+}^{n} L^1[0, b].$$

Proof. The result follows immediately, after applying n times the Fundamental Theorem of Calculus 1.12. Moreover, the previous sum is direct since $f \in I_{0+}^{n} L^1[0, b]$ implies $f(0) = f'(0) = f^{n-1)}(0) = 0$ and the unique polynomial of degree at most $n - 1$ satisfying such conditions is the zero one. $\square$

Remark 2.23. Observe how $AC^n[0,b]$ is decomposed in two direct summands: $\ker D^n_{0+} = \left\langle \left\{1,\dots,t^{n-2},t^{n-1}\right\}\right\rangle$ and $\operatorname{Im} I^n_{0+} = I^n_{0+}L^1[0,b]$.

Moreover, we have the following immediate relation between the spaces $AC^n[0,b]$, with respect to the inclusion.

Proposition 2.24. *If $n > m > 0$, we have that*

$$AC^n[0,b] \subset AC^m[0,b].$$

Proof. We note that

$$AC^n[0,b] = \left\langle\{1,t,\dots,t^{m-1}\}\right\rangle \oplus \left\langle\{t^m,\dots,t^{n-1}\}\right\rangle \oplus I^n_{0+}L^1[0,b],$$

and we make the following straightforward claims:

- $\left\langle\{1,t,\dots,t^{m-1}\}\right\rangle \subset AC^m[0,b]$,
- $I^n_{0+}L^1[0,b] \subset I^m_{0+}L^1[0,b] \subset AC^m[0,b]$.

Therefore, we only need to check $\left\langle\{t^m,\dots,t^{n-1}\}\right\rangle \subset AC^m[0,b]$. It suffices to prove that

$$\left\langle\{t^m,\dots,t^{n-1}\}\right\rangle \subset I^m_{0+}L^1[0,b],$$

but this is trivial since

$$\left\langle\{t^m,\dots,t^{n-1}\}\right\rangle = I^m_{0+}\left\langle\{t^0,\dots,t^{n-m-1}\}\right\rangle.$$

□

Although the previous result seems pretty immediate and irrelevant, it hides the key idea for a successful treatment of the fractional case. Now we will reproduce the natural elaboration of the fractional analogue of the spaces $AC^n[0,b]$. For this construction, already presented in [69], it is not true that the space of order α is contained in the space of order β if $\alpha > \beta$. Indeed, this particular behaviour will imply the existence of functions that can be differentiated α times, but not β times, which is surprising since we are assuming that $\alpha > \beta$. For instance, it will be shown that a constant function is differentiable α times if and only if $\alpha \in (0,1) \cup \mathbb{N}$.

Definition 2.25. For each $\alpha \in \mathbb{R}^+$, we construct the following space

$$\mathcal{X}_\alpha = \left(I^{\lceil\alpha\rceil-\alpha}_{0+}\right)^{-1}\left(AC^{\lceil\alpha\rceil}[0,b]\right),$$

which will be called the space of functions with summable fractional derivative of order α.

Remark 2.26. Therefore, functions in $\mathcal{X}_\alpha$ are the ones that produce a function in $AC^{\lceil\alpha\rceil}[0,b]$ after being integrated $\lceil\alpha\rceil-\alpha$ times. Each of the functions obtained after this "integration process" can be differentiated $\lceil\alpha\rceil$ times in the weak sense of Fundamental Theorem of Calculus 1.12. Therefore, $\mathcal{X}_\alpha$ is the most general space where we can define $D^\alpha_{0^+} := D^{\lceil\alpha\rceil} I^{\lceil\alpha\rceil-\alpha}_{0^+}$.

Of course, it is possible to describe the exact structure of the space $\mathcal{X}_\alpha$.

Lemma 2.27. *We have that*

$$\mathcal{X}_\alpha = \left\langle \left\{ t^{\alpha-\lceil\alpha\rceil}, \dots, t^{\alpha-2}, t^{\alpha-1} \right\} \right\rangle \oplus I^\alpha_{0^+} L^1[0,b].$$

Proof. If there exists a function f in both summands, then $I^{\lceil\alpha\rceil-\alpha}_{0^+} f$ will be simultaneously a polynomial of degree at most $\lceil\alpha\rceil - 1$, recall Example 2.17, and a function in $I^{\lceil\alpha\rceil}_{0^+} L^1[0,b]$. Therefore $I^{\lceil\alpha\rceil-\alpha}_{0^+} f$ has to be the zero function due to Lemma 2.22 and, since fractional integrals are injective (Proposition 2.16), $f \equiv 0$.

It is clear that, applying $I^{\lceil\alpha\rceil-\alpha}_{0^+}$ to the right hand side, we will produce a function in $AC^{\lceil\alpha\rceil}[0,b]$. Moreover, it is trivial that any function in $AC^{\lceil\alpha\rceil}[0,b]$ can be obtained in this way, in virtue of Example 2.17. Since the operator $I^{\lceil\alpha\rceil-\alpha}_{0^+}$ is injective, the result follows. □

From the previous lemma, we get this immediate corollary.

Corollary 2.28. *Given $f \in L^1[0,b]$, we have that $f \in I^\alpha_{0^+} L^1[0,b]$ if, and only if, $f \in \mathcal{X}_\alpha$ and also $D^s I^{\lceil\alpha\rceil-\alpha}_{0^+} f(0) = 0$ for each $s \in \{0, \dots, \lceil\alpha\rceil - 1\}$.*

Hence, we can upgrade Definition 2.20 in the following way, coinciding with Definition 2.4 in [69].

Definition 2.29. Consider $\alpha \geq 0$ and $f \in \mathcal{X}_\alpha$. We define the Riemann-Liouville fractional derivative of order α and base point 0 as

$$D^\alpha_{0^+} f := D^{\lceil\alpha\rceil}_{0^+} \circ I^{\lceil\alpha\rceil-\alpha}_{0^+} f,$$

where the last derivative may be understood in the weak sense exposed previously.

Remark 2.30. Observe that, if $f \in L^1[0,b]$, then $D^\alpha_{0^+} I^\beta_{0^+} f(0) = 0$ provided that $\beta \geq \alpha + 1$. This happens because it is a well known fact for the case $\alpha = 0$ and $\beta = 1$, but the previous expression equals $I^1_{0^+} \left(I^{\beta-\alpha-1}_{0^+} f \right)(0)$.

We attempt to fully understand how D_{0+}^{α} works over $\mathcal{X}_\alpha$ and the most natural way is to split the problem into two parts, as suggested by Lemma 2.27. We already know that D_{0+}^{α} is the left inverse for I_{0+}^{α}, so we should study how does it behave when applied to $\left\langle\left\{t^{\alpha-\lceil\alpha\rceil},\dots,t^{\alpha-2},t^{\alpha-1}\right\}\right\rangle$. It is a well known and straightforward computation that

$$D_{0+}^{\alpha}\left(\left\langle\left\{t^{\alpha-\lceil\alpha\rceil},\dots,t^{\alpha-2},t^{\alpha-1}\right\}\right\rangle\right)=\{0\},$$

and, hence, the kernel of D_{0+}^{α} has dimension $\lceil\alpha\rceil$ and is given by

$$\ker D_{0+}^{\alpha}=\left\langle\left\{t^{\alpha-\lceil\alpha\rceil},\dots,t^{\alpha-2},t^{\alpha-1}\right\}\right\rangle. \tag{2.9}$$

Moreover, we note that if $f(t)=a_0\,t^{\alpha-\lceil\alpha\rceil}+\cdots+a_{\lceil\alpha\rceil-1}\,t^{\alpha-1}$, with $a_j\in\mathbb{R}$ for each $j\in\{0,1,\dots,\lceil\alpha\rceil-1\}$, it is immediate to do the following calculations from Example 2.17, where $j\in\{1,\dots,\lceil\alpha\rceil-1\}$,

$$\begin{aligned}\left(I_{0+}^{\lceil\alpha\rceil-\alpha}f\right)(0)&=a_0\,\Gamma(\alpha-\lceil\alpha\rceil+1),\\ \left(D_{0+}^{\alpha-\lceil\alpha\rceil+j}f\right)(0)&=a_j\,\Gamma(\alpha-\lceil\alpha\rceil+j+1).\end{aligned} \tag{2.10}$$

The previous formula generalizes the expression of the Taylor coefficients for the fractional case, and it can be used to codify functions in $\mathcal{X}_\alpha$ modulo $I_{0+}^{\alpha}L^1[0,b]$, since

$$\begin{aligned}\left(I_{0+}^{\lceil\alpha\rceil-\alpha}g\right)(0)&=0,\\ \left(D_{0+}^{\alpha-\lceil\alpha\rceil+j}g\right)(0)&=0,\end{aligned} \tag{2.11}$$

for $g\in I_{0+}^{\alpha}L^1[0,b]$, due to Remark 2.30.

In general, fractional differentiation presents some extra problems that do not exist when dealing with fractional integrals. One of the most famous obstacles is that there is no Index Law for fractional differentiation. In our opinion, the main reason underlying all these difficulties is the following one, although this point of view, to the best of our knowledge, has not been extensively discussed in the literature.

Remark 2.31. The condition $\alpha>\beta$ does not ensure $\mathcal{X}_\alpha\subset\mathcal{X}_\beta$, unless $\alpha-\beta\in\mathbb{N}$. This makes Riemann-Liouville derivatives somehow tricky, since the differentiability for a higher order does not imply, necessarily, the differentiability for a lower order. In particular, this fact has critical implications when considering fractional differential equations, as we shall see in the manuscript, since the function has to be differentiable for each order involved in the equation. These problems give an idea of why can be a logical thought to work with fractional integrals instead, and try to inherit the results obtained for the case of fractional derivatives, as an alternative approach to prove them for fractional derivatives directly.

Consequently, it is interesting to describe the exact structure of a finite intersection of such spaces of different orders. Although the proof is not very complicated, we were not able to find this result in the classical literature. However, it is true that the main relevance of this result shows up after studying the basic theory of fractional integral equations and trying to translate it to the field of fractional differential equations. We emphasize the importance of this first result in the problem of attaching initial values to a fractional differential equation, in such a way that the existence and uniqueness of solution is ensured, at least for the "easy cases".

Theorem 2.32. *Consider $\beta_n > \ldots > \beta_1 \geq 0$, we have that*

$$\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j} = \left\langle \left\{ t^{\beta_n - \lceil \beta_n - \beta_* \rceil}, \ldots, t^{\beta_n - 1} \right\} \right\rangle \oplus I_{0^+}^{\beta_n} L^1[0,b],$$

where β_* is the maximum β_j such that $\beta_n - \beta_j \notin \mathbb{Z}^+$. If such a β_j does not exist, we define $\beta_* = 0$.

In particular, $I_{0^+}^{\beta_n} L^1[0,b] \subset \bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$ and it has codimension $\lceil \beta_n - \beta_* \rceil$.

Proof. It is obvious that $\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j} \subset \mathcal{X}_{\beta_n}$. Hence,

$$\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j} \subset \left\langle \left\{ t^{\beta_n - \lceil \beta_n \rceil}, \ldots, t^{\beta_n - 1} \right\} \right\rangle \oplus I_{0^+}^{\beta_n} L^1[0,b]. \tag{2.12}$$

It is clear that $I_{0^+}^{\beta_n} L^1[0,b]$ lies in $\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$, so the remaining question is to see when a linear combination of the $t^{\beta_n - k}$, where $k \in \{1, \ldots, \lceil \beta_n \rceil\}$, lies in $\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$.

The key point is to realize that, for any finite set $\mathcal{F} \subset (-1, +\infty)$,

$$\sum_{\gamma \in \mathcal{F}} c_\gamma \, t^\gamma \in \mathcal{X}_{\beta_j}, \text{ where } c_\gamma \neq 0 \text{ for each } \gamma \in \mathcal{F},$$

if, and only if, $\gamma - \beta_j > -1$ (implying $t^\gamma \in I_{0^+}^{\beta_j} L^1[0,b]$) or $\gamma - \beta_j \in \mathbb{Z}^-$ (implying $t^\gamma \in \left\{ t^{\beta_j - \lceil \beta_j \rceil}, \ldots, t^{\beta_j - 1} \right\}$) for every $\gamma \in \mathcal{F}$. Consequently, it is enough to study when $t^{\beta_n - k}$ lies in each $\mathcal{X}_{\beta_j}$, for $k \in \{1, \ldots, \lceil \beta_n \rceil\}$, and there are two options:

- If $\beta_n - \beta_j \in \mathbb{Z}^+$, we know that $t^{\beta_n - k} \in \mathcal{X}_{\beta_j}$ always. This is a mere corollary of Lemma 2.27, since:

  If $\beta_n - k \leq \beta_j - 1$, it is an immediate claim, since $t^{\beta_n - k} \in \ker D_{0^+}^{\beta_j}$.

  If $\beta_n - k > \beta_j - 1$, it is also direct, since $t^{\beta_n - k} \in I_{0^+}^{\beta_j} L^1[0,b]$, due to Example 2.17. Thus, $\mathcal{X}_{\beta_n} \subset \mathcal{X}_{\beta_j}$ if $\beta_n - \beta_j \in \mathbb{Z}^+$, and such orders β_j are not relevant for computing the intersection.

- In case $\beta_n - \beta_j \notin \mathbb{Z}^+$, we see that $t^{\beta_n - k} \notin \left\{t^{\beta_j - \lceil \beta_j \rceil}, \dots, t^{\beta_j - 1}\right\}$. Thus, $t^{\beta_n - k} \in \mathcal{X}_{\beta_j}$ implies that $t^{\beta_n - k} \in I_{0^+}^{\beta_j} L^1[0,b]$ and, hence, it is necessary and sufficient to impose the condition $\beta_n - k > \beta_j - 1$, which can be rewritten as

$$k < \beta_n - \beta_j + 1. \tag{2.13}$$

  We want to find the possible values for $k \in \{1, \dots, \lceil \beta_n \rceil\}$ ensuring that (2.13) holds for every j such that $\beta_n - \beta_j \notin \mathbb{Z}$. This condition is equivalent to $k < \beta_n - \beta_* + 1$, where β_* is the greatest β_j such that $\beta_n - \beta_j \notin \mathbb{Z}$. Indeed, the previous condition for k can be rewritten as $1 \leq k \leq \lceil \beta_n - \beta_* \rceil$.

Therefore, the coefficients c_γ which are not necessarily null are the ones associated to $t^\gamma = t^{\beta_n - k}$, where $k \in \{1, \dots, \lceil \beta_n - \beta_* \rceil\}$. $\square$

Remark 2.33. Due to Theorem 2.32, in any affine subspace of $\bigcap_{j=1}^n \mathcal{X}_{\beta_j}$ with dimension strictly higher than $\lceil \beta_n - \beta_* \rceil$, there are infinitely many pairs of distinct functions such that their difference lies in $I_{0^+}^{\beta_n} L^1[0,b]$.

Finally, we provide a result that combines the FC notions with the LaT. This is specially useful when dealing with some fractional differential equations, as we shall see later.

Proposition 2.34. *Consider a function u and a real number $\alpha \geq 0$ such that $u^{\lceil \alpha \rceil - 1)}$ is an absolutely continuous function on $(0, +\infty)$. The hypothesis $u^{\lceil \alpha \rceil)} \in D_{\mathcal{L}}$, together with the conditions $u(0^+) = 0, \dots, u^{\lceil \alpha \rceil - 1)}(0^+) = 0$, ensures that*

$$\mathcal{L}[D_{0^+}^{\alpha} u](s) = s^\alpha \cdot U(s) \in D_{\mathcal{L}}.$$

This statement arises as a very natural generalization of Proposition 1.34. The result corresponds with Equation (2.248) in [59]. We summarize the main idea with an example. Suppose that we are dealing with a differential equation and, in the transformed domain, we find an addend like $\sqrt{s} \cdot F(s)$.

Then, provided that some technical conditions hold ($\mathcal{L}^{-1}[F]$ is absolutely continuous, $\mathcal{L}^{-1}[F]' \in D_{\mathcal{L}}$ and $\mathcal{L}^{-1}[F](0^+) = 0$), when we return to the time domain, we will obtain $D_{0^+}^{\frac{1}{2}} f(t)$.

2.2.2 The Caputo fractional derivative

In the literature [59], it is common to find another possibilities for defining the fractional derivatives. Although we will focus on the Riemann-Liouville definition, specially when studying fractional differential equations, it is relevant to take into account that there are several other definitions with different properties. Moreover, some of them are equivalent over certain spaces of functions. In this sense, we introduce the Caputo derivative, which appears in many applications of FC, see [59].

At first, we construct the following family of spaces.

Definition 2.35. For each $\alpha \in \mathbb{R}^+$, we define

$$\mathcal{Y}_\alpha = AC^{\lceil \alpha \rceil}[0, b],$$

which will be called the space of functions with Caputo summable fractional derivative of order α.

Definition 2.36. For a function $f \in \mathcal{Y}_\alpha$, the Caputo derivative of order $\alpha \in \mathbb{R}^+$ is given by

$$D_{0^+}^{C,\alpha} f(t) := I_{0^+}^{\lceil \alpha \rceil - \alpha} D^{\lceil \alpha \rceil} f(t),$$

where the last derivative may be understood in the weak sense exposed previously.

Remark 2.37. It is obvious that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$, provided that $\alpha > \beta$.

Of course, it is straightforward to derive these results, which are analogous to the Riemann-Liouville case.

Lemma 2.38. *We have that*

$$\mathcal{Y}_\alpha = \left\langle \left\{ 1, \ldots, t^{\lceil \alpha \rceil - 2}, t^{\lceil \alpha \rceil - 1} \right\} \right\rangle \oplus I_{0^+}^{\lceil \alpha \rceil} L^1[0, b].$$

Corollary 2.39. *Given $f \in L^1[0, b]$, we have that $f \in I_{0^+}^{\alpha} L^1[0, b]$ if, and only if, $f \in \mathcal{Y}_\alpha$ and also $D^s f(0) = 0$ for each $s \in \{0, \ldots, \lceil \alpha \rceil - 1\}$.*

Remark 2.40. We observe that $\ker D_{0^+}^{C,\alpha} = \left\langle \left\{ 1, \ldots, t^{\lceil \alpha \rceil - 2}, t^{\lceil \alpha \rceil - 1} \right\} \right\rangle$.

Remark 2.41. If $\alpha \in \mathbb{Z}^+$, it is clear that $\mathcal{X}_\alpha = \mathcal{Y}_\alpha$ and $D^\alpha_{0^+} = D^{C,\alpha}_{0^+}$.

In other case, we should compute explicitly $\mathcal{X}_\alpha \cap \mathcal{Y}_\alpha$, before making any comparison.

Theorem 2.42. *For $\alpha \notin \mathbb{N}$, we have $\mathcal{X}_\alpha \cap \mathcal{Y}_\alpha = \langle \{t^{\lceil\alpha\rceil-1}\}\rangle \oplus I^{\lceil\alpha\rceil}_{0^+} L^1[0,b]$.*

Proof. It is obvious that $\mathcal{X}_\alpha \cap \mathcal{Y}_\alpha \subset \mathcal{Y}_\alpha$, and it is also straightforward to check that $I^{\lceil\alpha\rceil}_{0^+} L^1[0,b] \subset \mathcal{Y}_\alpha \cap \mathcal{X}_\alpha$. Thus, due to Lemma 2.38, the remaining question is what is the largest subspace in

$$\left\langle \left\{1, \dots, t^{\lceil\alpha\rceil-2}, t^{\lceil\alpha\rceil-1}\right\}\right\rangle$$

that is contained in $\mathcal{X}_\alpha$ or, equivalently, in $I^\alpha_{0^+} L^1[0,b]$.

To conclude, we just claim that the greater subspace of

$$\left\langle \left\{1, \dots, t^{\lceil\alpha\rceil-2}, t^{\lceil\alpha\rceil-1}\right\}\right\rangle$$

contained in $I^\alpha_{0^+} L^1[0,b]$ is $\langle \{t^{\lceil\alpha\rceil-1}\}\rangle$, since $t^\gamma \in I^\alpha_{0^+} L^1[0,b]$ implies $\gamma > \alpha - 1$, recall Example 2.17. $\square$

Remark 2.43. When $\alpha \notin \mathbb{Z}^+$, the intersection of the domains of the operators $D^\alpha_{0^+}$ and $D^{C,\alpha}_{0^+}$ is $\mathcal{X}_\alpha \cap \mathcal{Y}_\alpha$. Both operators are equal in the subspace of codimension one $I^{\lceil\alpha\rceil}_{0^+} L^1[0,b]$ and they differ on $\langle \{t^{\lceil\alpha\rceil-1}\}\rangle$:

$$D^\alpha_{0^+} t^{\lceil\alpha\rceil-1} = \frac{\Gamma(\lceil\alpha\rceil)}{\Gamma(\lceil\alpha\rceil-\alpha)} t^{\lceil\alpha\rceil-\alpha-1} \neq 0 = D^{C,\alpha}_{0^+} t^{\lceil\alpha\rceil-1}.$$

In particular, if $f \in X_\alpha \cap Y_\alpha$, the Riemann-Liouville and the Caputo derivative of order α coincide if, and only if, $D^{\lceil\alpha\rceil-1}_{0^+} f(0) = 0$.

2.2.3 The lack of an Index Law for fractional derivatives

Probably the first reference to FC, at least in an informal way, is the famous dialogue between L'Hôpital and Leibniz. The most natural answer that anybody would have expected to the question *"What happens if the order of differentiation is $n = \frac{1}{2}$?"* would be something like *"It has to be something that, when it is applied twice, it gives the usual derivative"*.

The previous answer invites us to open the following discussion. Let's think about derivatives as operators that, when applied to a function, they give another function. We would expect to have the property that, after composing two times the $\frac{1}{2}$ derivative $D^{\frac{1}{2}}$, we get $D^{\frac{1}{2}} \circ D^{\frac{1}{2}} = D$. In general, when we compose n times the derivative of order $\frac{1}{n}$, which will be called $D^{\frac{1}{n}}$, we would like to get the usual derivative D. An interesting discussion about what should be called "fractional derivative" is performed in [57].

We have already highlighted the role of the Index Law (Proposition 2.11) in fractional integration. In this sense, we have reproduced some results that show how the Index Law, together with some additional properties, gives an axiomatic characterization for the Riemann-Liouville fractional integral, recall Theorem 2.18. However, in the case of fractional derivatives, the situation is not so satisfactory.

We know that the usual definitions for fractional derivatives (Caputo, Riemann-Liouville,...), which are presented in the classical monographs [40, 55, 59, 69] do not satisfy the Index Law. In this section we will show, roughly speaking, that there are no fractional derivatives such that the Index Law holds. The content of this section is based on the results exposed in [18].

More specifically, we will prove that the following identity does not hold

$$H^n = D, \tag{2.14}$$

for any linear operator H, where the left hand side is obtained after composing H with itself $n > 1$ times. To provide some examples, before facing the general result, we compute explicitly the left hand side for some particular choices of fractional derivatives H. More specifically, we will compute the kernel of H^n in the case of the Riemann-Liouville fractional derivative and also for the Caputo derivative.

The case of the Riemann-Liouville derivative From Equation (2.9), we deduce the identity $\dim(\ker D^{\alpha}_{0+}) = \lceil \alpha \rceil$. In fact, from the previously mentioned proposition, it is not very difficult to check inductively from Example 2.17 that, for

$$D^{\alpha_n}_{0^+} \circ \cdots \circ D^{\alpha_1}_{0^+},$$

the kernel is conformed by any linear combination of the functions $f(t) = t^{\gamma}$, where

$$\gamma \in \bigcup_{j=1}^{n} \left\{ \sum_{i=1}^{j} \alpha_i - \lceil \alpha_j \rceil + k_j : k_j \in \{0, 1, \ldots, \lceil \alpha_j \rceil - 1\} \right\}.$$

The dimension of the kernel is exactly $\lceil \alpha_1 \rceil + \cdots + \lceil \alpha_n \rceil$. In particular, if all the $\alpha_i = \frac{1}{n}$, we get that $\gamma \in \{0, -\frac{1}{n}, -\frac{2}{n}, \ldots, -\frac{n-1}{n}\}$. Of course, for those values of γ (except for $\gamma = 0$), we have that $D\, t^{\gamma} \neq 0$. Hence, (2.14) is not possible for the Riemann-Liouville fractional derivative.

The case of the Caputo derivative From Remark 2.40, we see that $\dim\left(\ker D_{0^+}^{C,\alpha}\right) = \lceil\alpha\rceil$, and we know the explicit description of its kernel. From this information, it is straightforward to check inductively that the kernel of

$$D_{0^+}^{C,\alpha_n} \circ \cdots \circ D_{0^+}^{C,\alpha_1}$$

is conformed by any linear combination of the functions $f(t) = t^\gamma$, where

$$\gamma \in \bigcup_{j=1}^{n} \left\{ \sum_{i=1}^{j-1} \alpha_i + k_j : k_j \in \{0, 1, \ldots, \lceil\alpha_j\rceil - 1\} \right\}.$$

The dimension of the kernel is exactly $\lceil\alpha_1\rceil + \cdots + \lceil\alpha_n\rceil$. In particular, if all the $\alpha_i = \frac{1}{n}$, we get that $\gamma \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$. Of course, for those values of γ (except for $\gamma = 0$), we have that $D\, t^\gamma \neq 0$. Hence, (2.14) is not possible for the Caputo fractional derivative.

We have already checked the well known fact that the identity (2.14), where the left composition has n factors, does not hold for two of the most relevant fractional derivatives. In fact, in both cases, the left hand side has a kernel of dimension n that strictly contains the kernel of D, with dimension 1. This remark concerning the dimensions of the kernels is the key fact to show that, in a very general case, the previous identity (2.14) can not hold independently of the chosen fractional derivative.

More specifically, we will prove that (2.14) is impossible over any functional space S that contains the constant functions, where the existence of primitives is ensured and where the operators involved are well defined. For instance, one can think that S is some $C^k(\mathbb{R})$, with $k \in \mathbb{Z}^+ \cup \{\infty\} \cup \{\omega\}$, where $\mathcal{C}^\omega(\mathbb{R})$ denotes the analytical functions over $\mathbb{R}$. The only assumption is that the fractional derivative H is linear. Now, we state and prove the main result.

Theorem 2.44. *Suppose that S is a vector space inside the set of real-valued functions of real argument such that:*

i) For every element $f \in S$, there exists a differentiable function $F \in S$ such that $F' = f$.

ii) If c is a constant function, then $c \in S$.

Then, given $n \in \mathbb{Z}^+$, there is no linear map $H : S \longrightarrow S$ such that $H^n = D$ holds, where H^n denotes the composition of H with itself $n > 1$ times and D represents the classical derivative.

Proof. It is well known that the functions that lie in the kernel of D are the constant functions. By hypothesis *ii)*, these functions belong to the vector space S, and hence

$$\ker D = \{f \in S : f \text{ is constant}\}.$$

Now, we will assume that there is some H, which is a linear map from S to itself, such that $H^n = D$ holds, and we will achieve a contradiction for $n > 1$. The contradiction will follow by finding an element $f \in S$ such that $f \in \ker H^n$, but $f \notin \ker D$.

At first, we observe that $\ker H \neq 0$, since $\ker H = 0$ would imply that $\ker D = 0$, which is not true. Furthermore, it is evident from $H^n = D$ that $\ker H \subset \ker D$. Since $\dim(\ker D) = 1$, we have that $\ker H = \ker D$.

Moreover, H has to be surjective because of the combined fact that D is surjective, due to hypothesis *i)*, together with the identity $H^n = D$. Consequently, the operator H^{n-1}, which consists in $n-1$ compositions of H, is surjective.

This means that, if 1 denotes the constant function with value 1, there exists an $f \in S$ with $H^{n-1}(f) = 1$. Observe that $f \notin \ker H = \ker D$, since in other case we would have $H^{n-1}(f) = 0$. However, remembering that $1 \in \ker H = \ker D$, we have that

$$0 \neq D(f) = H(H^{n-1}(f)) = H(1) = 0,$$

which is clearly a contradiction. □

Remark 2.45. The role of D can be changed for any linear operator on S whose kernel is, exactly, the set of real constant functions.

Remark 2.46. The hypothesis involving the existence of primitives in S is rather general. In fact, this property holds for many common functional spaces like $C^k(\mathbb{R})$, with $k \in \mathbb{Z}^+ \cup \{\infty\} \cup \{\omega\}$, where the derivative of its elements is well defined.

