INTERNATIONAL DOCTORAL
SCHOOL OF THE USC

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PhD Thesis

SPECTRAL THEORY AND GREEN'S FUNCTIONS RELATED TO NONLOCAL DIFFERENTIAL EQUATIONS. APPLICATION TO NONLINEAR PROBLEMS

Santiago de Compostela, 2024

Doctoral Programme in Mathematics

ESCOLA DE DOUTORAMENTO
INTERNACIONAL DA USC

DOCTORAL THESIS

SPECTRAL THEORY AND GREEN'S FUNCTIONS RELATED TO NONLOCAL DIFFERENTIAL EQUATIONS. APPLICATION TO NONLINEAR PROBLEMS

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PHD PROGRAMME IN MATHEMATICS

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A mi familia,
a mis padres Malika y Ali
y a mi hermano Oussama.

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Agradecimientos

En primer lugar, deseo expresar mi agradecimiento a mis directores de tesis: Alberto Cabada Fernández y Lucía López Somoza. Me acompañaron y apoyaron cada día durante toda la etapa de realización de la tesis dando consejos y sugerencias, y sobre todo calma y seguridad ante cualquiera de los problemas encontrados. Para mi además de que sean directores de tesis han sidos también amigos míos. Sin ellos ha sido imposible llevar a cabo esta tesis hasta el final. Gracias de nuevo a los dos por haberlo hecho posible.

Gracias a mi familia que forma parte de mis estudios, mi carrera y toda mi vida. A mis padres, Malika y Ali, por guiarme por el buen camino de adquirir conocimiento e interesarse en las cosas más importantes. Gracias a mi hermano Oussama que me acompañó y apoyó siendo él, además de mi hermano, un gran amigo. Gracias a mi prima Asinat por formar parte de nuestra casa y aportarme motivación para seguir luchando.

Gracias a todos los compañeros y todas las compañeras del Área de Análisis Matemático por hacer de este sitio un lugar de aprendizaje y colaboración. Quiero agradecer a Juan José Nieto Roig, el director del grupo de investigación, "Ecuaciones diferenciales no lineales GI-1561" de USC, Sebastián Buedo Fernández, Daniel Cao Labora, Érika Diz Pita, Francisco Fernández Fernández, Adrián Fernández Tojo, Rodrigo López Pouso, Jorge Losada Rodríguez, María Victoria Otero Espinar, Jorge Rodríguez López, Rosana Rodríguez López y Rosa María Trinchet Soria,

Gracias a mis amigos: Noé, Raúl, Aziz, Hamid, Hamza y Mustafá por estar siempre a mi lado y pasando cada momento juntos.

Quiero animar a todos los doctorandos a seguir luchando por sus sueños y contra los obstáculos que suelen surgir a lo largo de la realización de sus tesis.

Vuelvo a agradecer a todos los han que estado ayudándome en cualquier momento de mi etapa doctoral y no he mencionado anteriormente.

Mouhcine Yousfi Khoumsi

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Glossary

Throughout the work we will use the notations defined below:

$\mathbb{N}$: Set of natural numbers.

$\mathbb{R}$: Set of real numbers.

$\mathbb{R}^+$: Set of positive real numbers.

$C(I) \equiv C(I, \mathbb{R})$: Space of continuous real functions defined on an interval I.

$C^n(I) \equiv C^n(I, \mathbb{R}),\ n \in \mathbb{N}$: Space of n-times differentiable real functions defined on an interval I such that the j-th derivative is continuous for all $j = 0, \ldots, n$.

$C^\infty(I) \equiv C^\infty(I, \mathbb{R})$: Space of infinitely differentiable real functions defined on an interval I.

$L^\alpha(I), 1 \leq \alpha < \infty$: Space of the Lebesgue measurable functions f on the interval I such that the Lebesgue integral $\int_I |f(t)|^\alpha \, dt$ is finite.

$\|f\|_\alpha, 1 \leq \alpha < \infty$: Norm of f in the space $L^\alpha(I)$, that is,

$$\|f\|_\alpha = \left(\int_I |f(t)|^\alpha \, dt \right)^{\frac{1}{\alpha}}.$$

$L^\infty(I), 1 \leq \alpha < \infty$: Space of the Lebesgue measurable functions f on the interval I such that are essentially bounded.

$\|f\|_\infty, 1 \leq \alpha < \infty$: Norm of f in the space $L^\infty(I)$, that is,

$$\|f\|_\infty = \operatorname{ess\,sup} \{|f(t)|,\ t \in I\}.$$

$AC(I)$: Space of absolutely continuous functions, that is,

$$AC(I) := \{u \in C(I) : \ \exists f \in L^1(I) \ \text{ such that } u(t) = u(t_0) + \int_{t_0}^{t} f(s)\, ds;\ t,\ t_0 \in I\}.$$

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$W^{k,p}(I), k, p \in \mathbb{N}$: Sobolev space on the set I, that is,

$$W^{k,p}(I) := \Big\{ u \in C^{k-1}(I) : \ \ u^{(k-1)} \in AC(I),\ u^{(k)} \in L^p(I) \Big\}.$$

$\mathcal{M}_{n\times n}(\mathbb{R})$: Set of the $n \times n$ matrices with real coefficients.

$\bar{\Omega}$: Closure of the set Ω.

$\partial\Omega$: Boundary of the set Ω.

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Introduction

The study of the existence and uniqueness of solutions of ordinary differential equations coupled to functional and two-point boundary conditions represents a fundamental line in the study of mathematical research. This is due to the fact that practically all physical phenomena, as well as many others in economics, biology or epidemiology, are modelled by equations of this type.

Our study will focus on the qualitative properties of the solutions of ordinary differential equations coupled to functional and two-point boundary conditions with applications to nonlinear problems, focusing on the study of solutions of constant sign in all of their interval of definition. This phenomenon owes its interest to the fact that many of the magnitudes that arise in the problems modelled by the equations considered can only take nonnegative values (pressure, power, temperature in Kelvin degrees, number of individuals affected by a disease,...). At the same time, in many engineering problems, models study the deviation from the equilibrium point of certain structures. This deformation, in order to keep the structure stable, must occur in the same direction in which the force is applied, which mathematically translates into a solution of constant sign.

The techniques that we will use to guarantee the existence of solutions to this type of problems are based on the abstract formulation of an equivalent problem, in such a way that the sought solutions coincide with the fixed points of the corresponding integral operators defined in abstract spaces. These integral operators will be determined by a kernel related to the linear part of the equation, called the Green's function, which contains much of the information of the linear part of the equation considered along with the studied boundary conditions. Green's functions have been extensively studied over the past few decades. Studies that refer to its basic properties of, among others, existence, uniqueness, sign, bounded or symmetry, can be consulted in the references [6, 39].

It is well-known that the constant sign of Green's functions is a fundamental tool to ensure the existence of constant sign solutions of the related nonlinear problems, since it is a basic assumption to apply some classical methods such as lower and upper solutions, monotone iterative techniques, Leray-Schauder degree theory (see [14, 55, 56]), or fixed point theorems on cones.

Moreover, in much of the fixed point theory, operators are defined in subspaces called cones, which allow the properties of Green's function to be transferred to the solution of the considered problem, at the same time, monotone iterative methods allow to obtain the sought solutions as a limit of sequences formed by solutions of associated linear systems. In this case, the sign of the Green's function will be decisive in order to ensure the convergence of

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the successions thus constructed. In this way, the more information we obtain from Green's function, the more properties we can deduce for the possible solutions of the problem under study. It is important to note that in general the expression of these functions is very complicated, being very difficult to obtain the optimal values of the different parameters involved in the equation and for which the Green's function has constant sign.

In most of the syllabus of the Thesis we will focus on ordinary differential equations with nonlocal functional boundary conditions. Inclusion in the boundary conditions of a functional condition will cause the solution of the equation of the problem studied at a point not to depend solely on what happens at that point, but also on what happens at other different points or even throughout the definition interval.

For example, in the case of considering an epidemiological model, in which the evolution of a pandemic in a population is studied (such as Coronavirus), it may happen that daily cases do not depend on the contagion produced on that same day, but on the contagion produced in the previous days. The inclusion of such variants in differential equations requires a functional condition.

Another biological example to consider might be to study the evolution of a fish population subject to fishing with possible prohibition times. In this sense, if the reproduction of the fish is seasonal, the number of fish at a given time depends on the number of fish that have reproduced at a previous time (which may be at a certain distance in time, depending on the time elapsed between the laying of the eggs and their hatching). In the same way, the number of fish that reproduce will depend on the fishing of a previous period of time. The study and analysis of this type of models, which are fundamental to determine the levels of fishing that allow for a sustainable situation, also involve differential equations with functional conditions.

Another application outside the biological field is in the infrastructure study area, it is worth pointing out some of the possible applications of the equations with functional conditions to the study of the oscillation of a suspension bridge subject to a certain load. In this case, the model would be represented by a system of differential equations that could also include functional type conditions, this is because the deformation of the beam at a given point depends on the forces affecting the entire beam and not just those that apply at that particular point.

So, we will focus our research on the study of ordinary differential equations coupled to nonlocal boundary conditions with applications to problems of order two subjects to functional conditions. We will extend this study to nonlinear systems with non-separated variables coupled to arbitrary nonlocal boundary conditions. In this sense, we will follow the ideas made in this line in [42, 69, 84] for integral operators.

We will study comparison results between Green's functions related to the Hill's equation subject to different boundary conditions. The relationship between the respective spectrums of the different considered problems will also be studied. It is important to note that in the works [9, 10] the authors found strong relationships between Green's functions related to various boundary value problems. Meanwhile, our study consists of finding relationships between Green's functions following a different approach than that developed in previous works.

Finally, the spectral characterization of the constant-sign derivatives of Green's function of a differential equation of order n coupled to two-point boundary conditions will be ad-

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dressed, following the ideas made in the works [24, 26].

The present Thesis is divided into eight chapters. The content of these chapters, except for the preliminary chapter, contain the works carried out by the author in recent years, which can be consulted in the references [17–23].

Objectives and Hypotheses

The main objective of the Thesis is to find results that allow the study of certain boundary value problems, from the theoretical framework, which can be later applied within the different sectors of the industry. At has been mentioned before, the study of these types of problems and their possible solutions is crucial due to their regular occurrences in various fields such as economics, physics, chemistry, biology, and epidemiology.

The specific objectives in which the work is framed are the following:

- Study of $n^{\text{th}}-$order differential equations coupled to nonlocal functional boundary conditions. We obtain results that prove the existence of a solution and, in particular, of constant sign solutions of the considered equations.
- Study of second order Dirichlet and periodic problems perturbed by integral boundary conditions.
- Relations and comparison results between the Green's functions of the Hill's equation subject to different boundary conditions.
- Obtaining the spectrum of the Hill's equation as a function of the related Green's functions coupled to different boundary conditions using the study made in the previous item.
- Deducing the existence of solutions to problems with certain boundary conditions from the existence of solutions to problems with other boundary conditions.
- Study of the existence of solutions to nonlinear systems subject to arbitrary nonlocal linear boundary conditions.
- Spectral characterization of the constant sign derivatives of the Green's function of a $n^{\text{th}}-$order differential equation coupled to two-point boundary conditions.

Methodology

To study the existence of constant sign solutions of functional nonlinear ordinary differential equations, we will use the classical results of fixed points of operators defined in Banach spaces [8, 43, 44, 62, 90]. This is due to the fact that, as has been indicated before, the considered problem can be transformed into another equivalent integral problem in which its kernel is the Green's function of the studied problem.

We obtain the formula and comparison results of Green's function of functional differential equations coupled to arbitrary nonlocal linear boundary conditions. In particular, the

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second order Dirichlet and periodic problems perturbed by integral boundary conditions are studied. The study of Dirichlet problem generalizes the work done in [13].

To deduce the existence of a solution to problems with boundary conditions from the existence of a solution to problems with different boundary conditions, we will use two approaches: the first one will be based on the superposition property of the solutions of differential problems. On the other hand, the second one will make use of a general formula of the Green's function, which allows to relate two different Green's functions as long as the boundary conditions of one of them can be expressed in terms of the other and both problems are nonresonant. This approach is different from the techniques developed in [9]. The spectrum of problems coupled to boundary conditions is also characterized by deducing inclusion relationships between different spectra.

As for linear systems with non-separated variables subject to arbitrary nonlocal linear boundary conditions, we will obtain a generalization of the formula of Green's function obtained in the scalar case and deduce the existence of solutions of the nonlinear problem following the techniques used in [69, 84].

Finally, by means of spectral theory, we will characterize the constant sign of the derivatives of the Green's function of a differential equation of order n subject to two-point boundary conditions. We will follow the approach developed in the papers [24, 26] to deduce more general results that include those obtained in these works, but, in this case, for a more particular differential operator.

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Resumo

A presente Tese, titulada "*Teoría Espectral e Funcións de Green Relacionadas con Ecuacións Diferenciais Non Locais. Aplicación a Problemas Non Lineares*" recolle o traballo realizado polo autor nos últimos anos durante os seus estudos de doutoramento na Universidade de Santiago de Compostela. Os traballos obtidos desenvolvéronse coa colaboración dos directores de tese Alberto Cabada (Universidade de Santiago de Compostela) e Lucía López-Somoza (Universidade de Santiago de Compostela).

O texto divídese en oito capítulos. O Capítulo 1 recolle os preliminares que necesitaremos para desenvolver o resto dos capítulos. Os Capítulos 2, 3 e 4 tratan problemas de ecuacións diferenciais ordinarias con condicións de fronteira funcionais non locais para o caso escalar. Os Capítulos 3 e 4 son unha aplicación do Capítulo 2 a problemas de Dirichlet e periódico de segunda orde, pertubados por condicións de fronteira integrais.

Nos Capítulos 5 e 6 estudaremos as relacións de comparación entre as expresións das funcións de Green e os espectros da ecuación de Hill con diferentes condicións de fronteira de dous puntos. Tamén se incluirá unha aplicación a problemas non lineares.

No Capítulo 7 xeneralízase a fórmula escalar obtida no Capítulo 2 a sistemas suxeitos a condicións de fronteira lineares non locais arbitrarias e inclúese unha aplicación a problemas non lineares.

Finalmente, o Capítulo 8 está adicado ao estudo, por medio da teoría espectral, do signo constante das derivadas da función de Green asociada a un operador de orde n con condicións de fronteira de dous puntos.

A continuación resumimos os contidos e os resultados obtidos en cada un dos capítulos mencionados anteriormente.

Capítulo 1: Preliminares

Este capítulo, como xa se indicou anteriormente, e co fin de que a memoria sexa o máis autocontida posible, está adicado á presentación de preliminares e resultados necesarios para desenvolver os capítulos restantes.

Na primeira parte comezaremos definindo o concepto de función de Green de operadores lineares xerais de orde n suxeitos a condicións de fronteira lineares. Preséntanse as propiedades destas funcións e a súa importancia na resolución de problemas lineares e non lineares. Tamén introduciremos o concepto de función de Green no contexto de sistemas de primeira orde con condicións de fronteira de dous puntos.

Tamén se introducen as condicións que deben cumprir as funcións de Green e o operador diferencial para asegurar a monotonía e o signo constante das funcións de Green con respecto

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a un parámetro real implicado na ecuación que define ao operador.

Na última parte recollemos os principais teoremas clásicos de punto fixo, entre os que destacamos os teoremas de punto fixo de Schaefer e Krasnoselskii, e o teorema de caracterización de operadores compactos de Ascoli-Arzelà.

Capítulo 2: Funcións de Green con Condicións de Fronteira Non Locais Arbitrarias

Neste capítulo obtemos a expresión explícita da función de Green relativa ao seguinte problema

$$\begin{cases} T_n[M]\,u(t) = \sigma(t), & t \in I := [a,b], \\ B_i(u) = \delta_i\, C_i(u), & i = 1,\dots,n, \end{cases} \tag{1}$$

sendo

$$T_n[M]\,u(t) := L_n u(t) + M\,u(t), \quad t \in I,$$

con

$$L_n u(t) := u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_n(t)u(t), \quad t \in I,$$

e σ e a_k funcións continuas para todo $k = 1,\dots,n$, $M \in \mathbb{R}$ e $\delta_i \in \mathbb{R}$ para todo $i = 1,\dots,n$.

Ademais, $C_i : C(I) \to \mathbb{R}$ é un operador linear continuo e B_i son as condicións xerais de fronteira linear de dous puntos, é dicir,

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1,\dots,n,$$

sendo $\alpha_j^i,\ \beta_j^i$ constantes reais para todo $i = 1,\dots,n$ e $j = 0,\dots,n-1$.

O problema (1) cobre unha gran familia de problemas de fronteira dependendo dos funcionais lineares continuos C_i, $i = 1,\dots,n$, considerados. Entre estes problemas podemos atopar problemas integrais e multipunto ou unha combinación de ambos.

Caracterizaremos o espectro do problema (1) baixo o suposto de que o problema homoxéneo relacionado

$$\begin{cases} T_n\,[M]\,u\,(t) = 0, & t \in I, \\ B_i\,(u) = 0, & i = 1,\dots n, \end{cases} \tag{2}$$

ten unicamente a solución trivial.

O noso principal resultado é o seguinte, que proba a existencia e unicidade da solución do problema (1) baixo hipóteses axeitadas:

Teorema 1. *(Theorem 2.2.2) Supoñamos que o problema* (2) *ten* $u = 0$ *como a súa única solución e sexa* g_M *a súa función de Green asociada. Sexan* $\sigma \in C(I)$ *e* $\delta_i \in \mathbb{R}$, $i = 1,\dots,n$ *tales que*

$$\det(I_n - A) \neq 0, \tag{3}$$

con I_n *a matriz identidade de orde* n *e* $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *dada por*

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1,\dots,n\}.$$

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Entón o problema (1) *ten unha única solución* $u \in C^n(I)$*, dada pola expresión*

$$u(t) = \int_a^b G(t, s, \delta_1, \ldots, \delta_n, M)\, \sigma(s)\, ds,$$

onde

$$G(t, s, \delta_1, \ldots, \delta_n, M) := g_M(t, s) + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j\left(g_M(\cdot, s)\right), \quad t,\ s \in I, \quad (4)$$

sendo ω_i*, para* $i \in \{1, \ldots, n\}$*, a única solución do problema*

$$\begin{cases} T_n[M]\, u(t) = 0, & t \in I, \\ B_j(u) = 0, & j \in \{1, \ldots, n\} \backslash \{i\}, \\ B_i(u) = 1, & \end{cases}$$

e $B = (b_{ij})_{n \times n} = (I_n - A)^{-1}$.

A demostración do resultado anterior baséase no principio de superposición de solucións, a linearidade dos funcionais continuos C_i e certos argumentos da álxebra linear.

Como consecuencia deste teorema obtemos a expresión exacta da función de Green relativa ao problema (1). Ademais, chegamos ao seguinte resultado de comparación:

Corolario 2. *(Corollary 2.2.7) Supoñamos que todos os operadores lineares* C_i *son iguais, é dicir,* $C_i = C$*,* $i = 1, \ldots, n$*. Sexan* G *e* g_M *as funcións de Green relativas aos problemas* (1) *e* (2)*, respectivamente. Supoñamos que o problema* (2) *ten* $u = 0$ *como a súa única solución e que se cumpre a condición* (3)*.*

Ademais, supoñamos que se satisfán as seguintes hipóteses:

(a) $\displaystyle\sum_{j=1}^{n} \delta_j\, C(\omega_j) < 1.$

(b) $\delta_i\, \omega_i(t) \geq 0$*,* $\forall t \in I, \quad i = 1, \ldots, n.$

(c) Se $u \geq 0$ *en* I*, entón* $C(u) \geq 0$*.*

Entón, cúmprense as seguintes afirmacións:

(i) Se $g_M(t, s) \geq 0$*, para todo* (t, s) *en* $I \times I$ *entón*

$$G(t, s, \delta_1, \ldots, \delta_n, M) \geq g_M(t, s) \geq 0$$

para todo (t, s) *en* $I \times I$*.*

(ii) Se $g_M(t, s) \leq 0$*, para todo* (t, s) *en* $I \times I$ *entón*

$$G(t, s, \delta_1, \ldots, \delta_n, M) \leq g_M(t, s) \leq 0$$

para todo (t, s) *en* $I \times I$*.*

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A fórmula (4) empregarase nos capítulos posteriores para obter a expresión da función de Green de problemas de segunda orde con condicións de fronteira funcionais sempre que o problema homoxéneo teña unicamente a solución trivial e se satisfaga a condición non espectral (3). En caso contrario, tal fórmula deixa de ser válida e a correspondente función de Green determínase por medio de integración directa. Tamén empregaremos a fórmula (4) para relacionar funcións de Green relativas á ecuación de Hill asociada a distintas condicións de fronteira.

Capítulo 3: Problema Periódico Perturbado de Segunda Orde

Neste capítulo estúdase o signo constante da función de Green G_{M,δ_1,δ_2} asociada ao seguinte problema periódico perturbado de segunda orde, con parámetros e condicións de fronteira integrais:

$$\begin{cases} u''(t) + M\,u(t) = \sigma(t), \quad t \in J := [0,1], \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{5}$$

onde $M, \delta_1, \delta_2 \in \mathbb{R}$.

Nótese que a formulación deste problema é un caso particular dos problemas considerados no capítulo anterior.

En primeiro lugar, calculamos o espectro do problema estudado e caracterizamos os puntos onde a función de Green se anula cando ten signo constante. Despois, empregando a fórmula (4) aplicada ao noso caso, con

$$C_1(u) = C_2(u) = \int_0^1 u(s)\,ds,$$

obténse a expresión exacta da función de Green para o caso $M \neq 0$, xa que este non é un autovalor do problema homoxéneo. Pola contra, obteremos a expresión para o caso $M = 0$ por integración directa, xa que estamos ante un autovalor do problema homoxéneo.

Para abordar o estudo do problema (5), considérase en primeiro lugar o caso particular $\delta_1 = 0$ e calcúlanse as rexións de signo constante da función de Green en dito caso. A continuación, estúdase o caso $\delta_1 \neq 0$ a partir do anterior.

Deste modo, próbase a existencia de funcións f, g e k que delimitan as rexións de signo contante da función de Green, tal e como se indica a continuación:

Teorema 3. *(Theorem 3.4.8) Sexan $M = 0$ e $\delta_2 \in (-8, 0)$, entón $G_{0,\delta_1,\delta_2} > 0$ en $J \times J$ se, e só se,*

$$|\delta_1| < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8, -4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Teorema 4. *(Theorem 3.4.4) Sexan $M < \pi^2, M \neq 0$ e $\delta_2 \in (M, f(M))$, entón G_{M,δ_1,δ_2} é estritamente negativa en $J \times J$ se, e só se,*

$$|\delta_1| < 2\,(M - \delta_2)\,G_{M,0,\delta_2}\left(1, \frac{1}{2}\right).$$

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Teorema 5. *(Theorem 3.4.6) Supoñamos que $M = m^2$, $m \in (0, \pi)$ e $\delta_2 \in (g(M), M)$. Entón, cúmprense as dúas propiedades seguintes:*

1. *Se $\delta_2 \in (k(M), M)$, entón G_{M,δ_1,δ_2} é estritamente positiva en $J \times J$ se, e só se,*

$$|\delta_1| < m \cot\left(\frac{m}{2}\right).$$

2. *Se $\delta_2 \in (g(M), k(M)]$, entón G_{M,δ_1,δ_2} é estritamente positiva en $J \times J$ se, e só se,*

$$|\delta_1| < \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)\left(m^2 - \delta_2\right)^2}}{m}.$$

Capítulo 4: Problemas de Dirichlet Non Lineares Perturbados por Condicións de Fronteira Integrais

Este capítulo é o primeiro en considerar os problemas de fronteira non lineares.

Seguiremos un estudo similar ao realizado co problema discutido no capítulo anterior para abordar o problema de Dirichlet non linear de segunda orde perturbado por condicións de fronteira integrais. A idea básica consiste en aplicar a fórmula (4) para deducir a expresión exacta da función de Green asociada ao problema linear.

En concreto, consideramos as seguintes ecuacións non lineares

$$u''(t) + \gamma\, u(t) + f(t, u(t)) = 0, \quad t \in J,$$

e

$$u''(t) + \gamma\, u(t) - f(t, u(t)) = 0, \quad t \in J,$$

suxeitos ás condicións de fronteira integrais

$$u(0) = \delta_1 \int_0^1 u(s)\, ds, \quad u(1) = \delta_2 \int_0^1 u(s)\, ds,$$

sendo γ, δ_1, δ_2 tales que $\gamma < \pi^2$, $\delta_1 \geq 0$ e $\delta_2 \geq 0$, e coa parte non linear f cumprindo hipóteses de regularidade e signo axeitadas.

Co fin de abordar a existencia de solucións positivas a estes problemas non lineares, realízase un estudo previo do signo constante da función de Green asociada, en función dos valores dos parámetros γ, δ_1 e δ_2.

A continuación, empregaremos o teorema do punto fixo de Krasnoselskii para deducir a existencia de solucións positivas destes problemas non lineares baixo a hipótese de que a parte non linear da ecuación $f : J \times [0, \infty) \to [0, \infty)$ sexa unha función continua. En particular, próbase que se a función de Green é de signo constante entón existe unha solución positiva do problema no cono

$$K = \{u \in X \,/\, \min\{u(t) : t \in J\} \geq \bar{c}\, \|u\|_\infty\},$$

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onde $X \equiv (C(J), \|\cdot\|_\infty)$ é o espazo real de Banach das funcións continuas en J dotado da norma do supremo $\|u\|_\infty$ e $\bar{c}$ é unha constante que depende do valor dos parámetros implicados no problema.

Capítulo 5: A Ecuación de Hill Asociada a Diferentes Condicións de Fronteira Linear de Dous Puntos

Este capítulo trata o estudo das propiedades e relacións entre as funcións de Green relativas á ecuación de Hill

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a.e. } t \in J,$$

onde $\lambda \in \mathbb{R}$, $J = [0,1]$, $a : J \to \mathbb{R}$ e $a \in L^1(J)$, suxeita a diferentes condicións de fronteira.

En particular, consideramos a ecuación de Hill suxeita a condicións de fronteira de tipo Neumann, Dirichlet, periódicas e mixtas. Denotamos por $G_D[\lambda]$, $G_P[\lambda]$, $G_N[\lambda]$, $G_{M_1}[\lambda]$ e $G_{M_2}[\lambda]$ á función de Green asociada aos problemas de Dirichlet, periódico, Neumann, mixto 1 e mixto 2, respectivamente. Ademais, denotamos por λ_0^D, λ_0^P, λ_0^N, $\lambda_0^{M_1}$ e $\lambda_0^{M_2}$ os primeiros autovalores de cada problema.

Dado que a ecuación estudada é de segunda orde, sabemos que é posible utilizar a teoría de Sturm-Liouville para analizar as propiedades de oscilación das solucións. En xeral, esta teoría non é válida para as ecuacións diferenciais de orde superior.

A comparación e as relacións entre as funcións de Green lévanse a cabo desde dous enfoques diferentes: o primeiro baséase no principio de superposición de solucións e no feito de reescribir as condicións dun problema como unha combinación linear do outro, mentres que o segundo enfoque consiste nunha aplicación directa da fórmula (4). Isto conduce a resultados xa coñecidos (aínda que probados de forma diferente) e a outros novos na literatura sobre a comparación entre as funcións de Green. Resumimos as comparacións obtidas no seguinte resultado:

Corolario 6. *Cúmprense as seguintes desigualdades:*

1) $G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^P.$

2) $G_N[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^N.$

3) $G_{M_1}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^{M_1}.$

4) $G_{M_2}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^{M_2}.$

5) $G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1]\times(0,1], \ \ \forall\,\lambda < \lambda_0^N.$

6) $G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in [0,1)\times[0,1), \ \ \forall\,\lambda < \lambda_0^N.$

En xeral, non existe comparación entre $G_P[\lambda]$ e $G_N[\lambda]$ nin entre $G_P[\lambda]$ e $G_{M_i}[\lambda]$ para $i = 1, 2$, e daremos algúns exemplos para comprobalo no caso de problemas periódicos e mixtos.

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Finalmente, a última sección do capítulo está dedicada á aplicación dos resultados obtidos co segundo enfoque a problemas non lineares. En concreto, baixo supostos axeitados sobre a parte non linear f, estúdase a existencia de solucións do seguinte problema xeral non linear

$$\begin{cases} L_n\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \\ B_i(u) = \delta_i\, C_i(u), \quad i = 1, \dots, n. \end{cases}$$

A existencia de solucións para este problema pódese deducir a partir da existencia de solucións do siguiente problema con condicóns de fronteira homoxéneas, aplicando o Teorema de punto fixo de Schaefer,

$$\begin{cases} L_n\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \\ B_i(u) = 0, \quad i = 1, \dots, n. \end{cases}$$

Tamén se obtén unha estimación da distancia entre as solucións de ambos os problemas.

Os resultados de existencia e a estimación de distancia entre solucións pódense aplicar a todos os diferentes problemas non lineares de segunda orde asociados ás diferentes condicións de contorno mencionadas anteriormente.

En particular, baixo certas condicións de regularidade sobre a función f, podemos deducir a existencia da solución do problema de Dirichlet non linear

$$L[\lambda]\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \quad u(0) = u(1) = 0,$$

despois de probar a existencia da solución do problema periódico non linear

$$L[\lambda]\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \quad u(0) = u(1),\ u'(0) = u'(1),$$

e viceversa.

Capítulo 6: Espectros da Ecuación de Hill con Diferentes Condicións de Fronteira

Este capítulo é un complemento do anterior que continúa co estudo da ecuación de Hill asociada aos problemas de fronteira de dous puntos máis habituais: Neumann, Dirichlet, mixtas e periódicas. En concreto, abordarase o estudo dos espectros de ditos problemas obtendo relacións notables entre eles.

A razón principal que motiva o estudo do espectro é o feito de que os autovalores aparecen na procura de solucións de signo constante, dado que adoitan definir os límites das rexións nas que as funcións de Green asociadas teñen un signo positivo ou negativo (ver [26] para máis detalles).

O resultado principal obtido neste capítulo é o seguinte, que mostra unha relación de inclusión entre os espectros dos problemas (1) e (2):

Teorema 7. *(Theorem 6.1.1) Cúmprese que*

$$S_{NH} \subset \Lambda_{NH} \subset \Lambda_H \sqcup S_{NH}$$

onde $\sqcup$ *denota a unión disxunta,* $S_{NH} := \left\{ \lambda \in \mathbb{R} \setminus \Lambda_H \ / \quad \det(I_n - A) = 0 \right\}$, *con* A *a matriz definida no Teorema 1, e* Λ_H *e* Λ_{NH} *son os espectros dos problemas* (1) *e* (2)*, respectivamente.*

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Centrarémonos en amosar algunas relacións entre os espectros de diferentes problemas empregando o teorema anterior e as relacións obtidas no segundo enfoque do capítulo anterior. A partir destas relacións, deduciremos algunhas conexións entre varios espectros e algúns outros conxuntos caracterizados como os ceros dunha certa función.

Finalmente, aplicamos os resultados obtidos para deducir a existencia de solucións de problemas non lineares da forma

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T],$$

con diferentes tipos de condicións de fronteira, onde a parte non linear da ecuación $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfai as seguintes condicións de regularidade:

(H_1) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ é unha función $L^\infty([0, T])-$Carathéodory.

(H_2) $\lim\limits_{|u|\to\infty} \dfrac{f(t, u)}{u} = -\bar{\lambda}$, uniformemente en $t \in [0, T]\backslash\mathcal{M}$, con $\mathcal{M}$ un conxunto de Lebesgue de medida nula.

Supoñemos ademais que se cumpre a seguinte condición:

(H_3^D) $\omega_I^2[\bar{\lambda}](T) := G_I[\bar{\lambda}](T, 0) \neq 0,$

onde $G_I[\bar{\lambda}]$ é a función de Green asociada ao problema inicial

$$u''(t) + (a(t) + \bar{\lambda})\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \quad u(0) = u'(0) = 0.$$

Baixo as hipóteses previas chégase ao seguinte resultado de existencia de solucións non triviais para o problema de Dirichlet.

Teorema 8. *Supoñamos que se cumpren as condicións (H_1), (H_2) e (H_3^D). Entón, o problema de Dirichlet non linear*

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \textit{a.e. } t \in [0, T], \quad u(0) = u(T) = 0,$$

ten polo menos unha solución $u \in W^{2,1}([0, T])$.

Como consecuencia do resultado anterior e baixo a hipótese (H_3^D) e a seguinte condición de regularidade da parte non linear:

$(\tilde{H}_2)$ $\lim\limits_{u\to\infty} \dfrac{f(t, u)}{u} = -\tilde{\lambda}$, uniformemente en $t \in [0, T]\backslash\mathcal{M}'$, con $\mathcal{M}'$ un conxunto de Lebesgue de medida nula,

chegamos ao siguinte corolario.

Corolario 9. *(Corollary 6.2.3) Supoñamos que $f : [0, T] \times [0, \infty) \to (-\infty, 0]$ é unha función $L^\infty([0, T])-$Carathéodory, cumprindo as condicións $(\tilde{H}_2)$ e (H_3^D). Entón, para todo $\tilde{\lambda} < \lambda_0^D$, ó problema de Dirichlet non linear ten unha solución maior ou igual que cero en $[0, T]$.*

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Analogamente, danse resultados de existencia de solución para problemas con condicións Neumann, periódicas e mixtas.

Capítulo 7: Sistemas Non Lineares Suxeitos a Condicións de Fronteira Lineares Non Locais Arbitrarias

Neste capítulo estudamos a existencia de solución dun problema integral incluído na familia das chamadas ecuacións xeneralizadas de Hammerstein.

En particular, ocupámonos da existencia de solucións de sistemas non lineares de ecuacións integrais da seguinte forma:

$$\begin{cases} u(t) = \displaystyle\int_a^b G_1(t,s)\, f_1(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_2(t,s)\, f_2(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds, \quad t \in I := [a,b], \\ v(t) = \displaystyle\int_a^b G_3(t,s)\, f_1(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_4(t,s)\, f_2(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds, \quad t \in I, \end{cases}$$

sendo $n, m \in \mathbb{N}$, $G_i : I \times I \to \mathbb{R}$, $i = 1,2,3,4$, os núcleos integrais que satisfán certas condicións de regularidade, tales que $G_r \in W^{n-1,1}(I \times I)$, $r = 1,2$, $G_l \in W^{m-1,1}(I \times I)$, $l = 3,4$, e $f_i : I \times \mathbb{R}^{n+m} \to [0,\infty)$ son funcións $L^1(I)-$Carathéodory.

En caso de que o sistema anterior proveña dun sistema de ecuacións diferenciais ordinarias, os catro núcleos obtéñense a través do cálculo da función de Green.

En concreto, partiremos de problemas diferenciais da forma

$$\begin{cases} L_1(u,v)(t) = \sigma_1(t), \quad t \in I, \\ L_2(u,v)(t) = \sigma_2(t), \\ \quad B_i(u,v) = \delta_i\, C_i(u,v), \quad i = 1,\ldots,n+m, \end{cases}$$

onde

$$L_1(u,v)(t) := u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} b_q(t)\, v^{(q)}(t),$$

$$L_2(u,v)(t) := v^{(m)}(t) + \sum_{k=0}^{n-1} c_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} d_q(t)\, v^{(q)}(t),$$

sendo σ_1, σ_2, a_k, c_k, para todo $k = 0,\ldots,n-1$, e b_q, d_q para todo $q = 0,\ldots,m-1$, funcións continuas en I. Ademais, $\delta_i \in \mathbb{R}$ para todo $i = 1,\ldots,n+m$.

Por outra parte, $C_i : C^n(I) \times C^m(I) \to \mathbb{R}$ son operadores lineares continuos e B_i cobre as condicións xerais de fronteira linear de dous puntos, é dicir,

$$B_i(u,v) = \sum_{j=0}^{n-1} \left(\alpha_j^i\, u^{(j)}(a) + \beta_j^i\, u^{(j)}(b)\right) + \sum_{l=0}^{m-1} \left(\tilde{\alpha}_l^i\, v^{(l)}(a) + \tilde{\beta}_l^i\, v^{(l)}(b)\right),\ i = 1,\ldots,n+m,$$

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onde α_j^i, β_j^i, $\tilde{\alpha}_l^i$ e $\tilde{\beta}_l^i$ son constantes reais para todo $j = 0, \ldots, n-1$, $l = 0, \ldots, m-1$ e $i = 1, \ldots, n+m$.

Cabe sinalar que este problema constitúe unha xeneralización do caso escalar do Capítulo 2 para sistemas.

Para obter a expresión da función de Green, facemos un cambio de variable ao problema linear asociado co fin de convertilo noutro sistema $(n+m)-$dimensional de primeira orde. Unha vez completado este proceso, aplicamos un razoamento similar ao dado no Capítulo 2 obtendo unha fórmula da función de Green do último sistema que xeneraliza a dada na expresión (4) do caso escalar.

A continuación, estudaremos o problema non linear. Entre as hipóteses pedidas sobre os catro núcleos G_i, $i = 1, 2, 3, 4$, inclúese que algunhas das súas derivadas (non necesariamente todas) sexan positivas nun subintervalo de I que pode ser dexenerado, é dicir, podería ser un único punto.

Baixo certas condicións, a existencia de solucións do problema non linear dedúcese demostrando a existencia de puntos fixos do operador integral asociado no cono $H = H_1 \times H_2$, onde

$$H_1 = \left\{ u \in C^{n-1}(I) : \begin{array}{c} u^{(i)}(t) \geq 0, \ \ t \in [m_i^{12}, n_i^{12}], \ \ i \in I_1, \\ \min\limits_{t \in [a_i^{12}, b_i^{12}]} u^{(i)}(t) \geq \xi^1 \|u^{(i)}\|_{[c_i^{12}, d_i^{12}]}, \ \ i \in K_1 \end{array} \right\}$$

e

$$H_2 = \left\{ v \in C^{m-1}(I) : \begin{array}{c} v^{(j)}(t) \geq 0, \ \ t \in [m_j^{34}, n_j^{34}], \ \ j \in I_2, \\ \min\limits_{t \in [a_j^{34}, b_j^{34}]} u^{(j)}(t) \geq \xi^2 \|u^{(j)}\|_{[c_j^{34}, d_j^{34}]}, \ \ j \in K_2 \end{array} \right\}.$$

Neste caso, as constantes que aparecen na definición de H_1 e H_2 definen os intervalos nos que a función de Green e as súas correspondentes derivadas teñen signo constante.

En canto á técnica para demostrar a existencia de puntos fixos, emprégase o Teorema do punto fixo de Krasnoselskii aplicado ao cono H.

O noso método é máis xeral e presenta vantaxes sobre os problemas integrais non lineares que consideran só dúas funcións de Green. En particular, hai problemas nos que o noso método é aplicable e que ao transformarse en problemas con dúas funcións de Green, os resultados previos probados na literatura para este tipo de problemas non son válidos.

Finalmente, presentamos un exemplo de aplicación do noso método e outro onde mostramos as vantaxes en comparación cos traballos previos que consideran só dous núcleos nas ecuacións integrais.

Capítulo 8: Signo Constante das Derivadas das Funcións de Green Vía Teoría Espectral

O obxectivo deste último capítulo é caracterizar o signo constante de certas derivadas da función de Green $g_M(t, s)$ asociada ao siguiente problema

$$T_n[M]\, u(t) \equiv u^{(n)}(t) + M\, u(t) = 0\,, \ t \in I := [a, b]\,,$$

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xunto coas condicións de fronteira de dous puntos:

$$\begin{aligned} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) \quad &= 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) &= 0, \end{aligned}$$

con $k \in \{1, \ldots, n-1\}$ e os conxuntos de índices

$$\{\sigma_1, \ldots, \sigma_k\}, \{\varepsilon_1, \ldots, \varepsilon_{n-k}\} \subset \{0, \ldots, n-1\},$$

tales que

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n-1\,, \quad 0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leq n-1\,.$$

Para estudar este operador suxeito ás condicións de contorno dos puntos consideradas anteriormente, definiremos os conceptos de carácter inverso positivo, inverso negativo, fortemente inverso positivo e fortemente inverso negativo nun certo espazo de funcións que verifican as condicións de contorno. Ademais, presentaremos as hipóteses necesarias sobre o operador $T_n[M]$ para caracterizar o conxunto de parámetros M para os que temos garantido o carácter fortemente inverso positivo e negativo, así como un resultado no que se obtén unha condición suficiente que permite asegurar o carácter inverso positivo e negativo do operador considerado.

Denotamos por $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ o espazo das funcións de clase $C^n(I)$ que cumpren as condicións de contorno consideradas.

A caracterización do signo constante das derivadas realízase a través da teoría espectral dos autovalores (seguindo as ideas presentadas nos traballos [24,26]) sen calcular a expresión explícita da función de Green, que en moitos casos é difícil de encontrar. Polo tanto, os resultados obtidos constitúen unha ferramenta moi poderosa para garantirmos que unha gran familia de problemas lineares con condicións de fronteira de dous puntos teña unha solución de signo constante ou con derivadas de signo constante.

En primeiro lugar, calculamos as condicións de fronteira que satisfai a $q-$ésima derivada de $g_M(t,s)$ con respecto a t, $\frac{\partial^q}{\partial t^q}\, g_M(t,s)$, con $1 \leq q \leq n-1$, e deducimos as relacións entre estas condicións e as condicións de fronteira orixinais do espazo $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$, así como a relación coas condicións de fronteira dos respectivos problemas adxuntos.

A continuación, defínense $\{\sigma_1^q, \ldots, \sigma_{c_q}^q\}, \{\varepsilon_1^q, \ldots, \varepsilon_{d_q}^q\} \subset \{0, \ldots, n-1\}$, da seguinte maneira:

$$\begin{aligned} \{\sigma_1^q, \ldots, \sigma_{c_q}^q\} &= \{\sigma_1 - q, \ldots, \sigma_k - q\} \cap \{0, \ldots, n-q-1\}, \\ \{\varepsilon_1^q, \ldots, \varepsilon_{d_q}^q\} &= \{\varepsilon_1 - q, \ldots, \varepsilon_{n-k} - q\} \cap \{0, \ldots, n-q-1\}, \end{aligned}$$

e caracterízanse os intervalos de signo constante da derivada q-ésima con respecto de t de g_M en tres casos distintos:

(a) $c_q + d_q = n - q$, $c_q \geq 1$ e $d_q \geq 1$,

(b) $c_q = n - q$ e $d_q = 0$,

(c) $c_q = 0$ e $d_q = n - q$.

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Na última sección probamos que no primeiro caso (a) a derivada $q-$ésima ten signo constante nun intervalo á dereita ou á esquerda do primeiro autovalor do problema, onde o outro extremo do intervalo se caracteriza como un autovalor doutro problema distinto. Non obstante, nos casos (b) e (c) próbase que o outro extremo dos intervalos de signo constante é o valor 0. Ademais, preséntanse exemplos de aplicación a problemas concretos nos que se aprecia a utilidade e o interese dos resultados alcanzados, así como a súa facilidade de aplicación.

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Resumen

La presente Tesis, titulada "*Teoría Espectral y Funciones de Green Relacionadas con Ecuaciones Diferenciales No Locales. Aplicación a Problemas no Lineales*" recoge el trabajo realizado por el autor en los últimos años durante sus estudios de doctorado en la Universidade de Santiago de Compostela. Los trabajos obtenidos han sido desarrollados con la colaboración de los directores de tesis Alberto Cabada (Universidade de Santiago de Compostela) y Lucía López-Somoza (Universidade de Santiago de Compostela).

El texto se divide en ochos capítulos. El Capítulo 1 recoge los preliminares que necesitaremos para desarrollar el resto de los capítulos. Los Capítulos 2, 3 y 4 tratan problemas de ecuaciones diferenciales ordinarias con condiciones de frontera funcionales no locales para el caso escalar. Los Capítulos 3 y 4 son una aplicación del Capítulo 2 a los problemas de Dirichlet y periódico de segundo orden, pertubados por condiciones de frontera integrales.

En los Capítulos 5 y 6 estudiaremos las relaciones de comparación entre las expresiones de las funciones de Green y los espectros de la ecuación de Hill con diferentes condiciones de frontera de dos puntos. También, se hará una aplicación a problemas no lineales.

En el Capítulo 7 se generaliza la fórmula escalar obtenida en el Capítulo 2 a sistemas sujetos a condiciones de frontera lineales no locales arbitrarias y se incluye una aplicación a problemas no lineales.

Finalmente, el Capítulo 8 está dedicado al estudio, por medio de la teoría espectral, del signo constante de las derivadas de la función de Green asociada a un operador de orden n con condiciones de frontera de dos puntos.

A continuación resumimos los contenidos y los resultados obtenidos en cada uno de los capítulos mencionados anteriormente.

Capítulo 1: Preliminares

Este capítulo, como ya se ha indicado anteriormente, y con el fin de que la memoria sea lo más autocontenida posible, está dedicado a la presentación de preliminares y resultados necesarios para desarrollar los capítulos restantes.

En la primera parte, comenzaremos definiendo el concepto de funciones de Green de operadores lineales generales de orden n sujetos a condiciones de frontera lineales. Se presentan las propiedades de estas funciones y su importancia en la resolución de problemas lineales y no lineales. También introduciremos el concepto de función de Green en el contexto de sistemas de primer orden con condiciones de frontera de dos puntos.

También se introducen las condiciones que deben cumplir las funciones de Green y el operador diferencial para asegurar la monotonía y el signo constante de las funciones de

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Green con respecto a un parámetro real implicado en la ecuación que define al operador.

En la última parte recogemos los principales teoremas clásicos de punto fijo, entre los que destacamos los teoremas de punto fijos de Schaefer y Krasnoselskii, y el teorema de caracterización de operadores compactos de Ascoli-Arzelà.

Capítulo 2: Funciones de Green con Condiciones de Frontera Lineales No Locales Arbitrarias

En este capítulo obtenemos la expresión explícita de la función de Green relativa al siguiente problema

$$\begin{cases} T_n[M]\,u(t) = \sigma(t), & t \in I := [a,b], \\ B_i(u) = \delta_i\, C_i(u), & i = 1,\dots,n, \end{cases} \tag{1}$$

siendo

$$T_n[M]\,u(t) := L_n u(t) + M\,u(t), \quad t \in I,$$

con

$$L_n u(t) := u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_n(t)u(t), \quad t \in I,$$

y σ y a_k funciones continuas para todo $k = 1,\dots,n$, $M \in \mathbb{R}$ y $\delta_i \in \mathbb{R}$ para todo $i = 1,\dots,n$.

Además, $C_i : C(I) \to \mathbb{R}$ es un operador lineal continuo y B_i son las condiciones generales de frontera lineal de dos puntos, es decir,

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1,\dots,n,$$

siendo α_j^i, β_j^i constantes reales para todo $i = 1,\dots,n$ y $j = 0,\dots,n-1$.

El problema (1) cubre una gran familia de problemas de frontera dependiendo de los funcionales lineales continuos C_i, $i = 1,\dots,n$, considerados. Entre estos problemas podemos encontrar problemas integrales y multipunto o una combinación de ambos.

Caracterizaremos el espectro del problema (1) bajo el supuesto de que el problema homogéneo relacionado

$$\begin{cases} T_n\,[M]\,u\,(t) = 0, & t \in I, \\ B_i\,(u) = 0, & i = 1,\dots n. \end{cases} \tag{2}$$

tiene únicamente la solución trivial.

Nuestro principal resultado es el siguiente, que demuestra la existencia y unicidad de la solución del problema (1) bajo hipótesis adecuadas:

Teorema 1. *(Theorem 2.2.2) Supongamos que el problema* (2) *tiene $u = 0$ como su única solución y sea g_M su función de Green asociada. Sean $\sigma \in C(I)$ y $\delta_i \in \mathbb{R}$, $i = 1,\dots,n$, tales que*

$$\det(I_n - A) \neq 0, \tag{3}$$

con I_n la matriz identidad de orden n y $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ dada por

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1,\dots,n\}.$$

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Entonces el problema (1) *tiene una única solución* $u \in C^n(I)$*, dada por la expresión*

$$u(t) = \int_a^b G(t, s, \delta_1, \ldots, \delta_n, M)\, \sigma(s)\, ds,$$

donde

$$G(t, s, \delta_1, \ldots, \delta_n, M) := g_M(t, s) + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j\left(g_M(\cdot, s)\right), \quad t,\ s \in I, \quad (4)$$

siendo ω_i*, para* $i \in \{1, \ldots, n\}$*, la única solución del problema*

$$\begin{cases} T_n[M]\, u(t) = 0, & t \in I, \\ B_j(u) = 0, & j \in \{1, \ldots, n\} \backslash \{i\}, \\ B_i(u) = 1, & \end{cases}$$

y $B = (b_{ij})_{n \times n} = (I_n - A)^{-1}$.

La demostración del resultado anterior se basa en el principio de superposición de soluciones, la linealidad de los funcionales continuos C_i y ciertos argumentos del álgebra lineal.

Como consecuencia de este teorema obtenemos la expresión exacta de la función de Green relativa al problema (1). Además, llegamos al siguiente resultado de comparación:

Corolario 2. *(Corollary 2.2.7) Supongamos que todos los operadores lineales* C_i *son iguales, es decir,* $C_i = C$*,* $i = 1, \ldots, n$*. Sean* G *y* g_M *las funciones de Green relativas a los problemas* (1) *y* (2)*, respectivamente. Supongamos que el problema* (2) *tiene* $u = 0$ *como su única solución y que se cumple la condición* (3)*.*

Además, supongamos que se satisfacen las siguientes hipótesis:

(a) $\displaystyle\sum_{j=1}^{n} \delta_j\, C(\omega_j) < 1.$

(b) $\delta_i\, \omega_i(t) \geq 0$*,* $\forall t \in I, \quad i = 1, \ldots, n.$

(c) Si $u \geq 0$ *en* I*, entonces* $C(u) \geq 0$*.*

Entonces, se cumplen las siguientes afirmaciones:

(i) Si $g_M(t, s) \geq 0$*, para todo* (t, s) *en* $I \times I$ *entonces*

$$G(t, s, \delta_1, \ldots, \delta_n, M) \geq g_M(t, s) \geq 0$$

para todo (t, s) *en* $I \times I$*.*

(ii) Si $g_M(t, s) \leq 0$*, para todo* (t, s) *en* $I \times I$ *entonces*

$$G(t, s, \delta_1, \ldots, \delta_n, M) \leq g_M(t, s) \leq 0$$

para todo (t, s) *en* $I \times I$*.*

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La fórmula (4) se utilizará en los capítulos posteriores para obtener la expresión de la función de Green de problemas de segundo orden con condiciones de frontera funcionales siempre que el problema homogéneo tenga únicamente la solución trivial y se satisfaga la condición no espectral (3). En caso contrario, dicha fórmula deja de ser válida y se determina la correspondiente función de Green por medio de integración directa. También emplearemos la fórmula (4) para relacionar funciones de Green relativas a la ecuación de Hill asociada a distintas condiciones de frontera.

Capítulo 3: Problema Periódico Perturbado de Segundo Orden

En este capítulo se estudia el signo constante de la función de Green G_{M,δ_1,δ_2} asociada al siguiente problema periódico perturbado de segundo orden, con parámetros y condiciones de frontera integrales:

$$\begin{cases} u''(t) + M\,u(t) = \sigma(t), \quad t \in J := [0,1], \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{5}$$

donde $M, \delta_1, \delta_2 \in \mathbb{R}$.

Nótese que la formulación de este problema es un caso particular de los problemas considerados en el capítulo anterior.

En primer lugar, calculamos el espectro del problema estudiado y caracterizamos los puntos donde la función de Green se anula cuando tiene signo constante. Después, usando la fórmula (4) aplicada en nuestro caso con

$$C_1(u) = C_2(u) = \int_0^1 u(s)\,ds,$$

se obtiene la expresión exacta de la función de Green para el caso $M \neq 0$, ya que este no es un autovalor del problema homogéneo. Por el contrario, obtendremos la expresión para el caso $M = 0$ por integración directa, ya que estamos ante un autovalor del problema homogéneo.

Para abordar el estudio del problema (5), se considera en primer lugar el caso particular $\delta_1 = 0$ y se calculan las regiones de signo constante de la función de Green en dicho caso. A continuación, se estudia el caso $\delta_1 \neq 0$ a partir del anterior.

De este modo, se demuestra la existencia de funciones f, g y k que delimitan las regiones de signo contante de la función de Green, tal y como se indica a continuación:

Teorema 3. *(Theorem 3.4.8) Sean $M = 0$ y $\delta_2 \in (-8, 0)$, entonces $G_{0,\delta_1,\delta_2} > 0$ en $J \times J$ si y solo si*

$$|\delta_1| < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8, -4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Teorema 4. *(Theorem 3.4.4) Sean $M < \pi^2$, $M \neq 0$ y $\delta_2 \in (M, f(M))$, entonces G_{M,δ_1,δ_2} es estrictamente negativa en $J \times J$ si y solo si*

$$|\delta_1| < 2\,(M - \delta_2)\,G_{M,0,\delta_2}\left(1, \frac{1}{2}\right).$$

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Teorema 5. *(Theorem 3.4.6) Supongamos que $M = m^2$, $m \in (0, \pi)$ y $\delta_2 \in (g(M), M)$. Entonces, se cumplen las dos propiedades siguientes:*

1. *Si $\delta_2 \in (k(M), M)$, entonces G_{M,δ_1,δ_2} es estrictamente positiva en $J \times J$ si y solo si*

$$|\delta_1| < m \cot\left(\frac{m}{2}\right).$$

2. *Si $\delta_2 \in (g(M), k(M)]$, entonces G_{M,δ_1,δ_2} es estrictamente positiva en $J \times J$ si y solo si*

$$|\delta_1| < \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)\left(m^2 - \delta_2\right)^2}}{m}.$$

Capítulo 4: Problemas de Dirichlet No Lineales Perturbados por Condiciones de Frontera Integrales

Este capítulo es el primero en considerar los problemas de frontera no lineales.

Seguiremos un estudio similar al realizado con el problema discutido en el capítulo anterior para abordar el problema de Dirichlet no lineal de segundo orden perturbado por condiciones de frontera integrales. La idea básica consiste en aplicar la fórmula (4) para deducir la expresión exacta de la función de Green asociada al problema lineal.

En concreto, consideramos las siguientes ecuaciones no lineales

$$u''(t) + \gamma\, u(t) + f(t, u(t)) = 0, \quad t \in J,$$

y

$$u''(t) + \gamma\, u(t) - f(t, u(t)) = 0, \quad t \in J,$$

sujetos a las condiciones de frontera integrales

$$u(0) = \delta_1 \int_0^1 u(s)\, ds, \quad u(1) = \delta_2 \int_0^1 u(s)\, ds,$$

siendo γ, δ_1, δ_2 tales que $\gamma < \pi^2$, $\delta_1 \geq 0$ y $\delta_2 \geq 0$, y con la parte no lineal f cumpliendo hipótesis de regularidad y signo adecuadas.

Con el fin de abordar la existencia de soluciones positivas a estos problemas no lineales, se realiza un estudio previo del signo constante de la función de Green asociada en función de los valores de los parámetros γ, δ_1 y δ_2.

A continuación, usaremos el teorema de punto fijo de Krasnoselskii para deducir la existencia de soluciones positivas de estos problemas no lineales bajo la hipótesis de que la parte no lineal de la ecuación $f : J \times [0, \infty) \to [0, \infty)$ sea una función continua. En particular se prueba que si la función de Green es de signo constante entonces existe una solución positiva del problema en el cono

$$K = \{u \in X \,/\, \min\{u(t) : t \in J\} \geq \bar{c}\, \|u\|_\infty\},$$

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donde $X \equiv (C(J), \|\cdot\|_\infty)$ es el espacio real de Banach de las funciones continuas en J dotado de la norma del supremo $\|u\|_\infty$ y $\bar{c}$ es una constante que depende del valor de los parámetros implicados en el problema.

Capítulo 5: La Ecuación de Hill Asociada a Diferentes Condiciones de Frontera Lineal de Dos Puntos

Este capítulo trata el estudio de las propiedades y relaciones entre las funciones de Green relativas a la ecuación de Hill

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a.e. } t \in J,$$

donde $\lambda \in \mathbb{R}$, $J = [0,1]$, $a : J \to \mathbb{R}$, $a \in L^1(J)$, sujeta a diferentes condiciones de frontera.

En particular, consideramos la ecuación de Hill sujeta a condiciones de frontera de tipo Neumann, Dirichlet, periódicas y mixtas. Denotamos por $G_D[\lambda]$, $G_P[\lambda]$, $G_N[\lambda]$, $G_{M_1}[\lambda]$ y $G_{M_2}[\lambda]$ a la función de Green asociada a los problemas de Dirichlet, periódico, Neumann, mixto 1 y mixto 2, respectivamente. Además, denotamos por λ_0^D, λ_0^P, λ_0^N, $\lambda_0^{M_1}$ y $\lambda_0^{M_2}$ los primeros autovalores de cada problema.

Dado que la ecuación estudiada es de orden dos, sabemos que es posible utilizar la teoría de Sturm-Liouville para analizar las propiedades de oscilación de las soluciones. En general, esta teoría no es válida para las ecuaciones diferenciales de orden superior.

La comparación y las relaciones entre las funciones de Green se llevan a cabo desde dos enfoques diferentes: el primero se basa en el principio de superposición de soluciones y en el hecho de reescribir las condiciones de un problema como una combinación lineal del otro, mientras que el segundo enfoque consiste en una aplicación directa de la fórmula (4). Esto conduce a resultados ya conocidos (pero probados de forma diferente) y otros nuevos en la literatura sobre la comparación entre las funciones de Green. Resumimos las comparaciones obtenidas en el siguiente resultado:

Corolario 6. *Se cumplen las siguientes desigualdades:*

1) $G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^P.$

2) $G_N[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^N.$

3) $G_{M_1}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^{M_1}.$

4) $G_{M_2}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^{M_2}.$

5) $G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in (0,1]\times(0,1], \ \ \forall\,\lambda < \lambda_0^N.$

6) $G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \ \ \forall\,(t,s) \in [0,1)\times[0,1), \ \ \forall\,\lambda < \lambda_0^N.$

En general, no existe comparación entre $G_P[\lambda]$ y $G_N[\lambda]$ ni entre $G_P[\lambda]$ y $G_{M_i}[\lambda]$ para $i = 1, 2$, y daremos algunos ejemplos para comprobarlo en el caso de problemas periódicos y mixtos.

Finalmente, la última sección del capítulo está dedicada a la aplicación de los resultados obtenidos con el segundo enfoque a problemas no lineales. En concreto, bajo supuestos

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adecuados sobre la parte no lineal f, se estudia la existencia de soluciones del siguiente problema general no lineal

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ B_i(u) = \delta_i\, C_i(u), & i = 1, \dots, n. \end{cases}$$

La existencia de soluciones para este problema se puede deducir a partir de la existencia de soluciones del siguiente problema con condicones de frontera homogéneas, aplicando el Teorema de punto fijo de Schaefer,

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ B_i(u) = 0, & i = 1, \dots, n. \end{cases}$$

También se obtiene una estimación de la distancia entre las soluciones de ambos problemas.

Los resultados de existencia y la estimación de distancia entre soluciones se pueden aplicar a todos los diferentes problemas no lineales de segundo orden asociados a las diferentes condiciones de contorno mencionadas anteriormente.

En particular, bajo ciertas condiciones de regularidad sobre la función f, podemos deducir la existencia de la solución del problema de Dirichlet no lineal

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) = u(1) = 0,$$

después de probar la existencia de la solución del problema periódico no lineal

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) = u(1),\ u'(0) = u'(1),$$

y viceversa.

Capítulo 6: Espectros de la Ecuación de Hill con Diferentes Condiciones de Frontera

Este capítulo es un complemento del anterior que continúa con el estudio de la ecuación de Hill asociada a los problemas de frontera de dos puntos más habituales: Neumann, Dirichlet, mixtas y periódicas. En concreto, se abordará el estudio de los espectros de dichos problemas obteniendo relaciones notables entre ellos.

La razón principal que motiva el estudio del espectro es el hecho de que los autovalores aparecen en la búsqueda de soluciones de signo constante, ya que suelen definir los límites de las regiones en las que las funciones de Green asociadas tienen un signo positivo o negativo (ver [26] para más detalles).

El resultado principal obtenido en este capítulo es el siguiente, que muestra una relación de inclusión entre los espectros de los problemas (1) y (2):

Teorema 7. *(Theorem 6.1.1) Se cumple que*

$$S_{NH} \subset \Lambda_{NH} \subset \Lambda_H \sqcup S_{NH}$$

donde $\sqcup$ *denota la unión disjunta,* $S_{NH} := \left\{ \lambda \in \mathbb{R} \setminus \Lambda_H \ / \ \det(I_n - A) = 0 \right\}$*, con* A *la matriz definida en el Teorema 1, y* Λ_H *y* Λ_{NH} *son los espectros de los problemas* (1) *y* (2)*, respectivamente.*

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Nos centraremos en mostrar algunas relaciones entre los espectros de diferentes problemas utilizando el teorema anterior y las relaciones obtenidas en el segundo enfoque del capítulo anterior. A partir de estas relaciones, deduciremos algunas conexiones entre varios espectros y algunos otros conjuntos caracterizados como los ceros de una cierta función.

Finalmente, aplicamos los resultados obtenidos para deducir la existencia de soluciones de problemas no lineales de la forma

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T],$$

con diferentes tipos de condiciones de frontera, donde la parte no lineal de la ecuación $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisface las siguientes condiciones de regularidad:

(H_1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ es una función $L^\infty([0,T])-$Carathéodory.

(H_2) $\lim\limits_{|u|\to\infty} \dfrac{f(t,u)}{u} = -\bar{\lambda}$, uniformemente en $t \in [0,T]\backslash\mathcal{M}$, con $\mathcal{M}$ un conjunto de Lebesgue de medida nula.

Suponemos además que se cumple la siguiente condición:

(H_3^D) $\omega_I^2[\bar{\lambda}](T) := G_I[\bar{\lambda}](T,0) \neq 0$,

donde $G_I[\bar{\lambda}]$ es la función de Green relacionada al problema inicial

$$u''(t) + (a(t) + \bar{\lambda})\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad u(0) = u'(0) = 0.$$

Bajo las hipótesis previas se llega al siguiente resultado de existencia de soluciones no triviales para el problema de Dirichlet.

Teorema 8. *Supongamos que se cumplen las condiciones (H_1), (H_2) y (H_3^D). Entonces, el problema de Dirichlet no lineal*

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \textit{a.e. } t \in [0,T], \quad u(0) = u(T) = 0,$$

tiene al menos una solución $u \in W^{2,1}([0,T])$.

Como consecuencia del resultado anterior, bajo la hipótesis (H_3^D) y la siguiente condición de regularidad de la parte no lineal:

$(\tilde{H}_2)$ $\lim\limits_{u\to\infty} \dfrac{f(t,u)}{u} = -\tilde{\lambda}$, uniformemente en $t \in [0,T]\backslash\mathcal{M}'$, con $\mathcal{M}'$ un conjunto de Lebesgue de medida nula,

llegamos al siguiente corolario.

Corolario 9. *(Corollary 6.2.3) Supongamos que $f : [0,T] \times [0,\infty) \to (-\infty, 0]$ es una función $L^\infty([0,T])-$Carathéodory cumpliendo las condiciones $(\tilde{H}_2)$ y (H_3^D). Entonces, para todo $\tilde{\lambda} < \lambda_0^D$, el problema de Dirichlet no lineal tiene una solución mayor o igual que cero en $[0,T]$.*

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Análogamente se dan resultados de existencia de solución para problemas con condiciones Neumann, periódicas y mixtas.

Capítulo 7: Sistemas No Lineales Sujetos a Condiciones de Frontera Lineales No Locales Arbitrarias

En este capítulo estudiamos la existencia de solución de un problema integral incluido en la familia de las llamadas ecuaciones generalizadas de Hammerstein.

En particular, nos ocupamos de la existencia de soluciones de sistemas no lineales de ecuaciones integrales de la siguiente forma:

$$\begin{cases} u(t) = \displaystyle\int_a^b G_1(t,s)\, f_1(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_2(t,s)\, f_2(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds, \quad t \in I := [a,b], \\ v(t) = \displaystyle\int_a^b G_3(t,s)\, f_1(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_4(t,s)\, f_2(s,u(s),\cdots,u^{(n-1)}(s),v(s),\cdots,v^{(m-1)}(s))\, ds, \quad t \in I, \end{cases}$$

siendo $n, m \in \mathbb{N}$, $G_i : I \times I \to \mathbb{R}$, $i = 1,2,3,4$, los núcleos integrales que satisfacen ciertas condiciones de regularidad, tales que $G_r \in W^{n-1,1}(I \times I)$, $r = 1,2$, $G_l \in W^{m-1,1}(I \times I)$, $l = 3,4$, y $f_i : I \times \mathbb{R}^{n+m} \to [0,\infty)$ son funciones $L^1(I)-$Carathéodory.

En caso de que el sistema anterior provenga de un sistema de ecuaciones diferenciales ordinarias, los cuatro núcleos se obtienen a través del cálculo de la función de Green.

En concreto, partiremos de problemas diferenciales de la forma

$$\begin{cases} L_1(u,v)(t) = \sigma_1(t), \quad t \in I, \\ L_2(u,v)(t) = \sigma_2(t), \\ \quad B_i(u,v) = \delta_i\, C_i(u,v), \quad i = 1,\ldots,n+m, \end{cases}$$

donde

$$L_1(u,v)(t) := u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} b_q(t)\, v^{(q)}(t),$$

$$L_2(u,v)(t) := v^{(m)}(t) + \sum_{k=0}^{n-1} c_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} d_q(t)\, v^{(q)}(t),$$

siendo σ_1, σ_2, a_k, c_k, para todo $k = 0,\ldots,n-1$, y b_q, d_q para todo $q = 0,\ldots,m-1$, funciones continuas en I. Además, $\delta_i \in \mathbb{R}$ para todo $i = 1,\ldots,n+m$.

Por otra parte, $C_i : C^n(I) \times C^m(I) \to \mathbb{R}$ son operadores lineales continuos y B_i cubre las condiciones generales de frontera lineal de dos puntos, es decir,

$$B_i(u,v) = \sum_{j=0}^{n-1} \left(\alpha_j^i\, u^{(j)}(a) + \beta_j^i\, u^{(j)}(b)\right) + \sum_{l=0}^{m-1} \left(\tilde{\alpha}_l^i\, v^{(l)}(a) + \tilde{\beta}_l^i\, v^{(l)}(b)\right),\ i = 1,\ldots,n+m,$$

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donde α_j^i, β_j^i, $\tilde{\alpha}_l^i$ y $\tilde{\beta}_l^i$ son constantes reales para todo $j = 0, \ldots, n-1$, $l = 0, \ldots, m-1$ e $i = 1, \ldots, n+m$.

Cabe señalar que este problema constituye una generalización del caso escalar del Capítulo 2 para sistemas.

Para obtener la expresión de la función de Green, hacemos un cambio de variable al problema lineal asociado con el fin de convertirlo en otro sistema $(n+m)-$dimensional de primer orden. Una vez completado este proceso, aplicamos un razonamiento similar al dado en el Capítulo 2, obteniendo una fórmula de la función de Green del último sistema que generaliza la dada en la expresión (4) del caso escalar.

A continuación, estudiaremos el problema no lineal. Entre las hipótesis pedidas sobre los cuatro núcleos G_i, $i = 1, 2, 3, 4$, se incluye que algunas de sus derivadas (no necesariamente todas) sean positivas en un subintervalo de I que puede ser degenerado, es decir, podría ser un único punto.

Bajo ciertas condiciones, la existencia de soluciones del problema no lineal se deduce demostrando la existencia de puntos fijos del operador integral asociado en el cono $H = H_1 \times H_2$, donde

$$H_1 = \left\{ u \in C^{n-1}(I) : \begin{array}{l} u^{(i)}(t) \geq 0, \quad t \in [m_i^{12}, n_i^{12}], \quad i \in I_1, \\ \min\limits_{t \in [a_i^{12}, b_i^{12}]} u^{(i)}(t) \geq \xi^1 \|u^{(i)}\|_{[c_i^{12}, d_i^{12}]}, \quad i \in K_1 \end{array} \right\}$$

y

$$H_2 = \left\{ v \in C^{m-1}(I) : \begin{array}{l} v^{(j)}(t) \geq 0, \quad t \in [m_j^{34}, n_j^{34}], \quad j \in I_2, \\ \min\limits_{t \in [a_j^{34}, b_j^{34}]} u^{(j)}(t) \geq \xi^2 \|u^{(j)}\|_{[c_j^{34}, d_j^{34}]}, \quad j \in K_2 \end{array} \right\}.$$

En este caso, las constantes que aparecen en la definición de H_1 y H_2 definen los intervalos en los que la función de Green y sus correspondientes derivadas tienen signo constante.

En cuanto a la técnica para demostrar la existencia de puntos fijos, se utiliza el Teorema del punto fijo de Krasnoselskii aplicado al cono H.

Nuestro método es más general y presenta ventajas sobre los problemas integrales no lineales que consideran solo dos funciones de Green. En particular, hay problemas en los que nuestro método es aplicable y que al transformarse en problemas con dos funciones de Green, los resultados previos probados en la literatura para este tipo de problemas no son válidos.

Finalmente, presentamos un ejemplo de aplicación de nuestro método y otro donde mostramos las ventajas en comparación con los trabajos previos que consideran solo dos núcleos en las ecuaciones integrales.

Capítulo 8: Signo Constante de las Derivadas de las Funciones de Green Vía Teoría Espectral

El objetivo de este último capítulo es caracterizar el signo constante de ciertas derivadas de la función de Green $g_M(t, s)$ asociada al siguiente problema

$$T_n[M]\, u(t) \equiv u^{(n)}(t) + M\, u(t) = 0\,, \; t \in I := [a, b]\,,$$

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junto con las condiciones de frontera de dos puntos:

$$\begin{aligned} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) \quad &= 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) &= 0, \end{aligned}$$

con $k \in \{1, \ldots, n-1\}$ y los conjuntos de índices

$$\{\sigma_1, \ldots, \sigma_k\}, \{\varepsilon_1, \ldots, \varepsilon_{n-k}\} \subset \{0, \ldots, n-1\},$$

tales que

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n-1\,, \quad 0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leq n-1\,.$$

Para estudiar este operador sujeto a las condiciones de contorno de dos puntos consideradas anteriormente, definiremos los conceptos de carácter inverso positivo, inverso negativo, fuertemente inverso positivo y fuertemente inverso negativo en un cierto espacio de funciones que verifican las condiciones de contorno. Además, presentaremos las hipótesis necesarias sobre el operador $T_n[M]$ para caracterizar el conjunto de parámetros M para los que tenemos garantizado el carácter fuertemente inverso positivo y negativo, así como un resultado en el que se obtiene una condición suficiente que permite asegurar el carácter inverso positivo y negativo del operador considerado.

Denotamos por $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$ el espacio de las funciones de clase $C^n(I)$ que cumplen las condiciones de contorno consideradas.

La caracterización del signo constante de las derivadas se realiza a través de la teoría espectral de los autovalores (siguiendo las ideas presentadas en los trabajos [24, 26]) sin calcular la expresión explícita de la función de Green, que en muchos casos es difícil de encontrar. Por lo tanto, los resultados obtenidos constituyen una herramienta muy poderosa para garantizar que una gran familia de problemas lineales con condiciones de frontera de dos puntos tenga una solución de signo constante o con derivadas de signo constante.

En primer lugar, calculamos las condiciones de frontera que satisface la $q-$ésima derivada de $g_M(t,s)$ con respecto a t, $\frac{\partial^q}{\partial t^q}\, g_M(t,s)$, con $1 \leq q \leq n-1$, y deducimos las relaciones entre estas condiciones y las condiciones de frontera originales del espacio $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$, así como la relación con las condiciones de frontera de los respectivos problemas adjuntos.

A continuación, se definen $\{\sigma_1^q, \ldots, \sigma_{c_q}^q\}, \{\varepsilon_1^q, \ldots, \varepsilon_{d_q}^q\} \subset \{0, \ldots, n-1\}$, de la siguiente manera:

$$\begin{aligned} \{\sigma_1^q, \ldots, \sigma_{c_q}^q\} &= \{\sigma_1 - q, \ldots, \sigma_k - q\} \cap \{0, \ldots, n-q-1\}, \\ \{\varepsilon_1^q, \ldots, \varepsilon_{d_q}^q\} &= \{\varepsilon_1 - q, \ldots, \varepsilon_{n-k} - q\} \cap \{0, \ldots, n-q-1\}, \end{aligned}$$

y se caracterizan los intervalos de signo constante de la derivada q-ésima con respecto de t de g_M en tres casos distintos:

(a) $c_q + d_q = n - q$, $c_q \geq 1$ y $d_q \geq 1$,

(b) $c_q = n - q$ y $d_q = 0$,

(c) $c_q = 0$ y $d_q = n - q$.

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En la última sección probamos que en el primer caso (a) la derivada $q-$ésima tiene signo constante en un intervalo a la derecha o la izquierda del primer autovalor del problema, donde el otro extremo del intervalo se caracteriza como un autovalor de otro problema distinto. Sin embargo, en los casos (b) y (c) se prueba que el otro extremo de los intervalos de signo constante es el valor 0. Además, se presentan ejemplos de aplicación a problemas concretos en los que se aprecia la utilidad y el interés de los resultados alcanzados, así como la facilidad de su aplicación.

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Summary

The present Thesis entitled "*Spectral Theory and Green's Functions Related to Nonlocal Differential Equations. Application to Nonlinear Problems*" collects the work done by the author in recent years during his PhD studies at University of Santiago de Compostela. The work has been developed with the collaboration of the thesis supervisors Alberto Cabada (University of Santiago de Compostela) and Lucía López-Somoza (University of Santiago de Compostela).

The text is divided into eight chapters. Chapter 1 contains some preliminaries that we will need to develop the rest of the chapters. Chapters 2, 3, and 4 deal with problems of ordinary differential equations coupled to nonlocal functional boundary conditions for the scalar case. Chapters 3 and 4 consist on an application of Chapter 2 to second order problems with perturbed periodic and Dirichlet integral boundary conditions.

In Chapters 5 and 6 we will study the comparison relationships between the expressions of the Green's functions and the spectra of the Hill's equation coupled to different two-point boundary conditions. An application to nonlinear problems will also be made.

Chapter 7 generalizes the scalar formula obtained in Chapter 2 to systems subject to arbitrary linear nonlocal boundary conditions and includes an application to nonlinear problems.

Finally, Chapter 8 is devoted to the study, via spectral theory, of the constant sign of the derivatives of the Green's function related to an operator of order n coupled to two-point boundary conditions.

Below we summarize the contents and the results obtained in each of the chapters mentioned above.

Chapter 1: Preliminaries

This chapter, as indicated above, and in order to make the work as self-contained as possible, is dedicated to the presentation of some preliminaries and results that we need to develop the remaining chapters.

In the first part, we will begin by defining the concept of Green's function of general linear operators of order n subject to boundary conditions. The properties of these functions and its importance in solving both linear and nonlinear problems are presented. We will also introduce the concept of Green's function in the context of first order systems with two-point boundary conditions.

In addition, we introduce the conditions that the Green's functions and the differential operator must satisfy in order to ensure the monotony and constant sign of the Green's functions with respect to a real parameter involved in the equation that defines the operator.

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In the last part, we collect the main classical fixed point theorems that will be used, among which we highlight the fixed point Theorems of Schaefer and Krasnoselskii, and the characterization theorem of compact operators of Ascoli-Arzelà.