Remark 2.47. The hypotheses involving the existence of constant functions in S is necessary in the following sense. If we consider a functional space S without constant functions and their primitives, $\langle\{1, t, t^2, \dots\}\rangle \cap S = \{0\}$, we can find a definition for fractional derivative such that the Index Law holds. Consider, for instance, the space

$$I_{0^+}^{\infty} L^1[0,b] := \bigcap_{\alpha \in \mathbb{R}^+} I_{0^+}^{\alpha} L^1[0,b].$$

In that space, the Riemann-Liouville fractional derivative, which is the left inverse of the corresponding fractional integral is, indeed, a right inverse too.

Chapter 3

Riemann-Liouville fractional integral equations

In the previous chapter, we have proved that there are no fractional derivative such that the Index Law holds, recall Theorem 2.44. Therefore, instead of studying fractional differential equations directly, we focus on fractional integral equations. In a few words, we provide a new method for solving linear fractional integral equations with constant coefficients and rational orders. In particular, this algorithm allows us to link the behaviour of fractional order integral equations to integer order integral equations. In the next chapter, some of these results will be applied to the case of fractional differential equations for the Riemann-Liouville derivative. The interest of this kind of equations will be motivated in the chapter involving applications.

3.1 Solutions for linear fractional integral equations with constant coefficients

During the last 30 years, several methods have been proposed for solving fractional differential equations. We highlight, because of their relevance and completeness, the references [40, 55]. In many references, this kind of equations are studied directly, trying to get a reasonable abstraction of the usual theory that fits well with our intuition for the fractional case. As we shall see in the next chapter, the literature is sometimes a bit vague in this sense, since there are details that are not clear as, for instance, the space where we should seek the solutions to fractional differential equations. Thus, from our point of view, it is better to study Riemann-Liouville fractional integral equations first, to later derive some consequences for fractional differential equations. In our opinion, there are several reasons that support this philosophy:

- It makes more sense to study fractional integral equations, since the Riemann-Liouville integral operator can be axiomatically defined in reasonable terms, Theorem 2.18. Indeed, it seems the most reasonable interpolation for classical integration. However, it is not so clear what definition for derivative should we follow if we were interested in studying directly fractional differential equations.

- Dealing directly with fractional differential equations is excessively complicated, specially if one wants to study the existence and uniqueness of solution while being careful about the space where solutions are defined. In this sense, we recall Remark 2.31 stating that being differentiable for a higher order does not imply the differentiability for lower orders. As we shall see, this difficulty causes a fractional differential equation to have less solutions than expected.

- Therefore, results involving fractional differential equations are stated in a more complicated way, sometimes with "strange" hypotheses. However, these conditions appear naturally when we consider the associated fractional integral problem and try to derive its differential version.

We shall see the specific meaning of the previously exposed philosophy during the development of this chapter. Thus, in this section, we will provide an original method described in [15] to solve some fractional integral equations, more precisely, the ones which are linear, have constant coefficients and such that all the involved integration orders are rational. The method essentially turns a fractional integral equation into an ordinary integral equation by applying a suitable fractional integral operator. Furthermore, it is shown that our construction is "optimal" in the sense that the ordinary integral equation that we obtain has the least possible order. Later, some technical upgrades will be discussed and we will use the results to face the case of fractional differential equations in the next chapter.

3.1.1 The main idea

We will illustrate the main idea underlying our method in a simple case. Due to its simplicity and its classical interest, Abel equation is one of the most extensively studied problems involving FC. There exists a clear physical interpretation of it, and it can be introduced in many diverse contexts.

This interpretation is quite clear in [20] and the solution to the equation is perfectly explained in [69], but for the current purpose it is enough to consider the famous and classical Abel integral equation as the fractional integral equation

$$I_{0+}^{\frac{1}{2}} v(t) = w(t), \tag{3.1}$$

where $w \in L^1[0,b]$, and we seek for a solution $v \in L^1[0,b]$. Obviously, the condition $w \in I_{0+}^{\frac{1}{2}} L^1[0,b]$ is necessary and sufficient for the existence of solution, so we will assume it. Moreover, the solution will be unique since $I_{0+}^{\frac{1}{2}}$ is injective, recall Proposition 2.16.

The classical argument to obtain the solution is to apply $D_{0+}^{\frac{1}{2}}$ to both sides of the equation. Since the Riemann-Liouville fractional derivative is the left inverse to the Riemann-Liouville fractional integral, there is no much more to say here because

$$v(t) = D_{0+}^{\frac{1}{2}} w(t).$$

However, we are going to show two additional methods to solve (3.1). The principal advantage of these methods is that they can be applied to more general equations, where it is not trivial how to find a left inverse for the involved integral operator. In this case, both of them exploit the identity $I_{0+}^{\frac{1}{2}} \circ I_{0+}^{\frac{1}{2}} = I_{0+}^{1}$.

The first method just deduces, from (3.1), the equation

$$I_{0+}^{1} v(t) = I_{0+}^{\frac{1}{2}} w(t). \tag{3.2}$$

Differentiation is allowed at both sides of the equation, since the right hand side lies on the space $I_{0+}^{\frac{1}{2}} I_{0+}^{\frac{1}{2}} L^1[0,b] = I_{0+}^{1} L^1[0,b]$. So, the problem is solved just by computing the solution to (3.2), which has natural integration orders because the right term is known, and checking that the solution satisfies (3.1), where a fractional integral was present.

The second method is focused on studying the equation

$$I_{0+}^{1} \widetilde{v}(t) = w(t).$$

Clearly, if it does exist a solution $\widetilde{v}$ to this problem, then the term $I_{0+}^{\frac{1}{2}} \widetilde{v}(t)$ satisfies

$$I_{0+}^{\frac{1}{2}} \left(I_{0+}^{\frac{1}{2}} \widetilde{v}(t) \right) = w(t).$$

So, in that case, $I_{0^+}^{\frac{1}{2}} \widetilde{v}(t)$ is the unique solution to (3.1).

The essential idea in the previous two methods is to construct an integral equation with integer orders from the initial fractional integral equation. Of course, the performance in the proposed example has been terribly simple. We have only used that $I_{0^+}^{\frac{1}{2}}$ acts as a "complementary" operator of $I_{0^+}^{\frac{1}{2}}$, in the sense that their composition is an integral operator of integer order. The question is under what conditions a fractional integral operator admits such a complement. The answer is satisfactory when the fractional operator is a linear combination, with constant coefficients, of fractional integrals of rational order.

3.1.2 Construction of the complementary fractional operator

Now, we face the proof of the previous claim. We consider the following fractional integral equation

$$\left(c_1\, I_{0^+}^{\alpha_1} + \cdots + c_n\, I_{0^+}^{\alpha_n}\right)\, v = w, \qquad 0 \leq \alpha_n < \alpha_{n-1} < \ldots < \alpha_1, \tag{3.3}$$

where $w \in L^1[0,b]$ and $c_i \in \mathbb{R}$, for all $i \in \{1, \ldots, n\}$, are known with $c_1 \neq 0$. The previous equation will be called linear fractional order equation with constant coefficients and it is evident that (3.1) and (3.2) are particular cases of (3.3). The key remark is that we can state a non-trivial assertion about equations in the form (3.3), provided that the orders of integration are all rational. Roughly speaking, equations like (3.3) are not more complicated than the similar equations with integrals of natural order. In the analogous case of fractional order differential equations, it is not possible to do this in general, due to the lack of an Index Law, remember Theorem 2.44.

As we have mentioned before, we want to reduce, in a quick and easy way, equations of the type (3.3) to similar ones with natural orders. Since the orders α_i, where $i \in \{1, \ldots, n\}$, are all rational, we can express them with a common denominator. We call this common denominator q, and we just rewrite the expression (3.3) as

$$\left(c_1\, I_{0^+}^{\frac{a_1}{q}} + \cdots + c_n\, I_{0^+}^{\frac{a_n}{q}}\right) v = w. \tag{3.4}$$

As we have already mentioned, we want to find an operator in

$$\langle \mathcal{I} \rangle := \left\langle \left\{ I_{0^+}^{\alpha} \in \mathrm{End}_B\left(L^1[0,b]\right) : \alpha \in \mathbb{Q}^+ \cup \{0\} \right\} \right\rangle$$

that converts the left hand side of (3.4) into an analogous expression, but with natural integration orders. The first question to arise is, obviously, if this procedure is always possible. The second one should be, if it is possible, how can we construct the operator explicitly.

We will deal with both questions from an algebraic and elegant point of view by introducing the definition of a generalized polynomial, which is an analogue to the ordinary case but with fractional degrees. It will be a trivial remark that the existence of the complementary linear operator for any equation of the type (3.4) is equivalent to the statement “for any generalized polynomial p, it does always exist another generalized polynomial $\widehat{p}$ that makes $p \cdot \widehat{p}$ a polynomial”. Finally, the truthfulness of the assertion concerning generalized polynomials will be proved, ensuring that the procedure shown to solve equations of the form (3.4) does always work.

Definition 3.1. A generalized polynomial (in one variable) is an algebraic expression of the type

$$p\,(X) = c_1\,X^{\alpha_1} + \cdots + c_n\,X^{\alpha_n}, \text{ where } \alpha_1 > \ldots > \alpha_n \geq 0,$$

and for every $i \in \{1, \ldots, n\}$ we have $c_i \in \mathbb{C}$ and $\alpha_i \in \mathbb{Q}^+ \cup \{0\}$. Of course, taking q as the least common multiple of the denominators, the previous expression can be rewritten as

$$p\,(X) = c_1\,X^{\frac{a_1}{q}} + \cdots + c_n\,X^{\frac{a_n}{q}}, \text{ where } a_1 > \ldots > a_n \geq 0.$$

The set of generalized polynomials will be denoted by $\mathcal{G}$.

Remark 3.2. By defining the sum and product of generalized polynomials in the usual way, it is immediate to see that $(\mathcal{G}, +, \cdot)$ is a $\mathbb{C}$-algebra isomorphic to $(\langle \mathcal{I} \rangle, +, \circ)$.

The previous isomorphism between $\mathbb{C}$-algebras allows us to reformulate our question in terms of generalized polynomials. Given any generalized polynomial p, does it always exist another generalized polynomial $\widehat{p}$ such that $p \cdot \widehat{p}$ is a polynomial?

Example 3.3. *For instance, given*

$$p\,(X) = c_1\,X^2 + c_2\,X^{\frac{3}{2}} + c_3,$$

the choice

$$\widehat{p}\,(X) = c_1\,X^2 - c_2\,X^{\frac{3}{2}} + c_3$$

ensures that their product

$$(p \cdot \widehat{p})\,(X) = c_1^2\,X^4 - c_2^2\,X^3 + 2\,c_1\,c_3\,X^2 + c_3^2$$

is a polynomial.

The natural question that we have stated before is if this construction is always possible. To solve this question, an algorithm to construct $\widehat{p}$, which depends only on finding some kind of "roots of p", will be given. Later we will improve this result, avoiding the necessity of the computation of those roots.

Our idea lies in the following well known remark.

Remark 3.4. Assume that $q \in \mathbb{Z}^+$ and $a \in \mathbb{C} \setminus \{0\}$, then the following decomposition holds

$$X^q - a^q = \prod_{j=0}^{q-1}(X - a\,\xi^j) = (X-a)(X-a\,\xi)\cdots(X-a\,\xi^{q-1}),$$

where ξ is a primitive q-root of 1. This decomposition follows immediately from the fact that, when $a \neq 0$, both polynomials of degree q have exactly the same n distinct roots and the same principal coefficient. When $a = 0$, the remark holds trivially.

Next, we state and prove our main result.

Theorem 3.5. *For every generalized polynomial $p \in \mathcal{G}$, there exists another generalized polynomial $\widehat{p} \in \mathcal{G}$ such that $p \cdot \widehat{p} \in \mathbb{C}[X]$.*

Proof. We begin with

$$p(X) = c_1\,X^{\frac{a_1}{q}} + \cdots + c_n\,X^{\frac{a_n}{q}} = c_1\left(X^{\frac{1}{q}}\right)^{a_1} + \cdots + c_n\left(X^{\frac{1}{q}}\right)^{a_n},$$

and we can consider the associated polynomial

$$\pi(X) = c_1X^{a_1} + \cdots + c_nX^{a_n},$$

which consists in changing the symbol $X^{\frac{1}{q}}$ by X. Note that the existence of $\widehat{p}$ such that $p \cdot \widehat{p}$ is a polynomial is equivalent to the existence of $\widehat{\pi}$ such that $\pi \cdot \widehat{\pi} \in \mathbb{C}\,[X^q]$. The Fundamental Theorem of Algebra allows a decomposition of the type

$$\pi(X) = c_1\,(X - r_1)\cdots(X - r_l)\,,$$

where we have renamed the degree as $l := a_1$. If we use the notation

$$\pi_i(X) = (X - r_i)\,, \quad i \in \{1, 2, \ldots, l\},$$

it is clear that a possible choice for $\widehat{\pi}$ is $\widehat{\pi} = \widehat{\pi_1} \cdot \widehat{\pi_2} \cdots \widehat{\pi_l}$. So, in fact, it is enough to compute a possibility for every $\widehat{\pi_i}$.

Now, Remark 3.4 shows that, when choosing

$$\widehat{\pi}_i(X) = (X - r_i\,\xi)\left(X - r_i\,\xi^2\right)\cdots\left(X - r_i\,\xi^{q-1}\right),$$

where ξ is a primitive q-root of 1, we have that

$$(\pi_i \cdot \widehat{\pi}_i)(X) = X^q - r_i^q.$$

So, as we claimed before, we have that the choice $\widehat{\pi} = \widehat{\pi_1}\cdots\widehat{\pi}_l$ gives

$$(\pi \cdot \widehat{\pi})(X) = c_1\left(X^q - r_1^q\right)\left(X^q - r_2^q\right)\cdots\left(X^q - r_l^q\right).$$

Finally, if we define $\widehat{p}\,(X) = \widehat{\pi}\left(X^{\frac{1}{q}}\right)$, we have that

$$(p \cdot \widehat{p})(X) = c_1\left(X - r_1^q\right)\left(X - r_2^q\right)\cdots\left(X - r_l^q\right),$$

which is a polynomial in X of degree l. □

Remark 3.6. It is also possible to compute

$$\widehat{p}\,(X) = \prod_{j=1}^{q-1} p(\xi^j X) = p(\xi X)\cdot p(\xi^2 X)\cdots p(\xi^{q-1}X), \tag{3.5}$$

which allows to find $\widehat{p}$ without computing the roots of π. It is straightforward to check that this definition for $\widehat{p}$ is the same than the one given in the proof, up to a possible minus sign.

Corollary 3.7. *If* (3.3) *is rewritten as*

$$T\,v = w, \text{ where } T \in \langle\mathcal{I}\rangle, \tag{3.6}$$

then it is possible to find $\widehat{T} \in \langle\mathcal{I}\rangle$ *that produces*

$$\widehat{T}\circ T = T\circ\widehat{T} \in \langle\{\mathrm{Id}, I_{0+}^1, I_{0+}^2, I_{0+}^3, \dots\}\rangle.$$

From this corollary, two methods that reduce (3.3) to a similar equation with natural orders are developed. The first method gives a set of possible solutions that contains the set of authentic solutions, the second one gives a set of solutions that is contained in the set of all the solutions.

Superset method: We can use Corollary 3.7 to deduce, from (3.6), that

$$\left(\widehat{T}\circ T\right)\,v = \left(T\circ\widehat{T}\right)\,v = \widehat{T}\,w, \text{ where } T, \widehat{T} \in \langle\mathcal{I}\rangle, \tag{3.7}$$

so every solution to (3.6) is a solution to (3.7). If we know how to solve equations with natural orders, we know how to solve (3.7). So, if we check the solutions to (3.7) in (3.6), we find all the solutions to (3.6).

Subset method: This method is a bit more tricky. We deal with the equation

$$\left(\widehat{T} \circ T\right) v = \left(T \circ \widehat{T}\right) v = w, \text{ where } T, \widehat{T} \in \langle \mathcal{I} \rangle. \tag{3.8}$$

If we know how to solve equations with integer orders, we know how to solve (3.8). If u is a solution to (3.8), it is trivial to deduce that $\widehat{T}(u)$ solves (3.6). The fundamental problem of this method is that it does not ensure that every solution is obtained. The advantage is that, in some cases, the right side of the equation can be more treatable in (3.8) than in (3.7). Since (3.6) will be proved to have at most one solution, if we find it with the "subset method" the procedure will be over.

3.2 Uniqueness and existence of solution

We have previously discussed that the general equation

$$T\, v(t) := c_1\, I_{0^+}^{\frac{a_1}{q}} v(t) + \cdots + c_n\, I_{0^+}^{\frac{a_n}{q}} v(t) = w(t),$$

is strongly linked to an analogous equation with integer orders. Hence, it will be nice to have some results for this kind of equations with integer orders. Although some of the results that we are going to develop can be obtained more or less directly, via Theorems 1.29 and 1.31, we will do it from an intuitive point of view without almost any tool. The main reason to choose this path is that, at the end of the day, we want to derive an algorithm that allows to compute the solution of any equation of the type (3.4) in a simple and explicit manner.

3.2.1 Uniqueness results

We will obtain the uniqueness of solution to Equation (3.4), after proving a sequence of suitable lemmas. Observe that the uniqueness of solution is an immediate consequence of Titchmarsh Theorem 1.29 applied to (3.4) for $a_n > 0$, and it can be also easily adapted to cover the case $a_n = 0$ too.

Remark 3.8. Assume $w \in L^1[0,b]$. If the problem

$$c_n\, I_{0^+}^{n} v(t) + \cdots + c_1\, I_{0^+}^{1} v(t) + c_0\, v(t) = w(t), \text{ where } c_n \neq 0, \tag{3.9}$$

has a solution $v \in L^1[0,b]$ and s is the least integer in $\{0,1,\ldots,n\}$ such that $c_s \neq 0$, then $w \in I_{0^+}^{s} L^1[0,b]$. Hence, we can rewrite the problem as

$$I_{0^+}^{s}\left(c_n\, I_{0^+}^{n-s} v(t) + \cdots + c_s\, I_{0^+}^{0} v(t)\right) = I_{0^+}^{s} \widetilde{w}(t), \text{ where } c_n, c_s \neq 0,$$

and $w = I^s_{0+}\widetilde{w}$, with $\widetilde{w} \in L^1[0,b]$. Moreover, since I^s_{0+} is an injective operator, the previous problem is equivalent to

$$c_n\, I^{n-s}_{0+} v(t) + \cdots + c_s\, I^0_{0+} v(t) = \widetilde{w}(t),\ \text{where } c_n,\, c_s \neq 0.$$

Hence, in case the reduction to a problem with $c_0 \neq 0$ is not possible, the original equation has no solution. Thus, when we consider (3.9), we will assume without loss of generality the additional hypothesis $c_0 \neq 0$.

Lemma 3.9. *Consider $w \in AC^m[0,b]$ for some $m \geq 1$. If the problem*

$$c_n\, I^n_{0+} v(t) + \cdots + c_1\, I^1_{0+} v(t) + c_0\, v(t) = w(t),\ \text{where } c_0,\, c_n \neq 0, \tag{3.10}$$

has solutions in $L^1[0,b]$, then the solutions are in $AC^m[0,b]$. Moreover, if $w \in \mathcal{C}^\infty[0,b]$ the solutions are also in $\mathcal{C}^\infty[0,b]$.

Proof. From (3.10), we deduce that

$$c_n\, I^n_{0+} v(t) + \cdots + c_1\, I^1_{0+} v(t) - w(t) = -c_0\, v(t),\ \text{where } c_0,\, c_n \neq 0.$$

Note that if k is the largest positive integer in $\{0,1,\ldots,m-1\}$ such that $v \in AC^k[0,b]$, where we interpret $AC^0[0,b] = L^1[0,b]$, the left hand side lies in $AC^{k+1}[0,b]$, leading to a contradiction.

If $w \in \mathcal{C}^\infty[0,b]$ and $v \notin \mathcal{C}^\infty[0,b]$, we arrive to the same kind of contradiction for any finite value of k, since the left hand side will be more regular than the right hand side. □

Lemma 3.10. *The problem* (3.10) *with $w = 0$ has only the trivial solution.*

Proof. Because of the previous development we already know that v is a solution to

$$c_n\, v(t) + \cdots + c_1\, D^{n-1} v(t) + c_0\, D^n v(t) = 0.$$

Hence, $v(t)$ lies, a priori, in a vector space of dimension n. By substituting $t = 0$ into the equation (3.10), we get the condition $v(0) = 0$. From that information, by differentiating the equation (3.10) one time and substituting $t = 0$, we deduce the condition $v'(0) = 0$. This procedure can be done until differentiating $n-1$ times and the obtained conditions are $v(0) = v'(0) = \cdots = v^{n-1)}(0) = 0$, which in fact show that the unique possible solution is $v = 0$. □

Corollary 3.11. *If $w \in L^1[0,b]$, then the problem* (3.10) *has at most one solution.*

Proof. The proof is trivial, since the existence of two different solutions would imply that their difference is a non trivial solution to (3.10) with $w = 0$, contradicting the previous lemma. □

This result extends in an obvious way to any fractional order integral equation of the type (3.4). To check this claim, note that if (3.4) had two different solutions for a certain source term w, their difference would be a non-trivial solution to the associated homogeneous problem. After applying the complementary fractional operator to both sides, we would deduce the existence of a non-trivial solution for (3.10) with zero source term.

Corollary 3.12. *Any linear rational order integral equation with constant coefficients of the type* (3.4) *has at most one solution in* $L^1[0,b]$.

3.2.2 Existence results

The first point that we have to take into consideration is that, in principle, the existence of solution is not guaranteed.

Example 3.13. *The set of solutions to the equation* $I^1_{0^+}v(t) = 1$*, when* $v \in L^1[0,b]$*, is empty since the equation leads to an obvious incoherence when* $t = 0$*. One problem here is that* $1 \notin I^1_{0^+}L^1[0,b]$*.*

The question is, obviously, if we can give a necessary and sufficient condition to ensure the existence of solution. We have already commented that, in case that the equation admits a solution, we could restrict our study to the case

$$c_n\, I^n_{0^+}v(t) + \cdots + c_1\, I^1_{0^+}v(t) + c_0\, v(t) = w(t), \text{ where } c_0,\, c_n \neq 0. \tag{3.11}$$

In this case, the existence of solution can be derived from Rust Theorem 1.31, even when the source term w is in $L^1[0,b]$. As we have done before, we do not use this to conclude, since we want to provide an explicit method to solve the equation. We will do it for a smooth enough source term, providing the result for $L^1[0,b]$ in a later subsection.

Consider

$$Tv(t) = c_n\, I^n_{0^+}v(t) + \cdots + c_1\, I^1_{0^+}v(t) + c_0\, v(t) = w(t).$$

If the right hand side is smooth enough, we differentiate several times, arriving to the ODE

$$c_n\, v(t) + \cdots + c_1\, D^{n-1}v(t) + c_0\, D^n v(t) = D^n w(t). \tag{3.12}$$

Moreover, we have the following conditions, which are obtained after differentiating k times in (3.11) and substituting $t = 0$, where $k \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} c_0\, v(0) &= w(0), \\ c_0\, v'(0) + c_1\, v(0) &= w'(0), \\ &\vdots \\ c_0\, v^{n-2)}(0) + \cdots + c_{n-2}\, v(0) &= w^{n-2)}(0), \\ c_0\, v^{n-1)}(0) + c_1\, v^{n-2)}(0) + \cdots + c_{n-1}\, v(0) &= w^{n-1)}(0). \end{aligned} \tag{3.13}$$

Observe that the initial conditions are given as the unique solution to a triangular system with no zeros on the diagonal and, hence, they exist and are unique. Thus, if v is a solution to the previous ordinary differential equation (ODE) (3.12), we integrate n times both sides of (3.12) to deduce that $c_n\, I_{0+}^n v(t) + \cdots + c_1\, I_{0+}^1 v(t) + c_0\, v(t) - w(t)$ is a polynomial of degree $n-1$. However, due to the initial conditions (3.13), it follows that the polynomial has to be identically null. Therefore, the solution to (3.12) fulfilling (3.13) is a solution to (3.10).

Corollary 3.14. *If the right hand side in* (3.10) *is smooth enough, then* (3.10) *has exactly one solution.*

Again, we can use this corollary to deduce an analogous result for the fractional case. If we want to ensure the existence of solution to $T\, v = w$, we built the equation $T\, \widehat{T}\, v = w$, which has integer orders of integration. Provided that w is smooth enough, the previous corollary ensures the existence of u such that $T\, \widehat{T}\, u = w$. Finally, we observe that this implies the existence of a solution for the fractional problem, since $T\left(\widehat{T}\, u\right) = w$.

Corollary 3.15. *Any linear rational order integral equation with constant coefficients of the type* (3.4) *with smooth source term and* $a_n = 0$ *has exactly one solution in* $L^1[0, b]$.

3.3 Optimality of the construction

In order to minimize the complexity of the previously shown process, it would be interesting to guarantee that the equation with integer orders has the least possible degree l. If we go back to Theorem 3.5, we can see that our construction seems pretty reasonable and it seems to be the minimal one, in the previous sense. In fact, we will show that it is the minimal construction when there is no pair of distinct roots $r_i, r_j \in \{r_1, ..., r_l\}$ of π and a primitive q-root of unity ξ such that $r_i = \xi\, r_j$. When this happens, it is obvious that there is no need to work separately with r_i and r_j, since π_j is one of the factors on $\widehat{\pi}_i$ and vice versa.

We will show this after providing several lemmas. We will suppose that $\widehat{P}$ is a generalized polynomial such that $p \cdot \widehat{P} \in \mathbb{C}\,[X]$ and we will deduce that in fact $\widehat{P}$ is a polynomial multiple of $\widehat{p}$.

Lemma 3.16. *The common denominator of the exponents in* $\widehat{P}$ *is the same as the one in* p.

Proof. The proof is trivial, since in other case we would immediately have an immediate absurd. If the statement were not true, there would be some addends in $\widehat{P}$ whose exponents are not integer multiples of $\frac{1}{q}$. Among these addends, the one with the biggest degree is chosen, let's call it $c\,X^r$. If we denote

$$p\,(X) = c_1\,X^{\frac{a_1}{q}} + \cdots + c_n\,X^{\frac{a_n}{q}},\ \text{where } a_1 > \ldots > a_n \geq 0,$$

it is clear that, in $(p \cdot \widehat{P})(X)$, there is going to be an addend with non-integer exponent $c\,c_1\,X^{r+\frac{a_1}{q}}$ that can not be cancelled with any other term, contradicting that $p \cdot \widehat{P}$ is a polynomial. $\square$

So, because of the previous lemma we can use, from now on, the notation

$$\widehat{P}\,(X) = d_1\,X^{\frac{a'_1}{q}} + \cdots + d_m\,X^{\frac{a'_m}{q}},$$

where $q > 0$ and $a'_1 > a'_2 > \ldots > a'_m \geq 0$. As we have done before, with the purpose of enlightening the notation, we consider

$$\widehat{\Pi}\,(X) = \widehat{P}\,(X^q) = d_1\,X^{a'_1} + \cdots + d_m\,X^{a'_m}.$$

Definition 3.17. Consider a polynomial $\rho \in \mathbb{C}[X]$ and a complex number $y_0 \in \mathbb{C}$, we denote by $\mathrm{ord}_\rho(y_0)$ the maximum natural number such that $(Y - y_0)^{\mathrm{ord}_\rho(y_0)}$ divides ρ. This number is usually called the "multiplicity of ρ at y_0".

Definition 3.18. Given a polynomial $\rho \in \mathbb{C}[X]$, we denote the set of roots of ρ as

$$R_\rho = \{y_0 \in \mathbb{C} : \rho(y_0) = 0\},$$

and the set of the q-powers of the roots as

$$R_\rho^q = \{y_0^q \in \mathbb{C} : \rho(y_0) = 0\}.$$

Remark 3.19. We note that $\#R_\rho^q \leq \#R_\rho$, since each element in R_ρ^q is obtained as the q-power of an element in R_ρ.