Chapter 2: Green's Function with Arbitrary Linear Nonlocal Boundary Conditions

In this chapter we obtain the explicit expression of the Green's function related to the following problem

$$\begin{cases} T_n[M]\,u(t) = \sigma(t), & t \in I := [a,b], \\ B_i(u) = \delta_i\, C_i(u), & i = 1,\dots,n, \end{cases} \tag{1}$$

where

$$T_n[M]\,u(t) := L_n u(t) + M\,u(t), \quad t \in I,$$

with

$$L_n u(t) := u^{(n)}(t) + a_1(t)\,u^{(n-1)}(t) + \cdots + a_n(t)\,u(t), \quad t \in I,$$

σ and a_k are continuous functions for all $k = 1,\dots,n$, $M \in \mathbb{R}$ and $\delta_i \in \mathbb{R}$ for all $i = 1,\dots,n$.

Moreover, $C_i : C(I) \to \mathbb{R}$ is a linear continuous operator and B_i covers the general two point linear boundary conditions, i.e.:

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1,\dots,n,$$

being α_j^i, β_j^i real constants for all $i = 1,\dots,n,\ \ j = 0,\dots,n-1$.

Problem (1) covers a large family of boundary value problems depending on the type of continuous linear functionals C_i considered. Among these problems we can find integral and multi-point problems or a combination of both.

We will characterize the spectrum of problem (1) under the assumption that the related homogeneous problem

$$\begin{cases} T_n\,[M]\,u\,(t) = 0, & t \in I, \\ B_i\,(u) = 0, & i = 1,\dots n, \end{cases} \tag{2}$$

has only the trivial solution.

Our main result is the following one which proves the existence and uniqueness of solution to problem (1) under suitable assumptions:

Theorem 1. *(Theorem 2.2.2) Assume that problem* (2) *has* $u = 0$ *as its unique solution and let* g_M *be its related Green's function. Let* $\sigma \in C(I)$ *and* $\delta_i \in \mathbb{R}$, $i = 1,\dots,n$, *be such that*

$$\det(I_n - A) \neq 0, \tag{3}$$

with I_n *the identity matrix of order* n *and* $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *given by*

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1,\dots,n\}.$$

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Then problem (1) *has a unique solution* $u \in C^n(I)$*, given by the expression*

$$u(t) = \int_a^b G(t, s, \delta_1, \ldots, \delta_n, M)\, \sigma(s)\, ds,$$

where

$$G(t, s, \delta_1, \ldots, \delta_n, M) := g_M(t, s) + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j\left(g_M(\cdot, s)\right), \quad t,\ s \in I, \quad (4)$$

with ω_i*, for* $i \in \{1, \ldots, n\}$*, the unique solution of the problem*

$$\begin{cases} T_n[M]\, u(t) = 0, & t \in I, \\ B_j(u) = 0, & j \in \{1, \ldots, n\} \backslash \{i\}, \\ B_i(u) = 1, & \end{cases}$$

and $B = (b_{ij})_{n \times n} = (I_n - A)^{-1}$.

The proof of the previous result is based on the principle of superposition of solutions, the linearity of continuous functionals C_i, and some arguments of linear algebra.

As a consequence of this theorem, we obtain the exact expression of the Green's function related to problem (1). Moreover, we arrive at the next comparison result:

Corollary 2. *(Corollary 2.2.7) Suppose that all the linear operators* C_i *are the same, that is,* $C_i = C$*,* $i = 1, \ldots, n$*. Assume that problem* (2) *has* $u = 0$ *as its unique solution and let* g_M *be its unique Green's function. Assume that condition* (3) *holds and let* G *be the Green's function related to problem* (1)*.*

Moreover, suppose that the following hypotheses are satisfied:

(a) $\displaystyle\sum_{j=1}^{n} \delta_j\, C(\omega_j) < 1.$

(b) $\delta_i\, \omega_i(t) \geq 0$*,* $\forall t \in I, \quad i = 1, \ldots, n.$

(c) If $u \geq 0$ *on* I*, then* $C(u) \geq 0$*.*

Then, the following assertions are fulfilled:

(i) If $g_M(t, s) \geq 0$*, for all* (t, s) *on* $I \times I$ *then* $G(t, s, \delta_1, \ldots, \delta_n, M) \geq g_M(t, s) \geq 0$ *for all* (t, s) *on* $I \times I$*.*

(ii) If $g_M(t, s) \leq 0$*, for all* (t, s) *on* $I \times I$ *then* $G(t, s, \delta_1, \ldots, \delta_n, M) \leq g_M(t, s) \leq 0$ *for all* (t, s) *on* $I \times I$*.*

The formula (4) will be used in later chapters to obtain the expression of the Green's functions of second-order problems with functional boundary conditions provided that the homogeneous problem has only the trivial solution and the nonspectral condition (3) is satisfied. Otherwise, this formula is no longer valid and the corresponding Green's function is

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determined by means of direct integration. We will also use the formula (4) to relate Green's functions associated to the Hill's equation coupled with different linear boundary conditions.

Chapter 3: Second-Order Perturbed Periodic Problem

In this chapter, we study the constant sign of the Green's function G_{M,δ_1,δ_2} related to the following perturbed second order periodic problem with parameters, coupled with integral conditions on the boundary:

$$\begin{cases} u''(t) + M\,u(t) = \sigma(t), \quad t \in J := [0,1], \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{5}$$

where $M, \delta_1, \delta_2 \in \mathbb{R}$.

Note that the formulation of this problem is a particular case of the problems discussed in previous chapter.

First, we find the spectrum of the studied problem and we characterize the points where the Green's function vanishes provided that it has a constant sign. After, using the formula (4) applied in our case with

$$C_1(u) = C_2(u) = \int_0^1 u(s)\,ds,$$

the exact expression of the related Green's function for $M \neq 0$ is obtained, since this is not an eigenvalue of the homogeneous problem. On the contrary, we will obtain the expression for the case $M = 0$ by direct integration, since we are dealing with an eigenvalue of the homogeneous problem.

To address the study of the problem (5), we first consider the particular case with $\delta_1 = 0$ and calculate the regions of constant sign of Green's function in that case. Next, the case with $\delta_1 \neq 0$ is studied from the previous one.

This way, the existence of functions f, g and k that delimit the regions of constant sign of the Green's function is proved as follows:

Theorem 3. *(Theorem 3.4.8) Let $M = 0$ and $\delta_2 \in (-8, 0)$, then $G_{0,\delta_1,\delta_2} > 0$ on $J \times J$ if and only if*

$$|\delta_1| < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8, -4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Theorem 4. *(Theorem 3.4.4) Let $M < \pi^2, M \neq 0$ and $\delta_2 \in (M, f(M))$, then G_{M,δ_1,δ_2} is strictly negative on $J \times J$ if and only if*

$$|\delta_1| < 2\,(M - \delta_2)\,G_{M,0,\delta_2}\left(1, \frac{1}{2}\right).$$

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Theorem 5. *(Theorem 3.4.6) Assume that $M = m^2$, $m \in (0, \pi)$, and $\delta_2 \in (g(M), M)$, then the two following properties are fulfilled:*

1. *If $\delta_2 \in (k(M), M)$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < m \cot\left(\frac{m}{2}\right).$$

2. *If $\delta_2 \in (g(M), k(M)]$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)\left(m^2 - \delta_2\right)^2}}{m}.$$

Chapter 4: Nonlinear Dirichlet Problem Perturbed by Integral Boundary Conditions

This chapter is the first one to consider nonlinear boundary value problems.

We will make a similar study to the one made with the problem discussed in the previous chapter to analyze the nonlinear second order Dirichlet problem perturbed by integral boundary conditions. The basic idea consists on applying the formula (4) to deduce the exact expression of the Green's function related to the linear problem.

To be concise, we consider the following nonlinear equations

$$u''(t) + \gamma\, u(t) + f(t, u(t)) = 0, \quad t \in J,$$

and

$$u''(t) + \gamma\, u(t) - f(t, u(t)) = 0, \quad t \in J,$$

subject to the integral boundary conditions

$$u(0) = \delta_1 \int_0^1 u(s)\, ds, \quad u(1) = \delta_2 \int_0^1 u(s)\, ds,$$

where γ, δ_1, δ_2 are such that $\gamma < \pi^2$, $\delta_1 \geq 0$ and $\delta_2 \geq 0$, and with the nonlinear part f fulfilling adequate regularity and sign assumptions.

In order to address the existence of positive solutions to these nonlinear problems, a previous study of the constant sign of the related Green's function as a function of the values of the parameters γ, δ_1 and δ_2 is carried out.

Next, we will use Krasnoselskii's fixed point Theorem to deduce the existence of positive solutions of nonlinear problems under the assumption that the nonlinear part of equation $f : J \times [0, \infty) \to [0, \infty)$ is a continuous function. In particular, it is proved that if the Green's function has constant sign then there is a positive solution of the problem in the cone

$$K = \{u \in X \,/\, \min\{u(t) : t \in J\} \geq \bar{c}\, \|u\|_\infty\},$$

where $X \equiv (C(J), \|\cdot\|_\infty)$ is the real Banach space of continuous functions in J endowed with the supremum norm $\|u\|_\infty$ and $\bar{c}$ is constant that depends on the value of the parameters involved in the problem.

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Chapter 5: Hill's Equation Subject to Different Two-Point Linear Boundary Conditions

This chapter contains the study of the properties and relationships between the Green's functions related to the Hill's equation

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a.e. } t \in J,$$

where $\lambda \in \mathbb{R}$, $J = [0,1]$, $a : J \to \mathbb{R}$, $a \in L^1(J)$, coupled to different boundary conditions.

In particular, we consider the Hill's equation subject to Neumann, Dirichlet, periodic and mixed boundary conditions. We denote by $G_D[\lambda]$, $G_P[\lambda]$, $G_N[\lambda]$, $G_{M_1}[\lambda]$ and $G_{M_2}[\lambda]$ the Green's function related to Dirichlet, periodic, Neumann, mixed 1 and mixed 2 problems, respectively. Moreover, we denote by λ_0^D, λ_0^P, λ_0^N, $\lambda_0^{M_1}$ and $\lambda_0^{M_2}$ the first eigenvalues of each problem.

Since the studied equation is of order two, we know that it is possible to use the Sturm-Liouville theory to analyze the oscillation properties of the solutions. In general, this theory does not hold true for higher order differential equations.

The comparison and the relationships between Green's functions are carried out from two different approaches: the first one is based on the principle of superposition of solutions and the fact of rewriting the conditions of one problem as a linear combination of another, while the second approach consists of a direct application of formula (4). This leads to results, some known (but proved differently) and some new in the literature on the comparison between Green's functions. We summarize the obtained relations in the following result:

Corollary 6. *The following inequalities hold:*

1) $G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\, (t,s) \in (0,1) \times (0,1), \ \ \forall\, \lambda < \lambda_0^P.$

2) $G_N[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\, (t,s) \in (0,1) \times (0,1), \ \ \forall\, \lambda < \lambda_0^N.$

3) $G_{M_1}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\, (t,s) \in (0,1) \times (0,1), \ \ \forall\, \lambda < \lambda_0^{M_1}.$

4) $G_{M_2}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall\, (t,s) \in (0,1) \times (0,1), \ \ \forall\, \lambda < \lambda_0^{M_2}.$

5) $G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) < 0, \ \ \forall\, (t,s) \in (0,1] \times (0,1], \ \ \forall\, \lambda < \lambda_0^N.$

6) $G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \ \ \forall\, (t,s) \in [0,1) \times [0,1), \ \ \forall\, \lambda < \lambda_0^N.$

In general, there is no comparison between $G_P[\lambda]$ and $G_N[\lambda]$ neither between $G_P[\lambda]$ and $G_{M_i}[\lambda]$ for $i = 1, 2$, and we will give some examples to check this in the case of periodic and mixed problems.

Finally, the last section of the chapter is devoted to an application to nonlinear problems of the results obtained in the second approach. Specifically, under suitable assumptions on the nonlinear part f, the existence of solutions to the following nonlinear problem is studied:

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ B_i(u) = \delta_i\, C_i(u), & i = 1, \ldots, n. \end{cases}$$

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The existence of solutions to this problem can be deduced from the existence of solutions to the problem with homogeneous boundary conditions

$$\begin{cases} L_n\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \\ B_i(u) = 0, \quad i = 1, \dots, n, \end{cases}$$

by applying Schaefer's fixed point Theorem of integral operators.

An estimate of the distance between the solutions of both problems is also obtained.

The results of existence and the estimation of the distance between the solutions can be applied to all the different second order nonlinear problems coupled to the various boundary conditions mentioned above.

In particular, under suitable assumptions on function f, we can deduce the existence of solution of the nonlinear Dirichlet problem

$$L[\lambda]\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \quad u(0) = u(1) = 0,$$

after proving the existence of solution of the nonlinear periodic problem

$$L[\lambda]\, u(t) = f(t, u(t)), \ \text{ a.e. } t \in J, \quad u(0) = u(1),\, u'(0) = u'(1),$$

and vice-versa.

Chapter 6: Spectra of the Hill's Equation Subject to Different Boundary Value Conditions

This chapter is a complement to the previous one that continues the study of the Hill's equation coupled to the usual two-point boundary conditions: Neumann, Dirichlet, mixed and periodic. Specifically, the study of the spectra of these problems will be addressed, obtaining notable relationships between them.

The main reason for the study of spectra is the fact that eigenvalues appear in the search of constant sign solutions, since they usually define the limits of regions in which the related Green's functions have a positive or negative sign (see [26] for details).

The main result obtained in this chapter, which shows an inclusion relationship between the spectra of problems (1) and (2), is the following one:

Theorem 7. *(Theorem 6.1.1) It holds that*

$$S_{NH} \subset \Lambda_{NH} \subset \Lambda_H \sqcup S_{NH},$$

where $\sqcup$ *denotes the disjoint union,* $S_{NH} := \left\{ \lambda \in \mathbb{R} \setminus \Lambda_H \ / \ \det(I_n - A) = 0 \right\}$, *with* A *the matrix defined on Theorem 1, and* Λ_H *and* Λ_{NH} *are the set of eigenvalues of problems* (1) *and* (2)*, respectively.*

We will focus on showing some relationships between the spectra of different problems using the previous theorem and the relationships obtained in the second approach of the previous chapter. From these relations, we will deduce some connections between various spectra and some other sets characterized as the zeros of a certain function.

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Finally, we apply the obtained results to deduce the existence of solutions to nonlinear problems of the form

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T],$$

with different types of boundary conditions, where the nonlinear part $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following regularity conditions:

(H_1) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a $L^\infty([0, T])-$Carathéodory function.

(H_2) $\lim\limits_{|u|\to\infty} \dfrac{f(t, u)}{u} = -\bar{\lambda}$, uniformly in $t \in [0, T]\backslash\mathcal{M}$, with $\mathcal{M}$ a set of Lebesgue measure equals to zero.

Moreover, we assume that the following condition is fulfilled:

(H_3^D) $\omega_I^2[\bar{\lambda}](T) := G_I[\bar{\lambda}](T, 0) \neq 0,$

where $G_I[\bar{\lambda}]$ is the Green's function related to the initial problem

$$u''(t) + (a(t) + \bar{\lambda})\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \quad u(0) = u'(0) = 0.$$

Under the previous hypotheses, the following result is reached for the existence of nontrivial solutions to the Dirichlet problem.

Theorem 8. *Suppose that (H_1), (H_2) and (H_3^D) hold. Then, the nonlinear Dirichlet problem*

$$u''(t) + a(t)\,u(t) = f(t, u(t)), \quad \textit{a.e. } t \in [0, T], \quad u(0) = u(T) = 0,$$

has at least one solution $u \in W^{2,1}([0, T])$.

As a consequence of the previous result and under the hypothesis (H_3^D) and the following regularity condition of the nonlinear part:

$(\tilde{H}_2)$ $\lim\limits_{u\to\infty} \dfrac{f(t, u)}{u} = -\tilde{\lambda}$, uniformly in $t \in [0, T]\backslash\mathcal{M}'$, with $\mathcal{M}'$ a set of Lebesgue measure equals to zero,

we arrive at the next corollary.

Corollary 9. *(Corollary 6.2.3) Assume that $f : [0, T] \times [0, \infty) \to (-\infty, 0]$ is a $L^\infty([0, T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and (H_3^D) hold. Then for all $\tilde{\lambda} < \lambda_0^D$ the nonlinear Dirichlet problem has a nonnegative solution on $[0, T]$.*

Similarly, some results are given for the existence of solutions for problems with Neumann, periodic and mixed conditions.

Chapter 7: Nonlinear Systems Subject to Arbitrary Linear Nonlocal Boundary Conditions

In this chapter we study the existence of solutions of an integral problem included in the family of so-called generalized Hammerstein equations.

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In particular, we deal with the existence of solutions of the nonlinear system of integral equations

$$\begin{cases} u(t) = \displaystyle\int_a^b G_1(t,s)\, f_1(s, u(s), \cdots, u^{(n-1)}(s), v(s), \cdots, v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_2(t,s)\, f_2(s, u(s), \cdots, u^{(n-1)}(s), v(s), \cdots, v^{(m-1)}(s))\, ds, \quad t \in I := [a,b], \\ v(t) = \displaystyle\int_a^b G_3(t,s)\, f_1(s, u(s), \cdots, u^{(n-1)}(s), v(s), \cdots, v^{(m-1)}(s))\, ds \\ \qquad + \displaystyle\int_a^b G_4(t,s)\, f_2(s, u(s), \cdots, u^{(n-1)}(s), v(s), \cdots, v^{(m-1)}(s))\, ds, \quad t \in I, \end{cases}$$

where $n, m \in \mathbb{N}$, $G_i : I \times I \to \mathbb{R}$, $i = 1, 2, 3, 4$, are the kernel functions satisfying certain regularity conditions such that $G_r \in W^{n-1,1}(I \times I)$, $r = 1, 2$, $G_l \in W^{m-1,1}(I \times I)$, $l = 3, 4$, and $f_i : I \times \mathbb{R}^{n+m} \to [0, \infty)$ are $L^1(I)-$Carathéodory functions.

In case that the above system comes from a system of differential equations, the four kernels are obtained through the calculation of Green's function.

Specifically, we will start with differential problems of the form

$$\begin{cases} L_1(u,v)(t) = \sigma_1(t), \quad t \in I, \\ L_2(u,v)(t) = \sigma_2(t), \\ \quad B_i(u,v) = \delta_i\, C_i(u,v), \quad i = 1, \ldots, n+m, \end{cases}$$

where

$$L_1(u,v)(t) := u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} b_q(t)\, v^{(q)}(t),$$

$$L_2(u,v)(t) := v^{(m)}(t) + \sum_{k=0}^{n-1} c_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} d_q(t)\, v^{(q)}(t),$$

being σ_1, σ_2, a_k, c_k, for all $k = 0, \ldots, n-1$, and b_q, d_q for all $q = 0, \ldots, m-1$, continuous functions on I. Moreover, $\delta_i \in \mathbb{R}$ for all $i = 1, \ldots, n+m$.

Furthermore, $C_i : C^n(I) \times C^m(I) \to \mathbb{R}$ are linear continuous operators and B_i covers the general two-point linear boundary conditions, that is,

$$B_i(u,v) = \sum_{j=0}^{n-1} \left(\alpha_j^i\, u^{(j)}(a) + \beta_j^i\, u^{(j)}(b) \right) + \sum_{l=0}^{m-1} \left(\tilde{\alpha}_l^i\, v^{(l)}(a) + \tilde{\beta}_l^i\, v^{(l)}(b) \right), \; i = 1, \ldots, n+m,$$

where α_j^i, β_j^i, $\tilde{\alpha}_l^i$ and $\tilde{\beta}_l^i$ are real constants for all $j = 0, \ldots, n-1$, $l = 0, \ldots, m-1$ and $i = 1, \ldots, n+m$.

It should be noted that this problem is a generalization to systems of the scalar case of Chapter 2.

To obtain the expression of the Green's function, we make a change of variable with the associated linear problem in order to convert it into another first-order $(n+m)-$dimensional system. Once this process has been completed, we apply a similar reasoning to that given in

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Chapter 2 by obtaining a formula of the Green's function of the last system that generalizes the one given in (4) of the scalar case.

Next, we will study the nonlinear problem. Among the required the hypotheses for the four kernels G_i, $i = 1, 2, 3, 4$, is that some of their derivatives (not necessarily all of them) are positive in a subinterval of I that can be degenerate, that is, it could be a unique point.

Under certain conditions, the existence of solutions to the nonlinear problem is deduced by proving the existence of fixed points of the related integral operator in the cone $H = H_1 \times H_2$, where

$$H_1 = \left\{ u \in C^{n-1}(I) : \begin{array}{l} u^{(i)}(t) \geq 0, \quad t \in [m_i^{12}, n_i^{12}], \quad i \in I_1, \\ \min\limits_{t \in [a_i^{12}, b_i^{12}]} u^{(i)}(t) \geq \xi^1 \|u^{(i)}\|_{[c_i^{12}, d_i^{12}]}, \quad i \in K_1 \end{array} \right\},$$

and

$$H_2 = \left\{ v \in C^{m-1}(I) : \begin{array}{l} v^{(j)}(t) \geq 0, \quad t \in [m_j^{34}, n_j^{34}], \quad j \in I_2, \\ \min\limits_{t \in [a_j^{34}, b_j^{34}]} u^{(j)}(t) \geq \xi^2 \|u^{(j)}\|_{[c_j^{34}, d_j^{34}]}, \quad j \in K_2 \end{array} \right\}.$$

In this case, the constants that appear in the definition of H_1 and H_2 define the intervals in which the Green's function and its corresponding derivatives have constant sign.

As for the technique to prove the existence of fixed points, the Krasnoselskii's fixed point Theorem applied to the cone H is used.

Our method is more general and presents advantages over non linear integral problems that consider only two Green's functions. In particular, there are problems in which our method is applicable but when they are transformed into problems with two Green's functions, the previous results given in the literature for this type of problem are not valid.

Finally, we introduce an example of the application of our method and another one where we show the advantages in comparison with previous work that considers only two kernels in the integral equations.

Chapter 8: Constant Sign Derivatives of Green's Functions Via Spectral Theory

The objective of this last chapter is to characterize the constant sign of certain derivatives of the Green's function $g_M(t, s)$ related to the problem

$$T_n[M]\, u(t) \equiv u^{(n)}(t) + M\, u(t) = 0\,, \; t \in I := [a, b]\,,$$

coupled to the two-point boundary conditions

$$\begin{aligned} u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) \quad &= 0, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) &= 0, \end{aligned}$$

with $k \in \{1, \ldots, n-1\}$ and the sets of indices

$$\{\sigma_1, \ldots, \sigma_k\}, \{\varepsilon_1, \ldots, \varepsilon_{n-k}\} \subset \{0, \ldots, n-1\},$$

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satisfy that

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n-1\,, \quad 0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leq n-1\,.$$

For studying this operator subject to the two-point boundary conditions considered above, we will define the concepts of inverse positive, inverse negative, strongly inverse positive and strongly inverse negative character in a certain function's space that verify the boundary conditions. Also, we will present the necessary assumptions on the operator $T_n[M]$ to characterize the set of parameters M for which the strongly inverse positive and negative character is guaranteed, and as well as a result in which we give a sufficient condition to ensure the inverse positive and negative character of the considered operator.

We denote by $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ the space of the functions of class $C^n(I)$ verifying the considered boundary conditions.

The characterization of the constant sign of the derivatives is done through the spectral theory of eigenvalues (following the ideas presented in the work [26]) without studying the expression of the Green's function, which, in many cases is not possible to find. Therefore, the obtained results constitute a very powerful tool to ensure that a large family of linear problems with two-point boundary conditions has a solution of constant sign or with constant-sign derivatives.

First, we compute the boundary conditions that satisfy the $q^{\text{th}}-$derivative of $g_M(t,s)$ with respect to t, $\frac{\partial^q}{\partial t^q} g_M(t,s)$, with $1 \leq q \leq n-1$, and deduce the relationships between these conditions and the original boundary conditions of the space $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$, as well as the relation to the boundary conditions of the respective adjoint problems.

Next, let us define $\{\sigma_1^q,\ldots,\sigma_{c_q}^q\}, \{\varepsilon_1^q,\ldots,\varepsilon_{d_q}^q\} \subset \{0,\ldots,n-1\}$, in the following way:

$$\{\sigma_1^q,\ldots,\sigma_{c_q}^q\} = \{\sigma_1 - q,\ldots,\sigma_k - q\} \cap \{0,\ldots,n-q-1\},$$
$$\{\varepsilon_1^q,\ldots,\varepsilon_{d_q}^q\} = \{\varepsilon_1 - q,\ldots,\varepsilon_{n-k} - q\} \cap \{0,\ldots,n-q-1\},$$

and the constant sign intervals of the $q^{th}-$derivative with respect to t of g_M are characterized in three different cases:

(a) $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$,

(b) $c_q = n - q$ and $d_q = 0$,

(c) $c_q = 0$ and $d_q = n - q$.

In the last section, we prove in the first case (a) that the $q^{th}-$derivative has constant sign on an interval to the right or left of the first eigenvalue of the problem, where the other extreme of the interval is characterized as an eigenvalue of another distinct problem. However, in cases (b) and (c) it is proven that the other extreme of the constant sign interval is the value 0. Throughout the development of this section, we present examples of application to particular problems in which the usefulness and interest of the achieved results are seen as well the ease of its application.

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Chapter 1
Preliminaries

In this first chapter we make a survey of the tools that we will need to develop the rest of the chapters.

Since one of our objectives is to study the constant sign of the Green's functions, in order to be able to use this information in the study of nonlinear problems, we will begin by introducing the axiomatic definition of Green's function and its properties. We will also discuss the existence and uniqueness of solution to nonhomogeneous problems and their relationship to Green's function. We will deal with these points in Section 1.1.

Next, in Section 1.2 we will present the results of the monotony of Green's functions with respect to a real parameter. Also, we will enunciate the results that characterize the real parameters sets where Green's functions maintain a constant sign under certain assumptions.

Finally, we will show in Section 1.3 that a nonlinear differential problem can be transformed into an equivalent integral one, in the sense that the solutions of the considered differential problem correspond with the fixed points of a suitable integral operator. Moreover, we will state some classical results that guarantee the existence of fixed points of certain operators defined in Banach spaces.

1.1 Green's Functions

In this section, we introduce the definition of the Green's function and its properties following references [6, 63].

Consider the $n^{\text{th}}-$order linear boundary value problem:

$$\begin{cases} L_n u(t) = \sigma(t), \quad t \in I := [a,b], \\ B_i(u) = 0, \quad i = 1, \dots, n, \end{cases} \tag{1.1.1}$$

where

$$L_n u(t) = u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_n(t)u(t), \quad t \in I,$$

and

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1, \dots, n,$$

being α_j^i, β_j^i real constants for all $i = 1, \dots, n$, $j = 0, \dots, n-1$, and $\sigma, a_k \in L^1(I)$ for all $k = 1, \dots, n$.

Next, we introduce the axiomatic definition of the Green's function related to problem (1.1.1).

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Definition 1.1.1. We say that g is a Green's function for problem (1.1.1) if it satisfies the following properties:

(G1) g is defined in the square $I \times I$ (except at the points with $t = s$ if $n = 1$).

(G2) For $k = 0, 1, \ldots, n-2$, the partial derivatives $\frac{\partial^k g}{\partial t^k}$ exist and are continuous in $I \times I$.

(G3) $\frac{\partial^{n-1} g}{\partial t^{n-1}}$ and $\frac{\partial^n g}{\partial t^n}$ exist and are continuous on the triangles $a \leq s < t \leq b$ and $a \leq t < s \leq b$.

(G4) For each fixed $s \in (a, b)$, there exist the lateral limits

$$\frac{\partial^{n-1} g}{\partial t^{n-1}}(t^-, t) = \frac{\partial^{n-1} g}{\partial t^{n-1}}(t, t^+) \quad \text{and} \quad \frac{\partial^{n-1} g}{\partial t^{n-1}}(t, t^-) = \frac{\partial^{n-1} g}{\partial t^{n-1}}(t^+, t)$$

and, moreover,

$$\frac{\partial^{n-1} g}{\partial t^{n-1}}(t^+, t) - \frac{\partial^{n-1} g}{\partial t^{n-1}}(t^-, t) = \frac{\partial^{n-1} g}{\partial t^{n-1}}(t, t^-) - \frac{\partial^{n-1} g}{\partial t^{n-1}}(t, t^+) = 1.$$

(G5) For each fixed $s \in (a, b)$, function $g(\cdot, s)$ is a solution of the differential equation $L_n u = 0$ a.e. on $[a, s) \cup (s, b]$, that is,

$$\frac{\partial^n g}{\partial t^n}(t, s) + a_1(t) \frac{\partial^{n-1} g}{\partial t^{n-1}}(t, s) + \cdots + a_{n-1}(t) \frac{\partial g}{\partial t}(t, s) + a_n(t)\, g(t, s) = 0,$$

for all $t \in I \setminus \{s\}$.

(G6) For each fixed $s \in (a, b)$, function $g(\cdot, s)$ satisfies the boundary conditions $B_i(g(\cdot, s)) = 0$, $i = 1, \ldots, n$, that is,

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j g}{\partial t^j}(a, s) + \beta_j^i \frac{\partial^j g}{\partial t^j}(b, s) \right) = 0, \quad i = 1, \ldots, n.$$

Consider the space

$$W^{n,1}(I) = \{u \in C^{n-1}(I) : \ u^{(n-1)} \in AC(I)\},$$

where $AC(I)$ denotes the set of absolutely continuous functions on I. In particular, we will consider $X \subset W^{n,1}(I)$ a Banach space defined in the following way

$$X = \{u \in W^{n,1}(I) : \ B_i(u) = 0,\ i = 1, \ldots, n\},$$

endowed with the usual norm

$$\|u\|_X = \max\{\|u^i\|,\ i = 0, \ldots, n-1\},$$

such that the next definition is satisfied.

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Definition 1.1.2. Given the Banach space X, operator L_n is said to be nonresonant in X if and only if the homogeneous equation

$$L_n\, u(t) = 0 \ \text{ a.e. } \ t \in I, \quad u \in X,$$

has only the trivial solution.

It is very well-known, see for instance [34, 48], that the integral operator, whose kernel is the Green's function g related to operator L_n on X, gives us a solution to the problem (1.1.1), as it is shown in the following result.

Theorem 1.1.3. *If operator L_n is nonresonant in X, then, for any $\sigma \in L^1(I)$, the non-homogeneous linear problem*

$$L_n\, u(t) = \sigma(t) \ \textit{ a.e. } \ t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t) = \int_a^b g(t,s)\, \sigma(s)\, ds, \quad \textit{for all } \ t \in I.$$

Recall that the set of solutions of the homogeneous linear equation $L_n u = 0$ is a vector space of dimension n of $W^{n,1}(I)$. Any base of this subspace will be called the fundamental system of solutions of the equation $L_n u = 0$.

Now, we formulate the following particular case of the result proved in [6, page 35]:

Theorem 1.1.4. *The following boundary value problem*

$$L_n\, u(t) = \sigma(t), \ \textit{ a.e. } \ , \quad B_i(u) = h_i, \ i = 1, \dots, n, \tag{1.1.2}$$

has a unique solution for any $\sigma \in L^1(I)$ and $h_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n\}$, if and only if

$$\det \begin{pmatrix} B_1(u_1) & \dots & B_1(u_n) \\ \vdots & \ddots & \vdots \\ B_n(u_1) & \dots & B_n(u_n) \end{pmatrix} \neq 0, \tag{1.1.3}$$

where $\big(u_1, \dots, u_n\big)$ is any set of linearly independent solutions of $L_n\, u(t) = 0$.

Remark 1.1.5. *It is immediate to verify that the fact that the determinant in previous result is different from zero does not depend on the chosen set of linearly independent solutions.*

Remark 1.1.6. *Notice that condition* (1.1.3) *is independent of the non homogeneous part of problem* (1.1.2)*: σ and h_i, $i \in \{1, \dots, n\}$.*

Remark 1.1.7. *One can see in [6, page 35] that the following property*

$$\textit{rank} \left(\begin{array}{ccc|ccc} \alpha_0^1 & \dots & \alpha_{n-1}^1 & \beta_0^1 & \dots & \beta_{n-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_0^n & \dots & \alpha_{n-1}^n & \beta_0^n & \dots & \beta_{n-1}^n \end{array} \right) = n,$$

is a necessary (but not sufficient) condition to ensure the uniqueness of solution of problem (1.1.2)*.*

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As a direct consequence of Theorem 1.1.4 we deduce the following result

Lemma 1.1.8. *There exists a unique Green's function related to problem* (1.1.1)*, g, if and only if for any $i \in \{1, \dots, n\}$, the following problem*

$$\begin{cases} L_n\, u(t) = 0, \ \ a.e. \ \ t \in I, \\ B_j(u) = 0, \ \ j \in \{1, \dots, n\} \setminus \{i\}, \\ B_i(u) = 1, \end{cases}$$

has a unique solution, that we denote as $\omega_i(t)$, $t \in I$.

In such a case, for any $\sigma \in L^1(I)$, the following problem

$$\begin{cases} L_n\, u(t) = \sigma(t), \ \ a.e. \ \ t \in I, \\ B_i(u) = \lambda_i, \ \ i = 1, \dots, n, \end{cases}$$

has a unique solution given by

$$u(t) = \int_a^b g(t,s)\, \sigma(s)\, ds + \sum_{i=1}^{n} \lambda_i\, \omega_i(t).$$

Let us see now how these results are adapted to the context of $n-$dimensional linear systems. Indeed, making a change of variable of type

$$x_1 = u, \ x_2 = x_1', \ \dots, \ x_n = x_{n-1}',$$

in equation (1.1.1), we obtain the following equivalent first order differential $n-$dimensional linear system

$$z'(t) = A(t)\, z(t) + f(t), \ \ t \in I, \tag{1.1.5}$$

together with the boundary conditions

$$B_i(z) = 0, \quad i = 1, \dots, n, \tag{1.1.6}$$

where

$$z(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{pmatrix}, \ A(t) = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -a_n(t) & -a_{n-1}(t) \cdots - a_1(t) \end{array}\right), \ f(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma(t) \end{pmatrix}, \ t \in I.$$

Remark 1.1.9. *Notice that the boundary conditions B_i, $i = 1, \dots, n$, do not only depend on u, but also on $u', \dots, u^{(n-1)}$. However, for the sake of simplicity, we will use the notation $B_i(u)$ instead of $B_i\left(u, \dots, u^{(n-1)}\right)$, $i = 1, \dots, n$.*

Remark 1.1.10. *We should note that once problem* (1.1.1) *transforms into problem* (1.1.5)–(1.1.6)*, the boundary conditions B_i, $i = 1, \dots n$, become of the form $E\, z(a) + F\, z(b)$ being $E, F \in \mathcal{M}_{n\times n}(\mathbb{R})$ given by:*

$$E = \begin{pmatrix} \alpha_0^1 & \cdots & \alpha_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \alpha_0^n & \cdots & \alpha_{n-1}^n \end{pmatrix} \quad \textit{and} \quad F = \begin{pmatrix} \beta_0^1 & \cdots & \beta_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \beta_0^n & \cdots & \beta_{n-1}^n \end{pmatrix}.$$

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Let us consider the homogeneous problem

$$z'(t) = A(t)\, z(t), \quad t \in I, \quad E\, z(a) + F\, z(b) = 0, \tag{1.1.7}$$

and let $\phi : I \to \mathcal{M}_{n\times n}(\mathbb{R})$ be its fundamental matrix, that is, ϕ is a regular functional matrix and a solution of the linear matrix equation

$$\phi'(t) = A(t)\, \phi(t), \quad t \in I.$$

Analogously to the results presented above, we formulate the following results of existence and uniqueness of solution and the characterization of the spectrum of problem (1.1.5)–(1.1.6).

Theorem 1.1.11. *Problem* (1.1.5)–(1.1.6) *has a unique solution* $z \in AC(I, \mathbb{R}^n)$ *for any* $f \in L^1(I, \mathbb{R}^n)$ *if and only if*

$$\det(M_\phi) \neq 0, \tag{1.1.8}$$

with ϕ *any fundamental matrix of system* (1.1.7) *and* $M_\phi := E\, \phi(a) + F\, \phi(b)$.

Remark 1.1.12. *The matrices* E *and* F *are not unique since we can take as* E *and* F *a multiple* $k\,E$ *and* $k\,F$ *with* k *a nonzero real number. It is immediate to verify that condition* (1.1.8) *is independent of the chosen fundamental matrix. Also, condition* (1.1.8) *holds if and only if problem* (1.1.7) *has only the trivial solution.*

Suppose that problem (1.1.7) has only the trivial solution. We know from Theorem 1.1.11 that problem (1.1.5)–(1.1.6) has a unique solution for any $f \in L^1(I, \mathbb{R}^n)$. In such a case, this solution is given by

$$z(t) = \int_a^b G(t,s)\, f(s)\, ds,$$

where G denotes the Green's function matrix related to problem (1.1.7) and is given by the expression

$$G(t,s) = \begin{cases} -\,\phi(t)\, M_\phi^{-1}\, F\, \phi(b)\, \phi^{-1}(s) + \phi(t)\, \phi^{-1}(s), & a \le s < t \le b, \\ -\,\phi(t)\, M_\phi^{-1}\, F\, \phi(b)\, \phi^{-1}(s), & a \le t < s \le b. \end{cases} \tag{1.1.9}$$

See [6, page 15] for more details.

Let

$$R(t) := G(t,a) = -\phi(t)\, M_\phi^{-1}\, F\, \phi(b)\, \phi^{-1}(a) + \phi(t)\, \phi^{-1}(a), \quad t \in (a,b].$$

We extend with continuity the function R to the interval I as $R(a) = \lim_{t\to a^+} R(t)$.

By definition, it is immediate to verify that

$$R'(t) = A(t)\, R(t), \quad \text{a.e. } t \in I.$$

Let us see that

$$E\, R(a) + F\, R(b) = E. \tag{1.1.10}$$

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Indeed, using expression (1.1.9) we have that

$$\begin{aligned} E\,R(a)+F\,R(b) =& -E\,\phi(a)\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(a)+E-F\,\phi(b)\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(a)\\ &+F\,\phi(b)\,\phi^{-1}(a)\\ =& -M_\phi\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(a)+E+F\,\phi(b)\,\phi^{-1}(a)=E. \end{aligned}$$

Now, defining

$$S(t):=G(t,b)=-\phi(t)\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(b),\quad t\in[a,b),$$

and extending it to I, by $S(b)=\lim_{t\to b^-}S(t)$, we have that,

$$S'(t)=A(t)\,S(t),\quad \text{a.e. } t\in I.$$

Now we verify that

$$E\,S(a)+F\,S(b)=-F. \tag{1.1.11}$$

Again, using the expression (1.1.9) we have that

$$\begin{aligned} E\,S(a)+F\,S(b) =& -E\,\phi(a)\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(b)-F\,\phi(b)\,M_\phi^{-1}\,F\,\phi(b)\,\phi^{-1}(b)\\ =& -(E\,\phi(a)+F\,\phi(b))\,M_\phi^{-1}\,F=-F. \end{aligned}$$

Analogously to Lemma 1.1.8, we have the following result for systems:

Lemma 1.1.13. *There exists a unique Green's function matrix related to Problem* (1.1.7) *if and only if for any $i\in\{1,\dots,n\}$, the following problem*

$$\begin{cases} z'(t)=A(t)\,z(t), & t\in I,\\ B_j(z)=0, & j\in\{1,\dots,n\}\setminus\{i\},\\ B_i(z)=1, \end{cases}$$

has a unique solution that we denote as $z_i(t)$, $t\in I$.

In such a case, for any $f\in L^1(I,\mathbb{R}^n)$, the following problem

$$\begin{cases} z'(t)=A(t)\,z(t)+f(t) & t\in I,\\ B_i(u)=\lambda_i, & i=1,\dots,n, \end{cases}$$

has a unique solution given by

$$z(t)=\int_a^b G(t,s)\,f(s)\,ds+\sum_{i=1}^n\lambda_i\,z_i(t).$$

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1.2 Monotony and Parameter Set of Constant Sign of Green's Functions

Let us consider the following $n^{\text{th}}-$order linear boundary value probblem

$$\tilde{T}_n[M]\,u(t) = \sigma(t), \quad t \in I, \;\; u \in U,$$

where $\tilde{T}_n[M]$ is defined from L_n as follows:

$$\tilde{T}_n[M]\,u(t) := L_n\,u(t) + M\,u(t), \quad t \in I,$$

with $M \in \mathbb{R}$ a real parameter and U a suitable Banach space, characterized by the considered boundary conditions.

When working with this operator, to stress the dependence of the Green's function on the parameter M, we will denote by g_M the Green's function realted to $\tilde{T}_n[M]$.

Now, we formulate the following results that prove the monotony of Green's functions with respect to the parameter M when these functions have constant sign.

Theorem 1.2.1. *[6, Theorems 1.8.1 and 1.8.6] Let M_1, $M_2 \in \mathbb{R}$ and suppose that operators $\tilde{T}_n[M_j]$, $j = 1, 2$, are invertible in U. Let g_{M_j}, $j = 1, 2$, be Green's functions related to operators $\tilde{T}_n[M_j]$ and suppose that both functions have the same constant sign on $\in I \times I$. Then, if $M_1 < M_2$, it is satisfied that $g_{M_2} \leq g_{M_2}$ on $I \times I$.*

Theorem 1.2.2. *[6, Theorems 1.8.5 and 1.8.9] Let $M_1 < \bar{M} < M_2$ be three real constants. Suppose that operator $\tilde{T}_n[M]$ is invertible in U for $M = M_j$, $j = 1, 2$, and that the corresponding Green's functions satisfy $g_{M_2} \leq g_{M_1} \leq 0$ (resp. $0 \leq g_{M_2} \leq g_{M_1}$) on $I \times I$. Then the operator $\tilde{T}_n[\bar{M}]$ is invertible in U and the related Green's function $g_{\bar{M}}$ satisfies $g_{M_2} \leq g_{\bar{M}} \leq g_{M_1} \leq 0$ (resp. $0 \leq g_{M_2} \leq g_{\bar{M}} \leq g_{M_1}$) on $I \times I$.*

The above theorem states that the set of parameters of M where g_M has a constant sign and it is monotone decreasing is a connected set.

Let us consider the following two conditions on g_M introduced in [6, pages 78 and 86] as follows:

(P_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1, k_2 \in L^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a.e. $s \in I$, satisfying

$$\phi(t)\,k_1(s) \leq g_M(t, s) \leq \phi(t)\,k_2(s)\,, \quad \text{for a.e. } (t, s) \in I \times I\,.$$

(N_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (a, b)$ and $k_1, k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a.e. $s \in I$, satisfying

$$\phi(t)\,k_1(s) \leq g_M(t, s) \leq \phi(t)\,k_2(s)\,, \quad \text{for a.e. } (t, s) \in I \times I\,.$$

It is obvious that if the Green's function g_M is strictly positive on $I \times I$, then condition (P_g) is trivially fulfilled. The same can be said of the condition (N_g) in the strictly negative case of g_M.

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Also, let us define the following sets that describe where the Green's function is of constant sign:

$$P_T = \{M \in \mathbb{R} \ \mid \ g_M(t,s) \geq 0 \quad \forall (t,s) \in I \times I\},$$
$$N_T = \{M \in \mathbb{R} \ \mid \ g_M(t,s) \leq 0 \quad \forall (t,s) \in I \times I\}.$$

We point out that, from Theorem 1.2.2, we can conclude that previous sets are real intervals, which may be empty in some situations.

Next, we state the results that characterize the properties of intervals N_T and P_T.

Lemma 1.2.3. *[6, Theorem 1.8.23 and Lemma 1.8.25] Let $\bar{M} \in \mathbb{R}$ be fixed. Suppose that operator $\tilde{T}_n[\bar{M}]$ is invertible in U, its related Green's function is nonpositive on $I \times I$ and it satisfies condition (N_g).*

Then $N_T = (-\infty, \bar{M} - \lambda_1)$ or $N_T = [\bar{M} - \bar{\mu}, \bar{M} - \lambda_1)$, where $\lambda_1 < 0$ is the first eigenvalue of operator $\tilde{T}_n[\bar{M}]$ in U, and $\bar{\mu} \geq 0$ is such that $\tilde{T}_n[\bar{M} - \bar{\mu}]$ is invertible in U and the related nonpositive Green's function $g_{\bar{M}-\bar{\mu}}$ vanishes at some points of the square $I \times I$.

Lemma 1.2.4. *[6, Theorem 1.8.31 and Lemma 1.8.33] Let $\bar{M} \in \mathbb{R}$ be fixed. Suppose that operator $\tilde{T}_n[\bar{M}]$ is invertible in U, its related Green's function is nonnegative on $I \times I$ and it satisfies condition (P_g).*

Then $P_T = (\bar{M} - \lambda_1, +\infty)$ or $P_T = (\bar{M} - \lambda_1, \bar{M} - \bar{\mu}]$, where $\lambda_1 > 0$ is the first eigenvalue of operator $\tilde{T}_n[\bar{M}]$ in U, and $\bar{\mu} \geq 0$ is such $\tilde{T}_n[\bar{M} - \bar{\mu}]$ is invertible in U and the related nonnegative Green's function $g_{\bar{M}-\bar{\mu}}$ vanishes at some points of the square $I \times I$.

1.3 Fixed Point Theorems

As we have mentioned before, in the study of nonlinear boundary value problems we need to determine fixed points of suitable integral operators defined in adequate Banach spaces. To this end, in this section we state some fixed point theorems that we will use later to ensure the existence of solutions to the following nonlinear boundary problem:

$$\begin{cases} L_n\, u(t) = f(t, u(t)), & t \in I, \\ B_i(u) = 0, & i = 1, \ldots, n, \end{cases} \tag{1.3.12}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies some regularity conditions.

Among these hypotheses, we sometimes ask that f to be $L^1(I)-$Carathéodory or $L^\infty(I)-$Carathéodory, which are defined in the following way:

Definition 1.3.1. We say that $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^1(I)-$Carathéodory function if the following statements are true:

- $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$.

- $f(t, \cdot)$ is continuous for a.e. $t \in I$.

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- For every $R > 0$ there exists $\phi_R \in L^1(I)$ such that

$$\Big|f(t,x)\Big| \leq \phi_R(t),$$

for all $x \in [-R, R]$ and a.e. $t \in I$.

Definition 1.3.2. We say that $f : I \times \mathbb{R} \to \mathbb{R}$ is a $L^\infty(I)-$Carathéodory function if the following statements are true:

- $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$.
- $f(t, \cdot)$ is continuous for a.e. $t \in I$.
- For every $r > 0$ there exists $h_r \in L^\infty(I)$ such that

$$\Big|f(t,x)\Big| \leq h_r(t),$$

for all $x \in [-r, r]$ and a.e. $t \in I$.

Consider now the operator $T : U \to U$ defined as follows:

$$T\,u(t) := \int_0^T g(t,s)\, f(s, u(s))\, ds, \quad t \in I,$$

where g is the Green's function related to problem (1.1.1) (assuming that operator L_n is nonresonant in U). It is very well-known that the fixed points of operator T coincide with the solutions of problem (1.3.12).

The following definition gives the concept of compact operator.

Definition 1.3.3. Given two Banach spaces Y and Z, we say that an operator $T : Y \to Z$ is compact if and only if T is continuous and maps bounded subsets of Y into relatively compact subsets of Z.

The following well-known Ascoli-Arzelà's Theorem (see [59, Chapter 7, Theorem 18]) characterizes compact operators.

Theorem 1.3.4 (Ascoli-Arzelà). *Let Y be a Hausdorff compact topological space and Z a complete metric space, and consider $C(Y, Z)$ endowed with the topology of the uniform convergence. Then $F \subset C(Y, Z)$ has compact closure if and only if the two following properties hold:*

- *$F(y)$ has compact closure for each $y \in Y$.*
- *F is equicontinuous.*

Next we will state Schaefer's fixed point Theorem [4, 43, 67].

Theorem 1.3.5 (Schaefer). *Let $T : Y \to Y$ be a compact mapping of a Banach space Y, such that the set*

$$\{y \in Y : \ y = \mu\, T\, y \ \text{ for some } \ 0 \leq \mu \leq 1\}$$

is bounded. Then T has a fixed point in Y.

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We will go on defining the concept of cone in normed Banach spaces and establishing a result of the existence of fixed points due to Krasnoselskii.

Definition 1.3.6. Given a normed Banach space Y, we say that $K \subset Y$ is a cone if it is a closed and convex subset of Y satisfying that $K + K \subset K$, $\lambda K \subset K$ for all real number $\lambda \geq 0$, and $K \cap \{-K\} = \{0\}$.

Every cone $K \subset Y$ induces a partial order $\preceq$ in Y as follows:

$$x \preceq y \quad \text{if and only if} \quad y - x \in K.$$

Next, we formulate the classic Kasnoselskii's fixed point Theorem (see [44, 90]) on cone compression or expansion. This theorem is a powerful tool for deducing the existence of positive solutions to nonlinear problems.

Theorem 1.3.7 (Krasnoselskii). *Let Y be a Banach space and $K \subset Y$ a cone in Y. Let $\Omega_1, \Omega_2 \subset Y$ be open bounded sets such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ a compact operator that satisfies one of the following properties:*

1. $\|T(u)\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ *and* $\|T(u)\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.
2. $\|T(u)\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ *and* $\|T(u)\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.

Then T has a fixed point at $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Definition 1.3.8. Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be two normed spaces. We say that a linear operator $T : Y \to Z$ is bounded if its norm $\|T\|$ defined by

$$\|T\| = \sup_{\|u\|_Y \neq 0} \frac{\|T\,u\|_Z}{\|u\|_Y},$$

is finite.

Definition 1.3.9. Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be two normed spaces. We say that λ is an eigenvalue of a linear operator $T : Y \to Z$, if there exists $u \in Y$, $u \neq 0$ such that $T\,u = \lambda\,u$. The function u is called the eigenfunction related to the eigenvalue λ.

We will also use the following Krasnoselskii-Zabreiko fixed point Theorem (see [62]) to deduce the existence of nontrivial solutions to nonlinear problems.

Theorem 1.3.10. *Let $(Y, \|\cdot\|)$ be a Banach space and $T : Y \to Y$ a compact operator. If there exists a bounded linear operator $A : Y \to Y$ such that 1 is not an eigenvalue of A and*

$$\lim_{\|u\| \to \infty} \frac{\|T(u) - A(u)\|}{\|u\|} = 0,$$

then T has a fixed point in Y.

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Chapter 2

Green's Function with Arbitrary Linear Nonlocal Boundary Conditions

In this chapter we obtain the explicit expression of the Green's function related to a general $n^{th}-$ order differential equation with parameter dependence coupled to nonlocal linear boundary conditions. In such boundary conditions, a n dimensional parameter dependence is also assumed. Moreover, some comparison principles are obtained.

The explicit expression depends on the value of the Green's function related to the two-point homogeneous problem, that is, we are assuming that when all the parameters involved on the boundary conditions take the value zero then the problem has a unique solution which is characterized by the corresponding Green's function g_M. The expression of the Green's function G of the general problem is given as a function of g_M and the real parameters considered at the boundary conditions.

It is important to indicate that, in order to ensure the uniqueness of solutions of the linear considered problem, we must assume a nonresonant additional condition on the considered problem, which depends on the nonlocal conditions and the corresponding parameters.

We point out that the assumption of the uniqueness of solutions of the two-point homogeneous problem is not a necessary condition to ensure the solution of the general case. Of course, in this situation the expression we are looking for must be obtained in a different manner.

This chapter is organized as follows. In the next section we define the problem to be studied and recall some preliminaries. In section 2.2 we obtain the expression of the Green's function of the studied problem. In Section 2.3 we present an example where the formula is used to obtain the corresponding expression and to describe the exact set of parameters for which its Green's function has constant sign on its square of definition.

The results of this chapter are included in the paper [17].

2.1 Preliminaries

Consider the following $n^{\text{th}}-$order linear boundary value problem with parameter dependence:

$$\begin{cases} \tilde{T}_n[M]\,u(t) = \sigma(t), & t \in I, \\ B_i(u) = \delta_i\, C_i(u), & i = 1, \ldots, n, \end{cases} \tag{2.1.1}$$

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where

$$\tilde{T}_n[M]u(t) := L_n\, u(t) + M\, u(t), \quad t \in I,$$

with

$$L_n u(t) := u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_n(t)\, u(t), \quad t \in I.$$

Here σ and a_k are continuous functions for all $k = 1, \ldots, n$, $M \in \mathbb{R}$ and $\delta_i \in \mathbb{R}$ for all $i = 1, \ldots, n$.

Moreover, $C_i : C(I) \to \mathbb{R}$ is a linear continuous operator and B_i covers the general two point linear boundary conditions, i.e.:

$$B_i(u) = \sum_{j=0}^{n-1} \left(\alpha_j^i u^{(j)}(a) + \beta_j^i u^{(j)}(b) \right), \quad i = 1, \ldots, n,$$

being α_j^i, β_j^i real constants for all $i = 1, \ldots, n$, $j = 0, \ldots, n-1$.

Remark 2.1.1. *Examples of operator C_i can be the integral operator*

$$C_i(u) = \int_W u(s)\, v(s)\, ds, \quad \textit{where } W \subset I \textit{ is an interval and } v \in C(W),$$

or the multi-point operator

$$C_i(u) = \sum_{k=1}^{r} \varepsilon_k u(\nu_k), \quad \nu_k \in I, \quad \varepsilon_k \in \mathbb{R}, \quad k = 1, \ldots, r.$$

We point out that problem (2.1.1) covers any $n^{\text{th}}-$order differential equation and that on the choice of δ_i any of them could vanish, so this problem may be interpreted as a perturbation of a two-point boundary value problem.

Also, we consider the following homogeneous problem related to the general equation (2.1.1):

$$\begin{cases} \tilde{T}_n[M]\, u(t) = 0, & t \in I, \\ B_i(u) = 0, & i = 1, \ldots n. \end{cases} \tag{2.1.2}$$

We will obtain the explicit expression of the Green's function related to the nonlocal problem (2.1.1) under the assumption that the corresponding homogeneous problem (2.1.2) has only the trivial solution. Moreover we will characterize the spectrum of problem (2.1.1) as a function of the value of the nonlocal operators over functions related to the Green's function of problem (2.1.1).

We notice that the nonlocal linear operators depend only on the values of the function that we are looking for, but they could depend on any of its derivatives, and by using analogous reasoning, we will obtain similar results.

If the homogeneous problem (2.1.2) has as a unique solution the trivial one, then, from Theorem 1.1.3, problem (2.1.1), with $\delta_i = 0$, $i = 0, \ldots, n$, has a unique solution for any given $\sigma \in C(I)$. Moreover, such solution is given by

$$v(t) = \int_a^b g_M(t, s)\sigma(s)\, ds, \tag{2.1.3}$$

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where g_M denotes the Green's function related to problem (2.1.2).

It should be noted that, in general, obtaining the Green's function g_M related to problem problem (2.1.2) is not an easy problem. If the coefficients of operator L_n are not constant, such function wolud be given, in the majority of the situations, as a series expansion of the eigenfunctions of the considered homogeneous problem, see [63, 73] and references therein. On the other hand, for the constant coefficients of L_n coupled with arbitrary two-point linear boundary conditions, it has been developed an algorithm in a Mathematica package that calculates the exact expression of the Green's function. This algorithm is avaliable at [11].

2.2 Explicit Expression of the Green's function

In this section we deduce the explicit expression of the solution of the general problem (2.1.1). Under suitable assumptions concerning the spectrum of the considered problem, we prove the existence and uniqueness of the solution of problem (2.1.1). Moreover, the expression of its related Green's function is obtained.

Similarly to the Lemma 1.1.8, we deduce the following result.

Lemma 2.2.1. *There exists the unique Green's function related to problem* (2.1.2) *if and only if for any $i \in \{1, \dots, n\}$, the following problem*

$$\begin{cases} \tilde{T}_n[M]\, u(t) = 0, & t \in I, \\ B_j(u) = 0, & j \in \{1, \dots, n\} \backslash \{i\}, \\ B_i(u) = 1, \end{cases} \tag{2.2.4}$$

has a unique solution, that we denote as $\omega_i(t)$, $t \in I$.

The main result is the following one.

Theorem 2.2.2. *Assume that problem* (2.1.2) *has $u = 0$ as its unique solution. Let $\sigma \in C(I)$, and δ_i, $i = 1, \dots, n$, be such that*

$$\det(I_n - A) \neq 0, \tag{2.2.5}$$

with I_n the identity matrix of order n and $A = (a_{ij})_{n \times n} \in \mathcal{M}_{n \times n}(\mathbb{R})$ given by

$$a_{ij} = \delta_j\, C_i(\omega_j), \quad i,\ j \in \{1, \dots, n\}.$$

Then problem (2.1.1) *has a unique solution $u \in C^n(I)$, given by the expression*

$$u(t) = \int_a^b G(t, s, \delta_1, \dots, \delta_n, M)\, \sigma(s)\, ds, \tag{2.2.6}$$

where

$$G(t, s, \delta_1, \dots, \delta_n, M) := g_M(t, s) + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j\left(g_M(\cdot, s)\right), \quad t,\ s \in I, \tag{2.2.7}$$

with ω_i, the unique solution of problem (2.2.4) *and $B = (b_{ij})_{n \times n} = (I_n - A)^{-1}$.*

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Proof. Since problem (2.1.1) has a unique solution when $\delta_i = 0$ for all $i = 1, \ldots, n$, from Lemma 2.2.1 we know that any solution of (2.1.1) satisfies the following expression

$$u(t) = v(t) + \sum_{i=1}^{n} \omega_i(t)\delta_i\, C_i(u), \quad t \in I, \tag{2.2.8}$$

with v given by (2.1.3).

Applying linear continuous operators C_j on both sides of (2.2.8), we infer that

$$C_j(u) = C_j(v) + C_j\left(\sum_{i=1}^{n} \delta_i\, \omega_i\, C(u)\right) = C_j(v) + \sum_{i=1}^{n} \delta_i\, C_j\left(\omega_i\right) C_i(u), \quad j = 1, \ldots, n,$$

from which we deduce that

$$C_j(u) - \sum_{i=1}^{n} \delta_i\, C_j(\omega_i)\, C_i(u) = C_j(v), \quad j = 1, \ldots, n.$$

Therefore, we arrive at the following systems of equations

$$(I_n - A) \begin{pmatrix} C_1(u) \\ C_2(u) \\ \vdots \\ C_n(u) \end{pmatrix} = \begin{pmatrix} C_1(v) \\ C_2(v) \\ \vdots \\ C_n(v) \end{pmatrix}. \tag{2.2.9}$$

From previous equality we deduce that

$$C_i(u) = \sum_{j=1}^{n} b_{ij}\, C_j(v), \quad i = 1, \ldots, n,$$

and substituting this expression in (2.2.8) we obtain that

$$u(t) = v(t) + \sum_{i=1}^{n} \delta_i\, \omega_i(t) \left(\sum_{j=1}^{n} b_{ij}\, C_j(v)\right), \quad t \in I.$$

To calculate $C_j(v)$ we use the fact that C_j is linear and continuous, so we get that

$$C_j(v) = C_j\left(\int_a^b g_M(\cdot, s)\sigma(s)\, ds\right) = \int_a^b C_j\left(g_M(\cdot, s)\right)\sigma(s)\, ds.$$

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Using the previous equality, we have that

$$\begin{aligned} u(t) &= \int_a^b g_M(t,s)\sigma(s)ds + \sum_{i=1}^n \delta_i\,\omega_i(t)\left(\sum_{j=1}^n b_{ij}\int_a^b C_j\left(g_M(\cdot,s)\right)\sigma(s)\,ds\right) \\ &= \int_a^b \left(g_M(t,s) + \sum_{i=1}^n \delta_i\,\omega_i(t)\left(\sum_{j=1}^n b_{ij}\,C_j\left(g_M(\cdot,s)\right)\right)\right)\sigma(s)\,ds \\ &= \int_a^b \left(g_M(t,s) + \sum_{i=1}^n\sum_{j=1}^n \delta_i\,b_{ij}\,\omega_i(t)\,C_j\left(g_M(\cdot,s)\right)\right)\sigma(s)\,ds \\ &= \int_a^b G(t,s,\delta_1,\ldots,\delta_n,M)\,\sigma(s)\,ds. \end{aligned}$$

Therefore, we have proved that under assumption (2.2.5) coupled with the uniqueness of solution of problem (2.1.2), problem (2.1.1) has at least one solution given by expression (2.2.6).

To conclude the proof, we must show the uniqueness of solution. To this end, suppose that u and v are two different solutions of problem (2.1.1). Then,

$$\begin{cases} T_n[M]\,(u-v)(t) = 0, & t\in I, \\ B_i(u-v) = \delta_i\,C_i(u-v), & i=1,\ldots,n. \end{cases} \tag{2.2.10}$$

As a consequence, we have that

$$(u-v)(t) = \sum_{i=1}^n \delta_i\,C_i(u-v)\,\omega_i(t), \quad t\in I.$$

Applying operator C_j in both sides again, we reach to

$$C_j(u-v) = \sum_{i=1}^n \delta_i\,C_i(u-v)\,C_j(\omega_i), \quad j=1,\ldots,n,$$

or, which is the same,

$$(I_n - A)\begin{pmatrix} C_1(u-v) \\ C_2(u-v) \\ \vdots \\ C_n(u-v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Condition (2.2.5) implies that $C_i(u-v)=0$ for $i=1,\ldots,n$. Hence, form (2.2.10) we deduce that $u-v$ is a solution of the homogeneous problem

$$\begin{cases} T_n[M]\,(u-v)(t) = 0, & t\in I, \\ B_i(u-v) = 0, & i=1,\ldots,n. \end{cases}$$

Since this problem has only the trivial solution, we deduce that $u=v$ on I, and the proof is concluded. □

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Remark 2.2.3. *We emphasize that in previous result, the condition that there is a unique Green's function g_M related to problem* (2.1.2) *is fundamental to construct function G on* (2.2.7) *but such condition is not necessary in order to deduce the existence and uniqueness of solution of problem* (2.1.1)*. Also, such condition is independent of the non homogeneous part of problem* (2.1.1)*: σ, δ_i and operators C_i, $i = 1, \ldots, n$. For practical purposes, the required assumptions guarantee the existence of a unique solution of problem* (2.1.1) *provided any parameter $(M, \delta_1, \ldots, \delta_n)$, such that M is not an eigenvalue of problem* (2.1.2)*. However, in the example of application to the first order periodic problem perturbed by integral boundary conditions that will be discussed in Section 2.3, problem* (2.1.1) *could have a unique solution for some choice of $(M, \delta_1, \ldots, \delta_n)$, with M an eigenvalue of problem* (2.1.2)*. This shows that such condition is not necessary to ensure the existence of a unique solution to the problem* (2.1.1)*.*

On the other hand, the nonspectral condition (2.2.5) *is equivalent to assume that* 1 *is not an eigenvalue of matrix A. When such condition fails we have that problem* (2.1.1) *has not a unique solution. So, this nonspectral condition characterizes the uniqueness of solution of problem* (2.1.1) *provided the existence of g_M is assumed. In case of M being an eigenvalue of problem* (2.1.2)*, condition* (2.2.5) *makes no sense because ω_i do not exist.*

Now, we will calculate the Green's function related to problem (2.1.1) for the particular case that all the functionals at boundary conditions are the same (that is, there is some linear continuous operator C such that $C_i = C$ for $i = 1, \ldots, n$).

As a direct consequence of the previous theorem, we obtain the following result for this particular case.

Corollary 2.2.4. *Assume that problem* (2.1.2) *has $u = 0$ as its unique solution. Let $\sigma \in C(I)$, and δ_i, $i = 1, \ldots, n$, be such that*

$$\sum_{i=1}^{n} \delta_i \, C(\omega_i) \neq 1. \tag{2.2.11}$$

holds.

Then problem

$$\begin{cases} T_n[M]\, u(t) = \sigma(t), & t \in I, \\ B_i(u) = \delta_i \, C(u), & i = 1, \ldots, n, \end{cases} \tag{2.2.12}$$

has a unique solution $u \in C^n(I)$, given by the expression

$$u(t) = \int_a^b G(t, s, \delta_1, \ldots, \delta_n, M)\, \sigma(s)\, ds,$$

where

$$G(t, s, \delta_1, \ldots, \delta_n, M) := g_M(t, s) + \frac{\displaystyle\sum_{i=1}^{n} \delta_i \, \omega_i(t)}{1 - \displaystyle\sum_{j=1}^{n} \delta_j C(\omega_j)} C\left(g_M(\cdot, s)\right). \tag{2.2.13}$$

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Proof. Since we have only $C(u)$ as a unique unknown variable, we can argue as in the proof of Theorem 2.2.2 and we have that system (2.2.9) reduces to the one dimensional equation

$$\left(1-\sum_{i=1}^{n}\delta_i\,C(\omega_i)\right)C(u)=C(v),$$

and condition (2.2.5) reduces to (2.2.11).

In this case, it is obvious that

$$C(u)=\frac{C(v)}{1-\sum_{i=1}^{n}\delta_i\,C(\omega_i)}$$

and we reach to

$$I_n-A\equiv(a_{11})=\left(1-\sum_{i=1}^{n}\delta_i\,C(\omega_i)\right)\in\mathcal{M}_{1\times1}(\mathbb{R})$$

and

$$B=(I_n-A)^{-1}\equiv(b_{11})=\left(\frac{1}{1-\sum_{i=1}^{n}\delta_i\,C(\omega_i)}\right).$$

As a consequence, we deduce that expression (2.2.7) is rewritten in this case as

$$\begin{aligned}G(t,s,\delta_1,\ldots,\delta_n,M)&=g_M(t,s)+\sum_{i=1}^{n}\delta_i\,b_{11}\,\omega_i(t)\,C\left(g_M(\cdot,s)\right)\\&=g_M(t,s)+\frac{\sum_{i=1}^{n}\delta_i\omega_i\left(t\right)}{1-\sum_{j=1}^{n}\delta_jC(\omega_j)}C\left(g_M(\cdot,s)\right).\end{aligned}$$

□

Now, given $M\in\mathbb{R}$ and $\delta_j, j\neq k$ fixed, by differentiating equality (2.2.13) with respect to δ_k we deduce that

$$\frac{\partial G}{\partial\delta_k}\left(t,s,\delta_1,\delta_2,\cdots,\delta_n,M\right)=\frac{\omega_k\left(t\right)\left(1-\sum_{j=1}^{n}\delta_jC\left(\omega_j\right)\right)+C\left(\omega_k\right)\sum_{j=1}^{n}\delta_j\omega_j\left(t\right)}{\left(1-\sum_{j=1}^{n}\delta_jC\left(\omega_j\right)\right)^2}C\left(g_M\left(\cdot,s\right)\right).\tag{2.2.14}$$

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Thus, we can study the monotony of the Green's function related to problem (2.2.12) with respect to any parameter δ_k. In Section 2.3 we will use this equality in the study of the perturbed first order periodic problem.

Among the examples of the linear operator C, we highlight the integral operator and the evaluation at an interior point of interval I as shown in the examples below.

Example 2.2.5. *If operator C is given by $C(u) = \int_a^b u(s)\,ds$, then*

$$C\left(\int_a^b g_M(t,s)\,\sigma(s)\,ds\right) = \int_a^b \left(\int_a^b g_M(t,s)\,\sigma(s)\,ds\right) dt$$
$$= \int_a^b \left(\int_a^b g_M(t,s)\,dt\right) \sigma(s)\,ds = \int_a^b C\left(g_M(\cdot,s)\right)\,\sigma(s)\,ds.$$

Example 2.2.6. *If C is defined as $C(u) = u(c)$, for some $c \in (a,b)$, we have that*

$$C\left(\int_a^b g_M(t,s)\,\sigma(s)\,ds\right) = \int_a^b g_M(c,s)\,\sigma(s)\,ds = \int_a^b C\left(g_M(\cdot,s)\right)\,\sigma(s)\,ds.$$

Finally, as a direct consequence of expression (2.2.13), we deduce the following comparison result:

Corollary 2.2.7. *Assume that problem* (2.1.2) *has $u = 0$ as its unique solution and let g_M be its unique Green's function. Assume that condition* (2.2.11) *holds and let G be the Green's function related to problem* (2.2.12)*. Suppose that the following hypotheses are satisfied:*

(a) $\sum_{j=1}^{n} \delta_j\, C(\omega_j) < 1.$

(b) $\delta_i\, \omega_i(t) \geq 0$, $\forall t \in I$, $= 1, \ldots, n$.

(c) If $u \geq 0$ on I, then $C(u) \geq 0$.

Then, the following assertions are fulfilled:

(i) If $g_M(t,s) \geq 0$ for all (t,s) on $I \times I$ then $G(t,s,\delta_1,\ldots,\delta_n,M) \geq g_M(t,s) \geq 0$ for all (t,s) on $I \times I$.

(ii) If $g_M(t,s) \leq 0$ for all (t,s) on $I \times I$ then $G(t,s,\delta_1,\ldots,\delta_n,M) \leq g_M(t,s) \leq 0$ for all (t,s) on $I \times I$.

Remark 2.2.8. *As we will see in next section, the conditions of previous corollary are sufficient but not necessary to ensure the positiveness of the related Green's function.*

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2.3 Example of a First Order Periodic Problem

In this section, we show the applicability of the expression (2.2.13) obtained in previous section for the perturbed first order periodic problem. Moreover, we show the validity of the assumptions of Theorem 2.2.2 and Corollary 2.2.7 as noted in the previous Remark 2.2.8.

To be concise, we study the sign of the Green's function related to the following perturbed first order periodic problem:

$$\begin{cases} u'(t) + M\,u(t) = \sigma(t), \quad t \in [0,1], \\ u(0) - u(1) = \delta \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{2.3.15}$$

with $M, \delta \in \mathbb{R}$.

It is immediate to verify that the spectrum of problem (2.3.15) is given by

$$\Lambda_{M\delta} = \{(M,\delta) \in \mathbb{R}^2 \ / \ M = \delta\}.$$

In particular, when we consider the homogeneous periodic problem ($\delta = 0$):

$$\begin{cases} u'(t) + Mu(t) = \sigma(t), \ \ t \in [0,1], \\ u(0) - u(1) = 0, \end{cases} \tag{2.3.16}$$

we have that $M = 0$ is the unique eigenvalue of the considered problem. That is, there is a unique g_M if and only if $M \neq 0$. Moreover, see [6], it is immediate to verify that the expression of the Green's function of problem (2.3.16) is given by the following expression

$$g_M(t,s) = \frac{1}{1-e^{-M}} \begin{cases} e^{-M(t-s)}, & 0 \leq s \leq t \leq 1, \\ e^{-M(t-s+1)}, & 0 < t < s \leq 1, \end{cases}$$

and the solution of problem

$$\begin{cases} u'(t) + Mu(t) = 0, \ \ t \in [0,1], \\ u(0) - u(1) = 1, \end{cases}$$

is given by

$$\omega_1(t) = g_M(t,0) = \frac{e^{-M\,t}}{1-e^{-M}}, \quad t \in [0,1].$$

In this case, we infer that condition (2.2.11) is written as $\delta \neq M$, $M \neq 0$. Thus, formula (2.2.13) can be applied to this set of (M,δ). We point out that, in this case, it is valid for all the values (M,δ) that are not on the spectrum of (2.3.15) except for the ones given by $(0,\delta)$, with $\delta \neq 0$. The expression for this last situation must be studied separately (by calculating the solution directly).

First, we deduce the following symmetric property of the Green's function related to problem (2.3.15).

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Lemma 2.3.1. *Assume that problem* (2.3.15) *has a unique solution and let* $G(t,s,\delta,M)$ *be its related Green's function. Then the following symmetric property holds:*

$$G(t,s,\delta,M) = -G(1-t,1-s,-\delta,-M). \tag{2.3.17}$$

Proof. Let

$$u(t) = \int_0^1 G(t,s,\delta,M)\,\sigma(s)\,ds$$

be the unique solution of problem (2.3.15).

It is immediate to verify that $v(t) := u(1-t)$ is the unique solution of the following problem:

$$\begin{cases} v'(t) - M\,v(t) = -\sigma(1-t), \quad t \in [0,1], \\ v(0) - v(1) = -\delta \displaystyle\int_0^1 v(s)\,ds. \end{cases}$$

Thus, we deduce that

$$v(t) = -\int_0^1 G(t,s,-\delta,-M)\,\sigma(1-s)\,ds.$$

On the other hand, we have that

$$v(t) = u(1-t) = \int_0^1 G(1-t,s,\delta,M)\,\sigma(s)\,ds = \int_0^1 G(1-t,1-s,\delta,M)\,\sigma(1-s)\,ds.$$

Therefore, the equality (2.3.17) is fulfilled directly by identifying the two previous equalities. □

With this symmetry result we only have to study the sign of the Green's function $G(t,s,\delta,M)$ for $M > 0$ and $\delta \neq M$. The case $M = 0$ and $\delta \neq 0$ will be considered further separately.

Applying the expression (2.2.13), we have that the Green's function $G(t,s,\delta,M)$ related to problem (2.3.15) is given by:

$$G(t,s,\delta,M) = g_M(t,s) + \frac{\delta\, g_M(t,0)}{1-\delta\displaystyle\int_0^1 g_M(r,0)\,dr}\int_0^1 g_M(r,s)\,dr.$$

Moreover, it is immediate that

$$C(g_M(\cdot,s)) := \int_0^1 g_M(r,s)\,dr = \frac{1}{M} \quad \text{for all } s \in [0,1] \text{ and } M \neq 0.$$

Therefore, the explicit expression of the Green's function $G(t,s,\delta,M)$ remains as:

$$G(t,s,\delta,M) = \frac{\delta}{M-\delta}\frac{e^{-Mt}}{1-e^{-M}} + \frac{1}{1-e^{-M}}\begin{cases} e^{-M(t-s)}, & 0 \le s \le t \le 1, \\ e^{-M(t-s+1)}, & 0 < t < s \le 1. \end{cases} \tag{2.3.18}$$

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Remark 2.3.2. *Using Definition 1.1.1, for all $s \in (0,1)$, it holds that*

$$\lim_{t\to s^+} g_M(t,s) \equiv g_M(s^+,s) = 1 + \lim_{t\to s^-} g_M(t,s) \equiv 1 + g_M(s^-,s).$$

So, we can define $g_M(s,s)$ as $g_M(s^+,s)$ or $g_M(s^-,s)$ at our convenience.

As a consequence, for the function $G(t,s,\delta,M)$ we have that

$$\lim_{t\to s^+} G(t,s,\delta,M) \equiv G(s^+,s,\delta,M) = 1 + \lim_{t\to s^-} G(t,s,\delta,M) \equiv 1 + G(s^-,s,\delta,M),$$

for all $s \in (0,1)$, and we can define $G(s,s,\delta,M)$ as $G(s^+,s,\delta,M)$ or $G(s^-,s,\delta,M)$. Also, the equality (2.3.17) *of symmetric property in Lemma 2.3.1 must be interpreted in this sense. As we will see in the sequel, this fact has no influence on the sign of the Green's function.*

It is immediate to check that $g_M(t,s) > 0$ for all $t,s \in [0,1]$ and $M > 0$. So, using Corollary 2.2.7, we have that $G(t,s,\delta,M) > 0$ for all $(t,s) \in I \times I$ and $0 \le \delta < M$.

Now let us see the range of $\delta < 0$ and $M > 0$ for which function $G(t,s,\delta,M)$ is positive on $I \times I$.

Since $G(t,s,0,M) = g_M(t,s) > 0$ for all $t,\ s \in I$ and $M > 0$, we know, from (2.3.18), that the Green's function $G(t,s,\delta,M)$ will be positive for some values of $\delta < 0$.

Moreover, in this case the expression (2.2.14) reduces to

$$\frac{\partial G}{\partial \delta}(t,s,\delta,M) = \frac{M}{(M-\delta)^2}\frac{e^{-Mt}}{1-e^{-M}} > 0, \quad \text{for all } M > 0,\ \delta \neq M \text{ and } t,s \in [0,1].$$

As a consequence of this equality, we deduce that for any $M > 0$ fixed, the Green's function $G(t,s,\delta,M)$ is strictly increasing with respect to δ and so we have that the optimal value, $\delta(M)$, will be either $-\infty$ or the biggest negative real value for which $G(t,s,\delta,M)$ attains the value zero at some point $(t_0,s_0) \in I \times I$.

The following result characterizes the strictly positive sign of $G(t,s,\delta,M)$ on $I \times I$ for $M > 0$.

Lemma 2.3.3. *Let $M > 0$, then the Green's function related to problem* (2.3.15) *is strictly positive on $I \times I$ if and only if*

$$\delta \in \left(\frac{M}{1-e^M}, M\right).$$

Moreover, if $\delta = \dfrac{M}{1-e^M}$ then $G(t,s,\delta,M) = 0$ for all $t \in [0,s)$ and $G(t,s,\delta,M) > 0$ for all $t \in [s,1]$.

Proof. To proof that this value of δ is optimal, using (2.3.18), we have that for any $s \in (0,1)$, the Green's function $G \in C^1([0,s) \cup (s,1])$ and there are two real constants, $k_1(s,\delta)$ and $k_2(s,\delta)$, such that

$$G(t,s,\delta,M) = k_1(s,\delta)\, e^{M\,t}, \quad \text{for all } t \in [0,s)$$

and

$$G(t,s,\delta,M) = k_2(s,\delta)\, e^{M\,t}, \quad \text{for all } t \in (s,1].$$

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So, we deduce that if $G(t_0, s_0, \delta, M) = 0$ for some $t_0 \in [0, s_0]$ ($t_0 \in [s_0, 1]$) then $G(t, s_0, \delta, M) = 0$ for all $t \in [0, s_0]$ ($t \in [s_0, 1]$).

On the other hand, since for any $s \in (0, 1)$, the Green's function satisfies the boundary condition

$$G(0, s, \delta, M) - G(1, s, \delta, M) = \delta \int_0^1 G(t, s, \delta, M)\, dt, \tag{2.3.19}$$

we deduce that, whenever $G > 0$ on $I \times I$ and $\delta < 0$, it holds that $G(0, s, \delta, M) < G(1, s, \delta, M)$. So, we only need to study the values of $G(t, s, \delta, M)$ at points of the form $(0, s)$.

Thus, we must look for the biggest value of $\delta < 0$ for which

$$h(s) := G(0, s, \delta, M) = \frac{\delta\, e^M + (M - \delta)\, e^{Ms}}{(e^M - 1)(M - \delta)} = 0 \quad \text{ at some } s \in (0, 1).$$

Since

$$h'(s) = \frac{M\, e^{Ms}}{e^M - 1} > 0 \quad \text{ for all } s \in I \text{ and } M > 0,$$

we conclude that the optimal value of δ comes from the first root of the equation

$$h(0) = \frac{\delta\, e^M + (M - \delta)}{(e^M - 1)(M - \delta)} = 0,$$

which, trivially, is given by

$$\delta = \frac{M}{1 - e^M}.$$

Therefore, h has a positive sign on I if and only if $h(0)$ is positive, that is $\delta > \dfrac{M}{1 - e^M}$. □

Using similar arguments, we obtain the following result that characterizes the strictly negative sign of $G(t, s, \delta, M)$ for $M > 0$.

Lemma 2.3.4. *Assume that $M > 0$, then the Green's function related to problem* (2.3.15) *is strictly negative on $I \times I$ if and only if*

$$\delta \in \left(M, \frac{M\, e^M}{1 - e^M} \right).$$

Moreover, if $\delta = \dfrac{M\, e^M}{1 - e^M}$ then $G(t, s, \delta, M) = 0$ for all $t \in [s, 1]$ and $G(t, s, \delta, M) < 0$ for all $t \in [0, s)$.

Proof. First, we know that if the set of values δ for which $G(t, s, \delta, M) < 0$ is not empty, necessarily $\delta > M > 0$.

To prove that the obtained value of δ is optimal, the same reasoning as the previous result is used.

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Again, using equation (2.3.19), we have that, if $G < 0$ on $I \times I$ and $\delta > 0$ then $G(0, s, \delta, M) < G(1, s, \delta, M)$. So, we only need to study the values of $G(t, s, \delta, M)$ at points of the form $(1, s)$ and the function to be studied is

$$k(s) := G\,(1, s, \delta, M) = \frac{\delta + (M - \delta)e^{Ms}}{(e^M - 1)\,(M - \delta)}.$$

Since

$$k'(s) = \frac{Me^{Ms}}{e^M - 1} > 0 \quad \text{for all } s \in I \text{ and } M > 0,$$

we conclude that the optimal value of δ comes from the first root of the equation

$$k(1) = \frac{\delta + (M - \delta)e^{M}}{(e^M - 1)\,(M - \delta)} = 0,$$

which, trivially, is given by

$$\delta = \frac{M\,e^M}{e^M - 1}.$$

Therefore, k has a negative sign on I if and only if $k(1)$ is negative, that is $\delta < \dfrac{M\,e^M}{e^M - 1}$. □

Using the property of symmetry (2.3.17), we get at the following characterization of the constant sign of $G\,(t, s, \delta, M)$ for $M < 0$.

Lemma 2.3.5. *If $M < 0$, the following properties are fulfilled:*

1. $G\,(t, s, \delta, M) < 0$ *for all* $(t, s) \in [0, 1] \times [0, 1]$, *if and only if* $\delta \in \left(M, \dfrac{Me^M}{e^M - 1}\right)$.

2. *If* $\delta = \dfrac{M\,e^M}{1 - e^M}$ *then* $G\,(t, s, \delta, M) = 0$ *for all* $t \in [s, 1]$ *and* $G\,(t, s, \delta, M) < 0$ *for all* $t \in [0, s)$.

3. $G\,(t, s, \delta, M) > 0$ *for all* $(t, s) \in [0, 1] \times [0, 1]$, *if and only if* $\delta \in \left(\dfrac{M}{1 - e^M}, M\right)$.

4. *If* $\delta = \dfrac{M}{1 - e^M}$ *then* $G\,(t, s, \delta, M) = 0$ *for all* $t \in [0, s)$ *and* $G\,(t, s, \delta, M) > 0$ *for all* $t \in [s, 1]$.

Finally, we deal with the case $M = 0$. As we have seen, we cannot use the formula (2.3.18) to calculate the Green's function since $M = 0$ is an eigenvalue of problem (2.3.16). So, we need to compute $G(t, s, \delta) := G(t, s, \delta, 0)$ directly.

By integration, a solution of $u'(t) = \sigma(t), t \in I$, is given by

$$u(t) = c + \int_0^t \sigma(s)\,ds. \tag{2.3.20}$$

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So, $u(0) - u(1) = -\int_0^1 \sigma(s)\, ds$. On the other hand,

$$\int_0^1 u(t)\, dt = \int_0^1 \left(c + \int_0^t \sigma(s)\, ds \right) dt = c + \int_0^1 \int_0^t \sigma(s)\, ds\, dt.$$

Applying Fubini's Theorem, we have that

$$\int_0^1 \int_0^t \sigma(s)\, ds\, dt = \int_0^1 \int_s^1 \sigma(s)\, dt\, ds = \int_0^1 (1-s)\, \sigma(s)\, ds.$$

Imposing the boundary conditions we obtain the following equality

$$c = \int_0^1 (s-1)\, \sigma(s)\, ds - \frac{1}{\delta} \int_0^1 \sigma(s)\, ds.$$

Substituting the value of c in (2.3.20) we have that

$$u(t) = \int_0^1 (s-1)\, \sigma(s)\, ds - \frac{1}{\delta} \int_0^1 \sigma(s)\, ds + \int_0^t \sigma(s)\, ds = \int_0^1 G(t,s,\delta)\, \sigma(s)\, ds,$$

being

$$G(t,s,\delta) = \begin{cases} s - \frac{1}{\delta}, & 0 \le s \le t \le 1, \\ \\ s - \frac{1}{\delta} - 1, & 0 < t < s \le 1. \end{cases}$$

It is easy to verify that $G(t,s,\delta) < 0$ for all $t, s \in [0,1]$ if and only if $0 < \delta < 1$, and that $G(t,s,\delta) > 0$ for all $t, s \in [0,1]$ if and only if $-1 < \delta < 0$.

Moreover,

$$G(1,1,1) = 0 \text{ and } G(t,s,1) < 0 \text{ for all } (t,s) \in (I \times I) \backslash \{(1,1)\}$$

and

$$G(0,0,-1) = 0 \text{ and } G(t,s,-1) > 0 \text{ for all } (t,s) \in (I \times I) \backslash \{(0,0)\}.$$

Figure 2.3.1 shows the regions where the function G maintains a constant sign.

As we have mentioned earlier, this example shows that the conditions of Corollary 2.2.7 are sufficient but not necessary to guarantee the constant sign of the related Green's function.

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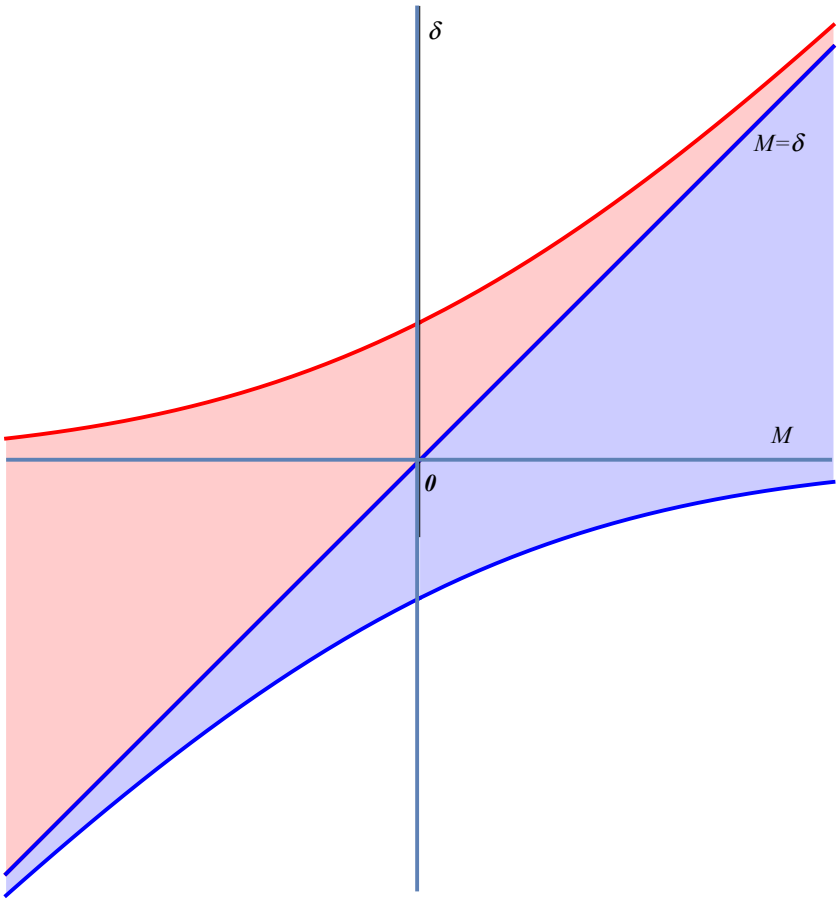

Figure 2.3.1: Regions of positive and negative sign of the Green's function in the plane $M \times \delta$. The blue region represents the positive sign of the Green's function while the red region corresponds which the negative sign of the Green's function.

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Chapter 3
Second Order Perturbed Periodic Problem

This chapter is devoted to characterize the constant sign of the Green's function related to a second order perturbed periodic problem coupled to integral boundary conditions. To do this, we will use the formula (2.2.7) proved in Theorem 2.2.2 of Chapter 2 to obtain the expression of the Green's function related to this problem.

The idea of the study is to determine the regions of constant sign of Green's functions related to the homogeneous problem and another integral problem with a unique parameter in the boundary conditions and, from this, to deduce the constant sign of the Green's function related to the main problem.

This chapter is organized as follows: In Section 3.1, we introduce the problem to be studied. In Section 3.2, we prove some properties of the Green's function that allow us to simplify the study of the main problem. Section 3.3 considers the particular case of the problem with a unique parameter in the boundary conditions. Finally, in Section 3.4 we calculate the regions of constant sign of Green's function related to the main problem using the study developed in Section 3.3 and the general properties proved in Section 3.2.

The current chapter includes the contents of the research article [18].

3.1 Introduction

Recently, integral boundary conditions have been widely considered in many works in the literature. Among the references related to this topic, see [13, 15, 54, 60, 71, 95] for integral boundary conditions in second and fourth order ODEs and references therein.

In this chapter we will study the regions of constant sign of the Green's function related to the following perturbed second order periodic problem with parameters, coupled with integral conditions on the boundary:

$$\begin{cases} u''(t) + M\,u(t) = \sigma(t), \quad t \in J := [0,1], \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{3.1.1}$$

where $M, \delta_1, \delta_2 \in \mathbb{R}$ and σ is a continuous function in J.

Note that this problem is a particular case of problem (2.1.1) with $n = 2$, $a_1 = a_2 = 0$, $[a,b] = [0,1]$, $B_1(u) = u(0) - u(1)$, $B_2(u) = u'(0) - u'(1)$, $C_1(u) = C_2(u) = \int_0^1 u(s)\,ds$.

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Following the notation introduced in Chapter 2, the Green's function related to problem (3.1.1) would be denoted by $G(t, s, \delta_1, \delta_2, M)$. However, since δ_1, δ_2 and M will be fixed parameters in each of the considered cases, for the sake of simplicity, we will denote $G_{M,\delta_1,\delta_2}(t, s) \equiv G(t, s, \delta_1, \delta_2, M)$ through this chapter.

We will make the study by distinguishing three cases: $M = 0$, $M > 0$ and $M < 0$. Each case will be analyzed separately and then we will characterize the optimal values of $M, \delta_1, \delta_2 \in \mathbb{R}$ for which the Green's function has constant sign.

In Chapter 2 we have proved in Theorem 2.2.2 the formula (2.2.7) that relates the Green's function of a differential problem coupled to funcional arbitrary linear nonlocal boundary conditions and the Green's function of the same differential problem coupled to homogeneous boundary conditions. Applying this formula to our problem, we will obtain the expression of Green's functions related to problem (3.1.1) for the cases $M > 0$ and $M < 0$. We will begin the study by analyzing the set of values of M and δ_2 for which the Green's function $G_{M,0,\delta_2}$ related to the following problem

$$\begin{cases} u''(t) + M\, u(t) = \sigma(t), \quad t \in J, \\ \quad u(0) - u(1) = 0, \\ \quad u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\, ds, \end{cases}$$

has constant sign on $J \times J$. Once we have found the expression of G_{M,δ_1,δ_2} as the sum of Green's function of the periodic problem and a function of t, we will use the known properties of the constant sign of Green's function of the periodic problem and properties of $G_{M,0,\delta_2}$ to characterize the regions of the constant sign of G_{M,δ_1,δ_2}.

For the case $M = 0$ we cannot apply this formula, since $M = 0$ is an eigenvalue of the periodic problem and, consequently, we will need to compute the expression of the Green's function of (3.1.1) by direct integration in this case.

3.2 Properties of the Green's Function

In this section, we deduce some properties of the Green's function G_{M,δ_1,δ_2}. First, we infer the following symmetric property of G_{M,δ_1,δ_2}.

Lemma 3.2.1. *Assume that problem* (3.1.1) *has a unique solution. Then the following symmetric property holds:*

$$G_{M,\delta_1,\delta_2}(t, s) = G_{M,-\delta_1,\delta_2}(1 - t, 1 - s), \quad t, s \in J. \tag{3.2.2}$$

Proof. Let

$$u(t) = \int_0^1 G_{M,\delta_1,\delta_2}(t, s)\, \sigma(s)\, ds$$

be the unique solution of problem (3.1.1).

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It is immediate to verify that $v(t) := u(1-t)$ is the unique solution of the following problem:

$$\begin{cases} v''(t) + Mv(t) = \sigma(1-t), & t \in J, \\ v(0) - v(1) = -\delta_1 \displaystyle\int_0^1 v(s)\, ds, \\ v'(0) - v'(1) = \delta_2 \displaystyle\int_0^1 v(s)\, ds. \end{cases}$$

As a direct consequence, we deduce that

$$v(t) = \int_0^1 G_{M,-\delta_1,\delta_2}(t,s)\, \sigma(1-s)\, ds.$$

On the other hand, we have that

$$v(t) = u(1-t) = \int_0^1 G_{M,\delta_1,\delta_2}(1-t,s)\, \sigma(s)\, ds = \int_0^1 G_{M,\delta_1,\delta_2}(1-t,1-s)\, \sigma(1-s)\, ds.$$

Therefore, the equality (3.2.2) is deduced directly by identifying the two previous equalities. □

Let us now characterize the points where a constant sign Grenn's function may vanish.

Lemma 3.2.2. *Let $M < \pi^2$. If G_{M,δ_1,δ_2} has constant sign on $J \times J$ and vanishes at some point (t_0, s_0), then either $t_0 = 0$, $t_0 = 1$ or $t_0 = s_0$.*

Proof. Let us suppose that $(t_0, s_0) \in (0,1) \times (0,1)$, with $t_0 > s_0$. In such a case, the function $u_1(t) := G_{M,\delta_1,\delta_2}(t, s_0)$ solves the problem

$$\begin{cases} u_1''(t) + M\, u_1(t) = 0, & t \in (s_0, 1], \\ u_1(t_0) = u_1'(t_0) = 0, \end{cases}$$

and so, $G_{M,\delta_1,\delta_2}(t, s_0) = 0$ for all $t \in (s_0, 1]$.

Now, using Condition $(G4)$ in Definition 1.1.1, we may extend u_1 to the interval J as the unique solution of the problem

$$\begin{cases} u_1''(t) + M\, u_1(t) = 0, & t \in [0, s_0), \\ u_1(s_0) = 0, \quad u_1'(s_0) = -1, \end{cases}$$

and from Condition $(G6)$ in Definition 1.1.1, this function must satisfy that:

$$u_1(0) = \delta_1 \int_0^1 u_1(t)\, dt = \delta_1 \int_0^{s_0} u_1(t)\, dt.$$

It can be easily seen that previous equality is only true for a particular value of δ_1:

$$\delta_1 = \begin{cases} \frac{\sqrt{M}\, \sin\left(\sqrt{M}\, s_0\right)}{1 - \cos\left(\sqrt{M}\, s_0\right)}, & M > 0, \\ \frac{2}{s_0}, & M = 0, \\ \frac{\sqrt{-M}\, \sinh\left(\sqrt{-M}\, s_0\right)}{-1 + \cosh\left(\sqrt{-M}\, s_0\right)}, & M < 0. \end{cases} \tag{3.2.3}$$

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Now, let us fix $t_1 \in (s_0, 1)$. As it was proved in [24], Section 3.2, it occurs that $v(s) = G_{M,\delta_1,\delta_2}(t_1, s)$ solves the problem

$$\begin{cases} v''(s) + M\, v(s) = 0, & s \in [0, t_1), \\ v(s_0) = v'(s_0) = 0, \end{cases} \tag{3.2.4}$$

and, as a consequence, $G_{M,\delta_1,\delta_2}(t_1, s) = 0$ for all $s \in [0, t_1)$.

Now, if we choose some $s_1 \in (0, t_1)$, $s_1 \neq s_0$, defining $u_2(t) = G_{M,\delta_1,\delta_2}(t, s_1)$ and reasoning analogously to the case with u_1, we deduce that $G_{M,\delta_1,\delta_2}(t, s_1) = 0$ for all $t \in (s_1, 1]$. As we have already seen, this is only possible if

$$\delta_1 = \begin{cases} \frac{\sqrt{M}\, \sin\left(\sqrt{M}\, s_1\right)}{1-\cos\left(\sqrt{M}\, s_1\right)}, & M > 0, \\ \frac{2}{s_1}, & M = 0, \\ \frac{\sqrt{-M}\, \sinh\left(\sqrt{-M}\, s_1\right)}{-1+\cosh\left(\sqrt{-M}\, s_1\right)}, & M < 0, \end{cases}$$

which contradicts (3.2.3).

In the case that the Green's function has constant sign and vanishes at some point $(t_0, 0)$, with $t_0 \in (0, 1)$, with the same arguments, we conclude that $G_{M,\delta_1,\delta_2}(t, 0) = 0$ for all $t \in J$. Now, for any $t_1 \in (0, 1)$, we have that $G_{M,\delta_1,\delta_2}(t_1, s)$ solves (3.2.4) with $v(0) = v'(0) = 0$. As a direct consequence, $G_{M,\delta_1,\delta_2}(t_1, s) = 0$ for all $s \in [0, t_1)$, and we arrive at a contradiction in a similar way to $s_0 \in (0, 1)$.

Finally, we note that the case $(t_0, s_0) \in (0, 1) \times (0, 1]$, with $t_0 < s_0$, can also be discarded as Lemma 3.2.1 implies that if G_{M,δ_1,δ_2} has constant sign and vanishes at $(t_0, s_0) \in (0, 1) \times (0, 1]$, with $t_0 < s_0$, then $G_{M,-\delta_1,\delta_2}$ will also have a constant sign and vanish at the point $(1 - t_0, 1 - s_0)$ which satisfies that $1 - t_0 > 1 - s_0 \geq 0$. □

Let us now compute the value the Green's function of problem (3.1.1). For $M \in \mathbb{R} \setminus \{0\}$, taking into account that

$$C(u) = C_1(u) = C_2(u) = \int_0^1 u(s)\, ds,$$

we have that, according to formula (2.2.7), the Green's function of problem (3.1.1) is given by the following expression

$$G_{M,\delta_1,\delta_2}(t, s) = G_{M,0,0}(t, s) + \frac{\delta_1\, \omega_1(t) + \delta_2\, \omega_2(t)}{1 - \left(\delta_1 \int_0^1 \omega_1(s)\, ds + \delta_2 \int_0^1 \omega_2(s)\, ds\right)} \int_0^1 G_{M,0,0}(t, s)\, dt, \tag{3.2.5}$$

where ω_1 is the unique solution to the problem

$$\begin{cases} u''(t) + M\, u(t) = 0, & t \in J, \\ u(0) - u(1) = 1, \\ u'(0) - u'(1) = 0, \end{cases}$$

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and ω_2 is the unique solution to the problem

$$\begin{cases} u''(t) + M\,u(t) = 0, & t \in J, \\ u(0) - u(1) = 0, \\ u'(0) - u'(1) = 1. \end{cases}$$

In this case, it is immediate to verify that $\omega_1(t) = \omega_2'(t)$, for all $t \in J$, therefore $\int_0^1 \omega_1(s)\,ds = 0$. Moreover, it is very well-known (see [6]) that $\omega_2(t) = G_{M,0,0}(t,0)$, and this function satisfies that

$$G_{M,0,0}(t,s) = \begin{cases} G_{M,0,0}(t-s,0), & 0 \le s \le t \le 1, \\ \\ G_{M,0,0}(1+t-s,0), & 0 \le t < s \le 1. \end{cases}$$

As a consequence,

$$\int_0^1 G_{M,0,0}(t,s)dt = \int_0^1 G_{M,0,0}(t,0)dt = \int_0^1 \omega_2(t)dt = -\frac{1}{M}\int_0^1 \omega_2''(t)dt = \frac{1}{M}, \quad \forall\, s \in J.$$

Then, the expression (3.2.5) can be rewritten as

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{M - \delta_2} = G_{M,0,\delta_2}(t,s) + \frac{\delta_1\,\omega_1(t)}{M - \delta_2}. \quad (3.2.6)$$

In the next section, taking into account the previous expression, we shall start with the study of the case $\delta_1 = 0$, and later on, we use (3.2.6) to study the general case the previous one.

3.3 Study of the Case $\boldsymbol{\delta_1 = 0}$

In this section, we deal with the study of the following linear perturbed periodic problem (considering $\delta_1 = 0$ in (3.1.1)):

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), & t \in J, \\ u(0) - u(1) = 0, \\ \\ u'(0) - u'(1) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \quad (3.3.7)$$

for $M, \delta_2 \in \mathbb{R}$. We deduce some suitable properties of the related Green's function in order to characterize the range of values M and δ_2 for which $G_{M,0,\delta_2}$ has constant sign on $J \times J$.

First of all, we note that the spectrum of problem (3.3.7) is given by

$$\Lambda_{M,\delta_2} = \left\{ (4k^2\pi^2, \delta_2),\ \delta_2 \in \mathbb{R},\ k = 1, 2, \dots \right\} \cup \left\{ (M, M),\ M \in \mathbb{R} \right\}.$$

On the other hand, the spectrum of the homogeneous periodic problem ($\delta_1 = \delta_2 = 0$)

$$\begin{cases} u''(t) + Mu(t) = \sigma(t), & t \in J, \\ u(0) - u(1) = 0, \\ u'(0) - u'(1) = 0, \end{cases}$$

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is given by

$$\Lambda_H = \{4k^2\pi^2 : \ k = 0, 1, 2 \ldots\},$$

that is, $G_{M,0,0}$ exists and is unique if and only if $M \neq 4k^2\pi^2$, $k = 0, 1, 2 \ldots$.

Thus, formula (3.2.5) is valid to compute $G_{M,0,\delta_2}$ for all $M \neq 4k^2\pi^2$, $k = 0, 1, \ldots$ and $\delta_2 \neq M$. The Green's function $G_{0,0,\delta_2}$, with $\delta_2 \neq 0$, exists, but it can not be calculated using (3.2.5), so we need to do this by means of direct integration.

In the following results we describe the points in $J \times J$ where a constant sign Green's function $G_{M,0,\delta_2}$ may vanish.

Lemma 3.3.1. *Let $M < \pi^2$. If $\delta_2 < 0$, $G_{M,0,\delta_2}$ is nonnegative on $J \times J$ and vanishes at some point $(t_0, s_0) \in J \times J$, then $t_0 = s_0$.*

Proof. From Lemma 3.2.2 we only need to discard the cases $(0, s_0)$ and $(1, s_0)$ with $s_0 \in (0, 1)$. We note that, since $G_{M,0,\delta_2}(0, s_0) = G_{M,0,\delta_2}(1, s_0)$, both cases are equivalent. Suppose then that

$$G_{M,0,\delta_2}(0, s_0) = G_{M,0,\delta_2}(1, s_0) = 0.$$

In such a case, since $G_{M,0,\delta_2}$ is nonnegative on $J \times J$, it must occur that $\frac{\partial}{\partial t} G_{M,0,\delta_2}(0, s_0) \geq 0$ and $\frac{\partial}{\partial t} G_{M,0,\delta_2}(1, s_0) \leq 0$, which contradicts the fact that

$$\frac{\partial}{\partial t} G_{M,0,\delta_2}(0, s) - \frac{\partial}{\partial t} G_{M,0,\delta_2}(1, s) = \frac{\delta_2}{M - \delta_2} < 0, \quad \forall\, s \in (0, 1).$$

As a consequence, the only possibility is that $t_0 = s_0$. □

Lemma 3.3.2. *Let $M < \pi^2$. If $\delta_2 > M$, $G_{M,0,\delta_2}$ is nonpositive on $J \times J$ and vanishes at some point $(t_0, s_0) \in J \times J$, then either $t_0 = 0$ or $t_0 = 1$.*

Proof. Again, from Lemma 3.2.2 we only need to discard the case $t_0 = s_0$. In such a case, since $G_{M,0,\delta_2}$ is nonpositive, it must occur that

$$\frac{\partial}{\partial t} G_{M,0,\delta_2}(t_0^-, t_0) \geq 0 \ \text{ and } \ \frac{\partial}{\partial t} G_{M,0,\delta_2}(t_0^+, t_0) \leq 0,$$

which contradicts the fact that

$$\frac{\partial}{\partial t} G_{M,0,\delta_2}(t^+, t) - \frac{\partial}{\partial t} G_{M,0,\delta_2}(t^-, t) = 1 \quad \forall\, t \in (0, 1),$$

by the property (G_4) of Definition 1.1.1.

As a consequence, the only possibility is that either $t_0 = 0$ or $t_0 = 1$. □

3.3.1 Expression of $G_{M,0,\delta_2}$

We calculate now the exact expression of $G_{M,0,\delta_2}$ by considering the different values of the parameters M and δ_2. We start with $M \neq 0$, that is, the situation in which problem (3.3.7) is uniquely solvable for $\delta_2 = 0$.

The exact expression of the function $G_{M,0,0}$ for $M \neq 0$ is given in [6], where, this expression was computed using the Mathematica package program [11].

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- **Case $M \neq 0$**

  In this case, from the expression (3.2.6) and since $w_2(t) = G_{M,0,0}(t,s)$, $t \in J$, we have

  $$G_{M,0,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_2}{M-\delta_2}\, G_{M,0,0}(t,0). \tag{3.3.8}$$

  We shall consider two different cases:

  1) $\boldsymbol{M = m^2 > 0}$, **with** $\boldsymbol{m \in (0,\infty)}$:
  In such a case, $G_{M,0,0}$ is given by the expression

  $$G_{M,0,0}(t,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m} \begin{cases} \cos\left(\dfrac{m}{2}\,(1+2s-2t)\right), & 0 \le s \le t \le 1, \\ \cos\left(\dfrac{m}{2}\,(1+2t-2s)\right), & 0 \le t < s \le 1, \end{cases}$$

  and so, (3.3.8) implies that

  $$\begin{aligned} G_{M,0,\delta_2}(t,s) =& \frac{\delta_2}{2m(m^2-\delta_2)}\, \csc\left(\frac{m}{2}\right) \cos\left(\frac{m}{2}\,(1-2t)\right) \\ &+ \frac{\csc\left(\frac{m}{2}\right)}{2m} \begin{cases} \cos\left(\dfrac{m}{2}\,(1+2s-2t)\right), & 0 \le s \le t \le 1, \\ \cos\left(\dfrac{m}{2}\,(1+2t-2s)\right), & 0 \le t < s \le 1. \end{cases} \end{aligned}$$

  2) $\boldsymbol{M = -m^2 < 0}$, **with** $\boldsymbol{m \in (0,\infty)}$:
  In this case, $G_{M,0,0}$ is given by

  $$G_{M,0,0}(t,s) = \frac{1}{2m\,(1-e^m)} \begin{cases} e^{m(1+s-t)} + e^{m(t-s)}, & 0 \le s \le t \le 1, \\ e^{m(1+t-s)} + e^{m(s-t)}, & 0 \le t < s \le 1. \end{cases}$$

  Thus,

  $$\begin{aligned} G_{M,0,\delta_2}(t,s) =& -\frac{\delta_2}{2m(1-e^m)(m^2+\delta_2)} \left(e^{m\,t} + e^{m\,(1-t)}\right) \\ &+ \frac{1}{2m\,(1-e^m)} \begin{cases} e^{m(1+s-t)} + e^{m(t-s)}, & 0 \le s \le t \le 1, \\ e^{m(1+t-s)} + e^{m(s-t)}, & 0 \le t < s \le 1. \end{cases} \end{aligned}$$

- **Case $M = 0$**

  In this case, as we have seen, we cannot use the formula (3.2.6) to calculate $G_{0,0,\delta_2}$, so we need compute it directly.

  By integration, the solution of equation $u''(t) = \sigma(t)$ is given by

  $$u(t) = c_1 + c_2 t + \int_0^t (t-s)\, \sigma(s)\, ds.$$

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Then,

$$u'(t) = c_2 + \int_0^t \sigma(s)\, ds.$$

Imposing now condition $u(0) = u(1)$, we have that $c_2 = -\int_0^1 (1-s)\,\sigma(s)\, ds$. Therefore, $u'(0) - u'(1) = -\int_0^1 \sigma(s)\, ds$, and since $u'(0) - u'(1) = \delta_2 \int_0^1 u(s)\, ds$, we deduce that

$$c_1 = -\frac{1}{\delta_2}\int_0^1 \sigma(s)\, ds - \frac{1}{2}\int_0^1 \left(1-s^2\right)\sigma(s)\, ds + \int_0^1 \left(s-s^2\right)\sigma(s)\, ds + \frac{1}{2}\int_0^1 (1-s)\,\sigma(s)\, ds.$$

Therefore,

$$\begin{aligned} u(t) = &-\frac{1}{\delta_2}\int_0^1 \sigma(s)\, ds - \frac{1}{2}\int_0^1 \left(1-s^2\right)\sigma(s)\, ds + \int_0^1 s\,(1-s)\,\sigma(s)\, ds \\ &+ \left(\frac{1}{2} - t\right)\int_0^1 (1-s)\,\sigma(s)\, ds + \int_0^t (t-s)\,\sigma(s)\, ds, \\ = &\int_0^1 G_{0,0,\delta_2}(t,s)\,\sigma(s)\, ds, \end{aligned}$$

where

$$G_{0,0,\delta_2}(t,s) = \begin{cases} -\dfrac{s}{2} - \dfrac{s^2}{2} + st - \dfrac{1}{\delta_2}, & 0 \le s \le t \le 1, \\ \dfrac{s}{2} - \dfrac{s^2}{2} - t + st - \dfrac{1}{\delta_2}, & 0 \le t < s \le 1. \end{cases}$$

3.3.2 Regions of Constant Sign of $G_{M,0,\delta_2}$

We shall study now the regions in which the functions that we have just calculated have constant sign on $J \times J$. First, we note that we can bound these regions in the following way.

Lemma 3.3.3. *$G_{M,0,\delta_2}$ will never have constant sign on $J \times J$ for any $M = m^2 > \pi^2$.*

Proof. From expression (3.3.8) and the fact that

$$G_{M,0,\delta_2}(t,0) = \left(1 + \frac{\delta_2}{M-\delta_2}\right) G_{M,0,0}(t,0) = \frac{m^2}{m^2-\delta_2}\,\frac{\csc\left(\frac{m}{2}\right)\cos\left(\frac{m}{2}(1-2t)\right)}{2m},$$

it is immediately deduced that $G_{M,0,\delta_2}(t,0)$ is sign-changing on J for any $m > \pi$. □

Lemma 3.3.4. *The following properties are fulfilled:*

- *If $M < \delta_2 \le 0$, then $G_{M,0,\delta_2}$ is negative on $J \times J$.*
- *If $0 \le \delta_2 < M \le \pi^2$, then $G_{M,0,\delta_2}$ is positive on $J \times J$.*
- *If $M = \pi^2$ and $0 \le \delta_2 < M$, then $G_{M,0,\delta_2}$ vanishes at the set*

  $$A := \{(0,0),(0,1),(1,0),(1,1)\}$$

  and is positive on $(J \times J)\backslash A$.

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Proof. It is immediately deduced from (3.3.8) and the fact that $G_{M,0,0}$ is negative on $J \times J$ for $M < 0$, positive on $J \times J$ for $0 < M < \pi^2$, and positive on $(J \times J) \backslash A$, vanishing at the set A, for $M = \pi^2$. □

Moreover, from (2.2.14) we have that

$$\frac{\partial}{\partial \delta_2} G_{M,0,\delta_2}(t,s) = \frac{M}{(M-\delta_2)^2}\, G_{M,0,0}(t,0) > 0,\ \forall\, M \in (-\infty, \pi^2)\backslash\{0\},\ \delta_2 \neq M,\ t,\ s \in J$$

and

$$\frac{\partial}{\partial \delta_2} G_{0,0,\delta_2}(t,s) = \left(\frac{1}{\delta_2}\right)^2 > 0 \quad \forall\, t,\ s \in J. \tag{3.3.9}$$

As a consequence, for any fixed $M < \pi^2$, $G_{M,0,\delta_2}$ is strictly increasing with respect to δ_2 and so we deduce the following facts:

- Since $G_{M,0,0} > 0$ on $J \times J$ for $M \in (0, \pi^2)$, we know that $G_{M,0,\delta_2}$ will be positive for some values of $\delta_2 < 0$. In particular, $G_{M,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(M), 0]$, where the optimal value $\delta_2(M)$ will be either $-\infty$ or the biggest negative real value for which $G_{M,0,\delta_2(M)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.
- Since $G_{M,0,0} < 0$ on $J \times J$ for $M < 0$, we know that $G_{M,0,\delta_2}$ will be negative for some values of $\delta_2 > 0$. In particular, $G_{M,0,\delta_2}$ will be negative for $\delta_2 \in [0, \delta_2(M))$, where the optimal value $\delta_2(M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,0,\delta_2(M)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.

Let us study now the range of values $\delta_2 < 0$ for which $G_{M,0,\delta_2}$ is positive. We get to the next result.

Theorem 3.3.5. *If $M = m^2$ with $m \in (0, \pi)$ and $\delta_2 \leq 0$, then $G_{M,0,\delta_2}(t,s) > 0$ for all $(t,s) \in J \times J$ if and only if*

$$-\frac{m^2 \cos\left(\frac{m}{2}\right)}{1 - \cos\left(\frac{m}{2}\right)} < \delta_2 \leq 0.$$

Proof. From Lemma 3.3.1, we only need to study the values of function $G_{M,0,\delta_2}$ at the diagonal of its square of definition, where we get the function

$$h(t) = G_{M,0,\delta_2}(t,t) = \frac{\coth\left(\frac{m}{2}\right)}{2m} + \frac{\delta_2 \cos\left(\frac{m}{2}(1-2t)\right) \csc\left(\frac{m}{2}\right)}{2m\,(m^2 - \delta_2)},\quad t \in J.$$

Furthermore,

$$h'(t) = \frac{\delta_2\ \csc\left(\frac{m}{2}\right) \sin\left(\frac{m}{2}(1-2t)\right)}{2\,(m^2 - \delta_2)},\quad t \in J,$$

and one can verify that, $h'(t) = 0$ if and only if $t = \frac{1}{2}$. Since

$$h''(t) = -\frac{m\,\delta_2\ \cos\left(\frac{m}{2}(1-2t)\right) \csc\left(\frac{m}{2}\right)}{2\,(m^2 - \delta_2)},$$

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it occurs that, $h''(\frac{1}{2}) > 0$ for $\delta_2 < 0$ and $m \in (0, \pi)$. Then, $t = \frac{1}{2}$ is an absolute minimum of h.

Thus, h has positive sign on I if and only if $h(\frac{1}{2}) > 0$, that is,

$$-\frac{m^2 \cos\left(\frac{m}{2}\right)}{1 - \cos\left(\frac{m}{2}\right)} < \delta_2 \leq 0, \quad m \in (0, \pi)\,.$$

□

Reasoning as before, let us see now the range of values of $\delta_2 > 0$ for which the function $G_{M,0,\delta_2}$ is negative.

Theorem 3.3.6. *If $M = -m^2$ with $m \in (0, \infty)$ and $\delta_2 \geq 0$, then $G_{M,0,\delta_2}(t, s) < 0$ for all $(t, s) \in J \times J$ if and only if*

$$0 \leq \delta_2 < \frac{2m^2 e^{\frac{m}{2}}}{1 + e^m - 2e^{\frac{m}{2}}}.$$

Proof. From Lemma 3.3.2, we only need to study the values of function $G_{M,0,\delta_2}$ at the points of the form $(0, s)$ and $(1, s)$. So, we have to study the function

$$r(s) = G_{M,0,\delta_2}(0, s) = G_{M,0,\delta_2}(1, s) = \frac{1}{2m\,(1 - e^m)} \left(e^{m(1-s)} + e^{ms} - \frac{\delta_2}{\delta_2 + m^2}\,(1 + e^m) \right).$$

Since $\frac{1}{2m(1-e^m)} < 0$ for all $m \in (0, \infty)$, it is enough to study the positive sign of the function

$$\beta(s) = e^{m\,(1-s)} + e^{ms} - \frac{\delta_2}{\delta_2 + m^2}(1 + e^m).$$

The derivative of β is

$$\beta'(s) = -me^{m\,(1-s)} + me^{ms},$$

which attains the value zero at $s = \frac{1}{2}$ and $\beta''\left(\frac{1}{2}\right) > 0$. Then, $s = \frac{1}{2}$ is a minimum of β. Since $\beta\,(0) = \beta\,(1) > 0$ we have that β is positive if and only if $\beta\left(\frac{1}{2}\right) > 0$, that is,

$$0 \leq \delta_2 < \frac{2m^2 e^{\frac{m}{2}}}{1 + e^m - 2e^{\frac{m}{2}}}, \quad m \in (0, \infty)\,.$$

□

From (3.3.9), we have that $G_{0,0,\delta_2}$ is strictly increasing with respect to δ_2. Then, using the previous results we deduce the following facts:

- Since $G_{M,0,\delta_2} > 0$ for $M \in (0, \pi^2)$ and $-\frac{m^2 \cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)} < \delta_2 \leq 0$, we know that $G_{0,0,\delta_2}$ will be positive for some values of $\delta_2 < 0$. In particular, $G_{0,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(0), 0)$, where the optimal value $\delta_2(0)$ will be either $-\infty$ or the biggest negative real value for which $G_{0,0,\delta_2(0)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.

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- Since $G_{M,0,\delta_2} < 0$ for $M < 0$ and $0 \leq \delta_2 < \frac{2m^2 e^{\frac{m}{2}}}{1+e^m-2e^{\frac{m}{2}}}$, we know that $G_{0,0,\delta_2}$ will be negative for some values of $\delta_2 > 0$. In particular, $G_{0,0,\delta_2}$ will be negative for $\delta_2 \in (0, \delta_2(0))$, where the optimal value $\delta_2(0)$ will be either $+\infty$ or the smallest positive real value for which $G_{0,0,\delta_2(0)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.

Let us study the sign of function $G_{0,0,\delta_2}$ according to the value of $\delta_2 \in \mathbb{R} \setminus \{0\}$.

Theorem 3.3.7. *$G_{0,0,\delta_2}$ is strictly positive on $J \times J$ if and only if $\delta_2 \in (-8, 0)$.*

Proof. For $\delta_2 < 0$, from Lemma 3.3.1, the function to study in this case is:

$$g(s) = G(0, s, \delta_2) = G(1, s, \delta_2) = -\frac{s}{2} + \frac{s^2}{2} - \frac{1}{\delta_2}, \quad s \in J.$$

Since, $g'(s) = -\frac{1}{2} + s$, $g'(s) = 0$ if and only if $s = \frac{1}{2}$, and $g''(\frac{1}{2}) > 0$. Thus, $s = \frac{1}{2}$ is an absolute minimum of g. On the other hand, since $g(0) = g(1) = -\frac{1}{\delta_2} > 0$ we deduce that g is positive on I if and only if $g(\frac{1}{2}) > 0$, that is, $-8 < \delta_2 < 0$. In conclusion, $G_{0,0,\delta_2}(t, s) > 0$ for all $(t, s) \in J \times J$ if and only if $-8 < \delta_2 < 0$. □

Using the same arguments, for the positive sign of δ_2 we arrive at the following result.

Theorem 3.3.8. *$G_{0,0,\delta_2}$ is strictly negative on $J \times J$ if and only if $\delta_2 \in (0, 8)$.*

Proof. For $\delta_2 > 0$, using Lemma 3.3.1, the function to study in this case is

$$r(s) = G_{0,0,\delta_2}(0, s) = G_{0,0,\delta_2}(1, s) = \frac{s}{2} - \frac{s^2}{2} - \frac{1}{\delta_2}, \quad s \in J,$$

which reaches its maximum at $s = \frac{1}{2}$. As a consequence, $G_{0,0,\delta_2}$ is negative if and only if $r(\frac{1}{2}) < 0$, that is, $0 < \delta_2 < 8$. □

Finally, as a consequence of the previous results we infer that:

- Since $G_{0,0,\delta_2} > 0$ on $J \times J$ for $\delta_2 \in (-8, 0)$, we know that $G_{M,0,\delta_2}$ will be positive for some values of $\delta_2 < M < 0$. In particular, $G_{M,0,\delta_2}$ will be positive for $\delta_2 \in (\delta_2(M), M)$, where the optimal value $\delta_2(M)$ will be either $-\infty$ or the biggest negative real value for which $G_{M,0,\delta_2(M)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.
- Since $G_{0,0,\delta_2} < 0$ on $J \times J$ for $\delta_2 \in (0, 8)$, we know that $G_{M,0,\delta_2}$ will be negative for some values of $\delta_2 > M > 0$. In particular, $G_{M,0,\delta_2}$ will be negative for $\delta_2 \in (M, \delta_2(M))$, where the optimal value $\delta_2(M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,0,\delta_2(M)}$ attains the value zero at some point $(t_0, s_0) \in J \times J$.

Let us study now the range of values $\delta > M = m^2 > 0$, $m \in (0, \pi)$ for which $G_{M,0,\delta_2}$ is negative.

Theorem 3.3.9. *If $M = m^2$ with $m \in (0, \pi)$, then $G_{M,0,\delta_2}(t, s) < 0$ for all $(t, s) \in J \times J$ if and only if*

$$m^2 < \delta_2 < \frac{m^2}{1 - \cos\left(\frac{m}{2}\right)}.$$

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Proof. Reasoning as before, using Lemma 3.3.2, we only need to consider $G_{M,0,\delta}$ at the points of the form $(0,s)$ and $(1,s)$, where the corresponding function to study is:

$$r(s) = G_{M,0,\delta_2}(0,s) = G_{M,0,\delta_2}(1,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m}\left[\cos\left(\frac{m}{2}\,(1-2t)\right) + \frac{\delta_2}{m^2-\delta_2}\cos\left(\frac{m}{2}\right)\right].$$

In this case, r has an absolute maximum at $s=\frac{1}{2}$. Thus, $r(s)<0$ for all $s\in J$ if and only if $r\left(\frac{1}{2}\right)<0$, that is, $\delta_2 < \frac{m^2}{1-\cos\left(\frac{m}{2}\right)}$. □

We will now make a study of the positive sign of $G_{M,0,\delta_2}$ for $\delta_2 < M = -m^2 < 0$ and $m\in(0,\infty)$.

Theorem 3.3.10. *If $M=-m^2$ with $m\in(0,\infty)$, then $G_{M,0,\delta_2}(t,s)>0$ for all $(t,s)\in J\times J$ if and only if $\delta_2 > -\dfrac{m^2\,(1+e^m)}{1+e^m-2e^{\frac{m}{2}}}$.*

Proof. From Lemma 3.3.1 we only must to study the behavior of the Green's function at the points of its diagonal. The function to consider

$$h(t) = G_{M,0,\delta_2}(t,t) = \frac{e^m+1}{2m\,(1-e^m)} - \frac{\delta_2}{\delta_2+m^2}\,\frac{e^{m(1-t)}+e^{mt}}{2m\,(1-e^m)},$$

has in this case an absolute minimum at $t=\frac{1}{2}$. So, h is positive on J if and only if $h\left(\frac{1}{2}\right)>0$, that is, $\delta_2 > -\dfrac{m^2\,(1+e^m)}{1+e^m-2e^{\frac{m}{2}}}$. □

Figure 3.3.1 shows the regions where the function $G_{M,0,\delta_2}$ maintains a constant sign.

3.4 Study of the Case $\delta_1 \neq 0$

In this section, from the study of the previous case, we calculate the regions of constant sign of Green's function related to problem (3.1.1) for the general situation of $\delta_1\neq 0$. As in the previous section, we divide the study in two cases depending on the fact that the parameter M is or not equal to zero.

First, we obtain the exact expression of G_{M,δ_1,δ_2} for the cases $M\neq 0$ and $M=0$.

- **Case $M\neq 0$**

  In this case, according to the main formula (3.2.6) and the fact that $\omega_1(t)=\omega_2'(t)$, $t\in J$, the Green's function related to problem (3.1.1) is given by

  $$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,0}(t,s) + \frac{\delta_1\,\omega_2'(t)+\delta_2\,\omega_2(t)}{M-\delta_2},$$

  where

  $$\omega_2(t) = \frac{1}{2m}\begin{cases} \cos\left(\frac{m}{2}\,(2\,t-1)\right)\csc\left(\frac{m}{2}\right), & M=m^2,\ M\neq 4k^2\pi^2,\ k=0,1,\ldots \\ -\cosh\left(\frac{m}{2}\,(2\,t-1)\right)\operatorname{csch}\left(\frac{m}{2}\right), & M=-m^2<0, \end{cases}$$

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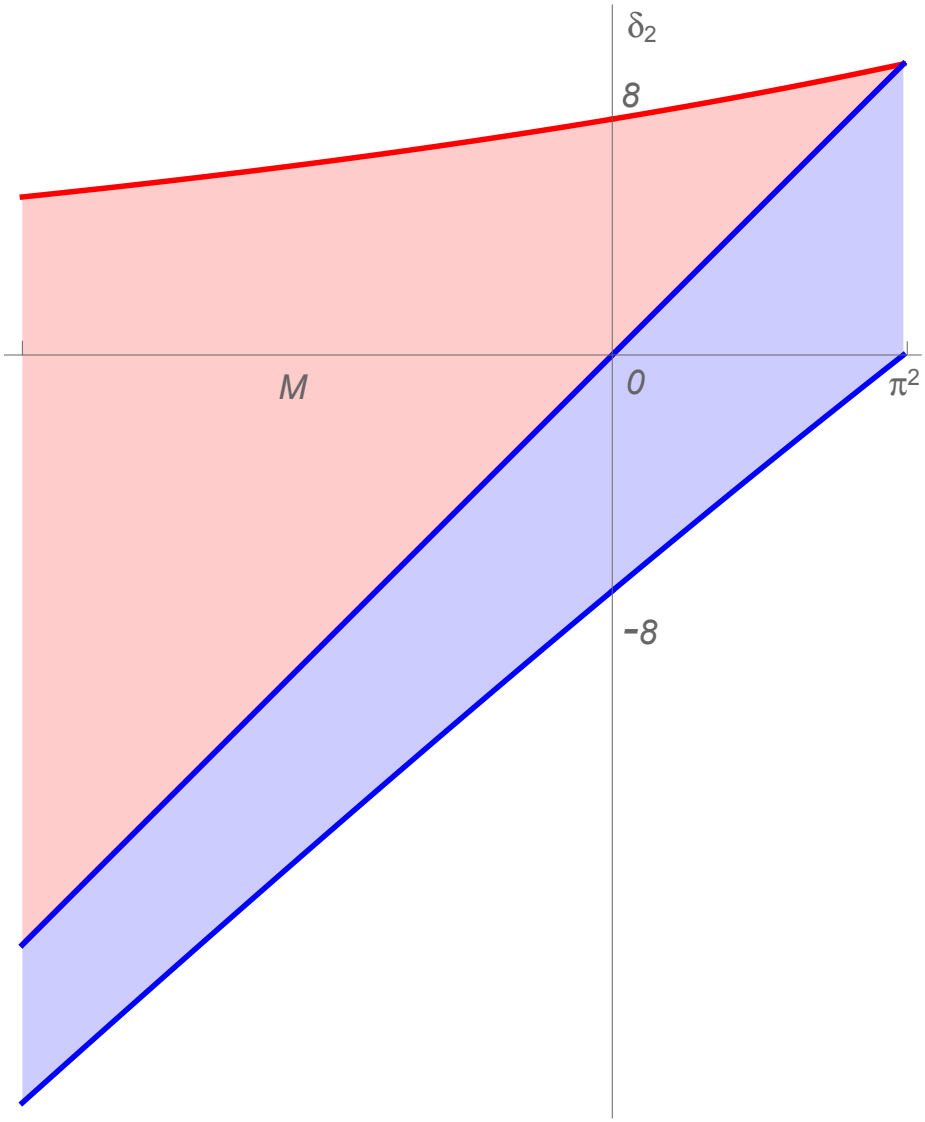

Figure 3.3.1: The blue region represents the positive sign of $G_{M,0,\delta_2}$ while the red region corresponds to the negative sign of $G_{M,0,\delta_2}$ at the the plane $M \times \delta_2$.

and the expression of $G_{M,0,0}$ has been given in Subsection 3.3.1.

We distinguish two cases:

1) $\boldsymbol{M = m^2 > 0, m > 0}$ **and** $\boldsymbol{M \neq 4k^2\pi^2, k = 0, 1, \ldots}$:
In such a case,

$$G_{M,\delta_1,\delta_2}(t,s) = \frac{\csc\left(\frac{m}{2}\right)}{2m}\left(\frac{\delta_2\,\cos\left(m\left(\frac{1}{2}-t\right)\right) + m\,\delta_1\,\sin\left(m\left(\frac{1}{2}-t\right)\right)}{m^2-\delta_2} + \begin{cases}\cos\left(m\left(\frac{1}{2}+s-t\right)\right), & 0\le s\le t\le 1\\ \\ \cos\left(m\left(\frac{1}{2}+t-s\right)\right), & 0\le t<s\le 1\end{cases}\right).$$

2) $\boldsymbol{M = -m^2, m > 0}$:
In this case,

$$G_{M,\delta_1,\delta_2}(t,s) = \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m}\left(\frac{\delta_2\cosh\left(m\left(\frac{1}{2}-t\right)\right) - m\,\delta_1\,\sinh\left(m\left(\frac{1}{2}-t\right)\right)}{m^2+\delta_2} + \begin{cases}-\cosh\left(m\left(\frac{1}{2}+s-t\right)\right), & 0\le s\le t\le 1\\ \\ -\cosh\left(m\left(\frac{1}{2}+t-s\right)\right), & 0\le t<s\le 1\end{cases}\right).$$

- **Case** $\boldsymbol{M = 0}$

In this case, we cannot apply formula (3.2.5), and so we need to compute G_{0,δ_1,δ_2} directly.

Again, a solution of $u''(t) = \sigma(t), t \in J$ is given by

$$u(t) = c_1 + c_2\,t + \int_0^t (t-s)\,\sigma(s)\,ds. \tag{3.4.10}$$

So, $u(0) - u(1) = -c_2 + \displaystyle\int_0^1 (s-1)\,\sigma(s)\,ds$. On the other hand,

$$\begin{aligned}\int_0^1 u(t)\,dt &= \int_0^1 \left(c_1 + c_2\,t + \int_0^t (t-s)\,\sigma(s)\,ds\right)dt\\ &= c_1 + \frac{c_2}{2} + \int_0^1\int_0^t (t-s)\,\sigma(s)\,ds\,dt.\end{aligned}$$

Applying Fubini's Theorem, we have that

$$\begin{aligned}\int_0^1\int_0^t (t-s)\,\sigma(s)\,ds\,dt &= \int_0^1\int_s^1 (t-s)\,\sigma(s)\,dt\,ds\\ &= \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)\,ds.\end{aligned}$$

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Imposing the boundary conditions in (3.1.1), we arrive at the following system of equations

$$\delta_1 c_1 + \left(\frac{\delta_1}{2} + 1\right) c_2 = \int_0^1 (s-1)\,\sigma(s)\,ds - \delta_1 \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)\,ds,$$
$$\delta_2 c_1 + \frac{\delta_2}{2} c_2 = -\int_0^1 \sigma(s)\,ds - \delta_2 \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)\,ds,$$

whose solutions are

$$c_1 = -\frac{1}{2}\int_0^1 (s-1)\,\sigma(s)\,ds - \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)\,ds - \frac{\delta_1+2}{2\,\delta_2}\int_0^1 \sigma(s)\,ds,$$
$$c_2 = \int_0^1 (s-1)\,\sigma(s)\,ds + \frac{\delta_1}{\delta_2}\int_0^1 \sigma(s)\,ds.$$

Substituting c_1 and c_2 in (3.4.10) we have that

$$\begin{aligned} u(t) = &-\frac{1}{2}\int_0^1 (s-1)\,\sigma(s)\,ds - \int_0^1 \left(\frac{s^2+1}{2} - s\right)\sigma(s)\,ds - \frac{\delta_1+2}{2\,\delta_2}\int_0^1 \sigma(s)\,ds \\ &+ \int_0^1 t\,(s-1)\,\sigma(s)\,ds + \frac{\delta_1}{\delta_2}\int_0^1 t\,\sigma(s)\,ds + \int_0^t (t-s)\,\sigma(s)\,ds \\ = &\int_0^1 G_{0,\delta_1,\delta_2}(t,s)\,\sigma(s)\,ds, \end{aligned}$$

being

$$G_{0,\delta_1,\delta_2}(t,s) = \frac{1}{2\,\delta_2}\begin{cases} -2 + \delta_1\,(-1+2t) - s\,\delta_2\,(1+s-2t), & 0 \le s \le t \le 1, \\ -2 + \delta_1\,(-1+2t) - \delta_2\,(s-1)\,(s-2t), & 0 \le t < s \le 1. \end{cases} \tag{3.4.11}$$

Now, we are in position to characterize the constant sign of G_{M,δ_1,δ_2} on $J \times J$ as a function of the parameters M, δ_1 and δ_2.

In this case, it occurs that

$$\frac{\partial}{\partial \delta_1} G_{M,\delta_1,\delta_2}(t,s) = \frac{\omega_1(t)}{M - \delta_2} \quad \text{for } M \neq 0,\ M \neq \delta_2$$

and

$$\frac{\partial}{\partial \delta_1} G_{0,\delta_1,\delta_2}(t,s) = \frac{1}{\delta_2}\left(t - \frac{1}{2}\right),$$

which implies that $\frac{\partial}{\partial \delta_1} G_{M,\delta_1,\delta_2}$ will change its sign depending on t. As a consequence, there will be some values of t for which G_{M,δ_1,δ_2} will increase with respect to δ_1 and some other values of t for which G_{M,δ_1,δ_2} will decrease with respect to δ_1. As a consequence, we deduce the following facts:

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- If M and δ_2 are such that $G_{M,0,\delta_2} > 0$ on $J \times J$, then G_{M,δ_1,δ_2} will be positive on $J \times J$ for some values (both positive and negative) of δ_1. In particular, by Lemma 3.2.1, we know that G_{M,δ_1,δ_2} will be positive on $J \times J$ for $\delta_1 \in (-\delta_1(\delta_2, M), \delta_1(\delta_2, M))$, where the optimal value $\delta_1(\delta_2, M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,\delta_1(\delta_2,M),\delta_2}$ attains the value zero at some point.
- If M and δ_2 are such that $G_{M,0,\delta_2} < 0$ on $J \times J$, then G_{M,δ_1,δ_2} will be negative on $J \times J$ for some values (both positive and negative) of δ_1. In particular, by Lemma 3.2.1, we know that G_{M,δ_1,δ_2} will be negative on $J \times J$ for $\delta_1 \in (-\delta_1(\delta_2, M), \delta_1(\delta_2, M))$, where the optimal value $\delta_1(\delta_2, M)$ will be either $+\infty$ or the smallest positive real value for which $G_{M,\delta_1(\delta_2,M),\delta_2}$ attains the value zero at some point.

Furthermore, the following result is fulfilled.

Corollary 3.4.1. *The two following properties hold:*

- *If M and δ_2 are such that $G_{M,0,\delta_2} > 0$ on $J \times J$, then G_{M,δ_1,δ_2} is either positive or changes its sign on $J \times J$.*
- *If M and δ_2 are such that $G_{M,0,\delta_2} < 0$ on $J \times J$, then G_{M,δ_1,δ_2} is either negative or changes its sign on $J \times J$.*

The following result states that G_{M,δ_1,δ_2} cannot have constant sign for any $\delta_1 \in \mathbb{R}$ if $G_{M,0,\delta_2}$ changes sign.

Lemma 3.4.2. *If M and δ_2 are such that $G_{M,0,\delta_2}$ changes sign on $J \times J$, then G_{M,δ_1,δ_2} also changes its sign on $J \times J$ for every $\delta_1 \in \mathbb{R}$.*

Proof. It is immediately verified using arguments similar to Theorems 3.3.5, 3.3.6, 3.3.9 and 3.3.10. In particular, it is obtained that:

1. If $M = m^2 \in (0, \pi^2)$ and $\delta_2 < -\frac{m^2 \cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2}, \frac{1}{2}\right) < 0$ and $G_{M,\delta_1,\delta_2}(0,0) > 0$.
2. If $M = m^2 \in (0, \pi^2)$ and $\delta_2 > \dfrac{m^2}{1 - \cos\left(\frac{m}{2}\right)}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2}, \frac{1}{2}\right) > 0$ and $G_{M,\delta_1,\delta_2}(0,0) < 0$.
3. If $M = -m^2$, with $m \in (0, \infty)$, and $\delta_2 > \dfrac{2m^2 e^{\frac{m}{2}}}{1 + e^m - 2e^{\frac{m}{2}}}$, then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2}, \frac{1}{2}\right) > 0$ and $G_{M,\delta_1,\delta_2}(0,0) < 0$.
4. If $M = -m^2$, with $m \in (0, \infty)$, and $\delta_2 < -\dfrac{m^2\left(1 + e^m\right)}{1 + e^m - 2e^{\frac{m}{2}}}$ then $G_{M,\delta_1,\delta_2}\left(\frac{1}{2}, \frac{1}{2}\right) < 0$ and $G_{M,\delta_1,\delta_2}(0,0) > 0$.
5. If $M > \pi^2$ then $G_{M,\delta_1,\delta_2}(t, 0)$ is sign-changing on J.

□

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Similarly to Lemmas 3.3.1 and 3.3.2, let us describe the points where a constant-sign Green's function G_{M,δ_1,δ_2} may vanish.

Lemma 3.4.3. *Let $M < \pi^2$ and $\delta_1 > 0$. If G_{M,δ_1,δ_2} has constant sign on $J \times J$ and vanishes at some point (t_0, s_0), then either $t_0 = 1$ or $t_0 = s_0$.*

Proof. Let us suppose that $G_{M,\delta_1,\delta_2} \geq 0$ (the case $G_{M,\delta_1,\delta_2} \leq 0$ would be analogous). From Lemma 3.2.2, we only need to discard the case $t_0 = 0$. Suppose then that $G_{M,\delta_1,\delta_2}(0, s_0) = 0$ for some $s_0 \in (0, 1)$. In such a case, from the equality

$$G_{M,\delta_1,\delta_2}(0, s_0) - G_{M,\delta_1,\delta_2}(1, s_0) = \delta_1 \int_0^1 G_{M,\delta_1,\delta_2}(t, s_0)\, dt > 0,$$

we deduce that $G_{M,\delta_1,\delta_2}(1, s_0) < 0$, which is a contradiction. Therefore, G_{M,δ_1,δ_2} cannot vanish at $(0, s_0)$ and the only possibility is that either $t_0 = 1$ or $t_0 = s_0$. □

We separate the study of G_{M,δ_1,δ_2} into two cases:

3.4.1 Negative Sign of G_{M,δ_1,δ_2}

We analyze the region where G_{M,δ_1,δ_2} is negative on its square of definition. In this case, we distinguish two situations.

3.4.1.1 Case $M \neq 0$

To do this, taking into account Corollary 3.4.1, we fix $M \neq 0$ and δ_2 for which $G_{M,0,\delta_2}$ is negative, that is, $\delta_2 \in (M, f(M))$, with

$$f(M) := \begin{cases} \dfrac{M}{1-\cos\left(\frac{\sqrt{M}}{2}\right)}, & 0 < M < \pi^2, \\ \dfrac{-2\,M\,e^{\frac{\sqrt{-M}}{2}}}{1+e^{\sqrt{-M}}-2e^{\frac{\sqrt{-M}}{2}}}, & M < 0. \end{cases}$$

Taking into account Lemma 3.2.1, it is only necessary to do the calculations for $\delta_1 > 0$ (since the case $\delta_1 < 0$ follows by symmetry).

The characterization of the set is given on the following result.

Theorem 3.4.4. *Let $M < \pi^2, M \neq 0$ and $\delta_2 \in (M, f(M))$, then G_{M,δ_1,δ_2} is strictly negative on $J \times J$ if and only if*

$$|\delta_1| < 2\,(M - \delta_2)\, G_{M,0,\delta_2}\left(1, \frac{1}{2}\right).$$

Proof. Since the function ω_1 is strictly decreasing on J, $\omega_1(0) = \frac{1}{2}$ and $\omega_1(1) = -\frac{1}{2}$, we infer that

$$\max_{t \in J} \frac{\delta_1 \omega_1(t)}{M - \delta_2} = \frac{\delta_1 \omega_1(1)}{M - \delta_2} = -\frac{\delta_1}{2\,(M - \delta_2)}.$$

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On the other hand,

$$\max_{t,s\in J} G_{M,0,\delta_2}(t,s) = G_{M,0,\delta_2}\left(1,\frac{1}{2}\right).$$

As a consequence, for $\delta_1 > 0$ we deduce from expression

$$G_{M,\delta_1,\delta_2}(t,s) = G_{M,0,\delta_2}(t,s) + \frac{\delta_1}{M-\delta_2}\,\omega_1(t),$$

that

$$\max_{t,s\in J} G_{M,\delta_1,\delta_2}(t,s) = G_{M,\delta_1,\delta_2}\left(1,\frac{1}{2}\right) = G_{M,0,\delta_2}\left(1,\frac{1}{2}\right) - \frac{\delta_1}{2\,(M-\delta_2)}.$$

Thus, G_{M,δ_1,δ_2} is negative if and only if $\delta_1 < 2\,(M-\delta_2)\,G_{M,0,\delta_2}\left(1,\frac{1}{2}\right)$.

Using the symmetry of G_{M,δ_1,δ_2} with respect to δ_1, we conclude the result. □

3.4.1.2 Case $M = 0$

As in the previous case, we compute the negative sign of G_{0,δ_1,δ_2} fixing the value $\delta_2 \in (0,8)$ for which $G_{0,0,\delta_2}$ is negative.

Theorem 3.4.5. *If $M = 0$ and $\delta_2 \in (0,8)$, then G_{0,δ_1,δ_2} is negative on $J \times J$ if and only if*

$$|\delta_1| < 2 - \tfrac{\delta_2}{4}.$$

Proof. Suppose that $\delta_1 > 0$ and let us calculate the maximum of G_{0,δ_1,δ_2} (whose expression is given by (3.4.11)). From Lemma 3.4.3 we know that such maximum is either at $t = 1$ or $t = s$.

At points of the form $(1,s)$ we have that

$$r(s) := G_{0,\delta_1,\delta_2}(1,s) = \frac{-2+\delta_1-\delta_2\,s\,(s-1)}{2\,\delta_2},\ s \in J,$$

has an absolute maximum at $s = \frac{1}{2}$. So, $r(s) < 0$ for all $s \in J$ if and only if $r\left(\frac{1}{2}\right) < 0$, that is, $0 < \delta_1 < 2 - \frac{\delta_2}{4}$.

Let us consider now the restriction to the diagonal, that is,

$$h(s) := G_{0,\delta_1,\delta_2}(s,s) = \frac{-2+\delta_1\,(-1+2\,s)+s\,\delta_2\,(s-1)}{2\,\delta_2},\ s \in J.$$

Given $c = \frac{1}{2} - \frac{\delta_1}{\delta_2} < \frac{1}{2}$, it holds that $h'(s) < 0$ for $s < c$, $h'(c) = 0$ and $h'(s) > 0$ for $s > c$. Thus, c is a minimum of h. If $c \in (0,\frac{1}{2})$, h attains its maximum either at $s = 1$ or at $s = 0$, while if $c \le 0$, then $h' > 0$ on $(0,1]$ and the maximum is attained at $s = 1$. In any case, $h(0) = \frac{-2-\delta_1}{2\delta_2} < 0$ and $h(1) = \frac{-2+\delta_1}{2\delta_2} > h(0)$. So, $h(s) < 0$ if and only if $h(1) < 0$, that is, $0 < \delta_1 < 2$.

Therefore, for $\delta_1 > 0$, $G_{M,\delta_1,\delta_2} < 0$ on $J \times J$ if and only if

$$\delta_1 < \min\left\{2 - \frac{\delta_2}{4},\, 2\right\} = 2 - \frac{\delta_2}{4}.$$

The case $\delta_1 < 0$ follows by symmetry. □

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3.4.2 Positive Sign of G_{M,δ_1,δ_2}

Let us calculate now the regions where G_{M,δ_1,δ_2} is positive.

3.4.2.1 Case $M \neq 0$

Taking into account Corollary 3.4.1, let us fix $M \neq 0$ and δ_2 for which $G_{M,0,\delta_2}$ is positive, that is, $\delta_2 \in (g(M), M)$, with

$$g(M) = \begin{cases} -\frac{M\cos\left(\frac{\sqrt{M}}{2}\right)}{1-\cos\left(\frac{\sqrt{M}}{2}\right)}, & 0 < M < \pi^2, \\ \frac{M\,\cosh\left(\frac{\sqrt{-M}}{2}\right)}{\cosh\left(\frac{\sqrt{-M}}{2}\right)-1}, & M < 0. \end{cases} \tag{3.4.12}$$

Now we define the function

$$k(M) := \begin{cases} -M\,\cot^2\left(\frac{\sqrt{M}}{2}\right), & 0 < M < \pi^2, \\ M\,\coth^2\left(\frac{\sqrt{-M}}{2}\right), & M < 0. \end{cases} \tag{3.4.13}$$

It is easy to verify that

$$g(M) < k(M) < M, \quad \text{for all} \quad M \neq 0.$$

Theorem 3.4.6. *Let functions g and k be defined in* (3.4.12) *and* (3.4.13) *respectively. Assume that $M = m^2$, $m \in (0, \pi)$ and $\delta_2 \in (g(M), M)$, then the two following properties are fulfilled:*

1. *If $\delta_2 \in (k(M), M)$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < m\cot\left(\frac{m}{2}\right).$$

2. *If $\delta_2 \in (g(M), k(M)]$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)\left(m^2-\delta_2\right)^2}}{m}.$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{M,δ_1,δ_2}. From Lemma 3.4.3 we know that such minimum is either at $t = 1$ or $t = s$.

Let us distinguish several cases:

1. $\delta_2 \geq 0$, that is, $\delta_2 \in [0, M)$:

   At the points of form $(1, s)$ the function to be studied is

$$r(s) = \frac{\cos\left(\frac{m}{2}(1-2s)\right)\csc\left(\frac{m}{2}\right)}{2m} + \frac{\delta_2\cot\left(\frac{m}{2}\right)}{2m\left(m^2-\delta_2\right)} - \frac{\delta_1}{2\left(m^2-\delta_2\right)}.$$

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It is immediate to verfiy that $r'(s) = 0$ if and only if $s = \frac{1}{2}$ and $s = \frac{1}{2}$ is an absolute maximum of r. Since $r(0) = r(1)$, then $r(s) > 0$ for all $s \in J$ if and only if $r(0) = r(1) > 0$, that is,

$$0 < \delta_1 < m \cot\left(\frac{m}{2}\right).$$

At the diagonal $t = s$ we obtain the following function

$$h(s) = \frac{\csc\left(\frac{m}{2}\right)}{2m}\left[\cos\left(\frac{m}{2}\right) + \frac{\delta_1\, m \sin\left(m\left(\frac{1}{2} - s\right)\right) + \delta_2 \cos\left(m\left(\frac{1}{2} - s\right)\right)}{m^2 - \delta_2}\right],$$

which attains its minimum at $s = 1$ and so $h(s) > 0$ on J if and only if $h(1) = r(1) > 0$.

Therefore, from Lemma 3.4.3, we have that $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ if and only if $0 < \delta_1 < m \cot\left(\frac{m}{2}\right)$.

2. $\delta_2 < 0$, that is, $\delta_2 \in (g(M), 0)$:

   At the points of the form $(1, s)$, analogously to the previous case, we obtain that $r(s) > 0$ on J if and only if $r(1) > 0$, that is,

$$0 < \delta_1 < m\, \cot\left(\frac{m}{2}\right).$$

   At the diagonal $t = s$, the derivative of h is

$$h'(s) = \frac{1}{2\left(m^2 - \delta_2\right)\sin\left(\frac{m}{2}\right)}\left[-m\,\delta_1 \cos\left(\frac{m}{2}\,(1 - 2s)\right) + \delta_2\, \sin\left(\frac{m}{2}\,(1 - 2s)\right)\right].$$

   Since $\sin\left(\frac{m}{2}\right) > 0$ for all $m \in (0, \pi)$, $h'(s) = 0$ if and only if

$$\tan\left(\frac{m}{2}\,(1 - 2s)\right) = \frac{m\delta_1}{\delta_2},$$

   that is, $s = \frac{1}{2} - \frac{1}{m}\arctan\left(\frac{m\delta_1}{\delta_2}\right)$.

   Futhermore,

$$h'(s) > 0 \text{ if and only if } s > \tfrac{1}{2} - \tfrac{1}{m}\arctan\left(\tfrac{m\delta_1}{\delta_2}\right),$$

   and

$$h'(s) < 0 \text{ if and only if } s > \tfrac{1}{2} - \tfrac{1}{m}\arctan\left(\tfrac{m\delta_1}{\delta_2}\right).$$

   So, $c = \frac{1}{2} - \frac{1}{m}\arctan\left(\frac{m\delta_1}{\delta_2}\right)$ is a minimum of h.

   Moreover, we note that $c \in J$ if and only if $\frac{1}{m}\arctan\left(\frac{m\delta_1}{\delta_2}\right) \in \left[-\frac{1}{2}, 0\right)$, that is,

$$0 < \delta_1 \leq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right).$$

   Therefore, we subdivide the case $\delta_2 < 0$ into two parts:

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(a) If $\delta_1 \geq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$, then $h'(s) < 0$ for all $s \in [0,1)$ and the minimum of h is attained at $s = 1$. Thus, $h(s) > 0$ on J if and only if $h(1) = r(1) > 0$, that is, $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$. As a consequence, $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ for $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$.

We note that the two previous conditions, $\delta_1 \geq -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$ and $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$, are compatible if and only if

$$\delta_2 > -m^2\cot^2\left(\frac{m}{2}\right) \equiv k(M).$$

(b) If $0 < \delta_1 < -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right)$, then h attains an absolute minimum at $c \in (0,1)$. In this case, $h(c) > 0$ if and only if

$$\delta_1 < \frac{\sqrt{m^4 - 2m^2\delta_2 - \delta_2^2 + (m^2-\delta_2)^2\cos(m)}}{\sqrt{2}\,m} = \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}.$$

We note that $-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2 > 0$ for $m \in (0,\pi)$ and $\delta_2 \in (g(M),0)$. Indeed, $-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2 > 0$ if and only if $\left|\cos\left(\frac{m}{2}\right)\right||m^2-\delta_2| \geq |\delta_2|$. Since $m \in (0,\pi)$ and $\delta_2 < 0$, previous inequality is equivalent to

$$\delta_2 \geq -\frac{m^2\cos\left(\frac{m}{2}\right)}{1-\cos\left(\frac{m}{2}\right)} \equiv g(M).$$

Moreover, we note that

$$\min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}\right\} = \begin{cases} \frac{\sqrt{-\delta_2^2+\cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}, & \delta_2 \in (g(M),k(M)), \\ -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), & \delta_2 \in (k(M),M). \end{cases}$$

As a consequence, we conclude that:

- If $\delta_2 \in (g(M),k(M)]$ then, from (b), $G_{M,\delta_1,\delta_2} > 0$ for

$$0 < \delta_1 < \min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}\right\} = \frac{\sqrt{-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)(m^2-\delta_2)^2}}{m}.$$

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- If $\delta_2 \in (k(M), M)$ then, from (a), $G_{M,\delta_1,\delta_2} > 0$ for

$$-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right) < \delta_1 < m\cot\left(\frac{m}{2}\right)$$

and, from (b), $G_{M,\delta_1,\delta_2} > 0$ for

$$\begin{aligned} 0 < \delta_1 &\leq \min\left\{-\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right), \frac{\sqrt{-\delta_2^2+\cos^2\left(\frac{m}{2}\right)\left(m^2-\delta_2\right)^2}}{m}\right\} \\ &= -\frac{\delta_2}{m}\tan\left(\frac{m}{2}\right). \end{aligned}$$

Thus, $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ for $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$.

The fact that the obtained bounds are optimal follows from Lemma 3.4.3.
Using the symmetry with respect to δ_1 we conclude the result. □

In the sequel, we will state the following result that characterize the strictly positive sign of G_{M,δ_1,δ_2} for the case $M < 0$.