Lemma 3.20. *If r_i is a root of $\pi(X)$ and ξ is a primitive q-root of unity, then, for any choice of $\widehat{\Pi}(X)$, we have that $r_i\,\xi^j$ is a root of $\left(\pi \cdot \widehat{\Pi}\right)(X)$ for every $j \in \{0, \ldots, q-1\}$.*

Furthermore, for any valid choice of $\widehat{\Pi}$ we always have that

$$\prod_{r_i^q \in R_\pi^q} (X^q - r_i^q)^{m(r_i)} \qquad \textit{divides} \qquad \left(\pi \cdot \widehat{\Pi}\right)(X),$$

where $m\,(r_i) = \max\limits_{0 \leq j \leq q-1} \left\{\mathrm{ord}_\pi \left(r_i\,\xi^j\right)\right\}$.

Proof. We have

$$\left(\pi \cdot \widehat{\Pi}\right)(X) = b_1 X^{s \cdot q} + \cdots + b_s X^q + b_{s+1}$$

and, as we have already mentioned before, there is a unique factorization

$$\left(\pi \cdot \widehat{\Pi}\right)(X) = b_1 (X^q - \beta_1) \cdots (X^q - \beta_s).$$

So, if r_i is a root of $\pi(X)$ with multiplicity k, $X - r_i$ divides at least k different factors $X^q - \beta_{i'}$ and $r_i\,\xi^j$ is necessarily a root of multiplicity k of $\left(\pi \cdot \widehat{\Pi}\right)(X)$, for every $j \in \{0, 1, \ldots, q-1\}$. Obviously, changing the role of r_i for each $r_i\,\xi^j$ implies that the minimum possible amount of factors in $\left(\pi \cdot \widehat{\Pi}\right)(X)$ that are divided by $X - r_i$ is $m(r_i)$. In particular, when $X - r_i$ appears in $\pi(X)$ with multiplicity k, the same factor appears in $\widehat{\Pi}\,(X)$ with multiplicity $m(r_i) - k$. $\square$

Example 3.21. *Now, we illustrate how does the algorithm work. For instance, consider the generalized polynomial*

$$p\,(X) = (X^{\frac{1}{4}} + 2)^2 (X^{\frac{1}{4}} - 2)(X^{\frac{1}{4}} + 3).$$

Our optimal proposal is

$$\widehat{p}\,(X) = (X^{\frac{1}{4}} - 2)(X^{\frac{1}{4}} + 2i)^2 (X^{\frac{1}{4}} - 2i)^2 (X^{\frac{1}{4}} - 3)(X^{\frac{1}{4}} + 3i)(X^{\frac{1}{4}} - 3i),$$

that makes

$$(p \cdot \widehat{p})(X) = \prod_{j=1}^{4} \left((X^{\frac{1}{4}} - 2i^j)^2 (X^{\frac{1}{4}} - 3i^j)\right) = (X-16)^2 (X-81).$$

3.4 An example

Example 3.22. *We show, as an example, how to solve the equation*

$$Tx(t) = I_0^1 x(t) + 5 I_0^{3/4} x(t) + 2 I_0^{1/2} x(t) - 20 I_0^{1/4} x(t) - 24 x(t) = e^t.$$

The associated generalized polynomial is

$$p\,(X) = (X^{\frac{1}{4}} + 2)^2 (X^{\frac{1}{4}} - 2)(X^{\frac{1}{4}} + 3).$$

Our optimal proposal is

$$\widehat{p}\,(X) = (X^{\frac{1}{4}} - 2)(X^{\frac{1}{4}} + 2i)^2 (X^{\frac{1}{4}} - 2i)^2 (X^{\frac{1}{4}} - 3)(X^{\frac{1}{4}} + 3i)(X^{\frac{1}{4}} - 3i),$$

which is associated to the integral operator

$$\begin{aligned} \widehat{T} =& I_0^2 - 5\, I_0^{7/4} + 23\, I_0^{3/2} - 85\, I_0^{5/4} + 190\, I_0^1 - 440\, I_0^{3/4} \\ &+ 672\, I_0^{1/2} - 720\, I_0^{1/4} + 864\, \mathrm{Id}. \end{aligned} \tag{3.14}$$

That produces

$$(p \cdot \widehat{p})(X) = (X - 16)^2 (X - 81) = X^3 - 113 X^2 + 2848 X - 20736.$$

The computing method deals with the equation

$$(T \circ \widehat{T}) y(t) = I_{0^+}^3 y(t) - 113\, I_{0^+}^2 y(t) + 2848\, I_{0^+} y(t) - 20736\, y(t) = e^t.$$

We have already shown that the solutions to the previous equation are analytical. So, if we differentiate the equation three times, we can get the equation

$$y(t) - 113\, y'(t) + 2848\, y''(t) - 20736\, y'''(t) = e^t. \tag{3.15}$$

The general solution to (3.15) *can be checked to be of the form*

$$y(t) = c_1\, e^{\frac{t}{81}} + c_2\, e^{\frac{t}{16}} + c_3\, e^{\frac{t}{16}} t - \frac{e^t}{18000},$$

but only one of those functions is indeed a solution to (3.15)*. From* (3.15) *we can derive the following system of equations*

$$\begin{aligned} -20736\, y(0) &= 1, \\ -20736\, y'(0) + 2848\, y(0) &= 1, \\ -20736\, y''(0) + 2848\, y'(0) - 113\, y(0) &= 1, \end{aligned}$$

which has, as unique solution,

$$y(0) = \frac{-1}{20736}, \quad y'(0) = \frac{-737}{13436928}, \quad y''(0) = \frac{-1932835}{34828517376}.$$

Now, we can use this information to compute the values of c_1, c_2 and c_3 from

$$\begin{aligned} y(0) &= c_1 + c_2 - \frac{1}{18000}, \\ y'(0) &= \frac{c_1}{81} + \frac{c_2}{16} + c_3 - \frac{1}{18000}, \\ y''(0) &= \frac{c_1}{81^2} + \frac{c_2}{16^2} + \frac{c_3}{8} - \frac{1}{18000}. \end{aligned}$$

Hence, we obtain

$$c_1 = \frac{1}{27378000}, \quad c_2 = \frac{71}{9734400}, \quad c_3 = \frac{1}{3993600}.$$

Therefore,

$$y(t) = \frac{1}{27378000}\, e^{\frac{t}{81}} + \left(\frac{71}{9734400} + \frac{1}{3993600}\, t\right) e^{\frac{t}{16}} - \frac{1}{18000}\, e^t.$$

Finally, we obtain that

$$x(t) = \widehat{T}\, y(t) = \widehat{T}\left(\frac{1}{27378000}\, e^{\frac{t}{81}} + \left(\frac{71}{9734400} + \frac{1}{3993600}\, t\right) e^{\frac{t}{16}} - \frac{1}{18000}\, e^t\right),$$

where $\widehat{T}$ is given by (3.14)*, so we can see that the solution is a sum of exponential functions and their fractional integrals, which are Mittag-Leffler functions, [55].*

3.5 Several upgrades

The previous approach turns any linear fractional integral equation with constant coefficients and rational orders into a similar one, but with integer orders. If the right hand side is smooth enough we can differentiate at both sides to get a linear ODE with constant coefficients and some initial conditions, that can be solved via an standard process.

In this algorithm, there are two major obstacles that did not allow to obtain a complete result. These are the assumptions over the smoothness of the source term and the assumption about the rationality of the orders. Consequently, we will describe now a modification of the previously presented process for the case where the source term is not smooth enough to differentiate the required amount of times. Furthermore, we will also study the fractional integral equations with non-rational orders by a limit process of fractional integral equations with rational orders.

3.5.1 The non-smooth case

We have explained that we can reduce our problem to an ODE if the right hand side is smooth enough, but, what happens if this is not the case? We will describe an idea that allows to split up the computation into two parts. The first one consists in trivial operations that turn the problem into a smoother one by iteration and they provide some non-smooth remainder. The second part consists in solving the smooth problem previously obtained. The addition of the solution to the smooth problem with the remainder gives the solution to the original problem.

We reconsider the previous equation (3.10)

$$c_n\, I^n_{0+} v_0(t) + \cdots + c_1\, I^1_{0+} v_0(t) + c_0\, v_0(t) = w_0(t), \text{ where } c_0,\, c_n \neq 0, \tag{3.16}$$

and recall that $c_0 \neq 0$ implies no loss of generality in the case of the existence of solution. Furthermore, we can assume that $c_0 = 1$ just by dividing both sides by c_0.

The argument of applying D^n to both sides of the equation (3.16) is valid, for instance, if $w_0 \in \mathcal{C}^n[0,b]$ and we are able to reduce our problem to a linear ODE with constant coefficients. If it is not the case, we show now how to proceed. We will assume the worst possible case, meaning $w_0 \in L^1[0,b]$ but not continuous. We begin with the following equation

$$c_n\, I^n_{0+} v_0(t) + \cdots + c_1\, I^1_{0+} v_0(t) + v_0(t) = w_0(t), \tag{3.17}$$

and we define $v_1(t) = v_0(t) - w_0(t)$. So we have the equation

$$c_n\, I^n_{0+} v_1(t) + \cdots + c_1\, I^1_{0+} v_1(t) + v_1(t) = -c_n\, I^n_{0+} w_0(t) - \cdots - c_1\, I^1_{0+} w_0(t),$$

which can be turned into

$$c_n I^n_{0+} v_1(t) + \cdots + c_1 I^1_{0+} v_1(t) + v_1(t) = w_1(t)$$

just by renaming the right hand side. Note that the new problem looks quite similar to (3.17), but the right hand side has gained one degree of regularity compared with w_0, and, now $w_1 \in AC[0,b]$. If we keep repeating this procedure, we can construct a finite sequence of problems where v_i are the unknowns and the w_i the known source terms of the type

$$c_n I^n_{0+} v_{i+1}(t) + \cdots + c_1 I^1_{0+} v_{i+1}(t) + v_{i+1}(t) = w_{i+1}(t), \tag{3.18}$$

with the inductive relations, for any $i \in \mathbb{N}$,

$$\begin{aligned} v_{i+1}(t) &= v_i(t) - w_i(t), \\ w_{i+1}(t) &= - c_n I^n_{0+} w_i(t) - \cdots - c_1 I^1_{0+} w_i(t), \end{aligned} \tag{3.19}$$

and the property $w_{i+1} \in AC^{i+1}[0,b] \subset \mathcal{C}^i[0,b]$ for each $i \in \mathbb{N}$. When $i = n$, the right hand side, which is $w_{n+1}(t)$, lies in $\mathcal{C}^n[0,b]$. So, when $i = n$, we differentiate in the equation (3.18) to compute $v_{n+1}(t)$ as the unique solution to an ODE with initial conditions, like we have already done previously. Finally, we can use the relations (3.19) to recover $v_n(t)$, $v_{n-1}(t), \ldots, v_1(t)$ and, finally, $v_0(t)$, which is the desired solution.

Thus, putting together Corollary 3.14 with the result described in this subsection, we obtain the following result

Corollary 3.23. *Consider the operator $T \in \mathrm{End}_B(L^1[0,b])$ given by*

$$T := c_1\, I_{0^+}^{\frac{a_1}{q}} + \cdots + c_{n-1}\, I_{0^+}^{\frac{a_{n-1}}{q}} + c_n\,\mathrm{Id}.$$

Then, we have that $T, T^{-1} \in \mathrm{Aut}_B(L^1[0,b])$

Proof. We have proved that $T \in \mathrm{End}_B(L^1[0,b])$ is bijective. This implies that $T \in \mathrm{Aut}_B(L^1[0,b])$. Furthermore, the Bounded Inverse Theorem, Theorem 1.6, ensures that $T^{-1} \in \mathrm{Aut}_B(L^1[0,b])$. □

3.5.2 The non-rational case

There is an interesting mathematical fact that has not been considered in [15], that was the possibility of including irrational orders. Pragmatically, one can think that it is not so important because of the density of $\mathbb{Q}$ in $\mathbb{R}$, if there is some kind of continuous correlation between the orders and the solutions. This idea is essentially correct and we will formalise it this subsection.

We consider the following family of problems, which depend on m,

$$T_m\, v(t) := \left(c_n\, I_{0^+}^{\alpha_{n,m}} + \cdots + c_1\, I_{0^+}^{\alpha_{1,m}} + c_0\, I_{0^+}^{0}\right)\, v(t) = w(t), \tag{3.20}$$

where, for each $m \in \mathbb{N}$, we have that $\alpha_{n,m} > \ldots > \alpha_{1,m} > 0$. Furthermore, assume that the orders of integration are positive rational numbers and that $c_i \neq 0$ for each $i \in \{0, \ldots, n\}$. Under these hypotheses we know that, for each $m \in \mathbb{N}$, because of Corollary 3.23, $T_m \in \mathrm{Aut}_B(L^1[0,b])$ and (3.20) has a unique solution. Furthermore, this solution can be obtained as $T_m^{-1}(w)$.

Obviously, we are not interested in any random choice for the family $(T_m)_{m\in\mathbb{N}}$. We want to use this family to study the problems with non-rational orders, so let's assume that, for each $i \in \{1, \ldots, n\}$, we have that $(\alpha_{i,m})_{m\in\mathbb{N}}$ tends to some $\alpha_i \in \mathbb{R}^+$. So, let's define

$$T = c_n\, I_{0^+}^{\alpha_n} + \cdots + c_1\, I_{0^+}^{\alpha_1} + c_0 I_{0^+}^{0}.$$

In this case, the continuity of the fractional integral operator with respect to the order, see Proposition 2.14, allows us to guarantee that the convergence condition $(\alpha_{i,m})_{m\in\mathbb{N}} \to \alpha_i$ when $m \to \infty$ for every $i \in \{1, \ldots, n-1\}$ implies that $T_m \to T$.

In this case, it is a natural question if the sequence of unique solutions to the rational problems, namely $(y_m)_{m\in\mathbb{N}}$, tends to a solution to the problem with the limit orders α_i. This is true but, at first, we have to establish the existence and uniqueness of solution to the limit problem, since this was only ensured, originally, for rational orders.

Lemma 3.24. *If $T_m \to T$ as $m \to \infty$ and $T_m \in \mathrm{Aut}_B(L^1[0,b])$ for any $m \in \mathbb{N}$, then $T \in \mathrm{Aut}_B(L^1[0,b])$ and $T_m^{-1} \to T^{-1}$ as $m \to \infty$.*

Proof. At first, we shall see that $T \in \mathrm{Aut}_B(L^1[0,b])$ using two results that were described in the section of preliminary results.

i) T **is bijective:** We can not use directly Titchmarsh Theorem (Theorem 1.29) to conclude that T is injective, since T is not a convolution operator. However, we observe that $I^1_{0^+} \circ T$ is a convolution operator whose kernel does not vanish at any interval $[0,\lambda]$ for $\lambda > 0$. Consequently, $I^1_{0^+} \circ T$ is injective, implying that T is injective too.

To ensure that T is surjective, we will use our adapted version of the Rust Theorem (Theorem 1.31). We write $T = \Upsilon + \mathrm{Id}$, making two immediate remarks, ensuring the applicability of this result:

- On the one hand, we know that Υ is an endomorphism in $\mathcal{C}[0,b]$, since fractional integration preserves continuity (Proposition 1.27).
- On the other hand, we know that the integral kernel associated to the operator Υ^m is eventually bounded, since we only need to consider $m \geq \frac{1}{\alpha_1}$, where α_1 is the least integral order in Υ, to ensure that Υ^m is a sum of fractional integral operators of orders greater than one.

Thus, we know that $T(L^1[0,b]) \supset \mathcal{C}[0,b] \supset I^1_{0^+}L^1[0,b]$ and we need to ensure that, in fact, $T(L^1[0,b]) = L^1[0,b]$.

For this purpose, we will use again the idea of "smoothing the source term". Although it is surprisingly simple, it is powerful enough for our purposes. We just observe that the equation $(\Upsilon + \mathrm{Id})\, v = w$ is equivalent to $(\Upsilon + \mathrm{Id})\,(v - w) = -\Upsilon\, w$, in the sense that v satisfies one of the equations if and only if it satisfies the other one. Then, if

$w \in L^1[0,b]$, the solvability of an equation with source term $w \in L^1[0,b]$ is equivalent to the solvability of an equation with source term $-\Upsilon\, w \in I_{0^+}^{\alpha_1} L^1[0,b]$. If we consider $m \geq \frac{1}{\alpha_1}$, we conclude that the equation with source term $w \in L^1[0,b]$ is solvable if and only if the equation with source term $(-1)^m\, \Upsilon^m\, w \in I_{0^+}^{m\cdot\alpha_1} L^1[0,b]$ is solvable. The hypothesis $m \geq \frac{1}{\alpha_1}$ implies that $(-1)^m\, \Upsilon^m\, w \in I_{0^+}^{1} L^1[0,b] \subset \mathcal{C}[0,b]$ and, hence, both equations are solvable. Since $w \in L^1[0,b]$ is arbitrary, we conclude that $T(L^1[0,b]) = L^1[0,b]$.

ii) **T^{-1} is the limit of T_m^{-1}:**

Due to the previous part of the proof, T^{-1} is well defined. Furthermore, we can ensure that $T^{-1} \in \mathrm{Aut}_B(L^1[0,b])$ because of the Bounded Inverse Theorem, Theorem 1.6. Of course, the same conclusion applies to any T_m^{-1} but, furthermore, we will show that the bound for $\|T_m^{-1}\|$ can be chosen to be uniform in m.

As T and T^{-1} are bounded, we can state that there exist $\mu_1, \mu_2 > 0$ such that $\|Tv\| \in (\mu_1, \mu_2)$ for any v with $\|v\| = 1$. The convergence $T_m \to T$ implies that, given $\varepsilon > 0$, it is possible to find $N \in \mathbb{N}$ such that $\|T_m - T\| < \varepsilon$ for $m \geq N$. This clearly implies that $\|T_m v\| \in (\mu_1 - \varepsilon, \mu_2 + \varepsilon)$ if $\|v\| = 1$, so, for instance, by choosing $\varepsilon = \frac{\mu_1}{2}$, we ensure that $\frac{2}{\mu_1}$ is an upper bound for $\|T_m^{-1}\|$ when m is large enough.

The identity $T^{-1}\,(T_m - T)\,T_m^{-1} = T^{-1} - T_m^{-1}$ implies that

$$\|T^{-1}\| \cdot \|T_m - T\| \cdot \|T_m^{-1}\| \geq \|T^{-1} - T_m^{-1}\|.$$

The left hand side tends to zero, as the norm of the difference tends to zero and the other factors are asymptotically bounded, so the right side tends to zero too. This means that the sequence $\left(T_m^{-1}\right)_{m\in\mathbb{N}}$ converges to T^{-1}.

This completes the proof of the result. $\square$

Remark 3.25. In the previous proof we have set that Corollary 3.23 holds also for irrational orders. In a few words, we have proved that the sequence of unique solutions for the problems $T_m\, v = w$ converges to the solution to the limit problem $T\, v = w$, which is unique too.

Therefore, we can restate the final result of this section as follows.

Corollary 3.26. *Consider the operator $T \in \mathrm{End}_B(L^1[0,b])$ given by*

$$T := c_1\, I_{0^+}^{\alpha_1} + \cdots + c_{n-1}\, I_{0^+}^{\alpha_{n-1}} + c_n\, \mathrm{Id}.$$

Then, we have that $T, T^{-1} \in \operatorname{Aut}_B(L^1[0,b])$. *Moreover, if*

$$T_m := c_1\, I_{0^+}^{\alpha_{1,m}} + \cdots + c_{n-1}\, I_{0^+}^{\alpha_{n-1,m}} + c_n\, \mathrm{Id},$$

and we have that $(\alpha_{j,m})_{m\in\mathbb{N}}$ *converges to* α_j *for each* $j \in \{1,\ldots,n-1\}$, *we have that* $(T_m)_{m\in\mathbb{N}}$ *converges to* T *and* $(T_m^{-1})_{m\in\mathbb{N}}$ *converges to* T^{-1} *as* $m \to \infty$.

Chapter 4

Riemann-Liouville fractional differential equations

The study of differential equations of fractional order obeys several relevant reasons. On the one hand, in theoretical terms, it is interesting to consider a fractional analogue of the classical differential equations and to study their main properties. On the other hand, there are many different applications of FC, specially to real world problems [51, 52, 53].

One of the main points of discussion is what derivative should be used for these applications. Several realistic applications have been established in terms of the Riemann-Liouville fractional derivative, since the models can be derived axiomatically from the analytical properties of the fractional derivative like Proposition 2.34. One of the most famous examples is the Bagley-Torvik equation, related to the motion of a thin plate in a viscous fluid, see [59, 76], which is derived rigorously from the one dimensional version of diffusion equation.

Generally, many of these fractional differential equations with Riemann-Liouville derivatives assume that the initial conditions are trivial, to ensure that the Riemann-Liouville fractional derivative is well defined. To consider their analogue with non-trivial initial values, it is quite common to substitute the Riemann-Liouville fractional derivative for the Caputo notion. In many papers, it is frequent to find the justification that this change does not carry a problem, since the equation is still the same for the case of trivial initial values, although it is possible to discuss if this generalization is valid for the real world applications. Observe that, in virtue of Remark 2.43, if a function $f \in \mathcal{X}_\alpha \cap \mathcal{Y}_\alpha$ fulfils $D_{0^+}^{\lceil \alpha \rceil - 1} f(0) = 0$, then $f \in I_{0^+}^{\lceil \alpha \rceil}[0, b]$ and $D_{0^+}^{C,\alpha} f = D_{0^+}^{\alpha} f$.

However, there are still some relevant papers and monographs that discuss the theory and applications of fractional differential equations in terms of Riemann-Liouville fractional derivatives, which is our main interest. We highlight the treatment presented in the classical reference [55], where the following theorem is a reinterpretation of the one originally established at the beginning of section 5.5, Theorem 1.

Theorem 4.1. *Consider a linear homogeneous fractional differential equation with constant coefficients and rational orders. If the highest order of differentiation is α, then the equation has $\lceil\alpha\rceil$ linearly independent solutions.*

It is relevant to note that many references for FC, even the classical ones, are not clear enough about the notion of solution to a fractional differential equation. With the previous sentence, we mean that it is desirable to introduce a suitable space of differentiable functions first, to later discuss about the solvability of the fractional differential equation. In one of the main results of this chapter, developed in [12], we will show that the previous theorem is true in some weak sense. However, after defining formally the notion of "strong solution", we will see that, in general, there are less than $\lceil\alpha\rceil$ linearly independent solutions. Indeed, only for those "strong" solutions it will be coherent to talk about initial values.

4.1 Intuitive lack of linearly independent solutions

We will discuss this lack of linearly independent solutions very briefly, and from an intuitive point of view. In the next section, a formal theory will be developed to explain this fact. The main point is that the spaces $\mathcal{X}_\alpha$ are contained in $L^1[0,b]$.

We have previously discussed that $\mathcal{X}_1 = AC[0,b]$ is the maximal space where we can define differentiation, in the sense that the Fundamental Theorem of Calculus holds. However, $\mathcal{X}_1$ does not contain every function that can be differentiated almost everywhere in $[0,b]$ in the usual sense. It is "good news" that being differentiable almost everywhere is not enough to belong to $\mathcal{X}_1$, since there is a very well known example of a continuous function f (Cantor devil's staircase) whose derivative is 0 almost everywhere in $[0,b]$ with $f(0) = 0$ and $f(b) = 1$. Thus, it is reasonable to state that such a function can not be a solution of $f' = 0$ in $[0,b]$, although f is differentiable almost everywhere with zero derivative. This is a good reason to consider $\mathcal{X}_1$ as the natural space where we should seek solutions to $f' = 0$ in $[0,b]$.

Moreover, we observe that functions like $t^{-\gamma}$, for $\gamma \geq 1$, are excluded of being in any $\mathcal{X}_\alpha \subset L^1[0,b]$, although they are differentiable at any point in $(0, b]$. It is relevant to point out that a function that is everywhere differentiable in $[0, b]$ lies automatically in $\mathcal{X}_1$. Note that it is reasonable to require fractional differentiable functions to be integrable on $[0, b]$, since the Riemann-Liouville fractional derivative is defined in terms of the Riemann-Liouville fractional integral, which is well defined in $L^1[0, b]$.

Example 4.2. *Consider the equation*

$$\left(D_{0^+}^{\frac{4}{3}} + D_{0^+}^{1}\right) u_1(t) = 0. \tag{4.1}$$

If u_1 solves Equation (4.1)*, then $u_1 \in \mathcal{X}_{\frac{4}{3}} \cap \mathcal{X}_1$, and can be written as*

$$u_1(t) = d_1\, t^{\frac{1}{3}} + I_{0^+}^{\frac{4}{3}} g_1(t), \tag{4.2}$$

for a certain $g_1 \in L^1[0, b]$, recall Theorem 2.32. Thus, each solution to Equation (4.1) *is in correspondence with a solution to*

$$\left(I_{0^+}^{\frac{1}{3}} + \mathrm{Id}\right) g_1(t) = -\frac{d_1}{3}\, t^{-\frac{2}{3}},$$

which depends on one parameter. Since the left operator is a bounded isomorphism in $L^1[0, b]$, due to Corollary 3.26, the vector space of solutions to Equation (4.1) *has dimension one.*

However, there is an intuitive way to face the problem, that may cause some confusion. Since, $D_{0^+}^{\frac{4}{3}} = D_{0^+}^{1} \circ D_{0^+}^{\frac{1}{3}}$, we can rewrite Equation (4.1) *as*

$$D_{0^+}^{1} \left(D_{0^+}^{\frac{1}{3}} + \mathrm{Id}\right) u_2(t) = 0. \tag{4.3}$$

Essentially, we need to solve

$$\left(D_{0^+}^{\frac{1}{3}} + \mathrm{Id}\right) u_2(t) = c,$$

for a constant c. We know that

$$u_2(t) = d_2\, t^{-\frac{2}{3}} + I_{0^+}^{\frac{1}{3}} g_2(t), \tag{4.4}$$

and we derive the family of problems

$$\left(I_{0^+}^{\frac{1}{3}} + \mathrm{Id}\right) g_2(t) = c - d_2\, t^{-\frac{2}{3}}.$$

The previous argument now gives that the vector space of solutions to (4.3) *has dimension two, which was the dimension predicted by Theorem 4.1. The main point is that not all of them lie in* $\mathcal{X}_{\frac{4}{3}} \cap \mathcal{X}_1$.

Note that, if we try to apply $D_{0+}^{\frac{4}{3}} + D_{0+}^{1}$ *to* (4.4), *there is a problem with the computation* $D_{0+}^{1}\, d_2\, t^{-\frac{2}{3}}$ *if* $d_2 \neq 0$, *since the function does not lie in* $\mathcal{X}_1$. *Thus,* $\left(D_{0+}^{\frac{4}{3}} + D_{0+}^{1}\right) u_2(t) \in L^1[0,b]$ *implies* $d_2 = 0$ *and, strictly speaking, the vector space of authentical solutions has dimension one.*

Remark 4.3. The moral of the previous example is the following one. Equations (4.1) and (4.3) can be rewritten as

$$D_{0+}^{\frac{4}{3}} \left(I_{0+}^{\frac{1}{3}} + \mathrm{Id}\right) u(t) = 0.$$

Thus, we need to solve the fractional integral problem

$$\left(I_{0+}^{\frac{1}{3}} + \mathrm{Id}\right) u(t) \in \ker D_{0+}^{\frac{4}{3}}. \tag{4.5}$$

The set of solutions has dimension two, due to Corollary 3.26. However, a solution to Equation (4.5) does not lie, necessarily, in the space where $D_{0+}^{\frac{4}{3}} + D_{0+}^{1}$ makes sense. In this case, the intersection between the solutions of (4.5) and the domain of definition of $D_{0+}^{\frac{4}{3}} + D_{0+}^{1}$, which was $\mathcal{X}_{\frac{4}{3}} \cap \mathcal{X}_1$, has dimension one.

4.2 The sets of "strong" and "weak" solutions to Riemann-Liouville fractional differential equations

We will connect the material of the previous chapter with the theory of fractional differential equations for Riemann-Liouville derivatives. These results will be later used to discuss some applications of fractional equations in different models. Recall that we have considered the equation

$$T\, v(t) := \left(c_1\, I_{0+}^{\alpha_1} + \cdots + c_{n-1}\, I_{0+}^{\alpha_{n-1}} + \mathrm{Id}\right) v(t) = w(t), \tag{4.6}$$

where $c_i \neq 0$ for $i \in \{1, \ldots, n-1\}$, $\alpha_1 > \ldots > \alpha_{n-1} \geq 0$ and $w \in L^1[0,b]$. The main result obtained in Corollary 3.26 is that this equation has a unique solution $v \in L^1[0,b]$ for any integrable source term $w \in L^1[0,b]$.

It would be nice that some of the previous results, as it is the case for Corollary 3.26, were inherited by fractional differential equations. We are interested in studying the solutions to this general class of linear problems with constant coefficients

$$L\,u(t) := \left(c_1\,D_{0^+}^{\beta_1} + \cdots + c_{n-1}\,D_{0^+}^{\beta_{n-1}} + D_{0^+}^{\beta_n}\right) u(t) = w(t), \tag{4.7}$$

where $\beta_n > \ldots > \beta_1 \geq 0$ and $w \in L^1[0,b]$. Of course, the first question is where should we seek the solution.

Remark 4.4. In the usual case of integer orders, we find the solutions in $\mathcal{X}_{\beta_n}$. Although it is quite common to forget it, the underlying reason to do this is that $\bigcap_{j=1}^{n}\mathcal{X}_{\beta_j} = \mathcal{X}_{\beta_n}$. This means that any function with summable derivative of order β_n has also a summable derivative of any lower order. However, in general, this does not necessarily happen when the involved orders are non-integer. Thus, we may have $\bigcap_{j=1}^{n}\mathcal{X}_{\beta_j} \neq \mathcal{X}_{\beta_n}$ and, of course, a solution to Equation (4.7) has to lie in $\bigcap_{j=1}^{n}\mathcal{X}_{\beta_j}$.

Consequently, it is convenient to know the structure of the set $\bigcap_{j=1}^{n}\mathcal{X}_{\beta_j}$, which has already been described in Theorem 2.32, to study the existence and uniqueness of solution. Of course, to expect uniqueness of solution, some initial conditions have to be added to Equation (4.7). The fundamental remark is that Equation (4.7) can be rewritten as

$$L\,u(t) := D_{0^+}^{\beta_n}\left(c_1 I_{0^+}^{\beta_n-\beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n-\beta_{n-1}} + \mathrm{Id}\right) u(t) = w(t). \tag{4.8}$$

In consequence, it is quite natural to make the following reflection. If $u(t)$ solves (4.8), it is because

$$\left(c_1 I_{0^+}^{\beta_n-\beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n-\beta_{n-1}} + \mathrm{Id}\right) u(t) \in I_{0^+}^{\beta_n} w(t) + \ker D_{0^+}^{\beta_n}. \tag{4.9}$$

We will refer to the set of solutions to Equation (4.9) as the set of weak solutions. The previous terminology obeys the following reason: although a solution to (4.8) solves (4.9), the converse does not hold in general. The key point is that a solution to (4.9) may not lie in $\bigcap_{j=1}^{n}\mathcal{X}_{\beta_j}$. The set of solutions to (4.8) will be called set of strong solutions.

At this point, we know two crucial things:

- We have already computed $\ker D_{0^+}^{\beta_n} = \left\langle \left\{ t^{\beta_n-1}, \ldots, t^{\beta_n-\lceil\beta_n\rceil} \right\} \right\rangle$ in Equation (2.9), which is a vector space of dimension $\lceil\beta_n\rceil$. Therefore, the set of weak solutions has also dimension $\lceil\beta_n\rceil$ too, since it is the image of the affine space $I_{0^+}^{\beta_n} w(t) + \ker D_{0^+}^{\beta_n}$ via the automorphism $T^{-1} \in \mathrm{Aut}_B(L^1[0,b])$, where T denotes the left hand side operator in (4.9).

- The dimension of the set of strong solutions is bounded from above by the least possible number $\lceil \beta_n - \beta_j \rceil$, where $\beta_n - \beta_j \notin \mathbb{Z}^+$. We define β_* as the β_j that minimizes the previous non-integer amount. If all the previous differences are integer, the bound is simply $\lceil \beta_n \rceil$ and we define $\beta_* = 0$.

From these remarks, there are some remaining points that need to be studied in detail. At first, we will show the motivation for the estimate for the dimension of the set of strong solutions. Moreover, we will prove that it is sharp by inspecting which elements in $\ker D_{0^+}^{\beta_n}$ guarantee that the weak solution associated to those elements is, indeed, a strong one. Finally, it will be immediately known how to codify the strong solutions in terms of the initial conditions and the relation between these initial conditions and the choice for the corresponding element in $\ker D_{0^+}^{\beta_n}$.

Remark 4.5. We have that $\bigcap_{j=1}^n \mathcal{X}_{\beta_j} \subset I_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil + 1 - \varepsilon} L^1[0,b]$ for every increment $\varepsilon > 0$, but not for $\varepsilon = 0$, due to Theorem 2.32. Moreover, note that, if $f \in \ker D_{0^+}^{\beta_n}$ is chosen as the right addend in the right hand side for (4.9), we have that $f \in I_{0^+}^{\gamma} L^1[0,b]$ if and only if $u \in I_{0^+}^{\gamma} L^1[0,b]$ for $\gamma \leq \beta_n$. Therefore, to have a strong solution, it is mandatory to select $f \in \left\langle \left\{ t^{\beta_n - 1}, \ldots, t^{\beta_n - \lceil \beta_n - \beta_* \rceil} \right\} \right\rangle$.

Lemma 4.6. *If $u \in L^1[0,b]$ is a solution to*

$$\left(c_1 I_{0^+}^{\beta_n - \beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n - \beta_{n-1}} + \mathrm{Id} \right) u(t) = I_{0^+}^{\beta_n} w(t) + f(t) \tag{4.10}$$

for $f \in \left\langle \left\{ t^{\beta_n - 1}, \ldots, t^{\beta_n - \lceil \beta_n - \beta_* \rceil} \right\} \right\rangle \subset \ker D_{0^+}^{\beta_n}$, then $u \in \bigcap_{j=1}^n \mathcal{X}_{\beta_j}$.

Proof. If we use the notation $\Upsilon := c_1 I_{0^+}^{\beta_n - \beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n - \beta_{n-1}}$, we deduce from Equation (4.10) that

$$(\Upsilon + \mathrm{Id})\,(u(t) - f(t)) = I_{0^+}^{\beta_n} w(t) - \Upsilon\, f(t).$$

Observe now that the addend $-\Upsilon\, f(t)$ can be decomposed in two parts, since two different situations may happen:

- If $\beta_n - \beta_j \notin \mathbb{Z}$, we see that $I_{0^+}^{\beta_n - \beta_j} t^{\beta_n - k}$ will be always in the space $I_{0^+}^{\beta_n + (\beta_n - \beta_* + 1) - \lceil \beta_n - \beta_* \rceil - \varepsilon} L^1[0,b]$ for every $\varepsilon > 0$. This simply occurs because the least exponent for t is achieved when $\beta_j = \beta_*$ and $k = \lceil \beta_n - \beta_* \rceil$. Indeed, for $\varepsilon > 0$ small enough, the previous space is contained in $I_{0^+}^{\beta_n} L^1[0,b]$.

- If $\beta_n - \beta_j \in \mathbb{Z}^+$, there are two options:

  If $\beta_n - \beta_j > \beta_n - \beta_*$, we have that $I_{0^+}^{\beta_n - \beta_j} t^{\beta_n - k}$ lies again in $I_{0^+}^{\beta_n} L^1[0,b]$, since the maximum value admitted for k is $\lceil \beta_n - \beta_* \rceil$.

  If $\beta_n - \beta_j < \beta_n - \beta_*$, we have that $I_{0^+}^{\beta_n - \beta_j} t^{\beta_n - k} \in \left\langle \left\{ t^{\beta_n - k'} \right\} \right\rangle$ with $k' < k$.

Thus, we can write $I_{0^+}^{\beta_n} w(t) - \Upsilon\, f(t) = I_{0^+}^{\beta_n} w_1(t) + f_1(t)$, and get the equation

$$\left(c_1 I_{0^+}^{\beta_n - \beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n - \beta_{n-1}} + \mathrm{Id} \right) (u(t) - f(t)) = I_{0^+}^{\beta_n} w_1(t) + f_1(t).$$

Note that f lied in a $\lceil \beta_n - \beta_* \rceil$ dimensional space, but f_1 lies in a (at most) $\lceil \beta_n - \beta_* \rceil - 1$ dimensional space.

If we repeat this process, we obtain

$$\left(c_1 I_{0^+}^{\beta_n - \beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n - \beta_{n-1}} + \mathrm{Id} \right) (u(t) - f(t) - f_1(t)) = I_{0^+}^{\beta_n} w_2(t) + f_2(t),$$

with f_2 lying in a (at most) $\lceil \beta_n - \beta_* \rceil - 2$ dimensional space. After a suitable number of iterations, the vector space has to be of dimension zero and we have the situation

$$(\Upsilon + \mathrm{Id})\, (u(t) - f(t) - \cdots - f_{r-1}(t)) = I_{0^+}^{\beta_n} w_r(t) \in I_{0^+}^{\beta_n} L^1[0,b].$$

Therefore, $u(t) - f(t) - \cdots - f_{r-1}(t) \in I_{0^+}^{\beta_n} L^1[0,b]$. Finally, if we use that

$$f(t) + f_1(t) + \cdots + f_{r-1}(t) \in \left\langle \left\{ t^{\beta_n - 1}, \ldots, t^{\beta_n - \lceil \beta_n - \beta_* \rceil} \right\} \right\rangle,$$

it follows that $u \in \left\langle \left\{ t^{\beta_n - 1}, \ldots, t^{\beta_n - \lceil \beta_n - \beta_* \rceil} \right\} \right\rangle \oplus I_{0^+}^{\beta_n} L^1[0,b] = \bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$. $\square$

Remark 4.7. Moreover, Equation (4.8) has at most one solution in the space $\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$, provided that $D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil} u(0), \ldots, D_{0^+}^{\beta_n - 1} u(0)$ are given. To check this, just observe that, if we have two solutions to (4.8) with the same initial values, their difference can be written as $I_{0^+}^{\beta_n} g$ with $g \in L^1[0,b]$ and g would fulfil $(\Upsilon + \mathrm{Id})\, g = 0$. However, the injectivity of the left hand side operator implies that $g \equiv 0$.

We highlight that the first initial value in the previous remark can be, in fact, a fractional integral if $-1 < \beta_n - \lceil \beta_n - \beta_* \rceil < 0$.

4.3 A suitable choice for the initial values

Up to this point we have checked that, a priori, there are more weak solutions (a $\lceil \beta_n \rceil$ dimensional space) than strong solutions (a $\lceil \beta_n - \beta_* \rceil$ dimensional space). We have also proved how weak solutions are encoded depending on the source term, more specifically depending on the element chosen in $\ker D_{0^+}^{\beta_n}$. Moreover, we know that, if the choice is made in a certain subspace of $\ker D_{0^+}^{\beta_n}$, then the obtained solution is a strong one. However, one could think about encoding strong solutions via initial conditions instead of using weak problems and selecting a strong source term. Therefore, the last task should consist in relating the choices for $\ker D_{0^+}^{\beta_n}$ that give a strong solution with the corresponding initial conditions for the strong problem. In this sense, we provide the following lemma.

Lemma 4.8. *Equation* (4.8) *has a unique solution in* $\bigcap_{j=1}^{n} \mathcal{X}_{\beta_j}$*, after providing the initial values* $D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil} u(0), \ldots, D_{0^+}^{\beta_n - 1} u(0)$*. This solution coincides with the unique solution to* (4.10)*, where* $f \in \left\langle \left\{ t^{\beta_n - 1}, \ldots, t^{\beta_n - m} \right\} \right\rangle$ *is the unique function fulfilling*

$$\begin{aligned} D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil} u(0) &= D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil} f(0), \\ D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil + 1} u(0) + c_{n-1} \, D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil} u(0) &= D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil + 1} f(0), \\ &\vdots \\ D_{0^+}^{\beta_n - 1} u(0) + \cdots + c_{n - \lceil \beta_n - \beta_* \rceil} \, D_{0^+}^{\beta_n - \lceil \beta_n - \beta_* \rceil - 1} u(0) &= D_{0^+}^{\beta_n - 1} f(0). \end{aligned}$$

Proof. Consider again the equation

$$\left(c_1 I_{0^+}^{\beta_n - \beta_1} + \cdots + c_{n-1} I_{0^+}^{\beta_n - \beta_{n-1}} + \mathrm{Id} \right) u(t) = I_{0^+}^{\beta_n} w(t) + f(t),$$

where we will suppose that each positive integer less than or equal to $\beta_n - \beta_1$ can be written as $\beta_n - \beta_j$ for some j. This does not imply a loss of generality, since we can assume that some $c_j = 0$, if needed. The only purpose of this assumption is to ease the notation in this proof in the way that is described in the following paragraph.

If $\beta_* = \beta_{n-m}$ and $\beta_n - \beta_{n-m}$ is the least possible non-integer difference with minuend β_n, we can use the previous notational assumption to check that $\beta_n - \beta_{n-m} = j$ for $j < m$ and $\beta_n - \beta_{n-j} \in (m-1, m)$. In consequence, $\lceil \beta_n - \beta_{n-m} \rceil = m$ and $c_{n-m+1}, \ldots, c_{n-1}$ are constants multiplying integrals of integer order in (4.10).

Recall that we look for strong solutions to (4.10) that lie in the space $\left\langle \left\{ t^{\beta_n-1}, \ldots, t^{\beta_n-m} \right\} \right\rangle \oplus I_{0^+}^{\beta_n} L^1[0,b]$, so we write

$$u(t) = d_1\, t^{\beta_n-m} + \cdots + d_m\, t^{\beta_n-1} + I_{0^+}^{\beta_n} \widetilde{u}(t).$$

Moreover, taking into account that $f \in \ker D_{0^+}^{\beta_n} \cap \mathcal{X}_{\beta_1} \cap \cdots \cap \mathcal{X}_{\beta_n}$ allows to describe

$$f(t) = b_1\, t^{\beta_n-m} + \cdots + b_m\, t^{\beta_n-1}.$$

Now, we will derive the initial conditions after applying $D_{0^+}^{\beta_n-k}$, for every $k \in \{1, \ldots, m\}$, and substituting $t = 0$ in (4.10).

At the right hand side of (4.10) this is done easily, since we deduce immediately that $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n} w(t) \in I_{0^+}^{1} L^1[0,b]$ and, thus, the substitution at $t = 0$ gives zero. The function $D_{0^+}^{\beta_n-k} f(t)$ can be computed trivially, due to the expression of f.

At the left hand side, there are two different addends to be considered. First, $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n-\beta_j} I_{0^+}^{\beta_n} \widetilde{u}(t) \in I_{0^+}^{1} L^1[0,b]$ for any $j \in \{1, \ldots, n\}$ and, thus, the substitution at $t = 0$ gives zero. Second, $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n-\beta_j} t^{\beta_n-l}$, where we recall that $l, k \in \{1, \ldots, m\}$, has three possibilities:

- If $\beta_n - \beta_j > l - k$, then $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n-\beta_j} t^{\beta_n-l}$ is 0 at $t = 0$.
- If $\beta_n - \beta_j = l - k$, then $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n-\beta_j} t^{\beta_n-l} = \Gamma(\beta_n - l + 1)$ is constant, and it is obviously defined for $t = 0$.
- If $\beta_n - \beta_j < l - k \leq m - 1$, then $\beta_n - \beta_j$ is necessarily an integer and $D_{0^+}^{\beta_n-k} I_{0^+}^{\beta_n-\beta_j} t^{\beta_n-l}$ is the zero function.

The interest of the previous trichotomy is that we never obtain some $t^{-\gamma}$ with $\gamma > 0$. In other case, we would have a huge trouble, since we could not evaluate the expression for $t = 0$. Fortunately, we can always apply $D_{0^+}^{\beta_n-k}$ to Equation (4.10), for every value $k \in \{1, \ldots, m\}$, and substitute at $t = 0$. We get the following system of linear equations

$$\begin{aligned}
D_{0^+}^{\beta_n-m} u(0) &= D_{0^+}^{\beta_n-m} f(0),\\
D_{0^+}^{\beta_n-m+1} u(0) + c_{n-1} D_{0^+}^{\beta_n-m} u(0) &= D_{0^+}^{\beta_n-m+1} f(0),\\
&\vdots\\
D_{0^+}^{\beta_n-1} u(0) + c_{n-1}\, D_{0^+}^{\beta_n-2} u(0) + \cdots + c_{n-m+1}\, D_{0^+}^{\beta_n-m} u(0) &= D_{0^+}^{\beta_n-1} f(0).
\end{aligned}$$

Note that all the involved derivatives have the same decimal part, since only the coefficients $c_{n-1}, \cdots, c_{n-m+1}$ appear in the system. We also highlight that the system has always a unique solution, since it is triangular and it has no zero element in the diagonal. Therefore, a strong choice for f determines a vector of initial values $(D_{0^+}^{\beta_n - m} u(0), \ldots, D_{0^+}^{\beta_n - 1} u(0))$ and vice-versa in a bijective way. $\square$

4.4 Some examples

We shall give two examples summarizing how to apply all the previous results.

Example 4.9. *Consider the following fractional differential equation (strong problem)*

$$\left(D_{0^+}^{\frac{7}{3}} + 3\, D_{0^+}^{\frac{4}{3}} + 4\, D_{0^+}^{\frac{1}{3}} \right) u(t) = t^3,$$

where $\beta_1 = \frac{1}{3}, \beta_2 = \frac{4}{3}, \beta_3 = \frac{7}{3}$. *In this case, note that* $\beta_* = 0$, *since all the differences* $\beta_3 - \beta_j$ *are integers. The strong solutions for the example lie in* $\bigcap_{j=1}^{3} \mathcal{X}_{\beta_j}$. *The dimension of the affine space of strong solutions is* $\lceil \beta_3 \rceil = 3$ *and the initial conditions that ensure the existence and uniqueness of solution are* $D_{0^+}^{\frac{4}{3}} u(0) = a_3$, $D_{0^+}^{\frac{1}{3}} u(0) = a_2$ *and* $I_{0^+}^{\frac{2}{3}} u(0) = a_1$.

Moreover, after left-factoring $D_{0^+}^{\frac{7}{3}}$, *we find that the associated family of weak problems is*

$$\left(4\, I_{0^+}^{2} + 3\, I_{0^+}^{1} + \mathrm{Id}\right) u(t) = I_{0^+}^{\frac{7}{3}} t^3 + f(t)$$

where $f(t) \in \left\langle \left\{ t^{\frac{4}{3}}, t^{\frac{1}{3}}, t^{-\frac{2}{3}} \right\} \right\rangle$, *which lies in a three dimensional space. The, a priori weak, obtained solution is always strong since* $\lceil \beta_3 - \beta_* \rceil = \lceil \beta_3 \rceil$.

Finally, the relation between a choice for $f(t) = b_3\, t^{\frac{4}{3}} + b_2\, t^{\frac{1}{3}} + b_1\, t^{-\frac{2}{3}}$ *providing a strong solution and the initial conditions* a_1, a_2 *and* a_3 *is*

$$\begin{aligned} a_1 = I_{0^+}^{\frac{2}{3}} f(0) &= b_1 \cdot \Gamma\left(1 - \frac{2}{3}\right), \\ a_2 + 3\, a_1 = D_{0^+}^{\frac{1}{3}} f(0) &= b_2 \cdot \Gamma\left(1 + \frac{1}{3}\right), \\ a_3 + 3\, a_2 + 4\, a_3 = D_{0^+}^{\frac{4}{3}} f(0) &= b_3 \cdot \Gamma\left(1 + \frac{4}{3}\right). \end{aligned}$$

Example 4.10. *Consider the following fractional differential equation (strong problem)*

$$\left(D_{0^+}^{\frac{13}{4}}+3\,D_{0^+}^{\frac{9}{4}}+D_{0^+}^{2}+D_{0^+}^{\frac{5}{4}}+D_{0^+}^{1}\right)u(t)=t,$$

where $\beta_1=1,\beta_2=\frac{5}{4},\beta_3=2,\beta_4=\frac{9}{4},\beta_5=\frac{13}{4}$. In this case, note that $\beta_*=\beta_3$, since it fulfils the property that $\beta_5-\beta_*$ is the least possible non-integer value among the differences $\beta_n-\beta_j$. The strong solutions for the example lie in $\bigcap_{j=1}^{5}\mathcal{X}_{\beta_j}$. The dimension of the affine space of strong solutions is $\lceil\beta_5-\beta_*\rceil=2$, and the initial conditions that ensure the existence and uniqueness of solution are $D_{0^+}^{\frac{9}{4}}u(0)=a_2$ and $D_{0^+}^{\frac{5}{4}}u(0)=a_1$.

Moreover, after extracting the factor $D_{0^+}^{\frac{13}{4}}$, we find that the associated family of weak problems is

$$\left(I_{0^+}^{\frac{9}{4}}+I_{0^+}^{2}+I_{0^+}^{\frac{5}{4}}+3\,I_{0^+}^{1}+\mathrm{Id}\right)u(t)=I_{0^+}^{\frac{13}{4}}t+f(t)$$

where $f(t)\in\left\langle\left\{t^{\frac{9}{4}},t^{\frac{5}{4}},t^{\frac{1}{4}},t^{-\frac{3}{4}}\right\}\right\rangle$, which lying in a four dimensional space. The, a priori weak, obtained solution will be strong if $f(t)\in\left\langle\left\{t^{\frac{9}{4}},t^{\frac{5}{4}}\right\}\right\rangle$.

Finally, the relations between a choice for $f(t)=b_2\,t^{\frac{9}{4}}+b_1\,t^{\frac{5}{4}}$ providing a strong solution and the initial conditions a_1 and a_2 are

$$a_1=D_{0^+}^{\frac{5}{4}}u(0)=D_{0^+}^{\frac{5}{4}}f(0)=b_1\cdot\Gamma\left(1+\frac{5}{4}\right),$$

$$a_2+3\,a_1=D_{0^+}^{\frac{9}{4}}u(0)+3\,D_{0^+}^{\frac{5}{4}}u(0)=D_{0^+}^{\frac{9}{4}}f(0)=b_2\cdot\Gamma\left(1+\frac{9}{4}\right).$$

Chapter 5

Applications

In this chapter, we develop several applications of our previous results.

In the first section, we study a particular class of fractional differential equations with Caputo derivatives via the previously obtained results for fractional differential equations in terms of Riemann-Liouville derivatives. Later, we apply this analysis to the Basset problem, described in [32].

In the second section, we develop a simple application of our results to Special Relativity. More specifically, we perform a mental experiment where the speed of a spacecraft can be obtained as the solution to a certain fractional integral equation.

Finally, in the last section, we emulate the ideas in [76] and we see how can we derive a fractional dependence for the shear force in a beam-type equation under some additional assumptions.

5.1 Fractional differential equations with Caputo derivative

In this section, we will see how to study some fractional differential equations with Caputo derivative that appear in the literature, using the results developed in the previous chapters. Note that the reasonable framework to do this is when the order of the equation is an integer, since the initial conditions will be described in terms of integer derivatives, recall Lemma 4.8.

More specifically, we consider a family of linear fractional differential equations with rational orders with Riemann-Liouville or Caputo derivatives and we will compute their solutions. The equation object of study is a generalization of Equations (4.1) and (4.2) in [32], which are the fractional relaxation equation and the fractional oscillation equation. Furthermore, Basset problem is a particular case of the fractional relaxation equation with $\alpha = \frac{1}{2}$. As we have commented, we will work simultaneously with the equation with Caputo derivatives, which are the ones used in [32], and with Riemann-Liouville derivatives, to which we have devoted the most relevant part of these study. For that reason, we will use the symbol δ^C, whose value is 1 if we are in the Caputo case and 0 if we are in the Riemann-Liouville context. However this duplication will not last too long, because we will reduce both problems to a single one soon by using Remark 2.43.

We want to solve the following equation, where $\alpha \in (0, m) \cap \mathbb{Q}$, $\alpha \notin \mathbb{Z}$,

$$L^{RL}\,\tilde{u}(t) := a_3 D^m_{0^+}\tilde{u}(t) + a_2\, D^\alpha_{0^+}\tilde{u}(t) + a_1\tilde{u}(t) = \tilde{w}(t),$$

or, in its Caputo version,

$$L^{C}\,\tilde{u}(t) := a_3 D^m_{0^+}\tilde{u}(t) + a_2\, D^{C,\alpha}_{0^+}\tilde{u}(t) + a_1\tilde{u}(t) = \tilde{w}(t).$$

Furthermore, we have the following initial conditions (for both cases)

$$\begin{aligned} D^j\tilde{u}(0^+) &= b_j \cdot \delta_C, && \text{where } 0 \le j \le \lfloor\alpha\rfloor - 1, \\ D^j\tilde{u}(0^+) &= b_j, && \text{where } \lfloor\alpha\rfloor \le j \le m-1, \end{aligned}$$

where b_j are some given real numbers. Observe that the prescribed initial values for the Riemann-Liouville case are, indeed, the ones indicated in Lemma 4.8. We can turn this problem into a similar one with homogeneous conditions by making

$$u(t) := \tilde{u}(t) - \sum_{j=\lfloor\alpha\rfloor}^{m-1} \frac{b_j}{j!}t^j - \delta_C \cdot \sum_{j=0}^{\lfloor\alpha\rfloor-1} \frac{b_j}{j!}t^j.$$

So, to solve our problem it is enough to find u obeying the following equation

$$L^{RL}\,u(t) := a_3 D^m_{0^+}u(t) + a_2 D^\alpha_{0^+}u(t) + a_1 u(t) = w^{RL}(t), \tag{5.1}$$

or

$$L^{C}\,u(t) := a_3 D^m_{0^+}u(t) + a_2 D^{C,\alpha}_{0^+}u(t) + a_1 u(t) = w^{C}(t),$$

with the initial conditions (for both cases)

$$D^j u(0^+) = 0, \quad \forall j \in \{0, \ldots, m-1\}, \tag{5.2}$$

where we have defined

$$w^{RL}(t) = \tilde{w}(t) - L^{RL}\left(\sum_{j=\lfloor\alpha\rfloor}^{m-1} \frac{b_j}{j!} t^j\right) \quad \text{and} \quad w^C(t) = \tilde{w}(t) - L^C\left(\sum_{j=0}^{m-1} \frac{b_j}{j!} t^j\right).$$

In fact, we can see that the initial conditions, specially in the Riemann-Liouville case, were imposed in such a way that they ensure $w^C, w^{RL} \in L^1[0,b]$. From now on, we will not make distinction between the two types of derivatives, because by (5.2) and Remark 2.43 both derivatives are undistinguishable. So, we will only take care about the Riemann-Liouville problem and we will use the notation $w(t) := w^{RL}(t)$ and $L := L^{RL}$.

Moreover, observe that $\alpha \in (0,m)\cap\mathbb{Q}$ allows to write $\alpha = \frac{p}{q}$ as irreducible fraction, and to assume $w \in \mathcal{C}^{mq}[0,b]$. As we have checked in the previous chapter, maybe w is not so smooth but, in that case, we know how to amend the problem, via trivial computations, to gain smoothness in the source term. So, just to avoid these computations, we will assume that $w \in \mathcal{C}^{mq}[0,b]$.