Theorem 3.4.7. *Let functions g and k be defined in* (3.4.12) *and* (3.4.13) *respectively. For any $M = -m^2$, with $m > 0$ and $\delta_2 \in (g(M), M)$, it holds that*

1. *If $\delta_2 \in (k(M), M)$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < m\coth\left(\frac{m}{2}\right).$$

2. *If $\delta_2 \in (g(M), k(M)]$, then G_{M,δ_1,δ_2} is strictly positive on $J \times J$ if and only if*

$$|\delta_1| < \frac{\sqrt{\delta_2^2-\left(m^2+\delta_2\right)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{M,δ_1,δ_2}. From Lemma 3.4.3 we know that such minimum is either at $t = 1$ or $t = s$.

At the points of the form $(1, s)$ we get the function

$$r(s) := \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m}\left[-\cosh\left(m\left(s-\frac{1}{2}\right)\right) + \frac{\delta_1 m\sinh\left(\frac{m}{2}\right)+\delta_2\cosh\left(\frac{m}{2}\right)}{m^2+\delta_2}\right],$$

whose minimum is attained at $s = 0$ and $s = 1$ (indeed, $r(0) = r(1)$). Thus, $r(s) > 0$ for all $s \in J$ if and only if $r(0) = r(1) > 0$, that is,

$$0 < \delta_1 < m\coth\left(\frac{m}{2}\right).$$

At the diagonal $t = s$ we obtain the following function

$$h(s) := \frac{\operatorname{csch}\left(\frac{m}{2}\right)}{2m}\left[-\cosh\left(\frac{m}{2}\right) + \frac{\delta_2\cosh\left(m\left(\frac{1}{2}-s\right)\right)-\delta_1 m\sinh\left(m\left(\frac{1}{2}-s\right)\right)}{m^2+\delta_2}\right].$$

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It occurs that $h'(s) = 0$ if and only if $\tanh\left(\frac{m}{2}(1-2s)\right) = \frac{m\delta_1}{\delta_2}$. Since $\delta_2 < 0$ and $\tanh^{-1}(x)$ exists for $x \in [-1,1]$, we have that h has a critical point

$$c = \frac{1}{2} - \frac{1}{m}\tanh^{-1}\left(\frac{m\delta_1}{\delta_2}\right)$$

if and only if $-1 \leq \frac{m\delta_1}{\delta_2} < 0$, that is, $\delta_1 \leq -\frac{\delta_2}{m}$. In such a case, it occurs that $h'(s) < 0$ for $s < c$ and $h'(s) > 0$ for $s > c$. Moreover, we can see that $c \in J$ if and only if $0 < \delta_1 \leq -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)\left(< -\frac{\delta_2}{m}\right)$. Therefore, we distinguish two cases:

(a) If $\delta_1 > -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$, then $h'(s) < 0$ for all $s \in J$ and h has a minimum at $s = 1$. Then $h(s) > 0$ for all $s \in I$ if and only if $h(1) = r(1) > 0$, that is, if and only if $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$. As a consequence, $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ for $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$.

We note that the two previous conditions, $\delta_1 > -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$ and $0 < \delta_1 < m\coth\left(\frac{m}{2}\right)$, are compatible if and only if $\delta_2 > -m^2\coth^2\left(\frac{m}{2}\right) \equiv k(M)$.

(b) If $0 < \delta_1 \leq -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right)$, then h attains an absolute minimum at $c \in (0,1)$. In this case, $h(c) > 0$ (and, consequently, $h(s) > 0$ for $s \in J$) if and only if

$$0 < \delta_1 < \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$

Note that, $-\delta_2^2 + \cos^2\left(\frac{m}{2}\right)\left(m^2-\delta_2\right)^2 > 0$ for $M = -m^2 < 0$, $m \in (0,\infty)$ and $\delta_2 \in (g(M), M)$. Indeed, $\delta_2^2 - \left(m^2+\delta_2\right)^2\cosh^2\left(\frac{m}{2}\right) > 0$ if and only if $|\delta_2| \geq \left|\cosh\left(\frac{m}{2}\right)\right||m^2+\delta_2|$. Since $m \in (0,\infty)$ and $\delta_2 < M < 0$, previous inequality is equivalent to

$$\delta_2 \geq \frac{m^2\cosh\left(\frac{m}{2}\right)}{1-\cosh\left(\frac{m}{2}\right)} \equiv g(M).$$

Therefore, since $h(c) < h(1) = r(1)$, $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ for

$$0 < \delta_1 < \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$

Moreover, we note that

$$\min\left\{-\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}\right\}$$
$$= \begin{cases} \frac{\sqrt{\delta_2^2-(m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}, & \delta_2 \in (g(M), k(M)), \\ -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), & \delta_2 \in (k(M), M). \end{cases}$$

As a consequence, we conclude that:

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- If $\delta_2 \in (g(M), k(M)]$ then, from (b), $G_{M,\delta_1,\delta_2} > 0$ for

$$0 < \delta_1 < \min\left\{ -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m} \right\}$$
$$= \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m}.$$

- If $\delta_2 \in (k(M), M)$ then, from (a), $G_{M,\delta_1,\delta_2} > 0$ for

$$-\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right) < \delta_1 < m\coth\left(\frac{m}{2}\right)$$

and, from (b), $G_{M,\delta_1,\delta_2} > 0$ for

$$0 < \delta_1 \leq \min\left\{ -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right), \frac{\sqrt{\delta_2^2 - (m^2+\delta_2)^2\cosh^2\left(\frac{m}{2}\right)}}{m} \right\}$$
$$= -\frac{\delta_2}{m}\tanh\left(\frac{m}{2}\right).$$

Thus, $G_{M,\delta_1,\delta_2} > 0$ on $J \times J$ for $0 < \delta_1 < m\cot\left(\frac{m}{2}\right)$.

Therefore, using symmetry with respect to δ_1, we conclude that the attained bounds are optimal from Lemma 3.4.3 and the result holds. □

3.4.2.2 Case $M = 0$

For the positive case, we set $\delta_2 \in (-8, 0)$ where $G_{0,0,\delta_2}$ is positive.

Theorem 3.4.8. *Let $M = 0$ and $\delta_2 \in (-8, 0)$, then $G_{0,\delta_1,\delta_2} > 0$ on $J \times J$ if and only if*

$$|\delta_1| < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8, -4), \\ 2, & \delta_2 \in [-4, 0). \end{cases}$$

Proof. Let us assume that $\delta_1 > 0$ and calculate the minimum of G_{0,δ_1,δ_2}. From Lemma 3.4.3 we know that such minimum is either at $t = 1$ or $t = s$.

As we have seen in the proof of Theorem 3.4.5, the function $r(s) = G_{0,\delta_1,\delta_2}(1, s)$ attains its maximum at $s = \frac{1}{2}$ and its minimum at $s = 0$ and $s = 1$. Hence, $r > 0$ if and only if $r(0) = r(1) > 0$, that is, $0 < \delta_1 < 2$.

On the other hand, as we have seen in the proof of Theorem 3.4.5 again, the function $h(s) = G_{0,\delta_1,\delta_2}(s, s)$ has an absolute minimum at $c = \frac{1}{2} - \frac{\delta_1}{\delta_2}$, $h'(s) < 0$ for $s < c$ and $h'(s) > 0$ for $s > c$.

Let us see for what values of δ_1 and δ_2 the value c belongs to J. Since $\delta_2 < 0$ we have that $-\frac{\delta_1}{\delta_2} > 0$. Hence, $c \in J$ if and only if $0 < \delta_1 \leq -\frac{\delta_2}{2}$.

We distinguish the following cases:

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- If $\delta_1 \geq -\frac{\delta_2}{2}$, then $c \geq 1$ and the minimum of h is attained at $s = 1$ and $h(1) > 0$ if and only if $\delta_1 < 2$. Since $h(1) = r(1)$, we deduce that if $-\frac{\delta_2}{2} \leq \delta_1 < 2$ then $G_{0,\delta_1,\delta_2} > 0$ on $J \times J$. We note that this is only possible when $\delta_2 > -4$.

- If $0 < \delta_1 < -\frac{\delta_2}{2}$, then $c \in (0,1)$ and $h(c) > 0$ if and only if $\delta_1 < \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}$. Since $r(1) = h(1) > h(c)$, we deduce that $G_{0,\delta_1,\delta_2} > 0$ for

$$0 < \delta_1 < \min\left\{-\frac{\delta_2}{2}, \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}\right\} = \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8,-4), \\ -\frac{\delta_2}{2}, & \delta_2 \in [-4,0). \end{cases}$$

In conclusion, $G_{0,\delta_1,\delta_2}(t,s) > 0$ on $J \times J$ if and only if

$$0 < \delta_1 < \begin{cases} \frac{1}{2}\sqrt{-8\delta_2 - \delta_2^2}, & \delta_2 \in (-8,-4), \\ 2, & \delta_2 \in [-4,0). \end{cases}$$

Using the symmetry with respect to δ_1 we conclude the result. □

3.4.3 A Particular Case: $\boldsymbol{\delta_2 = 0}$

Finally, as a consequence of Theorems 3.4.4 and 3.4.6, we arrive at the following corollary.

Corollary 3.4.9. *Let us consider the perturbed periodic problem*

$$\begin{cases} u''(t) + M\,u(t) = \sigma(t), \quad t \in J, \\ u(0) - u(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u'(0) - u'(1) = 0, \end{cases}$$

for $M \in \mathbb{R}\backslash\{0\}$. The following statements holds:

1. *If $M = m^2 > 0$, then $G_{M,\delta_1,0} > 0$ on $J \times J$ if and only if $m \in (0,\pi)$ and*

$$|\delta_1| < m \cot\left(\frac{m}{2}\right).$$

2. *If $M = -m^2 < 0$ and $m \in (0,\infty)$, then $G_{M,\delta_1,0} < 0$ on $J \times J$ if and only if*

$$|\delta_1| < \frac{2\,m\,e^{\frac{m}{2}}}{e^m - 1}.$$

In this case, the graph showing the sign of the Green's function $G_{M,\delta_1,0}$ on the plane $M \times \delta_1$ can be seen in Figure 3.4.1.

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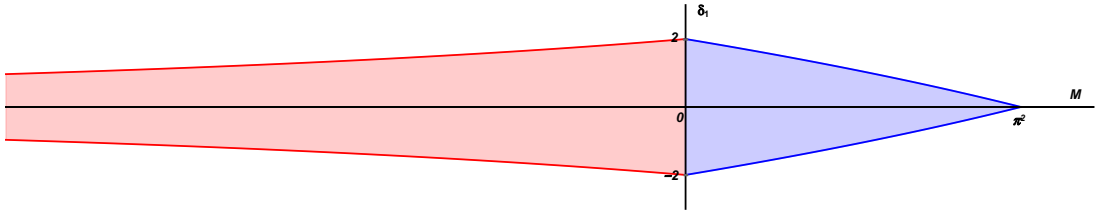

Figure 3.4.1: The blue and red areas represent the regions of positive and negative sign of the Green's function, respectively.

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Chapter 4
Nonlinear Dirichlet Problem Perturbed by Integral Boundary Conditions

In this chapter we make a study of the second order nonlinear perturbed Dirichlet problem with integral boundary conditions. We obtain the exact expression of the Green's function related to the perturbed problem in terms of the Green's function of the homogeneous Dirichlet problem. Moreover, we characterize the set of parameters where the Green's function has constant sign (which, contrary to the homogeneous case, can be either positive or negative) on its square of definition. Finally, as an application, the existence of positive solutions for the nonlinear problems is derived from fixed point theory applied to related operators defined on suitable cones in Banach spaces.

We mention that the interest in considering this boundary of conditions relies on the fact that they appear in different real phenomena as such, among others, problems of blood flow, chemical engineering, thermoelastic or population dynamics, see for instance [54, 95].

The chapter is divided in three sections. In the next section, we define the problem to be studied and demonstrate a symmetric property of its related Green's function. In Section 4.2 we will deal with the linear problem and prove the main properties of the related Green's function. Section 4.3 is devoted to ensure the existence of solutions of nonlinear problems. Moreover, some examples of the applicability of the obtained results are shown at the end of the chapter.

The results presented in this chapter are compiled in the paper [19].

4.1 Preliminaries

In this chapter, we consider the following nonlinear equations

$$u''(t) + \gamma\, u(t) + f(t, u(t)) = 0, \quad t \in J, \tag{4.1.1}$$

and

$$u''(t) + \gamma\, u(t) - f(t, u(t)) = 0, \quad t \in J, \tag{4.1.2}$$

subject to the integral boundary conditions

$$u(0) = \delta_1 \int_0^1 u(s)\, ds, \quad u(1) = \delta_2 \int_0^1 u(s)\, ds. \tag{4.1.3}$$

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We will study these nonlinear problems for $\gamma \leq \pi^2$, $\delta_1 \geq 0$ and $\delta_2 \geq 0$.

The existence of positive solutions will depend on the regularity and the sign of the function f and the different values of the real parameters γ, δ_1 and δ_2. More concisely, if the Green's function associated to the related linear part of the considered problem is positive, then we will deduce existence of positive solutions of problem (4.1.1), (4.1.3). We will ensure the existence of solutions of problem (4.1.2)–(4.1.3) when such function is negative. So, first of all, we will do an exhaustive study of the sign of the Green's function depending on the values of the parameters γ, δ_1 and δ_2. We mention that problem (4.1.1), (4.1.3) with either $\delta_1 = 0$ or $\delta_2 = 0$, has been studied in [13]. So our results generalize the ones obtained in that paper. The main difference with the properties of the Green's function in this general case, is that in [13] it is proved that, depending on the values of the real parameters, either the Green's function is positive on $(0,1) \times (0,1)$ or it changes its sign on $J \times J$. This property is, in some sense, natural and expected, because it is the situation for the homogeneous case $\delta_1 = \delta_2 = 0$.

However, as we will see on this chapter, we will prove that, in addition to being positive for small enough values of $\delta_1 + \delta_2$, the Green's function related to our problem is negative on $(0,1) \times (0,1)$ for some large enough values of $\delta_1 + \delta_2$ with $\delta_1 > 0$ and $\delta_2 > 0$. This is a, a priori, non expected situation, but it may be interpreted by the fact that in such a case we are considering a strong perturbation of the homogeneous case.

Let us consider the linear problem

$$u''(t) + \gamma\, u(t) + \sigma(t) = 0, \quad t \in J, \tag{4.1.4}$$

coupled to (4.1.3).

We note that problem (4.1.4), (4.1.3) is a particular case of problem (2.1.1) with $n = 2$, $[a,b] = [0,1]$, $a_1 = a_2 = 0$, $M = \gamma$, $B_1(u) = u(0)$, $B_2(u) = u(1)$, $C_1(u) = C_2(u) = \int_0^1 u(s)\, ds$ and σ on the other side of the equation (that is, σ plays in (4.1.4) the role of $-\sigma$ in (2.1.1)). Moreover, analogously to Chapter 3, we will denote by $G_{\gamma,\delta_1,\delta_2}(t,s) \equiv G(t,s,\gamma,\delta_1,\delta_2)$ the Green's function related to problem (4.1.4), (4.1.3).

We will obtain and study the sign of the Green's function related to problem (4.1.4), (4.1.3) by distinguishing three cases depending on the sign of the real parameter γ.

One we can verify that problem $u''(t) + \gamma\, u(t) = 0$, coupled to boundary conditions (4.1.3) has a nontrivial solution if and only if $\gamma = (2k\pi)^2$, $k = 1, 2, \ldots$ or

$$\delta_1 + \delta_2 = \Delta(\gamma) := \begin{cases} \dfrac{\sqrt{-\gamma}}{\tanh(\frac{\sqrt{-\gamma}}{2})}, & \gamma < 0, \\ 2, & \gamma = 0, \\ \dfrac{\sqrt{\gamma}}{\tan(\frac{\sqrt{\gamma}}{2})}, & \gamma > 0. \end{cases} \tag{4.1.5}$$

We point out that in the homogeneous case, that is, $\delta_1 = \delta_2 = 0$ the second condition $\delta_1 + \delta_2 = \Delta(\gamma)$ is rewritten as $\Delta(\gamma) = 0$, which is equivalent to $\gamma = (k\pi)^2$ with k odd. However, the eigenvalues of the homogeneous case are $\gamma = (k\pi)^2$, $k = 1, 2, \ldots$. Therefore, we have two cases regarding the eigenvalues of the homogeneous problem: those with k even remain eigenvalues of the perturbed problem for any value of δ_1 and δ_2, meanwhile those with k odd are eigenvalues only when $\delta_1 + \delta_2 = 0$.

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As we will see, we must take this into account when obtaining the expression of the Green's function for $\gamma > 0$.

Now, we deduce the following symmetric property with respect to the parameters δ_1 and δ_2.

Theorem 4.1.1. *Assume that the linear problem* (4.1.4) *coupled to the homogeneous boundary conditions* $u(0) = u(1) = 0$*, has a unique solution for any* $\sigma \in C(J)$*, that is,* $\gamma \neq (k\,\pi)^2$*,* $k = 1, 2, \ldots$ *and that* $\delta_1 + \delta_2 \neq \Delta(\gamma)$*. Then the following symmetry property holds*

$$G_{\gamma,\delta_1,\delta_2}(t,s) = G_{\gamma,\delta_2,\delta_1}(1-t,1-s), \quad \forall\,(t,s) \in J \times J. \tag{4.1.6}$$

Proof. From Theorem 2.2.2, taking $C(u) = \displaystyle\int_0^1 u(s)\,ds$, it follows that the Green's function of problem (4.1.4), (4.1.3) is given by the expression

$$G_{\gamma,\delta_1,\delta_2}(t,s) = G_{\gamma,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{1 - \delta_1\,C(\omega_1) - \delta_2\,C(\omega_2)} \int_0^1 G_{\gamma,0,0}(r,s)\,dr, \tag{4.1.7}$$

where $G_{\gamma,0,0}$ is the Green's function of the homogeneous Dirichlet problem

$$u''(t) + \gamma\,u(t) + \sigma(t) = 0, \;\; t \in J, \quad u(0) = u(1) = 0,$$

ω_1 is the unique solution of

$$u''(t) + \gamma\,u(t) = 0, \;\; t \in J, \quad u(0) = 1, \; u(1) = 0,$$

and ω_2 is the unique solution of

$$u''(t) + \gamma\,u(t) = 0, \;\; t \in J, \quad u(0) = 0, \; u(1) = 1.$$

The function $\omega(t) := \omega_1(1-t)$ solves the problem

$$\omega''(t) + \gamma\,\omega(t) = 0, \;\; t \in J, \quad \omega(0) = 0, \; \omega(1) = 1.$$

Therefore, by uniqueness of ω_2 we conclude that $\omega_2(t) = \omega_1(1-t)$ for all $t \in J$ and, as a consequence, we have that $C(\omega_1) = C(\omega_2)$.

Now, let us see that $G_{\gamma,0,0}(t,s) = G_{\gamma,0,0}(1-t,1-s)$ for all $(t,s) \in J \times J$. Indeed, if u is a solution of the homogeneous problem

$$u''(t) + \gamma\,u(t) + \sigma(t) = 0,\; t \in J, \quad u(0) = u(1) = 0,$$

then the function $v(t) := u(1-t)$ satisfies

$$v''(t) + \gamma\,v(t) + \sigma(1-t) = 0,\; t \in J, \quad v(0) = v(1) = 0.$$

Thus,

$$u(1-t) = \int_0^1 G_{\gamma,0,0}(t,s)\,\sigma(1-s)\,ds = \int_0^1 G_{\gamma,0,0}(t,1-s)\,\sigma(s)\,ds.$$

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As a consequence

$$u(t) = \int_0^1 G_{\gamma,0,0}(1-t, 1-s)\,\sigma(s)\,ds$$

or, which is the same,

$$G_{\gamma,0,0}(t,s) = G_{\gamma,0,0}(1-t, 1-s),\ \forall (t,s) \in J \times J.$$

Then, using equality (4.1.7), we infer that

$$\begin{aligned} G_{\gamma,\delta_2,\delta_1}(1-t,1-s) &= G_{\gamma,0,0}(1-t,1-s) + \frac{\delta_2\,\omega_1(1-t) + \delta_1\,\omega_2(1-t)}{1-\delta_2\,C(\omega_1) - \delta_1\,C(\omega_2)} \int_0^1 G_{\gamma,0,0}(r,1-s)\,dr \\ &= G_{\gamma,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{1-\delta_1\,C(\omega_1) - \delta_2\,C(\omega_2)} \int_0^1 G_{\gamma,0,0}(1-r,1-s)\,dr \\ &= G_{\gamma,0,0}(t,s) + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{1-\delta_1\,C(\omega_1) - \delta_2\,C(\omega_2)} \int_0^1 G_{\gamma,0,0}(r,s)\,dr \\ &= G_{\gamma,\delta_1,\delta_2}(t,s), \quad \text{for all } (t,s) \in J \times J. \end{aligned}$$

This concludes the proof of equality (4.1.6). □

Remark 4.1.2. *If $\gamma = (k\,\pi)^2$, for some $k \in \mathbb{N}$ odd, and $\delta_1 + \delta_2 \neq 0$, we have that the problem* (4.1.4) *coupled to* (4.1.3) *has a unique solution for any $\sigma \in C(J)$ and its Green's function $G_{(k\,\pi)^2,\delta_1,\delta_2}$ exists and satisfies that*

$$G_{(k\,\pi)^2,\delta_1,\delta_2}(t,s) = \lim_{\gamma \to (k\,\pi)^2} G_{\gamma,\delta_1,\delta_2}(t,s) \quad \forall\ (t,s) \in J \times J.$$

So, the equality (4.1.6) *of symmetric property in the previous theorem must be true in this case.*

4.2 Green's Function Related to Linear Problem

In this section, we will obtain the expression and study the sign of the Green's function related to problem (4.1.4), (4.1.3) distinguishing three cases depending on the sign of the parameter γ. Moreover, we will deduce some fundamental properties that we will use later to prove the existence of positive solutions of the nonlinear problems (4.1.1), (4.1.3) and (4.1.2)–(4.1.3).

4.2.1 Case $\gamma = 0$

In this subsection, we obtain the expression and deduce some suitable properties of the Green's function related to the linear problem

$$u''(t) + \sigma(t) = 0, \quad t \in J, \tag{4.2.8}$$

subject to the integral boundary conditions (4.1.3).

Also, we analyze the range of values of δ_1 and δ_2 for which the Green's function has constant sign on $J \times J$.

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Theorem 4.2.1. *Let $\delta_1 + \delta_2 \neq 2 (= \Delta(0))$ and $\sigma \in \mathcal{C}(J)$, then problem* (4.2.8)*,* (4.1.3) *has a unique solution $u \in C^2(J)$, which is given by the following expression*

$$u(t) = \int_0^1 G_{0,\delta_1,\delta_2}(t,s)\,\sigma(s)ds,$$

where

$$G_{0,\delta_1,\delta_2}(t,s) = \begin{cases} s\,\frac{(1-t)(2-\delta_2-\delta_1 s)+\delta_2(1-s)t}{2-\delta_1-\delta_2}, & 0 \leq s \leq t \leq 1, \\ (1-s)\,\frac{t(2-\delta_1-\delta_2+\delta_2 s)+\delta_1 s(1-t)}{2-\delta_1-\delta_2}, & 0 \leq t < s \leq 1. \end{cases} \tag{4.2.9}$$

Proof. According to equation (4.1.7) the Green's function of problem (4.2.8), (4.1.3) is given by

$$G_{0,\delta_1,\delta_2}(t,s) = G_{0,0,0}(t,s) + \frac{\delta_1\, w_1(t) + \delta_2\, w_2(t)}{1 - \delta_1\, C(w_1) - \delta_2\, C(w_2)} \int_0^1 G_{0,0,0}(t,s)\,dt, \tag{4.2.10}$$

where

$$G_{0,0,0}(t,s) = \begin{cases} s\,(1-t), & 0 \leq s \leq t \leq 1, \\ t\,(1-s), & 0 \leq t < s \leq 1. \end{cases}$$

is the Green's function related to the homogeneous Dirichlet problem (this expression is obtained using the Mathematica package [11])

$$u''(t) + \sigma(t) = 0, \;\; t \in J, \;\; u(0) = u(1) = 0,$$

$\omega_1(t) = 1 - t$ is the unique solution of

$$u''(t) = 0, \;\; t \in J, \;\; u(0) = 1, \; u(1) = 0,$$

and $\omega_2(t) = t$ is the unique solution of

$$u''(t) = 0, \;\; t \in J, \;\; u(0) = 0, \; u(1) = 1.$$

Moreover, we have that $C(w_1) = C(w_2) = \displaystyle\int_0^1 w_2(t)\,dt = \frac{1}{2}$. Using the equality (4.2.10) and the fact that $C(G_{\gamma,0,0}(\cdot,s)) = \frac{1}{2}s\,(1-s)$ we conclude the proof of the result. □

The following result characterizes the range of values of δ_1 and δ_2 for which the Green's function G_{0,δ_1,δ_2} has constant sign on $J \times J$.

Lemma 4.2.2. *Let G_{0,δ_1,δ_2} be the Green's function related to problem* (4.2.8)*, (4.1.3), given by expression* (4.2.9)*. Then, for all $\delta_1 + \delta_2 \neq 2\,(= \Delta(0))$ the following properties are fulfilled:*

1. *$G_{0,\delta_1,\delta_2}(t,0) = G_{0,\delta_1,\delta_2}(t,1) = 0$, for all $t \in J$.*
2. *$G_{0,\delta_1,\delta_2}(t,s)$ is continuous on $J \times J$.*
3. *$G_{0,\delta_1,\delta_2}(0,s) = 0$ for all $s \in (0,1)$ if and only if $\delta_1 = 0$.*

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4. $(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(0,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_1 > 0$.
5. $G_{0,\delta_1,\delta_2}(1,s) = 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_2 = 0$.
6. $(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(1,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_2 > 0$.
7. $(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_1 \leq 2$ *and* $\delta_2 \leq 2$.
8. $G_{0,\delta_1,\delta_2}(t,s) > 0$ *for all* $t, s \in (0,1)$ *if and only if* $0 \leq \delta_1+\delta_2 < 2$, $0 \leq \delta_1 \leq 2$ *and* $0 \leq \delta_2 \leq 2$.
9. $G_{0,\delta_1,\delta_2}(t,s) < 0$ *for all* $(t,s) \in (0,1)\times(0,1)$ *if and only if* $\delta_1+\delta_2 > 2$, $0 < \delta_1 \leq 2$ *and* $0 < \delta_2 \leq 2$.
10. G_{0,δ_1,δ_2} *changes its sign on* $(0,1)\times(0,1)$ *if and only if one of the following situations holds:*
    a) $\delta_1 < 0$.
    b) $\delta_2 < 0$.
    c) $\delta_1 > 2$.
    d) $\delta_2 > 2$.
11. *If* $0 \leq \delta_1+\delta_2 < 2$, $0 \leq \delta_1 \leq 2$ *and* $0 \leq \delta_2 \leq 2$, *then* $G_{0,\delta_1,\delta_2}(t,s) \leq \frac{1}{2(2-\delta_1-\delta_2)}$, *for all* $t,\ s \in J$.
12. *If* $\delta_1+\delta_2 > 2$, $0 < \delta_1 \leq 2$ *and* $0 < \delta_2 \leq 2$, *then* $G_{0,\delta_1,\delta_2}(t,s) \geq \frac{1}{2(2-\delta_1-\delta_2)}$, *for all* $t,\ s \in J$.

Proof. Using the expression of G_{0,δ_1,δ_2} given in (4.2.9) it is easy to verify that Properties 1. and 2. are fulfilled. For the rest of properties we have that:

3. It follows from the equality $G_{0,\delta_1,\delta_2}(0,s) = \delta_1 \frac{s\,(1-s)}{2-\delta_1-\delta_2}$.
4. This statement follows from the following fact
$$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(0,s) = \delta_1\, s\,(1-s), \quad \text{for all } s \in (0,1).$$
5. It is immediate from the equality $G_{0,\delta_1,\delta_2}(1,s) = \delta_2 \frac{s\,(1-s)}{2-\delta_1-\delta_2}$.
6. This statement follows from the fact that $G_{0,\delta_1,\delta_2}(1,s) = G_{0,\delta_2,\delta_1}(0,1-s)$ and Theorem 4.1.1.
7. The following equality is fulfilled for all $s \in (0,1)$:
$$(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) = s(1-s)\,(2-\delta_2+(\delta_2-\delta_1)s).$$
Therefore, $(2-\delta_1-\delta_2)\ G_{0,\delta_1,\delta_2}(s,s) > 0$ if and only if $2-\delta_2+(\delta_2-\delta_1)s > 0$ for all $s \in (0,1)$, which is true if and only if $\delta_1 \leq 2$ and $\delta_2 \leq 2$.

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8. Since $G_{0,\delta_1,\delta_2}(t,s)$ is linear on t, for all $s \in J$ fixed, $G_{0,\delta_1,\delta_2}(\cdot,s)$ attains its maximum and minimum either at $t=0, t=s$ or $t=1$.

   From Property 7., we have that $G_{0,\delta_1,\delta_2}(s,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_1 \leq 2$, $\delta_2 \leq 2$ and $\delta_1+\delta_2 < 2$.

   From Property 4., we have that $G_{0,\delta_1,\delta_2}(0,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_1 > 0$ and $\delta_1+\delta_2 < 2$.

   From Property 6., we have that $G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s \in (0,1)$ if and only if $\delta_2 > 0$ and $\delta_1+\delta_2 < 2$.

   As a consequence of the three previous assertions this property holds.

9. Again, for all $s \in J$ fixed, $G_{0,\delta_1,\delta_2}(\cdot,s)$ attains its maximum and minimum either at $t=0, t=s$ or $t=1$.

   From Property 7., we have that $G_{0,\delta_1,\delta_2}(s,s) < 0$ for all $s \in (0,1)$ if and only if $\delta_1 \leq 2$, $\delta_2 \leq 2$ and $\delta_1+\delta_2 > 2$.

   From Property 4., we have that $G_{0,\delta_1,\delta_2}(0,s) < 0$ for all $s \in (0,1)$ if and only if $\delta_1 > 0$ and $\delta_1+\delta_2 > 2$.

   From Property 6., we have that $G_{0,\delta_1,\delta_2}(1,s) < 0$ for all $s \in (0,1)$ if and only if $\delta_2 > 0$ and $\delta_1+\delta_2 > 2$.

   From these assertions, the property is proved.

10. The proof is a direct consequence of Properties 8. and 9.

11. From Property 8., we know that $G_{0,\delta_1,\delta_2}(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$. As in Property 8., we know that the maximum values will be attained at $G_{0,\delta_1,\delta_2}(s,s)$, $G_{0,\delta_1,\delta_2}(0,s)$ and/or $G_{0,\delta_1,\delta_2}(1,s)$. Now, since

$$\begin{aligned}(2-\delta_1-\delta_2)G_{0,\delta_1,\delta_2}(s,s) &= s(1-s)\left(2-\delta_2-\delta_1 s+\delta_2 s\right)\\ &= s(1-s)(2-\delta_1-\delta_2)+\delta_1\, s(1-s)^2+\delta_2\, s^2(1-s)\\ &\leq s(1-s)(2-\delta_1-\delta_2)+\delta_1\, s(1-s)+\delta_2\, s(1-s)\\ &= 2s(1-s) \leq \frac{1}{2},\end{aligned}$$

$$(2-\delta_1-\delta_2)\, G_{0,\delta_1,\delta_2}(0,s) = \delta_1 s(1-s) \leq \frac{\delta_1}{4} \leq \frac{1}{2},$$

   and

$$(2-\delta_1-\delta_2)\, G_{0,\delta_1,\delta_2}(1,s) = \delta_2 s(1-s) \leq \frac{\delta_2}{4} \leq \frac{1}{2},$$

   the proof is concluded.

12. Making the same argument for the negative case, that is, $\delta_1+\delta_2 > 2$, $0 < \delta_1 \leq 2$ and $0 < \delta_2 \leq 2$, we obtain that $G_{0,\delta_1,\delta_2}(t,s) \geq \frac{1}{2(2-\delta_1-\delta_2)}$. □

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Remark 4.2.3. *We mention that Properties 1, 2, 5, 6, 7, 8, 10 and 11 are a generalization of the ones given in [13, Lemma 4]. Despite this, it is important to mention that the proof of Property 10 here is completely different to the simpler one given in that reference.*

The rest of the cases have no sense for that situation.

Next, we deduce two inequalities for the positiveness of the Green's function that are used, as we will see in the last section, to construct the cone where the Krasnoselskii's fixed point Theorem 1.3.7 is applied.

Lemma 4.2.4. *Assume that* $0 < \delta_1 + \delta_2 < 2\,(= \Delta(0))$, $0 < \delta_1 < 2$ *and* $0 < \delta_2 < 2$. *Then, there are two real constants* $c_1 \leq 1 \leq c_2$ *such that:*

$$c_1\, G_{0,\delta_1,\delta_2}(1,s) \leq G_{0,\delta_1,\delta_2}(t,s) \leq c_2\, G_{0,\delta_1,\delta_2}(1,s), \quad \textit{for all } t,s \in J. \tag{4.2.11}$$

Proof. For $s = 0$ and $s = 1$ the inequalities follow immediately from Property 1. in Lemma 4.2.2.

Since there exist

$$\lim_{s\to 0^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = t + (1-t)\frac{2-\delta_1-\delta_2}{\delta_2} > 0, \quad \text{for all } t \in J$$

and

$$\lim_{s\to 1^-} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{t(2-\delta_1)+\delta_1(1-t)}{\delta_2} > 0, \quad \text{for all } t \in J,$$

we can continuously extend the strictly positive function $\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)}$ defined on $J\times(0,1)$ to the compact set $J \times J$. Thus, by Weierstrass's Theorem, we have that the continuous positive expansion $\tilde{H}_{0,\delta_1,\delta_2}(t,s)$ attains a positive maximum c_2 and a positive minimum c_1 on $J \times J$, that is

$$c_1 \leq \tilde{H}_{0,\delta_1,\delta_2}(t,s) \leq c_2, \quad \text{for all } t,\ s \in J.$$

Now, since $G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s \in (0,1)$, we deduce inequalities (4.2.11). □

Remark 4.2.5. *Let us see that the constants found in the previous result are*

$$c_1 = \min\left\{1, \frac{\delta_1}{\delta_2}\right\} \ \textit{ and } \ c_2 = \frac{2}{\delta_2}. \tag{4.2.12}$$

Indeed, on the one hand we have that

$$\lim_{t\to s^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{2+(s-1)\delta_2-\delta_1 s}{\delta_2} \leq \frac{2}{\delta_2},$$

and, on the other hand,

$$\lim_{t\to 0^+} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = \frac{\delta_1}{\delta_2}, \quad \lim_{t\to 1^-} \frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} = 1.$$

Moreover, since $\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)}$ *is a nonnegative, continuous and linear function with respect to t, we deduce*

$$\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} \leq \max\left\{1, \frac{\delta_1}{\delta_2}, \frac{2}{\delta_2}\right\} = \frac{2}{\delta_2},$$

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and

$$\frac{G_{0,\delta_1,\delta_2}(t,s)}{G_{0,\delta_1,\delta_2}(1,s)} \geq \min\left\{1, \frac{\delta_1}{\delta_2}, \frac{2}{\delta_2}\right\} = \min\left\{1, \frac{\delta_1}{\delta_2}\right\}.$$

As a consequence of the previous result we arrive at the following corollary.

Corollary 4.2.6. *Assume that* $0 < \delta_1 + \delta_2 < 2\,(= \Delta(0))$, $0 < \delta_1 < 2$ *and* $0 < \delta_2 < 2$. *Then, it holds that:*

$$\min_{t\in J} G_{0,\delta_1,\delta_2}(t,s) \geq \frac{c_1}{c_2} \max_{t\in J} G_{0,\delta_1,\delta_2}(t,s), \quad \textit{for all } s \in J.$$

Proof. From Lemma 4.2.4 we have that

$$c_1\, G_{0,\delta_1,\delta_2}(1,s) \leq G_{0,\delta_1,\delta_2}(t,s), \quad \text{for all } t, s \in J,$$

and

$$\frac{1}{c_2}\, G_{0,\delta_1,\delta_2}(t,s) \leq G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } t, s \in J.$$

Then,

$$\min_{t\in J} G_{0,\delta_1,\delta_2}(t,s) \geq c_1\, G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } s \in J,$$

and

$$\frac{1}{c_2} \max_{t\in J} G_{0,\delta_1,\delta_2}(t,s) \leq G_{0,\delta_1,\delta_2}(1,s), \quad \text{for all } s \in J.$$

Therefore,

$$\min_{t\in J} G_{0,\delta_1,\delta_2}(t,s) \geq c_1\, G_{0,\delta_1,\delta_2}(1,s) \geq \frac{c_1}{c_2} \max_{t\in J} G_{0,\delta_1,\delta_2}(t,s), \quad \text{for all } s \in J,$$

and the result holds. □

4.2.2 Case $\gamma > 0$

In this subsection, we will study the constant sign of the Green's function related to the following linear problem

$$u''(t) + \gamma\, u(t) + \sigma(t) = 0, \quad t \in J, \tag{4.2.13}$$

with $\gamma = m^2$, $m > 0$, subject to the integral boundary conditions (4.1.3). As in the previous subsection, we will deduce two inequalities when this function takes positive values.

The following result gives the expression of the Green's function $G_{\gamma,\delta_1,\delta_2}$ related to problem (4.2.13), (4.1.3).

Theorem 4.2.7. *Let* $\gamma = m^2$, $m > 0$, $m \neq 2\,k\pi$, $k = 1, 2, \ldots$, $\delta_1 + \delta_2 \neq \frac{m\cos\left(\frac{m}{2}\right)}{\sin\left(\frac{m}{2}\right)}\,(= \Delta(m^2))$, *and* $\sigma \in \mathcal{C}(J)$. *Then, problem* (4.2.13), (4.1.3) *has a unique solution* $u \in C^2(J)$, *which is given by the expression*

$$u(t) = \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\, \sigma(s)\, ds,$$

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where

$$G_{\gamma,\delta_1,\delta_2}(t,s)=\begin{cases}G^1_{\gamma,\delta_1,\delta_2}(t,s), & 0\le s\le t\le 1,\\ G^2_{\gamma,\delta_1,\delta_2}(t,s), & 0\le t<s\le 1.\end{cases} \tag{4.2.14}$$

Here, if $m\neq k\,\pi$, $k\in\mathbb{N}$ *odd,*

$$\begin{aligned}G^1_{\gamma,\delta_1,\delta_2}(t,s)=&\frac{\csc(m)\,\sin(m\,s)\,\sin(m\,(1-t))}{m}\\&+\frac{(-1+\omega_1(s)+\omega_2(s))(\delta_1\,\omega_1(t)+\delta_2\,\omega_2(t))}{m\,(m-(\delta_1+\delta_2)\tan(\frac{m}{2}))},\end{aligned}$$

and

$$\begin{aligned}G^2_{\gamma,\delta_1,\delta_2}(t,s)=&\frac{\csc(m)\,\sin(m\,t)\,\sin(m\,(1-s))}{m}\\&+\frac{(-1+\omega_1(s)+\omega_2(s))(\delta_1\,\omega_1(t)+\delta_2\,\omega_2(t))}{m\,(m-(\delta_1+\delta_2)\tan(\frac{m}{2}))},\end{aligned}$$

being $\omega_1(t)=\csc(m)\,\sin(m\,(1-t))$, $t\in J$ *and* $\omega_2(t)=\csc(m)\,\sin(m\,t)$, $t\in J$.

In the case of $m=k\,\pi$, *for some* $k\in\mathbb{N}$ *odd, we have that*

$$G^1_{(k\,\pi)^2,\delta_1,\delta_2}(t,s)=\lim_{\gamma\to(k\,\pi)^2}G^1_{\gamma,\delta_1,\delta_2}(t,s)$$

and

$$G^2_{(k\,\pi)^2,\delta_1,\delta_2}(t,s)=\lim_{\gamma\to(k\,\pi)^2}G^2_{\gamma,\delta_1,\delta_2}(t,s).$$

Proof. Consider the case $m\neq k\,\pi$, $k\in\mathbb{N}$, and let ω_1 be the unique solution of

$$\omega_1''(t)+m^2\,\omega_1(t)=0,\quad t\in J,\qquad \omega_1(0)=1,\quad \omega_1(1)=0,$$

and ω_2 be defined as the unique solution of

$$\omega_2''(t)+m^2\,\omega_2(t)=0,\quad t\in J,\qquad \omega_2(0)=0,\quad \omega_2(1)=1.$$

It is immediate to verify that

$$\omega_2(t)=\csc(m)\,\sin(m\,t),$$

and

$$\omega_1(t)=\omega_2(1-t)=\csc(m)\,\sin(m\,(1-t)).$$

It is very well-known, see [6], that the Green's function related to the homogeneous case (the case $\delta_1=\delta_2=0$) is given by the expression

$$G_{\gamma,0,0}(t,s)=\begin{cases}\dfrac{\csc(m)\sin(m\,s)\sin(m\,(1-t))}{m}, & 0\le s\le t\le 1,\\[2ex] \dfrac{\csc(m)\sin(m\,(1-s))\sin(m\,t)}{m}, & 0\le t<s\le 1.\end{cases}$$

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On the other hand, it is not difficult to verify that $C(\omega_1) = C(\omega_2) = \frac{\tan\left(\frac{m}{2}\right)}{m}$ and

$$C(G_{\gamma,0,0}(\cdot, s)) = \frac{-1 + \omega_1(s) + \omega_2(s)}{m^2}.$$

Substituting those values in (4.1.7), we obtain the first part of result.

Finally, for $\gamma = (k\,\pi)^2$ with k odd, it is easy to see, using the Remark 4.1.2 and the axiomatic Definition 1.1.1 of the Green's function, that the expression obtained by taking limits on G^1_γ and G^2_γ when γ tends to $(k\,\pi)^2$ is a Green's function of problem (4.2.13), (4.1.3).

□

Remark 4.2.8. *It should be noted that in the previous result we cannot apply the expression* (4.2.10) *to obtain the Green's function for the case $\gamma = (k\pi)^2$, with k odd and $\delta_1 + \delta_2 \neq 0\,(= \Delta(\gamma))$ (contrary to the case $\gamma = (k\pi)^2$, with k even) because in this situation, since $(k\pi)^2$ is an eigenvalue of problem*

$$u''(t) + \gamma\, u(t) = 0,\ t \in J,\ \ u(0) = u(1) = 0,$$

we have that $G_{\gamma,0,0}$ does not exist.

The following result describes the points where a constant sign Green's function of problem (4.2.13), (4.1.3) may vanish.

Lemma 4.2.9. *Let $\gamma = m^2$, $m > 0$, $\gamma < \pi^2$ and assume that problem* (4.2.13), (4.1.3) *has a unique solution for any $\sigma \in C(J)$. If $G_{\gamma,\delta_1,\delta_2}$ has constant sign on $J \times J$ and vanishes at some point (t_0, s_0), then either $s_0 = 0$, $s_0 = 1$, $t_0 = 0$, $t_0 = 1$ or $t_0 = s_0$.*

Proof. Let us suppose that $(t_0, s_0) \in (0,1) \times (0,1)$, with $t_0 > s_0$. From the definition of the Green's function, we have that $u(t) := G_{\gamma,\delta_1,\delta_2}(t, s_0)$, $t \in J$, is the unique solution of the problem

$$u''(t) + \gamma\, u(t) = 0, \quad t \in (s_0, 1], \qquad u(t_0) = u'(t_0) = 0,$$

and so $G_{\gamma,\delta_1,\delta_2}(t, s_0) = 0$ for all $t \in (s_0, 1]$.

Now, from the properties of the Green's function given in Definition 1.1.1, we have that

$$u(1) = \delta_2 \int_0^1 u(s)\, ds = 0$$

which implies that either $u(t) = 0$ for all $t \in J$, which contradicts that $u'(s_0^-) = 1 < 0 = u'(s_0^+)$, or u changes its sign on J, which contradicts the constant sign of the Green's function.

In the case $(t_0, s_0) \in (0,1) \times (0,1)$, with $t_0 < s_0$, from Theorem 4.1.1 and Remark 4.1.2 we deduce that if $G_{\gamma,\delta_1,\delta_2}$ has constant sign on $J \times J$ and vanishes at (t_0, s_0), then $G_{\gamma,\delta_2,\delta_1}$ will also have constant sign on $J \times J$ and vanish at $(1 - t_0, 1 - s_0)$ with $1 - t_0 > 1 - s_0$. Therefore, we are in the previous case and the proof is concluded. □

In the sequel we deduce some properties of the Green's function $G_{\gamma,\delta_1,\delta_2}$.

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Lemma 4.2.10. *Let $\gamma = m^2$, $m > 0$, $m \neq 2\,k\,\pi$, $k = 1, 2, \ldots$ (we understand that $\Delta(\pi^2)\left(\equiv \frac{\pi}{\tan(\pi/2)}\right) = 0$). Then for all $\delta_1 + \delta_2 \neq \frac{m}{\tan\left(\frac{m}{2}\right)}\,(= \Delta(m^2))$, the following properties hold:*

1. $G_{\gamma,\delta_1,\delta_2}(t,1) = G_{\gamma,\delta_1,\delta_2}(t,0) = 0$, *for all* $t \in J$.
2. $G_{\gamma,\delta_1,\delta_2}(t,s)$ *is continuous at* $(t,s) \in J \times J$.
3. $G_{\gamma,\delta_1,\delta_2}(0,s) = 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_1 = 0$.
4. $\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(0,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $m \in (0,\pi)$ *and* $\delta_1 > 0$.
5. $G_{\gamma,\delta_1,\delta_2}(1,s) = 0$ *for all* $s \in (0,1)$ *if and only if* $\delta_2 = 0$.
6. $\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(1,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $m \in (0,\pi)$ *and* $\delta_2 > 0$.
7. *If* $m \in (0,\pi)$, *then* $\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(s,s) > 0$ *for all* $s \in (0,1)$, $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ *and* $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$.
8. $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ *for all* $t,s \in (0,1)$ *if and only if* $0 \leq \delta_1+\delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$, $m \in (0,\pi)$, $\delta_1 \geq 0$ *and* $\delta_2 \geq 0$.
9. $G_{\gamma,\delta_1,\delta_2}(t,s) < 0$ *for all* $t,s \in (0,1)$ *if and only if* $\delta_1+\delta_2 > \frac{m}{\tan\left(\frac{m}{2}\right)}$, $0 < \delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$, $0 < \delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ *and* $m \in (0,\pi)$.
10. $G_{\gamma,\delta_1,\delta_2}$ *changes its sign on* $(0,1)\times(0,1)$ *if and only if one the following properties is fulfilled:*
    - *(a)* $m > \pi$,
    - *(b)* $\delta_1 > \frac{m}{\tan\left(\frac{m}{2}\right)}$,
    - *(c)* $\delta_2 > \frac{m}{\tan\left(\frac{m}{2}\right)}$,
    - *(d)* $\delta_1 < 0$,
    - *(e)* $\delta_2 < 0$.

Proof. Properties 1. and 2. are immediate using the expression (4.2.14). Let us now prove the remaining properties:

3. Let $s \in (0,1)$, then $G_{\gamma,\delta_1,\delta_2}(0,s) = 0$ if and only if

$$\frac{\delta_1\left(-1 + w_1(s) + w_2(s)\right)}{m\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)} = 0,$$

which is equivalent to

$$\delta_1(-1 + w_1(s) + w_2(s)) = 0.$$

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It is easy to see that the function

$$r_m(s) := -1 + w_1(s) + w_2(s),$$

is positive on $(0,1)$ if and only if $m \in (0,\pi)$. Moreover, it is not identically zero on $(0,1)$ for any $m \neq 2\,k\,\pi$, $k = 1, 2, \ldots$.

Therefore $G_{\gamma,\delta_1,\delta_2}(0,s) = 0$ if and only if $\delta_1 = 0$.

4. From (4.2.14), we infer that

$$\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(0,s) = \frac{\delta_1}{m} r_m(s) > 0,$$

for all $s \in (0,1)$, if and only if $m \in (0,\pi)$ and $\delta_1 > 0$.

5. 6. These properties are immediately deduced from Properties 3. and 4. and the following equality

$$G_{\gamma,\delta_1,\delta_2}(1,s) = G_{\gamma,\delta_2,\delta_1}(0,1-s), \quad \forall s \in J,$$

deduced from (4.1.6) and Remark 4.1.2.

7. Let us define the function

$$h(s,m,\delta_1,\delta_2) := m\,\sin(m)\,\left(-m + (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(s,s).$$

It occurs that

$$\frac{\partial}{\partial\delta_1} h(s,m,\delta_1,\delta_2) = (1-\cos(m))\sin(m-ms) > 0,$$

and

$$\frac{\partial}{\partial\delta_2} h(s,m,\delta_1,\delta_2) = 2\sin(ms)\sin^2\left(\frac{1}{2}(m-ms)\right) > 0,$$

for all $s \in (0,1)$ and $m \in (0,\pi)$.

Moreover,

$$h\left(s,m,\frac{m}{\tan\left(\frac{m}{2}\right)},\frac{m}{\tan\left(\frac{m}{2}\right)}\right) = -4\sin^2\left(\frac{ms}{2}\right)\sin^2\left(\frac{m(1-s)}{2}\right) < 0,$$

for all $s \in (0,1)$ and $m \in (0,\pi)$.

Therefore, $h(s,m,\delta_1,\delta_2) < 0$ for all $s \in (0,1)$, $m \in (0,\pi)$, $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$, and the property is fulfilled.

8. From Lemma 4.2.9, for any $s \in (0,1)$ fixed, if function $G_{\gamma,\delta_1,\delta_2}(\cdot,s)$ has constant sign, then it attains its maximum and minimum values either at $t = 0$, $t = s$ or $t = 1$.

From Property 4., we have that $G_{\gamma,\delta_1,\delta_2}(0,s) > 0$ for all $s \in (0,1)$, if and only if $\delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_1 > 0$.

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From Property 6., we have that $G_{0,\delta_1,\delta_2}(1,s) > 0$ for all $s \in (0,1)$, if and only if $\delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 > 0$.
From Property 7. we have that if $m \in (0,\pi)$ and $\delta_1+\delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$ then $G_{\gamma,\delta_1,\delta_2}(s,s) > 0$ for all $s \in (0,1)$, provided that $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$.

Notice that if $\delta_1 > 0$ and $\delta_2 > 0$, the fact that $\delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)}$ implies that $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$. So, we deduce that $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$ for such values of the parameters.

If $G_{\gamma,\delta_1,\delta_2}(s,s)$ is positive for other range of values we have that either $G_{\gamma,\delta_1,\delta_2}(0,s)$ or $G_{\gamma,\delta_1,\delta_2}(1,s)$ will be negative.

As a consequence of the previous assertions this property holds.

9. The proof is analogous to the proof of Property 8.

10. For any fixed values of t, s, δ_1 and δ_2, from Theorems 1.2.1 and 1.2.2, we know that $G_{\gamma,\delta_1,\delta_2}(t,s)$ is monotone increasing with respect to γ on the intervals of γ where such function has constant sign.

    We must take into account that, $\Delta(m^2)$ is strictly positive and strictly decreasing for $m \in (0,\pi)$.

    Let $\delta_1 > 0$ and $\delta_2 > 0$ be such that $\delta_1 + \delta_2 < \frac{m_1}{\tan\left(\frac{m_1}{2}\right)}$ for some $m_1 \in (0,\pi)$. Thus, there is a unique value $m_0 \in (m_1,\pi)$ such that

    $$\delta_1 + \delta_2 = \frac{m_0}{\tan\left(\frac{m_0}{2}\right)} = \Delta(m_0^2)$$

    and

    $$0 < \delta_1 + \delta_2 < \frac{m}{\tan\left(\frac{m}{2}\right)} \quad \text{for all } m \in (0,m_0).$$

    Thus $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ on $(0,1)\times(0,1)$ for all $m \in (0,m_0)$.

    Therefore, if $\delta_1 > 0$ and $\delta_2 > 0$, we have that there is $m_2 \in (m_0,\pi)$ such that $\delta_1 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ and $\delta_2 \leq \frac{m}{\tan\left(\frac{m}{2}\right)}$ for all $m \in (m_0,m_2]$. In consequence, $G_{\gamma,\delta_1,\delta_2}(t,s) < 0$ on $(0,1)\times(0,1)$ if and only if $m \in (m_0,m_2]$ (see Figure 4.2.1).

    For $m > m_2$ we have that $G_{\gamma,\delta_1,\delta_2}$ must change its sign on $J \times J$, on the contrary it contradicts the monotone increasing property of this function with respect to m.

    In the case of $\delta_1 = 0$, we will have $\delta_2 = \frac{m_0}{\tan\left(\frac{m_0}{2}\right)}$ and so, the Green's function $G_{\gamma,\delta_1,\delta_2}$ can never be negative on $(0,1)\times(0,1)$ and must change its sign for all $m > m_0$. The same holds for $\delta_2 = 0$ (see Figure 4.2.2). Both cases have been considered in the work [13].

□

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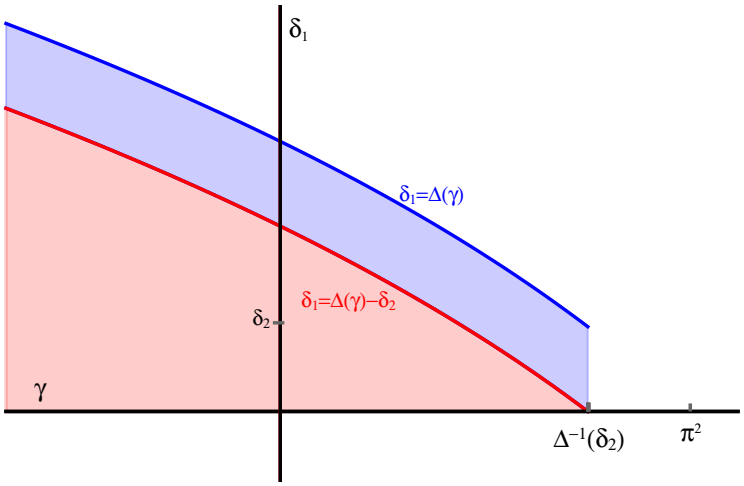

Figure 4.2.1: Given $\delta_2 > 0$, the Green's function is positive on the red region, negative on the blue one, and changes its sign otherwise.

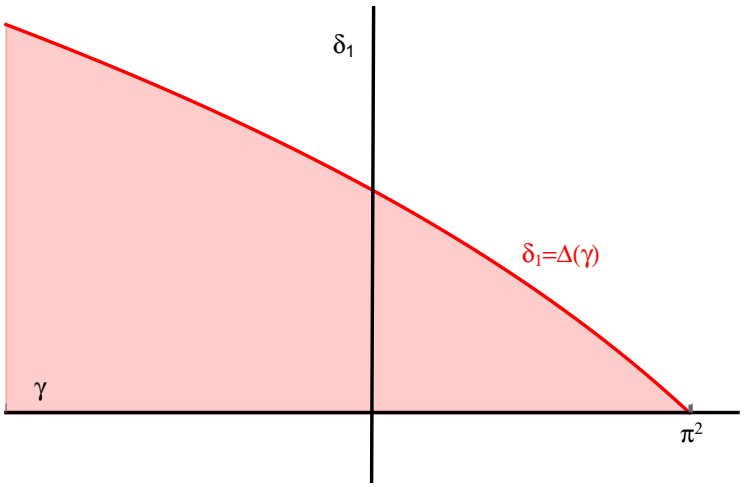

Figure 4.2.2: If $\delta_2 = 0$, the Green's function is positive on the red region and changes its sign otherwise.

Remark 4.2.11. *We mention that Properties 1, 2, 5, 6, 8 and 10 are a generalization of the ones given in [13, Lemma 8]. Despite this, it is important to mention that the proof of Property 10 here is completely different to the simpler one given in that reference.*

The rest of the cases have no sense for that situation.

Analogously to Lemma 4.2.4, we get the following result.

Lemma 4.2.12. *Let $\gamma = m^2$, $0 < m < \pi$, $0 < \delta_1 + \delta_2 < \frac{m}{\tan(\frac{m}{2})}$ $(= \Delta(m^2))$. Then, there are real constants $c_3 > 0$ and $c_4 > 0$ such that*

$$c_3\, G_{\gamma,\delta_1,\delta_2}(1,s) \leq G_{\gamma,\delta_1,\delta_2}(t,s) \leq c_4\, G_{\gamma,\delta_1,\delta_2}(1,s), \quad \textit{for all } t,\, s \in J. \tag{4.2.15}$$

Proof. If $s = 0$ or $s = 1$ the result follows from Lemma 4.2.10. Let $t \in J$ be fixed. On the one hand, we have that

$$\lim_{s\to 0^+} \frac{G_{\gamma,\delta_1,\delta_2}(t,s)}{G_{\gamma,\delta_1,\delta_2}(1,s)} = \frac{\omega_1(t)\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)}{\delta_2 \csc(m)\,(1-\cos(m))} + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{\delta_2} > 0.$$

On the other hand, since

$$\omega_1'(1) + \omega_2'(1) < 0,$$

we have that

$$\lim_{s\to 1^-} \frac{G_{\gamma,\delta_1,\delta_2}(t,s)}{G_{\gamma,\delta_1,\delta_2}(1,s)} = \frac{-\omega_1(t)\left(m - (\delta_1+\delta_2)\tan\left(\frac{m}{2}\right)\right)}{\delta_2(\omega_1'(1)+\omega_2'(1))} + \frac{\delta_1\,\omega_1(t) + \delta_2\,\omega_2(t)}{\delta_2} > 0.$$

Taking into account Properties 6. and 8. of Lemma 4.2.10, we have that the strictly positive function $\frac{G_{\gamma,\delta_1,\delta_2}(\cdot,s)}{G_{\gamma,\delta_1,\delta_2}(1,s)}$ defined on $J \times (0,1)$ extends continuously to the compact set $J \times J$. Therefore, there exist $c_3 > 0$ and $c_4 > 0$, the minimum and the maximum, respectively, of such extension on $J \times J$. Therefore the inequalities (4.2.15) are fulfilled and we conclude the proof. □

As a consequence of the previous result, we arrive at the next corollary.

Corollary 4.2.13. *Let $\gamma = m^2$, $0 < m < \pi$, $0 < \delta_1 + \delta_2 < \frac{m}{\tan(\frac{m}{2})}$ $(= \Delta(m^2))$. Then, it holds that:*

$$\min_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s) \geq \frac{c_3}{c_4} \max_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s), \quad \textit{for all } t,\, s \in J.$$

Proof. From Lemma 4.2.12 we have that

$$c_3\, G_{\gamma,\delta_1,\delta_2}(1,s) \leq G_{\gamma,\delta_1,\delta_2}(t,s), \quad \text{for all } t, s \in J,$$

and

$$\frac{1}{c_4}\, G_{\gamma,\delta_1,\delta_2}(t,s) \leq G_{\gamma,\delta_1,\delta_2}(1,s), \quad \text{for all } t, s \in J.$$

Then,

$$\min_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s) \geq c_3\, G_{\gamma,\delta_1,\delta_2}(1,s), \quad \text{for all } s \in J,$$

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and

$$\frac{1}{c_4}\max_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s) \le G_{\gamma,\delta_1,\delta_2}(1,s), \quad \text{for all } s\in J.$$

Therefore,

$$\min_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s) \ge c_3\, G_{\gamma,\delta_1,\delta_2}(1,s) \ge \frac{c_3}{c_4}\max_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s), \quad \text{for all } s\in J,$$

and the result holds. □

4.2.3 Case $\gamma < 0$

Finally, we deal in this subsection with the study of the constant sign of the Green's function related to the following linear problem

$$u''(t) - \gamma\, u(t) + \sigma(t) = 0, \quad t\in J, \tag{4.2.16}$$

with $\gamma = -m^2$, $m>0$, coupled to (4.1.3).

For the construction of the Green's function we obtain the following result.

Theorem 4.2.14. *Let $\gamma = -m^2$, $m>0$, $\delta_1+\delta_2 \neq \frac{m}{\tanh(\frac{m}{2})}$ $(=\Delta(-m^2))$ and $\sigma\in\mathcal{C}(J)$, then problem* (4.2.16)*,* (4.1.3) *has a unique solution $u\in C^2(J)$, which is given by the expression*

$$u(t) = \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,\sigma(s)\,ds,$$

where

$$G_{\gamma,\delta_1,\delta}(t,s) = \begin{cases} G^1_{\gamma,\delta_1,\delta_2}(t,s), & 0\le s\le t\le 1,\\ G^2_{\gamma,\delta_1,\delta_2}(t,s), & 0\le t<s\le 1,\end{cases}$$

with

$$G^1_{\gamma,\delta_1,\delta_2}(t,s) = \frac{\sinh(m\,s)\,\sinh(m\,(1-t))}{m\sinh(m)} - \frac{(-1+\omega_1(s)+\omega_2(s))(\delta_1\,\omega_1(t)+\delta_2\,\omega_2(t))}{m\,(m-(\delta_1+\delta_2)\tanh(m/2))},$$

and

$$G^2_{\gamma,\delta_1,\delta_2}(t,s) = \frac{\sinh(m\,t)\,\sinh(m\,(1-s))}{m\sinh(m)} - \frac{(-1+\omega_1(s)+\omega_2(s))(\delta_1\,\omega_1(t)+\omega_2\,w_2(t))}{m\,(m-(\delta_1+\delta_2)\tanh(\frac{m}{2}))},$$

being $\omega_1(t) = \frac{\sinh(m\,(1-t))}{\sinh(m)}$, $t\in J$ and $\omega_2(t) = \frac{\sinh(m\,t)}{\sinh(m)}$, $t\in J$.

Proof. It is very well-known, see [6], that the Green's function related to the homogeneous case ($\delta_1=\delta_2=0$) is given by the expression

$$G_{\gamma,0,0}(t,s) = \begin{cases} \frac{\sinh(ms)\,\sinh(m-mt)}{m\sinh(m)}, & 0\le s\le t\le 1,\\ \frac{\sinh(mt)\,\sinh(m-ms)}{m\sinh(m)}, & 0\le t<s\le 1,\end{cases}$$

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One can easly verify that $\omega_1(t) = \frac{\sinh(m-mt)}{\sinh(m)}$ is the unique solution of

$$\omega_1''(t) - m^2\,\omega_1(t) = 0, \quad t \in J, \qquad \omega_1(0) = 1, \quad \omega_1(1) = 0,$$

and $\omega_2(t) = \frac{\sinh(mt)}{\sinh(m)}$ is the unique solution of

$$\omega_1''(t) - m^2\,\omega_1(t) = 0, \quad t \in J, \qquad \omega_1(0) = 0, \quad \omega_1(1) = 1,$$

and, moreover, $\omega_1(t) = \omega_2(1-t)$ for all $t \in J$.

The result is obtained by applying the equality (4.2.10) and taking into account that in this case $C(w_1) = C(w_2) = \frac{\tanh(\frac{m}{2})}{m}$ and

$$C(G_{\gamma,0,0}(\cdot,s)) = \frac{1 - w_1(s) - w_2(s)}{m^2}.$$

□

We will now state the following results that characterize the constant sign of the Green's function on $J \times J$. We omit the proofs because they are analogous to the case $\gamma = m^2$ discussed in the previous section.

Lemma 4.2.15. *Let $\gamma = -m^2$, $m > 0$. Then for all $\delta_1 + \delta_2 \neq \frac{m}{\tanh(\frac{m}{2})}$ $(= \Delta(-m^2))$, the following properties hold:*

1. $G_{\gamma,\delta_1,\delta_2}(t,0) = G_{\gamma,\delta_1,\delta_2}(t,1) = 0$, *for all* $t \in J$.
2. $G_{\gamma,\delta_1,\delta_2}(t,s)$ *is continuous on* $J \times J$.
3. $G_{\gamma,\delta_1,\delta_2}(0,s) = 0$, *for all* $s \in (0,1)$ *if and only if* $\delta_1 = 0$.
4. $\left(m - (\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(0,s) > 0$ *for all* $s \in (0,1)$ *and* $m > 0$ *if and only if* $\delta_1 > 0$.
5. $G_{\gamma,\delta_1,\delta_2}(1,s) = 0$, *for all* $s \in (0,1)$ *if and only if* $\delta_2 = 0$.
6. $\left(m - (\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(1,s) > 0$ *for all* $s \in (0,1)$ *and* $m > 0$ *if and only if* $\delta_2 > 0$.
7. $\left(m - (\delta_1+\delta_2)\tanh\left(\frac{m}{2}\right)\right) G_{\gamma,\delta_1,\delta_2}(s,s) > 0$ *for all* $s \in (0,1)$, $m > 0$, $\delta_1 \leq \frac{m}{\tanh(\frac{m}{2})}$ *and* $\delta_2 \leq \frac{m}{\tanh(\frac{m}{2})}$.
8. $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ *for all* $t,s \in (0,1)$ *if and only if* $0 \leq \delta_1 + \delta_2 < \frac{m}{\tanh(\frac{m}{2})}$, $0 \leq \delta_1$, $0 \leq \delta_2$ *and* $m > 0$.
9. $G_{\gamma,\delta_1,\delta_2}(t,s) < 0$ *for all* $t,s \in (0,1)$ *if and only if* $\delta_1 + \delta_2 > \frac{m}{\tanh(\frac{m}{2})}$, $0 < \delta_1 \leq \frac{m}{\tanh(\frac{m}{2})}$, $0 < \delta_2 \leq \frac{m}{\tanh(\frac{m}{2})}$ *and* $m > 0$.
10. $G_{\gamma,\delta_1,\delta_2}$ *changes its sign on* $(0,1) \times (0,1)$ *if and only if one the following properties is fulfilled:*

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(a) $\delta_1 > \frac{m}{\tanh\left(\frac{m}{2}\right)}$,

(b) $\delta_2 > \frac{m}{\tanh\left(\frac{m}{2}\right)}$,

(c) $\delta_1 < 0$,

(d) $\delta_2 < 0$.

Remark 4.2.16. *We point out that in order to prove Property 10 in previous Lemma, we must argue in a similar manner to Property 10 in Lemma 4.2.10. In this case, we must take into account that function Δ is strictly positive and strictly decreasing in $(-\infty, 0)$ and (see Figures 4.2.1 and 4.2.2) that*

$$\lim_{\gamma \to -\infty} \Delta(\gamma) = +\infty.$$

Analogously to Lemmas 4.2.4 and 4.2.12, we can deduce the following result and the next corollary.

Lemma 4.2.17. *Let $\gamma = -m^2$, $m > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $0 < \delta_1 + \delta_2 < \frac{m}{\tanh\left(\frac{m}{2}\right)}$ $(= \Delta(-m^2))$. Then there are two positive real constants c_5 and c_6 such that:*

$$c_5\, G_{\gamma,\delta_1,\delta_2}(1,s) \leq G_{\gamma,\delta_1,\delta_2}(t,s) \leq c_6\, G_{\gamma,\delta_1,\delta_2}(1,s), \quad \textit{for all } t,\ s \in J. \tag{4.2.17}$$

Corollary 4.2.18. *Let $\gamma = -m^2$, $m > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $0 < \delta_1 + \delta_2 < \frac{m}{\tanh\left(\frac{m}{2}\right)}$ $(= \Delta(-m^2))$. Then, it holds that:*

$$\min_{t \in J} G_{\gamma,\delta_1,\delta_2}(t,s) \geq \frac{c_5}{c_6} \max_{t \in J} G_{\gamma,\delta_1,\delta_2}(t,s), \quad \textit{for all } t,\ s \in J.$$

4.3 Nonlinear Problem

In this section we will study the existence of positive solutions of the nonlinear problems (4.1.1), (4.1.3) and (4.1.2)–(4.1.3). The existence of positive solutions of problem (4.1.1), (4.1.3) when the related Green's function is positive and of problem (4.1.2)–(4.1.3) if it is negative are deduced from Krasnoselskii's fixed point Theorem.

We suppose that the nonlinear part of the equation satisfies the following regularity and constant sign condition:

(f) $\quad f : J \times [0,\infty) \to [0,\infty)$ is a continuous function.

Let $X \equiv (C(J), \|\cdot\|_\infty)$ be the real Banach space equipped with the supremum norm

$$\|u\|_\infty = \sup_{t \in J} |u(t)|, \quad \text{for all } u \in X.$$

4.3.1 Case of Positive Green's Function

We consider the situation in which the related Green's function $G_{\gamma,\delta_1,\delta_2}$ is positive, that is

$$\gamma < \pi^2 \text{ and } \delta_1,\ \delta_2 > 0 \text{ are such that } 0 < \delta_1 + \delta_2 < \Delta(\gamma), \tag{4.3.18}$$

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where $\Delta : (-\infty, \pi^2) \to \mathbb{R}$ is the function defined by expression (4.1.5).

We will apply Krasnoselskii's fixed point Theorem to the integral operator $T_{\gamma,\delta_1,\delta_2} : X \to X$, defined as

$$T_{\gamma,\delta_1,\delta_2}u(t) := \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\, f(s,u(s))\, ds, \quad t \in J. \tag{4.3.19}$$

It is very well-known that the fixed points of operator $T_{\gamma,\delta_1,\delta_2}$ coincide with the solutions of problem (4.1.1), (4.1.3).

Next we define the cone K where we will apply Krasnoselskii's fixed point Theorem:

$$K = \{u \in X \,/\, \min\{u(t) : t \in J\} \geq \bar{c}\, \|u\|_\infty\},$$

where

$$\bar{c} = \begin{cases} \dfrac{c_1}{c_2}, & \gamma = 0, \\ \dfrac{c_3}{c_4}, & \gamma > 0, \\ \dfrac{c_5}{c_6}, & \gamma < 0, \end{cases}$$

with c_1, c_2, c_3, c_4, c_5 and c_6 given in inequalities (4.2.11), (4.2.15) and (4.2.17).

From Corollaries 4.2.6, 4.2.13 and 4.2.18 we obtain that the following inequality is satisfied:

$$\min_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s) \geq \bar{c} \max_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s), \quad \text{for all } s,\, t \in J. \tag{4.3.20}$$

Using inequalities (4.2.11), (4.2.15) and (4.2.17) again, we can guarantee that there are two real positive constants

$$h = \begin{cases} c_1, & \gamma = 0, \\ c_3, & \gamma > 0, \\ c_5, & \gamma < 0, \end{cases} \quad \text{and} \quad k = \begin{cases} c_2, & \gamma = 0, \\ c_4, & \gamma > 0, \\ c_6, & \gamma < 0, \end{cases}$$

such that

$$h\, G_{\gamma,\delta_1,\delta_2}(1,s) \leq G_{\gamma,\delta_1,\delta_2}(t,s) \leq k\, G_{\gamma,\delta_1,\delta_2}(1,s), \quad \text{for all } t,\, s \in J. \tag{4.3.21}$$

Moreover, we will denote by

$$R = \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, ds = \begin{cases} \dfrac{\delta_2}{6(2-\delta_1-\delta_2)}, & \gamma = 0, \\ \dfrac{\delta_2\left(2\tan\left(\frac{\sqrt{\gamma}}{2}\right) - \sqrt{\gamma}\right)}{\gamma\left(\sqrt{\gamma} - (\delta_1+\delta_2)\tan\left(\frac{\sqrt{\gamma}}{2}\right)\right)}, & \gamma > 0, \\ \dfrac{\delta_2\left(2\tanh\left(\frac{\sqrt{-\gamma}}{2}\right) - \sqrt{-\gamma}\right)}{\gamma\left(\sqrt{-\gamma} - (\delta_1+\delta_2)\tanh\left(\frac{\sqrt{-\gamma}}{2}\right)\right)}, & \gamma < 0. \end{cases} \tag{4.3.22}$$

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We note that in the above definitions, although $\bar{c}$, h, k and R depend on γ, δ_1 and δ_2, we will omit such dependence in the notation for the sake of simplicity. Moreover, it is clear that, when γ, δ_1 and δ_2 are fixed, then $\bar{c}$, h, k and R are constants too.

In the sequel, with all these considerations we prove the following theorem that guarantees the existence of positive solutions.

Theorem 4.3.1. *Assume that the positiveness condition of the Green's function* (4.3.18) *is fulfilled. Moreover, suppose that the following conditions hold:*

1. *There exists $p > 0$ such that*

$$f(t,u) \leq \frac{p}{k\,R}, \quad \text{for all } t \in J \text{ and } u \in [0,p].$$

2. *There exists $q > 0$, with $q \neq p$ such that*

$$f(t,u) \geq \frac{q}{h\,R}, \quad \text{for all } t \in J \text{ and } u \in [\bar{c}\,q, q].$$

Then problem (4.1.1), (4.1.3) *has at least one positive solution $u \in K$, such that $\|u\|_\infty$ lies between p and q.*

Proof. First, since $G_{\gamma,\delta_1,\delta_2}(t,s) > 0$ for all $t,s \in (0,1)$, $f \geq 0$ and the fixed points of the operator $T_{\gamma,\delta_1,\delta_2}$ defined in (4.3.19) coincide with the solutions of problem (4.1.1), (4.1.3), we deduce that these solutions are nonnegative.

Now, we show that $T_{\gamma,\delta_1,\delta_2}$ is a compact operator and maps K into K.

Let us see that $T_{\gamma,\delta_1,\delta_2}$ maps K into K. For all $u \in K$, using inequality (4.3.20) we infer that, for all $t \in J$,

$$\begin{aligned} T_{\gamma,\delta_1,\delta_2}\,u(t) &= \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds \geq \bar{c}\int_0^1 \max_{t\in J} G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds \\ &\geq \bar{c}\,\max_{t\in J}\int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds. \end{aligned}$$

So, $T_{\gamma,\delta_1,\delta_2}\,u(t) \geq \bar{c}\,\|T_{\gamma,\delta_1,\delta_2}\,u\|_\infty$ for all $t \in J$, that is, $T_{\gamma,\delta_1,\delta_2}\,u \in K$.

On the other hand, since $G_{\gamma,\delta_1,\delta_2}$ and f are continuous, we have that operator $T_{\gamma,\delta_1,\delta_2}$ is continuous too.

Finally, we will prove that $T_{\gamma,\delta_1,\delta_2}$ maps bounded sets into relatively compact sets. Let $H \subset K$ be a bounded set. Then, there exists $r \in \mathbb{R}$, $r > 0$ such that $\|u\|_\infty \leq r$ for all $u \in H$. Let us take

$$M = \max_{t\in J, 0\leq u\leq r} |f(t,u)|.$$

So, for all $t \in J$ and $u \in H$, using the inequality (4.3.21) we infer that

$$|T_{\gamma,\delta_1,\delta_2}\,u(t)| = \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\,f(s,u(s))\,ds \leq \tilde{M} := k\,R\,M.$$

Thus, $T_{\gamma,\delta_1,\delta_2}(H)$ is bounded.

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Let us show then the equicontinuity of $T_{\gamma,\delta_1,\delta_2}(H)$. Indeed, for all $t \in J$ and $u \in H$, we have that

$$|(T_{\gamma,\delta_1,\delta_2}\, u)'(t)| = \left| \int_0^1 \frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s)\, f(s,u(s))\, ds \right| \leq \int_0^1 \left| \frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s) \right| |f(s,u(s))|\, ds$$
$$\leq M \int_0^1 \left| \frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s) \right| ds.$$

Using the regularity of the Green's function $G_{\gamma,\delta_1,\delta_2}$ we deduce that there exists $N \in \mathbb{R}$, $N > 0$ such that,

$$M \int_0^1 \left| \frac{\partial G_{\gamma,\delta_1,\delta_2}}{\partial t}(t,s) \right| ds \leq N.$$

So, for all $t_1, t_2 \in J,\ \ t_1 < t_2$, the following inequality holds

$$|(T_{\gamma,\delta_1,\delta_2}u)(t_2) - (T_{\gamma,\delta_1,\delta_2}u)(t_1)| = \left| \int_{t_1}^{t_2} (T_{\gamma,\delta_1,\delta_2}u)'(s) ds \right|$$
$$\leq \int_{t_1}^{t_2} |(T_{\gamma,\delta_1,\delta_2}u)'(s)| ds \leq N(t_2 - t_1).$$

Therefore, $T_{\gamma,\delta_1,\delta_2}(H)$ is an equicontinuous set in X. By Arzelà-Ascoli's Theorem, we deduce that $T_{\gamma,\delta_1,\delta_2}(H)$ is relatively compact, that is, $T_{\gamma,\delta_1,\delta_2} : K \to K$ is compact operator.