The initial conditions (5.2) and the fact that $u \in \mathcal{X}_m$ allow us to consider, due to Corollary 2.28, that $u \in I_{0^+}^m L^1[0,b]$, so we put

$$u(t) = I_{0^+}^m \tilde{v}(t).$$

So, using this integral representation and that $\alpha = \frac{p}{q}$, the previous Equation (5.1) turns now into

$$T\tilde{v}(t) = \left(a_1\, I_{0^+}^m + a_2\, I_{0^+}^{\frac{mq-p}{q}} + a_3\,\mathrm{Id}\right)\tilde{v}(t) = w(t), \tag{5.3}$$

where we have used the notation

$$T := a_1\, I_{0^+}^m + a_2\, I_{0^+}^{m-\alpha} + a_3\,\mathrm{Id}.$$

Now, we consider the following fractional integral operator, where ξ is a primitive root of the unity of order q and the circular symbol denotes the composition of the $q-1$ endomorphisms described at its right,

$$\widehat{T} = \bigcirc_{j=1}^{q-1}\left(a_1\, I_{0^+}^m + a_2\,\xi^j\, I_{0^+}^{\frac{mq-p}{q}} + a_3\,\mathrm{Id}\right).$$

It is easy to check, via Vieta's formulae for the polynomial $X^q - 1$, that its composition with the operator T gives

$$T \circ \widehat{T} = \widehat{T} \circ T = (-1)^{q-1}\, a_2^q\, I_{0^+}^{mq-p} + \sum_{j=0}^{q} \binom{q}{j} a_1^j\, a_3^{q-j}\, I_{0^+}^{mj}.$$

Up to now, we have seen that the unique solution to (5.1) with homogeneous conditions (5.2) is the m-th order integral, with base point zero, of the unique solution to (5.3). Now, we consider the equation

$$\left(T \circ \widehat{T}\right) v(t) = (-1)^{q-1}\, a_2^q\, I_{0^+}^{mq-p}\, v(t) + \sum_{j=0}^{q} \binom{q}{j}\, a_1^j\, a_3^{q-j}\, I_{0^+}^{mj}\, v(t) = w(t). \tag{5.4}$$

It is trivial to check that we can apply $\widehat{T}$ to a solution of (5.4) to obtain a solution of (5.3). Furthermore, using the smoothness of w, we can differentiate mq times at both sides of (5.4) leading to

$$(-1)^{q-1}\, a_2^q\, D^p\, v(t) + \sum_{j=0}^{q} \binom{q}{j}\, a_1^j\, a_3^{q-j}\, D^{m(q-j)}\, v(t) = D^{mq}\, w(t),$$

which can be rewritten, via the symmetry of the combinatorial numbers, as

$$(-1)^{q-1}\, a_2^q\, D^p\, v(t) + \sum_{j=0}^{q} \binom{q}{j}\, a_1^{q-j}\, a_3^j\, D^{mj}\, v(t) = D^{mq}\, w(t). \tag{5.5}$$

The integral problem (5.4) and the differential problem (5.5) are equivalent if we give some conditions on (5.5). Note that (5.5) has, as solutions, an affine space of dimension mq. In contrast, (5.4) has a unique solution because of Corollary 3.26. So, as we already know, what actually happens is that (5.5) inherits some initial conditions from (5.4) that reduce the mq dimensional affine space to a single function. These mq conditions can be deduced by differentiating j times in (5.4) and substituting with $t = 0$, where $j \in \{0, \ldots, mq-1\}$. If we denote by c_s, where $s \in \{0, \ldots, mq-1\}$, a symbol such that $c_s = 0$ when $s + p - mq < 0$ and $c_s = 1$ when $s + p - mq \geq 0$, we obtain the following triangular system with mq equations for the initial conditions, which has trivially a unique solution

$$c_s (-1)^{q-1} a_2^q D^{s+p-mq} v(0) + \sum_{j=0}^{\left\lfloor \frac{s}{m} \right\rfloor} \binom{q}{j} a_1^j a_3^{q-j} D^{s-mj} v(0) = D^s w(0), \tag{5.6}$$

for $s \in \{0, 1, \ldots, mq-1\}$. Thus, the unique solution to (5.1) with conditions (5.2) can be computed as the integral operator $I_{0^+}^m \circ \widehat{T}$ applied to the unique solution of the ODE (5.5) with conditions (5.6). In the following subsections, under additional assumptions on (5.1), more explicit calculations will be made.

5.1.1 The Basset problem

If we want to use our previous theory for the case of the Basset problem, described in [32] and [72], we can begin by considering $m = 1$, $a_1 = a_3 = 1$ and $a := a_2$ in (5.1), so we get the fractional relaxation equation, which corresponds to equation (4.1) in [32],

$$L\,u(t) := Du(t) + a\,D^{\frac{p}{q}}_{0+}u(t) + u(t) = w(t), \quad \text{with} \quad u(0) = 0. \tag{5.7}$$

Note that if we had chosen $m = 2$ in (5.1) we would be dealing with a generalization of the Bagley-Torvik equation, see [24] and [59], and also of the fractional oscillation equation, see [23] and [32]. As we have explained at the beginning of the previous subsection, there is no loss of generality in assuming homogeneous initial conditions for u or some smoothness for w after a suitable change in the source term. We have already commented that, if u is the unique solution to (5.7), then $u(t) = \hat{T} \circ I^1_{0+}v(t)$, where v fulfils the following ODE

$$(-1)^{q-1}a^q D^p v(t) + \sum_{j=0}^{q} \binom{q}{j} D^j v(t) = D^q w(t),$$

with the following initial conditions

$$\begin{aligned} \sum_{j=0}^{s} \binom{q}{j} D^{s-j}v(0) &= D^s w(0), \quad \text{if } 0 \le s < q-p, \\ (-1)^{q-1}a^q D^{s+p-q}v(0) + \sum_{j=0}^{s} \binom{q}{j} D^{s-j}v(0) &= D^s w(0), \quad \text{if } q-p \le s < q. \end{aligned}$$

In particular, if we are interested in the Basset problem, we have to take $p = 1$ and $q = 2$, which gives that the unique solution to the Basset problem is obtained by application of $(I^1_{0+} - aI^{\frac{1}{2}}_{0+} + I^0_{0+}) \circ I^1_{0+}$, which is a fractional integral operator, to the solution of the following ODE

$$v''(t) + \left(2 - a^2\right) v'(t) + v(t) = w''(t), \tag{5.8}$$

with the following initial conditions

$$v(0) = w(0) \quad \text{and} \quad v'(0) + \left(2 - a^2\right) v(0) = w'(0),$$

that can be easily rewritten as

$$v(0) = w(0) \quad \text{and} \quad v'(0) = \left(a^2 - 2\right) w(0) + w'(0). \tag{5.9}$$

With the assumptions previously mentioned in (5.7), about homogeneous initial conditions for u and smoothness for w, the solution for the Basset problem is the result of applying the operator $(I^1_{0+} - aI^{\frac{1}{2}}_{0+} + I^0_{0+}) \circ I^1_{0+}$ to the unique solution of the ODE (5.8) with conditions (5.9). Depending on the values of a and, specially, on the choice of the function $w(t)$, the computations can be easy or a bit hard but, in any case, standard.

It is interesting to choose a small q, just to make more explicit computations and a graphical representation. Imagine that, in (5.7), we choose $\alpha = \frac{1}{2}$, $a = \sqrt{2}$ and $w(t) = e^t$. From (5.8) and (5.9), we get the ODE

$$v''(t) + v(t) = e^t, \text{ with } v(0) = 1 \text{ and } v'(0) = 1.$$

It can be easily checked that the previous problem has the function $v(t) = \frac{1}{2}(e^t + \cos t + \sin t)$ as unique solution. For the particular data described previously, the solution for the Basset problem will be obtained as the following function described below

$$\begin{aligned}
&\left(I^1_{0+} - \sqrt{2}I^{\frac{1}{2}}_{0+} + I^0_{0+}\right) \circ I^1_{0+}\, v(t) \\
&= \frac{1}{2}\left(I^1_{0+} - \sqrt{2}I^{\frac{1}{2}}_{0+} + I^0_{0+}\right)(e^t + \sin t - \cos t) \\
&= e^t - \cos t - \frac{1}{\sqrt{2}}\, I^{\frac{1}{2}}_{0+}\, (e^t + \sin t - \cos t) \\
&= e^t - \cos t - \frac{\operatorname{erf}\left(\sqrt{t}\right)}{\sqrt{2}}\, e^t \\
&\quad + \sin t \cdot \left(S\left(\sqrt{\frac{2t}{\pi}}\right) - C\left(\sqrt{\frac{2t}{\pi}}\right)\right) \\
&\quad + \cos t \cdot \left(S\left(\sqrt{\frac{2t}{\pi}}\right) + C\left(\sqrt{\frac{2t}{\pi}}\right)\right),
\end{aligned} \tag{5.10}$$

where S and C are the Fresnel integrals given by

$$S(t) = \int_0^t \sin\left(\frac{\pi s^2}{2}\right) ds, \qquad C(t) = \int_0^t \cos\left(\frac{\pi s^2}{2}\right) ds,$$

and the error function is defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds.$$

The identities used in the last step of (5.10) are the equations (3.11), (3.12), (3.13) in [55]. We could have also used Table 9.1 in [69], which is more general, and involves Mitagg-Leffler and Kummer functions. However, for $\alpha = \frac{1}{2}$, they can be simplified via the identities presented in [55].

Finally, we make a representation of (5.10). It is clear that it has to look quite similar to the exponential function, because in (5.10) the non-exponential addends are, in comparison, insignificant for big values of t.

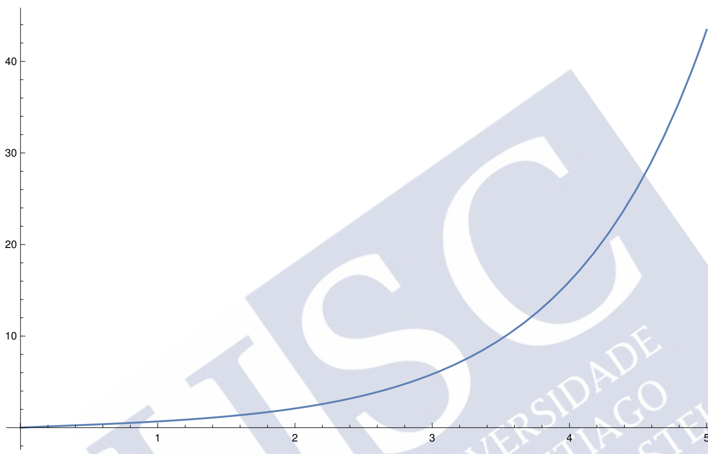

Figure 5.1: Representation of the solution (5.10).

5.2 Special Relativity

In this section, we will give an application of FC to Special Relativity (SR). The results described here are based on the work [14], developed in collaboration with J. Tenreiro Machado and A. Mendes Lopes.

5.2.1 Introduction

The main consequence of the Special Relativity Theory (SR), formulated in several steps by Lorentz [48], Poincaré [60] and Einstein [27] more than one hundred years ago, is that time or distance can not be measured in absolute terms. This means that, in general, two distinct observers can measure different values for the elapsed time between two events, or for the distance between two points. Essentially, the SR states that, if the observers are moving relatively at some constant velocity v, then the difference in the physical measurements is a consequence of v. In fact, if v is much smaller than the speed of light c, then this relativistic effect can be neglected. After the SR was formulated, some apparent paradoxes emerged. It is relevant to clarify that these "paradoxes" are not real, since they can be solved inside the framework of the SR, or using other generalizations of the theory, such as the General Relativity (GR), see [22], [70].

From a mathematical point of view, the SR states that it is impossible to assign absolute coordinates to the set of events. Indeed, since the times of Galileo and Descartes, the implicit assumption of identifying the space with $\mathbb{R}^3$ was considered. If one wants to add time to this description, then it is enough to consider $\mathbb{R} \times \mathbb{R}^3$, where the first coordinate t measures time and the other three (x, y, z) give the position in space. The essential point of the Cartesian model is that the notion of distance is independent of where the observer is located. The theory of Galilean transformations (GT) states that this still holds when one observer is moving at constant speed v with respect to the other [36]. However, the SR argues that this claim of the GT theory is not true. One of the main accomplishments of the SR is that it gives precise formulae showing how the measurements change with v. These expressions are called the Lorentz transformations (LT). The SR reveals that the difference predicted by the LT can be easily detected if v is large enough in comparison with the speed of light c.

As already mentioned, the theoretical basis of the SR was formulated in the early twentieth century. This theory explained the experimental results obtained by Michelson and Morley in 1887 [54]. Furthermore, the SR provided some additional predictions that were confirmed, mainly, in the experiences of Ives and Stilwell [37], and Kennedy and Thorndike [39], during the decade of 1930. Since then, several other experiments have supported the SR and its generalizations for wider contexts, like the GR, so that both theories are widely accepted nowadays. One of the most famous confirmation experiments, involving airplanes and atomic clocks, was performed by Hafele and Keating in 1971, with procedures and conclusions described in [34]. In fact, the detection of some objects predicted by the theory, like gravitational waves, is very recent [28].

In a different area of knowledge, the FC was formulated by Leibniz three hundred years ago. The fundamentals of FC, from the point of view of Physics, can be consulted, for instance, in [35, 77]. Recently, it was verified that FC has many different applications [68, 30, 31, 52, 61, 74], and it emerges naturally in several physical models. It is specially interesting how it may arise even when a fractional behaviour is not imposed initially [76]. In this section, under the framework of the SR, we study under what circumstances any magnitude, which is computed as a temporal integral (of integer order) from the point of view of one observer, can be measured as a fractional integral from the point of view of the other observer. When the magnitude is expressed as a linear combination of fractional integrals, we obtain a satisfactory result under very general hypotheses. Moreover, we provide numerical computations and graphical representations for a particular example. Finally, we show how to compute the velocity through the solution of a fractional integral equation involving these magnitudes.

5.2.2 Special Relativity theory

In this subsection, we introduce the fundamental concepts about the SR and we deduce the mathematical equations for the LT, under suitable hypotheses concerning the position and orientation of the edges. We provide rigorous mathematical arguments during this deduction, avoiding the implicit application of physical principles that were not stated previously in terms of mathematical properties. Our main line of thought to deduce the expression of the LT is described in [4]. Nonetheless, we examine some definitions, properties and arguments that, from our point of view, are not totally clear in the literature. In particular, we pay special attention to the notions of set of events, inertial frame of reference, and to the exact mathematical formulation of the physical principles in the SR.

Events and coordinates from a mathematical point of view

The notion of event has played a key role in the scientific and philosophical knowledge during the twentieth century. In fact, many authors of different disciplines have given their own arguments and ideas regarding the subject. A specially interesting point of view is provided by Russell in [64], where it is discussed if two observers can assure that they have perceived the same event. Despite what it may look, the previous debate is quite interesting since, in mathematical terms, it leads to the following question: What is the set of events, which will be denoted by Ω, and what properties does it have? We discuss why it is reasonable to assume that Ω is, at least, a topological space, although the notion of event will be a primordial non-defined concept. Therefore, in this language, one of the main claims of the SR is that there are many different reasonable ways to endow Ω with coordinates that do not produce the same metric, but still generate the same topology.

One of the main tasks in the SR is to study how coordinates differ when two frames that describe the same set of events are moving, with respect to each other, at constant velocity v. Intuitively, an event is a punctual fact that can be given temporal and spatial coordinates from the point of view of any observer, although these coordinates are not the same for each distinct frame.

Definition 5.1. We say that K is a system of coordinates if

$$K : \Omega \longrightarrow \mathbb{R} \times X$$

is a bijection where X is a metric space. Given any element $(t, x) \in \mathbb{R} \times X$, we will say that t is the time and x the spatial coordinate, respectively.

Definition 5.2. We say that a pair of systems of coordinates (K_1, K_2) is compatible if the composed map

$$K_2 \circ K_1^{-1} : \mathbb{R} \times X_1 \longrightarrow \mathbb{R} \times X_2$$

is a homeomorphism.

Note that once we restrict the possible metric spaces in Definition 5.1 to a certain set, like $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, we can consider the set $\mathcal{K}$ of all possible coordinate systems, avoiding Russell's paradox [65]. Definition 5.2 induces an equivalence relation in $\mathcal{K}$ and, hence, $\mathcal{K}$ may be partitioned into equivalence classes of compatible systems of coordinates. Moreover, each equivalence class $\mathcal{J} \subset \mathcal{K}$ induces a topology in Ω and the topologies induced by two different classes are distinct. For interested readers, some facts about equivalence relations and set theory can be found in [56]. Note that, essentially, a topology represents a qualitative way of defining the notions of near and far. Furthermore, although the SR states that Ω has not a canonical structure as metric space, it is metrizable and, therefore, the convergent sequences determine its topology [56].

The implication of the last paragraph in the real world is that there is a natural topology for Ω, provided that we assume that there is a universal way, independent of the observer, of deciding when a sequence of events $(\omega_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ converges to an event $\omega \in \Omega$. The previous assumption fits well with the human intuition. For our final purposes, it will be convenient to restrict our attention to the case of an Euclidean universe $X = \mathbb{R}^n$, for a fixed $n \in \mathbb{N}$.

Note that not any homeomorphism $K_2 \circ K_1^{-1}$ is what we understand as a reasonable change of coordinates. For instance, if $K_2 \circ K_1^{-1}$ mapped a spatial line into a temporal line, then it would not represent, intuitively, our idea of space-time coordinates. Therefore, it is not enough to work in

$\mathcal{J}$, since we need to make additional restrictions, in order to avoid some pathologies like the previous one. The solution in the SR is to work with inertial frames. The definition of inertial frame is controversial, since there exist several discrepancies about whether it is either absolute or relative. However, we will not enter that discussion, since it is not the aim of this subsection. In any case, we want to set clear why, independently of the chosen answer, it is not necessary to study the frames themselves and it is enough to examine the transformations between them:

- If we assume that being inertial is a relative property, then we can not decide if K_1 or K_2 are inertial by themselves. Nonetheless, we can establish when K_1 and K_2 are mutually inertial. Mathematically, this notion will be represented by a property P on the map $K_2 \circ K_1^{-1}$.

- If we assume that being inertial is an absolute property, then we can decide when a frame is inertial. However, if K_1 is inertial in this absolute sense, then K_2 is inertial in the same absolute sense if and only if the property P holds for $K_2 \circ K_1^{-1}$.

Hence, independently of the scope chosen, the relevant question is what is property P and what are the changes of coordinates $K_2 \circ K_1^{-1}$ that satisfy P. Thus, since the transformations $K_2 \circ K_1^{-1}$ will be our main object of study, we will establish the special notation

$$f_{K_1,K_2} := K_2 \circ K_1^{-1}. \tag{5.11}$$

We will denote by $\mathcal{A}_n$ the set of transformations f_{K_1,K_2} from $\mathbb{R} \times \mathbb{R}^n$ to itself that satisfy the property P. We will say, indistinguishably, that

i) $f_{K_1,K_2} \in \mathcal{A}_n$,

ii) f_{K_1,K_2} is an inertial transformation,

iii) the pair of frames (K_1, K_2) is inertial,

iv) K_1 and K_2 are mutually inertial.

Now, we have to give conditions on the elements of $\mathcal{A}_n$ for characterizing precisely the elements in $\mathcal{A}_n$, or, equivalently, what property (or collection of properties) P is exhibited by an inertial transformation in the context of the SR. Note that we have already required, as first property, that the elements in $\mathcal{A}_n$ are homeomorphisms.

To conclude this part, we state two additional conditions, often considered implicitly, that are complementary to the axioms in the SR and that will be necessary to deduce the expression of the LT.

We will assume that $\mathcal{A}_n$ is a path-connected set. This represents the physical idea that it is always possible to find a path of intermediate inertial frames between two elements of $\mathcal{A}_n$. Moreover, Physics state that being mutually inertial is an equivalence relation, meaning that:

- The pair (K_1, K_1) is inertial.
- If (K_1, K_2) is inertial, so it is (K_2, K_1).
- If (K_1, K_2) and (K_2, K_3) are inertial, so it is (K_1, K_3).

From the point of view of the maps, this is equivalent to state that:

- $\mathrm{Id} \in \mathcal{A}_n$, where Id denotes the identity operator.
- If $f_{K_1,K_2} \in \mathcal{A}_n$, then $f_{K_2,K_1} \in \mathcal{A}_n$.
- If $f_{K_1,K_2}, f_{K_2,K_3} \in \mathcal{A}_n$, then $f_{K_1,K_3} \in \mathcal{A}_n$.

In this form, it seems that there is some natural underlying mathematical structure in $\mathcal{A}_n$. The problem is that the previous structure is not enough to derive further mathematical properties about $\mathcal{A}_n$. The task of the three principles in the SR is to give additional restrictions on the set $\mathcal{A}_n$. This allows us to consider a smaller subset $\mathscr{L}$ that can be studied more easily and has a richer algebraic structure. In fact, in $\mathscr{L}$ it will be possible to give an explicit description of the LT in terms of mathematical equations, which is relevant to develop any applications.

The three principles in Special Relativity

Now, we describe the three main principles in the SR. These laws will be formulated in mathematical form, under the previously introduced language and tools. They have to be thought as experimental axioms in the context of the SR, which are formulated in terms of mathematical identities, allowing us to characterize the possibilities for a generic change between inertial frames $f_{K_1,K_2} \in \mathcal{A}_n$. These three principles establish how an arbitrary change of inertial frames of reference behaves under three different types of transformations.

1. Relativity principle: It allows the definition of relative velocity v between two inertial frames, and establishes that the measurements of time and distance between two events depends only on v, up to translations in space-time and rotations in space. The first part of the principle states that there is a constant $v \in \mathbb{R}^n$ such that

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = f_{K_1,K_2} \begin{pmatrix} t_1 \\ v \cdot t_1 \end{pmatrix} - f_{K_1,K_2} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.12}$$

   Since being mutually inertial is an equivalence relation, (K_2, K_1) is also inertial and there will be another constant $w \in \mathbb{R}^n$ such that

$$\begin{pmatrix} t_1 \\ 0 \end{pmatrix} = f_{K_2,K_1} \begin{pmatrix} t_2 \\ w \cdot t_2 \end{pmatrix} - f_{K_2,K_1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.13}$$

   This constant w is called the reciprocal velocity of v.

   The second part of the principle states that, if (K_1, K_2) and (K_3, K_4) are inertial pairs with the same velocity v, then there is another inertial pair (K_5, K_6) obtained via translations in space-time and rotations in space from (K_3, K_4), such that

$$f_{K_1,K_2} \begin{pmatrix} t \\ x \end{pmatrix} = f_{K_5,K_6} \begin{pmatrix} t \\ x \end{pmatrix}.$$

   Remark 5.3. In short terms, the principle states that, for an unknown pair of inertial frames (K_1, K_2), the map f_{K_1,K_2} can be determined once we know the velocity v, modulo different translations in space-time and rotations in space applied to K_1 and K_2.

2. Principle of homogeneity of space-time: The measurement of time and distance between two events is independent of translations in space-time. In mathematical terms, if a pair of frames (K_3, K_4) is obtained by arbitrary translations T and T' from the inertial pair (K_1, K_2), respectively, as

$$\begin{aligned} f_{K_3,K_1} \begin{pmatrix} t \\ x \end{pmatrix} &= T \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t + \tau \\ x + \xi \end{pmatrix}, \\ f_{K_4,K_2} \begin{pmatrix} t \\ x \end{pmatrix} &= T' \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t + \tau' \\ x + \xi' \end{pmatrix}, \end{aligned} \tag{5.14}$$

   then the homogeneity principle states that the pair (K_3, K_4) is still inertial and, furthermore,

$$f_{K_1,K_2} \begin{pmatrix} t \\ x \end{pmatrix} - f_{K_1,K_2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = f_{K_3,K_4} \begin{pmatrix} t \\ x \end{pmatrix} - f_{K_3,K_4} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.15}$$

Remark 5.4. We verified how homogeneity principle implies that the difference in coordinates between two fixed events is invariant under spatio-temporal translations. Therefore, given any $\omega \in \Omega$, it is possible to select the translated inertial frames K_3 and K_4 such that

$$K_3(\omega) = K_4(\omega) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The main implication of homogeneity principle is that f_{K_1,K_2} is an affine map. Specifically, let us apply the homogeneity principle by selecting the frames $K_4 = K_2$ and $K_3 = T \circ K_1$, where T is an arbitrary translation. If we use the notation in (5.14), then, from (5.15), we get

$$f_{K_1,K_2}\begin{pmatrix} t \\ x \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f_{K_3,K_2}\begin{pmatrix} t \\ x \end{pmatrix} - f_{K_3,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.16}$$

If we recall that $K_3 = T \circ K_1$, then Equation (5.16) becomes

$$f_{K_1,K_2}\begin{pmatrix} t \\ x \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f_{K_1,K_2}\begin{pmatrix} t+\tau \\ x+\xi \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} \tau \\ \xi \end{pmatrix}. \tag{5.17}$$

Finally, we deduce from (5.17) that

$$\begin{aligned} f_{K_1,K_2}\begin{pmatrix} t \\ x \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + f_{K_1,K_2}\begin{pmatrix} \tau \\ \xi \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ = f_{K_1,K_2}\begin{pmatrix} t+\tau \\ x+\xi \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{5.18}$$

Hence, since f_{K_1,K_2} is continuous, we can apply Theorem 1.1 to (5.18) to conclude that

$$f_{K_1,K_2}\begin{pmatrix} \lambda\, t \\ \lambda\, x \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda\left(f_{K_1,K_2}\begin{pmatrix} t \\ x \end{pmatrix} - f_{K_1,K_2}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \tag{5.19}$$

for every $\lambda \in \mathbb{R}$.

Remark 5.5. The conditions (5.18) and (5.19) together imply that $f_{K_1,K_2} \in \mathcal{A}_n$ is an affine map and, consequently, f_{K_1,K_2} is determined by a matrix and the image of the origin. In the follow-up, we will use Remark 5.4 to consider only changes of coordinates that map the origin into itself, since the other ones can be achieved after a suitable translation. We will denote this new set of maps, with the additional restriction about preserving the origin, as $\mathcal{B}_n$.

Remark 5.6. It is obvious that the elements in $\mathcal{B}_n$ are still continuous maps. Moreover, the equivalence relation still holds in $\mathcal{B}_n$ since mapping the origin into itself is preserved under composition and inversion of maps. Finally, from the continuity of the translations, it is straightforward to see that $\mathcal{B}_n$ is still path-connected.

3. Isotropy principle: The notion of inertial pair of frames is independent of rotations. In mathematical terms, if a pair of frames (K_3, K_4) is obtained by arbitrary rotations R and R' from the inertial pair (K_1, K_2) as

$$\begin{aligned} f_{K_3,K_1}\begin{pmatrix} t \\ x \end{pmatrix} &= (\mathrm{Id}\times R)\begin{pmatrix} t \\ x \end{pmatrix} := \begin{pmatrix} t \\ R(x) \end{pmatrix}, \\ f_{K_4,K_2}\begin{pmatrix} t \\ x \end{pmatrix} &= (\mathrm{Id}\times R')\begin{pmatrix} t \\ x \end{pmatrix} := \begin{pmatrix} t \\ R'(x) \end{pmatrix}, \end{aligned} \tag{5.20}$$

then the isotropy principle states that the pair (K_3, K_4) is still inertial.

The main implication of isotropy principle is that we can assume that the velocities v and w are parallel with opposite directions and, if we only want to study the mutual movement of the observers, then we can restrict our study to the case where f_{K_1,K_2} is a map from $\mathbb{R}\times\mathbb{R}$ to itself.

Since $f_{K_3,K_4} = f_{K_2,K_4}\circ f_{K_1,K_2}\circ f_{K_3,K_1}$, from (5.12), (5.13) and (5.20), it is straightforward to check that

$$\begin{aligned} \begin{pmatrix} t_2 \\ 0 \end{pmatrix} &= f_{K_3,K_4}\begin{pmatrix} t_1 \\ t_1\cdot R^{-1}(v) \end{pmatrix}, \\ \begin{pmatrix} t_1 \\ 0 \end{pmatrix} &= f_{K_4,K_3}\begin{pmatrix} t_2 \\ t_2\cdot R'^{-1}(w) \end{pmatrix}. \end{aligned} \tag{5.21}$$

Hence, it is possible to make a different rotation in each of the previous frames to construct a new pair of inertial frames such that v and w are parallel to the canonical vector $(1, 0, \ldots, 0)$ and have opposite directions.

Remark 5.7. Condition (5.21) and the fact that f_{K_3,K_4} is a linear map imply that f_{K_3,K_4} maps $\mathbb{R}\times\left(\mathbb{R}\times\{0\}^{n-1}\right)$ to itself and that v and w have opposite directions. We consider $\mathcal{C}_n\subset\mathcal{B}_n$ the new set of maps with these additional restrictions.

Remark 5.8. Similar arguments to the ones used in Remark 5.6 show that any map in $\mathcal{C}_n$ is a homeomorphism, that the composition and inversion of maps in $\mathcal{C}_n$ are still in $\mathcal{C}_n$, and that $\mathcal{C}_n$ is again path-connected. Note that the set $\mathcal{A}_n$ is the result of applying arbitrary spatial rotations and space-time translations to the elements of $\mathcal{C}_n$.

Remark 5.9. If we only study the change of coordinates between the line that connects the two frames, i.e., the first spatial coordinate, then it is enough to restrict our study to the case $n = 1$, because of Remark 5.7. We will establish the notation $\mathscr{L} := \mathcal{C}_1$.

Further consequences of the principles

We have verified that the three physical principles in the SR are, in fact, mathematical hypotheses on the map f_{K_1,K_2}, where (K_1, K_2) is a pair of inertial frames. From them, we have been able to establish additional assumptions for the maps f_{K_1,K_2} to simplify its study, while preserving the generality up to translations in space-time and rotations in space. Up to now, we have used the three physical axioms in the SR to justify that it is possible to choose appropriate coordinates to ensure that f_{K_1,K_2} is a linear map. We must also take into account that, apart from the continuity of the maps f_{K_1,K_2}, there are two assumptions that have not been used substantially up to now, namely, that $\mathscr{L}$ is closed for inversion and composition and that $\mathscr{L}$ is path-connected.