Next, let us define the following subsets of X

$$K_p = \{u \in X \,/\, \|u\|_\infty < p\} \quad \text{and} \quad K_q = \{u \in X \,/\, \|u\|_\infty < q\}.$$

From inequality (4.3.21) we infer that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty = \max_{t\in J} \int_0^1 G_{\gamma,\delta_1,\delta_2}(t,s)\, f(s,u(s))\, ds \leq k \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds.$$

Using Condition 1. we have, for all $u \in K \cap \partial K_p$, that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty \leq k \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds \leq p = \|u\|_\infty.$$

On the other hand, for $u \in K \cap \partial K_q$, using Condition 2. we have that

$$q = \|u\|_\infty \geq u(t) \geq \bar{c}\|u\|_\infty = \bar{c}\, q$$

for all $t \in J$ and, from (4.3.21), we deduce that

$$\|T_{\gamma,\delta_1,\delta_2}\, u(t)\|_\infty \geq h \int_0^1 G_{\gamma,\delta_1,\delta_2}(1,s)\, f(s,u(s))\, ds \geq q = \|u\|_\infty.$$

Therefore, applying Krasnoselskii's Theorem 1.3.7, we conclude that problem (4.1.1), (4.1.3) has at least one positive solution $u \in K$ such that $\|u\|_\infty$ lies between p and q. □

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As an application of the previous result, we present the following example.

Example 4.3.2. *Consider the following problem*

$$\begin{cases} u''(t) + u(t)\, e^{u^2(t)} = 0,\ t \in J, \\ u(0) = \dfrac{1}{3}\displaystyle\int_0^1 u(s)\, ds, \\ u(1) = \dfrac{1}{6}\displaystyle\int_0^1 u(s)\, ds. \end{cases} \tag{4.3.23}$$

In this case, $\gamma = 0$, $\delta_1 = \frac{1}{3}$, $\delta_2 = \frac{1}{6}$ *and* $\delta_1 + \delta_2 = \frac{1}{2} < 2$. *Using* (4.2.12) *we have that* $h = c_1 = 1$ *and* $k = c_2 = 12$ *and, consequently,* $\bar{c} = \frac{1}{12}$. *Moreover, from* (4.3.22) *we know that*

$$R = \int_0^1 G_{0,\frac{1}{3},\frac{1}{6}}(1, s)\, ds = \frac{1}{54}.$$

The function $f(t, u) = u\, e^{u^2}$, $(t, u) \in J \times [0, \infty)$ *satisfies the assumption* (f). *Now, for* $0 < p \leq \sqrt{\ln\left(\frac{9}{2}\right)} \approx 1.226$ *it holds that*

$$f(t, u) \leq p\, e^{p^2} \leq \frac{9}{2}\, p \left(= \frac{p}{k\, R}\right) \quad \forall\, t \in J,\ u \in [0, p].$$

Similarly, for $q \geq 12\sqrt{\log 648} \approx 30.533$ *it holds that*

$$f(t, u) \geq \frac{1}{12}\, q\, e^{\frac{q^2}{144}} \geq 54\, q \left(= \frac{q}{h\, R}\right) \quad \forall\, t \in J,\ u \in \left[\frac{q}{12}, q\right].$$

Thus, from Theorem 4.3.1 we have that problem (4.3.23) *has at least one positive solution.*

4.3.2 Case of Negative Green's Function

In this last subsection we study the nonlinear problem (4.1.2)– (4.1.3).

For this, we must use the Green's function related to the following linear problem

$$u''(t) + \gamma\, u(t) - \sigma(t) = 0, \quad t \in J, \tag{4.3.24}$$

coupled to (4.1.3).

It is immediate to verify that the Green's function related to problem (4.3.24), (4.1.3) is given by the expression $-G_{\gamma,\delta_1,\delta_2}$.

So, we consider the situation in which the related Green's function $G_{\gamma,\delta_1,\delta_2}$ is negative, that is, the following condition is fulfilled:

$$\gamma < \pi^2,\ \delta_1 + \delta_2 > \Delta(\gamma),\ \delta_1 \leq \Delta(\gamma) \text{ and } \delta_2 \leq \Delta(\gamma). \tag{4.3.25}$$

Obviously, previous condition characterizes the set of parameters for which $-G_{\gamma,\delta_1,\delta_2}$ is positive.

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Therefore, the inequalities that we have used in the previous case, in Lemmas 4.2.4, 4.2.12 and 4.2.17, remain valid since it is clear that

$$c_1 \leq \frac{-G_{0,\delta_1,\delta_2}(t,s)}{-G_{0,\delta_1,\delta_2}(1,s)} \leq c_2, \qquad \text{for all } t \in J \text{ and } s \in (0,1),$$

$$c_3 \leq \frac{-G_{\gamma,\delta_1,\delta_2}(t,s)}{-G_{\gamma,\delta_1,\delta_2}(1,s)} \leq c_4, \quad \text{for } \gamma > 0 \qquad \text{and all } t \in J \text{ and } s \in (0,1),$$

and

$$c_5 \leq \frac{-G_{\gamma,\delta_1,\delta_2}(t,s)}{-G_{\gamma,\delta_1,\delta_2}(1,s)} \leq c_6, \quad \text{for } \gamma < 0 \qquad \text{and all } t \in J \text{ and } s \in (0,1).$$

Thus, we may define the constants $\bar{c}$, h, k and $R\,(<0)$ in the same way than before and inequalities (4.3.20) and (4.3.21) hold by simply changing $G_{\gamma,\delta_1,\delta_2}$ by $-G_{\gamma,\delta_1,\delta_2}$.

Analogously to Theorem 4.3.1, the following result can be proved in the case that the values of γ, δ_1, and δ_2 satisfy condition (4.3.25) (that is, $-G_{\gamma,\delta_1,\delta_2}$ is positive).

Theorem 4.3.3. *Assume that the negativeness condition* (4.3.25) *of the Green's function* $G_{\gamma,\delta_1,\delta_2}$ *is fulfilled . Moreover, suppose that the following conditions hold:*

1. *There exists* $p' > 0$ *such that*

$$f(t,u) \leq \frac{p'}{-k\,R}, \quad \textit{for all } t \in J \textit{ and } u \in [0,p'].$$

2. *There exists* $q' > 0$*, with* $q' \neq p'$ *such that*

$$f(t,u) \geq \frac{q'}{-h\,R}, \quad \textit{for all } t \in J \textit{ and } u \in [\bar{c}\,q', q'].$$

Then problem (4.1.2)*–(4.1.3) has at least one positive solution* $u \in K$*, such that* $||u||_\infty$ *lies between* p' *and* q'*.*

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Chapter 5
Hill's Equation Subject to Different Two-Point Linear Boundary Conditions

In this chapter we will deduce several properties of the Green's functions related to the Hill's equation

$$u''(t) + a(t)\,u(t) = 0$$

coupled to various boundary value conditions. In particular, the idea is to study the Green's functions of the second order differential operator coupled to Neumann, Dirichlet, periodic, mixed, initial and final boundary conditions, by expressing the Green's function of a given problem as a linear combination of the Green's function of the other ones.

A preliminary work in this line can be found in [9], where the authors develop a method which allows to write the Green's functions related to Neumann, Dirichlet and Mixed problems defined on the interval $[0, T]$ as a linear combination of Green's functions of some extended periodic problem (that is, the periodic problem was considered either on the interval $[0, 2T]$ or on $[0, 4T]$ and the potentials for these problems were the even extension $\widetilde{a}$ to $[0, 2T]$ of the potential $a(t)$ considered on $[0, T]$ and the even extension of $\widetilde{a}$ to $[0, 4T]$, respectively). As a consequence of such decomposition, the authors were able to deduce some comparison results between the solutions of the aforementioned problems. Moreover, they were able to relate the constant sign of the corresponding Green's functions.

This chapter can be regarded then as a continuation of the work developed in [9] as our main objective will also be the decomposition of some Green's functions in terms of other ones. However, the techniques used in this paper are completely different from those mentioned for [9]. More concretely, we will consider two different ways of making the decomposition of Green's functions.

The first one will be based on the superposition property of the solutions of a differential problem. On the other hand, the second one will make use of formula (2.2.7), which allows to relate two different Green's functions as long as the boundary value conditions of one of them can be rewritten in terms of the other one and both problems are nonresonant.

This way, we will consider periodic, Neumann, Dirichlet, mixed, initial and final conditions and relate their corresponding Green's functions two by two. One of the differences between these approaches and the one considered in [9] is the fact that here we are able to find a relation between any pair of the aforementioned Green's functions, not only between any of them and the periodic one. Another difference is that in the present chapter we are able to connect the Green's function related to the periodic problem on $[0, T]$ with the Green's

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function related to any of the other cited boundary condition on $[0, T]$, which was not possible with the techniques used in [9].

As a consequence of the expressions relating the Green's functions, we are able to find some connections between their constant sign. Some of the results were already proved in [9] (although, the proof was different) and some others are, as far as we know, new in the literature.

The chapter is organized as follows. In Section 5.1 we compile some preliminary results. Sections 5.2 and 5.3 include the decomposition of Green's functions using the two different approaches mentioned before. Finally, Section 5.4 includes an application to ensure the existence and find some bounds for the solutions of nonlinear problems. Also, an example of the application of the results is given.

The results of this chapter are compiled in the paper [20], except for Subsections 5.3.7 and 5.3.8 which can be found in [22].

5.1 Preliminaries

Consider the second order linear operator

$$L\,u(t) \equiv u''(t) + a(t)\,u(t) = 0, \quad \text{a.e. } t \in J,$$

with $a : J \to \mathbb{R}$, $a \in L^1(J)$.

Such equation models the linear approach of the pendulum movement, the classical spring mass system equation or, among many others, the Airy's and the Mathieus's equation, see references [7, 37, 70, 82, 85, 96] for more information on these applications.

Furthermore, it is important to note that the results obtained for Hill's equation can be easily extended (with a suitable change of variable, see [82]) to a general second order linear differential equation of the form

$$u''(t) + a_1(t)\,u'(t) + a_0(t)\,u(t) = 0, \quad \text{a.e. } t \in J,$$

provided that the functions a_0 and a_1 have enough regularity.

Moreover, the nonhomogeneous problem related to Hill's equation

$$u''(t) + a(t)\,u(t) = \sigma(t), \quad \text{a.e. } t \in J,$$

has also been extensively studied (see [1, 10, 31, 37, 46, 79, 85, 87, 89, 93, 94, 96] and the references therein), especially coupled to periodic conditions. In this sense, a particularly interesting case happens when σ has constant sign, which can be interpreted as the action of an external force acting over the system on a certain direction (positive or negative). In such a case, the solutions of constant sign of the equation can be interpreted as situations in which the deviation caused by the force is produced only in one direction (that is, the object oscillates only above or below the equilibrium point of the system).

It is in this context when the study of Green's functions gains importance, since the existence of solutions of differential equations with constant sign is directly related to the constant sign of the Green's functions. In particular, the fact that the Green's function related to a differential problem does not change its sign allows the application of several topological and iterative methods to deduce existence results for suitable nonlinear problems.

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We will work with the space

$$W^{2,1}(J) = \{u \in C(J) : u' \in AC(J)\}.$$

For any $\lambda \in \mathbb{R}$, consider operator $L\,[\lambda]$ defined from operator L as follows:

$$L[\lambda]\,u(t) \equiv u''(t) + (a(t) + \lambda)\,u(t), \quad t \in J. \tag{5.1.1}$$

In particular, note that $L \equiv L[0]$. When working with this operator, to stress the dependence of the Green's function on the parameter λ, we will denote by $G[\lambda]$ the Green's function related to $L[\lambda]$.

We will deal with some problems related to operator $L[\lambda]$, which we describe in the sequel:

- Neumann problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in J, \ \ u \in X_N = \{u \in W^{2,1}(I) : u'(0) = u'(1) = 0\}. \tag{5.1.2}$$

- Dirichlet problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in J, \ \ u \in X_D = \{u \in W^{2,1}(I) : u(0) = u(1) = 0\}. \tag{5.1.3}$$

- Mixed 1 problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in J, \ \ u \in X_{M_1} = \{u \in W^{2,1}(I) : u'(0) = u(1) = 0\}. \tag{5.1.4}$$

- Mixed 2 problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in J, \ u \in X_{M_2} = \{u \in W^{2,1}(I) : u(0) = u'(1) = 0\}. \tag{5.1.5}$$

- Periodic problem:

$$L[\lambda]\,u(t) \ = \sigma(t), \ \text{ a.e. } t \in J, \ \ u \in X_P = \{u \in W^{2,1}(I) : u(0) = u(1), \ u'(0) = u'(1)\}. \tag{5.1.6}$$

- Initial problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in [0,T], \ \ u \in X_I = \{u \in W^{2,1}(I) : u(0) = u'(0) = 0\}. \tag{5.1.7}$$

- Final problem:

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in [0,T], \ \ u \in X_F = \{u \in W^{2,1}(I) : u(T) = u'(T) = 0\}. \tag{5.1.8}$$

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We denote by $G_N[\lambda]$, $G_D[\lambda]$, $G_{M_1}[\lambda]$, $G_{M_2}[\lambda]$, $G_P[\lambda]$, $G_I[\lambda]$ and $G_F[\lambda]$ the Green's function related to Neumann, Dirichlet, mixed 1 and mixed 2, periodic, initial and final problems, respectively. Moreover, we denote by u_N, u_D, u_{M_1}, u_{M_2}, u_P, u_I and u_F the solutions of the corresponding problems, and by λ_0^N, λ_0^D, $\lambda_0^{M_1}$, $\lambda_0^{M_2}$ and λ_0^P the first eigenvalues of each problem (notice that initial and final problems has not eigenvalues).

We recall that the equation

$$L[\lambda]\,u(t) = \sigma(t), \quad \text{a.e. } t \in J \tag{5.1.9}$$

can be rewritten as a system of type (1.1.5) as follows

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a(t) - \lambda & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma(t) \end{pmatrix}. \tag{5.1.10}$$

In this case, we have that

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) - \lambda & 0 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} 0 \\ \sigma(t) \end{pmatrix}.$$

Now, we will give the expression of the different problems related to operator $L[\lambda]$ mentioned above, by giving the corresponding matrices E and F (see Remark 1.1.10) of each case:

- Neumann problem:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Dirichlet problem:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- Mixed 1 problem:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- Mixed 2 problem:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Periodic problem:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Initial problem:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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- Final problem:

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 5.1.1. *Remember that, as we said in Remark 1.1.12, the matrices E and F are not unique.*

Using [24, page 11] we know that the matrix function

$$g[\lambda](t,s) = \begin{pmatrix} -\frac{\partial}{\partial s}G[\lambda](t,s) & G[\lambda](t,s) \\ -\frac{\partial^2}{\partial s \partial t}G[\lambda](t,s) & \frac{\partial}{\partial t}G[\lambda](t,s) \end{pmatrix}$$

is the Green's function related to system (5.1.10) associated with the differential equation (5.1.9), coupled to the boundary conditions

$$E\,x(0) + F\,x(1) = 0 \tag{5.1.11}$$

where $G[\lambda]$ is the Green's function of the linear equation (5.1.9) coupled to boundary conditions (5.1.11) under the notation $x = \begin{pmatrix} u \\ u' \end{pmatrix}$.

Now we introduce some auxiliary functions that we are going to use throughout this chapter to relate the different problems that we have defined above.

Let us define $r_1[\lambda]$ as the unique solution to the problem

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u(0) = 1, \;\; u(1) = 0, \tag{5.1.12}$$

$r_2[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u(0) = 0, \;\; u(1) = 1, \tag{5.1.13}$$

$r_3[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u(0) - u(1) = 1, \;\; u'(0) - u'(1) = 0,$$

$r_4[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u(0) - u(1) = 0, \;\; u'(0) - u'(1) = 1, \tag{5.1.14}$$

$r_5[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u'(0) = 1, \;\; u'(1) = 0,$$

$r_6[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u'(0) = 0, \;\; u'(1) = 1,$$

$r_7[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \;\; \text{a.e. } t \in J, \;\; u(0) = 1, \;\; u'(1) = 0,$$

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$r_8[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u(0) = 0, \quad u'(1) = 1,$$

$r_9[\lambda]$ as the unique solution to

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u'(0) = 1, \quad u(1) = 0,$$

$r_{10}[\lambda]$ as the unique solution

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u'(0) = 0, \quad u(1) = 1,$$

$r_{11}[\lambda]$ as the unique solution

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u(0) = 1, \quad u'(0) = 0,$$

$r_{12}[\lambda]$ as the unique solution

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u(0) = 0, \quad u'(0) = 1,$$

$r_{13}[\lambda]$ as the unique solution

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u(1) = 1, \quad u'(1) = 0,$$

and $r_{14}[\lambda]$ as the unique solution of the problem

$$L[\lambda]\,u(t) = 0, \quad \text{a.e. } t \in J, \quad u(1) = 0, \quad u'(1) = 1.$$

Note that the uniqueness of the functions $r_k[\lambda]$, $k \in \{1, \ldots, 14\}$, is a direct consequence of Lemma 1.1.8.

Now, we will find the expression of $r_1[\lambda]$ as a function of the Green's function of the Dirichlet problem using the equalities (1.1.10) and (1.1.11) given in the preliminary chapter.

For the Dirichlet problem, equation (1.1.10) becomes the following equality

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial}{\partial s} G_D[\lambda](0,0) & G_D[\lambda](0,0) \\ -\frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0) & \frac{\partial}{\partial t} G_D[\lambda](0,0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial}{\partial s} G_D[\lambda](1,0) & G_D[\lambda](1,0) \\ -\frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,0) & \frac{\partial}{\partial t} G_D[\lambda](1,0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$\begin{array}{cc} -\frac{\partial}{\partial s} G_D[\lambda](0,0) = 1, & \frac{\partial}{\partial s} G_D[\lambda](1,0) = 0, \\ G_D[\lambda](0,0) = 0, & G_D[\lambda](1,0) = 0. \end{array}$$

By the uniqueness of the function $r_1[\lambda]$, from the Definition 1.1.1 of the Green's function it follows that

$$r_1[\lambda](t) = -\frac{\partial}{\partial s} G_D[\lambda](t,0).$$

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Making similar arguments, we can deduce that

$$\begin{array}{lll} r_2[\lambda](t) = \frac{\partial}{\partial s} G_D[\lambda](t,1), & r_3[\lambda](t) = -\frac{\partial}{\partial s} G_P[\lambda](t,0), & r_4[\lambda](t) = G_P[\lambda](t,0), \\ r_5[\lambda](t) = G_N[\lambda](t,0), & r_6[\lambda](t) = -G_N[\lambda](t,1), & r_7[\lambda](t) = -\frac{\partial}{\partial s} G_{M_2}[\lambda](t,0), \\ r_8[\lambda](t) = -G_{M_2}[\lambda](t,1), & r_9[\lambda](t) = G_{M_1}[\lambda](t,0), & r_{10}[\lambda](t) = -\frac{\partial}{\partial s} G_{M_1}[\lambda](t,1) \\ r_{11}[\lambda](t) = -\frac{\partial}{\partial s} G_I[\lambda](t,0), & r_{12}[\lambda](t) = G_I[\lambda](t,0), & r_{13}[\lambda](t) = \frac{\partial}{\partial s} G_F[\lambda](t,1) \\ r_{14}[\lambda](t) = -G_F[\lambda](t,1). & & \end{array}$$

Next, we study the oscillation of the functions $r_1[\lambda]$ and $r_2[\lambda]$ using the Sturm-Liouville theory of eigenvalues.

In particular we will use Sturm's comparison Theorem, which compares the oscillation of solutions relative to two given potentials, one greater than the other almost everywhere on J. The proof of this result can be seen in [9, 92].

Theorem 5.1.2. *Let u and v be nontrivial solutions of*

$$u''(t) + q(t)\, u(t) = 0, \quad \textit{a.e. } t \in J$$

and

$$v''(t) + a(t)\, v(t) = 0, \quad \textit{a.e. } t \in J$$

respectively, with $q, r \in L^1(J)$ such that $q > r$ a.e. on J. Then u vanishes at least once between two consecutive zeros of v.

Let $\{\lambda_n^D\}_{n=0}^{\infty}$ be the sequence of eigenvalues of the Dirichlet problem:

$$(D_\lambda) \quad L[\lambda]\, u(t) = 0, \quad \text{a.e. } t \in J, \quad u(0) = u(1) = 0.$$

It is very well-known that, $\lim_{n\to\infty} \lambda_n^D = \infty$ (see [92, Theorem 4.3.1]), and that any of the eigenvalues has a single associated eigenvector v_n, such that

$$(D_n) \quad L[\lambda_n^D]\, v_n(t) = 0, \quad \text{a.e. } t \in J, \quad v_n(0) = v_n(1) = 0,$$

with exactly n zeros in $(0,1)$. Moreover, this eigenfunction satisfies that $v_n'(0) \neq 0$, since otherwise $v_n(t) = 0$ for all $t \in J$.

Lemma 5.1.3. *The set of solutions of the problem*

$$\begin{cases} L[\lambda_n^D]\, w(t) = 0, & t \in J, \\ w(0) = 0. \end{cases} \tag{5.1.15}$$

is given by the set $\{c\, v_n(t) : \ t \in J,\, c \in \mathbb{R}\}$.

Proof. Let $w \neq 0$ be a solution of problem (5.1.15). It occurs that, $w'(0) \neq 0$ (if not $w = 0$ on J), and thus there exists $c \in \mathbb{R}$, $c \neq 0$ such that $w'(0) = c\, v_n'(0)$. Moreover, the function $x(t) = w(t) - c\, v_n(t)$, $t \in J$, fulfills that

$$\begin{cases} L[\lambda_n^D]\, x(t) = 0, & t \in J, \\ x(0) = 0, \\ x'(0) = 0. \end{cases}$$

Then, $x(t) = 0$ for all $t \in J$, that is, $w(t) = c\, v_n(t)$, for all $t \in J$. □

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Lemma 5.1.4. *Problem* (5.1.13) *has a unique solution if and only if* $\lambda \neq \lambda_n^D$, $n = 0, 1, \ldots$.

Proof. If $\lambda = \lambda_n^D$ then, from Lemma 5.1.3, any solution $r_2[\lambda]$ of (5.1.13) will be a multiple of the eigenfunction v_n, which implies that $r_2[\lambda](1) = 0$ and this is a contradiction. Thus, problem (5.1.13) has no solutions for $\lambda = \lambda_n^D$.

If $\lambda \neq \lambda_n^D$ for all $n = 0, 1, \ldots$, then for any $c \neq 0$, let $u_c[\lambda]$ be the unique solution to the initial value problem

$$\begin{cases} L[\lambda]\, u(t) = 0, \quad t \in J, \\ u(0) = 0, \quad u'(0) = c, \end{cases}$$

which satisfies that $u_c[\lambda](1) \neq 0$ (since, otherwise, λ would be an eigenvalue of Dirichlet problem).

Fixed $c \in \mathbb{R} \setminus \{0\}$. We have just shown that $u_c[\lambda](1) \neq 0$, then, the function

$$r_2[\lambda](t) := \frac{u_c[\lambda](t)}{u_c[\lambda](1)}$$

fulfills that

$$\begin{cases} L[\lambda]\, r_2[\lambda](t) = 0, \quad t \in J, \\ r_2[\lambda](0) = 0, \\ r_2[\lambda](1) = 1. \end{cases}$$

Thus, we have proved that problem (5.1.13) has a solution for any $\lambda \neq \lambda_n^D$.

If there are two different solutions, u_1 and u_2, of problem (5.1.13) we have that $w = u_1 - u_2$ will be a nontrivial solution of problem

$$\begin{cases} L[\lambda]\, w(t) = 0, \quad t \in J, \\ w(0) = w(1), \end{cases}$$

or, which is the same, λ is an eigenvalue of the Dirichlet problem, a contradiction. Thus, $r_2[\lambda] = u_1 = u_2$ on J, and this completes the proof. □

Lemma 5.1.5. *The unique solution* $r_2[\lambda]$ *of problem* (5.1.13) *has exactly* n *zeros in* $(0, 1)$ *if and only if* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \ldots$ *and* $r_2[\lambda] > 0$ *on* $(0, 1]$ *if and only if* $\lambda < \lambda_0^D$. *Moreover,* $(-1)^n r_2'[\lambda](0) > 0$ *for all* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \ldots$ *and* $r_2'[\lambda](0) > 0$, *for all* $\lambda < \lambda_0^D$.

Proof. If $\lambda < \lambda_0^D$ we have that there is no $t_0 \in (0, 1)$ such that $r_2[\lambda](t_0) = 0$, since in such a case $r_2[\lambda](0) = r_2[\lambda](t_0) = 0$, with $t_0 - 0 < 1$, which contradicts the definition of λ_0^D and Sturm's comparison Theorem 5.1.2: the distance between two consecutive zeros of any solution of $L[\lambda]\, u(t) = 0$ is greater than 1. Moreover, since $r_2[\lambda](1) = 1 > 0$ we have that $r_2[\lambda](t) > 0$ for all $t \in (0, 1]$ and $r_2'[\lambda](0) > 0$.

Let us see now that if $\lambda_0^D < \lambda < \lambda_1^D$, then the function $r_2[\lambda]$ has a unique zero in $(0, 1)$.

The function $r_2[\lambda]$ cannot have constant sign as a consequence of Sturm's comparison Theorem. Indeed, since for $\lambda = \lambda_0^D$ the distance between two consecutive zeros of the solution is equal to 1, then for $\lambda > \lambda_0^D$ it will always be less than 1. As $r_2[\lambda](0) = 0$ we have that $r_2[\lambda](t_0) = 0$ for some $t_0 \in (0, 1)$.

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Similarly, the function $r_2[\lambda]$ cannot have 2 zeros in $(0,1)$ because $\lambda < \lambda_1^D$ and for $\lambda = \lambda_1^D$ we have that the solutions have exactly one zero in $(0,1)$ and vanish at 0 and 1. Thus, for $\lambda_0^D < \lambda < \lambda_1^D$ we have that the first zero of $r_2[\lambda]$ will be $t_1 \in (t_0, 1)$.

Moreover, if there exists $t_2 \in (t_1, 1)$ such that $r_2[\lambda](t_2) = 0$, then by Sturm's comparison Theorem the eigenfunction of λ_1^D would have a zero in (t_1, t_2), which is false. Therefore, $r_2[\lambda]$ has exactly one zero in $(0,1)$ for $\lambda \in (\lambda_0^D, \lambda_1^D)$.

In particular $r_2'[\lambda](0) < 0$, since otherwise $r_2[\lambda]$ must have 2 zeros in $(0,1)$, which is not true.

Following the same argument, it can be proved that, in general, $r_2[\lambda]$ has exactly n zeros in $(0,1)$ if and only if $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \ldots$ and $(-1)^n\, r_2'[\lambda](0) > 0$. □

Doing a similar study to the previous one, we have the following results for $r_1[\lambda]$.

Lemma 5.1.6. *Problem* (5.1.12) *has a unique solution if and only if* $\lambda \neq \lambda_n^D$, $n = 0, 1, \ldots$.

Lemma 5.1.7. *The unique solution* $r_1[\lambda]$ *of problem* (5.1.12) *has exactly* n *zeros in* $(0,1)$ *if and only if* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \ldots$, *and* $r_1[\lambda] > 0$ *on* $[0,1)$ *if and only if* $\lambda < \lambda_0^D$. *Moreover,* $(-1)^n\, r_1'(1) < 0$ *for all* $\lambda \in (\lambda_{n-1}^D, \lambda_n^D)$, $n = 1, 2, \ldots$, *and* $r_1'[\lambda](1) < 0$, *for all* $\lambda < \lambda_0^D$.

5.2 Decomposing Green's Functions

In this section we deal with the first approach to obtain the relationships between the expressions of the Green's functions related to problems (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6). It should be noted that this approach is different from the one used in [9], where expressions are obtained for the corresponding extensions of the potential $a(t)$ to the intervals $[0,2]$ and $[0,4]$. In addition, the relationships obtained allow us to compare some of Green's functions.

Remark 5.2.1. *We emphasize that we can obtain our conclusions given above about the relationship between Green's functions at any arbitrary real interval of the form* $[a,b]$.

5.2.1 Dirichlet and Periodic Problems

In this subsection, we give an expression that allows us to obtain the expression of the Green's function related to the periodic problem as a function of the Green's function of the Dirichlet problem.

Theorem 5.2.2. *If operator* $L[\lambda]$ *is nonresonant both in* X_D *and* X_P, *then it holds that*

$$\begin{aligned} G_P[\lambda](t,s) &= G_D[\lambda](t,s) + (r_1[\lambda](t) + r_2[\lambda](t))\; G_P[\lambda](1,s) \\ &= G_D[\lambda](t,s) + \left(\frac{\partial}{\partial s} G_D[\lambda](t,1) - \frac{\partial}{\partial s} G_D[\lambda](t,0)\right) G_P[\lambda](1,s),\ \forall\,(t,s) \in J \times J. \end{aligned} \tag{5.2.16}$$

Proof. We express the periodic problem (5.1.6) in function of the Dirichlet one (5.1.3) as follows

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in J, \quad u(0) = u(1), \quad u(1) = u(1) + u'(0) - u'(1), \tag{5.2.17}$$

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with $\sigma \in L^1(J)$.

Then, using Lemma 1.1.8, we have that the solution of problem (5.2.17) is given by the following expression

$$\begin{aligned} u_P(t) &= \int_0^1 G_P[\lambda](t,s)\,\sigma(s)\,ds \\ &= \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)u_P(1) + r_2[\lambda](t)\,(u_P(1) + u_P'(0) - u_P'(1)) \\ &= \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)\int_0^1 G_P[\lambda](1,s)\,\sigma(s)\,ds \\ &\quad + r_2[\lambda](t)\int_0^1 \left[G_P[\lambda](1,s) + \frac{\partial}{\partial t}G_P[\lambda](0,s) - \frac{\partial}{\partial t}G_P[\lambda](1,s)\right]\,\sigma(s)\,ds \\ &= \int_0^1 \left[G_D[\lambda](t,s) + (r_1[\lambda](t) + r_2[\lambda](t))\,G_P[\lambda](1,s)\right]\,\sigma(s)\,ds, \end{aligned}$$

where the last equality follows from Definition 1.1.1, condition $(G6)$:

$$\frac{\partial}{\partial t}G_P[\lambda](0,s) = \frac{\partial}{\partial t}G_P[\lambda](1,s), \quad \forall s \in (0,1).$$

Since previous equalities hold for every $\sigma \in L^1(J)$, we obtain (5.2.16). □

As a direct consequence of equality (5.2.16) and the oscilation properties proved in Lemmas 5.1.5 and 5.1.7, we deduce the following comparison between the values of the Green's functions related to the Dirichlet and periodic problems.

Corollary 5.2.3. *The following inequality holds*

$$G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall (t,s) \in (0,1)\times(0,1), \quad \forall\, \lambda < \lambda_0^P. \tag{5.2.18}$$

Proof. It is immediate to verify that the function

$$r[\lambda](t) := r_1[\lambda](t) + r_2[\lambda](t) \tag{5.2.19}$$

solves the following problem

$$L[\lambda]\,r[\lambda](t) = 0, \quad \text{a.e. } t \in J, \quad r[\lambda](0) = r[\lambda](1) = 1.$$

From Lemmas 5.1.5 and 5.1.7, it is obvious that if $\lambda < \lambda_0^D$ then $r[\lambda](t) > 0$ for all $t \in J$.

Moreover, we know that $G_P[\lambda]$ is negative on $J \times J$ for all $\lambda < \lambda_0^P$ and $G_D[\lambda]$ is negative on $(0,1)\times(0,1)$ for all $\lambda < \lambda_0^D$ (see [9, Lemma 2.9]). In addition, $\lambda_0^P < \lambda_0^D$ (see [10, page 44]).

As $r[\lambda](t) > 0$ for all $t \in J$ when $\lambda < \lambda_0^D$, using (5.2.16), we obtain the result. □

Remark 5.2.4. *Let $K_\lambda = r'[\lambda](0) - r'[\lambda](1)$, with $r[\lambda]$ defined in* (5.2.19)*. On the one hand, if $K_\lambda = 0$, then λ is an eigenvalue of the periodic problem. In this case, $r'[\lambda](0) = r'[\lambda](1)$ or, equivalently,*

$$r_1'[\lambda](0) - r_1'[\lambda](1) = r_2'[\lambda](1) - r_2'[\lambda](0). \tag{5.2.20}$$

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Since the Wronskian of $r_1[\lambda]$ *and* $r_2[\lambda]$, $W[\lambda](t) = r_1[\lambda](t)\, r_2'[\lambda](t) - r_1'[\lambda](t)\, r_2[\lambda](t)$, *is constant on* J,

$$\begin{aligned} W'[\lambda](t) &= r_1'[\lambda](t)\, r_2'[\lambda](t) + r_1[\lambda](t)\, r_2''[\lambda](t) - r_1''[\lambda](t)\, r_2[\lambda](t) - r_1'[\lambda](t)\, r_2'[\lambda](t) \\ &= -(a(t)+\lambda)\, r_1[\lambda](t)\, r_2[\lambda](t) + (a(t)+\lambda)\, r_1[\lambda](t)\, r_2[\lambda](t) = 0, \end{aligned}$$

we have that $W[\lambda](0) = W[\lambda](1)$ *and, consequently by* (5.1.12) *and* (5.1.13),

$$r_2'[\lambda](0) = -r_1'[\lambda](1).$$

Therefore, using (5.2.20) *we obtain that the eigenvalues of the periodic problem are given by*

$$r_1'[\lambda](0) - r_1'[\lambda](1) = r_2'[\lambda](1) + r_1'[\lambda](1),$$

or, which is the same,

$$2\, r_1'[\lambda](1) = r_1'[\lambda](0) - r_2'[\lambda](1).$$

On the other hand, if $K_\lambda \neq 0$, *then* $\frac{r[\lambda](t)}{K_\lambda}$ *is the unique solution of*

$$L[\lambda]\, u(t) = 0, \quad t \in J, \quad u(0) = u(1), \quad u'(0) = u'(1) + 1,$$

from which we deduce (see [10, Lemma 21]) that

$$\frac{r[\lambda](t)}{K_\lambda} = G_P[\lambda](t,0) < 0, \quad \forall \lambda < \lambda_0^P.$$

Substituting in equation (5.2.16) *with* $s = 0$ *and taking into account that* $G_D[\lambda](t,0) = 0$ *we infer the following equality*

$$G_P[\lambda](t,0) = r[\lambda](t)\, G_P[\lambda](1,0). \tag{5.2.21}$$

Therefore,

$$\frac{1}{K_\lambda} = G_P[\lambda](1,0).$$

Note that $K_\lambda < 0$ *for all* $\lambda < \lambda_0^P$ *because* $r[\lambda] > 0$ *for all* $\lambda < \lambda_0^D$ *and* $\lambda_0^P < \lambda_0^D$.

Remark 5.2.5. *From equality* (5.2.16) *it follows that*

$$\frac{G_D[\lambda](t,s) - G_P[\lambda](t,s)}{G_P[\lambda](1,s)} = -r[\lambda](t),$$

if λ *is not a eigenvalue of the Dirichlet and periodic problems.*

Deriving the above equality with respect to s, *we obtain the following identity*

$$\frac{\partial}{\partial s}\left(G_D[\lambda](t,s) - G_P[\lambda](t,s)\right) G_P[\lambda](1,s) = \left(G_D[\lambda](t,s) - G_P[\lambda](t,s)\right) \frac{\partial}{\partial s} G_P[\lambda](1,s),$$

for all $(t,s) \in J \times J$.

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Now we will do a similar study on the other way, that is, we will express the Dirichlet conditions as a function of the periodic ones as follows

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in J,\ \ u(0)-u(1)=-u(1),\ \ u'(0)-u'(1)=u'(0)-u'(1)+u(1).$$

Taking into account that $r_4[\lambda](t)=G_P[\lambda](t,0)$ solves (5.1.14), performing the calculations in an analogous way than before, using Lemma 1.1.8 the following result is attained.

Theorem 5.2.6. *Assume that operator $L[\lambda]$ is nonresonant both in X_D and X_P, then it holds that*

$$\begin{aligned}G_D[\lambda](t,s)=&G_P[\lambda](t,s)+r_4[\lambda](t)\left(\frac{\partial}{\partial t}G_D[\lambda](0,s)-\frac{\partial}{\partial t}G_D[\lambda](1,s)\right)\\ =&G_P[\lambda](t,s)+G_P[\lambda](t,0)\left(\frac{\partial}{\partial t}G_D[\lambda](0,s)-\frac{\partial}{\partial t}G_D[\lambda](1,s)\right),\ \ \forall(t,s)\in J\times J.\end{aligned}\tag{5.2.22}$$

Remark 5.2.7. *We do another alternative proof of* (5.2.18) *using the previous result. Indeed, if $\lambda<\lambda_0^D$ we have that $G_D[\lambda]<0$ on $(0,1)\times(0,1)$ and, as a consequence,*

$$\frac{\partial}{\partial t}G_D[\lambda](0,s)<0<\frac{\partial}{\partial t}G_D[\lambda](1,s),\quad s\in(0,1).$$

Moreover, if $\lambda<\lambda_0^P$ then $G_P[\lambda]<0$ on $J\times J$. Thus, from (5.2.22) *and the fact that $\lambda_0^P<\lambda_0^D$ we deduce the inequality* (5.2.18) *again.*

5.2.2 Dirichlet and Neumann Problems

In this section we will consider the Dirichlet and Neumann problems. We obtain some expressions that allow us to connect both Green's functions.

Theorem 5.2.8. *Assume that operator $L[\lambda]$ is nonresonant in the spaces X_D and X_N. Then the following equality is satisfied*

$$\begin{aligned}G_N[\lambda](t,s)&=G_D[\lambda](t,s)+r_1[\lambda](t)\,G_N[\lambda](0,s)+r_2[\lambda](t)\,G_N[\lambda](1,s)\\ &=G_D[\lambda](t,s)-\frac{\partial}{\partial s}G_D[\lambda](t,0)\,G_N[\lambda](0,s)+\frac{\partial}{\partial s}G_D[\lambda](t,1)\,G_N[\lambda](1,s),\end{aligned}\tag{5.2.23}$$

for all $(t,s)\in J\times J$.

Proof. Let us rewrite Neumann problem in the following way

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in J,\ \ u(0)=u(0)+u'(0),\ \ u(1)=u(1)+u'(1).$$

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Using Lemma 1.1.8, the solution to the above problem is

$$\begin{aligned} u_N(t) &= \int_0^1 G_N[\lambda](t,s)\,\sigma(s)\,ds \\ &= \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)\,u_N(0) + r_2[\lambda](t)\,u_N(1) \\ &= \int_0^1 G_D[\lambda](t,s)\,\sigma(s)\,ds + r_1[\lambda](t)\int_0^1 G_N[\lambda](0,s)\,\sigma(s)\,ds \\ &\quad + r_2[\lambda](t)\int_0^1 G_N[\lambda](0,s)\,\sigma(s)\,ds. \end{aligned}$$

Therefore, since previous equalities hold for every $\sigma \in L^1(J)$, we obtain (5.2.23). □

As a consequence of the previous result we arrive at the following one.

Corollary 5.2.9. *For all $\lambda < \lambda_0^N$, the following inequality holds*

$$G_N[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\,(t,s) \in (0,1)\times(0,1). \tag{5.2.24}$$

Proof. From Lemmas 5.1.5 and 5.1.7, we know that $r_1[\lambda]$ and $r_2[\lambda]$ are positive on $(0,1)$ for all $\lambda < \lambda_0^D$. In addition, $\lambda_0^N \leq \lambda_0^P < \lambda_0^D$ (see [10, page 44]), $G_N[\lambda] < 0$ on $J \times J$ for all $\lambda < \lambda_0^N$ (see [9, Corollary 4.5]) and $G_D[\lambda] < 0$ on $(0,1)\times(0,1)$ for all $\lambda < \lambda_0^D$ (see [9, Lemma 2.9]). Then, for all $\lambda < \lambda_0^N$, $r_1[\lambda]$ and $r_2[\lambda]$ are positive on $(0,1)$. Hence, using (5.2.23) we obtain the result. □

Remark 5.2.10. *The above result can be deduced from [9, Corollaries 4.5, 4.8 and 4.10]. As we mentioned at the beginning of the chapter, the argument used in this reference is based on considering the even extension of the solution to the interval $[0,2]$, which is a different approach than the one we have used here. Also, the expression* (5.2.23) *relating $G_N[\lambda]$ and $G_D[\lambda]$ is different from the one obtained in such reference.*

For the reverse process, writing the Dirichlet problem as a function of Neumann problem as

$$L[\lambda]\,u(t) = \sigma(t), \quad \text{a.e. } t \in J, \quad u'(0) = u(0) + u'(0), \quad u'(1) = u(1) + u'(1),$$

we reach to the next result as a consequence of Lemma 1.1.8.

Theorem 5.2.11. *Assume that operator $L[\lambda]$ is nonresonant in the spaces X_D and X_N. Then, the following equalities are satisfied*

$$\begin{aligned} G_D[\lambda](t,s) &= G_N[\lambda](t,s) + r_5[\lambda](t)\,\frac{\partial}{\partial t}G_D[\lambda](0,s) + r_6[\lambda](t)\,\frac{\partial}{\partial t}G_D[\lambda](1,s) \\ &= G_N[\lambda](t,s) + G_N[\lambda](t,0)\,\frac{\partial}{\partial t}G_D[\lambda](0,s) - G_N[\lambda](t,1)\,\frac{\partial}{\partial t}G_D[\lambda](1,s), \end{aligned} \tag{5.2.25}$$

for all $(t,s) \in J \times J$.

Remark 5.2.12. *Notice that, the equality* (5.2.25) *implies* (5.2.24) *by taking into account that for $\lambda < \lambda_0^N$ we have that $G_N[\lambda] < 0$ on $J \times J$ and $G_D[\lambda] < 0$ on $(0,1)\times(0,1)$ (which implies $\frac{\partial}{\partial t}G_D(0,s) < 0 < \frac{\partial}{\partial t}G_D(1,s)$).*

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5.2.3 Dirichlet and Mixed Problems

In this case we carry out an analysis of the relationship between the Green's functions of Dirichlet and mixed 1 problems, assuming that $L[\lambda]$ is nonresonant in X_D and X_{M_1}. Let us rewrite mixed 1 problem in the following way

$$L[\lambda]\,u(t) = \sigma(t), \quad \text{a.e. } t \in J, \quad u(0) = u(0) + u'(0), \quad u(1) = 0,$$

and following the same steps than before in the previous subsection, the next result is attained.

Theorem 5.2.13. *Assume that $L[\lambda]$ is nonresonant both in X_D and X_{M_1}, then it holds that*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_D[\lambda](t,s) + r_1[\lambda](t)\, G_{M_1}[\lambda](0,s) \\ &= G_D[\lambda](t,s) - \frac{\partial}{\partial s} G_D[\lambda](t,0)\, G_{M_1}[\lambda](0,s), \quad \forall (t,s) \in J \times J. \end{aligned} \tag{5.2.26}$$

Using the equality (5.2.26), we deduce the following corollary.

Corollary 5.2.14. *For all $\lambda < \lambda_0^{M_1}$, the following inequality holds*

$$G_{M_1}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall\, (t,s) \in (0,1) \times (0,1). \tag{5.2.27}$$

Proof. We know from [9, Remark 4.19] that $\lambda_0^{M_1} < \lambda_0^D$. Moreover, we have that $G_{M_1}[\lambda] < 0$ on $[0,1) \times [0,1)$ if and only if $\lambda < \lambda_0^{M_1}$ (see [9, Corollary 4.7]) and $G_D[\lambda] < 0$ on $(0,1) \times (0,1)$ if and only if $\lambda < \lambda_0^D$, which implies that $\frac{\partial}{\partial s} G_D(t,0) < 0$ for all $\lambda < \lambda_0^D$ and $t \in (0,1)$.

Therefore, from (5.2.26) we obtain inequality (5.2.27). □

Similarly, for mixed 2 problem we arrive at the following results.

Theorem 5.2.15. *If operator $L[\lambda]$ is nonresonant in X_D and X_{M_2}, then the following equality holds*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_D[\lambda](t,s) + r_2[\lambda](t)\, G_{M_2}[\lambda](1,s) \\ &= G_D[\lambda](t,s) + \frac{\partial}{\partial s} G_D[\lambda](t,1)\, G_{M_2}[\lambda](1,s), \quad \forall\, (t,s) \in J \times J, \end{aligned} \tag{5.2.28}$$

Corollary 5.2.16. *The following inequality holds*

$$G_{M_2}[\lambda](t,s) < G_D[\lambda](t,s) < 0, \quad \forall (t,s) \in (0,1) \times (0,1), \quad \forall\, \lambda < \lambda_0^{M_2}.$$

Remark 5.2.17. *The above inequality between $G_{M_2}[\lambda]$ and $G_D[\lambda]$ can be deduced from [9, Corollaries 4.7, 4.8, 4.13]. Moreover the expression* (5.2.28) *relating $G_{M_2}[\lambda]$ and $G_D[\lambda]$ is different from the one obtained in that reference.*

However, as far as we know, there is no expression in the literature that relate G_{M_1} and G_D and, as a consequence, equality (5.2.26) *and inequality* (5.2.27) *are new in the literature.*

By the reciprocal process, we can relate expressions of the Green's function of the Dirichlet problem and the ones of the mixed problems.

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Theorem 5.2.18. *If operator $L[\lambda]$ is nonresonant in X_D and X_{M_2}, then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_{M_2}[\lambda](t,s) + r_8[\lambda](t)\,\frac{\partial}{\partial t}G_D[\lambda](1,s) \\ =& G_{M_2}[\lambda](t,s) - G_{M_2}[\lambda](t,1)\,\frac{\partial}{\partial t}G_D[\lambda](1,s), \quad \forall\,(t,s)\in J\times J. \end{aligned}$$

Theorem 5.2.19. *If operator $L[\lambda]$ is nonresonant in X_D and X_{M_1}, then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_{M_1}[\lambda](t,s) + r_9[\lambda](t)\,\frac{\partial}{\partial t}G_D[\lambda](0,s) \\ =& G_{M_1}[\lambda](t,s) + G_{M_1}[\lambda](t,0)\,\frac{\partial}{\partial t}G_D[\lambda](0,s), \quad \forall\,(t,s)\in J\times J. \end{aligned}$$

Remark 5.2.20. *Reasoning in a similar way to the previous sections, it can be verified that from the equalities of the two previous results we can deduce the Corollaries 5.2.14 and 5.2.16.*

5.2.4 Neumann and Mixed Problems

Now, we study the relationships between the Green's functions of the Neumann and mixed problems.

First, let us rewrite the mixed 2 problem as a function of the Neumann one as

$$L[\lambda]\,u(t) = \sigma(t), \quad \text{a.e. } t\in J, \quad u'(0) = u(0) + u'(0), \quad u'(1) = 0,$$

and reasoning as in the previous sections, we arrive at the next results.

Theorem 5.2.21. *Assume that operator $L[\lambda]$ is nonresonant in X_N and X_{M_1}. Then it holds that*

$$\begin{aligned} G_{M_2}[\lambda](t,s) =& G_N[\lambda](t,s) + r_5[\lambda](t)\,\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s) \\ =& G_N[\lambda](t,s) + G_N[\lambda](t,0)\,\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s), \quad \forall\,(t,s)\in J\times J. \end{aligned} \tag{5.2.29}$$

Corollary 5.2.22. *The following inequality holds*

$$G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s) < 0, \quad \forall\,(t,s)\in(0,1]\times(0,1], \quad \forall\,\lambda<\lambda_0^N. \tag{5.2.30}$$

Proof. We know that, $G_{M_2}[\lambda](t,s) < 0$ for all $(t,s)\in(0,1]\times(0,1]$ if and only if $\lambda<\lambda_0^{M_2}$ (see [9, Corollary 4.6]). Since $G_{M_2}[\lambda](0,s) = 0$ we deduce that $\dfrac{\partial}{\partial t}G_{M_2}[\lambda](0,s) < 0$ for such λ. In addition, $\lambda_0^N < \lambda_0^{M_1}$ (see [9, Remark 4.19]). Therefore, using equality (5.2.29) we obtain the result. □

The Neumann problem from the mixed 1 problem is expressed as follows

$$L[\lambda]\,u(t) = \sigma(t), \quad \text{a.e. } t\in J, \quad u'(0) = 0, \quad u(1) = u'(1) + u(1).$$

Analogously, for mixed 1 problem we have the following results.

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Theorem 5.2.23. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and X_N, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) =& G_N[\lambda](t,s) + r_6[\lambda](t)\,\frac{\partial}{\partial t} G_{M_1}[\lambda](1,s) \\ =& G_N[\lambda](t,s) - G_N[\lambda](t,1)\,\frac{\partial}{\partial t} G_{M_1}[\lambda](1,s), \quad \forall\,(t,s)\in J\times J. \end{aligned} \tag{5.2.31}$$

Corollary 5.2.24. *The following inequality is fulfilled*

$$G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \ \ \forall\,(t,s)\in[0,1)\times[0,1), \ \ \forall\,\lambda<\lambda_0^N. \tag{5.2.32}$$

Remark 5.2.25. *Inequality* (5.2.32) *can be deduce from [6, Corollaries 4.5, 4.8, 4.13]. Moreover, the identities* (5.2.29) *and* (5.2.31) *together with inequality* (5.2.30) *are new in the literature.*

For the reverse process, we can obtain additional relations between the Green's function of the Neumann problem and the ones of the mixed problems. The Neumann problem from the mixed 2 problem is expressed as follows

$$L[\lambda]\,u(t)=\sigma(t), \ \ \text{a.e. } t\in J, \ \ u(0)=u(0)+u'(0), \ \ u'(1)=0.$$

Theorem 5.2.26. *If operator $L[\lambda]$ is nonresonant in X_N and X_{M_2}, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_2}[\lambda](t,s) + r_7[\lambda](t)\,G_N[\lambda](0,s) \\ &= G_{M_2}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)\,G_N[\lambda](0,s), \ \ \forall\,(t,s)\in J\times J. \end{aligned}$$

Arguing as for the mixed 1 problem, we consider de Neumann one

$$L[\lambda]\,u(t)=\sigma(t), \ \ \text{a.e. } t\in J, \ \ u'(0)=0, \ \ u(1)=u(1)+u'(1),$$

expressed from the mixed 2, and arrive at the next theorem.

Theorem 5.2.27. *If operator $L[\lambda]$ is nonresonant in X_N and X_{M_1}, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_{M_1}[\lambda](t,s) + r_{10}[\lambda](t)\,G_N[\lambda](1,s) \\ &= G_{M_1}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\,G_N[\lambda](1,s), \ \ \forall\,(t,s)\in J\times J. \end{aligned}$$

Remark 5.2.28. *We point out that from the two previous theorems we can deduce Corollaries 5.2.22 and 5.2.24.*

5.2.5 Periodic and Neumann Problems

Concerning the Neumann and periodic problems, we may rewrite the periodic problem by means of the Neumann one as follows:

$$L[\lambda]\,u(t)=\sigma(t), \ \ \text{a.e. } t\in J, \ \ u'(0)=u(0)-u(1)+u'(0), \ \ u'(1)=u'(0).$$

So, arguing as before, we arrive at the next theorem.

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Theorem 5.2.29. *If operator $L[\lambda]$ is nonresonant both in X_N and X_P, the following equality is fulfilled*

$$\begin{aligned} G_P[\lambda](t,s) =& G_N[\lambda](t,s) + (r_5[\lambda](t) + r_6[\lambda](t))\ \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ =& G_N[\lambda](t,s) + (G_N[\lambda](t,0) - G_N[\lambda](t,1))\ \frac{\partial}{\partial t} G_P[\lambda](0,s), \quad \forall\, (t,s) \in J \times J. \end{aligned} \tag{5.2.33}$$

Remark 5.2.30. *From* (5.2.33) *and due to the symmetry of $G_P[\lambda]$ and $G_N[\lambda]$, we deduce that*

$$(G_N[\lambda](t,0) - G_N[\lambda](t,1))\ \frac{\partial}{\partial t} G_P[\lambda](0,s) = (G_N[\lambda](s,0) - G_N[\lambda](s,1))\ \frac{\partial}{\partial t} G_P[\lambda](0,t),$$

for all $(t,s) \in J \times J$.

If $\frac{\partial}{\partial t} G_P[\lambda](0,t) \neq 0$ and $\frac{\partial}{\partial t} G_P[\lambda](0,s) \neq 0$, then

$$\frac{G_N[\lambda](t,0) - G_N[\lambda](t,1)}{\frac{\partial}{\partial t} G_P[\lambda](0,t)} = \frac{G_N[\lambda](s,0) - G_N[\lambda](s,1)}{\frac{\partial}{\partial t} G_P[\lambda](0,s)} = c_1 \in \mathbb{R}.$$

We know that $\frac{\partial}{\partial t} G_P[\lambda](t,s) = \frac{\partial}{\partial s} G_P[\lambda](s,t)$ and $\frac{\partial}{\partial s} G_P[\lambda](t,s) = \frac{\partial}{\partial t} G_P[\lambda](s,t)$. Then

$$G_N(t,0) - G_N(t,1) = c_1 \frac{\partial}{\partial t} G_P[\lambda](0,t) = c_1 \frac{\partial}{\partial s} G_P[\lambda](t,0).$$

As a consequence, the equality (5.2.33) *becomes:*

$$G_P[\lambda](t,s) - G_N[\lambda](t,s) = c_1 \frac{\partial}{\partial t} G_P[\lambda](0,t) \frac{\partial}{\partial t} G_P[\lambda](0,s)$$

for all $(t,s) \in J \times J$.

For the reverse process by writing the Neumann problem as a function of the periodic one:

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in J, \quad u(0) - u(1) = u(0) - u(1) + u'(1), \quad u'(0) - u'(1) = -u'(1),$$

we arrive at the next result.

Theorem 5.2.31. *Assume that $L[\lambda]$ is nonresonant in X_N and X_P, then*

$$\begin{aligned} G_N[\lambda](t,s) &= G_P[\lambda](t,s) + r_3[\lambda](t)\, (G_N[\lambda](0,s) - G_N[\lambda](1,s)) \\ &= G_P[\lambda](t,s) - \frac{\partial}{\partial s} G_P[\lambda](t,0)\, (G_N[\lambda](0,s) - G_N[\lambda](1,s)), \quad \forall\, (t,s) \in J \times J. \end{aligned}$$

5.2.6 Periodic and Mixed Problems

The same arguments of the previous subsections are applicable to the periodic and mixed problems.

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For the mixed 1, we will assume that $L[\lambda]$ is nonresonant in X_P and X_{M_1}. We rewrite the periodic problem as a function of the mixed 1 one as follows:

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in J,\ \ u(0)-u(1)=u(0),\ \ u'(0)-u'(1)=-u'(1),$$

and we obtain the follwing result.

Theorem 5.2.32. *Assume that $L[\lambda]$ is nonresonant in X_P and X_{M_1}. Then it holds that*

$$\begin{aligned}G_{M_1}[\lambda](t,s)=&G_P[\lambda](t,s)+r_3[\lambda](t)\,G_{M_1}[\lambda](0,s)-r_4[\lambda](t)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s)\\=&G_P[\lambda](t,s)-\frac{\partial}{\partial s}G_P[\lambda](t,0)\,G_{M_1}[\lambda](0,s)-G_P[\lambda](t,0)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s),\end{aligned}$$

for all $(t,s)\in J\times J$.

Next example shows that, in general, the Green's functions of periodic and mixed 1 problems are not comparable.

Example 5.2.33. *We consider the differential equation $u''(t)-m^2\,u(t)=0$, $t\in J$, and $m\in(0,\infty)$. In this case, $a(t)=-m^2$, $t\in J$ and $\lambda=0$.*

Using the Mathematica Software package (see [11]) to calculate the expressions of the Green's functions, we have that the Green's function of the periodic problem is given by

$$G_P(t,s)=\begin{cases}\dfrac{e^{m\,(s-t)+m}+e^{m\,(t-s)}}{2\,m\,(1-e^m)},\ \ 0\le s\le t\le 1,\\[2ex]\dfrac{e^{m\,(t-s)+m}+e^{m\,(s-t)}}{2\,m\,(1-e^m)},\ \ 0\le t<s\le 1,\end{cases}$$

and that of the mixed 1 *problem is*

$$G_{M_1}(t,s)=\begin{cases}\dfrac{(e^{2\,m\,s}+1)\,(e^{2\,m\,t}-e^{2\,m})\,e^{-m\,(s+t)}}{2\,m\,(e^{2\,m}+1)},\ \ 0\le s\le t\le 1,\\[2ex]\dfrac{(e^{2\,mt}+1)\,(e^{2\,ms}-e^{2\,m})\,e^{-m\,(t+s)}}{2\,m\,(e^{2\,m}+1)},\ \ 0\le t<s\le 1.\end{cases}$$

Note that both functions G_P and G_{M_1} are negative on $J\times J$ and $m\in(0,\infty)$. Green's functions G_P and G_{M_1} are comparable for small values of m. Figure 5.2.1 represents the Green's functions G_P and G_{M_1} for $m=1$ that are comparable (in which case $G_p<G_{M_1}$) and Figure 5.2.2 represents them for the value $m=2$, showing that they are not comparable.

Similarly, we study the relationship between Green's functions of periodic and mixed 2. We rewrite the periodic problem as a function of the mixed 2 one as follows

$$L[\lambda]\,u(t)=\sigma(t),\ \text{ a.e. } t\in J,\ \ u(0)-u(1)=-u(1),\ \ u'(0)-u'(1)=u'(0).$$

We arrive at the next theorem.

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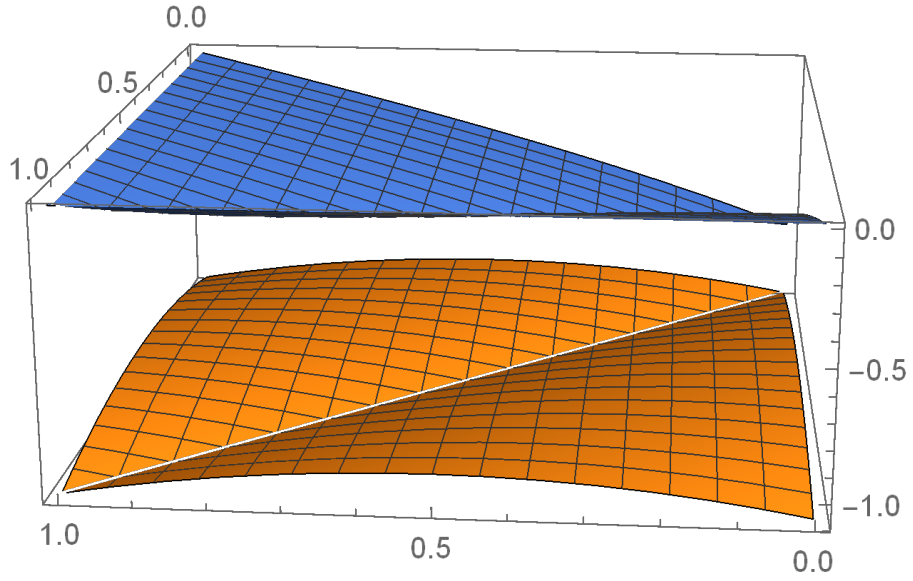

Figure 5.2.1 : The blue graph corresponds to function G_{M_1} and the orange graph represents the function G_P on $J \times J$ for the value $m = 1$.

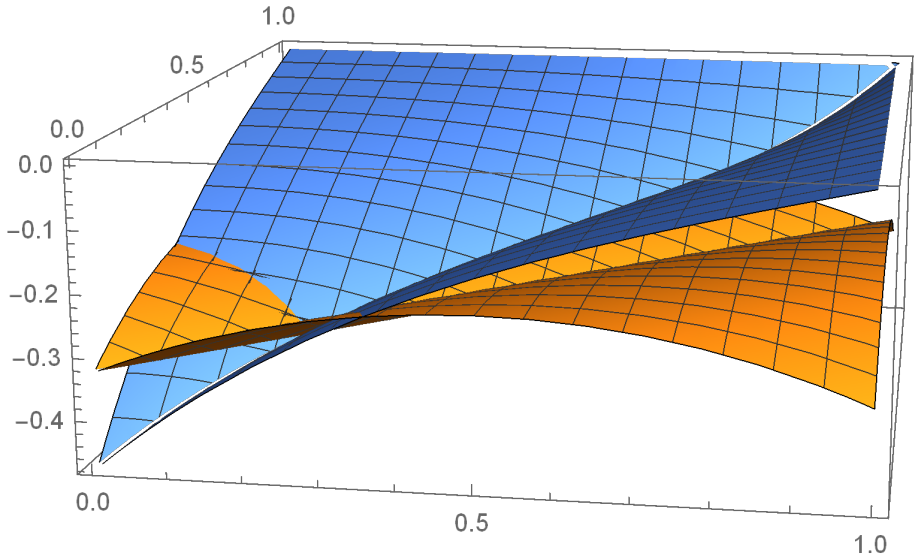

Figure 5.2.2 : The blue graph corresponds to function G_{M_1} and the orange graph represents the function G_P on $J \times J$ for the value $m = 2$.

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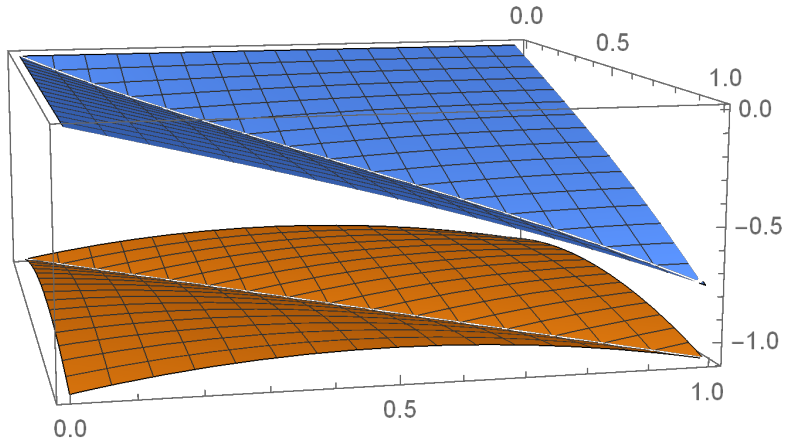

Figure 5.2.3: The blue graph corresponds to the function G_{M_2} and the orange graph represents the function G_P on $J \times J$ for the value $m = 1$.

Theorem 5.2.34. *Assume that $L[\lambda]$ is nonresonant in X_P and X_{M_2}, then it holds that*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_P[\lambda](t,s) - r_3[\lambda](t)\, G_{M_2}[\lambda](1,s) + r_4[\lambda](t)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ &= G_P[\lambda](t,s) + \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_{M_2}[\lambda](1,s) + G_P[\lambda](t,0)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

The above equation is analogous to the equation relating periodic to mixed 1. So, in general, Green's functions of the periodic and mixed 2 problems will not be comparable either.

Example 5.2.35. *We use in this example the same equation as in Example 5.2.33. The Green's function of the mixed 2 problem is given by*

$$G_{M_2}(t,s) = \begin{cases} -\dfrac{(e^{2\,m\,s}+1)\,(e^{2\,m\,t}+e^{2\,m})\,e^{-m\,(s+t)}}{2\,m\,(e^{2\,m}+1)}, & 0 \le s \le t \le 1, \\ -\dfrac{(e^{2\,m\,t}+1)\,(e^{2\,m\,s}+e^{2\,m})\,e^{-m\,(t+s)}}{2\,m\,(e^{2\,m}+1)}, & 0 \le t < s \le 1, \end{cases}$$

and is negative on $J \times J$ for all $m \in (0,\infty)$. Green's functions G_P and G_{M_2} are comparable for small values of m. Figure 5.2.3 represents the Green's functions G_P and G_{M_2} for $m = 1$ (in which case $G_P < G_{M_2}$). Figure 5.2.4 represents these functions for the value $m = 3$, in which case they are not comparable.

Finally for the reverse process, arguing in a similar manner than in previous sections, we can relate the expression of the Green's functions of the periodic problem and the ones related to mixed problems.

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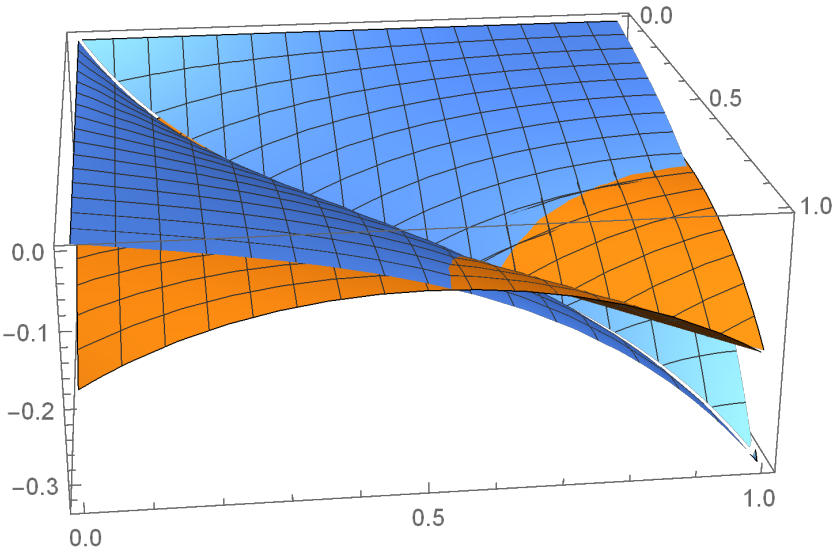

Figure 5.2.4: The blue graph corresponds to the function G_{M_2} and the orange graph represents the function G_P on $J \times J$ for the value $m = 3$.

Theorem 5.2.36. *If operator $L[\lambda]$ is nonresonant both in X_P and X_{M_2}, then*

$$\begin{aligned} G_P[\lambda](t,s) =& G_{M_2}[\lambda](t,s) + r_7[\lambda](t)\, G_P[\lambda](1,s) + r_8[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ =& G_{M_2}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)\, G_P[\lambda](1,s) - G_{M_2}[\lambda](t,1)\, \frac{\partial}{\partial t} G_P[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.37. *If operator $L[\lambda]$ is nonresonant both in X_P and X_{M_1}, then*

$$\begin{aligned} G_P[\lambda](t,s) =& G_{M_1}[\lambda](t,s) + r_9[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](1,s) + r_{10}[\lambda](t)\, G_P[\lambda](0,s) \\ =& G_{M_1}[\lambda](t,s) + G_{M_1}[\lambda](t,0)\, \frac{\partial}{\partial t} G_P[\lambda](1,s) - \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\, G_P[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

5.2.7 Initial Problem

Reasoning as in the previous cases, we include in this subsection the relationships between Green's function of the initial problem and all the other considered boundary conditions.

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Theorem 5.2.38. *If operator $L[\lambda]$ is nonresonant in X_P, the following equality is fulfilled*

$$\begin{aligned} G_I[\lambda](t,s) =& G_P[\lambda](t,s) - r_3[\lambda](t)\, G_I[\lambda](1,s) - r_4[\lambda](t)\, \frac{\partial}{\partial t} G_I[\lambda](1,s) \\ =& G_P[\lambda](t,s) + \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_I[\lambda](1,s) - G_P[\lambda](t,0)\, \frac{\partial}{\partial t} G_I[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.39. *Assume that $L[\lambda]$ is nonresonant in X_D, then*

$$\begin{aligned} G_P[\lambda](t,s) =& G_I[\lambda](t,s) + r_{11}[\lambda](t)\, G_P[\lambda](1,s) + r_{12}[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](1,s) \\ =& G_I[\lambda](t,s) - \frac{\partial}{\partial s} G_I[\lambda](t,0)\, G_P[\lambda](1,s) + G_I[\lambda](t,0)\, \frac{\partial}{\partial t} G_P[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.40. *If operator $L[\lambda]$ is nonresonant in X_N, the following equality is fulfilled*

$$\begin{aligned} G_I[\lambda](t,s) =& G_N[\lambda](t,s) + r_6[\lambda](t)\, \frac{\partial}{\partial t} G_I[\lambda](1,s) \\ =& G_N[\lambda](t,s) - G_N[\lambda](t,1)\, \frac{\partial}{\partial t} G_I[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.41. *Assume that $L[\lambda]$ is nonresonant in X_N, then*

$$\begin{aligned} G_N[\lambda](t,s) =& G_I[\lambda](t,s) + r_{11}[\lambda](t)\, G_N[\lambda](0,s) \\ =& G_I[\lambda](t,s) - \frac{\partial}{\partial s} G_I[\lambda](t,0)\, G_N[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.42. *If operator $L[\lambda]$ is nonresonant in X_D, the following equality is fulfilled*

$$\begin{aligned} G_I[\lambda](t,s) =& G_D[\lambda](t,s) + r_2[\lambda](t)\, G_I[\lambda](1,s) \\ =& G_D[\lambda](t,s) + \frac{\partial}{\partial s} G_D[\lambda](t,1)\, G_I[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.43. *Assume that $L[\lambda]$ is nonresonant in X_D, then*

$$\begin{aligned} G_D[\lambda](t,s) &= G_I[\lambda](t,s) + r_{12}[\lambda](t)\, \frac{\partial}{\partial s} G_D[\lambda](0,s) \\ &= G_I[\lambda](t,s) + G_I[\lambda](t,0)\, \frac{\partial}{\partial s} G_D[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

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Theorem 5.2.44. *Assume that operator $L[\lambda]$ is nonresonant in X_{M_1}. Then it holds that*

$$\begin{aligned} G_I[\lambda](t,s) =& G_{M_1}[\lambda](t,s) + r_{10}[\lambda](t)\, G_I[\lambda](1,s) \\ =& G_{M_1}[\lambda](t,s) - \frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)\, G_I[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.45. *If operator $L[\lambda]$ is nonresonant in X_{M_1}, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_I[\lambda](t,s) + r_{11}[\lambda](t)\, G_{M_1}[\lambda](1,s) \\ &= G_I[\lambda](t,s) - \frac{\partial}{\partial s} G_I[\lambda](t,0)\, G_{M_1}[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.46. *Assume that $L[\lambda]$ is nonresonant in X_{M_2}, then*

$$\begin{aligned} G_I[\lambda](t,s) =& G_{M_2}[\lambda](t,s) + r_8[\lambda](t)\, \frac{\partial}{\partial t} G_I[\lambda](1,s) \\ =& G_{M_2}[\lambda](t,s) - G_{M_2}[\lambda](t,1)\, \frac{\partial}{\partial t} G_I[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.47. *If operator $L[\lambda]$ is nonresonant in X_{M_2}, then*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_I[\lambda](t,s) + r_{12}[\lambda](t)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ &= G_I[\lambda](t,s) + G_I[\lambda](t,0)\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

5.2.8 Final Problem

Finally, arguing as before, we include in this subsection the relationships between Green's function of the final problem and all the other considered problems.

Theorem 5.2.48. *If operator $L[\lambda]$ is nonresonant in X_P, the following equality is fulfilled*

$$\begin{aligned} G_F[\lambda](t,s) =& G_P[\lambda](t,s) + r_3[\lambda](t)\, G_F[\lambda](0,s) + r_4[\lambda](t)\, \frac{\partial}{\partial t} G_F[\lambda](0,s) \\ =& G_P[\lambda](t,s) - \frac{\partial}{\partial s} G_P[\lambda](t,0)\, G_F[\lambda](0,s) + G_P[\lambda](t,0)\, \frac{\partial}{\partial t} G_F[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

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Theorem 5.2.49. *Assume that $L[\lambda]$ is nonresonant in X_D, then*

$$\begin{aligned} G_P[\lambda](t,s) =& G_F[\lambda](t,s) + r_{13}[\lambda](t)\, G_P[\lambda](0,s) + r_{14}[\lambda](t)\, \frac{\partial}{\partial t} G_P[\lambda](0,s) \\ =& G_I[\lambda](t,s) + \frac{\partial}{\partial s} G_F[\lambda](t,1)\, G_P[\lambda](0,s) - G_F[\lambda](t,1)\, \frac{\partial}{\partial t} G_P[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.50. *If operator $L[\lambda]$ is nonresonant in X_N, the following equality is fulfilled*

$$\begin{aligned} G_F[\lambda](t,s) =& G_N[\lambda](t,s) + r_5[\lambda](t)\, \frac{\partial}{\partial t} G_F[\lambda](0,s) \\ =& G_N[\lambda](t,s) + G_N[\lambda](t,0)\, \frac{\partial}{\partial t} G_F[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.51. *Assume that $L[\lambda]$ is nonresonant in X_N, then*

$$\begin{aligned} G_N[\lambda](t,s) =& G_F[\lambda](t,s) + r_{13}[\lambda](t)\, G_N[\lambda](1,s) \\ =& G_F[\lambda](t,s) + \frac{\partial}{\partial s} G_F[\lambda](t,1)\, G_N[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.52. *If operator $L[\lambda]$ is nonresonant in X_D, the following equality is fulfilled*

$$\begin{aligned} G_F[\lambda](t,s) =& G_D[\lambda](t,s) + r_1[\lambda](t)\, G_F[\lambda](0,s) \\ =& G_D[\lambda](t,s) - \frac{\partial}{\partial s} G_D[\lambda](t,0)\, G_F[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.53. *Assume that $L[\lambda]$ is nonresonant in X_D, then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_F[\lambda](t,s) + r_{14}[\lambda](t)\, \frac{\partial}{\partial s} G_D[\lambda](1,s) \\ =& G_F[\lambda](t,s) - G_F[\lambda](t,1)\, \frac{\partial}{\partial s} G_D[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.54. *Assume that operator $L[\lambda]$ is nonresonant in X_{M_1}. Then it holds that*

$$\begin{aligned} G_F[\lambda](t,s) =& G_{M_1}[\lambda](t,s) + r_9[\lambda](t)\, \frac{\partial}{\partial t} G_F[\lambda](0,s) \\ =& G_{M_1}[\lambda](t,s) + G_{M_1}[\lambda](t,0)\, \frac{\partial}{\partial t} G_F[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

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Theorem 5.2.55. *If operator $L[\lambda]$ is nonresonant in X_{M_1}, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) &= G_F[\lambda](t,s) + r_{14}[\lambda](t)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s) \\ &= G_F[\lambda](t,s) - G_F[\lambda](t,1)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.56. *Assume that $L[\lambda]$ is nonresonant in X_{M_2}, then*

$$\begin{aligned} G_F[\lambda](t,s) =& G_{M_2}[\lambda](t,s) + r_7[\lambda](t)\,G_F[\lambda](0,s) \\ =& G_{M_2}[\lambda](t,s) - \frac{\partial}{\partial s}G_{M_2}[\lambda](t,0)\,G_F[\lambda](0,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

Theorem 5.2.57. *If operator $L[\lambda]$ is nonresonant in X_{M_2}, then*

$$\begin{aligned} G_{M_2}[\lambda](t,s) &= G_F[\lambda](t,s) + r_{13}[\lambda](t)\,G_{M_2}[\lambda](1,s) \\ &= G_F[\lambda](t,s) + \frac{\partial}{\partial s}G_F[\lambda](t,1)\,G_{M_2}[\lambda](1,s), \end{aligned}$$

for all $(t,s) \in J \times J$.

5.3 Alternative Decomposition of Green's Functions

In this section we give another way to relate Green's functions coupled to the different value conditions studied in previous section. The main difference consists on the fact that in this case, instead of Lemma 1.1.8, as in previous section, we will use Theorem 2.2.2. The obtained expressions will be different to the ones deduced in previous section.