At first, let us reformulate the main conclusion of the relativity principle for the case $f_{K_1,K_2} \in \mathscr{L}$. If $f_{K_1,K_2} \in \mathscr{L}$, then, from the relativity principle, see Remark 5.3, it is possible to define the map

$$L_v \begin{pmatrix} t \\ x \end{pmatrix} = f_{K_1,K_2} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{5.22}$$

since f_{K_1,K_2} fixes the origin and the direction of w is determined, once v is known.

Remark 5.10. Note that, in principle, it is not necessarily true that any $v \in \mathbb{R}$ is the velocity associated to a pair of inertial frames. Hence, we will denote the set of possible velocities as Ξ. However, it is a direct claim that if $v \in \Xi$, then $-v \in \Xi$. This happens because, if v is the velocity associated to (K_1, K_2), then by selecting $R = -\mathrm{Id}$ in (5.20) and (5.21), we construct a new pair of inertial frames with velocity $-v$.

Besides, we will use several times the following argument.

Remark 5.11. If (K_3, K_4) is a pair of inertial frames, where K_3 is already fixed and $v \in \Xi$, then there is always a choice for K_4 such that v is the relative velocity of $f_{K_3,K_4} \in \mathscr{L}$. The argument is straightforward. If $v \in \Xi$, then it will be the relative velocity associated to some $f_{K_1,K_2} \in \mathscr{L}$ and defining $K_4 = f_{K_1,K_2} \circ K_3$, the conclusion follows immediately.

After introducing this background, we are in conditions to present to one of the main points concerning the LT.

Theorem 5.12. *The set* $\mathscr{L} = \{L_v \in \mathcal{M}_{2\times 2}(\mathbb{R}) : v \in \Xi\}$ *is a group.*

Proof. We have to check the following four axioms: well-defined, associativity, existence of a neutral element and existence of an inverse for each element.

- If $L_v, L_{v'} \in \mathscr{L}$, then $L_v = f_{K_1,K_2}$ and $L_{v'} = f_{K_3,K_4}$. If we use Remark 5.11, then we can find K_5 such that $f_{K_3,K_4} = f_{K_2,K_5}$, where (K_2, K_5) is inertial. Hence, since (K_1, K_2) is inertial, (K_1, K_5) will be also inertial Furthermore, $L_{v'} \cdot L_v = f_{K_1,K_5}$, so there will be $v'' \in \Xi$ such that $f_{K_1,K_5} = L_{v''}$.
- The composition of maps is associative, so the operation in $\mathscr{L}$ will be associative.
- The neutral element is L_0, since any frame K_1 is inertial with respect to itself and we have $L_0 = f_{K_1,K_1} = \mathrm{Id}$.
- Each transformation f_{K_1,K_2} has an inverse, namely f_{K_2,K_1}, since their composition is $f_{K_1,K_1} = f_{K_2,K_2} = \mathrm{Id}$.

□

The structure given by Theorem 5.12 will be useful to study the set $\mathscr{L}$. However, to complete that task, we also need further information about the relation between v and w. In Theorem 5.12, we have checked that L_0 is the neutral element of the group $\mathscr{L}$ and that $L_v^{-1} = L_w$, where v is the velocity of the pair (K_1, K_2) and w is the velocity of the pair (K_2, K_1). Hence, we can consider a function $\varphi : \Xi \longrightarrow \Xi$ defined by $\varphi(v) = w$. It is vital to determine how this function works in order for being able to apply the group postulates properly and conclude a closed expression for L_v.

The first claim is that Ξ is an interval. Unlike to what is assumed in [4], where this property is required as an axiom, we deduce it from the path-connectedness of $\mathscr{L}$. To see this, note that $\mathscr{L}$ is path-connected and v is obtained from L_v via a continuous map $\mu : \mathscr{L} \longrightarrow \Xi$ defined as

$$v = \mu(L_v) := \frac{\pi_2\left(\mathrm{Inv}(L_v)\begin{pmatrix}1\\0\end{pmatrix}\right)}{\pi_1\left(\mathrm{Inv}(L_v)\begin{pmatrix}1\\0\end{pmatrix}\right)}, \tag{5.23}$$

where π_j, $j \in \{1,2\}$, is the projection to the j coordinate and Inv computes the inverse of a matrix. Since projections and inversions of matrices are continuous maps, μ is continuous. Then, it follows that Ξ is path-connected and, therefore, is an interval.

The second claim is that φ is continuous. We already know that

$$\varphi(\varphi(v)) = v, \tag{5.24}$$

so φ represents a bijection from the interval Ξ into itself. Since $w = \varphi(v) = (\mu \circ \mathrm{Inv} \circ \mu^{-1})(v)$ and by the continuity of μ and Inv, we only need to show that μ^{-1} is continuous. Since μ is under the hypotheses of Theorem 1.49, this is a direct claim. Note that here, contrarily to what is written in [4], the continuity of φ is not required as an axiom, because it has now been deduced mathematically from the other postulates. Hence, φ is a continuous bijective function between two intervals and a very well known result in Real Analysis ensures that it is either increasing or decreasing.

Assume that φ is increasing and that there is some $v \in \Xi$ with $v < \varphi(v)$. Then, by applying φ, we get the contradiction $\varphi(v) < \varphi(\varphi(v)) = v$, where the last identity comes from (5.24). We obtain a similar contradiction if we suppose that there is some $v \in \Xi$ such that $v > \varphi(v)$. Therefore, if φ is increasing the only possibility left is to have

$$\varphi(v) = v, \tag{5.25}$$

for every $v \in \Xi$.

If φ is decreasing, then $-\varphi$ is an increasing function from Ξ to Ξ, because of Remark 5.10, and we repeat the previous argument to conclude

$$\varphi(v) = -v, \tag{5.26}$$

for every $v \in \Xi$. Since we have chosen frames such that v and $\varphi(v)$ have opposite signs, (5.25) is an absurd and the reciprocal velocity of v is given by $\varphi(v) = -v$.

Now, we use (5.26) and the group axioms to give an specific expression for the LT

$$L_v = \begin{pmatrix}\gamma_v & \delta_v\\ \beta_v & \alpha_v\end{pmatrix}, \tag{5.27}$$

using our previous partial results.

Since the velocity of the pair (K_1, K_2) is v, we know by (5.26) that the velocity of the pair (K_2, K_1) is $-v$. Hence, we obtain

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v & \delta_v \\ \beta_v & \alpha_v \end{pmatrix} \begin{pmatrix} t_1 \\ v \cdot t_1 \end{pmatrix} \Longrightarrow \beta_v = -v\,\alpha_v,$$
$$\begin{pmatrix} t_2 \\ -v \cdot t_2 \end{pmatrix} = \begin{pmatrix} \gamma_v & \delta_v \\ \beta_v & \alpha_v \end{pmatrix} \begin{pmatrix} t_1 \\ 0 \end{pmatrix} \Longrightarrow \beta_v = -v\,\gamma_v.$$

These conditions together imply that $\alpha_v = \gamma_v$ and $\beta_v = -v \cdot \gamma_v$. Therefore, we can rewrite (5.27) as follows

$$L_v = \begin{pmatrix} \gamma_v & \delta_v \\ -v\,\gamma_v & \gamma_v \end{pmatrix}. \tag{5.28}$$

Moreover, since L_v is a bijection for any $v \in \Xi$, it follows that

$$\gamma_v \neq 0, \tag{5.29}$$

for any $v \in \Xi$. Suppose now two arbitrary velocities $v, v' \in \Xi$. Then, we will have $L_{v'} L_v = L_u$ for some $u \in \Xi$. The key observation is that both diagonal elements in the matrix L_u are identical, since their common value is γ_u and, consequently, the same happens for $L_{v'} L_v$. If we compute $L_{v'} L_v$, then we get

$$L_{v'} L_v = \begin{pmatrix} \gamma_{v'}\gamma_v - \gamma_v \delta_{v'} v & \gamma_{v'}\delta_v + \gamma_v \delta_{v'} \\ -\gamma_{v'}\gamma_v (v' + v) & \gamma_{v'}\gamma_v - \gamma_{v'}\delta_v v' \end{pmatrix},$$

and the coincidence of the diagonal elements implies

$$\gamma_v \delta_{v'} v = \delta_v \gamma_{v'} v'. \tag{5.30}$$

If v or v' are 0, then we already know that $\delta_0 = 0$, since L_0 is the identity matrix and δ_0 is a non-diagonal element. If v and v' are non-null, then we rearrange the identity (5.30), finding a constant value k independent of the velocity, namely

$$k := \frac{\delta_{v'}}{\gamma_{v'} v'} = \frac{\delta_v}{\gamma_v v}, \tag{5.31}$$

where the division is allowed because of (5.29). This definition (5.31) allows us to derive the identity

$$\delta_v = k\,\gamma_v v, \tag{5.32}$$

which is valid for any $v \in \Xi$, since the identity also holds for $v = 0$. Moreover, the description in (5.28) and (5.32) allows us to write

$$L_v = \gamma_v \begin{pmatrix} 1 & kv \\ -v & 1 \end{pmatrix}. \tag{5.33}$$

Now, we focus on γ_v. First, we observe that $\gamma_v = \gamma_{-v}$, since γ_v is the factor involving time dilation and, due to the isotropy principle, it does not depend on the sign of v. Combining the previous observation with (5.33), and noting that $L_v \cdot L_v^{-1} = \mathrm{Id}$, we conclude that

$$\gamma_v^2 = \frac{1}{1 + kv^2}. \tag{5.34}$$

Since $\mathscr{L}$ is path-connected and the projection to a coordinate is a continuous map, the set of values of γ_v is also path-connected and, thus, is an interval. Moreover, this interval can not contain 0, because of (5.29) or (5.34), and, consequently, it has constant sign. This sign is positive because $\gamma_0 = 1$, since L_0 is the identity matrix. Therefore, from (5.34), we obtain

$$\gamma_v = \frac{1}{\sqrt{1 + kv^2}}. \tag{5.35}$$

Using (5.35), we rewrite (5.33) as

$$L_v = \frac{1}{\sqrt{1 + kv^2}} \begin{pmatrix} 1 & kv \\ -v & 1 \end{pmatrix}. \tag{5.36}$$

The final question is: what can be said about this universal constant k? The first point is that there are purely mathematical arguments that show that $k \leq 0$. This happens because, if k is assumed to be strictly greater than 0, then it can be shown that $\mathscr{L}$ is not a group anymore. This argument is developed in [4]. The other cases give rise to the GT and the LT:

- If we take $k = 0$, then we recover the GT and the set of allowed velocities is $\Xi = \mathbb{R}$.
- If we consider $k < 0$, then we have the LT and the set of allowed velocities is $\Xi = (-c, c)$, where $k = -\frac{1}{c^2}$.

As we already mentioned at the introduction, during the last century, several experiments confirmed the different predictions of the LT for a certain value of c. This particular value, that allows to match the predictions of the LT and the experimental observations, corresponds to the speed of light in vacuum and, in SI units, is approximately $c = 299\,792\,458\, m \cdot s^{-1}$.

Remark 5.13. The maximum known speed that has been reached by the fastest present day spacecraft is $200\,000\,m \cdot s^{-1}$ [29]. If we compute the associated value of γ_v, then we get

$$\gamma_v \approx \left(1 - \left(\frac{200}{300\,000}\right)^2\right)^{-\frac{1}{2}} = 1.000000222.$$

This shows that, unless we need very accurate measurements or that we are dealing with movements of small particles that achieve much higher speeds than human macroscopic means of transport, we can ignore the relativistic effect and use the GT instead of the LT.

5.2.3 Emergence of Fractional Calculus in the Lorentz transformations

In this subsection, we relate the FC and the LT. We start by generalizing the previous (classical) construction of the LT. This formulation allows to consider accelerated frames, instead of inertial ones, see [2]. For instance, this description is used in [44] to compare the measurement of time on Earth with respect to an accelerated spaceship. Finally, we use this new expression to derive conditions helpful to decide whether a magnitude is perceived as ea usual integral or as a fractional integral depending on the considered observer.

Accelerating frames

We construct a general version of the LT that allows us to consider accelerated frames. This idea is discussed in [2]. Suppose, at first, that the velocity of frame K_2 perceived from K_1 is piecewise constant. This means that we can consider a bounded time domain $[0, t_1]$ associated to K_1, that can be subdivided into m intervals $(s_0, s_1), (s_1, s_2), \ldots, (s_{m-1}, s_m)$, where $s_0 = 0$ and $s_m = t_1$, such that the velocity of the pair (K_1, K_2) is constant on each subinterval (s_{j-1}, s_j) with value v_j. Mathematically, it yields

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = \sum_{j=1}^{m} L_{v_j} \begin{pmatrix} s_j - s_{j-1} \\ v_j \cdot (s_j - s_{j-1}) \end{pmatrix}. \tag{5.37}$$

Consider now an integrable function $v : [0, t_1] \longrightarrow \Xi$ and suppose that (5.37) represents a Riemann sum. Since v is integrable, it is possible to take limits in (5.37), leading to the integral expression

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = \int_0^{t_1} L_{v(s)} \begin{pmatrix} 1 \\ v(s) \end{pmatrix} ds,$$

or, in an equivalent way, to

$$\begin{pmatrix} t_2 \\ 0 \end{pmatrix} = \int_0^{t_1} \frac{1}{\sqrt{1+kv(s)^2}} \begin{pmatrix} 1 & k\,v(s) \\ -v(s) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ v(s) \end{pmatrix} ds.$$

In particular, the time t_2 is given by the integral

$$t_2 = \psi(t_1) := \int_0^{t_1} \sqrt{1+kv^2(s)}\, ds. \tag{5.38}$$

Now, if we consider a more general framework, where the function $R(t_2)$ gives the total amount of a magnitude with respect to K_2 time, this means

$$R(t_2) = \int_0^{t_2} \rho(s)\, ds. \tag{5.39}$$

If we use (5.38) to make a change of variables in (5.39) via the function ψ, and using the notation $f = \rho \circ \psi$, then we conclude that

$$R(t_2) = \int_0^{t_1} f(s)\sqrt{1+kv^2(s)}\, ds. \tag{5.40}$$

Intuitively, (5.40) states that $R(t_2)$ can be also obtained as the temporal integral of the corresponding density in K_1 time, just making the change $f = \rho \circ \psi$. However, f is not a true temporal density in K_1 time. In fact, although it has $[0, t_1]$ as domain, it is not a rate of change in K_1 time, since it still expresses the infinitesimal growth of $R(\psi(s))$ with respect to an infinitesimal increment in K_2 time $\psi(s)$. Nonetheless, an infinitesimal increment in K_2 time $\psi(s)$ corresponds to an infinitesimal growth in K_1 time s amplified by the factor $\psi'(s) = \sqrt{1+kv^2(s)}$. This explains, intuitively, why the correction factor $\sqrt{1+kv^2(s)}$ appears in the integral.

A relationship with Fractional Calculus

We explore some connections between the previous expressions and FC. It is known that FC has several applications [68] and it is a relevant modelling tool in many types of phenomena [33, 53, 81], even if we do not impose any fractional behaviour a priori [51, 76]. Moreover, the effects of SR have already been studied in some other disciplines, for instance, in finances in [50]. In our case, the link appears naturally when we wish to compute an accumulated magnitude, whose rate of change with respect to time is known, and we have two different observers. From the point of view of one observer, the global magnitude is obtained as a time integral (5.39), whereas from the point of view of the other observer the magnitude is computed as a fractional integral (5.40), under suitable hypotheses. Moreover, when v is negligible with respect to c, the order of this fractional behaviour is close to 1, recovering the standard case.

Assume that f is symmetric on its domain. This means that, if we work on the interval $[0, t_1]$, then $f(s) = f(t_1 - s)$ for any $s \in [0, t_1]$. At first, suppose the very particular case where we have the additional property

$$\sqrt{1 + kv^2(s)} = \frac{s^{\alpha-1}}{\Gamma(\alpha)}, \tag{5.41}$$

with $\alpha \geq 1$. Equation (5.41) allows us to perform some calculations to consider later the more general case formulated in Equation (5.44). Since $k = -\frac{1}{c^2} < 0$, we have to restrict our interest to an interval $[0, t_1]$, where the right hand side in (5.41) is less than or equal to 1. Equation (5.41) just means that the velocity follows the particular expression

$$v(t) = c\sqrt{1 - \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^2}. \tag{5.42}$$

Under these hypotheses, it is already possible to show a first example of the fractional dependence.

Example 5.14. *In* (5.40)*, let us consider the function* $f(s) = 1$ *as the symmetric rate of change. In fact, this selection for* $f(s)$ *means that* $\rho(s) = 1$ *and that we are simply working with the rate of change of time in* K_2 *with respect to time in* K_2*. Hence, Equation* (5.40)*, where* $f(s) = 1$*, is just* (5.38)*. Since we are assuming also* (5.41)*, we get*

$$t_2 = \psi(t_1) = (I^{\alpha}_{0^+}1)(t_1) = \frac{t_1^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.43}$$

This means that if the velocity follows the expression (5.42) *for some* $\alpha \geq 1$*, then the time elapsed in one frame can be computed, from the point of view of the other one, as the fractional integral of order* α *of the constant function* 1*. When* $\alpha \to 1^+$*, the velocity tends to the null function in* (5.42) *and, simultaneously, the two different measured times* t_1 *and* t_2 *in* (5.43) *tend to the same value.*

The assumption (5.41) is quite restrictive, and we will consider a more general case. Instead of (5.41), we suppose that $\sqrt{1+kv^2(s)}$ can be expressed as a linear combination, maybe infinitely countable, of addends similar to the ones in the right hand side of (5.41). Mathematically, we have

$$\sqrt{1+kv^2(s)}=\sum_{j=0}^{\infty} a_j \frac{s^{\alpha_j-1}}{\Gamma(\alpha_j)}, \tag{5.44}$$

where $\alpha_j \geq 1$ and $a_j \in \mathbb{R}$ are fixed real coefficients such that the previous series converges in $[0,t_1]$.

Remark 5.15. Note that this hypothesis is reasonably general, since any continuous (or even integrable) function can be approximated by an expression like (5.44) in $[0,t_1]$. Furthermore, the assumption (5.44) is less exigent than requiring $\sqrt{1+kv^2(s)}$ to be an analytical function in $[0,t_1]$, since the analytic case is a consequence of the particular choice $\alpha_j = j$. Indeed, we allow $\sqrt{1+kv^2(s)}$ to be a composition of analytical functions with fractional powers like square or cubic roots. Moreover, it is straightforward to check that $\sqrt{1+kv^2(s)}$ is analytical if and only if there is a positive analytical function $z(s)$ such that $(z(s)-1)(z(s)+1)=kv(s)^2$.

In this case, if we rewrite (5.40) using (5.44), we conclude

$$R(t_2)=\int_0^{t_1}\sum_{j=0}^{\infty} a_j \frac{s^{\alpha_j-1}}{\Gamma(\alpha_j)} f(t_1-s)\,ds=\sum_{j=0}^{\infty} a_j I_{0^+}^{\alpha_j} f(t_1), \tag{5.45}$$

where the last identity holds provided that we apply a theorem that ensures the interchangeability of the possibly infinite sum and the integral. In practice, it will be enough to show that the functional series (5.44) converges uniformly in the compact $[0,t_1]$, which can be done via the Weierstrass M-test [62].

The same arguments can be applied interchanging the assumptions between v and f. That is, instead of the previous hypotheses, we can assume that $v^2(s)=v^2(t_1-s)$ for any $s\in[0,t_1]$ and that $f(s)=\sum_{j=1}^{\infty} b_j \frac{s^{\beta_j-1}}{\Gamma(\beta_j)}$. In fact, the condition $v^2(s)=v^2(t_1-s)$ can be very reasonable for a travel that involves going to a place and coming back in a symmetric way. In this case, if we use the notation $g(s)=\sqrt{1+kv^2(s)}$, then we obtain the analogous result

$$R(t_2)=\sum_{j=0}^{\infty} b_j I_{0^+}^{\beta_j} g(t_1). \tag{5.46}$$

5.2.4 Discussion of the results

Here, we illustrate the previous results. The main purpose of this subsection is to show, for a particular example, that the results given by (5.40) and (5.45) are identical. This fact is already known, but it will be interesting to give a numerical example of the convergence of the partial sums in (5.45) to (5.40). Hence, let us consider the selection

$$\sqrt{1+kv^2(s)} = 1 - \theta e^{-\sqrt{s}}, \tag{5.47}$$

depending on the parameter $\theta \in [0,1]$ and $\rho \equiv 1$. In terms of the velocity, Equation (5.47) can be rewritten as

$$v(s) = c\,\sqrt{\theta}\,e^{-\sqrt{s}}\sqrt{2\,e^{\sqrt{s}} - \theta}.$$

In particular, if we compute the integral in (5.38) or, equivalently, in (5.40), we get

$$R(t_2) = t_2 = \psi(t_1) = 2\theta e^{-\sqrt{t_1}}(1+\sqrt{t_1}) + t_1 - 2\theta. \tag{5.48}$$

In Fig. 5.2, we represent $t_2 = \psi(t_1)$ for different values of the parameter θ.

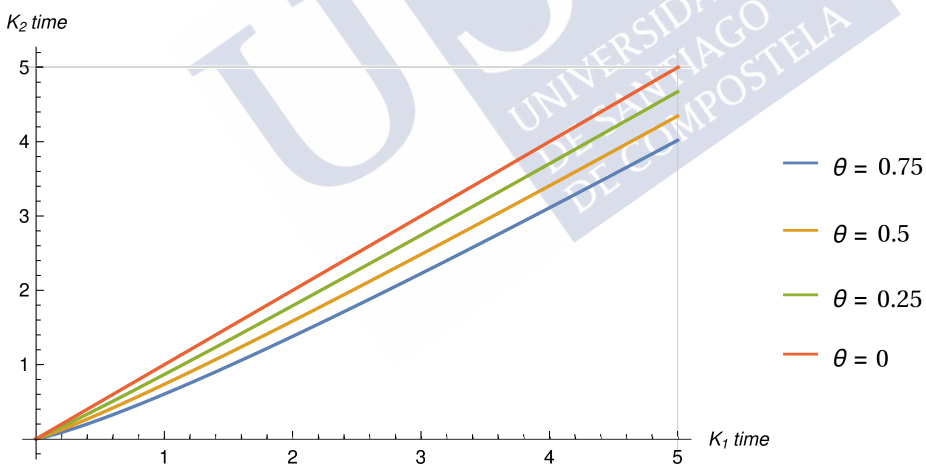

Figure 5.2: Representation of the elapsed time t_2 in Equation (5.48), where $\theta \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$.

The measurements of the times t_1 and t_2 would be approximately identical if the graph of the associated function is close to the diagonal ($\theta = 0$). We see how the distance between the diagonal (red line) and the graph increases with θ. Therefore, when the velocity increases, the temporal dilation is more visible. We show now, for this particular case, the specific expression of t_2 as an infinite linear combination of fractional integrals.

To express the perceived values of time as a combination of fractional integrals we recall that, since

$$e^s = \sum_{j=0}^{\infty} \frac{s^j}{j!}, \tag{5.49}$$

we have, from (5.47),

$$\sqrt{1+kv^2(s)} = 1 - \theta e^{-\sqrt{s}} = 1 - \theta + \sum_{j=1}^{\infty} (-1)^{j+1} \theta \, \frac{s^{\frac{j}{2}}}{j!}. \tag{5.50}$$

In fact, the previous series converges uniformly on any compact interval $[0, t_1]$ by direct application of the Weierstrass M-test, together with the identity (5.49), for $s = t_1$. Now, from (5.45) and (5.50), we get that

$$R(t_2) = t_2 = \psi(t_1) = t_1 + \sum_{j=0}^{\infty} (-1)^{j+1} \theta \, \frac{\Gamma(1+\frac{j}{2})}{j!} \left(I_{0^+}^{1+\frac{j}{2}} 1 \right)(t_1). \tag{5.51}$$

In fact, for the numerical approximation, we can consider partial sums in (5.51) like

$$S_m(t_1) := t_1 + \sum_{j=0}^{m} (-1)^{j+1} \theta \, \frac{\Gamma(1+\frac{j}{2})}{j!} \left(I_{0^+}^{1+\frac{j}{2}} 1 \right)(t_1). \tag{5.52}$$

In this line of thought, Fig. 5.3 shows how the partial sums involving fractional integrals in (5.52) converge to the true value (5.48) as m increases.

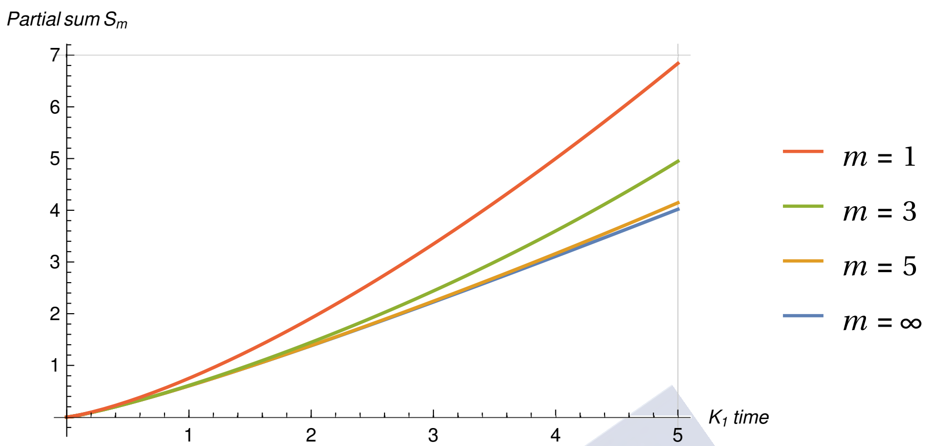

Figure 5.3: Representation of the partial sums S_m in Equation (5.52) when $m \in \{1, 3, 5, \infty\}$.

In synthesis, we have given the explicit expression of the perceived time from the second observer as a series of fractional integrals and we have represented the convergence of the partial sums in (5.52).

Determination of the velocity from a fractional integral equation

The previous framework allows to obtain the relative velocity as the solution to a fractional integral equation, provided that we know the other involved data. To see this, consider the *gedankenexperiment*[1] of a spacecraft that departs from a base, and suppose that it sends back information very frequently to its departure point. Suppose that the spacecraft sends back small nano-spaceships continuously and that they replicate backwards the original trajectory of the main spacecraft. Moreover, imagine that some component of the nano-spaceships gets quantitatively worse with time and that its ratio of degradation, where the temporal ratio is computed in the sense of (5.40), and is given by $f(s) = 2\left(s - \frac{12}{\sqrt{\pi}} + 4\right)$. Finally, assume that, when each nano-spaceship returns to the base, we measure its accumulated time

[1] German word that describes a "conceptual or thought experiment"

degradation R by means of

$$\begin{aligned} R(s) =& \frac{1}{16}\Bigg(-36C\left(\frac{2\sqrt{s}}{\sqrt{\pi}}\right)\sin(2s) + 36S\left(\frac{2\sqrt{s}}{\sqrt{\pi}}\right)\cos(2s) \\ &+ 10s^2 - \frac{160s^{3/2}}{\sqrt{\pi}} + 80s + 3 - 3\cos(2s) + 24\sin(2s)\Bigg), \end{aligned}$$

where

$$C(z) := \int_0^z \cos\left(\frac{t^2\pi}{2}\right) dt \text{ and } S(z) := \int_0^z \sin\left(\frac{t^2\pi}{2}\right) dt$$

denote the Fresnel integrals.