In order to make the calculations, we must keep in mind that the choice of the C_1 and C_2 operators and the values δ_1 and δ_2 is not unique.

5.3.1 Dirichlet and Mixed Problems

First, we express the Green's function of mixed 2 problem in terms of the Green's function of Dirichlet problem.

Theorem 5.3.1. *If operator $L[\lambda]$ is nonresonant in X_D and $r_2'[\lambda](1) \neq 0$, then the following equality holds*

$$\begin{aligned} G_{M_2}[\lambda](t,s) =& G_D[\lambda](t,s) - \frac{r_2[\lambda](t)}{r_2'[\lambda](1)}\,\frac{\partial}{\partial t}G_D[\lambda](1,s) \\ =& G_D[\lambda](t,s) - \frac{\frac{\partial}{\partial s}G_D\lambda](t,1)}{\frac{\partial^2}{\partial s \partial t}G_D[\lambda](1,1)}\,\frac{\partial}{\partial t}G_D[\lambda](1,s), \quad \forall\,(t,s) \in J \times J. \end{aligned} \tag{5.3.34}$$

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Proof. We write the mixed 2 problem based on the Dirichlet problem as follows

$$L[\lambda]\,u(t)=\sigma(t),\quad \text{a.e. } t\in J,\quad u(0)=0,\quad u(1)=u(1)+u'(1). \tag{5.3.35}$$

Using the notation of Theorem 2.2.2, we have that in this case $B_1(u)=u(0)$, $B_2(u)=u(1)$, $C_1(u)=0$, $C_2(u)=u(1)+u'(1)$ and $\delta_1=\delta_2=1$. Moreover, $\omega_1(t)=r_1[\lambda](t)$, $\omega_2(t)=r_2[\lambda](t)$ and the matrix $A_D^1[\lambda]$ in this case is

$$A_D^1[\lambda]=\begin{pmatrix}0 & 0\\ r_1'[\lambda](1) & 1+r_2'[\lambda](1)\end{pmatrix}$$

and $|I-A_D^1[\lambda]|=-r_2'[\lambda](1)\neq 0$. So,

$$b_D^1=(I-A_D^1[\lambda])^{-1}=\begin{pmatrix}1 & 0\\ -\frac{r_1'[\lambda](1)}{r_2'[\lambda](1)} & -\frac{1}{r_2'[\lambda](1)}\end{pmatrix}.$$

In consequence, as a direct application of the equality (2.2.7) we obtain the result. □

Corollary 5.3.2. *For all $\lambda<\lambda_0^{M_2}$, we infer that $r_2'[\lambda](1)>0$.*

Proof. From Corollary 5.2.16 we have that $G_{M_2}[\lambda]<G_D[\lambda]<0$ for all $\lambda<\lambda_0^{M_2}$ and, as a direct consequence, $\frac{\partial}{\partial t}G_D[\lambda](1,s)>0$. Lemma 5.1.5 says us that $r_2[\lambda]>0$ on $(0,1]$ for all $\lambda<\lambda_0^D$. Since $\lambda_0^{M_2}<\lambda_0^D$ (see [10, page 108]) we deduce, from equality (5.3.34) that $r_2'[\lambda](1)>0$. □

Similarly, we study the mixed 1 problem as a function of the Dirichlet one.

Theorem 5.3.3. *If operator $L[\lambda]$ is nonresonant in X_D and $r_1'[\lambda](0)\neq 0$, then the following equality holds*

$$\begin{aligned}G_{M_1}[\lambda](t,s)=&G_D[\lambda](t,s)-\frac{r_1[\lambda](t)}{r_1'[\lambda](0)}\,\frac{\partial}{\partial t}G_D[\lambda](0,s)\\ =&G_D[\lambda](t,s)-\frac{\frac{\partial}{\partial s}G_D\lambda](t,0)}{\frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,0)}\,\frac{\partial}{\partial t}G_D[\lambda](0,s),\quad \forall\,(t,s)\in J\times J.\end{aligned} \tag{5.3.36}$$

Proof. Let us rewrite mixed 1 problem in the following way

$$L[\lambda]\,u(t)=\sigma(t),\quad \text{a.e. } t\in J,\quad u(0)=u(0)+u'(0),\quad u(1)=0. \tag{5.3.37}$$

In this case, we have that $B_1(u)=u(0)$, $B_2(u)=u(1)$, $C_1(u)=u(0)+u'(0)$, $C_2(u)=0$ and $\delta_1=\delta_2=1$. Moreover, $\omega_1(t)=r_1[\lambda](t)$, $\omega_2(t)=r_2[\lambda](t)$ and the matrix $A_D^2[\lambda]$ is

$$A_D^2[\lambda]=\begin{pmatrix}1+r_1'[\lambda](0) & r_2'[\lambda](0)\\ 0 & 0\end{pmatrix}$$

and $|I-A_D^2[\lambda]|=-r_1'[\lambda](0)\neq 0$. So,

$$b_D^2=(I-A_D^2[\lambda])^{-1}=\begin{pmatrix}-\frac{1}{r_1'[\lambda](0)} & -\frac{r_2'[\lambda](0)}{r_1'[\lambda](0)}\\ 0 & 1\end{pmatrix}.$$

Therefore, using (2.2.7), we deduce the equality. □

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Corollary 5.3.4. *For all $\lambda < \lambda_0^{M_1}$, we infer that $r_1'[\lambda](0) < 0$.*

Proof. We know from Corollary 5.2.14 that $G_{M_1}[\lambda] < G_D[\lambda]$ for all $\lambda < \lambda_0^{M_1}$ and from Lemma 5.1.7 that $r_1[\lambda] > 0$ on $[0,1)$ for all $\lambda < \lambda_0^D$. Since $\lambda_0^{M_1} < \lambda_0^D$ (see [10, page 108]) using the equality (5.3.36) we arrive at the result. □

Remark 5.3.5. *In problem* (5.3.35) *we can do the calculations in a simpler way by taking $C_1(u) = C_2(u) = u(1) + u'(1)$, $\delta_1 = 0$ and $\delta_2 = 1$. The same can be done with problem* (5.3.37) *by taking $C_1(u) = C_2(u) = u(0) + u'(0)$, $\delta_1 = 1$ and $\delta_2 = 0$.*

We now do the reciprocal process by writing the Dirichlet problem based on the mixed ones. We arrive at the following results.

Theorem 5.3.6. *If operator $L[\lambda]$ is nonresonant in X_{M_2} and $r_8[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_{M_2}[\lambda](t,s) - \frac{r_8[\lambda](t)}{r_8[\lambda](1)}\, G_{M_2}[\lambda](1,s) \\ =& G_{M_2}[\lambda](t,s) - \frac{G_{M_2}[\lambda](t,1)}{G_{M_2}[\lambda](1,1)}\, G_{M_2}[\lambda](1,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

Theorem 5.3.7. *If operator $L[\lambda]$ is nonresonant in X_{M_1} and $r_9[\lambda](0) \neq 0$, then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_{M_1}[\lambda](t,s) - \frac{r_9[\lambda](t)}{r_9[\lambda](0)}\, G_{M_1}[\lambda](0,s) \\ =& G_{M_1}[\lambda](t,s) - \frac{G_{M_1}[\lambda](t,0)}{G_{M_1}[\lambda](0,0)}\, G_{M_1}[\lambda](0,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

5.3.2 Neumann and Dirichlet Problems

In this case we study the relationships between the Green's function of the Neumann and Dirichlet problems. Reasoning as in the previous subsection we have the next result.

Theorem 5.3.8. *If operator $L[\lambda]$ is nonresonant in X_D and*

$$|I - A_D^3[\lambda]| := r_1'[\lambda](0)\, r_2'[\lambda](1) - r_2'[\lambda](0)\, r_1'[\lambda](1) \neq 0,$$

then it holds that

$$\begin{aligned} G_N[\lambda](t,s) =& G_D[\lambda](t,s) - \frac{r_2'[\lambda](1)}{|I - A_D^3[\lambda]|}\, r_1[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) + \frac{r_1'[\lambda](1)}{|I - A_D^3[\lambda]|}\, r_2[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &+ \frac{r_2'[\lambda](0)}{|I - A_D^3[\lambda]|}\, r_1[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) - \frac{r_1'[\lambda](0)}{|I - A_D^3[\lambda]|}\, r_2[\lambda](t)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ =& G_D[\lambda](t,s) + \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,1)\, \frac{\partial}{\partial s} G_D[\lambda](t,0)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &- \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](1,0)\, \frac{\partial}{\partial s} G_D[\lambda](t,1)\, \frac{\partial}{\partial t} G_D[\lambda](0,s) \\ &- \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,1)\, \frac{\partial}{\partial s} G_D[\lambda](t,0)\, \frac{\partial}{\partial t} G_D[\lambda](1,s) \\ &+ \frac{1}{|I - A_D^3[\lambda]|}\, \frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0)\, \frac{\partial}{\partial s} G_D[\lambda](t,1)\, \frac{\partial}{\partial t} G_D[\lambda](1,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

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Proof. Let us rewrite Neumann problem in the following form

$$L[\lambda]\,u(t)=\sigma(t),\ \ \text{a.e. } t\in J,\ \ u(0)=u(0)+u'(0),\ \ u(1)=u(1)+u'(1).$$

As the previous subsection, using the notation of Theorem 2.2.2, we have that in this case $B_1(u)=u(0)$, $B_2(u)=u(1)$, $C_1(u)=u(0)+u'(0)$, $C_2(u)=u(1)+u'(1)$, $\delta_1=1$ and $\delta_2=1$. Moreover, $\omega_1(t)=r_1[\lambda](t)$, $\omega_2(t)=r_2[\lambda](t)$ and the matrix $A_D^3[\lambda]$ in this case is given by

$$A_D^3[\lambda]=\begin{pmatrix}1+r_1'[\lambda](0) & r_2'[\lambda](0)\\ r_1'[\lambda](1) & 1+r_2'[\lambda](1)\end{pmatrix}$$

with $|I-A_D^3[\lambda]|=r_1'[\lambda](0)\,r_2'[\lambda](1)-r_2'[\lambda](0)\,r_1'[\lambda](1)\neq 0$. So,

$$b_D^3=(I-A_D^3[\lambda])^{-1}=\begin{pmatrix}-\frac{r_2'[\lambda](1)}{|I-A_D^3[\lambda]|} & \frac{r_2'[\lambda](0)}{|I-A_D^3[\lambda]|}\\ \frac{r_1'[\lambda](1)}{|I-A-D^3[\lambda]|} & -\frac{r_1'[\lambda](0)}{|I-A_D^3[\lambda]|}\end{pmatrix}.$$

Hence, applying equality (2.2.7) for this case the result is concluded. □

We now reverse the process by studying the Dirichlet problem as a function of the Neumann one as follows

$$L[\lambda]\,u(t)=\sigma(t),\ \ \text{a.e. } t\in J,\ \ u'(0)=u(0)+u'(0),\ \ u'(1)=u(1)+u'(1).$$

and applying analogous calculations.

Theorem 5.3.9. *Assume that $L[\lambda]$ is nonresonant in X_N and*

$$|I-A_N^1[\lambda]|:=r_5[\lambda](0)\,r_6[\lambda](1)-r_6[\lambda](0)\,r_5[\lambda](1)\neq 0,$$

then

$$\begin{aligned}G_D[\lambda](t,s)=&G_N[\lambda](t,s)-\frac{1}{|I-A_N^1[\lambda]|}\Big(r_5[\lambda](t)\Big(-r_6[\lambda](1)\,G_N[\lambda](0,s)+r_6[\lambda](0)\,G_N[\lambda](1,s)\Big)\\&+r_6[\lambda](t)\Big(r_5[\lambda](1)\,G_N[\lambda](0,s)-r_5[\lambda](0)\,G_N[\lambda](1,s)\Big)\Big)\\=&G_N[\lambda](t,s)-\frac{1}{|I-A_N^1[\lambda]|}\Big(G_N[\lambda](t,0)\Big(G_N[\lambda](1,1)\,G_N[\lambda](0,s)-G_N[\lambda](0,1)\,G_N[\lambda](1,s)\Big)\\&-G_N[\lambda](t,1)\Big(G_N[\lambda](1,0)\,G_N[\lambda](0,s)-G_N[\lambda](0,0)\,G_N[\lambda](1,s)\Big)\Big),\ \ \forall\,(t,s)\in J\times J.\end{aligned}$$

5.3.3 Periodic and Dirichlet Problems

In this section we give a relationship between $G_P[\lambda]$ and $G_D[\lambda]$ following the same steps than in previous sections.

Theorem 5.3.10. *Assume that $L[\lambda]$ is nonresonant in X_D and*

$$|I-A_D^4[\lambda]|:=2r_1'[\lambda](1)+r_2'[\lambda](1)-r_1'[\lambda](0)\neq 0,$$

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then

$$\begin{aligned} G_P[\lambda](t,s) =& G_D[\lambda](t,s) + \frac{(r_1[\lambda](t)+r_2[\lambda](t))}{|I-A_D^4[\lambda]|}\left(\frac{\partial}{\partial t}G_D[\lambda](0,s) - \frac{\partial}{\partial t}G_D[\lambda](1,s)\right) \\ =& G_D[\lambda](t,s) + \frac{\left(\frac{\partial}{\partial s}G_D[\lambda](t,1) - \frac{\partial}{\partial s}G_D[\lambda](t,0)\right)}{|I-A_D^4[\lambda]|}\left(\frac{\partial}{\partial t}G_D[\lambda](0,s) - \frac{\partial}{\partial t}G_D[\lambda](1,s)\right), \end{aligned}$$

for all $(t,s) \in J \times J$.

Remark 5.3.11. *Notice that, using* (5.2.16) *and the equality of the previous theorem we have that*

$$G_P[\lambda](1,s) = \frac{1}{|I-A_D^4[\lambda]|}\left[\frac{\partial}{\partial t}G_D[\lambda](0,s) - \frac{\partial}{\partial t}G_D[\lambda](1,s)\right].$$

It is very well-known that the Green's function of periodic problem (5.1.6) *is symmetric, that is,* $G_P[\lambda](t,s) = G_P[\lambda](s,t)$ *for all* $t,\ s \in J$. *So, since* $G_P[\lambda](0,s) = G_P[\lambda](1,s)$ *from Definition 1.1.1 we have that* $G_P[\lambda](s,0) = G_P[\lambda](1,s)$.

We know, from (5.2.21)*, that* $G_P[\lambda](s,0) = \frac{r[\lambda](s)}{K_\lambda}$ *if* $K_\lambda \neq 0$. *Therefore,*

$$r[\lambda](s) = \frac{K_\lambda}{|I-A_D^4[\lambda]|}\left(\frac{\partial G_D[\lambda]}{\partial t}(0,s) - \frac{\partial G_D[\lambda]}{\partial t}(1,s)\right),$$

where $K_\lambda = r'[\lambda](0) - r'[\lambda](1)$, *with* $r[\lambda]$ *defined in* (5.2.19).

Finally, doing the process backwards by studying the Dirichlet problem as a function of the periodic one as follows

$$L[\lambda]\,u(t) = \sigma(t), \ \text{ a.e. } t \in J, \ \ u(0)-u(1) = -u(1), \ \ u'(0)-u'(1) = u'(0)-u'(1)+u(1),$$

we obtain the next theorem.

Theorem 5.3.12. *If operator* $L[\lambda]$ *is nonresonant in* X_P *and* $r_4[\lambda](1) \neq 0$, *then*

$$\begin{aligned} G_D[\lambda](t,s) =& G_P[\lambda](t,s) - \frac{r_4[\lambda](t)}{r_4[\lambda](1)}\,G_P[\lambda](1,s) \\ =& G_P[\lambda](t,s) - \frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\,G_P[\lambda](1,s), \ \ \forall\,(t,s) \in J \times J. \end{aligned}$$

Remark 5.3.13. *From Theorem 5.3.12 we deduce Theorem 5.2.3:*

$$G_P[\lambda](t,s) < G_D[\lambda](t,s) < 0, \ \ \forall (t,s) \in (0,1)\times(0,1), \ \ \forall\,\lambda < \lambda_0^P.$$

5.3.4 Neumann and Mixed Problems

We do the same as before to study the relationship between the Green's functions of Neumann and mixed 1 and 2 problems.

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Theorem 5.3.14. *Assume that $L[\lambda]$ is nonresonant X_N and $r_5[\lambda](0) \neq 0$, then it holds that*

$$\begin{aligned} G_{M_2}[\lambda](t,s) =& G_N[\lambda](t,s) - \frac{r_5[\lambda](t)}{r_5[\lambda](0)}\, G_N[\lambda](0,s) \\ =& G_N[\lambda](t,s) - \frac{G_N[\lambda](t,0)}{G_N[\lambda](0,0)}\, G_N[\lambda](0,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

Remark 5.3.15. *Using previous expression, we have another proof of Corollary 5.2.22. Indeed, we know that $G_N[\lambda] < 0$ for all $\lambda < \lambda_0^N$ and $r_5[\lambda](t) = G_N[\lambda](t,0) < 0$, using the above equality we deduce for all $\lambda < \lambda_0^N$ that*

$$G_N[\lambda](t,s) < G_{M_2}[\lambda](t,s), \quad \textit{for all } (t,s) \in J \times J.$$

Analogously, doing the calculations for the mixed 1 problem as a function of the Neumann problem, we deduce the relationship between Green's functions given in the next theorem.

Theorem 5.3.16. *Assume that $L[\lambda]$ is nonresonant in X_N and $r_6[\lambda](1) \neq 0$, then*

$$\begin{aligned} G_{M_1}[\lambda](t,s) =& G_N[\lambda](t,s) - \frac{r_6[\lambda](t)}{r_6[\lambda](1)}\, G_N[\lambda](1,s) \\ =& G_N[\lambda](t,s) - \frac{G_N[\lambda](t,1)}{G_N[\lambda](1,1)}\, G_N[\lambda](1,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

Remark 5.3.17. *As a consequence of the last equality we will give another proof of Corollary 5.2.24. Indeed, considering that $G_N[\lambda] < 0$ for all $\lambda < \lambda_0^N$ and $r_6[\lambda](t) > 0$, $t \in J$, it follows that for all $\lambda < \lambda_0^N$*

$$G_N[\lambda](t,s) < G_{M_1}[\lambda](t,s) < 0, \quad \textit{for all } (t,s) \in [0,1) \times [0,1).$$

With the reverse process, we can obtain additional realtions for the Green's function of the Neumann and the ones related to mixed problems.

Theorem 5.3.18. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and $r'_{10}[\lambda](1) \neq 0$, then it holds that*

$$\begin{aligned} G_N[\lambda](t,s) =& G_{M_1}[\lambda](t,s) - \frac{r_{10}[\lambda](t)}{r'_{10}[\lambda](1)}\, \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s) \\ =& G_{M_1}[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_{M_1}[\lambda](t,1)}{\frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](1,1)}\, \frac{\partial}{\partial t} G_{M_1}[\lambda](1,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

Theorem 5.3.19. *Assume that $L[\lambda]$ is nonresonant in X_{M_2} and $r_7[\lambda](0) \neq 0$, then it holds that*

$$\begin{aligned} G_N[\lambda](t,s) =& G_{M_2}[\lambda](t,s) - \frac{r_7[\lambda](t)}{r'_7[\lambda](0)}\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s) \\ =& G_{M_2}[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_{M_2}[\lambda](t,0)}{\frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0)}\, \frac{\partial}{\partial t} G_{M_2}[\lambda](0,s), \quad \forall\,(t,s) \in J \times J. \end{aligned}$$

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5.3.5 Periodic and Neumann Problems

In this section we look for a relationship between the Green's functions $G_P[\lambda]$ and $G_N[\lambda]$ following the same steps than in previous sections.

Theorem 5.3.20. *Assume that $L[\lambda]$ is nonresonant in X_P and $r_3'[\lambda](1) \neq 0$, then it holds that*

$$\begin{aligned} G_N[\lambda](t,s) =& G_P[\lambda](t,s) - \frac{r_3[\lambda](t)}{r_3'[\lambda](1)} \frac{\partial}{\partial t} G_P[\lambda](1,s) \\ =& G_P[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_P[\lambda](t,0)}{\frac{\partial^2}{\partial s \partial t} G_P[\lambda](1,0)} \frac{\partial}{\partial t} G_P[\lambda](1,s), \quad \forall\, (t,s) \in J \times J. \end{aligned}$$

Finally doing the reverse process by studying the periodic problem as a function of the Neumann one as follows

$$L[\lambda]\, u(t) = \sigma(t), \quad \text{a.e. } t \in J, \quad u'(0) = u(0) - u(1) + u'(0), \quad u'(1) = u'(0),$$

we deduce the following result.

Theorem 5.3.21. *If operator $L[\lambda]$ is nonresonant in X_N and*

$$|I - A_N^2[\lambda]| := r_5[\lambda](1) - r_5[\lambda](0) + r_6[\lambda](1) - r_6[\lambda](0) \neq 0,$$

then the next equality is fulfilled

$$\begin{aligned} G_P[\lambda](t,s) =& G_N[\lambda](t,s) + \frac{1}{|I - A_N^2[\lambda]|} \Big(r_5[\lambda](t) + r_6[\lambda](t) \Big) \Big(G_N[\lambda](0,s) - G_N[\lambda](1,s) \Big) \\ =& G_N[\lambda](t,s) + \frac{1}{|I - A_N^2[\lambda]|} \Big(G_N[\lambda](t,0) - G_N[\lambda](t,1) \Big) \Big(G_N[\lambda](0,s) - G_N[\lambda](1,s) \Big), \end{aligned}$$

for all $(t,s) \in J \times J$.

5.3.6 Periodic and Mixed Problems

The same arguments of the previous subsections are applicable to the periodic and mixed problems.

Theorem 5.3.22. *If operator $L[\lambda]$ is nonresonant in X_P and*

$$|I - A_P^2[\lambda]| := \Big(1 - r_3[\lambda](0) \Big) \Big(1 + r_4'[\lambda](1) \Big) + r_4[\lambda](0)\, r_3'[\lambda](1) \neq 0,$$

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then the next equality is fulfilled

$$\begin{aligned}
G_{M_1}[\lambda](t,s) =& G_P[\lambda](t,s) + \frac{r_3[\lambda](t)}{|I - A_P^2[\lambda]|}\left(\left(1 + r_4'[\lambda](1)\right) G_P[\lambda](0,s) - r_4[\lambda](0)\,\frac{\partial}{\partial t}G_P[\lambda](1,s)\right)\\
&- \frac{r_4[\lambda](t)}{|I - A_P^2[\lambda]|}\left(r_3'[\lambda](1)\,G_P[\lambda](0,s) + \left(1 - r_3[\lambda](0)\right)\frac{\partial}{\partial t}G_P[\lambda](1,s)\right)\\
=& G_P[\lambda](t,s) - \frac{1}{|I - A_P^2[\lambda]|}\left(1 + \frac{\partial}{\partial t}G_P[\lambda](1,0)\right)\frac{\partial}{\partial s}G_P[\lambda](t,0)\,G_P[\lambda](0,s)\\
&+ \frac{G_P[\lambda](0,0)}{|I - A_P^2[\lambda]|}\,\frac{\partial}{\partial s}G_P[\lambda](t,0)\,\frac{\partial}{\partial t}G_P[\lambda](1,s)\\
&+ \frac{1}{|I - A_P^2[\lambda]|}\,\frac{\partial^2}{\partial s \partial t}G_P(1,0)\,G_P[\lambda](t,0)\,G_P[\lambda](0,s)\\
&- \frac{1}{|I - A_P^2[\lambda]|}\left(1 + \frac{\partial}{\partial s}G_P[\lambda](0,0)\right)G_P[\lambda](t,0)\,\frac{\partial}{\partial t}G_P[\lambda](1,s), \quad \forall\,(t,s) \in J \times J.
\end{aligned}$$

Similarly, we do a similar study of mixed 2 problem as a function of the periodic problem.

Theorem 5.3.23. *If operator $L[\lambda]$ is nonresonant in X_P and*

$$|I - A_P^3[\lambda]| = \Big(1 + r_3[\lambda](1)\Big)\Big(1 - r_4'[\lambda](0)\Big) + r_3'[\lambda](0)\,r_4[\lambda](1) \neq 0,$$

then the next equality is fulfilled

$$\begin{aligned}
G_{M_2}[\lambda](t,s) =& G_P[\lambda](t,s) - \frac{r_3[\lambda](t)}{|I - A_P^3[\lambda]|}\left(\left(1 - r_4'[\lambda](0)\right) G_P[\lambda](1,s) + r_4[\lambda](1)\,\frac{\partial}{\partial t}G_P[\lambda](0,s)\right)\\
&- \frac{r_4[\lambda](t)}{|I - A_P^3[\lambda]|}\left(r_3'[\lambda](0)\,G_P[\lambda](1,s) - \left(1 + r_3[\lambda](1)\right)\frac{\partial}{\partial t}G_P[\lambda](0,s)\right)\\
=& G_P[\lambda](t,s) + \frac{1}{|I - A_P^3[\lambda]|}\left(1 - \frac{\partial}{\partial t}G_P[\lambda](0,0)\right)\frac{\partial}{\partial s}G_P[\lambda](t,0)\,G_P[\lambda](1,s)\\
&+ \frac{G_P[\lambda](1,0)}{|I - A_P^3[\lambda]|}\,\frac{\partial}{\partial s}G_P[\lambda](t,0)\,\frac{\partial}{\partial t}G_P[\lambda](0,s)\\
&+ \frac{1}{|I - A_P^3[\lambda]|}\,\frac{\partial^2}{\partial s \partial t}G_P[\lambda](0,0)\,G_P[\lambda](t,0)\,G_P[\lambda](1,s)\\
&+ \frac{1}{|I - A_P^3[\lambda]|}\left(1 - \frac{\partial}{\partial s}G_P[\lambda](1,0)\right)G_P[\lambda](t,0)\,\frac{\partial}{\partial t}G_P[\lambda](0,s), \quad \forall\,(t,s) \in J \times J.
\end{aligned}$$

Finally for the reverse process, we obtain the next theorems.

Theorem 5.3.24. *Assume that $L[\lambda]$ is nonresonant in X_{M_2} and*

$$|I - A_{M_2}[\lambda]| := \Big(1 - r_7[\lambda](1)\Big)\Big(1 - r_8'(0)\Big) - r_8[\lambda](1)\,r_7'[\lambda](0) \neq 0,$$

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then

$$\begin{aligned}
G_P[\lambda](t,s) =& G_{M_2}[\lambda](t,s) + \frac{1}{|I-A_{M_2}[\lambda]|}\Big(\Big(1-r_8'[\lambda](0)\Big)\, r_7[\lambda](t)\, G_{M_2}[\lambda](1,s) \\
&+ r_8[\lambda](1)\, r_7[\lambda](t)\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s) + r_7'[\lambda](0)\, r_8[\lambda](t)\, G_{M_2}(1,s) \\
&+\Big(1-r_7[\lambda](1)\Big)\, r_8[\lambda](t)\,\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s)\Big) \\
=& G_{M_2}[\lambda](t,s) + \frac{1}{|I-A_{M_2}[\lambda]|}\Big(-\Big(1+\frac{\partial}{\partial t}G_{M_2}[\lambda](0,1)\Big)\,\frac{\partial}{\partial s}G_{M_2}[\lambda](t,0)\, G_{M_2}[\lambda](1,s) \\
&+ G_{M_2}[\lambda](1,1)\,\frac{\partial}{\partial s}G_{M_2}[\lambda](t,0)\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s) \\
&+ \frac{\partial^2}{\partial s\partial t}G_{M_2}[\lambda](0,0)\, G_{M_2}[\lambda](t,1)\, G_{M_2}(1,s) \\
&- \Big(1+\frac{\partial}{\partial s}G_{M_2}[\lambda](1,0)\Big)\, G_{M_2}[\lambda](t,1)\,\frac{\partial}{\partial t}G_{M_2}[\lambda](0,s)\Big), \quad \forall\,(t,s)\in J\times J.
\end{aligned}$$

Theorem 5.3.25. *Assume that $L[\lambda]$ is nonresonant in X_{M_1} and*

$$|I-A_{M_1}[\lambda]| := \Big(1-r_9'[\lambda](1)\Big)\Big(1-r_{10}[\lambda](0)\Big) - r_9[\lambda](0)\, r_{10}'[\lambda](1) \neq 0,$$

then

$$\begin{aligned}
G_P[\lambda](t,s) =& G_{M_1}[\lambda](t,s) + \frac{r_9[\lambda](t)}{|I-A_{M_1}[\lambda]|}\Big((1-r_{10}[\lambda](0))\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s) + r_{10}'[\lambda](1)\, G_{M_1}[\lambda](0,s)\Big) \\
&+ \frac{r_{10}[\lambda](t)}{|I-A_{M_1}[\lambda]|}\Big(r_9[\lambda](0)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s) + \big(1-r_9'[\lambda](1)\big)\, G_{M_1}[\lambda](0,s)\Big) \\
=& G_{M_1}[\lambda](t,s) + \frac{1}{|I-A_{M_1}[\lambda]|}\Big(1+\frac{\partial}{\partial s}G_{M_1}[\lambda](0,1)\Big)\, G_{M_1}[\lambda](t,0)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s) \\
&- \frac{1}{|I-A_{M_1}[\lambda]|}\,\frac{\partial^2}{\partial s\partial t}G_{M_1}[\lambda](1,1)\, G_{M_1}[\lambda](t,0)\, G_{M_1}[\lambda](0,s) \\
&- \frac{G_{M_1}[\lambda](0,0)}{|I-A_{M_1}[\lambda]|}\,\frac{\partial}{\partial s}G_{M_1}[\lambda](t,1)\,\frac{\partial}{\partial t}G_{M_1}[\lambda](1,s) \\
&- \frac{1}{|I-A_{M_1}[\lambda]|}\Big(1-\frac{\partial}{\partial t}G_{M_1}[\lambda](1,0)\Big)\,\frac{\partial}{\partial s}G_{M_1}[\lambda](t,1)\, G_{M_1}[\lambda](0,s), \quad \forall\,(t,s)\in J\times J.
\end{aligned}$$

5.3.7 Initial Problem

We include in this subsection the relationships between Green's function of the initial problem and all the other considered boundary conditions.

Theorem 5.3.26. *Suppose that*

$$|I-A_I^1[\lambda]| := \Big(1+\frac{\partial}{\partial s}G_I[\lambda](1,0)\Big)\,\Big(1-\frac{\partial}{\partial t}G_I[\lambda](1,0)\Big) + G_I[\lambda](1,0)\,\frac{\partial^2}{\partial s\partial t}G_I[\lambda](1,0) \neq 0.$$

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Then the following equality holds

$$\begin{aligned}G_P[\lambda](t,s) =& G_I[\lambda](t,s) - \frac{G_I[\lambda](1,s)}{|I-A_I^1[\lambda]|}\left(\left(1-\frac{\partial}{\partial t}G_I[\lambda](1,0)\right)\frac{\partial}{\partial s}G_I[\lambda](t,0)\right.\\ &+ G_I[\lambda](t,0)\,\frac{\partial^2}{\partial s\partial t}G_I[\lambda](1,0)\Big) + \frac{\frac{\partial}{\partial t}G_I[\lambda](1,s)}{|I-A_I^1[\lambda]|}\left(G_I[\lambda](t,0)\left(1+\frac{\partial}{\partial s}G_I[\lambda](1,0)\right)\right.\\ &\left.- G_I[\lambda](1,0)\,\frac{\partial}{\partial s}G_I[\lambda](t,0)\right),\end{aligned}$$

for all $(t,s)\in J\times J$.

Theorem 5.3.27. *Assume that* $\frac{\partial^2}{\partial s\partial t}G_I[\lambda](1,0)\neq 0$. *Then, the next equality holds*

$$G_N[\lambda](t,s) = G_I[\lambda](t,s) - \frac{\frac{\partial}{\partial s}G_I[\lambda](t,0)}{\frac{\partial^2}{\partial s\partial t}G_I[\lambda](1,0)}\,\frac{\partial}{\partial t}G_I[\lambda](1,s),\quad \forall\,(t,s)\in J\times J.$$

Theorem 5.3.28. *Assume that* $G_I[\lambda](1,0)\neq 0$. *Then, the following equality holds*

$$G_D[\lambda](t,s) = G_I[\lambda](t,s) - \frac{G_I[\lambda](t,0)}{G_I[\lambda](1,0)}\,G_I[\lambda](1,s),\quad \forall\,(t,s)\in J\times J.$$

Theorem 5.3.29. *Assume that* $\frac{\partial}{\partial s}G_I[\lambda](1,0)\neq 0$. *Then, the following equality is fulfilled*

$$G_{M_1}[\lambda](t,s) = G_I[\lambda](t,s) - \frac{\frac{\partial}{\partial s}G_I[\lambda](t,0)}{\frac{\partial}{\partial s}G_I[\lambda](1,0)}\,G_I[\lambda](1,s),\quad \forall\,(t,s)\in J\times J.$$

Theorem 5.3.30. *Assume that* $\frac{\partial}{\partial t}G_I[\lambda](1,0)\neq 0$. *Then, the following equality is fulfilled*

$$G_{M_2}[\lambda](t,s) = G_I[\lambda](t,s) - \frac{G_I[\lambda](t,0)}{\frac{\partial}{\partial t}G_I[\lambda](1,0)}\,\frac{\partial}{\partial t}G_I[\lambda](1,s),\quad \forall\,(t,s)\in J\times J.$$

5.3.8 Final Problem

Finally, we include in this subsection the relationships between Green's function of the final problem and all the other considered problems.

Theorem 5.3.31. *Assume that*

$$|I-A_F^1[\lambda]| := \left(1-\frac{\partial}{\partial s}G_F[\lambda](0,1)\right)\left(1+\frac{\partial}{\partial t}G_F[\lambda](0,1)\right)+G_F[\lambda](0,1)\,\frac{\partial^2}{\partial s\partial t}G_F[\lambda](0,1)\neq 0,$$

then the following equality holds

$$\begin{aligned}G_P[\lambda](t,s) =& G_F[\lambda](t,s) + \frac{G_F[\lambda](0,s)}{|I-A_F^1[\lambda]|}\left(\left(1+\frac{\partial}{\partial t}G_F[\lambda](0,1)\right)\frac{\partial}{\partial s}G_F[\lambda](t,1)\right.\\ &- G_F[\lambda](t,1)\,\frac{\partial^2}{\partial s\partial t}G_F[\lambda](0,1)\Big) - \frac{\frac{\partial}{\partial t}G_F[\lambda](0,s)}{|I-A_F^1[\lambda]|}\left(G_F[\lambda](t,1)\left(1-\frac{\partial}{\partial s}G_F[\lambda](0,1)\right)\right.\\ &\left.+ G_F[\lambda](0,1)\,\frac{\partial}{\partial s}G_F[\lambda](t,1)\right),\end{aligned}$$

for all $(t,s)\in J\times J$.

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Theorem 5.3.32. *Assume that* $\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,1) \neq 0$. *Then, the following equality holds*

$$G_N[\lambda](t,s) = G_F[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_F[\lambda](t,1)}{\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,1)} \frac{\partial}{\partial t} G_F[\lambda](0,s), \quad \forall\, (t,s) \in J \times J.$$

Theorem 5.3.33. *Assume that* $G_F[\lambda](0,1) \neq 0$. *Then, the next equality is fulfilled*

$$G_D[\lambda](t,s) = G_F[\lambda](t,s) - \frac{G_F[\lambda](t,1)}{G_F[\lambda](0,T)} G_F[\lambda](0,s), \quad \forall\, (t,s) \in J \times J.$$

Theorem 5.3.34. *Assume that* $\frac{\partial}{\partial t} G_F[\lambda](0,1) \neq 0$. *Then, it holds that*

$$G_{M_1}[\lambda](t,s) = G_F[\lambda](t,s) - \frac{G_F[\lambda](t,1)}{\frac{\partial}{\partial t} G_F[\lambda](0,1)} \frac{\partial}{\partial t} G_F[\lambda](0,s), \quad \forall\, (t,s) \in J \times J.$$

Theorem 5.3.35. *Assume that* $\frac{\partial}{\partial s} G_F[\lambda](0,1) \neq 0$. *Then, it holds that*

$$G_{M_2}[\lambda](t,s) = G_F[\lambda](t,s) - \frac{\frac{\partial}{\partial s} G_F[\lambda](t,1)}{\frac{\partial}{\partial s} G_F[\lambda](0,1)} G_F[\lambda](0,s), \quad \forall\, (t,s) \in J \times J.$$

5.4 Nonlinear Problem

This section is devoted to studying the existence of solutions of the nonlinear problem

$$\begin{cases} L_n\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \\ B_i(u) = \delta_i\, C_i(u), \quad i = 1, \ldots, n, \end{cases} \tag{5.4.38}$$

where L_n and B_I are defined in (1.1.1) and $C_i : C(I) \to \mathbb{R}$ are linear and continuous.

Such result can be deduced from the existence of solutions of the related homogeneous problem:

$$\begin{cases} L_n\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \\ B_i(u) = 0, \quad i = 1, \ldots, n. \end{cases} \tag{5.4.39}$$

Under certain conditions on the data of the nonlinear problem, the results of existence of solutions will be deduced from Schaefer's fixed point Theorem 1.3.5 of integral operators defined in Banach spaces. Moreover, an estimate of the distance between the solutions of problems (5.4.38) and (5.4.39) is also given.

We will assume that the nonlinear part of problem (5.4.38) satisfies the following regularity conditions:

(H_1) For $n \geq 2$, the function $f : J \times \mathbb{R} \to \mathbb{R}$ is a $L^1(J) -$ Carathéodory function.

For $n = 1$, the function $f : J \times \mathbb{R} \to \mathbb{R}$ is $L^\infty(J)-$Carathéodory function.

(H_2) $\exists K \in L^1(J), K \geq 0$ such that

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$$|f(t,x) - f(t,y)| \leq K(t)\,|x-y|, \quad \text{for all } x,y \in \mathbb{R} \text{ and } \text{ a.e. } t \in J.$$

Let us define $X \equiv (C(J), \|\cdot\|_\infty)$ the real Banach space endowed with the supremum norm

$$\|u\|_\infty = \sup_{t\in J} |u(t)|, \quad \text{for all } u \in X.$$

We will denote by u_A and u_B the solutions of problems (5.4.38) and (5.4.39) respectively.

As it has been said in Section 1.3, these solutions are given by the following expressions

$$u_A(t) = \int_0^1 G(t,s,\delta_1,\ldots,\delta_n)\, f(s,u_A(s))\, ds,$$
$$u_B(t) = \int_0^1 g(t,s)\, f(s,u_B(s))\, ds,$$

where G and g are the Green's functions related to the linear problems

$$\begin{cases} L_n\, u(t) = 0, \ \text{ a.e.} & t \in J, \\ B_i(u) = \delta_i\, C_i(u), & i = 1,\ldots,n, \end{cases}$$

and

$$\begin{cases} L_n\, u(t) = 0, \ \text{ a.e. } t \in J, \\ B_i(u) = 0, & i = 1,\ldots,n, \end{cases}$$

respectively. In addition, from Theorem 2.2.2 we have that both Green's functions are linked by the following equality

$$G(t,s,\delta_1,\ldots,\delta_n) = g(t,s) + \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i\, b_{ij}\, \omega_i(t)\, C_j(g(\cdot,s)), \quad t,\ s \in J, \tag{5.4.40}$$

relating the Green's functions G and g. This expression is a particular case of (2.2.7) with $M = 0$.

Let us define

$$K^1 = \max_{t\in J} \int_0^1 |g(t,s)|\, K(s)\, ds,$$
$$K^2_{ij} = \max_{t\in J} |\omega_i(t)| \int_0^1 |C_j(g(\cdot,s))|\, K(s)\, ds, \quad \forall i,j = 1,\ldots,n,$$
$$K^3_{ij} = \max_{t\in J} |\omega_i(t)| \int_0^1 |C_j(g(\cdot,s))\, f(s,0)|\, ds, \quad \forall i,j = 1,\ldots,n,$$
$$P = \max_{t\in J} \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)|\, K(s)\, ds,$$
$$Q = \max_{t\in J} \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)\, f(s,0)|\, ds.$$

We assume that the following condition is fulfilled:

(H_3) $\quad K^1 < 1.$

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Theorem 5.4.1. *If conditions (H_2) and (H_3) hold, then the following inequality is fulfilled*

$$\|u_B - u_A\|_\infty \leq \frac{1}{1-K^1}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|K_{ij}^2\,\|u_A\|_\infty + \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,K_{ij}^3\right). \quad (5.4.41)$$

Proof. Using (5.4.40) we have that

$$\begin{aligned} u_B(t) - u_A(t) &= \int_0^1 g(t,s)\, f(s,u_B(s))\, ds - \int_0^1 G(t,s,\delta_1,\ldots,\delta_n)\, f(s,u_A(s))\, ds \\ &= \int_0^1 g(t,s)\ \left(f(s,u_B(s)) - f(s,u_A(s))\right)\ ds \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\, b_{ij}\,\omega_i(t)\ \int_0^1 C_j(g(\cdot,s))\,\left(f(s,u_A(s)) - f(s,0)\right)\ ds \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\, b_{ij}\,\omega_i(t)\ \int_0^1 C_j(g(\cdot,s))\, f(s,0)\, ds. \end{aligned}$$

Then, for all $t \in J$, from (H_2), we infer that

$$\begin{aligned} |u_B(t) - u_A(t)| \leq & \|u_B - u_A\|_\infty \int_0^1 |g(t,s)|\, K(s)\, ds \\ & + \|u_A\|_\infty \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,|\omega_i(t)|\ \int_0^1 |C_j(g(\cdot,s))|\, K(s)\, ds \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,|\omega_i(t)|\ \int_0^1 |C_j(g(\cdot,s))\, f(s,0)|\, ds. \end{aligned}$$

Therefore,

$$\|u_B - u_A\|_\infty \leq K^1\,\|u_B - u_A\|_\infty + \|u_A\|_\infty \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,K_{ij}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,K_{ij}^3,$$

that is, using (H_3),

$$\|u_B - u_A\|_\infty \leq \frac{1}{1-K^1}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|K_{ij}^2\,\|u_A\|_\infty + \sum_{i=1}^{n}\sum_{j=1}^{n}|\delta_i\, b_{ij}|\,K_{ij}^3\right).$$

□

As a consequence, we arrive at the next corollary.

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Corollary 5.4.2. *If conditions* (H_2) *and* (H_3) *hold, then the following inequalities are fulfilled*

$$\|u_B\|_\infty \leq \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}| K_{ij}^2 - K^1 + 1}{1-K^1}\, \|u_A\|_\infty + \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}|\, K_{ij}^3}{1-K^1},$$

$$\|u_B\|_\infty \geq \frac{1 - K^1 - \sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}| K_{ij}^2}{1-K^1}\, \|u_A\|_\infty - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} |\delta_i\, b_{ij}|\, K_{ij}^3}{1-K^1}.$$

Proof. The proof is an immediate consequence of (5.4.41) and the following inequality

$$\Big| \|u_B\|_\infty - \|u_A\|_\infty \Big| \leq \|u_B - u_A\|_\infty.$$

□

We will apply Schaefer's fixed point Theorem 1.3.5 to operator $T : X \to X$ given by

$$T\,u(t) := \int_0^1 G(t, s, \delta_1, \ldots, \delta_n)\, f(s, u(s))\, ds, \quad t \in J, \tag{5.4.42}$$

to guarantee the existence of a solution of problem (5.4.38).

Theorem 5.4.3. *Assume that* (H_1) *and* (H_2) *hold and* $P < 1$. *Then problem* (5.4.38) *has at least one solution* $u \in X$.

Proof. First, it is very well-known that the fixed points of operator T defined in (5.4.42) coincide with the solutions of problem (5.4.38).

Let us see, using standard techniques, that the operator $T : X \to X$ is well-defined. We suppose that $n \geq 2$. For the case $n = 1$ the proof is analogous.

Let $u \in X$ and $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} t_k = \tilde{t} \in J$. By the Definition 1.1.1 we have that the function $G(\cdot, s, \delta_1, \ldots, \delta_n)$ is uniformly continuous in J. Thus,

$$\lim_{k\to\infty} G(t_k, s, \delta_1, \ldots, \delta_n)\, f(s, u(s)) = G(\tilde{t}, s, \delta_1, \ldots, \delta_n)\, f(s, u(s)).$$

Since f is a $L^1(J)-$Carathéodory function (Definition 1.3.1) we obtain

$$|\, G(t, s, \delta_1, \ldots, \delta_n)\, f(s, u(s))\, | \leq |\, G(t, s, \delta_1, \ldots, \delta_n)\, |\ \phi_{\|u\|_\infty}(s).$$

Moreover, since $G(t, s, \delta_1, \ldots, \delta_n)$ is continuous on $J\times J$ (Definition 1.1.1), it is bounded on $J \times J$ by some constant $\tilde{M}$. Thus,

$$|G(t, s, \delta_1, \ldots, \delta_n)\, f(s, u(s))| \leq \tilde{M}\phi_{\|u\|_\infty}(s), \quad \text{a.e. } s \in J.$$

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Therefore, by Lebesgue's Dominated Convergence Theorem we obtain that

$$\begin{aligned}\lim_{k\to\infty} T\,u(t_k) &= \lim_{k\to\infty}\int_0^1 G(t_k,s,\delta_1,\ldots,\delta_n)\,f(s,u(s))\,ds\\ &= \int_0^1 \lim_{k\to\infty} G(t_k,s,\delta_1,\ldots,\delta_n)\,f(s,u(s))\,ds\\ &= \int_0^1 G(\tilde{t},s,\delta_1,\ldots,\delta_n)\,f(s,u(s))\,ds\\ &= T\,u(\tilde{t}).\end{aligned}$$

Thus, $T\,u \in X$.

Now, we show that operator T is compact.

Let $(u_k)_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$ in $C(J)$. Then, $\|u_k\|_\infty \leq R$ for all $k \in \mathbb{N}$ for some $R \in \mathbb{R}^+$.

Using Definition 1.3.1, we deduce that

$$\lim_{k\to\infty} f(s,u_k(s)) = f(s,u(s)), \quad \text{a.e. } s \in J.$$

Moreover,

$$|\,G(t,s,\delta_1,\ldots,\delta_n)\,f(s,u_k(s))\,| \leq \tilde{M}\phi_R(s), \quad \text{a.e. } s \in J.$$

Again, by Lebesgue's Dominated Convergence Theorem we deduce that

$$\begin{aligned}\lim_{k\to\infty} T\,u_k(t) &= \lim_{k\to\infty}\int_0^1 G(t,s,\delta_1,\ldots,\delta_n)\,f(s,u_k(s))\,ds\\ &= \int_0^1 \lim_{k\to\infty} G(t,s,\delta_1,\ldots,\delta_n)\,f(s,u_k(s))\,ds\\ &= \int_0^1 G(t,s,\delta_1,\ldots,\delta_n)\,f(s,u(s))\,ds\\ &= T\,u(t).\end{aligned}$$

Thus, operator T is continuous.

Next, we will prove that T maps bounded sets into relatively compact sets. Let $H \subset X$ be a bounded set. Since H is bounded, there exists $r \in \mathbb{R}$, $r > 0$ such that $\|u\|_\infty \leq r$ for all $u \in H$. Then,

$$\begin{aligned}|T\,u(t)| \leq& \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)||f(s,u(s)) - f(s,0)|\,ds\\ &+ \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)||f(s,0)|\,ds\\ \leq& \|u\|_\infty \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)|\,K(s)\,ds + \int_0^1 |G(t,s,\delta_1,\ldots,\delta_n)||f(s,0)|\,ds.\end{aligned}$$

So, for all $u \in H$, we have that

$$\|T\,u\|_\infty \leq r\,P + Q, \tag{5.4.43}$$

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that is, $T(H)$ is bounded.

Let us show now the equicontinuity of T. For all $t \in J$ and $u \in H$, we have that

$$\begin{aligned}|(T\,u)'(t)| &= \left| \int_0^1 \frac{\partial}{\partial t} G(t,s,\delta_1,\ldots,\delta_n)\, f(s,u(s))\, ds \right| \\ &\leq \int_0^1 \left| \frac{\partial}{\partial t} G(t,s,\delta_1,\ldots,\delta_n) \right| |f(s,u(s))|\, ds \\ &\leq \int_0^1 \left| \frac{\partial}{\partial t} G(t,s,\delta_1,\ldots,\delta_n) \right| \phi_r(s)\, ds.\end{aligned}$$

If $n \geq 2$, then the regularity of the Green's function $G(t,s,\delta_1,\ldots,\delta_n)$ allows us guarantee that there exists $M \in \mathbb{R}$, $M > 0$, such that $\left|\frac{\partial}{\partial t} G(t,s,\delta_1,\ldots,\delta_n)\right| \leq M$. Therefore,

$$\int_0^1 \left| \frac{\partial}{\partial t} G(t,s,\delta_1,\ldots,\delta_n) \right| \phi_r(s)\, ds \leq M \int_0^1 \phi_r(s)\, ds.$$

So, for all $t_1, t_2 \in J,\ \ t_1 < t_2$, we infer that

$$|(T\,u)(t_2) - (T\,u)(t_1)| = \left| \int_{t_1}^{t_2} (T\,u)'(s) ds \right| \leq \int_{t_1}^{t_2} |(T\,u)'(s)| ds \leq N(t_2 - t_1).$$

If $n = 1$, then the regularity of the Green's function $G(t,s,\delta_1)$ allows us to ensure that there exists $\tilde{N} \in \mathbb{R}$, $\tilde{N} > 0$ such that $\int_0^1 |G(t,s,\delta_1)|\, \phi_r(s)\, ds \leq \tilde{N}$. Therefore,

$$\int_0^1 \left| \frac{\partial}{\partial t} G(t,s,\delta_1) \right| \phi_r(s)\, ds = \int_0^1 |a_1(t)||G(t,s,\delta_1)|\, \phi_r(s)\, ds \leq \tilde{N}\, |a_1(t)|.$$

Then, for all $t_1, t_2 \in J,\ \ t_1 < t_2$, we have that

$$|(T\,u)(t_2) - (T\,u)(t_1)| = \left| \int_{t_1}^{t_2} (T\,u)'(s) ds \right| \leq \int_{t_1}^{t_2} |(T\,u)'(s)| ds \leq \tilde{N} \int_{t_1}^{t_2} |a_1(s)|\, ds.$$

Thus, $T(H)$ is an equicontinuous set in X. By Arzelà-Ascoli's Theorem 1.3.4, we deduce that $T(H)$ is relatively compact, that is, T is a compact operator.

Let $u \in X$ be such that $u = \mu\, T\, u$ for some $0 \leq \mu \leq 1$. Then, using (5.4.43) we have that

$$\|u\|_\infty = \mu \|T\,u\|_\infty \leq \|T\,u\|_\infty \leq \|u\|_\infty\, P + Q.$$

Thus

$$\|u\|_\infty \leq \frac{Q}{1-P}.$$

Therefore, applying Schaefer's Theorem 1.3.5, we conclude that problem (5.4.38) has at least one solution $u \in X$. □

Remark 5.4.4. *We note that by definition of X, $(T\,u)$ is not necessarily derivable. However, $(T\,u)'$ always exists because of the regularity of the Green's function.*

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The previous existence and estimation results can be applied to all the different second order nonlinear problems coupled to the boundary conditions considered along the chapter. In particular, we will apply the above results to the nonlinear second order Dirichlet probelm.

Suppose that, there exists u_D a solution of the nonlinear Dirichlet problem

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) = u(1) = 0 \tag{5.4.44}$$

and u_P a solution of the nonlinear periodic problem

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) - u(1) = u'(0) - u'(1) = 0.$$

By definition of the Green's functions, we have that

$$u_D(t) = \int_0^1 G_D[\lambda](t,s)\, f(s, u_D(s))\, ds$$

and

$$u_P(t) = \int_0^1 G_P[\lambda](t,s)\, f(s, u_P(s))\, ds.$$

We know from Theorem 5.3.12 that

$$G_D[\lambda](t,s) = G_P[\lambda](t,s) - \frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\, G_P[\lambda](1,s), \quad \forall\, (t,s) \in J \times J.$$

Let us define

$$\begin{aligned}
K_1^P &= \max_{t\in J} \int_0^1 |G_P[\lambda](t,s)|\, K(s)\, ds,\\
K_2^P &= \max_{t\in J} \left|\frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\right| \int_0^1 |G_P[\lambda](1,s))|\, K(s)\, ds,\\
K_3^P &= \max_{t\in J} \left|\frac{G_P[\lambda](t,0)}{G_P[\lambda](1,0)}\right| \int_0^1 |G_P[\lambda](1,s)\, f(s,0)|\, ds,\\
P_D &= \max_{t\in J} \int_0^1 |G_D[\lambda](t,s)|\, K(s)\, ds,\\
Q_D &= \max_{t\in J} \int_0^1 |G_D[\lambda](t,s)\, f(s,0)|\, ds.
\end{aligned}$$

As a direct consequence of Theorem 5.4.1 and Corollary 5.4.2 we arrive at the following results:

Theorem 5.4.5. *Suppose that (H_2) holds and $K_1^P < 1$, then the following inequality is fulfilled*

$$\|u_D - u_P\|_\infty \leq \frac{1}{1 - K_1^P} \left(K_2^P\, \|u_D\|_\infty + K_3^P\right). \tag{5.4.45}$$

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Corollary 5.4.6. *Assume that* (H_2) *holds and* $K_1^P < 1$. *Then, the following inequalities are fulfilled*

$$\|u_P\|_\infty \leq \frac{K_2^P - K_1^P + 1}{1 - K_1^P} \|u_D\|_\infty + \frac{K_3^P}{1 - K_1^P},$$
$$\|u_P\|_\infty \geq \frac{1 - K_1^P - K_2^P}{1 - K_1^P} \|u_D\|_\infty - \frac{K_3^P}{1 - K_1^P}.$$

Theorem 5.4.7. *Assume that* (H_1) *and* (H_2) *hold and* $P_D < 1$. *Then the nonlinear Dirichlet problem* (5.4.44) *has at least one solution.*

Analogously, we can deduce the existence of the nonlinear mixed 2 problem

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) = u'(1) = 0 \tag{5.4.46}$$

after proving the existence of solution of the nonlinear Neumann problem

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u'(0) = u'(1) = 0.$$

From Theorem 5.3.14 we know that the following equality holds:

$$G_{M_2}[\lambda](t,s) = G_N[\lambda](t,s) - \frac{G_N[\lambda](t,0)}{G_N[\lambda](0,0)}\, G_N[\lambda](0,s), \quad \forall\, (t,s) \in J \times J.$$

Let us define

$$K_1^N = \max_{t \in J} \int_0^1 |G_N[\lambda](t,s)|\, K(s)\, ds,$$
$$K_2^N = \max_{t \in J} \left| \frac{G_N[\lambda](t,0)}{G_N[\lambda](0,0)} \right| \int_0^1 |G_N[\lambda](0,s)|\, K(s)\, ds,$$
$$K_3^N = \max_{t \in J} \left| \frac{G_N[\lambda](t,0)}{G_N[\lambda](0,0)} \right| \int_0^1 |G_N[\lambda](0,s)\, f(s,0)|\, ds,$$
$$P_{M_2} = \max_{t \in J} \int_0^1 |G_{M_2}[\lambda](t,s)|\, K(s)\, ds,$$
$$Q_{M_2} = \max_{t \in J} \int_0^1 |G_{M_2}[\lambda](t,s)\, f(s,0)|\, ds.$$

Theorem 5.4.8. *Suppose that* (H_2) *holds and* $K_1^N < 1$, *then the following inequality is fulfilled*

$$\|u_{M_2} - u_N\|_\infty \leq \frac{1}{1 - K_1^N} \left(K_2^N \|u_{M_2}\|_\infty + K_3^N \right).$$

Corollary 5.4.9. *Assume that* (H_2) *holds and* $K_1^N < 1$. *Then, the following inequalities are fulfilled*

$$\|u_N\|_\infty \leq \frac{K_2^N - K_1^N + 1}{1 - K_1^N} \|u_{M_2}\|_\infty + \frac{K_3^N}{1 - K_1^N},$$
$$\|u_N\|_\infty \geq \frac{1 - K_1^N - K_2^N}{1 - K_1^N} \|u_{M_2}\|_\infty - \frac{K_3^N}{1 - K_1^N}.$$

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Theorem 5.4.10. *Assume that* (H_1) *and* (H_2) *hold and* $P_{M_2} < 1$. *Then the nonlinear mixed 2 problem* (5.4.46) *has at least one solution.*

The same previous arguments can be applied to the following nonlinear problems

- Nonlinear Neumann problem:

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u'(0) = u'(1) = 0,$$

- Nonlinear mixed 1 problem:

$$L[\lambda]\, u(t) = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u'(0) = u(1) = 0,$$

- Nonlinear periodic problem:

$$L[\lambda]\, u(t) \; = f(t, u(t)), \quad \text{a.e. } t \in J, \quad u(0) = u(1), \; u'(0) = u'(1),$$

using the formulas that relate Green's functions obtained in the previous section.

In the sequel, we present an example to illustrate our results.

Example 5.4.11. *Consider the following equation*

$$u''(t) - \, u(t) = \frac{c}{\sqrt{t}}\, e^{-u^2(t)}, \quad \textit{a.e. } t \in J, \quad \textit{and } \; c > 0.$$

In this case, $f(t,u) = \frac{c}{\sqrt{t}}\, e^{-u^2}$ *is* $L^1(J)$*– Carathéodory function and* $f(t,0) = \frac{c}{\sqrt{t}} \neq 0$ *for all* $t \in (0,1]$. *Moreover,*

$$\Big| \frac{\partial}{\partial u} f(t,u) \Big| = \frac{2c}{\sqrt{t}} |u| e^{-u^2} \leq c \sqrt{\frac{2}{e\, t}}, \quad \textit{a.e. } t \in J.$$

Therefore, f *satisfies condition* (H_2) *with* $K(t) = c\sqrt{\frac{2}{e\, t}}$ *for a.e.* $t \in J$.

We have that the Green's function of the periodic problem is given by

$$G_P(t,s) = \begin{cases} \dfrac{e^{s-t+1} + e^{t-s}}{2\,(1-e)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{e^{t-s+1} + e^{s-t}}{2\,(1-e)}, & 0 \leq t < s \leq 1, \end{cases}$$

and that of the Dirichlet problem is

$$G_D(t,s) = \begin{cases} -\dfrac{(e^{2\,s} - 1)\,(e^2 - e^{2\,t})\, e^{-(s+t)}}{2\,(e^2-1)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(e^{2\,(s-1)} - 1)\,(e^{2\,t} - 1)\, e^{-(s+t-2)}}{2\,(e^2-1)}, & 0 \leq t < s \leq 1. \end{cases}$$

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With the notation used in Theorem 5.4.5 and by numerical approach, it can be seen that

$$
\begin{aligned}
K_1^P &= \max_{t\in J} \int_0^1 |G_P(t,s)|\, K(s)\, ds \approx 1.7472\, c,\\
K_2^P &= \max_{t\in J} \left|\frac{G_P(t,0)}{G_P(1,0)}\right| \int_0^1 |G_P(1,s))|\, K(s)\, ds \approx 1.744\, c,\\
K_3^P &= \max_{t\in J} \left|\frac{G_P(t,0)}{G_P(1,0)} \int_0^1 G_P[\lambda](1,s)\, f(s,0)\, ds\right| \approx 2.033\, c,\\
P_P &= \max_{t\in J} \int_0^1 |G_P(t,s)|\, K(s)\, ds \approx 1.7472\, c,\\
Q_P &= \max_{t\in J} \int_0^1 |G_P(t,s)\, f(s,0)|\, ds \approx 2.0369\, c,\\
P_D &= \max_{t\in J} \int_0^1 |G_D(t,s)|\, K(s)\, ds \approx 0.1651\, c,\\
Q_D &= \max_{t\in J} \int_0^1 |G_D(t,s)\, f(s,0)|\, ds \approx 0.179\, c.
\end{aligned}
$$

Then, the conditions $K_1^P < 1$ *and* $P_D < 1$ *are fulfilled if and only if*

$$
0 < c < \min\left\{\frac{1}{0.1651}, \frac{1}{1.7472}\right\} \approx 0.572344.
$$

Therefore if $0 < c < 0.572344$*, then by Theorem 5.4.7 there is at least one solution* u_D *of the Dirichlet problem*

$$
u''(t) - u(t) = \frac{c}{\sqrt{t}}\, e^{-u^2(t)}, \quad \textit{a.e. } t \in J, \quad u(0) = u(1) = 0.
$$

By the proof of Theorem 5.4.3 we have that

$$
\|u_P\|_\infty \leq \frac{Q_P}{1 - P_P} \approx \frac{2.0369\, c}{1 - 1.7472\, c}
$$

and

$$
\|u_D\|_\infty \leq \frac{Q_D}{1 - P_D} \approx \frac{0.179\, c}{1 - 0.1651\, c}.
$$

As a consequence, we deduce that

$$
\begin{aligned}
\|u_P - u_D\|_\infty \leq \|u_P\|_\infty + \|u_D\|_\infty &\leq \frac{2.0369\, c}{1 - 1.7472\, c} + \frac{0.179\, c}{1 - 0.1651\, c}\\
&= \frac{c\,(7.68176 - 2.25\, c)}{c^2 - 6.62928\, c + 3.46665} := \gamma(c).
\end{aligned}
\tag{5.4.47}
$$

On the other hand, if $0 < c < 0.572344$ *applying inequality* (5.4.45) *we obtain the following*

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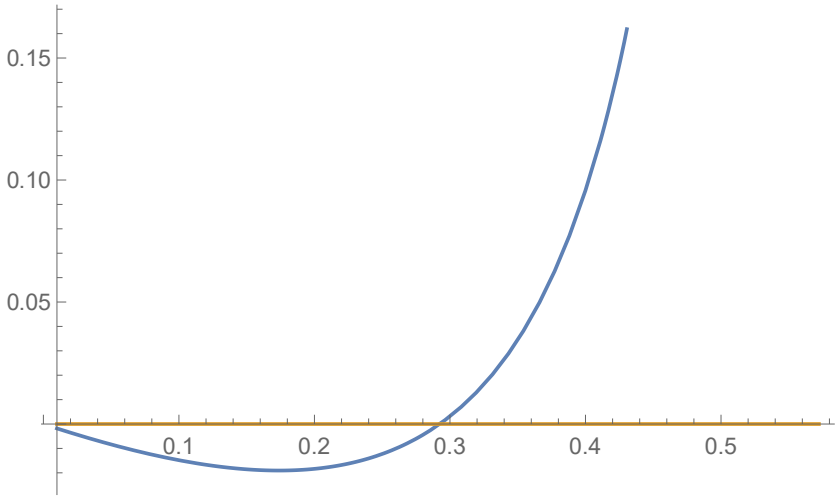

Figure 5.4.1: Representation of the function $\psi - \gamma$ on the interval $(0, 0.572344)$.

estimate of the distance between the solutions:

$$\begin{aligned}\|u_P - u_D\|_\infty &\leq \frac{1}{1-K_1^P}\left(K_2^P\,\|u_D\|_\infty + K_3^P\right) \leq \frac{1}{1-K_1^P}\left(K_2^P\,\frac{0.179\,c}{1-0.1651\,c} + K_3^P\right)\\ &\approx \frac{1}{1-1.7472\,c}\left(1.744\,c\,\frac{0.179\,c}{1-0.1651\,c} + 2.033\,c\right)\\ &= \frac{c\,(7.0477 - 0.0813703\,c)}{c^2 - 6.62928\,c + 3.46665} := \psi(c).\end{aligned} \tag{5.4.48}$$

Comparing (5.4.47) *and* (5.4.48) *(see Figure 5.4.1), we have that the estimate* (5.4.48) *is better than* (5.4.47) *for* $0 < c < 0.2878$ *and worse for* $0.2878 < c < 0.572344$.

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Chapter 6
Spectra of the Hill's Equation Subject to Different Boundary Value Conditions

In this chapter we will continue with the study of the Hill's equation

$$u''(t) + a(t)\,u(t) = 0$$

coupled to the different boundary conditions considered in previous chapter, and we will characterize its spectrum in terms of the corresponding Green's functions.

In particular, from the relations between Green's functions proved in previous chapter, we will deduce some connections between various spectra and some other sets characterized as the zeros of certain functions. More concisely, for each pair of boundary conditions, we will prove a relation of the following form

$$S_{1,2} \subset \Lambda_1 \subset \Lambda_2 \sqcup S_{1,2},$$

where Λ_1 and Λ_2 will be the spectra of the Hill's equation related to two different boundary conditions and $S_{1,2}$ will be some appropriate set, characterized as the zeros of a suitable function. Here, by $\sqcup$ we denote the disjoint union of two sets. In many of the considered cases, when the spectra Λ_1 and Λ_2 are disjoint, we immediately characterize $\Lambda_1 = S_{1,2}$. This last characterization shows to be very useful from a numerical point of view, as it is simple to compute the zeros of the function which defines the set $S_{1,2}$. Some of the characterizations proved using this technique were very previously known (some of them can be found, for instance, in [70]) although some others are, as far as we are concerned, new in the literature.

Moreover, as an application, we will deduce the existence of positive solutions to a given nonlinear problem. The results are derived from Krasnoselskii-Zabreico fixed point Theorem 1.3.10.

The topic of nonlinear boundary value problems has been widely considered since long time in the literature, with special attention to the existence of solutions for such problems. It is in this context where the corresponding spectral theory arises, due to the fact that topological methods, such as degree theory, fixed point index in cones or lower and upper functions method, are mainly based on the invertibility of certain linear operators, for which the eigenvalues of such operators need to be considered.

The reader can find some classical and recent references, in which the spectral theory is considered, among others, the following ones [2, 9, 29, 30, 32, 45, 49, 51, 53, 57, 58, 70, 72, 75, 81, 92, 94, 97].

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This chapter continues the work of the lines initiated in Chapter 2 and [9]. It is important to point out that the results obtained here are completely different to the ones used in the reference [9], where the technique used by the authors is based in the extension (even or odd, depending on the boundary conditions) of the potential $a(t)$ defined on $[0,T]$ with $T>0$ to the intervals $[0,2T]$ and $[0,4T]$.

The chapter is divided in the following way: Section 6.1 contains the decomposition and characterization of all the aforementioned spectra. Last Section 6.2 shows an application of the obtained results to prove the existence of solution of a nonlinear problems related to the Hill's equation coupled to different boundary conditions.

All the results given in this chapter are collected in [22].

6.1 Decomposition of the Spectra

In this section, using Theorem 2.2.2 we will show the relation between the spectra of the problems

$$\begin{cases} T_n[\lambda]\,u(t)=\sigma(t), \ \text{ a.e. } \ t\in[0,T], \\ \qquad B_i(u)=\delta_i\,C_i(u), \quad i=1,\dots,n, \end{cases} \tag{6.1.1}$$

and

$$\begin{cases} T_n[\lambda]\,u(t)=\sigma(t), \ \text{ a.e. } \ t\in[0,T], \\ \qquad B_i(u)=0, \qquad\qquad i=1,\dots n, \end{cases} \tag{6.1.2}$$

with $\sigma\in L^1([0,T])$, $T_n[\lambda]$, $B_i(u)$, δ_i and $C_i(u)$ defined in Section 2.1. In this case, let us suppose that the coefficients of the operator $T_n[\lambda]$ are such that $a_i:[0,T]\to\mathbb{R}$, $a_i\in L^\alpha([0,T])$, $\alpha\geq 1$, for $i=1,\dots,n$.

After proving an inclusion for the general operator $T_n[\lambda]$, we apply it to the problems discussed in Chapter 5 related to operator $L[\lambda]$ defined in (5.1.1).

We will denote by Λ_H, Λ_{NH}, Λ_D, Λ_P, Λ_N, Λ_{M_1} and Λ_{M_2}, the set of eigenvalues of problems (6.1.2), (6.1.1), (5.1.3), (5.1.6), (5.1.2), (5.1.4) and (5.1.5), respectively.

Note that the spectra of problems (5.1.7) and (5.1.8) (denoted by Λ_I and Λ_F) are empty, that is, $\Lambda_I=\Lambda_F=\varnothing$.

Also, we denote by $\omega_I^i[\lambda]$, $\omega_F^i[\lambda]$, $\omega_D^i[\lambda]$, $\omega_P^i[\lambda]$, $\omega_N^i[\lambda]$, $\omega_{M_1}^i[\lambda]$ and $\omega_{M_2}^i[\lambda]$, $i=1,2$, the functions ω_i, $i=1,2$ of Lemma 1.1.8 of the initial, final, Dirichlet, periodic, Neumann, mixed 1 and mixed 2 problems, respectively. Moreover, it has been shown in Section 5.1, that the expressions of functions $\omega_D^i[\lambda]$, $\omega_P^i[\lambda]$, $\omega_N^i[\lambda]$, $\omega_{M_1}^i[\lambda]$ and $\omega_{M_2}^i[\lambda]$, $i=1,2$ are given by

$$\begin{aligned}
&\omega_D^1[\lambda](t) \equiv r_1[\lambda](t)=-\tfrac{\partial}{\partial s}G_D[\lambda](t,0), && \omega_D^2[\lambda](t) \equiv r_2[\lambda](t)=\tfrac{\partial}{\partial s}G_D[\lambda](t,T),\\
&\omega_P^1[\lambda](t) \equiv r_3[\lambda](t)=-\tfrac{\partial}{\partial s}G_P[\lambda](t,0), && \omega_P^2[\lambda](t) \equiv r_4[\lambda](t)=G_P[\lambda](t,0),\\
&\omega_N^1[\lambda](t) \equiv r_5[\lambda](t)=G_N[\lambda](t,0), && \omega_N^2[\lambda](t) \equiv r_6[\lambda](t)=-G_N[\lambda](t,T),\\
&\omega_{M_2}^1[\lambda](t) \equiv r_7[\lambda](t)=-\tfrac{\partial}{\partial s}G_{M_2}[\lambda](t,0), && \omega_{M_2}^2[\lambda](t) \equiv r_8[\lambda](t)=-G_{M_2}[\lambda](t,T),\\
&\omega_{M_1}^1[\lambda](t) \equiv r_9[\lambda](t)=G_{M_1}[\lambda](t,0), && \omega_{M_1}^2[\lambda](t) \equiv r_{10}[\lambda](t)=-\tfrac{\partial}{\partial s}G_{M_1}[\lambda](t,T),\\
&\omega_I^1[\lambda](t) \equiv r_{11}[\lambda](t)=-\tfrac{\partial}{\partial s}G_I[\lambda](t,0), && \omega_I^2[\lambda](t) \equiv r_{12}[\lambda](t)=G_I[\lambda](t,0),\\
&\omega_F^1[\lambda](t) \equiv r_{13}[\lambda](t)=\tfrac{\partial}{\partial s}G_F[\lambda](t,T), && \omega_F^2[\lambda](t) \equiv r_{14}[\lambda](t)=-G_F[\lambda](t,T).
\end{aligned}$$

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Let us define

$$S_{NH} = \Big\{\lambda \in \mathbb{R}\backslash\Lambda_H \ / \ \ \det(I_n - A) = 0\Big\},$$

with A the matrix defined on Theorem 2.2.2.

Next, we formulate the main result that relates the spectra of problems (6.1.1) and (6.1.2) through an inclusion relationship.

Theorem 6.1.1. *It holds that*

$$S_{NH} \subset \Lambda_{NH} \subset \Lambda_H \sqcup S_{NH}$$

where $\sqcup$ *denotes the disjoint union.*

Proof. First, let us see that $S_{NH} \subset \Lambda_{NH}$. Suppose that $\lambda \in S_{NH}$. Since $\lambda \notin \Lambda_H$ we have, using formula (2.2.8), that for $\sigma = 0$ the solutions of problem (6.1.1) satisfy the following expression

$$u(t) = \sum_{i=1}^{n} \omega_i(t)\delta_i\, C_i(u), \quad t \in [0,T]. \tag{6.1.3}$$

Applying linear continuous operators C_j on both sides of (6.1.3) we infer that

$$C_j\,(u) = C_j\left(\sum_{i=1}^{n} \delta_i\, \omega_i(t)\, C_i\,(u)\right) = \sum_{i=1}^{n} \delta_i\, C_j\,(\omega_i)\, C_i\,(u)\,, \quad j = 1,\ldots,n,$$

from which we deduce that

$$C_j(u) - \sum_{i=1}^{n} \delta_i\, C_j(\omega_i)\, C_i(u) = 0, \quad j = 1,\ldots,n.$$

Therefore, we arrive at the following system of equations

$$(I_n - A)\begin{pmatrix} C_1(u) \\ C_2(u) \\ \vdots \\ C_n(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\lambda \in S_{NH}$ the previous system has infinite solutions. Then, problem (6.1.1) has infinite solutions for $\sigma = 0$ and so $\lambda \in \Lambda_{NH}$. Therefore, $S_{NH} \subset \Lambda_{NH}$.

On the other hand, Theorem 2.2.2 states that if $\lambda \notin \Lambda_H$ and $\lambda \notin S_{NH}$ then $\lambda \notin \Lambda_{NH}$, that is,

$$\Lambda_{NH} \subset \Lambda_H \sqcup S_{NH},$$

being the union disjoint by definition of S_{NH}.

This completes the proof. □

Note that, in general, the inclusions given in Theorem 6.1.1 are not equalities. In particular, it is easy to prove the following results.

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Corollary 6.1.2. *Assume that $\lambda \notin \Lambda_H$, then it holds that*

$$\lambda \in \Lambda_{NH} \text{ if and only if } \lambda \in S_{NH}.$$

Corollary 6.1.3. *The following properties hold:*

1. $S_{NH} = \Lambda_{NH}$ *if and only if* $\Lambda_H \cap \Lambda_{NH} = \varnothing$.
2. $\Lambda_{NH} = \Lambda_H \sqcup S_{NH}$ *if and only if* $\Lambda_H \subset \Lambda_{NH}$.
3. $\Lambda_{NH} \subset \Lambda_H$ *if and only if* $S_{NH} = \varnothing$.

Later we will show an example with strict inclusions.

We divide this section into several subsections, obtaining the spectrum of each problem from those mentioned above. These spectra are characterized as suitable sets of real values that verify an equality involving the Green's function of another problem.

6.1.1 Spectral Characterization of the Periodic Problem

In this subsection, we will obtain the expression for spectrum of periodic problem (5.1.6) using Theorem 6.1.1 and the inclusions obtained in Corollary 6.1.3.

Next result is the characterization of the spectrum of the periodic problem (5.1.6).