The question is whether, only with this data, we are able to recover the velocity of the spacecraft. It is possible to obtain this information by means of a fractional integral equation, where the unknown function is $g(s) = \sqrt{1 + kv^2(s)}$. In this case, from (5.46), we know that the following equation holds

$$\left(2I_{0^+}^{2} - 12I_{0^+}^{\frac{3}{2}} + 8I_{0^+}^{1}\right) g(s) = R(s), \tag{5.53}$$

and, therefore, it is possible to determine g as the solution to a fractional integral equation. To ensure the existence and uniqueness of solution on the interval $[0, t_1]$ to (5.53), it is a necessary and sufficient condition that $R(s) \in I_{0^+}^{1} L^1[0, t_1]$, see [15], [16]. This condition holds, since it is straightforward to show that $R(0) = 0$. Hence, we consider the simplified problem obtained after differentiating both sides in (5.53)

$$\left(2I_{0^+}^{1} - 12I_{0^+}^{\frac{1}{2}} + 8\,\mathrm{Id}\right) g(s) = R'(s), \tag{5.54}$$

where the source term now is given by

$$\begin{aligned} R'(s) =& -\frac{9}{2}C\left(\frac{2\sqrt{s}}{\sqrt{\pi}}\right)\cos(2s) - \frac{9}{2}S\left(\frac{2\sqrt{s}}{\sqrt{\pi}}\right)\sin(2s) \\ &+ \frac{5s}{4} - \frac{15\sqrt{s}}{\sqrt{\pi}} + 5 + 3\cos(2s) + \frac{3}{8}\sin(2s). \end{aligned}$$

We apply the operator $\widehat{T} := 2I_{0^+}^{1} + 12I_{0^+}^{\frac{1}{2}} + 8\,\mathrm{Id}$ at both sides of (5.54), getting

$$\left(4I_{0^+}^{2} - 112I_{0^+}^{1} + 64\,\mathrm{Id}\right) g(s) = F(s), \tag{5.55}$$

where $F(s)$ is the result of applying $\widehat{T}$ to $R'(s)$. Using the formulae for fractional integrals available in [55], we conclude

$$F(s) = \frac{5s^2}{4} - 70s + \frac{323}{8} - 21\sin(2s) + \frac{189}{8}\cos(2s).$$

Moreover, g solves (5.55) if, and only if, it is the solution to the ODE

$$64\,g''(s) - 112\,g'(s) + 4\,g(s) = F''(s), \tag{5.56}$$

where

$$F''(s) = 84\sin(2s) - \frac{189}{2}\cos(2s) + \frac{5}{2},$$

together with the restrictions

$$\begin{aligned} 64\,g(0) &= F(0),\\ -112\,g(0) + 64\,g'(0) &= F'(0). \end{aligned}$$

Since, $F(0) = 4$ and $F'(0) = -112$, we get the following initial values for (5.56)

$$\begin{aligned} g(0) &= 1,\\ g'(0) &= 0. \end{aligned} \tag{5.57}$$

Finally, after solving the ODE (5.56) under the conditions (5.57), we conclude that its solution is given by $g(s) = 1 - \frac{3}{4}\sin^2(s) = \sqrt{1 + kv^2(s)}$. From that expression, we obtain that the velocity of the spacecraft is given by

$$|v(s)| = \frac{3}{4}\,c\,|\sin(s)|\sqrt{2 - \frac{3}{4}\sin^2(s)}. \tag{5.58}$$

We represent the normalized quotient $\frac{|v(s)|}{c}$ in Fig. 5.4.

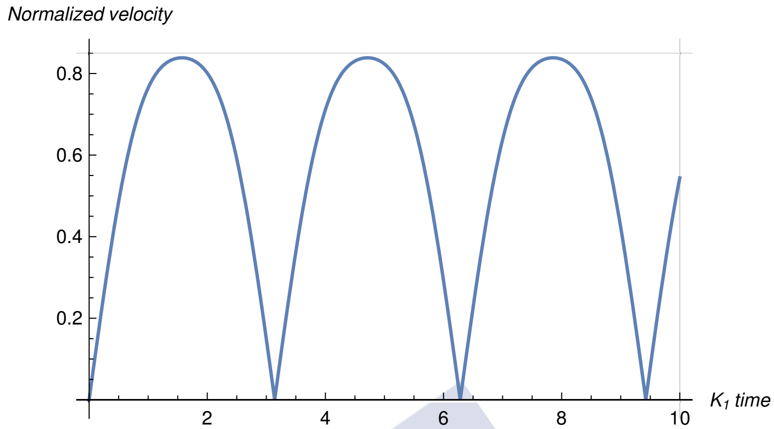

Figure 5.4: Representation of $\frac{|v(s)|}{c}$ from Equation (5.58).

In synthesis, we verify that the classical SR embeds important results when thinking on some generalizations, and that FC emerges as an important tool for characterizing some kind of memory effects in space-time.

5.3 Beam theory

In this section, with the the Euler-Lagrange equation, we show a fractional dependence for the shear force in a beam in the absence of a transverse load. The main assumption is that the beam is composed by a material with a negative Young modulus. The existence of such materials is documented in several recent references, [38, 46, 45, 80, 79]. We use, mainly, analytical calculations involving LaT to conclude that, up to a known constant, the shear force can be computed as the Riemann-Liouville fractional derivative of order $\frac{3}{2}$ of the deflection of the beam. This identity presents an interesting connection with the main result in [76]. The results described here are based on the work [11], developed in collaboration with J. Tenreiro Machado and A. Mendes Lopes.

5.3.1 Introduction

As we have mentioned along the chapters, FC has a considerable potential of application in many areas of the human knowledge [68]. For instance, the interest of FC in the theory of viscoelastic materials is described in the classical paper [73]. These implications have been studied up to the present time, see [47]. Moreover, there are several other applications of FC that justify its interest: from models of heat transport in nanofluids [58] to descriptions of the motion of projectiles [26].

One seminal FC model was proposed in [76], with a problem that involves the movement of a fluid in horizontal layers, in such a way the velocity field only depends on the time and depth. It is well known that the shear stress at any point and time can be computed as the product of the viscosity and the derivative of the velocity with respect to the depth. However, under certain assumptions, and using the properties of the LaT (LaT), which will be denoted by $\mathcal{L}$, it is possible to conclude that the shear stress can be also computed as a multiple of the temporal derivative of the velocity with order $\frac{1}{2}$. Moreover, the equation can be rewritten in terms of the derivative of displacement with order $\frac{3}{2}$.

This FC application is of considerable interest, since it shows a fractional dependence for a physical magnitude without imposing any fractional derivative in the original equation. This means that it is not necessary to consider a partial differential equation (PDE) with fractional orders to deduce that some magnitudes depend on fractional derivatives [51]. In other words, fractional operators may arise naturally in phenomena governed by differential equations that only involve standard derivatives in their formulation.

Hereafter, we obtain a similar result with a PDE, derived from the Euler-Lagrange beam equation, that reveals a fractional dependence. In this case, under some assumptions that will be introduced in the sequel, we conclude that the shear force depends on the fractional derivative of the deflection of the beam of order $\frac{3}{2}$ with respect to time.

The rest of the introduction includes explicit calculations that are used in the follow-up.

Some explicit computations

To conclude this subsection, we describe explicitly some examples of LaT that will be useful. We always assume that the considered functions for application of LaT are zero for $t < 0$ and that they follow the given expression only for $t \geq 0$. This is not a problem for the subsequent application, since t will measure time and the model will be only considered for $t \geq 0$. For interested readers it is possible to find a wide survey of useful formulae in [78].

The deduction of the first formula can be found as Example 3.11 in [25], where the constants L and ζ are assumed to be positive, and is given by

$$\mathcal{L}\left(\frac{\zeta L}{2\sqrt{\pi}}\frac{e^{\frac{-\zeta^2 L^2}{4t}}}{t^{\frac{3}{2}}}\right) = e^{-\zeta\sqrt{s}L}. \tag{5.59}$$

We will also use the following expression, see [78],

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi}}\frac{e^{\frac{-\zeta^2 L^2}{4t}}}{\sqrt{t}}\right) = \frac{e^{-\zeta\sqrt{s}L}}{\sqrt{s}}. \tag{5.60}$$

Another useful result for our calculations is described as Example 3.12 in [25], namely

$$\mathcal{L}\left(\operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right)\right) = \frac{e^{-\zeta\sqrt{s}L}}{s}, \tag{5.61}$$

where the complementary error function $\operatorname{erfc}(\cdot)$ is defined as

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t), \text{ where } \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-r^2}\,dr.$$

The previous result (5.61) can be derived immediately from the Convolution theorem (Theorem 1.35) and (5.59), since the LaT of the unit step is $1/s$. Finally, with the same argument, we can also deduce that

$$\mathcal{L}\left(\left(t + \frac{\zeta^2 L^2}{2}\right)\operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right) - \frac{\zeta L}{\sqrt{\pi}}\sqrt{t}\,e^{-\frac{\zeta^2 L^2}{4t}}\right) = \frac{e^{-\zeta\sqrt{s}L}}{s^2}. \tag{5.62}$$

5.3.2 Euler-Lagrange equation without a transverse load

We focus on the study of the deflection of a beam with length L. The beam may have some transverse loads or it could be already deflected at the initial time and, for sure, there will be restrictions at its two endpoints. A possible model for the deflection can be derived via standard procedures from Lagrangian mechanics, see [67]. In the absence of transverse loads and if the Young modulus E and the moment of area I of the beam are assumed to be constant, then we get to the description

$$EI\frac{\partial^4 w}{\partial x^4}(t,x) = -\mu\frac{\partial^2 w}{\partial t^2}(t,x), \tag{5.63}$$

where $w(t,x)$ represents the deflection at time $t \in [0,\infty)$ and length $x \in [0,L]$. Furthermore, μ is the mass density of the beam per unit length, that is assumed to be constant. An obvious remark is that the beam is modelled as a one dimensional object and, hence, the only relevant spatial coordinate is x.

Before going further, it is important to discuss the role of the Young modulus. The Young modulus of a material is, essentially, a measure of its stiffness and, in general, is assumed to be a positive quantity. If we have a block made of a given material and we apply some pressure in two opposite faces then, usually, the distance between them will be reduced. Furthermore, the proportion between the reduction and the original distance will be proportional to the pressure, at least when the pressure is "small" [49]. This proportion is the Young modulus. However, it was recently discovered [38, 46, 45, 80, 79], that it is possible to construct materials with $E < 0$. This means that the material responds with a dilatation to an external pressure acting on its surface, instead of a contraction.

To simplify our modelling, we rewrite (5.63) in an equivalent way. We use the following parameter ζ, which is always positive,

$$\zeta = \sqrt[4]{\frac{\mu}{|E|I}}. \tag{5.64}$$

If the Young modulus is positive, then the PDE (5.63) can be rewritten as

$$\frac{\partial^4 w}{\partial x^4}(t,x) = -\zeta^4 \frac{\partial^2 w}{\partial t^2}(t,x). \tag{5.65}$$

Otherwise, if the Young modulus is negative, then the PDE (5.63) is rewritten as

$$\frac{\partial^4 w}{\partial x^4}(t,x) = \zeta^4 \frac{\partial^2 w}{\partial t^2}(t,x). \tag{5.66}$$

It is necessary to impose boundary conditions such that the problem makes physical sense. The most common assumptions in literature is to consider:

- Two initial conditions at $t = 0$, namely $w(0,x)$ and $w_t(0,x)$
  - $w(0,x)$ gives the shape of the beam at time zero,
  - $w_t(0,x)$ gives the initial velocity of the beam at any point.
- Four boundary conditions at $x = 0$ and $x = L$, usually two for each endpoint, between the following eight options that are listed below:
  - $w(t,0)$ and $w(t,L)$, related to the height of the endpoints,

- $\frac{\partial w}{\partial x}(t,0)$ and $\frac{\partial w}{\partial x}(t,L)$, related to the slope at the endpoints,
- $\frac{\partial^2 w}{\partial x^2}(t,0)$ and $\frac{\partial^2 w}{\partial x^2}(t,L)$, related to the bending moment at the endpoints,
- $\frac{\partial^3 w}{\partial x^3}(t,0)$ and $\frac{\partial^3 w}{\partial x^3}(t,L)$, related to the shear force at the endpoints.

Among all possible combinations for selecting four conditions, some are more interesting from the physical point of view. To have a more detailed analysis of the physical interpretation of the boundary conditions, interested readers can check Section 2.1 in [67]. Furthermore, we will devote a part of this section to the choice of boundary conditions, selecting the most relevant cases for a real problem.

Remark 5.16. In our case, we always assume that the beam is, initially, in its natural position and at rest, that is, such that $w(0,x) = w_t(0,x) = 0$.

Finally, we introduce the shear force τ, which will exhibit a fractional dependence with respect to time. It is clear that, when the beam gets deflected, it experiments a bending moment, denoted by the symbol Q. This bending moment is given by the expression $Q(t,x) := -EI\frac{\partial^2}{\partial x^2}w(t,x)$ [67]. Its spatial rate of change is called the shear force and, since EI is assumed to be a constant, the shear force is given by

$$\tau(t,x) = -EI\frac{\partial^3 w}{\partial x^3}(t,x). \tag{5.67}$$

5.3.3 A beam with negative Young modulus

In this subsection, we verify that the shear force depends on the time fractional derivative of order $\frac{3}{2}$ of the deflection in a beam modelled by (5.63) with negative Young modulus. We start with the fourth order PDE (5.66) and, after applying the LaT, we set an ODE. From this transformed ODE, we deduce an expression for the shear force in the transformed domain. Finally, we invert the LaT in the expression that involves the shear force to obtain the final result. During the process, we include some restrictions in the boundary conditions at the endpoints of the beam to allow the final step of inverting the LaT.

Derivation of the fractional expression for the shear force

Since we are considering that the beam is composed of a material with negative Young modulus, we deal with Equation (5.66). The boundary conditions in Remark 5.16, together with Proposition 1.34, imply that for any x

the function $w_{tt}(t,x)$ is transformed into $s^2 W(s,x)$. In consequence, after applying LaT, Equation (5.66) is written as

$$\frac{\partial^4 W}{\partial x^4}(s,x) = \zeta^4 s^2 W(s,x). \tag{5.68}$$

Of course, one has to take into account the LaT of the boundary conditions, as shall be analysed in a latter subsection. The general solution for the ODE (5.68) is

$$\begin{aligned} W(s,x) =& C_1(s) e^{-\zeta\sqrt{s}x} + C_2(s) e^{\zeta\sqrt{s}x} \\ &+ C_3(s)\sin(\zeta\sqrt{s}x) + C_4(s)\cos(\zeta\sqrt{s}x), \end{aligned}$$

where the $C_j(s)$ coefficients are to be determined by means of the LaT of the boundary conditions. It is necessary to guarantee that $W(s,x)$ has an Inverse Laplace transform (ILaT) and, therefore, we assume that the three last coefficients are null. Since $C_2 = C_3 = C_4 = 0$, the solution to (5.68) is given by:

$$W(s,x) = C_1(s)\, e^{-\zeta\sqrt{s}x}. \tag{5.69}$$

It is clear that, if we apply the Convolution theorem (Theorem 1.35) backwards, and if we know the ILaT of $C_1(s)$, called $c_{1,0}(t)$, then, knowing (5.59), the solution to the problem is just

$$w(t,x) = \frac{\zeta\, x}{2\sqrt{\pi}}\, \frac{e^{\frac{-\zeta^2 x^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,0}(t), \tag{5.70}$$

since the right factor in the convolution is the ILaT of $e^{-\zeta\sqrt{s}x}$, described in (5.59). Nonetheless, instead of computing the solution, we are interested in showing the fractional behaviour for its associated shear force. From the transformed solution in (5.69), it is straightforward to derive the identity

$$\frac{\partial^3 W}{\partial x^3}(s,x) = -\zeta^3 \sqrt{s}^{\,3} W(s,x). \tag{5.71}$$

Moreover, Remark 5.16 ensures that we can apply Proposition 2.34 to (5.71), getting

$$\frac{\partial^3 w}{\partial x^3}(t,x) = -\zeta^3 \frac{\partial^{\frac{3}{2}} w}{\partial t^{\frac{3}{2}}}(t,x). \tag{5.72}$$

Finally, we recall the definition of the shear force (5.67) and apply it to (5.72), obtaining

$$\tau(t,x) = \zeta^3 EI \frac{\partial^{\frac{3}{2}} w}{\partial t^{\frac{3}{2}}}(t,x) = \sqrt[4]{\mu^3 |E| I}\, \frac{\partial^{\frac{3}{2}} w}{\partial t^{\frac{3}{2}}}(t,x), \tag{5.73}$$

where the last identity recalls the definition of ζ in Equation (5.64). Furthermore, it is possible to confirm that (5.73) is correct in terms of dimensional analysis, since the physical units at the left and the right hand sides are coherent.

Why not a positive Young modulus?

Before ending the section, we will make some considerations about Equations (5.65) and (5.66). More specifically, we show the limitations of adapting this result for a positive Young modulus. Up to now, we have been able to prove a clear fractional dependence for the shear force with respect to time when the Young modulus of the beam is negative. If one compares (5.66) with (5.72), then one can see that there is some underlying proportionality.

On the one hand, Equation (5.66) tells that, for any solution $w(t,x)$, "four derivatives in space equal two derivatives in time with an amplification of ζ^4". On the other hand, (5.72) states that "three derivatives in space are equivalent to one and a half derivative in time together with an amplification of ζ^3".

In the case of a positive Young modulus, we deal with (5.65) instead of (5.66). The problem of obtaining a result, similar to (5.72), preserving that proportionality is the minus sign in (5.65). More specifically, we would need to define what is the meaning of $\frac{3}{4}$ of a sign change. This question has no satisfactory answer on the set of real numbers. One possible solution is to use complex numbers, but this leads to a non-clear physical meaning of the shear force. Consequently, we focus only in the case that considers a negative Young modulus, where it is possible to extract a physical interpretation.

5.3.4 The boundary conditions for different type of beams

There are several possible options for the boundary conditions and, depending on the choice, there are different physical interpretations. The most common examples are, probably, the simply supported, namely the cantilever and the doubly-clamped beam structures. Readers can find more information concerning the boundary conditions for the beams in the Section 2.1 of [67].

It is important to note that, if we consider that the deflection of the beam is a time-depending function, then we have to choose non-trivial conditions for the endpoints. Let us recall that we have already imposed $w(x,0) = w_t(x,0) = 0$. Therefore, if the other conditions are also null, then the solution is the zero function which is merely the trivial case.

The main purpose of the section is to derive the boundary conditions for some particular cases of structures, ensuring that the transformed solution is (5.69). If this occurs, then we must have in mind that we describe the shear force in terms of a fractional derivative of the deflection with respect to time. This consequence was described in (5.73).

Suitable modification for a simply supported beam

A simply supported beam consists of a bar structure that is supported only at its endpoints. In this configuration, the endpoints are fixed (i.e., $w(t,0) = w(t,L) = 0$) and they have null bending moment (i.e., $w_{xx}(t,0) = w_{xx}(t,L) = 0$). We can consider a more general context of this framework, where the position of the endpoints is known, possibly changing in time. Furthermore, we suppose that it is possible to produce some extra bending moment at the endpoints.

In this case, the boundary conditions are

$$\begin{aligned} w(t,0) &= f_1(t), \quad w_{xx}(t,0) = f_2(t), \\ w(t,L) &= f_3(t), \quad w_{xx}(t,L) = f_4(t). \end{aligned}$$

Consequently, the transformed conditions are given by

$$\begin{aligned} W(s,0) &= F_1(s), \quad W_{xx}(s,0) = F_2(s), \\ W(s,L) &= F_3(s), \quad W_{xx}(s,L) = F_4(s). \end{aligned}$$

If the transformed solution is (5.69), then it is clear that the following identities hold

$$\begin{aligned} W(s,0) &= C_1(s), \\ W_{xx}(s,0) &= C_1(s)\,\zeta^2 s, \\ W(s,L) &= C_1(s)\,e^{-\zeta\sqrt{s}L}, \\ W_{xx}(s,L) &= C_1(s)\,e^{-\zeta\sqrt{s}L}\zeta^2 s. \end{aligned}$$

If $C_1(s)$ and $C_1(s)\cdot s$ have ILaT $c_{1,0}(t)$ and $c_{1,2}(t)$, respectively, then, by

(5.59), we conclude that

$$\begin{aligned} w(t,0) &= c_{1,0}(t), \\ w_{xx}(t,0) &= \zeta^2 c_{1,2}(t), \\ w(t,L) &= \frac{\zeta\, L}{2\sqrt{\pi}}\, \frac{e^{\frac{-\zeta^2\, L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,0}(t), \\ w_{xx}(t,L) &= \frac{\zeta^3\, L}{2\sqrt{\pi}}\, \frac{e^{\frac{-\zeta^2\, L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,2}(t). \end{aligned} \tag{5.74}$$

Suitable modification for a cantilevered beam

A cantilevered beam is a bar with one fixed and another free endpoint. In the standard case, the origin is clamped in such a way that it has no vertical movement or slope (i.e., $w(0,t)=0$ and $w_x(0,t)=0$). The other endpoint is completely free and, in the absence of other external factors, there will not occur any bending moments or shear forces (i.e., $w_{xx}(L,t)=0$ and $w_{xxx}(L,t)=0$). With the aim of generality, we assume now that the previous conditions are known functions of t, that is,

$$\begin{aligned} w(0,t) &= g_1(t), \quad w_x(0,t) = g_2(t), \\ w_{xx}(L,t) &= g_3(t), \quad w_{xxx}(L,t) = g_4(t). \end{aligned}$$

In this case, the transformed conditions are given by

$$\begin{aligned} W(s,0) &= G_1(s), \quad W_x(s,0) = G_2(s), \\ W_{xx}(s,L) &= G_3(s), \quad W_{xxx}(s,L) = G_4(s). \end{aligned}$$

If the transformed solution is given by (5.69), then the following identities hold

$$\begin{aligned} W(s,0) &= C_1(s), \\ W_x(s,0) &= -C_1(s)\, \zeta \sqrt{s}, \\ W_{xx}(s,L) &= C_1(s)\, e^{-\zeta\sqrt{s}L} \zeta^2 s, \\ W_{xxx}(s,L) &= -C_1(s)\, e^{-\zeta\sqrt{s}L} \zeta^3 s^{\frac{3}{2}}. \end{aligned}$$

If $C_1(s) \cdot s^{\frac{\alpha}{2}}$ has ILaT $c_{1,\alpha}(s)$ for $\alpha \in \{0,1,2,3\}$, then, by (5.59), we have

$$\begin{aligned} w(t,0) &= c_{1,0}(t), \\ w_x(t,0) &= -\zeta\, c_{1,1}(t), \\ w_{xx}(t,L) &= \frac{\zeta^3 L}{2\sqrt{\pi}} \frac{e^{\frac{-\zeta^2 L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,2}(t), \\ w_{xxx}(t,L) &= -\frac{\zeta^4 L}{2\sqrt{\pi}} \frac{e^{\frac{-\zeta^2 L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,3}(t). \end{aligned} \tag{5.75}$$

Suitable modification for a doubly-clamped beam

A doubly-clamped beam is a bar structure with both endpoints clamped. Usually, the position of each endpoint is fixed (i.e., $w(0,t) = c_1$ and $w(0,t) = c_2$) and the local slope does not change with time (i.e., $w_x(0,t) = c_3$ and $w_x(L,t) = c_4$). If we allow these restrictions to be some known functions of t, then, we get the conditions

$$\begin{aligned} w(0,t) &= h_1(t), \quad w_x(0,t) = h_2(t), \\ w(L,t) &= h_3(t), \quad w_x(L,t) = h_4(t). \end{aligned}$$

In consequence, the transformed conditions are given by

$$\begin{aligned} W(s,0) &= H_1(s), \quad W_x(s,0) = H_2(s), \\ W(s,L) &= H_3(s), \quad W_x(s,L) = H_4(s). \end{aligned}$$

If we want the transformed solution to be (5.69), then the following identities hold

$$\begin{aligned} W(s,0) &= C_1(s), \\ W_x(s,0) &= -C_1(s)\, \zeta\sqrt{s}, \\ W(s,L) &= C_1(s)\, e^{-\zeta\sqrt{s}L}, \\ W_x(s,L) &= -C_1(s)\, e^{-\zeta\sqrt{s}L}\zeta\sqrt{s}. \end{aligned}$$

If $C_1(s) \cdot s^{\frac{\alpha}{2}}$ has ILaT $c_{1,\alpha}(s)$ for $\alpha \in \{0, 1\}$, then, again due to (5.59), we can rewrite the conditions as

$$\begin{aligned} w(t,0) &= c_{1,0}(t), \\ w_x(t,0) &= -\zeta\, c_{1,1}(t), \\ w(t,L) &= \frac{\zeta\, L}{2\sqrt{\pi}}\, \frac{e^{\frac{-\zeta^2 L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,0}(t), \\ w_x(t,L) &= -\frac{\zeta^2\, L}{2\sqrt{\pi}}\, \frac{e^{\frac{-\zeta^2 L^2}{4t}}}{t^{\frac{3}{2}}} * c_{1,1}(t). \end{aligned} \tag{5.76}$$

5.3.5 Some examples

In this subsection, we present particular examples of the previous formulations. As we have verified, once the physical parameters μ, L, E and I are fixed, the result only depends on the boundary conditions. Furthermore, we have shown that, from any choice for $C_1(s)$, it is possible to determine the boundary conditions such that (5.73) holds. In particular, the shear force in Equation (5.67) is determined by a fractional derivative of the solution (5.70) with respect to t, which is the ILaT of (5.69).

In our examples, the ILaT can always be calculated analytically. However, if this was not the case, there are numerical methods to approximate the ILaT, for instance [1, 21]. There are several upgrades for these methods such as [41, 43]. Furthermore, we find some research [7, 71] where the numerical problem of estimating the ILaT is studied for specific examples in FC.

Calculations for the modified simply supported case

The particular choice $C_1(s) = \frac{1}{s^2}$ (or $c_{1,0}(t) = t$) eases the mathematical calculations, since we can invert (5.69) directly, avoiding the application of the Convolution theorem (Theorem 1.35). In that case, the solution (5.70) can be rewritten in terms of elementary and error functions. Finally, we define the boundary conditions. In this first example, we consider the conditions related to the modification of the simply supported beam. In this structure, the boundary conditions (5.74) can be computed directly, and we have that

$$C_1(s) = \frac{1}{s^2}, \quad C_1(s) \cdot s = \frac{1}{s}, \quad c_{1,0}(t) = t, \quad c_{1,2}(t) = 1.$$

Therefore, if we use (5.61) and (5.62) in the LaT version of (5.74), then we can rewrite conditions (5.74) as

$$\begin{aligned} w(t,0) &= t, \\ w_{xx}(t,0) &= \zeta^2, \\ w(t,L) &= \left(t + \frac{\zeta^2 L^2}{2}\right) \operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right) - \frac{\zeta L}{\sqrt{\pi}} \sqrt{t}\, e^{-\frac{\zeta^2 L^2}{4t}}, \\ w_{xx}(t,L) &= \zeta^2 \operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right). \end{aligned} \tag{5.77}$$

Since the derivative of any order of the functions $\operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right)$ and $e^{-\frac{\zeta^2 L^2}{4t}}$ at $t = 0^+$ is null, we can establish the following remark.

Remark 5.17. For small values of t, the previous conditions can be approximated by the following ones

$$\begin{aligned} w(t,0) &= t, \quad w(t,L) = 0, \\ w_{xx}(t,0) &= \zeta^2, \quad w_{xx}(t,L) = 0. \end{aligned}$$

The deflection $w(t,x)$ can be computed analogously to $w(t,L)$ in (5.77), since the inversion of the LaT does not depend on the spatial variable. We obtain

$$w(t,x) = \left(t + \frac{\zeta^2 x^2}{2}\right) \operatorname{erfc}\left(\frac{\zeta x}{2\sqrt{t}}\right) - \frac{\zeta x}{\sqrt{\pi}} \sqrt{t}\, e^{-\frac{\zeta^2 x^2}{4t}}. \tag{5.78}$$

The shear force can be computed directly from (5.67) and (5.78), so that

$$\tau(t,x) = \frac{\zeta^3 EI}{\sqrt{\pi}} \frac{1}{\sqrt{t}}\, e^{-\frac{\zeta^2 x^2}{4t}}. \tag{5.79}$$

Let us show some representations of the previous results. For the physical constants, we select the values $EI = -2$ and $\zeta = 2$. First, we fix the time variable t at some instants. Then, for each time instant, we represent the deflection and the shear force as functions of x.

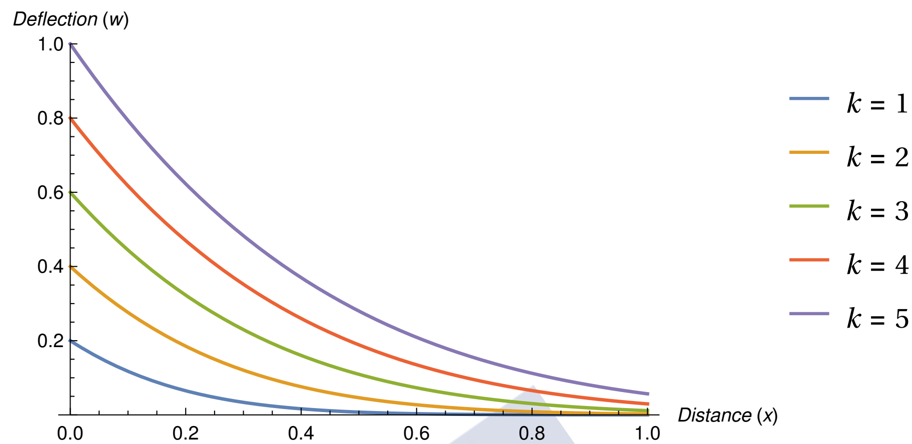

Figure 5.5: Representation of $w\left(\frac{k}{5}, x\right)$ in Equation (5.78), for the values $k \in \{1, 2, 3, 4, 5\}$.

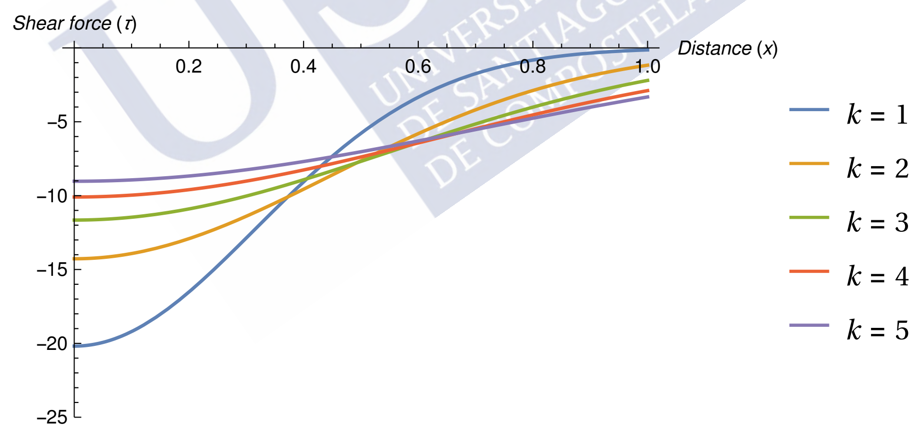

Figure 5.6: Representation of $\tau\left(\frac{k}{5}, x\right)$ in Equation (5.79), for the values $k \in \{1, 2, 3, 4, 5\}$.

Second, we fix the variable x for some specific distances and we represent the deflection and the shear force as function of t.

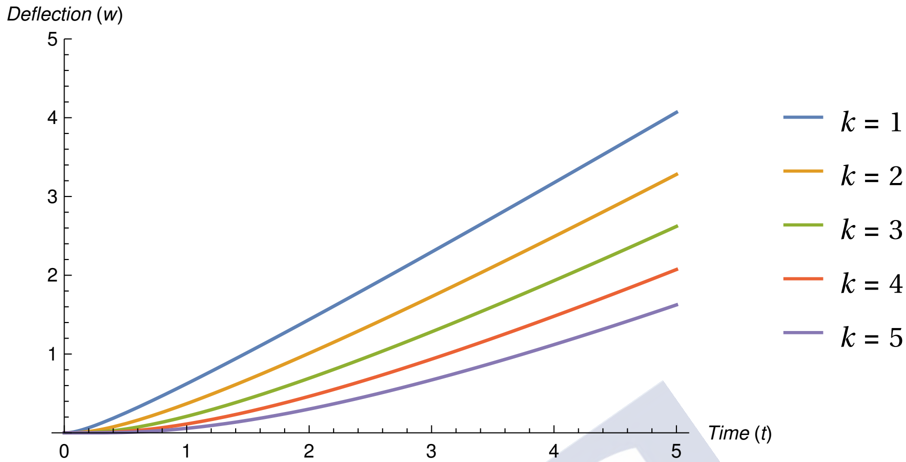

Figure 5.7: Representation of $w\left(t, \frac{k}{5}\right)$ in Equation (5.78), for the values $k \in \{1,2,3,4,5\}$.

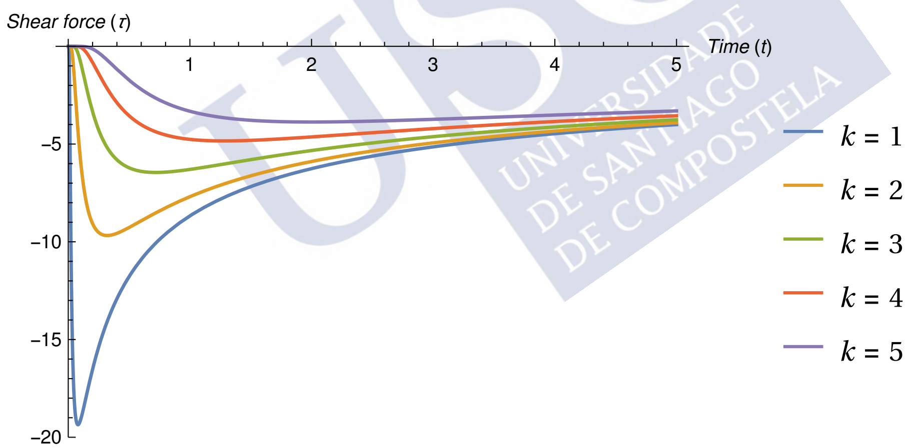

Figure 5.8: Representation of $\tau\left(t, \frac{k}{5}\right)$ in Equation (5.79), for the different values $k \in \{1,2,3,4,5\}$.

Fig. 5.5 represents the deflection of the beam for different values of time. Fig. 5.6 shows the shear force at each point of the beam for different values of time. The main observation is that, initially, the shear force is concentrated mainly at the origin $x = 0$, but, as time goes by, the shear is distributed along the beam. In fact, the charts reveal that the accumulated shear force is independent of the time. This can be proved analytically since, from (5.79), we know that

$$\int_0^\infty \tau(t,x)\,dx = \zeta^2 EI \operatorname{erf}\left(\frac{\zeta x}{2\sqrt{t}}\right)\Big|_0^\infty = \zeta^2 EI.$$

Fig. 5.7 depicts how the deflection at any point evolves versus time. Fig. 5.8 illustrates that, at the selected points, the absolute value of the shear force increases initially until some peak and later it decays slowly as time passes. It can be shown that this behaviour happens for any fixed x and, in fact, the associated peak occurs at $t = \frac{\zeta^2 x^2}{2}$. For this minimum, due to (5.79), the shear force is

$$\tau\left(\frac{\zeta^2 x^2}{2}, x\right) = \frac{2EI\zeta^2}{\sqrt{e\,\pi}} \cdot \frac{1}{x}.$$

Calculations for the modified doubly-clamped case

We select the functional coefficient $C_1(s) = \frac{1}{s}$ or, equivalently, after inverting the LT, $c_{1,0}(t) = 1$. If we work with the modification of the doubly-clamped structure, the boundary conditions (5.76) can be computed directly, since

$$\begin{aligned} C_1(s) &= \frac{1}{s}, \quad C_1(s)\cdot\sqrt{s} = \frac{1}{\sqrt{s}}, \\ c_{1,0}(t) &= 1, \quad c_{1,1}(t) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t}}. \end{aligned}$$

More specifically, if we use (5.60) and (5.61) in the LaT version of (5.76), we can rewrite the conditions (5.76) as

$$\begin{aligned} w(t,0) = 1, &\quad w(t,L) = \operatorname{erfc}\left(\frac{\zeta L}{2\sqrt{t}}\right), \\ w_x(t,0) = -\frac{\zeta}{\sqrt{\pi}}\frac{1}{\sqrt{t}}, &\quad w_x(t,L) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t}}\, e^{-\frac{\zeta^2 L^2}{4t}}. \end{aligned} \tag{5.80}$$

Remark 5.18. Provided that the beam is long enough and if we only consider small values of t, the previous conditions can be approximated by

$$\begin{aligned} w(t,0) = 1, &\quad w(t,L) = 0, \\ w_{xx}(t,0) = -\frac{\zeta}{\sqrt{\pi}}\frac{1}{\sqrt{t}}, &\quad w_{xx}(t,L) = 0. \end{aligned}$$

The deflection $w(t,x)$ can be computed analogously to $w(t,L)$ in (5.80), since we only have to invert the LaT in t, getting

$$w(t,x) = \operatorname{erfc}\left(\frac{\zeta x}{2\sqrt{t}}\right). \tag{5.81}$$

Finally, the shear force can be computed directly from (5.67) and (5.81),

$$\tau(t,x) = -\frac{EI\zeta^3}{4\sqrt{\pi}}\frac{2t-\zeta^2x^2}{t^2\sqrt{t}}e^{-\frac{\zeta^2x^2}{4t}}. \tag{5.82}$$

Let us show some representations of the results. We keep the previous physical constants $EI = -2$ and $\zeta = 2$, that were used for the simply supported structure. First, we fix the time variable t at some particular time instants. Then, for each fixed time, we represent the deflection and the shear force as function of x.

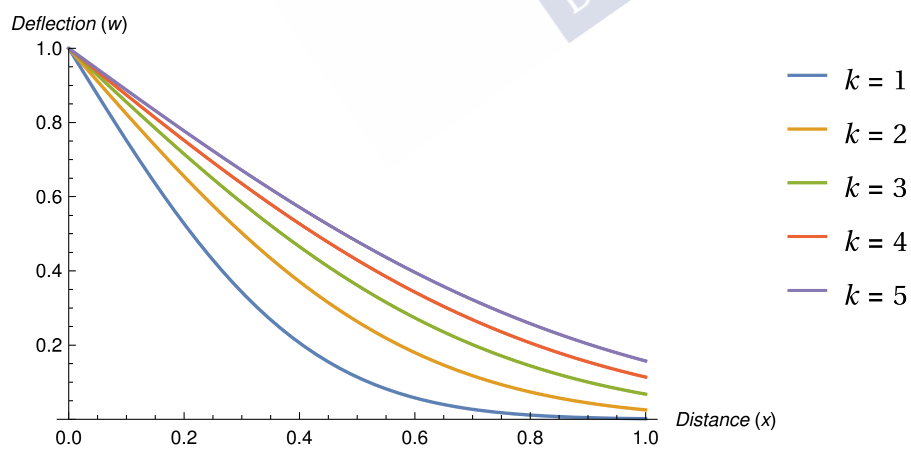

Figure 5.9: Representation of $w\left(\frac{k}{5},x\right)$ in Equation (5.81), for the values $k \in \{1,2,3,4,5\}$.

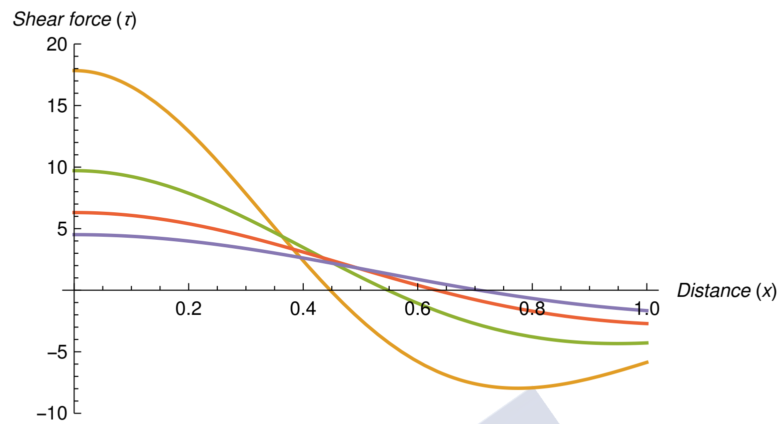

Figure 5.10: Representation of $\tau\left(\frac{k}{5},x\right)$ in Equation (5.82), for the values $k \in \{2,3,4,5\}$.

Since the solution associated with the value $k=1$ has a wider range than the others, we represent it in Fig. 5.11. We keep showing the solution for the value $k=2$ to see the relative size with respect to Fig. 5.10.

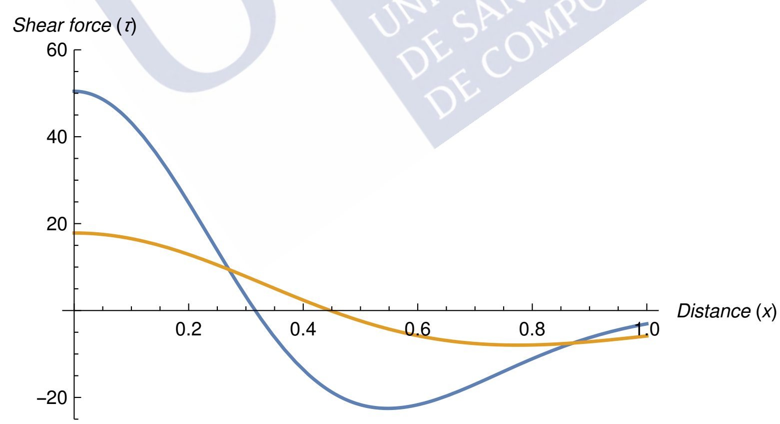

Figure 5.11: Representation of $\tau\left(\frac{k}{5},x\right)$ in Equation (5.82), where $k \in \{1,2\}$.

Now we fix the variable x for some particular distances and we represent the deflection and the shear force as function of t.

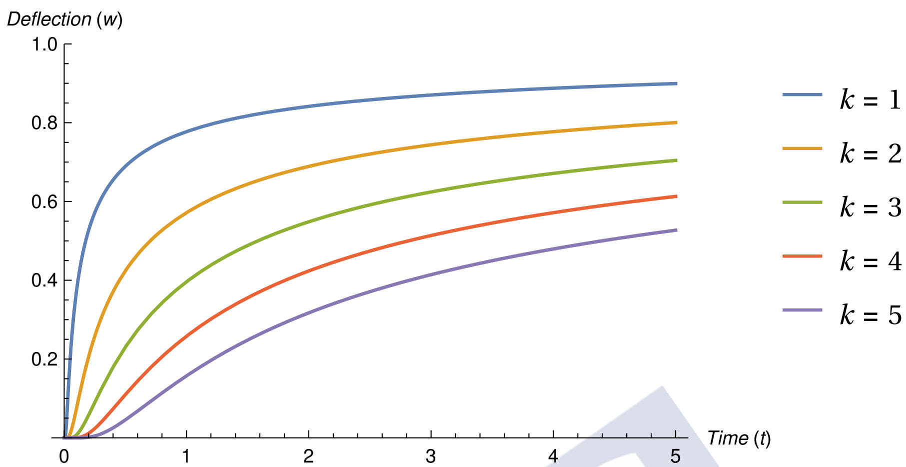

Figure 5.12: Representation of $w\left(t, \frac{k}{5}\right)$ in Equation (5.81), for the values $k \in \{1, 2, 3, 4, 5\}$.

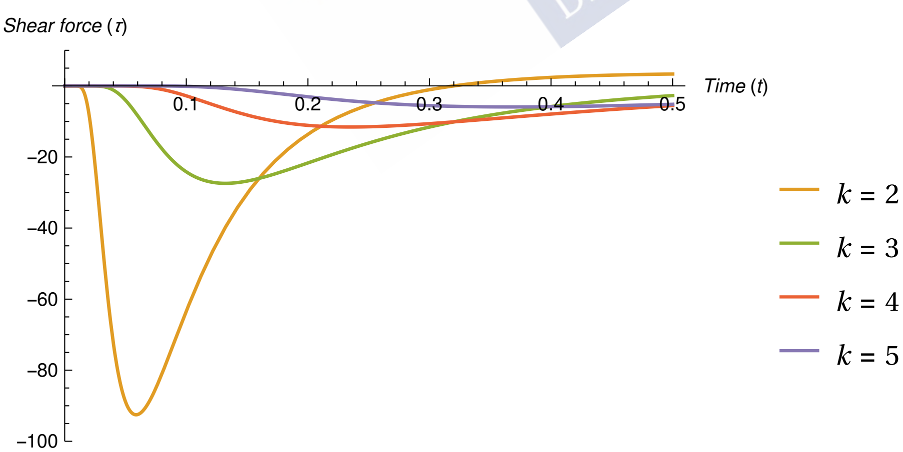

Figure 5.13: Representation of $\tau\left(t, \frac{k}{5}\right)$ in Equation (5.82), for the values $k \in \{2, 3, 4, 5\}$.

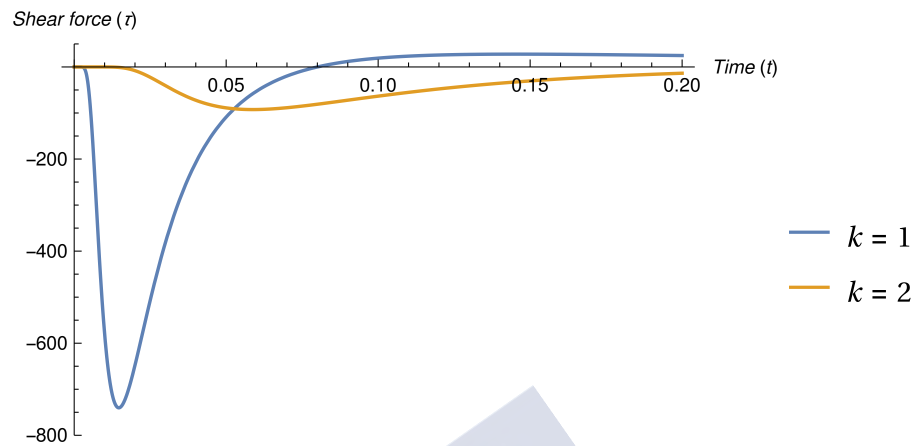

Figure 5.14: Representation of $\tau\left(t, \frac{k}{5}\right)$ in Equation (5.82), where $k \in \{1, 2\}$.

Fig. 5.9 shows the deflection of the beam (5.81) for different values of time. Figs. 5.10 and 5.11 show the shear force (5.82) at each point of the beam for different values of time. The main observation is that, for small times, the zone near the origin of the beam has a high positive shear force. The shear decays in space, it is zero at some point depending on t and we find a latter region with a high negative shear force. Furthermore, provided that the beam is long enough, the absolute value of the shear force becomes small for the points far, in space, from the origin.

It is possible to see in the charts that, as time passes, this extreme behaviour diminishes and the shear force remains close to zero. We can compute the maximum and the minimum value of the shear force for a fixed instant. As we have commented, both values tend to zero when the time tends to infinity.

For any fixed t, the maximum for the expression (5.82) is achieved at $x = 0$ with a value of

$$\tau(t, 0) = -\frac{EI\zeta^3}{2\sqrt{\pi}} \frac{1}{t\sqrt{t}}.$$

The minimum is reached at $x = \frac{\sqrt{6t}}{\zeta}$ and it is given by

$$\tau\left(t, \frac{\sqrt{6t}}{\zeta}\right) = \frac{EI\zeta^3}{e\sqrt{e\pi}} \frac{1}{t\sqrt{t}}.$$

It is also straightforward to compute the point where the shear force vanishes, getting

$$\tau\left(t, \sqrt{\frac{2\,t}{\zeta^2}}\right) = 0.$$

Furthermore, the accumulated shear force in the beam is zero for any time. This can be proved analytically since, from (5.82), we know that

$$\int_0^\infty \tau(t,x)\,dx = -\frac{xEI\zeta^3}{2\sqrt{\pi}} \frac{e^{-\frac{\zeta^2 x^2}{4t}}}{t^{3/2}}\Bigg|_0^\infty = 0.$$

Fig. 5.12 depicts how the deflection at any point tends to one when time increases. Figs. 5.13 and 5.14 reveal that, at the selected points, the shear force is initially zero and decreases until it reaches a negative minimum. Later, it increases until it achieves a positive maximum and, finally, it decreases slowly to zero.

The minimum occurs at $t = \zeta^2 x^2 \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)$. We obtain the value

$$\tau(t,x) = \frac{18EI\, e^{-\sqrt{\frac{3}{2}}-\frac{3}{2}}}{\left(3-\sqrt{6}\right)^{\frac{5}{2}}\sqrt{\pi}} \cdot \frac{1}{x^3}.$$

The maximum is found at $t = \zeta^2 x^2 \left(\frac{1}{2} + \frac{1}{\sqrt{6}}\right)$ and it is given by

$$\tau(t,x) = -\frac{18EI\, e^{\sqrt{\frac{3}{2}}-\frac{3}{2}}}{\left(3+\sqrt{6}\right)^{\frac{5}{2}}\sqrt{\pi}} \cdot \frac{1}{x^3}.$$

Finally, the accumulated shear force that suffers any fixed point different from the origin is zero because, provided that $x > 0$, we have

$$\int_0^\infty \tau(t,x)\,dt = \frac{EI\zeta^3}{\sqrt{\pi}} \frac{e^{-\frac{\zeta^2 x^2}{4t}}}{\sqrt{t}}\Bigg|_0^\infty = 0.$$

Chapter 6

Conclusions and future work

We end this Thesis by summarizing the main results that have been achieved and stating some others that would be interesting to deal with in the near future.

6.1 Conclusions

The most important conclusions are the following:

- In Section 1.3, we prove an original result. It ensures that a continuous bijection from a path-connected topological space to a topological space with the order topology is always a homeomorphism.

- In Subsection 2.1.5, we prove that there is a unique continuous extension for the Stieltjes integration with smooth integrator to fractional orders, such that the Index Law holds.

- In Subsection 2.2.3, we prove that it is not possible to find a definition for a fractional derivative such that the Index Law holds, provided that some reasonable assumptions are made.

- In Chapter 3, we have constructed a new method to solve linear fractional integral equations of constant coefficients. This method turns the fractional integral problem into an integer order problem, after using some properties concerning polynomials and roots of unity. This problem is solved after being transformed to an ODE. To achieve this result, we set some extra hypotheses involving the smoothness of the source term or that the integral orders are rational numbers. However, we are able to avoid them in posterior results that allow to turn non-smooth problems into smooth ones, after some iterations of a "smoothing" algorithm, and to put problems with irrational orders as the limit of the rational ones.

- In Chapter 4, we have illustrated the implications of our previous study on fractional integral equations in terms of fractional differential equations for Riemann-Liouville derivatives. We have shown how a linear fractional differential equation with constant coefficients of order β has $\lceil \beta \rceil$ linearly independent solutions only in a "weak" sense. In general, the space of solutions in a strong sense may have lower dimension than $\lceil \beta \rceil$. This dimension is, specifically, $\lceil \beta - \beta_* \rceil$, where β_* is the highest order in the equation such that $\beta - \beta_*$ is not an integer. In case that such a β_* does not exist, we simply define $\beta_* = 0$. We also conclude that, to deduce the uniqueness of solution, it makes sense to impose the initial values for the derivatives of orders $\beta - 1, \dots, \beta - \lceil \beta - \beta_* \rceil$.

- In Section 5.1, we have used the previous results to study a family of fractional differential equations involving Caputo derivatives. We have transformed the Caputo problem into a similar one, but with Riemann-Liouville derivatives. After applying some of our results, we have shown how our method applies to solve analytically a particular example of the Basset problem.

- In Section 5.2, we have established a relationship between LT and FC. Most specifically, we considered a magnitude R that can be described as a time integral from the point of view of one observer. Then, we have given a explicit description of the choices for the relative velocity v, between the previous observer and a new one, implying that R is described as a fractional integral from the point of view of the second observer. More generally, we have given conditions on v, see (5.44), so that R is described as a linear combination of fractional integrals, see (5.45). A particular case occurs when $\sqrt{1 + kv^2(s)}$ is an analytical function, where the linear combination of fractional integrals is, in fact, a combination of integer order repeated integrals. Moreover, we have discussed and illustrated specific numerical examples showing this behaviour and the convergence of the partial sums in the linear combination. Finally, we have presented a thought experiment where the, a priori unknown, relative velocity between two observers can be computed through the solution to a fractional integral equation involving other known data.

- In Section 5.3 we have shown a time fractional dependence for the shear force in a beam in the absence of a transverse load. The main assumptions are that the beam is modelled by the Euler-Lagrange equation, and that it is composed by a material with a negative Young modulus. More specifically, we find restrictions on the boundary conditions ensuring that the equation stating the fractional dependence, which is Equation 5.73, holds. Finally, we give some graphical representations and perform different calculations for several types of beams.

6.2 Future work

In this section, we include some possible future lines of research:

- The same idea used in Chapter 3, could be extended for integration of several variables. However, additional technical questions arise that need to be studied in detail. An interesting work for the future would be to extend the results in Chapter 3 for several variables.

- Analogously to the previous point, it seems that similar results to the ones obtained in Chapter 4 could be obtained for fractional PDE. We hope to explore this idea more deeply in the near future.

- It has been shown that, under certain hypotheses, the shear force in a beam can be described in terms of the fractional derivative of order $\frac{3}{2}$ of the deflection with respect to time. In a previous study [76], addressing the shear stress in a fluid system, it was shown that the stress depends on a second order spatial derivative of the displacement. However, when deriving its expression with respect to a time derivative, the order was found to be $\frac{3}{2}$, similarly to what occurs in our case. We recall that this conclusion was obtained after some restrictions on C_2, C_3 and C_4 were imposed, in order to ensure the applicability of the LT. Overcoming this limitation, via other techniques, may be an interesting research problem for the future.

We also expect to combine some of the previous results with other already developed work, which was not deeply discussed in this dissertation [8, 9, 17].

Glossary

$I^{\alpha}_{a^+}$ Left Riemann-Liouville fractional integral of order α and base point a. 41

Aut(X) Set of bijective linear maps from the Banach space X to itself (automorphisms of X). 16

Aut$_B(X)$ Set of bounded bijective linear maps from the Banach space X to itself (automorphisms of X). 16

End(X) Set of linear maps from the Banach space X to itself (endomorphisms of X). 16

End$_B(X)$ Set of continuous linear maps from the Banach space X to itself (bounded endomorphisms of X). 16

Mor(X, Y) Set of linear maps from the Banach space X to the Banach space Y (morphisms from X to Y). 15

Mor$_B(X, Y)$ Set of continuous linear maps from the Banach space X to the Banach space Y (bounded morphisms from X to Y). 15

$*$ Convolution operation. 26

$AC[a, b]$ Set of absolutely continuous functions. 19

$AC^n[0, b]$ Set of absolutely continuous functions of order n. 57

B Beta function. 31

C_a Convolution operator with base point a. 25

$D^{C,\alpha}_{0^+}$ Left Caputo fractional derivative of order α and base point a. 63

$D^{\alpha}_{0^+}$ Left Riemann-Liouville fractional derivative of order α and base point a. 59

$I_{a^+}^n$ Integral operator with base point a applied n times. 40

I_{h,a^+}^α Left Riemann-Liouville fractional integral of order α and base point a with respect to the function h. 53

$L^1[a, b]$ Set of integrable functions on $[a, b]$. 18

Γ Gamma function. 31

$\mathbb{N}$ The set of natural numbers $\{0, 1, \dots\}$. 31

$\mathbb{R}^+$ The set of positive real numbers. 14

$\mathbb{R}^n$ The set of vectors with n real coordinates. 14

$\mathbb{Z}^+$ The set of positive integer numbers $\{1, 2, \dots\}$. 29

$\mathbb{Z}$ The set of integer numbers. 31

$\mathcal{C}[a, b]$ Set of continuous functions on $[a, b]$. 27

$\mathcal{C}^{n-1}[a, b]$ Set of $n - 1$ times differentiable functions on $[a, b]$. 47

$\mathcal{L}$ Laplace transform. 30

$\mathcal{X}_\alpha$ Set of functions with summable Riemann-Liouville derivative of order α. 58

$\mathcal{Y}_\alpha$ Set of functions with summable Caputo derivative of order α. 63

$\mathrm{Comp}_{(\cdot,f)}$ Left composition operator with f. 17

$\mathrm{Comp}_{(g,\cdot)}$ Right composition operator with g. 17

Comp Composition operator ($\mathrm{Comp}(g, f) = f \circ g$). 16

Ind_a Index map defined as $\mathrm{Ind}_a(\alpha) = I_{a^+}^\alpha$. 47

FC Fractional Calculus. 13

GR General Relativity. 108

GT Galilean transformation. 108

ILaT Inverse Laplace transform. 135

LaT Laplace transform. 29

LT Lorentz transformations. 108

ODE Ordinary differential equation. 83

PDE Partial differential equation. 131

SR Special Relativity. 107

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