Theorem 6.1.4. *The spectrum of the periodic problem* (5.1.6) *has the following properties:*

1)
$$\begin{aligned}\Lambda_P &= \left\{\lambda \in \mathbb{R} \,\middle/\, \left(1+\frac{\partial}{\partial s}G_I[\lambda](T,0)\right)\left(1-\frac{\partial}{\partial t}G_I[\lambda](T,0)\right)+G_I[\lambda](T,0)\,\frac{\partial^2}{\partial s\partial t}G_I[\lambda](T,0)=0\right\}\\ &= \left\{\lambda \in \mathbb{R} \,/\, 2-\omega_I^1[\lambda](T)-(\omega_I^2)'[\lambda](T)=0\right\}.\end{aligned}$$

2)
$$\begin{aligned}\Lambda_P &= \left\{\lambda \in \mathbb{R} \,\middle/\, \left(1-\frac{\partial}{\partial s}G_F[\lambda](0,T)\right)\left(1+\frac{\partial}{\partial t}G_F[\lambda](0,T)\right)+G_F[\lambda](0,T)\,\frac{\partial^2}{\partial s\partial t}G_F[\lambda](0,T)=0\right\}\\ &= \left\{\lambda \in \mathbb{R} \;/\; 2-\omega_F^1[\lambda](0)-(\omega_F^2)'[\lambda](0)=0\right\}.\end{aligned}$$

3) $S_N^1 \subset \Lambda_P \subset \Lambda_N \sqcup S_N^1$ *where*

$$\begin{aligned}S_N^1 &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_N \,/\, G_N[\lambda](T,0)-G_N[\lambda](0,0)-G_N[\lambda](T,T)+G_N[\lambda](0,T)=0\right\}\\ &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_N \,/\, \omega_N^1[\lambda](T)-\omega_N^1[\lambda](0)+\omega_N^2[\lambda](T)-\omega_N^2[\lambda](0)=0\right\}.\end{aligned}$$

4) $S_D^1 \subset \Lambda_P \subset \Lambda_D \sqcup S_D^1$ *where*

$$\begin{aligned}S_D^1 &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \,\middle/\, \frac{\partial^2}{\partial s\partial t}G_D[\lambda](T,T)+\frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,0)-2\frac{\partial^2}{\partial s\partial t}G_D[\lambda](T,0)=0\right\}\\ &= \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \,/\, 2(\omega_D^1)'[\lambda](T)+(\omega_D^2)'[\lambda](T)-(\omega_D^1)'[\lambda](0)=0\right\}.\end{aligned}$$

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5) $S^1_{M_1} \subset \Lambda_P \subset \Lambda_{M_1} \sqcup S^1_{M_1}$ *where*

$$\begin{aligned} S^1_{M_1} =& \Big\{ \lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ \Big(1 - \frac{\partial}{\partial t} G_{M_1}[\lambda](T,0)\Big)\Big(1 + \frac{\partial}{\partial s} G_{M_1}[\lambda](0,T)\Big) \\ & + G_{M_1}[\lambda](0,0)\, \frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](T,T) = 0 \Big\} \\ =& \Big\{ \lambda \in \mathbb{R} \setminus \Lambda_{M_1} \ / \ \Big(1 - (\omega^1_{M_1})'[\lambda](T)\Big)\Big(1 - \omega^2_{M_1}[\lambda](0)\Big) - \omega^1_{M_1}[\lambda](0)\, (\omega^2_{M_1})'[\lambda](T) = 0 \Big\}. \end{aligned}$$

6) $S^1_{M_2} \subset \Lambda_P \subset \Lambda_{M_2} \sqcup S^1_{M_2}$ *where*

$$\begin{aligned} S^1_{M_2} =& \Big\{ \lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ \Big(1 + \frac{\partial}{\partial s} G_{M_2}[\lambda](T,0)\Big)\Big(1 + \frac{\partial}{\partial t} G_{M_2}(0,T)\Big) \\ & - G_{M_2}[\lambda](T,T)\, \frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0) = 0 \Big\} \\ =& \Big\{ \lambda \in \mathbb{R} \setminus \Lambda_{M_2} \ / \ \Big(1 - \omega^1_{M_2}[\lambda](T)\Big)\Big(1 - (\omega^2_{M_2})'(0)\Big) - \omega^2_{M_2}[\lambda](T)\, (\omega^1_{M_2})'[\lambda](0) = 0 \Big\}. \end{aligned}$$

Proof. It follows from Theorem 6.1.1 together with the relationships obtained in Theorem 5.3.26, Theorem 5.3.31, Theorem 5.3.21, Theorem 5.3.10, Theorem 5.3.25 and Theorem 5.3.24. Moreover, equalities 1) and 2) follow from the fact that $\Lambda_I = \varnothing$ and $\Lambda_F = \varnothing$. □

Remark 6.1.5. *The characterization* 1) *given in previous result is very well-known and has been widely used in the literature, see for instance [10, 70, 94]. Moreover, characterization* 2) *is new in the literature.*

Remark 6.1.6. *Note that, in cases* 3) *to* 6) *in general there is no characterization of* Λ_P *since* $\Lambda_P \cap \Lambda_N$, $\Lambda_P \cap \Lambda_D$, $\Lambda_P \cap \Lambda_{M_1}$ *and* $\Lambda_P \cap \Lambda_{M_2}$ *can be non-empty (in which case it would not be true that* $S^1_N = \Lambda_P$, $S^1_D = \Lambda_P$, $S^1_{M_1} = \Lambda_P$ *and* $S^1_{M_2} = \Lambda_P$*).*

The following examples show some cases where $\Lambda_P \cap \Lambda_N$ and $\Lambda_P \cap \Lambda_D$ are nonempty. In the case of intersections $\Lambda_P \cap \Lambda_{M_1}$ and $\Lambda_P \cap \Lambda_{M_2}$, we will see later in Examples 6.1.28 and 6.1.29 a case of nonempty intersections once we have characterized the spectrum of mixed problems.

Example 6.1.7. *Consider the differential equation* $u''(t) + \lambda\, u(t) = 0$, $t \in [0,1]$. *In this case,* $a(t) = 0$, $t \in [0,1]$.

It is very well known that $\Lambda_P = \{(2k\pi)^2 : k = 0, 1 \ldots\}$ *and* $\Lambda_N = \{(k\pi)^2 : k = 0, 1, \ldots\}$. *In this case, it is evident that the intersection* $\Lambda_P \cap \Lambda_N$ *is nonempty.*

Example 6.1.8. *We consider again the same equation as in Example 6.1.7 with* $a(t) = 0$, $t \in [0,1]$.

Since $\Lambda_P = \{(2k\pi)^2 : k = 0, 1 \ldots\}$ *and* $\Lambda_D = \{(k\pi)^2 : k = 1, 2, \ldots\}$ *we have that* $\Lambda_P \cap \Lambda_D \neq \varnothing$. *Therefore, we cannot affirm a stronger information for the spectral relationship of Dirichlet and periodic problem, without any additional assumption on function* a.

Moreover, as a consequence of the previous theorem, we obtain the following results.

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Corollary 6.1.9. *If $\lambda \in \Lambda_P$, then*

$$\left(1-\frac{\partial}{\partial s}G_F[\lambda](0,T)\right)\left(1+\frac{\partial}{\partial t}G_F[\lambda](0,T)\right)+G_F[\lambda](0,T)\,\frac{\partial^2}{\partial s\partial t}G_F[\lambda](0,T)=0$$

if and only if

$$\left(1+\frac{\partial}{\partial s}G_I[\lambda](T,0)\right)\left(1-\frac{\partial}{\partial t}G_I[\lambda](T,0)\right)+G_I[\lambda](T,0)\,\frac{\partial^2}{\partial s\partial t}G_I[\lambda](T,0)=0.$$

Corollary 6.1.10. *The following inclusions hold:*

1) $S_N^1 \cup S_D^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_D \sqcup S_D^1)$.

2) $S_N^1 \cup S_{M_1}^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_{M_1} \sqcup S_{M_1}^1)$.

3) $S_N^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_N \sqcup S_N^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.

4) $S_D^1 \cup S_{M_1}^1 \subset \Lambda_P \subset (\Lambda_D \sqcup S_D^1) \cap (\Lambda_{M_1} \sqcup S_{M_1}^1)$.

5) $S_D^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_D \sqcup S_D^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.

6) $S_{M_1}^1 \cup S_{M_2}^1 \subset \Lambda_P \subset (\Lambda_{M_1} \sqcup S_{M_1}^1) \cap (\Lambda_{M_2} \sqcup S_{M_2}^1)$.

6.1.2 Spectral Characterization of the Neumann Problem

In this subsection, we will obtain the characterization of the spectrum of the Neumann problem (5.1.2) following an argument analogous to the previous section.

Theorem 6.1.11. *The spectrum of the Neumann problem* (5.1.2) *has the following properties:*

1) $\Lambda_N = \left\{\lambda \in \mathbb{R} \,\middle/\, \frac{\partial^2}{\partial s\partial t}G_I[\lambda](T,0)=0\right\} = \left\{\lambda \in \mathbb{R} \,/\, (\omega_I^1)'[\lambda](T)=0\right\}$.

2) $\Lambda_N = \left\{\lambda \in \mathbb{R} \,\middle/\, \frac{\partial^2}{\partial s\partial t}G_F[\lambda](0,T)=0\right\} = \left\{\lambda \in \mathbb{R} \,/\, (\omega_F^1)'[\lambda](0)=0\right\}$.

3) $S_D^2 \subset \Lambda_N \subset \Lambda_D \sqcup S_D^2$ *where*

$$\begin{aligned} S_D^2 =& \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \,\middle/\, \frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,0)\,\frac{\partial^2}{\partial s\partial t}G_D[\lambda](T,T)\right. \\ &\left. + \frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,T)\,\frac{\partial^2}{\partial s\partial t}G_D[\lambda](T,0)=0\right\} \\ =& \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \,/\, (\omega_D^1)'[\lambda](0)\,(\omega_D^2)'[\lambda](T)-(\omega_D^2)'[\lambda](0)\,(\omega_D^1)'[\lambda](T)=0\right\}. \end{aligned}$$

4) $S_P^1 \subset \Lambda_N \subset \Lambda_P \sqcup S_P^1$ *where*

$$S_P^1 = \left\{\lambda \in \mathbb{R}\setminus\Lambda_P \,\middle/\, \frac{\partial^2}{\partial s\partial t}G_P[\lambda](T,0)=0\right\} = \left\{\lambda \in \mathbb{R}\setminus\Lambda_P \,/\, (\omega_P^1)'[\lambda](T)=0\right\}.$$

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5) $\Lambda_N = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \Big/ \frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](T,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \,/\, (\omega^2_{M_1})'[\lambda](T) = 0\Big\}.$

6) $\Lambda_N = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \Big/ \frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \,/\, (\omega^1_{M_2})'[\lambda](0) = 0\Big\}.$

Proof. It follows from Theorem 6.1.1 together with the relationships obtained in Theorem 5.3.27, Theorem 5.3.32, Theorem 5.3.8, Theorem 5.3.20, Theorem 5.3.18 and Theorem 5.3.19.

Moreover, as in the previous theorem, equalities 1) and 2) follow from the fact that $\Lambda_I = \varnothing$ and $\Lambda_F = \varnothing$ while equalities in 5) and 6) hold by the fact that $\Lambda_{M_1} \cap \Lambda_N = \varnothing$ and $\Lambda_{M_1} \cap \Lambda_N = \varnothing$ (see [9] and [10, Corollary 4.20]) and Corollary 6.1.3. □

As a consequence of the previous theorem, we obtain the following results.

Corollary 6.1.12. *If $\lambda \in \Lambda_N$, then the following assertions are equivalent:*

1. $\frac{\partial^2}{\partial s \partial t} G_F[\lambda](0,T) = 0.$
2. $\frac{\partial^2}{\partial s \partial t} G_I[\lambda](T,0) = 0.$
3. $\frac{\partial^2}{\partial s \partial t} G_{M_2}[\lambda](0,0) = 0.$
4. $\frac{\partial^2}{\partial s \partial t} G_{M_1}[\lambda](T,T) = 0.$

Corollary 6.1.13. *The following inclusions hold*

$$S^2_D \cup S^1_P \subset \Lambda_N \subset (\Lambda_D \sqcup S^2_D) \cap (\Lambda_P \sqcup S^1_P).$$

Below there is an example of an application of the above theorem.

Example 6.1.14. *Let us consider the equation $L[\lambda]\,u(t) = 0$, $t \in [0,6]$ with $a(t) = \ln(1+t)$, $t \in [0,6]$. By numerical approach we obtain the following approximations of the eigenvalues of Neumann problem:*

$$\lambda^N_0 \approx -0.80533, \;\; \lambda^N_1 \approx -0.02394 \;\; \lambda^N_2 \approx 1.25492 \;\; \textit{and} \;\; \lambda^N_3 \approx 3.14451.$$

These values have been obtained by numerical approximation as the zeros of the function $(\omega^1_{M_2})'[\lambda](0)$ for $\lambda \in [-1,4]$ according the characterization 6) of Theorem 6.1.11. Figure 6.1.1 represents the functions $(\omega^1_I)'[\lambda](6)$, $(\omega^1_F)'[\lambda](0)$, $(\omega^2_{M_1})'[\lambda](6)$ and $(\omega^1_{M_2})'[\lambda](0)$ that have the same zeros in the interval $[-1,4]$.

6.1.3 Spectral Characterization of the Dirichlet Problem

The same arguments of the previous subsections are applicable to the Dirichlet problem (5.1.3). We attain the following results.

Theorem 6.1.15. *The spectrum of the Dirichlet problem* (5.1.3) *satisfies the following properties:*

1) $\Lambda_D = \Big\{\lambda \in \mathbb{R} \,/\, G_I[\lambda](T,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \,/\, \omega^2_I[\lambda](T) = 0\Big\}.$

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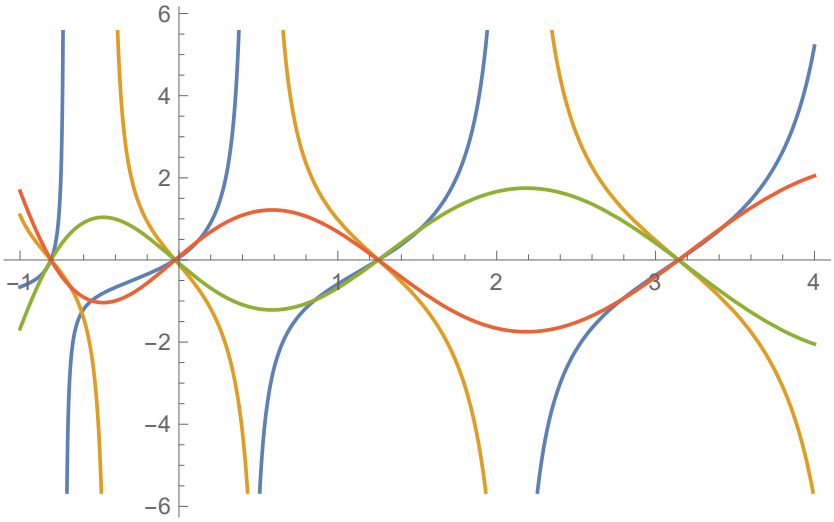

Figure 6.1.1: The blue graph represents the function $(\omega^1_{M_2})'[\lambda](0)$, the orange graph represents the function $(\omega^2_{M_1})'[\lambda](6)$, the green graph represents the function $(\omega^1_F)'[\lambda](0)$ and the red graph represents the $(\omega^1_I)'[\lambda](6)$ for λ in $[-1, 4]$.

2) $\Lambda_D = \Big\{\lambda \in \mathbb{R} \,/\, G_F[\lambda](0,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \,/\, \omega^2_F[\lambda](0) = 0\Big\}$.

3) $S^2_N \subset \Lambda_D \subset \Lambda_N \sqcup S^2_N$ *where*

$$\begin{aligned} S^2_N &= \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \,/\, G_N[\lambda](0,T)\, G_N[\lambda](T,0) - G_N[\lambda](0,0)\, G_N[\lambda](T,T) = 0\Big\} \\ &= \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \,/\, \omega^1_N[\lambda](0)\, \omega^2_N[\lambda](T) - \omega^2_N[\lambda](0)\, \omega^1_N[\lambda](T) = 0\Big\}. \end{aligned}$$

4) $S^2_P \subset \Lambda_D \subset \Lambda_P \sqcup S^2_P$ *where*

$$S^2_P = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \,/\, G_P[\lambda](T,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \,/\, \omega^2_P[\lambda](T) = 0\Big\}.$$

5) $\Lambda_D = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \,/\, G_{M_1}[\lambda](0,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_1} \,/\, \omega^1_{M_1}[\lambda](0) = 0\Big\}$.

6) $\Lambda_D = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \,/\, G_{M_2}[\lambda](T,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_{M_2} \,/\, \omega^2_{M_2}[\lambda](T) = 0\Big\}$.

Proof. It is proved using Theorem 6.1.1 and the relationships obtained in Theorem 5.3.28, Theorem 5.3.33, 5.3.9, 5.3.12, 5.3.7 and 5.3.6.

As in the previous cases, equalities 1) and 2) follow from the fact that $\Lambda_I = \Lambda_F = \varnothing$ while the equalities 5) and 6) hold by the fact that $\Lambda_{M_1} \cap \Lambda_D = \varnothing$ and $\Lambda_{M_1} \cap \Lambda_D = \varnothing$ (see [9] and [10, Corollary 4.20]) and Corollary 6.1.3. □

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Remark 6.1.16. *In general, the intersection $\Lambda_D \cap \Lambda_N$ can be nonempty, so we cannot ensure that $S_N^2 = \Lambda_D$. Next example shows a case in which $\Lambda_D \cap \Lambda_N \neq \varnothing$.*

Example 6.1.17. *Let us consider the differential equation $L[\lambda]\,u(t) = 0$, $t \in [0,1]$ with $a(t) = 0$, $t \in [0,1]$.*

It is very well known that $\Lambda_D = \{(k\pi)^2 : k = 1, 2, \ldots\}$ and $\Lambda_N = \{(k\pi)^2 : k = 0, 1, \ldots\}$. In this case, we have that $\Lambda_N = \{0\} \cup \Lambda_D$ and the intersection $\Lambda_D \cap \Lambda_N$ is nonempty and, in particular, $S_N^2 \neq \Lambda_D$.

Remark 6.1.18. *In Example 6.1.8 it has been seen that, in general, the intersection $\Lambda_P \cap \Lambda_D$ can be nonempty, so that the spectrum Λ_D cannot be characterized as the set S_P^2.*

Example 6.1.19. *Consider the following problem with constant coefficients on $[0,1]$ related to the operator*

$$L\,u(t) \equiv u''(t) + \lambda\,u(t), \quad t \in [0,1] \ \text{ and } \ \lambda > 0.$$

It is easy to see that $\Lambda_{M_1} = \Lambda_{M_2} = \left\{(k\pi + \frac{1}{2})^2 : k = 0, 1, \ldots\right\}$ and

$$G_{M_1}[\lambda](0,0) = G_{M_2}[\lambda](1,1) = -\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}.$$

Therefore,

$$\Lambda_D = \left\{\lambda \in \mathbb{R} / \ \ \tan\sqrt{\lambda} = 0\right\} = \left\{(k\pi)^2 : k = 1, \ldots\right\}.$$

The inclusions proved in 3) and 4) of Theorem 6.1.15 can be strict as it is shown in the following example.

Example 6.1.20. *Consider the differential equation $L[\lambda]\,u(t) = 0$, $t \in [0, 2\pi]$, with $a(t) = \cos(2\,t)$. In this case, it holds that*

$$S_P^2 \subset \Lambda_D \subset \Lambda_P \sqcup S_P^2$$

and

$$S_N^2 \subset \Lambda_D \subset \Lambda_N \sqcup S_N^2,$$

and all the inclusions are strict, as it can be seen in Figures 6.1.2 and 6.1.3. Indeed, the fourth eigenvalue of Λ_D is not in S_P^2 and the second eigenvalue of Λ_P is neither in S_P^2 nor in Λ_D. Similarly, the fifth eigenvalue of Λ_D is not in S_N^2 and the first eigenvalue of Λ_N is neither in S_N^2 nor in Λ_D.

As a consequence of Theorem 6.1.15, we deduce the next corollaries.

Corollary 6.1.21. *If $\lambda \in \Lambda_D$, then the following assertions are equivalent:*

1. $G_F[\lambda](0,T) = 0$.
2. $G_I[\lambda](T,0) = 0$.
3. $G_{M_2}[\lambda](T,T) = 0$.

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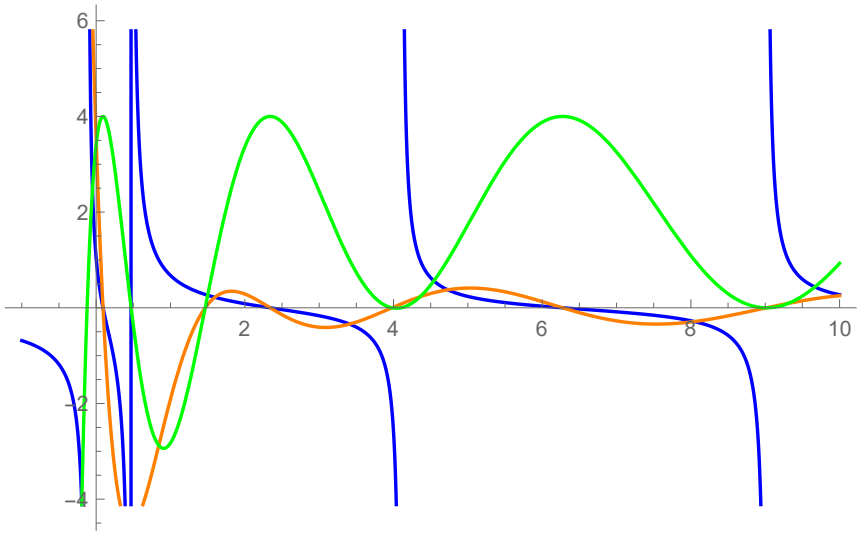

Figure 6.1.2: S_P^2 is the set of zeros of the blue graphic, Λ_D is the set of zeros of the orange graphic and Λ_P is the set of zeros of the green graphic.

4. $G_{M_1}[\lambda](0,0) = 0$.

Corollary 6.1.22. *The following inclusions hold*

$$S_P^2 \cup S_N^2 \subset \Lambda_D \subset (\Lambda_P \sqcup S_P^2) \cap (\Lambda_N \sqcup S_N^2).$$

In next example, we show that the relationship of inclusions given in the above result can be strict.

Example 6.1.23. *Let us consider the same differential equation than in Example 6.1.20. Taking into account Figures 6.1.2 and 6.1.3 and by numerical approach we have that* $\lambda \approx 9.0176$ *belongs to* Λ_D *but it is neither in* S_P^2 *nor* S_N^2 *and so* $S_P^2 \cup S_N^2 \subsetneq \Lambda_D$. *Note that this value of* λ *is a vertical asymptote of the blue curves whose zeros are* S_P^2 *and* S_N^2, *respectively.*

On the other hand, using [10, pages 94–96] we know that

$$\Lambda_N[a,T] \subset \Lambda_N[\tilde{a},2T] \cap \Lambda_P[\tilde{a},2T],$$

for all $a \in L^\alpha([0,T])$, $\alpha \geq 1$, *where* $\Lambda_N[a,T]$ *and* $\Lambda_N[\tilde{a},2T]$ *denote the spectrum of Neumann's problem at the intervals* $[0,T]$ *and* $[0,2T]$ *respectively,* $\Lambda_P[\tilde{a},2T]$ *denotes the spectrum of the Periodic problem in the interval* $[0,2T]$ *and function* $\tilde{a}$ *is the even extension of* a *to* $[0,2T]$.

In our case, since the function $a(t) = \cos(2\,t)$, $t \in [0,\pi]$, *is even we have that it coincides with its even extension on* $[0,2\pi]$. *Therefore,*

$$\Lambda_N[a,\pi] \subset \Lambda_N[a,2\pi] \cap \Lambda_P[a,2\pi] = \Lambda_N \cap \Lambda_P.$$

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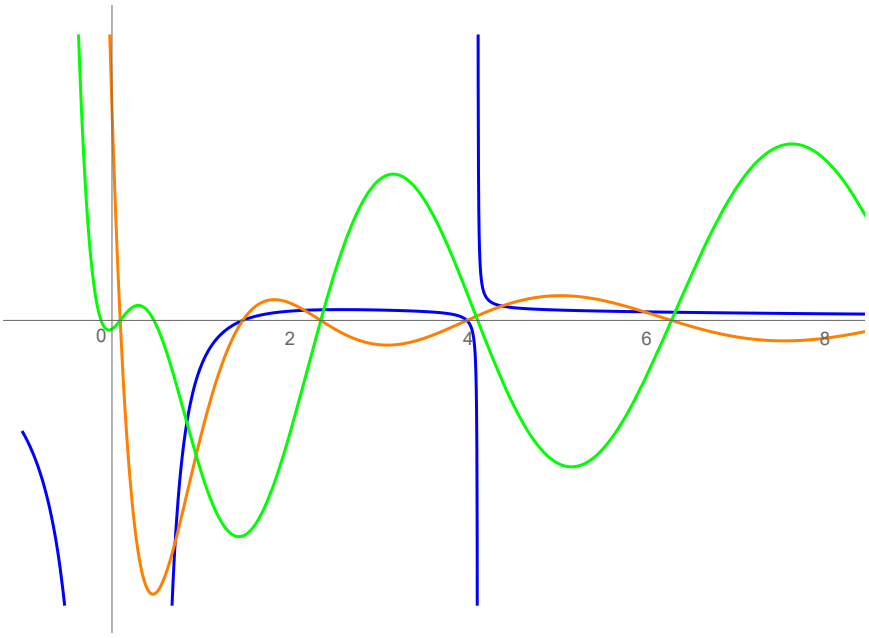

Figure 6.1.3: S_N^2 is the set of zeros of the blue graphic, Λ_D is the set of zeros of the orange graphic and Λ_N is the set of zeros of the green graphic.

By numerical approach, we have that $\lambda^* \approx 4.1009$ *belongs to* $\Lambda_N[a,\pi]$ *and so is in* $\Lambda_N \cap \Lambda_P$*, as can be seen in Figure 6.1.4.*

Since Λ_P *and* Λ_N *have in common* λ^**, which does not belong to* Λ_D*, we deduce that* $\Lambda_D \subsetneq (\Lambda_P \sqcup S_P^2) \cap (\Lambda_N \sqcup S_N^2)$*. Thus, in this case,*

$$S_P^2 \cup S_N^2 \subsetneq \Lambda_D \subsetneq (\Lambda_P \sqcup S_P^2) \cap (\Lambda_N \sqcup S_N^2).$$

6.1.4 Spectral Characterization of the Mixed Problems

Similarly, arguing as the previous sections we can characterize the spectrum of mixed problems.

For the mixed 1 problem, we arrive at the next results.

Theorem 6.1.24. *The spectrum of the mixed 1 problem* (5.1.4) *satisfies the following properties:*

1) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R} \,\middle/\, \frac{\partial}{\partial s} G_I[\lambda](T,0) = 0\right\} = \left\{\lambda \in \mathbb{R} \,/\, \omega_I^1[\lambda](T) = 0\right\}.$

2) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R} \,\middle/\, \frac{\partial}{\partial t} G_F[\lambda](0,T) = 0\right\} = \left\{\lambda \in \mathbb{R} \,/\, (\omega_F^2)'[\lambda](0) = 0\right\}.$

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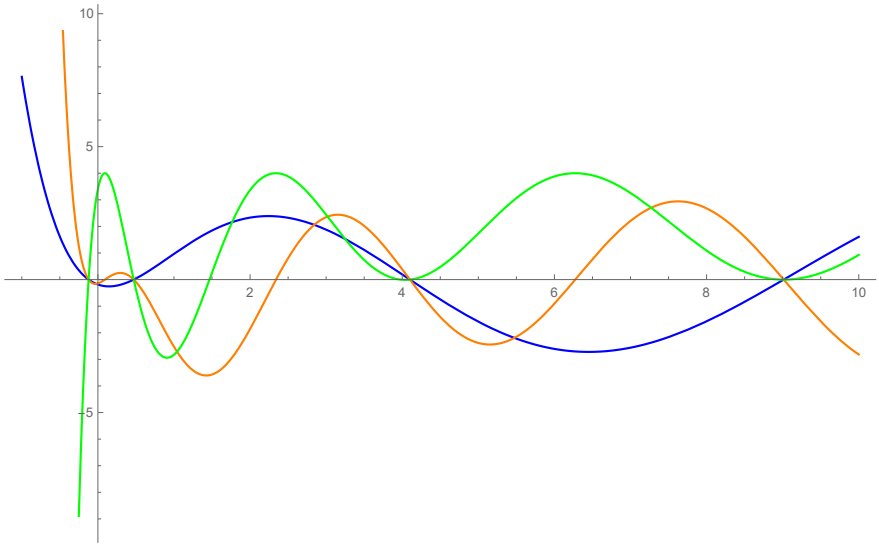

Figure 6.1.4: $\Lambda_N[a,\pi]$ is the set of zeros of the blue graphic, Λ_N is the set of zeros of the orange graphic and Λ_P is the set of zeros of the green graphic.

3) $\Lambda_{M_1} = \left\{\lambda \in \mathbb{R}\setminus\Lambda_D \;\middle/\; \frac{\partial^2}{\partial s\partial t}G_D[\lambda](0,0)=0\right\} = \left\{\lambda\in\mathbb{R}\setminus\Lambda_D \;/\; (\omega_D^1)'[\lambda](0)=0\right\}$.

4) $S_P^3 \subset \Lambda_{M_1} \subset \Lambda_P \sqcup S_P^3$ *where*

$$\begin{aligned} S_P^3 =& \left\{\lambda\in\mathbb{R}\setminus\Lambda_P \;\middle/\; \left(1+\frac{\partial}{\partial s}G_P[\lambda](0,0)\right)\left(1+\frac{\partial}{\partial t}G_P[\lambda](T,0)\right)\right. \\ &\left. - G_P[\lambda](0,0)\,\frac{\partial^2}{\partial s\partial t}G_P[\lambda](T,0)=0\right\} \\ =& \left\{\lambda\in\mathbb{R}\setminus\Lambda_P \;/\; \left(1-\omega_P^1[\lambda](0)\right)\left(1+(\omega_P^2)'[\lambda](T)\right)+\omega_P^2[\lambda](0)\,(\omega_P^1)'[\lambda](T)=0\right\}. \end{aligned}$$

5) $\Lambda_{M_1} = \left\{\lambda\in\mathbb{R}\setminus\Lambda_N \;/\; G_N[\lambda](T,T)=0\right\} = \left\{\lambda\in\mathbb{R}\setminus\Lambda_N \;/\; \omega_N^2[\lambda](T)=0\right\}$.

Proof. Using Theorem 6.1.1 with the relations obtained in Theorem 5.3.29, Theorem 5.3.34, Theorem 5.3.3, Theorem 5.3.22 and 5.3.16 the result yields.

As we know, equalities 1) and 2) are a direct consequence of the fact that $\Lambda_I = \Lambda_F = \varnothing$. On the other hand, the equalities 3) and 5) follow from the fact that $\Lambda_{M_1}\cap\Lambda_D = \varnothing$ and $\Lambda_{M_1}\cap\Lambda_N = \varnothing$ (see [9] and [10, Corollary 4.20]) and Corollary 6.1.3. □

Corollary 6.1.25. *If $\lambda\in\Lambda_{M_1}$, then the following assertions are equivalent:*

1. $\frac{\partial}{\partial t}G_F[\lambda](0,T)=0$.
2. $\frac{\partial}{\partial s}G_I[\lambda](T,0)=0$.
3. $G_N[\lambda](T,T)=0$.

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4. $\frac{\partial^2}{\partial s \partial t} G_D[\lambda](0,0) = 0$.

Finally, for mixed 2 problem (5.1.5) we obtain the following results.

Theorem 6.1.26. *The spectrum of the mixed 2 problem* (5.1.5) *satisfies the following properties:*

1) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \;\Big/\; \frac{\partial}{\partial t} G_I[\lambda](T,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \;/\; (\omega_I^2)'[\lambda](T) = 0\Big\}$.

2) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \;\Big/\; \frac{\partial}{\partial s} G_F[\lambda](0,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \;/\; \omega_F^1[\lambda](0) = 0\Big\}$.

3) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \;\Big/\; \frac{\partial^2}{\partial s \partial t} G_D[\lambda](T,T) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_D \;/\; (\omega_D^2)'[\lambda](T) = 0\Big\}$.

4) $S_P^4 \subset \Lambda_{M_2} \subset \Lambda_P \sqcup S_P^4$ *where*

$$\begin{aligned} S_P^4 =& \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \;\Big/\; \Big(1 - \frac{\partial}{\partial s} G_P[\lambda](T,0)\Big)\Big(1 - \frac{\partial}{\partial t} G_P[\lambda](0,0)\Big) \\ & - G_P[\lambda](T,0)\, \frac{\partial^2}{\partial s \partial t} G_P[\lambda](0,0) = 0\Big\} \\ =& \Big\{\lambda \in \mathbb{R} \setminus \Lambda_P \;/\; \Big(1 + \omega_P^1[\lambda](T)\Big)\Big(1 - (\omega_P^2)'[\lambda](0)\Big) + (\omega_P^1)'[\lambda](0)\, \omega_P^2[\lambda](T) = 0\Big\}. \end{aligned}$$

5) $\Lambda_{M_2} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \;/\; G_N[\lambda](0,0) = 0\Big\} = \Big\{\lambda \in \mathbb{R} \setminus \Lambda_N \;/\; \omega_N^1[\lambda](0) = 0\Big\}$.

Proof. It is proved using Theorem 6.1.1 with the relations obtained in Theorem 5.3.30, Theorem 5.3.35, Theorem 5.3.1, Theorem 5.3.23 and Theorem 5.3.14.

Again, as we know, equalities 1) and 2) are fulfilled because $\Lambda_I = \Lambda_F = \varnothing$ while the equalities 3) and 5) follow from the fact that $\Lambda_{M_1} \cap \Lambda_D = \varnothing$ and $\Lambda_{M_1} \cap \Lambda_N = \varnothing$ (see [9] and [10, Corollary 4.20]) and Corollary 6.1.3. □

Corollary 6.1.27. *If $\lambda \in \Lambda_{M_1}$, then the following assertions are equivalent:*

1. $\frac{\partial}{\partial s} G_F[\lambda](0,T) = 0$.
2. $\frac{\partial}{\partial t} G_I[\lambda](T,0) = 0$.
3. $G_N[\lambda](0,0) = 0$.
4. $\frac{\partial^2}{\partial s \partial t} G_D[\lambda](T,T) = 0$.

The following examples show that in general equalities $S_{M_1}^1 = \Lambda_P$ and $S_{M_2}^1 = \Lambda_P$ are not true.

Example 6.1.28. *Consider the mixed 1 problem and periodic problems on the same interval* $[0,1]$ *related to operator* $L[\lambda]\, u(t)$, $t \in [0,1]$, *where*

$$a(t) = \begin{cases} t, & 0 \le t < \dfrac{1}{2}, \\ p\, t, & \dfrac{1}{2} \le t \le 1, \end{cases}$$

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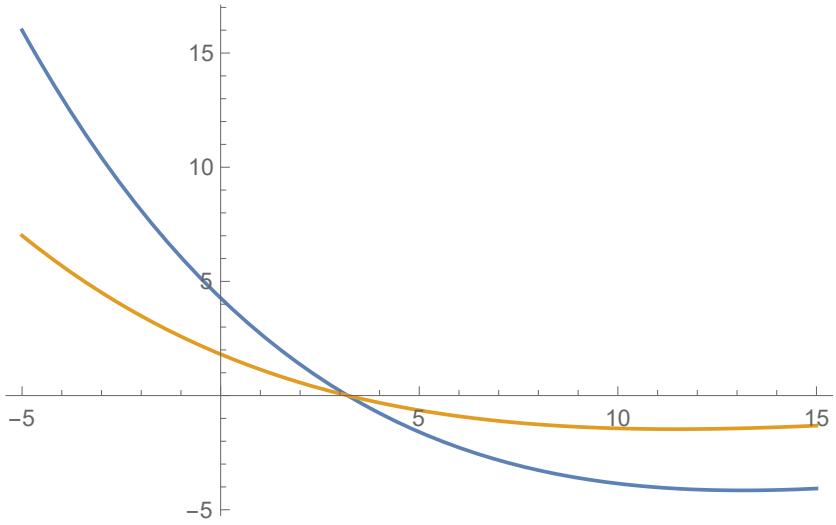

Figure 6.1.5: The orange graph represents the curve of the function $\omega_I^1[\lambda](1)$ and the blue graph corresponds to the curve of the function $\omega_I^1[\lambda](1) + (\omega_I^2)'[\lambda](1) - 2$ for λ in $[-5, 15]$ and $p = -9.6809$.

with $p \in \mathbb{R}$.

If we move the value of the parameter p*, it can be seen that the first eigenvalue of the mixed 1 problem goes from greater to less than the first eigenvalue of the periodic problem. Therefore, by continuity, there exists some value of* p *for which the first eigenvalues of both problems coincide.*

By numerical approach, we can see that such value of p *is approximately* -9.6809 *(see Figure 6.1.5). Moreover for this value, we obtain that* $\lambda_0^{M_1} = \lambda_0^P \approx 3.19907$.

As a consequence we have that Λ_P *may intersects* Λ_{M_1} *and so we cannot ensure that, in general,* $S_{M_1}^1 = \Lambda_P$.

Example 6.1.29. *Let us consider in this example the same equation as in Example 6.1.28.*

As in Example 6.1.28, there exists $p_0 \in \mathbb{R}$ *for which* $\lambda_0^{M_2} = \lambda_0^P$*. It can be seen, by numerical approach, that* $p_0 \approx 9.32$*. So, we cannot ensure that, in general, both spectra are disjoint.*

Graphically, this situation would be represented in Figure 6.1.6.

6.2 Application to Nonlinear Problems

In this last section, through the characterizations of the spectra seen in the previous section, we will deduce the existence of solutions to the nonlinear problem

$$L\,u(t) = f(t, u(t)), \text{ a.e. } t \in [0, T],$$

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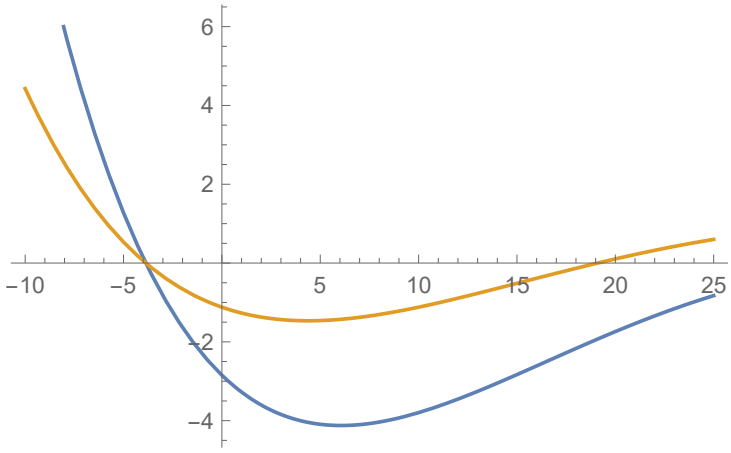

Figure 6.1.6: The orange graph represents the curve of the function $(\omega_I^2)'[\lambda](1)$ and the blue graph corresponds to the curve of the function $\omega_I^1[\lambda](1) + (\omega_I^2)'[\lambda](1) - 2$ for λ in $[-10, 25]$ and $p = 9.32$.

with different types of boundary conditions, where $L\,u(t) := u''(t) + a(t)\,u(t)$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies certain regularity conditions that we will give later.

The existence results will be deduced from Krasnoselskii-Zabreico fixed point Theorem 1.3.10.

In particular, when the Banach space is either X_D, X_P, X_N, X_{M_1} and X_{M_2} defined in (5.1.2)–(5.1.6) we will deal respectively with the following nonlinear problems:

- Dirichlet problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad u \in X_D. \tag{6.2.4}$$

- Periodic problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad u \in X_P. \tag{6.2.5}$$

- Neumann problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad u \in X_N. \tag{6.2.6}$$

- Mixed 1 problem:

$$L\,u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad u \in X_{M_1}. \tag{6.2.7}$$

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- Mixed 2 problem:

$$L\,u(t) = f(t, u(t)), \;\; \text{a.e. } \; t \in [0,T], \; u \in X_{M_2}. \tag{6.2.8}$$

where L is assumed to be nonresonant in X_D, X_P, X_N, X_{M_1} or X_{M_2} as appropiate.

For $\lambda = 0$, we denote by G_D, G_P, G_N, G_{M_1} and G_{M_2} the Green's function related to operator L defined in X_D, X_P, X_N, X_{M_1} and X_{M_2}, respectively.

We will begin to study the existence of solutions of the nonlinear second order Dirichlet problem (6.2.4).

We assume that the following conditions are fulfilled, for some $\bar{\lambda} \in \mathbb{R}$ fixed:

(H_1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a $L^\infty([0,T])-$Carathéodory function.

(H_2) $\lim\limits_{|u|\to\infty} \dfrac{f(t,u)}{u} = -\bar{\lambda}$, uniformly in $t \in [0,T]\backslash\mathcal{M}$, with $\mathcal{M}$ a set of Lebesgue measure equals to zero.

(H_3^D) $\omega_I^2[\bar{\lambda}](T) \neq 0$.

Remark 6.2.1. *Note that the condition (H_3^D) is equivalent to the fact that $\bar{\lambda} \notin \Lambda_D$ by the characterization of the spectrum Λ_D given in the Theorem 6.1.15. Moreover, the condition (H_3^D) is also equivalent to either $\omega_F^2[\bar{\lambda}](0) \neq 0$, $\omega_{M_1}^1[\bar{\lambda}](0) \neq 0$ or $\omega_{M_2}^2[\bar{\lambda}](T) \neq 0$ as a consequence of Theorem 6.1.15.*

Let $X \equiv (C([0,T]), \|\cdot\|_\infty)$ be the real Banach space endowed with the supremum norm

$$\|u\|_\infty = \max_{t\in[0,T]} \; |u(t)|, \quad \text{for all } \; u \in X,$$

and define the operator $T_D : X \to X$ as follows:

$$T_D\,u(t) := \int_0^T G_D(t,s)\, f(s, u(s))\, ds, \;\; t \in [0,T]. \tag{6.2.9}$$

Now, we prove the main result of existence of solutions of the nonlinear problem (6.2.4) following the line of [52].

Theorem 6.2.2. *Suppose that (H_1), (H_2) and (H_3^D) hold. Then, the nonlinear Dirichlet problem* (6.2.4) *has at least one solution $u \in W^{2,1}([0,T])$.*

Proof. First, note that the fixed points of operator T_D defined in (6.2.9) coincide with the solutions of problem (6.2.4).

By assumption (H_1) and using standard techniques we show that the operator T_D is well-defined.

Indeed, let $u \in X$ and $(t_k)_{k\in\mathbb{N}}$ be such that $\lim\limits_{k\to\infty} t_k = \tilde{t} \in [0,T]$.

On the one hand, using the Definition 1.1.1 (property (G_2)), we have that the function $G_D(\cdot, s)$ is uniformly continuous in $[0,T]$. Thus,

$$\lim_{k\to\infty} G_D(t_k, s)\, f(s, u(s)) = G_D(\tilde{t}, s)\, f(s, u(s)), \;\; \text{for all } s \in [0,T].$$

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On the other hand, since f is a $L^\infty([0,T])-$Carathéodory function we obtain

$$| G_D(t,s)\, f(s,u(s)) | \leq | G_D(t,s) | \ h_{\|u\|_\infty}(s), \quad \text{a.e. } s \in [0,T].$$

where $h_r \in L^\infty([0,T])$, $r>0$, is given in Definition 1.3.2.

Again, from property (G_2) in Definition 1.1.1, G_D is continuous on $[0,T]\times[0,T]$ and so it is bounded on $[0,T]\times[0,T]$ by some constant $\tilde{M}$. Thus,

$$|G_D(t,s)|\,|f(s,u(s))| \leq \tilde{M}\, h_{\|u\|_\infty}(s) := \tilde{h}(s), \quad \text{a.e. } s \in [0,T].$$

Therefore, since $\tilde{h} \in L^\infty([0,T])$ we deduce from Lebesgue's Dominated Convergence Theorem that

$$\lim_{k\to\infty} T_D\, u(t_k) = \lim_{k\to\infty} \int_0^T G_D(t_k,s)\, f(s,u(s))\, ds = \int_0^T G_D(\tilde{t},s)\, f(s,u(s))\, ds = T_D\, u(\tilde{t}).$$

Thus, $T_D\, u \in X$.

Now, we show that operator T is compact.

Let $(u_k)_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$ in $C([0,T])$. Then, $\|u_k\|_\infty \leq R$ for all $k \in \mathbb{N}$ for some $R \in \mathbb{R}^+$.

Using Definition 1.3.2, we deduce that

$$\lim_{k\to\infty} f(s,u_k(s)) = f(s,u(s)), \quad \text{a.e. } s \in [0,T],$$

and

$$| G(t,s)\, f(s,u_k(s)) | \leq \tilde{M}\, h_R(s) := \hat{h}(s), \quad \text{a.e. } s \in [0,T].$$

Again, since $\hat{h} \in L^\infty([0,T])$ from Lebesgue's Dominated Convergence Theorem we obtain that

$$\begin{aligned}\lim_{k\to\infty} |T_D\, u_k(t) - T_D\, u(t)| &\leq \lim_{k\to\infty} \int_0^T |G_D(t,s)|\,|f(s,u_k(s)) - f(s,u(s))|\, ds \\ &\leq \tilde{M} \int_0^T |f(s,u_k(s)) - f(s,u(s))|\, ds = 0.\end{aligned}$$

Since the last limit is independent of t, we have that

$$\lim_{k\to\infty} \|T_D\, u_k - T_D\, u\|_\infty = 0,$$

that is, $\lim_{k\to\infty} T_D\, u_k = T_D\, u$ in X. Thus, the operator T is continuous.

Let

$$B = \{u \in X : \ \|u\|_\infty \leq r\}$$

be a bounded set. We show that $T_D(B)$ is bounded and this implies that T maps bounded sets into relatively compact sets. Since,

$$\|T_D\|_\infty \leq \sup_{t\in I} \int_0^T |G_D(t,s)\, f(s,u(s))|\, ds \leq \tilde{M} \int_0^T h_r(s)\, ds,$$

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for all $u \in B$ and $h_r \in L^\infty([0,T])$, we have that $T_D(B)$ is bounded.

Now, we will prove the equicontinuity of T_D. For all $u \in B$, we have that

$$\begin{aligned}|T_D\,u(t_1) - T_D\,u(t_2)| &\leq \int_0^T |G_D(t_1,s) - G_D(t_2,s)|\,|f(s,u(s))|\,ds \\ &\leq \int_0^T |G_D(t_1,s) - G_D(t_2,s)|\,h_r(s)\,ds.\end{aligned}$$

Since G_D is uniformly continuous on $[0,T]\times[0,T]$, we infer that for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $t_1, t_2 \in [0,T]$, $|t_1 - t_2| < \delta$,

$$|T_D\,u(t_1) - T_D\,u(t_2)| \leq \varepsilon \int_0^T h_r(s)\,ds.$$

Thus, $T_D(B)$ is an equicontinuous set in X. By Arzelà-Ascoli's Theorem, we deduce that $T_D(B)$ is a compact operator.

Let us now consider the following linear problem

$$u''(t) + a(t)\,u(t) = -\bar{\lambda}\,u(t), \quad \text{a.e. } t \in [0,T], \quad u \in X_D, \tag{6.2.10}$$

and define the compact operator $A : X \to X$ by

$$A\,u(t) := -\bar{\lambda} \int_0^T G_D(t,s)\,u(s)\,ds, \quad t \in [0,T].$$

Again, the fixed points of the operator A coincide with the solutions of problem (6.2.10).

From the assumption (H_3^D), we know that $\bar{\lambda} \notin \Lambda_D$ and this implies that problem (6.2.10) has only the trivial solution $u = 0$. Thus, we have that 1 is not an eigenvalue of A, because, on the contrary, $u(t) = A\,u(t)$ would be a nontrivial solution of problem (6.2.10).

Finally, it remains for us to prove that

$$\lim_{\|u\|_\infty \to \infty} \frac{\|T(u) - A(u)\|_\infty}{\|u\|_\infty} = 0.$$

Let $\varepsilon > 0$ be arbitrarily fixed. By assumption (H_2) we have that there exists $M > 0$ such that if $|u| > M$ then

$$\left| f(t,u) + \bar{\lambda}\,u \right| < \varepsilon\,|u|, \quad \text{for all } t \in [0,T]\backslash\mathcal{M}. \tag{6.2.11}$$

Take

$$\hat{M} = \operatorname*{ess\,sup}_{|u|\leq M,\ t\in[0,T]} |f(t,u)|$$

and let $\sigma > M$ be large enough such that $\frac{M|\bar{\lambda}| + \hat{M}}{\sigma} < \varepsilon$.

Let $u \in X$ be such that $\|u\|_\infty > \sigma$. We distinguish two cases for all $s \in [0,T]\backslash\mathcal{M}$:

1) If $|u(s)| \leq M$ we have that

$$\left| f(s,u(s)) + \bar{\lambda}\,u(s) \right| \leq |f(s,u(s))| + |\bar{\lambda}|\,|u(s)| \leq \hat{M} + M\,|\bar{\lambda}| < \varepsilon\,\sigma \leq \varepsilon\,\|u\|_\infty.$$

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2) If $|u(s)| > M$ then, from (6.2.11), we deduce that

$$\left| f(s,u(s)) + \bar{\lambda}\, u(s) \right| < \varepsilon\, |u(s)| \leq \varepsilon\, \|u\|_\infty.$$

Therefore, in both cases we infer that

$$\left| f(s,u(s)) + \bar{\lambda}\, u(s) \right| \leq \varepsilon\, \|u\|_\infty, \quad \text{for a.e. } s \in [0,T]. \tag{6.2.12}$$

Since $G_D(t,s)$ is bounded on $[0,T]\times[0,T]$ we deduce that there exists $N \in \mathbb{R}$, $N > 0$, such that

$$\max_{t\in[0,T]} \int_0^T |G_D(t,s)|\, ds \leq N.$$

Now, let $u \in X$ with $\|u\|_\infty > \sigma$. From (6.2.12) we have that

$$\begin{aligned}
\|T_D(u) - A(u)\|_\infty &= \max_{t\in[0,T]} \left| \int_0^T G_D(t,s)\, (f(s,u(s)) + \bar{\lambda}\, u(s))\, ds \right| \\
&\leq \max_{t\in[0,T]} \int_0^T |G_D(t,s)| \left| f(s,u(s)) + \bar{\lambda}\, u(s) \right| ds \\
&\leq \varepsilon\, \|u\|_\infty \max_{t\in[0,T]} \int_0^T |G_D(t,s)|\, ds \leq \varepsilon\, N\, \|u\|_\infty.
\end{aligned}$$

Thus,

$$\lim_{\|u\|_\infty \to \infty} \frac{\|T_D(u) - A(u)\|_\infty}{\|u\|_\infty} = 0.$$

Therefore, from Theorem 1.3.10, we conclude that T_D has a fixed point $u \in X$, that is, problem (6.2.4) has at least one solution $u \in X$. □

We assume that the following condition is fulfilled:

$(\tilde{H}_2)$ $\lim\limits_{u\to\infty} \dfrac{f(t,u)}{u} = -\tilde{\lambda}$, uniformly in $t \in [0,T]\backslash\mathcal{M}'$, with $\mathcal{M}'$ a set of Lebesgue measure equals to zero.

As a consequence of previous theorem, to guarantee the existence of nonnegative solutions of the nonlinear problem (6.2.4), we arrive at the following corollary.

Corollary 6.2.3. *Assume that $f : [0,T]\times[0,\infty) \to (-\infty,0]$ is a $L^\infty([0,T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and (H_3^D) hold. Then for all $\tilde{\lambda} < \lambda_0^D$ the nonlinear Dirichlet problem* (6.2.4) *has a nonnegative solution u on $[0,T]$, $u \in W^{2,1}([0,T])$.*

Proof. Let us define the even extension of f to $[0,T]\times\mathbb{R}$, $\tilde{f} : [0,T]\times\mathbb{R} \to (-\infty,0]$, by

$$\tilde{f}(t,u) = \begin{cases} f(t,u), & u \geq 0, \\ f(t,-u), & u < 0. \end{cases}$$

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It is clear that $\tilde{f}$ is $L^\infty([0,T])-$Carathéodory function and

$$\lim_{|u|\to\infty} \frac{\tilde{f}(t,u)}{u} = -\tilde{\lambda}, \text{ uniformly in } t \in [0,T]\backslash\mathcal{M}',$$

with $\mathcal{M}'$ a set of Lebesgue measure equals to zero.

Note that $G_D[\tilde{\lambda}] < 0$ on $(0,T)\times(0,T)$ if and only if $\tilde{\lambda} < \lambda_0^D$ [10, Lemma 36]. From Theorem 6.2.2 we deduce that problem

$$L\,u(t) = \tilde{f}(t,u(t)), \text{ a.e. } t \in [0,T], \quad u \in X_D,$$

has a solution u. In such case, the solution u is given by

$$u(t) = \int_0^T G_D(t,s)\,\tilde{f}(s,u(s))\,ds,$$

and so $u \geq 0$ on $[0,T]$. Since $\tilde{f}(s,u(s)) = f(s,u(s))$, for all $s \in [0,T]$, we have that u is solution of (6.2.4), that is, problem (6.2.4) has at least one nonnegative solution u. □

The same previous argument can be applied to obtain similar results to prove the existence of solutions of nonlinear periodic, Neumann, mixed 1 and mixed 2 problems. We omit the proofs, which is analogous to those of the previous case.

Theorem 6.2.4. *Suppose that* (H_1) *and* (H_2) *hold. Moreover, assume that the following assumption is fulfilled:*

(H_3^P) $\omega_I^1[\bar{\lambda}](T) + (\omega_I^2)'[\bar{\lambda}](T) \neq 2.$

Then, the nonlinear periodic problem (6.2.5) *has at least one solution* $u \in W^{2,1}([0,T])$.

Corollary 6.2.5. *Assume that* $f : [0,T]\times[0,\infty) \to (-\infty,0]$ *is a* $L^\infty([0,T])-$*Carathéodory function and suppose that* $(\tilde{H}_2)$ *and* (H_3^P) *hold. Then for all* $\tilde{\lambda} < \lambda_0^P$ *the nonlinear periodic problem* (6.2.5) *has a nonnegative solution* u *on* $[0,T]$, $u \in W^{2,1}([0,T])$.

Theorem 6.2.6. *Suppose that* (H_1) *and* (H_2) *hold. Moreover, assume that the following assumption is fulfilled:*

(H_3^N) $(\omega_I^1)'[\bar{\lambda}](T) \neq 0.$

Then, the nonlinear Neumann Problem (6.2.6) *has at least one solution* $u \in W^{2,1}([0,T])$.

Corollary 6.2.7. *Assume that* $f : [0,T]\times[0,\infty) \to (-\infty,0]$ *is a* $L^\infty([0,T])-$*Carathéodory function and suppose that* $(\tilde{H}_2)$ *and* (H_3^N) *hold. Then, the nonlinear Neumann problem* (6.2.6) *has a nonnegative solution* u *on* $[0,T]$, $u \in W^{2,1}([0,T])$, *for all* $\tilde{\lambda} < \lambda_0^N$.

Theorem 6.2.8. *Suppose that* (H_1) *and* (H_2) *hold. Moreover, assume that the following assumption is fulfilled:*

$(H_3^{M_1})$ $\omega_I^1[\bar{\lambda}](T) \neq 0.$

Then, the nonlinear mixed 1 problem (6.2.7) *has at least one solution* $u \in W^{2,1}([0,T])$.

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Corollary 6.2.9. *Assume that $f : [0,T] \times [0,\infty) \to (-\infty, 0]$ is a $L^\infty([0,T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and $(H_3^{M_1})$ hold. Then for all $\tilde{\lambda} < \lambda_0^{M_1}$ the mixed 1 problem* (6.2.7) *has a nonnegative solution u on $[0,T]$, $u \in W^{2,1}([0,T])$.*

Theorem 6.2.10. *Suppose that (H_1) and (H_2) hold. Moreover, assume that the following assumption is fulfilled:*

$(H_3^{M_2})$ $(\omega_I^2)'[\bar{\lambda}](T) \neq 0$.

Then the nonlinear mixed 2 problem (6.2.8) *has at least one solution $u \in W^{2,1}([0,T])$.*

Corollary 6.2.11. *Assume that $f : [0,T] \times [0,\infty) \to (-\infty, 0]$ is a $L^\infty([0,T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and $(H_3^{M_2})$ hold. Then for all $\tilde{\lambda} < \lambda_0^{M_2}$ the mixed 2 problem* (6.2.8) *has a nonnegative solution u on $[0,T]$, $u \in W^{2,1}([0,T])$.*

Remark 6.2.12. *It should be noted that, as in Remark 6.2.1, the assumptions (H_3^P), (H_3^N), $(H_3^{M_1})$ and $(H_3^{M_2})$ can be replaced by their equivalents using the characterizations of Theorems 6.1.4, 6.1.11, 6.1.24 and 6.1.26, respectively.*

So, the condition (H_3^P) is equivalent to $\omega_F^1[\bar{\lambda}](0) + (\omega_F^2)'[\bar{\lambda}](0) \neq 2$, the condition (H_3^N) is equivalent to either $(\omega_F^1)'[\bar{\lambda}](0) \neq 0$, $(\omega_{M_1}^2)'[\bar{\lambda}](T) \neq 0$ or $(\omega_{M_2}^1)'[\bar{\lambda}](0) \neq 0$, the condition $(H_3^{M_1})$ is equivalent to either $(\omega_F^2)'[\bar{\lambda}](0) \neq 0$, $(\omega_D^1)'[\bar{\lambda}](0) \neq 0$ or $\omega_N^2[\bar{\lambda}](T) \neq 0$ and the condition $(H_3^{M_2})$ is equivalent to either $\omega_F^1[\bar{\lambda}](0) \neq 0$, $(\omega_D^2)'[\bar{\lambda}](T) \neq 0$ or $\omega_N^1[\bar{\lambda}](0) \neq 0$.

Remark 6.2.13. *We must note that conditions $\tilde{\lambda} < \lambda_0^P$, $\tilde{\lambda} < \lambda_0^N$, $\tilde{\lambda} < \lambda_0^{M_1}$ and $\tilde{\lambda} < \lambda_0^{M_2}$ imply that $G_P[\tilde{\lambda}] < 0$, $G_N[\tilde{\lambda}] < 0$, $G_{M_1}[\tilde{\lambda}] < 0$ and $G_{M_2}[\tilde{\lambda}] < 0$, as it can be seen in [10, Lemma 37], [10, Theorem 18], [10, Corollary 14] and [9, Corollary 13], respectively.*

Similarly, having in mind that $\lambda_0^P < \tilde{\lambda} \leq \lambda_0^A$, where λ_0^A denotes the smallest eigenvalue of the antiperiodic problem

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a.e., } t \in [0,T], \quad u(0) = -u(T), \quad u'(0) = -u'(T),$$

we have that $\lambda_0^P < \tilde{\lambda} \leq \min\{\lambda_0^{M_1}, \lambda_0^{M_2}\}$ imply that $G_P[\tilde{\lambda}] \geq 0$ and $G_N[\tilde{\lambda}] > 0$ on $(0,T) \times (0,T)$ as can be seen in [10, Lemma 37] and [10, Theorem 18], respectively, we arrive at the following results.

Corollary 6.2.14. *Assume that $f : [0,T] \times [0,\infty) \to [0,\infty)$ is a $L^\infty([0,T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and (H_3^P) hold. Then, the nonlinear periodic problem* (6.2.5) *has a nonnegative solution u on $[0,T]$, $u \in W^{2,1}([0,T])$, for all $\tilde{\lambda}$ such that $\lambda_0^P < \tilde{\lambda} \leq \lambda_0^A$.*

Corollary 6.2.15. *Assume that $f : [0,T] \times [0,\infty) \to [0,\infty)$ is a $L^\infty([0,T])-$Carathéodory function and suppose that $(\tilde{H}_2)$ and (H_3^N) hold. Then, the nonlinear Neumann problem* (6.2.6) *has a nonnegative solution u on $[0,T]$, $u \in W^{2,1}([0,T])$, for all $\tilde{\lambda}$ such that $\lambda_0^P < \tilde{\lambda} \leq \min\{\lambda_0^{M_1}, \lambda_0^{M_2}\}$.*

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Chapter 7
Nonlinear Systems Subject to Arbitrary Linear Non Local Boundary Conditions

In this chapter we will study the existence of solutions of the following system of integral equations

$$\begin{cases} u(t) = \displaystyle\int_a^b G_1(t,s)\, f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\, ds \\ \qquad\quad + \displaystyle\int_a^b G_2(t,s)\, f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\, ds, \quad t\in I, \\ v(t) = \displaystyle\int_a^b G_3(t,s)\, f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\, ds \\ \qquad\quad + \displaystyle\int_a^b G_4(t,s)\, f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\, ds, \quad t\in I, \end{cases} \tag{7.0.1}$$

where $n, m \in \mathbb{N}$, $G_i : I\times I \to \mathbb{R}$, $i = 1,2,3,4$, are the kernel functions satisfying certain regularity conditions such that $G_r \in W^{n-1,1}(I\times I)$, $r = 1,2$, $G_l \in W^{m-1,1}(I\times I)$, $l = 3,4$, and $f_i : I\times\mathbb{R}^{n+m} \to [0,\infty)$ are $L^1(I)-$ Carathéodory functions.

We emphasize that this type of problems resemble the Hammerstein equations that have been studied by some authors regarding the existence, non-existence and multiplicity of solutions, and applications to real phenomena of integral equations (see [3, 8, 33, 41, 42, 47, 56, 68, 88] and references therein).

The problems of existence and multiplicity of solutions of nonlinear differential systems with separate variables subject to boundary conditions can be posed as a problem of existence and multiplicity of solutions of a system of integral equations. In particular, if we have the following nonlinear problem with linear separate boundary conditions

$$\begin{cases} u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_n(t)\, u(t) \\ \quad = f_1(t,u(t),\dots,u^{(n-1)}(t),v(t),\dots,v^{(m-1)}(t)), \quad t\in I, \\ v^{(m)}(t) + b_1(t)\, v^{(m-1)}(t) + \cdots + b_m(t)\, v(t) \\ \quad = f_2(t,u(t),\dots,u^{(n-1)}(t),v(t),\dots,v^{(m-1)}(t)), \quad t\in I, \\ B_i\,(u) = h_i, \quad i = 1,\dots,n, \\ B_j\,(v) = q_j, \quad j = 1,\dots,m, \end{cases} \tag{7.0.2}$$

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where a_k, b_r are continuous functions for all $k = 1, \dots, n$, $r = 1, \dots, m$, $h_i, q_j \in \mathbb{R}$ for all $i = 1, \dots, n$, $j = 1, \dots, m$, $f_l : I \times \mathbb{R}^{n+m} \to \mathbb{R}$, $l = 1, 2$, satisfy certain asymptotic conditions and B_i (the same goes for B_j) covers the general two point linear boundary conditions, i.e.:

$$B_i(u) = \sum_{k=0}^{n-1} \left(\alpha_k^i u^{(k)}(a) + \beta_k^i u^{(k)}(b) \right), \quad i = 1, \dots, n,$$

being α_k^i, β_k^i real constants for all $i = 1, \dots, n$, $k = 0, \dots, n-1$, then its equivalent integral problem consists in studying the existence and multiplicity of fixed points of the following integral system

$$\begin{cases} u(t) = \displaystyle\int_a^b G_1(t,s)\, f_1(s, u(s), \dots, u^{n-1}(s), v(s), \dots, v^{m-1}(s))\, ds, \\ v(t) = \displaystyle\int_a^b G_2(t,s)\, f_2(s, u(s), \dots, u^{n-1}(s), v(s), \dots, v^{m-1}(s))\, ds, \end{cases}$$

where G_p, $p = 1, 2$, denote the Green's functions related to problem (7.0.2), verifying some adequate regularity properties. In particular, G_1 will be the Green's function related to problem

$$\begin{cases} u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_n(t)\, u(t) = 0, \quad t \in I, \\ B_i\,(u) = h_i, \quad i = 1, \dots, n, \end{cases}$$

and G_2 will be the Green's function related to

$$\begin{cases} v^{(m)}(t) + b_1(t)\, v^{(m-1)}(t) + \cdots + b_m(t)\, v(t) = 0, \quad t \in I, \\ B_j\,(v) = q_j, \quad j = 1, \dots, m. \end{cases}$$

In a more general framework, problem (7.0.1) covers a coupled system of two linear differential equations, one with order n and the other one with order m, both of them depending on u and v, and where the non homogeneous parts f_i, $i = 1, 2$, are nonlinear functions together with $n + m$ nonlocal linear boundary conditions. That is, fixed points of problem (7.0.1) correspond with solutions of

$$\begin{cases} L_1(u,v)(t) = \sigma_1(t), \quad t \in I, \\ L_2(u,v)(t) = \sigma_2(t), \\ \quad B_i(u,v) = \delta_i\, C_i(u,v), \quad i = 1, \dots, n+m, \end{cases} \tag{7.0.3}$$

where

$$\begin{aligned} L_1(u,v)(t) &:= u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} b_q(t)\, v^{(q)}(t), \\ L_2(u,v)(t) &:= v^{(m)}(t) + \sum_{k=0}^{n-1} c_k(t)\, u^{(k)}(t) + \sum_{q=0}^{m-1} d_q(t)\, v^{(q)}(t). \end{aligned} \tag{7.0.4}$$

Here σ_1, σ_2, a_k, c_k, for all $k = 0, \dots, n-1$, and b_q, d_q for all $q = 0, \dots, m-1$, are continuous functions on I. Moreover, $\delta_i \in \mathbb{R}$ for all $i = 1, \dots, n+m$.

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Furthermore, $C_i : C^n(I) \times C^m(I) \to \mathbb{R}$ is a linear continuous operator and B_i covers the general two point linear boundary conditions, i.e.:

$$B_i(u,v) = \sum_{j=0}^{n-1} \left(\alpha_j^i \, u^{(j)}(a) + \beta_j^i \, u^{(j)}(b) \right) + \sum_{l=0}^{m-1} \left(\tilde{\alpha}_l^i \, v^{(l)}(a) + \tilde{\beta}_l^i \, v^{(l)}(b) \right), \quad i = 1, \ldots, n+m,$$

being α_j^i, β_j^i, $\tilde{\alpha}_l^i$ and $\tilde{\beta}_l^i$ real constants for all $j = 1, \ldots, n-1$, $l = 0, \ldots, m-1$ and $i = 1, \ldots, n+m$. Note that in this case (contrary to problem (7.0.2)) the left hand side of the equations (that is, operators L_1 and L_2) may depend both on u and v and its derivatives.

In [84], the authors studied the existence of the following coupled system of integral equations of the Hammerstein type

$$\begin{cases} u_1(t) = \displaystyle\int_a^b k_1(t,s)\, g_1(s)\, f_1(s, u(s), \ldots, u^{(m_1)}(s), v(s), \ldots, v^{(n_1)}(s))\, ds, \\ u_2(t) = \displaystyle\int_a^b k_2(t,s)\, g_2(s)\, f_2(s, u(s), \ldots, u^{(m_2)}(s), v(s), \ldots, v^{(n_2)}(s))\, ds, \end{cases} \tag{7.0.5}$$

where $k_i : [0,1] \times [0,1] \to \mathbb{R}$, $i = 1, 2$, are the kernel functions such that $k_i \in W^{r_i,1}([0,1] \times [0,1])$, $r_1 = \max\{m_1, m_2\}$, $r_2 = \max\{n_1, n_2\}$, with $m_i, n_i \geq 0$, $g_i \in L^1([0,1])$ with $g_i(t) \geq 0$ for a.e. $t \in [0,1]$, and $f_i : [0,1] \times \mathbb{R}^{m_i+n_i+2} \to [0,\infty)$ are $L^\infty([0,1])-$ Carathéodory functions.

The solutions of (7.0.5) have been obtained by Krasnoselskii's fixed point Theorem. In this case, the integral kernels k_1 and k_2 considered, and their partial derivatives with respect to the first variable up to certain orders can be discontinuous and change its sign in the square $[0,1] \times [0,1]$, since a condition is required that they are positive in subintervals of $[0,1]$ that can be degenerated.

In [66] it is studied the existence and multiplicity of positive solutions of the following non linear differential system with perturbed integral boundary conditions

$$\begin{cases} x''(t) + f_1(t, x(t), y(t)) = 0, \quad t \in (0,1), \\ y''(t) + f_2(t, x(t), y(t)) = 0, \quad t \in (0,1), \\ \qquad x(0) = 0, \quad x(1) = \lambda \displaystyle\int_0^1 x(s)\, ds, \\ \qquad y(0) = 0, \quad y(1) = \lambda \displaystyle\int_0^1 y(s)\, ds, \end{cases}$$

where $\lambda \in (0,2)$ and $f_1, f_2 : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous functions which may admit a singularity at the origin, that is,

$$\lim_{(x,y)\to(0,0)} f_i(t,x,y) = +\infty \quad \text{uniformly in} \quad t, \quad i = 1, 2.$$

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This problem can be seen as a particular case of Hammerstein-type equations. By constructing two classes of cones and based on the Leray-Schauder Theorem and a well-known fixed point theorem in cones, the existence of at least one positive solution was obtained.

In [74], Minhós and Coxe investigated the fourth order coupled system

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)), & t \in [0,1], \\ v^{(4)}(t) = h(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)), & t \in [0,1], \end{cases}$$

with $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ some $L^1([0,1])-$ Carathéodory functions and the boundary conditions

$$\begin{cases} u(0) = u'(0) = u''(0) = u''(1) \\ v(0) = v'(0) = v''(0) = v''(1). \end{cases}$$

The existence of at least one solution of the previous problem was obtained by transforming it into an equivalent integral system with a certain kernel G and the construction of the lower and upper coupled solutions.

Motivated by the works above we consider the problem (7.0.1), and we will try to study the existence of its solution. The difference between our work and those mentioned above is that our case covers the case of a system of ordinary differential equations with four Green's functions and the variables may be not separated, while the previous works consider only two Green's functions with separate variables.

It should be noted that there are problems of type (7.0.1) in which our results are applicable considering four Green's functions and that can be transformed into coupled systems with separate variables like the previous works with two Green's functions, by passing linear terms to the right hand side of the equation and subtracting them from the f_i, $i = 1, 2$. However, in such cases, the results of the previous works are not applicable. This is due to the fact that the two new Green's functions do not satisfy the regularity conditions they ask for or the new functions of the nonlinear part do not satisfy the required asymptotic conditions. This will be illustrated in Example 7.3.6.

The existence of cases of this type of problems verifying what we have just noticed is what makes our method more general than the previous ones and presents advantages over the one that considers only two Green's functions, since this is not always applicable. This shows the novelty and interest of our method.

This chapter is organized in the following way: Section 7.1 contains some preliminaries that we need to develop the chapter. In Section 7.2, we will obtain the main result of existence and uniqueness of solution of the linear problem with an example of application of the result. Finally, Section 7.3 deals with the existence of solutions of the nonlinear problem under certain conditions of regularity of the kernel functions and asymptotic conditions of the nonlinear parts. An example is also given that illustrates our result of existence and an example in which considering a system with two Green's functions it is not possible to apply the results of the previous works as we mentioned earlier.

The results included in this chapter have been published in [21].

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7.1 Preliminaries

First, we begin with the study of the linear system (7.0.3) coupled to non-local boundary conditions involving certain parameters.

Recall that in the preliminary chapter using the following variable change

$$x_1 = u, \;\; x_2 = x_1', \ldots, x_n = x_{n-1}',$$
$$y_1 = v, \;\; y_2 = y_1', \ldots, y_m = y_{m-1}',$$

in system (7.0.3), we obtain the following equivalent first-order differential $(n+m)-$dimensional linear system

$$z'(t) = A(t)\, z(t) + f(t), \;\; t \in I, \tag{7.1.6}$$

together with the boundary conditions

$$B_i(z) = \delta_i\, C_i(z), \quad i = 1, \ldots, n+m, \tag{7.1.7}$$

where

$$z(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \\ y_1(t) \\ \vdots \\ y_{m-1}(t) \\ y_m(t) \end{pmatrix}, \quad A(t) = \left(\begin{array}{c|c} A_1(t) & A_2(t) \\ \hline A_3(t) & A_4(t) \end{array}\right), \quad f(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_1(t) \\ 0 \\ \vdots \\ 0 \\ \sigma_2(t) \end{pmatrix}, \; t \in I,$$

with

$$A_1(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_2(t) & -a_3(t) & \cdots & -a_{n-1}(t) \end{pmatrix},$$

$$A_2(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -b_0(t) & -b_2(t) & -b_3(t) & \cdots & -b_{m-1}(t) \end{pmatrix},$$

$$A_3(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -c_0(t) & -c_2(t) & -c_3(t) & \cdots & -c_{n-1}(t) \end{pmatrix},$$

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and

$$A_4(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0(t) & -d_2(t) & -d_3(t) & \cdots & -d_{m-1}(t) \end{pmatrix}.$$

Moreover, the boundary conditions B_i, $i = 1, \dots n+m$, become of the form $E\,z(a) + F\,z(b)$ with $E, F \in \mathcal{M}_{(n+m)\times(n+m)}(\mathbb{R})$ given by

$$E = \begin{pmatrix} \alpha_0^1 & \cdots & \alpha_{n-1}^1 & \tilde{\alpha}_0^1 & \cdots & \tilde{\alpha}_{m-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_0^{n+m} & \cdots & \alpha_{n-1}^{n+m} & \tilde{\alpha}_0^{n+m} & \cdots & \tilde{\alpha}_{m-1}^{n+m} \end{pmatrix}$$

and

$$F = \begin{pmatrix} \beta_0^1 & \cdots & \beta_{n-1}^1 & \tilde{\beta}_0^1 & \cdots & \tilde{\beta}_{m-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_0^{n+m} & \cdots & \beta_{n-1}^{n+m} & \tilde{\beta}_0^{n+m} & \cdots & \tilde{\beta}_{m-1}^{n+m} \end{pmatrix}.$$

Let us consider the homogeneous problem

$$z'(t) = A(t)\,z(t), \quad t \in I, \quad E\,z(a) + F\,z(b) = 0, \tag{7.1.8}$$

and let $\phi : I \to \mathcal{M}_{(n+m)\times(n+m)}(\mathbb{R})$ be its fundamental matrix.

In particular, if $\{u_1, \dots, u_{n+m}\}$ denotes the set of linearly independent solutions of $z'(t) = A(t)\,z(t)$ formed by the columns of ϕ we obtain that

$$M_\phi = E\,\phi(a) + F\,\phi(b) = \begin{pmatrix} B_1(u_1) & \cdots & B_1(u_{n+m}) \\ \vdots & \ddots & \vdots \\ B_{n+m}(u_1) & \cdots & B_{n+m}(u_{n+m}) \end{pmatrix},$$

and from Theorem 1.1.11 we know that problem (7.1.8) has only the trivial solution if and only if the determinant of the above matrix is not zero.

Example 7.1.1. *We consider the following system*

$$\begin{cases} u'(t) + \lambda\,u(t) = \sigma_1(t), \quad t \in [0,1], \\ v'(t) + \mu\,v(t) = \sigma_2(t), \\ u(0) + v(1) = \delta_1 \displaystyle\int_0^1 u(s)\,ds, \\ u(1) + v(0) = \delta_2 \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{7.1.9}$$

with σ_1, σ_2 two continuous functions, $\delta_i \in \mathbb{R}$, $i = 1, 2$ and $\lambda, \mu \in \mathbb{R}$. In this case, $B_1(u, v) = u(0) + v(1)$, $B_2(u, v) = u(1) + v(0)$ and $C_1(u, v) = C_2(u, v) = \int_0^1 u(s)\,ds$. In addition,

$$u_1(t) = \begin{pmatrix} e^{-\lambda t} \\ 0 \end{pmatrix}, \quad v_1(t) = \begin{pmatrix} 0 \\ e^{-\mu t} \end{pmatrix},$$

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are two linearly independent solutions of

$$\begin{cases} u'(t) + \lambda\, u(t) = 0, & t \in [0,1], \\ v'(t) + \mu\, v(t) = 0, & t \in [0,1], \end{cases}$$

that are the columns of the fundamental matrix $\phi(t) = \begin{pmatrix} e^{-\lambda t} & 0 \\ 0 & e^{-\mu t} \end{pmatrix}$. *Thus, problem* (7.1.9) *for* $\delta_1 = \delta_2 = 0$ *has a unique solution if and only if*

$$\det \begin{pmatrix} B_1(u_1) & B_1(v_1) \\ B_2(u_1) & B_2(v_1) \end{pmatrix} = \det \begin{pmatrix} 1 & e^{-\mu} \\ e^{-\lambda} & 1 \end{pmatrix} = 1 - e^{-(\lambda+\mu)} \neq 0,$$

that is, $\lambda + \mu \neq 0$.

7.2 Explicit Expression of the Green's Function

To obtain the explicit expression of the Green's function of the considered linear system with nonlocal boundary conditions (7.1.6)–(7.1.7), it is assumed that the Green's function of the homogeneous problem (7.1.8) exists and is unique.

Next, we enunciate and prove the following result of the existence and uniqueness of solution to problem (7.1.6)–(7.1.7) which generalizes Theorem 2.2.2 of the scalar case seen in Chapter 2.

Theorem 7.2.1. *Suppose that problem* (7.1.8) *has only the trivial solution and let* g *be its related Green's function. Let* $f \in C(I, \mathbb{R}^{n+m})$, *and* $\delta_i \in \mathbb{R}$, $i = 1, \ldots, n+m$, *be such that*

$$\det(I_{n+m} - D) \neq 0, \tag{7.2.10}$$

with I_{n+m} *the identity matrix of order* $n+m$ *and* $D = (d_{ij}) \in \mathcal{M}_{(n+m)\times(n+m)}(\mathbb{R})$ *given by*

$$d_{ij} = \delta_j\, C_i(z_j), \quad i,\ j \in \{1, \ldots, n+m\}.$$

Then problem (7.1.6)–(7.1.7) *has a unique solution* $z \in C(I, \mathbb{R}^{n+m})$, *given by the expression*

$$z(t) = \int_a^b G(t, s, \delta_1, \cdots, \delta_{n+m})\, f(s)\, ds, \tag{7.2.11}$$

where

$$G(t, s, \delta_1, \cdots, \delta_n) := g(t,s) + \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \delta_i\, r_{ij}\, z_i(t)\, h_j(s), \tag{7.2.12}$$

with $z_j \in C(I, \mathbb{R}^{n+m})$ *defined in Lemma* 1.1.13, $R = (r_{ij})_{(n+m)\times(n+m)} = (I_{n+m} - D)^{-1}$, *and*

$$h_j(s) := \begin{pmatrix} C_j(g^1(\cdot, s)) & C_j(g^2(\cdot, s)) & \cdots & C_j(g^{n+m}(\cdot, s)) \end{pmatrix} \in C(I, \mathcal{M}_{1\times(n+m)}(\mathbb{R})),$$

for all $j = 1, \ldots, n+m$, *where* g^k, $k = 1, \ldots, n+m$, *denotes the* $k^{\text{th}}-$*column of the matrix* g.

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Proof. Since problem (7.1.8) has only the trivial solution, we know that problem (7.1.6)–(7.1.7) with $\delta_i = 0$, $i = 1, \ldots, n+m$, has a unique solution given by

$$w(t) = \int_a^b g(t,s)\, f(s)\, ds.$$

Thus, from Lemma 1.1.13, any solution to problem (7.1.6)–(7.1.7) has the following form

$$z(t) = w(t) + \sum_{i=1}^{n+m} z_i(t)\, \delta_i\, C_i(z), \quad t \in I. \tag{7.2.13}$$

For $j = 1, \ldots, n+m$, applying the linear operator C_j to both sides of the previous equality we obtain that

$$C_j(z) = C_j(w) + C_j\left(\sum_{i=1}^{n+m} \delta_i\, z_i(t)\, C_i(z)\right) = C_j(w) + \sum_{i=1}^{n+m} \delta_i\, C_j(z_i)\, C_i(z),$$

or, equivalently,

$$C_j(z) - \sum_{i=1}^{n+m} \delta_i\, C_j(z_i)\, C_i(z) = C_j(w).$$

Therefore, the above equalities are equivalent to the following system

$$(I_{n+m} - D) \begin{pmatrix} C_1(z) \\ C_2(z) \\ \vdots \\ C_{n+m}(z) \end{pmatrix} = \begin{pmatrix} C_1(w) \\ C_2(w) \\ \vdots \\ C_{n+m}(w) \end{pmatrix}.$$

Since $I_{n+m} - D$ is a invertible matrix, we have that

$$C_i(z) = \sum_{j=1}^{n+m} r_{ij}\, C_j(w), \quad i = 1, \ldots, n+m.$$

Thus, equality (7.2.13) becomes

$$z(t) = w(t) + \sum_{i=1}^{n+m} \delta_i\, z_i(t) \left(\sum_{j=1}^{n+m} r_{ij}\, C_j(w)\right) = w(t) + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \delta_i\, r_{ij}\, z_i(t)\, C_j(w), \quad t \in I.$$

Due to the fact that C_j is a continuous linear operator, we have that

$$C_j(w) = C_j\left(\int_a^b g(\cdot,s)\, f(s)\, ds\right) = \int_a^b C_j\left(g(\cdot,s)\, f(s)\right)\, ds. \tag{7.2.14}$$

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Since

$$g(\cdot,s)\,f(s) = \begin{pmatrix} g^1(\cdot,s) & g^2(\cdot,s) & \cdots & g^{n+m}(\cdot,s) \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \\ \vdots \\ f_{n+m}(s) \end{pmatrix}$$
$$= f_1(s)\,g^1(\cdot,s) + \cdots + f_{n+m}(s)\,g^{n+m}(\cdot,s),$$

we have that

$$\begin{aligned} C_j\left(g(\cdot,s)\,f(s)\right) =& f_1(s)\,C_j(g^1(\cdot,s)) + \cdots + f_{n+m}(s)\,C_j(g^{n+m}(\cdot,s)) \\ =& \begin{pmatrix} C_j(g^1(\cdot,s)) & C_j(g^2(\cdot,s)) & \cdots & C_j(g^{n+m}(\cdot,s)) \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \\ \vdots \\ f_{n+m}(s) \end{pmatrix} \\ =& h_j(s)\,f(s). \end{aligned}$$

Substituting the previous expression into (7.2.14), we obtain that $C_j(w) = \displaystyle\int_a^b h_j(s)\,f(s)\,ds$. Therefore,

$$\begin{aligned} z(t) &= w(t) + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \delta_i\,r_{ij}\,z_i(t)\,C_j(w) \\ &= \int_a^b g(t,s)\,f(s)\,ds + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \delta_i\,r_{ij}\,z_i(t) \int_a^b h_j(s)\,f(s)\,ds \\ &= \int_a^b \left(g(t,s) + \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \delta_i\,r_{ij}\,z_i(t)\,h_j(s) \right) f(s)\,ds \\ &= \int_a^b G(t,s,\delta_1,\cdots,\delta_n)\,f(s)\,ds. \end{aligned}$$

Thus, we have proved the existence of at least one solution to problem (7.1.6)–(7.1.7) given by (7.2.11).

Finally, we prove the uniqueness of the solution of problem (7.1.6)–(7.1.7). Suppose that problem (7.1.6)–(7.1.7) has two different solutions w_1 and w_2. Therefore, $w_1 - w_2$ solves the following problem

$$\begin{cases} (w_1 - w_2)'(t) = A(t)\,(w_1 - w_2)(t), & t \in I, \\ B_i(w_1 - w_2) = \delta_i\,C_i(w_1 - w_2), & i = 1,\ldots,n+m. \end{cases} \tag{7.2.15}$$

So, from (7.2.13), we know that the solution $w_1 - w_2$ is given by

$$(w_1 - w_2)(t) = \sum_{i=1}^{n+m} \delta_i\,z_i(t)\,C_i(w_1 - w_2), \quad t \in I.$$

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By applying the operator C_j to both sides of the previous equality we obtain that

$$C_j(w_1 - w_2) = \sum_{i=1}^{n+m} \delta_i \, C_j(z_i) \, C_i(w_1 - w_2), \quad j = 1, \ldots, n+m,$$

or, equivalently,

$$(I_{n+m} - D) \begin{pmatrix} C_1(w_1 - w_2) \\ C_2(w_1 - w_2) \\ \vdots \\ C_{n+m}(w_1 - w_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Again, since $I_{n+m} - D$ is a invertible matrix, we infer that $C_i(w_1 - w_2) = 0$ for $i = 1, \ldots, n+m$. Therefore, by (7.2.15) we have that $w_1 - w_2$ is a solution of the homogeneous problem

$$\begin{cases} (w_1 - w_2)'(t) = A(t) \, (w_1 - w_2)(t), \ \ t \in I, \\ B_i(w_1 - w_2) = 0, \ \ i = 1, \ldots, n+m. \end{cases}$$

By hypothesis, we know that the homogeneous problem (7.1.8) has only the trivial solution so it follows that $w_1 = w_2$ on I, and this completes the proof. □

As a direct consequence of the previous theorem, we obtain the following result for this particular case that all the functionals at the boundary conditions are the same, that is, there is some linear continuous operator C such that $C_i = C$ for $i = 1, \ldots, n+m$.

Corollary 7.2.2. *Assume that problem* (7.1.8) *has* $u = 0$ *as its unique solution and let* g *be its unique Green's function. Let* $f \in C(I, \mathbb{R}^{n+m})$*, and* δ_i, $i = 1, \ldots, n+m$*, be such that*

$$\sum_{i=1}^{n+m} \delta_i \, C(z_i) \neq 1.$$

holds.

Then problem

$$z'(t) = A(t) \, z(t) + f(t), \ \ t \in I, \ \ B_i(z) = \delta_i \, C(z), \ \ i = 1, \ldots, n+m,$$

has a unique solution $z \in C(I, \mathbb{R}^{n+m})$*, given by the expression*

$$z(t) = \int_a^b G(t, s, \delta_1, \ldots, \delta_n) \, f(s) \, ds,$$

where

$$G(t, s, \delta_1, \ldots, \delta_n) := g(t, s) + \frac{\displaystyle\sum_{i=1}^{n} \delta_i \, z_i(t)}{1 - \displaystyle\sum_{j=1}^{n} \delta_j C(z_j)} \, h(s),$$

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with

$$h(s) := \begin{pmatrix} C(g^1(\cdot,s)) & C(g^2(\cdot,s)) & \cdots & C(g^{n+m}(\cdot,s)) \end{pmatrix} \in C(I, \mathcal{M}_{1\times(n+m)}(\mathbb{R})),$$

where g^k, $k = 1, \ldots, n+m$, denotes the $k^{\text{th}}-$column of the matrix g.

In the following result, once the expression for the solution z is obtained, we can retrieve the solutions u and v of problem (7.0.3).

Corollary 7.2.3. *Assume that problem* (7.1.8) *has only the trivial solution and let g be its related Green's function. Let $\sigma_1, \sigma_2 \in C(I)$, and $\delta_i \in \mathbb{R}$, $i = 1, \ldots, n+m$, be such that condition* (7.2.10) *holds.*

Then system (7.0.3) *has a unique solution $(u, v) \in C^n(I) \times C^m(I)$ given by*

$$\begin{cases} u(t) = \displaystyle\int_a^b \Big(G_{1,n}(t,s,\delta_1,\cdots,\delta_n)\,\sigma_1(s) + G_{1,n+m}(t,s,\delta_1,\cdots,\delta_n)\,\sigma_2(s) \Big)\, ds, \quad t \in I, \\ v(t) = \displaystyle\int_a^b \Big(G_{n+1,n}(t,s,\delta_1,\cdots,\delta_n)\,\sigma_1(s) + G_{n+1,n+m}(t,s,\delta_1,\cdots,\delta_n)\,\sigma_2(s) \Big)\, ds, \end{cases} \tag{7.2.16}$$

where $G_{i,j}$ denotes the (i,j) entries, of the matrix G defined in (7.2.12).

Proof. The proof it is an immediate consequence of Theorem 7.2.1 taking into account that $u(t) = z_1(t)$ and $v(t) = z_{n+1}(t)$, $t \in I$, where z is the expression of the solution of the problem (7.1.6)–(7.1.7) given in (7.2.11). □

In the sequel, we present an example to ilustrate the previous result.

Example 7.2.4. *Consider the system*

$$\begin{cases} u''(t) - v(t) = e^t, \quad t \in [0,1], \\ v'(t) - u'(t) = t, \quad t \in [0,1], \\ u(0) = 0, \\ u'(0) = u'(1), \\ v(0) = 0. \end{cases} \tag{7.2.17}$$

For this system, we have that $\sigma_1(t) = e^t$, $\sigma_2(t) = t$, $t \in [0,1]$, $\delta_1 = 0$, $\delta_2 = 1$, $\delta_3 = 0$, $B_1(u,v) = u(0)$, $B_2(u,v) = u'(0)$, $B_3(u,v) = v(0)$, $C_1(u,v) = 0$, $C_2(u,v) = u'(1)$, and, $C_3(u,v) = 0$.

Performing the calculation of the elements of Theorem 7.2.1 *and substituting them in the expression* (7.2.12)*, we have*

$$G(t,s) = \begin{pmatrix} 1 & \sinh(t-s) + \frac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)} & \cosh(t-s) - \frac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)} - 1 \\ 0 & \cosh(t-s) + \frac{\cosh(s-1)\,\cosh(t)}{1-\cosh(1)} & \sinh(t-s) - \frac{\cosh(t)\,\sinh(s-1)}{1-\cosh(1)} \\ 0 & \sinh(t-s) + \frac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)} & \cosh(t-s) - \frac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)} \end{pmatrix},$$

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for $0 \leq s \leq t \leq 1$, *and*

$$G(t,s) = \begin{pmatrix} 0 & \frac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)} & -\frac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)} \\ 0 & \frac{\cosh(s-1)\,\cos(t)}{1-\cosh(1)} & -\frac{\cosh(t)\,\sinh(s-1)}{1-\cosh(1)} \\ 0 & \frac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)} & -\frac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)} \end{pmatrix},$$

for $0 \leq t < s \leq 1$.

Now, making use of the expressions (7.2.16) *we have that the solution of problem* (7.2.17) *is given by*

$$u(t) = \frac{e^{-t}\left(e\,(4e-5) + e^{2t}\,(2+e+e^2\,(-2+t)+t-2\,e\,t) - (-1+e)^2\,e^t\,(2+t^2)\right)}{2(e-1)^2},$$
$$v(t) = \frac{e^{-t}\left(e\,(4e-5) + e^{2t}\,(2+e+e^2\,(t-2)+t-2\,e\,t)\right)}{2(e-1)^2} - 1, \quad t \in [0,1].$$

Remark 7.2.5. *We emphasize that results similar to those obtained above can be deduced for the general problem:*

$$\begin{cases} L_i(u_1,\cdots,u_n)(t) = \sigma_i(t), \quad t \in I, \quad i = 1,\ldots,n, \\ \\ B_p(u_1,\cdots,u_n) = \delta_p\, C_p(u_1,\cdots,u_n), \quad p = 1,\ldots,\displaystyle\sum_{i=1}^{n} r_i, \end{cases}$$

where

$$L_i(u_1,\cdots,u_n)(t) := u_i^{(r_i)}(t) + \sum_{k=0}^{r_i-1} a_k^i(t)\,u_i^{(k)}(t) + \sum_{j\neq i}\sum_{l=0}^{r_j-1} b_{lj}^i(t)\,u_j^{(l)}(t).$$

Here, σ_i, a_k^i *and* b_{lj}^i *for all* $i = 1,\ldots,n$, $k = 0,\ldots,r_i-1$, *and* $l = 0,\ldots,r_j-1$, *with* $j \in \{1,\ldots,n\}$, $j \neq i$, *are continuous functions on* I, *and* $r_i \in \mathbb{N}$ *for all* $i = 1,\ldots,n$ *and* $\delta_p \in \mathbb{R}$ *for all* $p = 1,\ldots,\displaystyle\sum_{i=1}^{n} r_i$.

Moreover, $C_p : \displaystyle\prod_{k=1}^{n} C^{r_k}(I) \to \mathbb{R}$ *is a linear continuous operator and* B_p *covers the general two point linear boundary conditions, i.e.,*

$$B_p(u_1,\cdots,u_n) = \sum_{i=1}^{n}\sum_{j=0}^{r_i-1}\left(\alpha_{ji}^p\,u_i^{(j)}(a) + \beta_{ji}^p\,u_i^{(j)}(b)\right), \quad p = 1,\ldots,\sum_{i=1}^{n} r_i,$$

with α_{ji}^p *and* β_{ji}^p *being real constants for all* $j = 1,\ldots,r_i-1$, $i = 1,\ldots,n$ *and* $p = 1,\ldots,\displaystyle\sum_{i=1}^{n} r_i$.

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7.3 Nonlinear Problem

In this section, we will study the existence of a nontrivial solution of the system of integral equation (7.0.1).

In the sequel, we will set out the conditions we need to ensure the existence of at least a nontrivial solution to problem (7.0.1):

(H_1) The kernel functions $G_r : I \times I \to \mathbb{R}$, $r = 1, 2$, are such that $G_r \in W^{n-1,1}(I \times I)$.

The $(n-1)$– th derivative of G_r, $r = 1, 2$, satisfies that for every $\varepsilon > 0$ and every fixed $\mu \in I$, there exists some $\delta > 0$ such that $|t - \mu| < \delta$ then

$$\left| \frac{\partial^{n-1} G_r}{\partial t^{n-1}}(t, s) - \frac{\partial^{n-1} G_r}{\partial t^{n-1}}(\mu, s) \right| < \varepsilon,$$

for a.e. $s < \min\{t, \mu\}$ and for a.e. $s > \max\{t, \mu\}$.

Moreover, if $n \geq 2$, then for $i = 0, \ldots, n-2$, it holds that for every $\varepsilon > 0$ and every fixed $\mu \in I$, there exists some $\delta > 0$ such that $|t - \mu| < \delta$ implies that

$$\left| \frac{\partial^{i} G_r}{\partial t^{i}}(t, s) - \frac{\partial^{i} G_r}{\partial t^{i}}(\mu, s) \right| < \varepsilon, \quad \text{for a.e. } s \in I.$$

The kernel functions $G_l : I \times I \to \mathbb{R}$, $l = 3, 4$, are such that $G_l \in W^{m-1,1}(I \times I)$.

The $(m-1)$– th derivative of G_l, $l = 3, 4$, satisfies that for every $\varepsilon > 0$ and every fixed $\mu \in I$, there exists some $\delta > 0$ such that $|t - \mu| < \delta$ then

$$\left| \frac{\partial^{m-1} G_l}{\partial t^{m-1}}(t, s) - \frac{\partial^{m-1} G_l}{\partial t^{m-1}}(\mu, s) \right| < \varepsilon,$$

for a.e. $s < \min\{t, \mu\}$ and for a.e. $s > \max\{t, \mu\}$.

Moreover, if $m \geq 2$, then for $j = 0, \ldots, m-2$, it holds that for every $\varepsilon > 0$ and every fixed $\mu \in I$, there exists some $\delta > 0$ such that $|t - \mu| < \delta$ implies that

$$\left| \frac{\partial^{j} G_l}{\partial t^{j}}(t, s) - \frac{\partial^{j} G_l}{\partial t^{j}}(\mu, s) \right| < \varepsilon, \quad \text{for a.e. } s \in I,$$

(H_2) For $r = 1, 2$, and for each $i \in I_1$, $I_1 \subset J_1 := \{0, \ldots, n-1\}$, $I_1 \neq \varnothing$, there exists a subinterval $[m_i^r, n_i^r] \subset I$ such that

$$\frac{\partial^{i} G_r}{\partial t^{i}}(t, s) \geq 0, \quad \text{for all } \ t \in [m_i^r, n_i^r], \quad s \in I.$$

For $l = 3, 4$, and for each $j \in I_2$, $I_2 \subset J_2 := \{0, \ldots, m-1\}$, $I_2 \neq \varnothing$, there exists a subinterval $[m_j^l, n_j^l] \subset I$ such that

$$\frac{\partial^{j} G_l}{\partial t^{j}}(t, s) \geq 0, \quad \text{for all } \ t \in [m_j^l, n_j^l], \quad s \in I.$$

We also admit that these intervals could be degenerate, that is, it is possible that $m_i^r = n_i^r$ and $m_j^l = n_j^l$.

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(H_3) For $r = 1, 2$ and for each $i \in J_1$, there exist positive functions $h_i^r \in L^1(I)$ such that

$$\left|\frac{\partial^i G_r}{\partial t^i}(t,s)\right| \leq h_i^r(s) \ \text{ for all } \ t \in I \ \text{ and a.e. } \ s \in I.$$

For $l = 3, 4$ and for each $j \in J_2$, there exist positive functions $h_j^l \in L^1(I)$ such that

$$\left|\frac{\partial^j G_l}{\partial t^j}(t,s)\right| \leq h_j^l(s) \ \text{ for all } \ t \in I \ \text{ and a.e. } \ s \in I.$$

(H_4) For $r = 1, 2$, and for each $i \in K_1$, $K_1 \subset I_1$, $K_1 \neq \varnothing$, there exist subintervals $[a_i^r, b_i^r] \subset [m_i^r, n_i^r]$ and $[c_i^r, d_i^r] \subset I$, nonnegative functions $\phi_i^r : I \to [0, \infty)$ and $\xi_i^r \in (0, 1)$ such that

$$\left|\frac{\partial^i G_r}{\partial t^i}(t,s)\right| \leq \phi_i^r(s), \ \text{ for all } \ t \in [c_i^r, d_i^r] \ \text{ and a.e. } \ s \in I,$$

$$\frac{\partial^i G_r}{\partial t^i}(t,s) \geq \xi_i^r \, \phi_i^r(s), \ \text{ for all } \ t \in [a_i^r, b_i^r] \ \text{ and a.e. } \ s \in I.$$

For $l = 3, 4$, and for each $j \in K_2$, $K_2 \subset I_2$, $K_2 \neq \varnothing$, there exist subintervals $[a_j^l, b_j^l] \subset [m_j^l, n_j^l]$ and $[c_j^l, d_j^l] \subset I$, nonnegative functions $\phi_j^l : I \to [0, \infty)$ and $\xi_j^l \in (0, 1)$ such that

$$\left|\frac{\partial^j G_l}{\partial t^j}(t,s)\right| \leq \phi_j^l(s), \ \text{ for all } \ t \in [c_j^l, d_j^l] \ \text{ and a.e. } \ s \in I,$$

$$\frac{\partial^j G_l}{\partial t^i}(t,s) \geq \xi_j^l \, \phi_j^l(s), \ \text{ for all } \ t \in [a_j^l, b_j^l] \ \text{ and a.e. } \ s \in I.$$

Moreover, one of the following conditions must be met:

(H_4^1) There exists some $i_0 \in K_1$ such that

$$S_1 := [a_{i_0}^1, b_{i_0}^1] \cap [a_{i_0}^2, b_{i_0}^2] \cap [c_{i_0}^1, d_{i_0}^1] \cap [c_{i_0}^2, d_{i_0}^2] \neq \varnothing$$

and $[a_{i_0}^{12}, b_{i_0}^{12}] := [a_{i_0}^1, b_{i_0}^1] \cap [a_{i_0}^2, b_{i_0}^2]$ is a nondegenerate interval.

(H_4^2) There exists some $j_0 \in K_2$ such that

$$S_2 := [a_{j_0}^3, b_{j_0}^3] \cap [a_{j_0}^4, b_{j_0}^4] \cap [c_{j_0}^3, d_{j_0}^3] \cap [c_{j_0}^4, d_{j_0}^4] \neq \varnothing$$

and $[a_{j_0}^{34}, b_{j_0}^{34}] := [a_{j_0}^3, b_{j_0}^3] \cap [a_{j_0}^4, b_{j_0}^4]$ is a nondegenerate interval.

(H_5) There exists $i_0 \in I_1$ such that either $[m_{i_0}^1, n_{i_0}^1] \cap [m_{i_0}^2, n_{i_0}^2] \equiv I$ or $[c_{i_0}^1, d_{i_0}^1] \cap [c_{i_0}^2, d_{i_0}^2] \equiv I$ and moreover, $\{0, \ldots, i_0\} \subset J_1$.

There exists $j_0 \in I_2$ such that either $[m_{j_0}^3, n_{j_0}^3] \cap [m_{j_0}^4, n_{j_0}^4] \equiv I$ or $[c_{j_0}^3, d_{j_0}^3] \cap [c_{j_0}^4, d_{j_0}^4] \equiv I$ and moreover, $\{0, \ldots, j_0\} \subset J_2$.

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(H_6) The functions $f_i : I \times \mathbb{R}^{n+m} \to [0,\infty)$, $i = 1,2$, are $L^1(I)-$Carathéodory functions, that is,

- $f_i(\cdot, x_1, \cdots, x_n, y_1, \cdots, y_m)$ is measurable for each $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ fixed.
- $f_i(t, \cdots, \cdots, \cdots)$ is continuous for a.e. $t \in I$.
- For each $R > 0$ there exists $\mu_R^1,\ \mu_R^2 \in L^1(I)$ such that

$$f_i(t, x_1, \ldots, x_n, y_1, \ldots, y_m) \leq \mu_R^i(t),$$

for all $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in (-R, R)^{n+m}$, a.e. $t \in I$, and $i = 1, 2$.

(H_7) For each $r = 1, 2$ and $i \in J_1$ we have that $h_i^r\, \mu_R^1 \in L^1(I)$ for every $R > 0$.
For each $l = 3, 4$ and $j \in J_2$ we have that $h_j^l\, \mu_R^2 \in L^1(I)$ for every $R > 0$.

(H_8) For each $i = 1, 2$ we have that

$$\limsup_{\min\{|x_1|,\ldots,|x_n|,|y_1|,\ldots,|y_m|\}\to\infty} \max_{t\in I} \frac{f_i(t, x_1, \ldots, x_n, y_1, \ldots, y_m)}{|x_1| + \ldots + |x_n| + |y_1| + \ldots + |y_m|} = 0,$$

$$\liminf_{|x_1|,\ldots,|x_n|,|y_1|,\ldots,|y_m|\to 0} \min_{t\in I} \frac{f_i(t, x_1, \ldots, x_n, y_1, \ldots, y_m)}{|x_1| + \ldots + |x_n| + |y_1| + \ldots + |y_m|} = +\infty.$$

Let us consider $X := C^{n-1}(I) \times C^{m-1}(I)$ the Banach space equipped with the norm

$$\|(u, v)\|_X = \max\{\|u\|_{C^{n-1}(I)}, \|v\|_{C^{m-1}(I)}\},$$

where $\|u\|_{C^{n-1}(I)} = \max\{\|u^{(i)}\|_\infty,\ \ i \in J_1\}$ and $\|v\|_{C^{m-1}(I)} = \max\{\|v^{(j)}\|_\infty,\ \ j \in J_2\}$.

Let us denote

$$\begin{array}{ll} [m_i^{12}, n_i^{12}] := [m_i^1, n_i^1] \cap [m_i^2, n_i^2],\ i \in I_1, & [m_j^{34}, n_j^{34}] := [m_j^3, n_j^3] \cap [m_j^4, n_j^4],\ j \in I_2, \\ [a_i^{12}, b_i^{12}] := [a_i^1, b_i^1] \cap [a_i^2, b_i^2],\ i \in K_1, & [a_j^{34}, b_j^{34}] := [a_j^3, b_j^3] \cap [a_j^4, b_j^4],\ j \in K_2, \\ [c_i^{12}, d_i^{12}] := [c_i^1, d_i^1] \cap [c_i^2, d_i^2],\ i \in K_1, & [c_j^{34}, d_i^{34}] := [c_j^3, d_j^3] \cap [c_j^4, d_j^4],\ j \in K_2. \end{array}$$

Once we have defined the above conditions, we will define the following cones:

$$H_1 = \left\{ u \in C^{n-1}(I) : \begin{array}{l} u^{(i)}(t) \geq 0,\ \ t \in [m_i^{12}, n_i^{12}],\ \ i \in I_1, \\ \min\limits_{t\in[a_i^{12}, b_i^{12}]} u^{(i)}(t) \geq \xi^1 \|u^{(i)}\|_{[c_i^{12}, d_i^{12}]},\ \ i \in K_1 \end{array} \right\},$$

and

$$H_2 = \left\{ v \in C^{m-1}(I) : \begin{array}{l} v^{(j)}(t) \geq 0,\ \ t \in [m_j^{34}, n_j^{34}],\ \ j \in I_2, \\ \min\limits_{t\in[a_j^{34}, b_j^{34}]} u^{(j)}(t) \geq \xi^2 \|u^{(j)}\|_{[c_j^{34}, d_j^{34}]},\ \ j \in K_2 \end{array} \right\},$$

where $\xi^1 = \min\{\xi_i^r :\ \ r \in \{1, 2\},\ i \in K_1\}$ and $\xi^2 = \min\{\xi_j^l :\ \ l \in \{3, 4\},\ j \in K_2\}$.

Moreover, the product space $H := H_1 \times H_2$ is a cone in the Banach space X with the norm $\|(u, v)\|_X$ as seen in the result below.

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Lemma 7.3.1. *Condition* (H_5) *guarantees that* H *is a cone in* X.

Proof. We show that every H_i, $i = 1, 2$ is a cone and therefore H is also a cone. We do the proof for H_1 and for H_2 it is done in an analogous way.

Let us see that H_1 is a closed and convex subset of $C^{n-1}(I)$ and satisfies the properties of the Definition 1.3.6.

From the definition of H_1 it follows that it is a closed set. Let us show that H_1 is convex. Let $u, w \in H_1$ and $\lambda \in (0, 1)$. Then

$$(1-\lambda)\, u^{(i)}(t) + \lambda\, w^{(i)}(t) \geq 0, \quad \text{for } t \in [m_i^{12}, n_i^{12}], \quad i \in I_1,$$

and, for $i \in K_1$,

$$\begin{aligned}
\min_{t\in[a_i^{12},b_i^{12}]} \Big((1-\lambda)\, u^{(i)}(t) + \lambda\, w^{(i)}(t)\Big) &\geq (1-\lambda) \min_{t\in[a_i^{12},b_i^{12}]} u^{(i)}(t) + \lambda \min_{t\in[a_i^{12},b_i^{12}]} w^{(i)}(t) \\
&\geq (1-\lambda)\, \xi^1 \|u^{(i)}\|_{[c_i^{12},d_i^{12}]} + \lambda\, \xi^1 \|w^{(i)}\|_{[c_i^{12},d_i^{12}]} \\
&\geq \xi^1 \Big((1-\lambda)\, \|u^{(i)}\|_{[c_i^{12},d_i^{12}]} + \lambda\, \|w^{(i)}\|_{[c_i^{12},d_i^{12}]}\Big) \\
&\geq \xi^1 \|(1-\lambda)\, u^{(i)} + \lambda\, w^{(i)}\|_{[c_i^{12},d_i^{12}]}.
\end{aligned}$$

Thus, $(1-\lambda)\, u + \lambda\, w \in H_1$.

Furthermore, using the definition of H_1 it is immediate to verify that if $u \in H_1$ then $\lambda\, u \in H_1$ for all $\lambda \geq 0$.

Finally, we show that $H_1 \cap \{-H_1\} = \{0\}$. To proof this, from assumption (H_5) we distinguish two cases:

- There exists $i_0 \in I_1$ such that $[m_{i_0}^{12}, n_{i_0}^{12}] \equiv I$.

  Suppose that $u, -u \in H_1$. Then, $u^{(i_0)}(t) \geq 0$ and $-u^{(i_0)}(t) \geq 0$ for all $t \in I$ and this implies that $u^{(i_0)}(t) = 0$ on I. Therefore, $u^{(i_0-1)}$ is constant on I if $i_0 \geq 1$.

  Since $i_0 - 1 \in I_1$ we have that $u^{(i_0-1)}(t) \geq 0$ and $-u^{(i_0-1)}(t) \geq 0$ for all $t \in [m_{i_0-1}^{12}, n_{i_0-1}^{12}]$, that is $u^{(i_0-1)}(t) = 0$ on $[m_{i_0-1}^{12}, n_{i_0-1}^{12}]$. Thus, since $u^{(i_0-1)}$ is constant on I we deduce that $u^{(i_0-1)} \equiv 0$ on I.

  Repeating the same argument by induction we obtain that $u \equiv 0$ on I. Therefore, $H_1 \cap \{-H_1\} = \{0\}$.

- There exists $i_0 \in I_1$ such that $[c_{i_0}^{12}, d_{i_0}^{12}] \equiv I$.

  Suppose that $u, -u \in H_1$. In this case, we have that

  $$\min_{t\in[a_{i_0}^{12},b_{i_0}^{12}]} u^{(i_0)}(t) \geq \xi^1 \|u^{(i_0)}\|_I \text{ and } \min_{t\in[a_{i_0}^{12},b_{i_0}^{12}]} (-u^{(i_0)}(t)) \geq \xi^1 \|u^{(i_0)}\|_I.$$

  So, $\|u^{(i_0)}\|_I = 0$, which implies that $u^{(i_0)} \equiv 0$ on I. Again, using the argument of the previous case we deduce that $u \equiv 0$ on I and this completes the proof.

□

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We will apply Krasnoselskii's fixed point Theorem 1.3.7 to operator $T : H \to H$ given by $T = (T_1, T_2)$ with $T_1 : H \to H_1$ defined as

$$\begin{aligned} T_1(u,v)(t) := &\int_a^b G_1(t,s)\, f_1(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds \\ &+ \int_a^b G_2(t,s)\, f_2(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds, \quad t \in I, \end{aligned} \tag{7.3.18}$$

and $T_2 : H \to H_2$ defined as

$$\begin{aligned} T_2(u,v)(t) := &\int_a^b G_3(t,s)\, f_1(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds \\ &+ \int_a^b G_4(t,s)\, f_2(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds, \quad t \in I, \end{aligned} \tag{7.3.19}$$

to deduce the existence of a fixed point of that operator.

Lemma 7.3.2. *If assumptions $(H_1) - (H_7)$ hold, then the operator $T : H \to H$ is compact.*

Proof. We will divide the proof in three steps:

Step 1. T is well-defined in H.

Let $(u, v) \in H$. We show that $T(u, v) \in X$, that is, $T_1(u, v) \in C^{n-1}(I)$ and $T_2(u, v) \in C^{m-1}(I)$. We do the proof for T_1 (for T_2 is analogous).

For all $i \in J_1$ and $t \in I$, we have that

$$\begin{aligned} (T_1(u,v))^{(i)}(t) = &\int_a^b \frac{\partial^i}{\partial t^i} G_1(t,s)\, f_1(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds \\ &+ \int_a^b \frac{\partial^i}{\partial t^i} G_2(t,s)\, f_2(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s))\, ds. \end{aligned}$$

Now, let $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} t_k = \tilde{t} \in I$. By the assumption (H_1) we have that

$$\begin{aligned} &\lim_{k\to\infty} \frac{\partial^i}{\partial t^i} G_1(t_k,s)\, f_1(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s)) \\ =&\frac{\partial^i}{\partial t^i} G_1(\tilde{t},s)\, f_1(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s)), \end{aligned}$$

and

$$\begin{aligned} &\lim_{k\to\infty} \frac{\partial^i}{\partial t^i} G_2(t_k,s)\, f_2(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s)) \\ =&\frac{\partial^i}{\partial t^i} G_2(\tilde{t},s)\, f_2(s, u(s), \dots, u^{(n-1)}(s), v(s), \dots, v^{(m-1)}(s)), \end{aligned}$$

for a.e. $s \in I$.

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Moreover, since f_r, $r=1,2$, is a $L^1(I)-$Carathéodory function we obtain

$$\begin{aligned}&\left|\frac{\partial^i}{\partial t^i}G_r(t,s)\,f_r(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\right|\\ \leq&\left|\frac{\partial^i}{\partial t^i}G_r(t,s,\delta_1,\dots,\delta_n)\right|\mu^r_{\|(u,v)\|_X}(s)\leq h_j(s)\,\mu^r_{\|(u,v)\|_X}(s).\end{aligned}$$

Therefore, by (H_7) and Lebesgue's Dominated Convergence Theorem we obtain that

$$\begin{aligned}\lim_{k\to\infty}(T_1(u,v))^{(i)}(t_k)=&\lim_{k\to\infty}\int_a^b\frac{\partial^i}{\partial t^i}G_1(t_k,s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\lim_{k\to\infty}\int_a^b\frac{\partial^i}{\partial t^i}G_2(t_k,s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ =&(T_1(u,v))^i(\tilde{t}).\end{aligned}$$

Thus, $T_1(u,v)\in C^{n-1}(I)$.

We show that $T(u,v)\in H$, that is, $T_1(u,v)\in H_1$ and $T_2(u,v)\in H_2$.

For $i\in I_1$ we have that $(T_1(u,v))^{(i)}(t)\geq 0$ for all $t\in[m_i^{12},n_i^{12}]$. Moreover, for $i\in K_1$ and $t\in[c_i^{12},d_i^{12}]$ we have that

$$\begin{aligned}|(T_1(u,v))^{(i)}(t)|\leq&\int_a^b\left|\frac{\partial^i}{\partial t^i}G_1(t,s)\right|f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\int_a^b\left|\frac{\partial^i}{\partial t^i}G_2(t,s)\right|f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ \leq&\int_a^b\phi_i^1(s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\int_a^b\phi_i^2(s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\end{aligned}$$

Therefore,

$$\begin{aligned}\|(T_1(u,v))^{(i)}\|_{[c_i^{12},d_i^{12}]}\leq&\int_a^b\phi_i^1(s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\int_a^b\phi_i^2(s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\end{aligned}$$

On the other hand, for $t\in[a_i^{12},b_i^{12}]$, we have that

$$\begin{aligned}(T_1(u,v))^{(i)}(t)=&\int_a^b\frac{\partial^i}{\partial t^i}G_1(t,s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\int_a^b\frac{\partial^i}{\partial t^i}G_2(t,s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ \geq&\int_a^b\xi_i^1\,\phi_i^1(s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\ &+\int_a^b\xi_i^2\,\phi_i^2(s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\end{aligned}$$

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So, for $i \in K_1$ we have that

$$\min_{t\in[a_i^{12},b_i^{12}]}(T_1(u,v))^{(i)}(t) \geq \xi^1 \|(T_1(u,v))^{(i)}\|_{[c_i^{12},d_i^{12}]}.$$

Therefore, $T_1(u,v) \in H_1$. Analogously, it is proved that $T_2(u,v) \in H_2$.

Step 2. T is continuous in X.

This is equivalent to show that T_1 and T_2 are continuous. Let $\{(u_k,v_k)\}_{k\in\mathbb{N}} \subset X$ such that $u_k \to u$ in $C^{n-1}(I)$ and $v_k \to v$ in $C^{m-1}(I)$. Then, $u_k^{(i)} \to u^{(i)}$, $i \in J_1$, and $v_k^{(j)} \to u^{(j)}$, $j \in J_2$, in $C(I)$. Moreover, $\|(u_k,v_k)\|_X \leq R$ for all $k \in \mathbb{N}$ for some $R \in \mathbb{R}^+$.

From (H_6), we have that

$$\begin{aligned}&\lim_{k\to\infty} f_r(s,u_k(s),\ldots,u_k^{(n-1)}(s),v_k(s),\ldots,v_k^{(m-1)}(s))\\ =&f_r(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s)),\end{aligned}$$

for a.e $s \in I$ and $r = 1, 2$. So, it is clear that

$$\begin{aligned}&\lim_{k\to\infty} h_i^r(s)\, f_r(s,u_k(s),\ldots,u_k^{(n-1)}(s),v_k(s),\ldots,v_k^{(m-1)}(s))\\ =&h_i^r(s)\, f_r(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s)),\end{aligned}$$

a.e $s \in I$, $r = 1, 2$ and $i \in J_1$. Moreover, for all $k \in \mathbb{N}$ we have that

$$\left|h_i^r(s)\, f_r(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s))\right| \leq h_i^r(s)\,\mu_R^r(s), \quad \text{a.e. } s \in I.$$

From (H_7), applying Lebesgue's Dominated Convergence Theorem we deduce that

$$\begin{aligned}\lim_{k\to\infty}\int_a^b h_i^r(s)\Big|&f_r(s,u_k(s),\ldots,u_k^{(n-1)}(s),v_k(s),\ldots,v_k^{(m-1)}(s))-\\ &f_r(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s))\Big|\,ds = 0.\end{aligned}$$

On the other hand, for $t \in I$, we have that

$$\begin{aligned}&|(T_1(u_k,v_k))^{(i)}(t) - (T_1(u,v))^{(i)}(t)|\\ \leq&\int_a^b h_i^1(s)\Big|f_1(s,u_k(s),\ldots,u_k^{(n-1)}(s),v_k(s),\ldots,v_k^{(m-1)}(s))-\\ &f_1(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s))\Big|\,ds\\ &+\int_a^b h_i^2(s)\Big|f_2(s,u_k(s),\ldots,u_k^{(n-1)}(s),v_k(s),\ldots,v_k^{(m-1)}(s))-\\ &f_2(s,u(s),\ldots,u^{(n-1)}(s),v(s),\ldots,v^{(m-1)}(s))\Big|\,ds.\end{aligned}$$

So, $\lim_{k\to\infty} \|(T_1(u_k,v_k))^{(i)} - (T_1(u,v))^{(i)}\|_\infty = 0$ which implies that $T_1(u_k,v_k) \to T_1(u,v)$ uniformly on I. Thus, operator T_1 is continuous. Similarly, we obtain that operator T_2 is continuous.

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Step 3. T is compact.

We will prove that T maps bounded sets into relatively compact sets. Let $B \subset X$ be a bounded set. Since B is bounded, there exists $L \in \mathbb{R}$, $L > 0$ such that $\|(u,v)\|_X \leq L$ for all $(u,v) \in B$. Then, for $i \in J_1$ we have that

$$\begin{aligned}|(T_1(u,v))^{(i)}(t)| \leq & \int_a^b \left|\frac{\partial^i}{\partial t^i} G_1(t,s)\right| f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds \\ & + \int_a^b \left|\frac{\partial^i}{\partial t^i} G_2(t,s)\right| f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds \\ \leq & \int_a^b \left(h_i^1(s)\,\mu_L^1(s) + h_i^2(s)\,\mu_L^2(s)\right)\,ds := M_i, \quad \text{for all } \ t \in I,\end{aligned}$$

where $M_i > 0$. So, for all $(u,v) \in B$, we have that

$$\|T_1(u,v)\|_{C^{n-1}(I)} \leq \max\{M_i : \ i \in J_1\},$$

that is, $T_1(B)$ is bounded. Using the same argument we obtain that $T_2(B)$ is bounded. Therefore, $T(B)$ is bounded.

Let us show now the equicontinuity of T. Let $t_2 \in I$ be fixed. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ given in (H_1) and for $i = 0,\dots,n-2$ with $n \geq 2$, it holds that $|t_1 - t_2| < \delta$ implies that for all $(u,v) \in B$, we have that

$$\begin{aligned}|(T_1(u,v))^{(i)}(t_1) - (T_1(u,v))^{(i)}(t_2)| \leq & \int_a^b \left|\frac{\partial^i}{\partial t^i} G_1(t_1,s) - \frac{\partial^i}{\partial t^i} G_1(t_2,s)\right| \mu_L^1(s)\,ds \\ & + \int_a^b \left|\frac{\partial^i}{\partial t^i} G_2(t_1,s) - \frac{\partial^i}{\partial t^i} G_2(t_2,s)\right| \mu_L^2(s)\,ds \\ \leq & \varepsilon \int_a^b \left(\mu_L^1(s) + \mu_L^2(s)\right)\,ds.\end{aligned}$$

Since $\mu_L^1, \mu_L^2 \in L^1(I)$, we have that there exists $N_1 > 0$ such that

$$|(T_1(u,v))^{(i)}(t_1) - (T_1(u,v))^{(i)}(t_2)| \leq \varepsilon\, N_1,$$

for all $(u,v) \in B$.

For the $(n-1)-$derivative, form assumption (H_1), given $\varepsilon > 0$ there exists $\delta > 0$ such

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that if $|t_1 - t_2| < \delta,\ \ t_1 < t_2$, we have that

$$\begin{aligned}
|(T_1(u,v))^{(n-1)}(t_1) - (T_1(u,v))^{(n-1)}(t_2)| \leq & \int_a^b \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,s) \Big| \mu_L^1(s)\, ds \\
& + \int_a^b \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,s) \Big| \mu_L^2(s)\, ds \\
= & \int_a^{t_1} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,s) \Big| \mu_L^1(s)\, ds \\
& + \int_{t_1}^{t_2} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,s) \Big| \mu_L^1(s)\, ds \\
& + \int_{t_2}^{b} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,s) \Big| \mu_L^1(s)\, ds \\
& + \int_a^{t_1} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,s) \Big| \mu_L^2(s)\, ds \\
& + \int_{t_1}^{t_2} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,s) \Big| \mu_L^2(s)\, ds \\
& + \int_{t_2}^{b} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,s) \Big| \mu_L^2(s)\, ds.
\end{aligned}$$

From (H_1), we deduce that first, third, fourth, and sixth integrals in the above expression can be arbitrarily small when $|t_1 - t_2| < \delta$. On the other hand, since

$$\Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,\cdot) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,\cdot) \Big| \mu_L^1(\cdot),\ \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,\cdot) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,\cdot) \Big| \mu_L^2(\cdot) \in L^1[t_1,t_2],$$

there exists $\delta' > 0$ such that $|t_1 - t_2| < \delta'$ implies that

$$\int_{t_1}^{t_2} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t_2,s) \Big| \mu_L^1(s)\, ds + \int_{t_1}^{t_2} \Big| \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_1,s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G_2(t_2,s) \Big| \mu_L^2(s)\, ds < \varepsilon.$$

Thus, for $|t_1 - t_2| < \min\{\delta, \delta'\}$, $t_1 < t_2$ there exists $N_2 > 0$ such that

$$|(T_1(u,v))^{(n-1)}(t_1) - (T_1(u,v))^{(n-1)}(t_2)| \leq \varepsilon\, N_2$$

for all $(u,v) \in B$.

Similarly, for $|t_1 - t_2| < \delta$, $t_1 > t_2$, we infer that there exists $N_3 > 0$ such that

$$|(T_1(u,v))^{(n-1)}(t_1) - (T_1(u,v))^{(n-1)}(t_2)| \leq \varepsilon\, N_3$$

for all $(u,v) \in B$.

This demonstrates the pointwise equicontinuity of T_1 in I. Since I is compact, the pointwise equicontinuity is equivalent to uniform equicontinuity (see [78, page 30 and problem 31]). Thus, $T_1(B)$ is an equicontinuous set in $C^{n-1}(I)$. In analogous way, we can show that $T_2(B)$ is also an equicontinuous set in $C^{m-1}(I)$. Therefore, $T(B)$ is an equicontinuous set in X. By Ascoli-Arzelà's Theorem 1.3.4, we deduce that $T(B)$ is relatively compact, that is, T is a compact operator. □

Next, we will prove the following main result of the existence of solutions of problem (7.0.1).

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Theorem 7.3.3. *Suppose that the conditions* $(H_1)-(H_8)$ *hold. Then problem* (7.0.1) *has at least one nontrivial solution* $(u,v)\in X$.

Proof. First, we note that the solutions of problem (7.0.1) coincide with fixed points of the operator $T=(T_1,T_2)$ with T_1 and T_2 defined in (7.3.18)–(7.3.19).

We know from Lemma 7.3.2 that $T:H\to H$ is a compact.

Now, we show that $T(u,v)\not\leq(u,v)$, for all $(u,v)\in H\cap\partial\Omega_1$ with

$$\Omega_1=\{(u,v)\in H:\ \|(u,v)\|_H<\rho_1\},$$

for some $\rho_1>0$.

From assumption (H_4), assuming that the first option (H_4^1) is satisfied, there exists $i_0\in K_1$ such that $S_1\neq\varnothing$ and $[a_{i_0}^{12},b_{i_0}^{12}]$ is a nondegenerate interval (the case for j_0 is proved analogously). Let us define

$$\varepsilon_1=\frac{1}{(\xi^1)^2\displaystyle\int_{a_{i_0}^{12}}^{b_{i_0}^{12}}\left(\phi_{i_0}^1(s)+\phi_{i_0}^2(s)\right)ds}.$$

By the second condition of hypothesis (H_8), there exists $\rho_1>0$ such that when $\|(u,v)\|_H<\rho_1$ we have that

$$\begin{aligned}&f_i(t,u(t),\dots,u^{(n-1)}(t),v(t),\dots,v^{(m-1)}(t))>\\&\varepsilon_1\left(|u(t)|+\ldots+|u^{(n-1)}(t)|+|v(t)|+\ldots+|v^{(m-1)}(t)|\right),\end{aligned}$$

for all $t\in I$ and $i=1,2$.

Consider $(u,v)\in H\cap\partial\Omega_1$. Then, we have that for $t\in S_1$,

$$\begin{aligned}(T_1(u,v))^{(i_0)}(t)&=\int_a^b\frac{\partial^{i_0}}{\partial t^{i_0}}G_1(t,s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\&\quad+\int_a^b\frac{\partial^{i_0}}{\partial t^{i_0}}G_2(t,s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\&\geq\int_{a_{i_0}^{12}}^{b_{i_0}^{12}}\frac{\partial^{i_0}}{\partial t^{i_0}}G_1(t,s)\,f_1(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\&\quad+\int_{a_{i_0}^{12}}^{b_{i_0}^{12}}\frac{\partial^{i_0}}{\partial t^{i_0}}G_2(t,s)\,f_2(s,u(s),\dots,u^{(n-1)}(s),v(s),\dots,v^{(m-1)}(s))\,ds\\&>\varepsilon_1\,\xi^1\int_{a_{i_0}^{12}}^{b_{i_0}^{12}}(\phi_{i_0}^1(s)+\phi_{i_0}^2(s))(|u(s)|+\ldots+|u^{(n-1)}(s)|+|v(s)|+\ldots+|v^{(m-1)}(s)|)\,ds\\&\geq\varepsilon_1\,(\xi^1)^2\,\|u^{(i_0)}\|_{[c_{i_0}^{12},d_{i_0}^{12}]}\int_{a_{i_0}^{12}}^{b_{i_0}^{12}}(\phi_{i_0}^1(s)+\phi_{i_0}^2(s))\,ds=\|u^{(i_0)}\|_{[c_{i_0}^{12},d_{i_0}^{12}]}.\end{aligned}$$

Therefore, $(T_1(u,v))^{(i_0)}(t)>u^{(i_0)}(t)$ for all $t\in S_1$ and so it is proved that $T(u,v)\not\leq(u,v)$.

Next, we prove that $T(u,v)\not\geq(u,v)$, for all $(u,v)\in H\cap\partial\Omega_2$ where Ω_2 is an open set that is defined later.

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Let us define

$$\varepsilon_2 = \min\left\{ \frac{1}{(n+m)\int_a^b (h_i^1(s)+h_i^2(s))\,ds} : \ i \in J_1 \right\}.$$

By the first condition of hypothesis (H_8), there exists $M > 0$ such that when $\min\{|u^{(i)}(t)| : \ i \in J_1\} \geq M$ and $\min\{|v^{(j)}(t)| : \ j \in J_2\} \geq M$ we have that

$$\begin{aligned} & f_i(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(m-1)}(t)) \\ \leq & \varepsilon_2 \left(|u(t)| + \ldots + |u^{(n-1)}(t)| + |v(t)| + \ldots + |v^{(m-1)}(t)|\right) \\ \leq & \varepsilon_2 \,(n+m)\, \|(u,v)\|_H, \end{aligned}$$

for all $t \in I$ and $i = 1, 2$.

Consider $\rho_2 > \{\rho_1, M\}$. Using similar arguments to [69, Theorem 3] it can be proved that the set

$$\Omega_2 = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{m-1} \Big\{ (u,v) \in H : \ \min_{t\in I} |u^{(i)}(t)| < \rho_2, \ \ \min_{t\in I} |v^{(j)}(t)| < \rho_2 \Big\},$$

is unbounded in the cone H and the fixed point index of operator T with respect to Ω_2 is only defined in the case that the set of fixed points of operator T in Ω_2, that is, $(id - T)^{-1}\{0\} \cap \Omega_2$, is compact.

Now, let us see that $\|T(u,v)\|_H \leq \|(u,v)\|_H$ for all $(u,v) \in H \cap \partial\Omega_2$. This is equivalently to show that for all $(u,v) \in H \cap \partial\Omega_2$,

$$\|T_1(u,v)\|_{C^{n-1}(I)} \leq \|(u,v)\|_H \ \text{ and } \ \|T_2(u,v)\|_{C^{m-1}(I)} \leq \|(u,v)\|_H.$$

The proof will be done only for T_1, as the other case is similar.

Consider $(u,v) \in H \cap \partial\Omega_2$. Therefore,

$$\min\left\{ \min_{t\in I} |u^{(i)}(t)| : \ i \in J_1 \right\} = \rho_2 \ \text{ and } \ \min\left\{ \min_{t\in I} |v^{(j)}(t)| : \ j \in J_2 \right\} = \rho_2.$$

Then, for all $i \in J_1$, we have that

$$\begin{aligned} (T_1(u,v))^{(i)}(t) = & \int_a^b \frac{\partial^i}{\partial t^i} G_1(t,s)\, f_1(s, u(s), \ldots, u^{(n-1)}(s), v(s), \ldots, v^{(m-1)}(s))\, ds \\ & + \int_a^b \frac{\partial^i}{\partial t^i} G_2(t,s)\, f_2(s, u(s), \ldots, u^{(n-1)}(s), v(s), \ldots, v^{(m-1)}(s))\, ds \\ \leq & \int_a^b h_i^1(s)\, f_1(s, u(s), \ldots, u^{(n-1)}(s), v(s), \ldots, v^{(m-1)}(s))\, ds \\ & + \int_a^b h_i^2(s)\, f_2(s, u(s), \ldots, u^{(n-1)}(s), v(s), \ldots, v^{(m-1)}(s))\, ds \\ \leq & (n+m)\, \varepsilon_2\, \|(u,v)\|_H \int_a^b (h_i^1(s) + h_i^2(s))\, ds \leq \|(u,v)\|_H. \end{aligned}$$

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Thus, $\|T_1(u,v)\|_{C^{n-1}(I)} \leq \|(u,v)\|_H$, for all $(u,v) \in H \cap \partial\Omega_2$, and so, $T(u,v) \not\geq (u,v)$, for all $(u,v) \in H \cap \partial\Omega_2$.

Therefore, from Theorem 1.3.7, we deduce that the operator T has a fixed point in $H \cap (\bar{\Omega}_2 \setminus \Omega_1)$ which is a solution of problem (7.0.1). □

As a consequence, we may study the existence of solution of the following nonlinear problem:

$$\begin{cases} L_1(u,v)(t) = f_1(t,u(t),\ldots,u^{(n-1)}(t),v(t),\ldots,v^{(m-1)}(t)), & t \in I, \\ L_2(u,v)(t) = f_2(t,u(t),\ldots,u^{(n-1)}(t),v(t),\ldots,v^{(m-1)}(t)), & t \in I, \\ B_i(u,v) = \delta_i\, C_i(u,v), \quad i = 1,\ldots,n+m, \end{cases} \tag{7.3.20}$$

where $f_i : I \times \mathbb{R}^{n+m} \to [0,\infty)$, $i = 1,2$, are $L^1(I)-$Carathéodory functions and L_1 and L_2 are introduced in (7.0.4).

As a notation, we will denote

$$\begin{aligned} G_1(t,s) &\equiv G_{1,n}(t,s,\delta_1,\ldots,\delta_{n+m}), & G_2(t,s) &\equiv G_{1,n+m}(t,s,\delta_1,\ldots,\delta_{n+m}), \\ G_3(t,s) &\equiv G_{n+1,n}(t,s,\delta_1,\ldots,\delta_{n+m}), & G_4(t,s) &\equiv G_{n+1,n+m}(t,s,\delta_1,\ldots,\delta_{n+m}). \end{aligned}$$

It is easy to verify that u and v are solutions of problem (7.3.20) if and only if they satisfy the system of integral equations (7.0.1). Therefore, we arrive at the following result.

Theorem 7.3.4. *Suppose that the conditions $(H_1)-(H_8)$ hold. Then problem* (7.3.20) *has at least one nontrivial solution $(u,v) \in X$.*

In the sequel, we present an example to illustrate the previous result.

Example 7.3.5. *Consider the following nonlinear problem*

$$\begin{cases} u''(t) - v(t) + (t^2+1)\, e^{-|u(t)|-|v(t)|} = 0, & t \in [0,1], \\ v'(t) - u'(t) + \dfrac{e^t}{\ln(e+(u'(t))^2+(v(t))^2)} = 0, & t \in [0,1], \\ u(0) = 0, \\ u'(0) = u'(1), \\ v(0) = 0. \end{cases} \tag{7.3.21}$$

In this case $n = 2$, $m = 1$, $J_1 = \{0,1\}$, $J_2 = \{0\}$ and the functions $f_1(t,x,y,z) = (t^2+1)\, e^{-|x|-|z|}$, $f_2(t,x,y,z) = \frac{e^t}{\ln(e+y^2+z^2)}$ satisfy conditions (H_6) and (H_8).

Since the functions f_1 and f_2 are positive and appear by adding in the equations of problem (7.3.21)*, we have to change the sign of Green's functions. Using the calculus of*

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Example 7.2.4, we obtain that

$$G_1(t,s) = \begin{cases} \sinh(s-t) - \dfrac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)}, & 0 \le s \le t \le 1, \\ -\dfrac{\cosh(s-1)\,\sinh(t)}{1-\cosh(1)}, & 0 \le t < s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} -\cosh(s-t) + \dfrac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)} + 1, & 0 \le s \le t \le 1, \\ \dfrac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)}, & 0 \le t < s \le 1, \end{cases}$$

$$G_3(t,s) = G_1(t,s), \ \textit{for all} \ \ t,s \in [0,1],$$

and

$$G_4(t,s) = \begin{cases} -\cosh(s-t) + \dfrac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)}, & 0 \le s \le t \le 1, \\ \dfrac{\sinh(s-1)\,\sinh(t)}{1-\cosh(1)}, & 0 \le t < s \le 1. \end{cases}$$

The first derivative of G_1 is

$$\frac{\partial}{\partial t}G_1(t,s) = \begin{cases} -\cosh(s-t) - \dfrac{\cosh(s-1)\,\cosh(t)}{1-\cosh(1)}, & 0 \le s \le t \le 1, \\ -\dfrac{\cosh(s-1)\,\cosh(t)}{1-\cosh(1)}, & 0 \le t < s \le 1, \end{cases}$$

The functions G_1 and $\frac{\partial}{\partial t}G_1$ are positive on $[0,1]\times[0,1]$. So, we can take $[m_0^1, n_0^1] = [0,1]$ and $[m_1^1, n_1^1] = [0,1]$.

Let $\mu \in I$ be fixed. The function G_1 is continuous on $[0,1]\times[0,1]$, so the hypothesis (H_1) is immediate for $i = 0$. Moreover, the first derivative $\frac{\partial}{\partial t}G_1$ is continuous on $([0,1]\times[0,1]) \setminus \{(t,t) : t \in [0,1]\}$, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-\mu| < \delta$ implies that

$$\left|\frac{\partial G_1}{\partial t}(t,s) - \frac{\partial G_1}{\partial t}(\mu,s)\right| < \varepsilon, \ \ \textit{for all} \ \ s \neq \mu.$$

Therefore, the hypothesis (H_1) holds for G_1.

Moreover, it can be checked that

$$|G_1(t,s)| \le h_0^1(s) = G_1(1,s) = \sinh(s-1) - \frac{\cosh(s-1)\,\sinh(1)}{1-\cosh(1)},$$

and

$$\left|\frac{\partial}{\partial t}G_1(t,s)\right| \le h_1^1(s) = \frac{\cosh(s-1)\,\cosh(s)}{\cosh(1)-1},$$

for all $t \in [0,1]$ and $s \in [0,1]$.

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If we take $\phi_0^1(s) = h_0^1(s)$, $[c_0^1, d_0^1] = [0,1]$, *and* $[a_0^1, b_0^1] = [0,1]$, *it can be verified that for the constant* $\xi_0^1 = \frac{1}{n_1}$,

$$G_1(t,s) \geq \xi_0^1 \, \phi_0^1(s) \;\; \textit{for all} \;\; t \in [0,1], \;\; s \in [0,1],$$

being n_1 *a sufficiently large constant such that* $\xi_0^1 \in (0,1)$.

Analogously, if we choose $\phi_1^1(s) = h_1^1(s)$, $[c_1^1, d_1^1] = [0,1]$, *and* $[a_1^1, b_1^1] = [0,1]$, *it can be shown that verified for the constant* $\xi_1^1 \approx 0.5728$,

$$\frac{\partial}{\partial t} G_1(t,s) \geq \xi_1^1 \, \phi_1^1(s) \;\; \textit{for all} \;\; t \in [0,1], \;\; s \in [0,1].$$

The first derivative of G_2 *is*

$$\frac{\partial}{\partial t} G_2(t,s) = \begin{cases} \sinh(s-t) + \dfrac{\sinh(s-1)\,\cosh(t)}{1-\cosh(1)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{\sinh(s-1)\,\cosh(t)}{1-\cosh(1)}, & 0 \leq t < s \leq 1, \end{cases}$$

The functions G_2 *and* $\frac{\partial}{\partial t} G_2$ *are positive on* $[0,1]\times[0,1]$. *So, we can take* $[m_0^2, n_0^2] = [0,1]$ *and* $[m_1^2, n_1^2] = [0,1]$.

The function G_2 *belongs to* $C^1([0,1]\times[0,1])$, *so it trivially satisfies hypothesis* (H_1).

It can be seen that

$$|G_2(t,s)| \leq h_0^2(s) = G_2(1,s) = \cosh(s) - \frac{(1+e)\,\sinh(s)}{e-1} + 1,$$

and

$$\left| \frac{\partial}{\partial t} G_2(t,s) \right| \leq h_1^2(s) = \frac{\partial}{\partial t} G_2(s,s) = \frac{\cosh(s)\,\sinh(s-1)}{1-\cosh(1)},$$

for all $t \in [0,1]$, $s \in [0,1]$.

In this case, if we take $\phi_0^2(s) = h_0^2(s)$, $\phi_1^2(s) = h_1^2(s)$ *and* $[c_0^2, d_0^2] = [c_1^2, d_1^2] = [a_0^2, b_0^2] = [a_1^2, b_1^2] = [0,1]$, *then it is verified for the constants* $\xi_0^2 = \frac{1}{n_2}$ *and* $\xi_1^2 \approx 0.6481$ *that*

$$G_2(t,s) \geq \xi_0^2 \, \phi_0^2(s) \;\; \textit{for all} \;\; t \in [0,1], \;\; s \in [0,1],$$

and

$$\frac{\partial}{\partial t} G_2(t,s) \geq \xi_1^2 \, \phi_1^2(s) \;\; \textit{for all} \;\; t \in [0,1], \;\; s \in [0,1],$$

being n_2 *a sufficiently large constant such that* $\xi_0^2 \in (0,1)$.

Moreover, for $i_0 = 0, 1$ *we have that* $[a_{i_0}^{12}, b_{i_0}^{12}] = [0,1]$ *are nondegenerate intervals. So, the assumption* (H_4^1) *is fulfilled.*

The function G_3 *fulfills the same assumptions as the function* G_1 *because they are equal. So,* G_3 *satisfies the hypothesis* (H_2) *and we can take* $[m_0^3, n_0^3] = [0,1]$.

Also, by taking $\phi_0^3(s) = h_0^1(s)$, $[c_0^3, d_0^3] = [0,1]$, *and* $[a_0^3, b_0^3] = [0,1]$, *we have for the constant* $\xi_0^3 := \xi_0^1$ *that*

$$G_3(t,s) \geq \xi_0^1 \, \phi_0^3(s) \;\; \textit{for all} \;\; t \in [0,1], \;\; s \in [0,1].$$

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Finally, the function G_4 is continuous on $([0,1]\times[0,1])\setminus\{(t,t):\ t\in[0,1]\}$ and is nonnegative on the triangle $\{(t,s)\in[0,1]\times[0,1]:\ t<s\}$. Thus, the hypothesis (H_2) is fulfilled for G_4 and we can take $[m_0^4,n_0^4]=\{0\}$.

Analogously, it holds that

$$|G_4(t,s)|\le h_0^4(s),\ \ \textit{for all}\ \ t\in[0,1],\ s\in[0,1],$$

with

$$h_0^4(s):=\max\left\{\frac{\sinh(s-1)\,\sinh(s)}{1-\cosh(1)},1-\frac{\sinh(s-1)\,\sinh(s)}{1-\cosh(1)}\right\}.$$

If we choose $\phi_0^4(s)=h_0^4(s)$, $[c_0^4,d_0^4]=[0,1]$, and $[a_0^4,b_0^4]=\{0\}$, it is satisfied for the constant $\xi_0^4=\frac{1}{n_4}$ that

$$G_4(t,s)\ge\xi_0^4\,\phi_0^4(s)\ \ \textit{for all}\ \ t=0,\ s\in[0,1],$$

being n_4 a sufficiently large constant such that $\xi_0^4\in(0,1)$.

As a consequence, we deduce that $I_1=K_1=\{0,1\}$, $I_2=K_2=\{0\}$ and the assumptions $(H_1)-(H_8)$ hold. Moreover, the cone $H=H_1\times H_2$ that we use to localize the solutions is given by

$$H_1=\left\{u\in C^1(I):\ \begin{array}{c}u(t)\ge0,\ u'(t)\ge0\ \ t\in[0,1],\\ \min\limits_{t\in[0,1]}u(t)\ge\xi^1\|u\|_{[0,1]},\ \min\limits_{t\in[0,1]}u'(t)\ge\xi^1\|u'\|_{[0,1]}\end{array}\right\},$$

and

$$H_2=\left\{v\in C(I):\ \ v(0)\ge\xi^2\|v\|_{[0,1]}\right\},$$

where $\xi^1=\min\{\frac{1}{n_1},\frac{1}{n_2}\}$ and $\xi^2=\min\{\frac{1}{n_1},\frac{1}{n_4}\}$.

Thus, from Theorem 7.3.4, there is at least one nontrivial solution $(u,v)\in C^1(0,1)\times C(0,1)$ of problem (7.3.21)*.*

As we mentioned in the introduction of this chapter, the following example shows the non-applicability of results of the previous works by transforming the problem of the above example into one with separate variables and two Green's functions.

Example 7.3.6. *Let's consider again the problem of Example 7.3.5. By transforming this problem into one with separate variables, we have the following equivalent problem*

$$\begin{cases}u''(t)-u(t)+\tilde f_1(t,u(t),u'(t),v(t))=0,\ \ t\in[0,1],\\ v'(t)-\tilde f_2(t,u(t),u'(t),v(t))=0,\ \ t\in[0,1],\\ u(0)=0,\\ u'(0)=u'(1),\\ v(0)=0,\end{cases}$$

where $\tilde f_1(t,x,y,z)=x-z+(t^2+1)\,e^{-|x|-|z|}$ and $\tilde f_2(t,x,y,z)=y-\frac{e^t}{\ln(e+y^2+z^2)}$.

Note that the functions $\tilde f_1$ and $\tilde f_2$ do not satisfy the hypothesis (H_8) (the positive sign of $\bar f_1$ and $\bar f_2$ is not guaranteed either) since the limit of the first condition of (H_8) does no exist.

However, by passing the linear terms to the right hand side of the equation and subtracting them from the f_i, $i=1,2$, the new non linearities $\tilde f_i$:

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1) *They could lose the constant sign.*

2) *They would not have a sublinear behavior at the limit.*

These two statements are common hypotheses in previous works which consider only two Green's functions, and so this problem could not be solved using such methods. This shows the benefits of our approach to deal with nonlinear systems where the linear operators depend on two non-separate variables u *and* v*.*

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Chapter 8
Constant Sign Derivatives of Green's Functions Via Spectral Theory

Many papers in the literature have studied the qualitative properties of solutions of differential equations coupled with different boundary conditions. Several techniques have been used to ensure the existence, non-existence and multiplicity of solutions. Among them, stands out the method of lower and upper solutions, coupled with monotone iterative techniques which, as it is very well-known, are equivalent to the constant sign of the Green's function of the corresponding linear problem. Moreover the method of constructing suitable cones in Banach spaces where to apply the well-known Krasnoselskii's fixed-point theorem, see for instance [5,36,50,61,64,85,86] is a powerful tool to deduce the existence of positive solutions of the nonlinear problem. Such cones are constructed using constant sign properties of the Green's function and some of its derivatives.

To be concise, in this chapter, we consider the following $\mathrm{n}^{\mathrm{th}}-$order linear differential equation

$$T_n[M]\,u(t) \equiv u^{(n)}(t) + M\,u(t) = 0\,,\ t \in I\,, \tag{8.0.1}$$

where $M \in \mathbb{R}$ a is real parameter, coupled to the two-point boundary conditions

$$u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) \quad = 0, \tag{8.0.2}$$

$$u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0, \tag{8.0.3}$$

with $k \in \{1, \ldots, n-1\}$ and the sets of indices

$$\{\sigma_1, \ldots, \sigma_k\}, \{\varepsilon_1, \ldots, \varepsilon_{n-k}\} \subset \{0, \ldots, n-1\},$$

satisfying that

$$0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n-1\,, \quad 0 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-k} \leq n-1\,.$$

This chapter is devoted to study the set of parameters M in which certain partial derivatives of the Green's function g_M, related to problem (8.0.1)–(8.0.3), are of constant sign without using the explicit expression of Green's function. It should be noted that, in general, it is not possible to obtain the explicit expression of the Green's function, specially for non constant coefficients.

For the particular case of equation (8.0.1), together with boundary conditions (8.0.2)–(8.0.3), the interval of parameters for which the Green's function has constant sign has been characterized in [26, Theorem 8.1]. Also for boundary conditions of the type $(k, n-k)$, i.e., $\sigma_j = j-1$ for all $j = 1, \ldots, k$, and $\varepsilon_j = j-1$ for all $j = 1, \ldots, n-k$ which are a particular

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case of the previous ones, a characterization of the constant sign intervals of the Green's function is given in [24]. Moreover the calculation of the eigenvalues that characterizes these intervals up to $n = 8$ is obtained in that reference.

In [26], the authors studied the strongly inverse positive (negative) character of the general n^{th}- order operator

$$\hat{T}_n[M]\,u(t) = u^{(n)}(t) + p_1(t)\,u^{(n-1)}(t) + \cdots + (p_n(t) + M)\,u(t) = 0\,,\ t \in I\,,$$

where $p_j \in C^{n-j}(I)$ are given functions, coupled with different boundary conditions (8.0.2)–(8.0.3). This work generalizes the ones given in [24] for the problems with the $(k, n-k)$ boundary conditions.

In [27], it is studied the fourth order operator

$$T[M]\,u(t) \equiv u^{(4)}(t) + p_1(t)\,u'''(t) + p_2(t)\,u''(t) + M\,u(t),\ \ t \in I,$$

coupled with the two point boundary conditions:

$$u(a) = u(b) = u''(a) = u''(b) = 0,$$

where, $p_1 \in C^3(I)$ and $p_2 \in C^2(I)$. Assuming that the second linear differential equation

$$L_2\,u(t) \equiv u''(t) + p_1(t)\,u'(t) + p_2(t)\,u(t) = 0,\ \ t \in I,$$

is disconjugate on I. The authors characterized the parameter's set where the Green's function related to operator $T[M]$ defined in the space

$$X = \{u \in C^4(I) :\ \ u(a) = u(b) = u''(a) = u''(b) = 0\},$$

is of constant sign on $I \times I$.

The main assumption is the disconjugacy character of the operator $T_n[M]$, which, as we know, is disconjugate for the value $M = 0$ (see [24]) at any arbitrary interval $[a, b]$. In the literature, sufficient conditions have been given to ensure the disconjugacy character of linear operators, we refer to [35, 40, 65, 76, 80, 83, 91] and the references therein. A characterization of this property is proved in [24], in which the optimal extremes of the intervals are given as the eigenvalues of suitable $(k, n-k)$ problems.

It is in this context that the corresponding spectral theory arises: we will prove that the constant sign interval is characterized by the first eigenvalues of the studied operator, related to suitable boundary conditions.

The study starts from conditions (8.0.2)–(8.0.3) and constructs the conditions satisfied by the derivatives of g_M, defining the necessary hypotheses that we need to obtain the main results, and deducing the relationships between the adjoint spaces of the respective related spaces. This way, we will obtain results that characterize the constant sign interval of the corresponding derivatives.

The novelty of the chapter is that the results generalize [26, Theorem 8.1] for the particular operator $T_n[M]$ defined in (8.0.1). Also, as a consequence of these results, we will deduce some cases in which the derivatives of the Green's function cannot be of constant sign. Furthermore, we will obtain a necessary condition that allow us to ensure that the corresponding derivatives are strictly positive (strictly negative) on the interior of the square of definition.

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The chapter is organized as follows: in a preliminary Section 8.1 we introduce the fundamental concepts, the results and the main hypotheses that are needed in the development of the chapter. The next Section 8.2 is devoted to prove the results that characterize the constant sign of the derivatives of the Green's function by distinguishing three cases. The main results are proved through spectral theory. This section includes also a corollary that provides sufficient conditions for derivatives to be nonpositive or nonnegative, a necessary condition to ensure the negative (positive) sign of certain derivatives with respect to t of g_M, and examples of application of our main results.

The results of this chapter are included in the work [23].

8.1 Preliminaries, Hypotheses and Main Assumptions

In this section, we will make a survey of several results and properties that will be used along the paper. The main part of these results are proven in [26], and many of them are given in [24].

Let us consider the $n^{\text{th}}-$order linear differential equation (8.0.1) coupled to the two-point boundary conditions (8.0.2)–(8.0.3).

Definition 8.1.1. Let us say that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy property (N_a) if

$$\sum_{\sigma_j<h} 1+\sum_{\varepsilon_j<h} 1\geq h\,,\quad \forall h\in\{1,\dots,n-1\}\,.$$

Notation 8.1.2. *Let us denote*

$$\alpha=\min\{i\in\{0,\dots,n-1\}:\ \ i\notin\{\sigma_1,\dots,\sigma_k\}\}\,, \tag{8.1.4}$$
$$\beta=\min\{i\in\{0,\dots,n-1\}:\ \ i\notin\{\varepsilon_1,\dots,\varepsilon_{n-k}\}\}\,. \tag{8.1.5}$$

We define the following set of functions related to the boundary conditions (8.0.2)–(8.0.3):

$$X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}=\left\{u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_k)}(a)=u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0\right\},$$

Also, we define the following sets of functions:

$$\begin{aligned}X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_{k-1}|\alpha\}}=\Big\{&u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_{k-1})}(a)=0\,,\ u^{(\alpha)}(a)=0\,,\\ &u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k})}(b)=0\Big\}\,,\\ X^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}\}}_{\{\sigma_1,\dots,\sigma_k|\alpha\}}=\Big\{&u\in C^n(I)\ \mid\ u^{(\sigma_1)}(a)=\cdots=u^{(\sigma_k)}(a)=0\,,\ u^{(\alpha)}(a)=0\,,\\ &u^{(\varepsilon_1)}(b)=\cdots=u^{(\varepsilon_{n-k-1})}(b)=0\Big\}\,,\end{aligned}$$

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$$X_{\{\sigma_1,\dots,\sigma_{k-1}\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}|\beta\}} = \Big\{ u \in C^n(I) \;\mid\; u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_{k-1})}(a) = 0\,, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k})}(b) = 0\,,\; u^{(\beta)}(b) = 0 \Big\}\,,$$

$$X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k-1}|\beta\}} = \Big\{ u \in C^n(I) \;\mid\; u^{(\sigma_1)}(a) = \cdots = u^{(\sigma_k)}(a) = 0\,, \\ u^{(\varepsilon_1)}(b) = \cdots = u^{(\varepsilon_{n-k-1})}(b) = 0\,,\; u^{(\beta)}(b) = 0 \Big\}\,.$$

Definition 8.1.3. Let us say that the operator $T_n[M]$ satisfies the property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ if, and only if, there exists the following decomposition:

$$T_0\,u(t) = u(t)\,, \quad T_l\,u(t) = \frac{d}{dt}\left(\frac{T_{l-1}\,u(t)}{v_l(t)}\right)\,,\; l = 1,\dots,n\,,\; t \in I, \tag{8.1.6}$$

where $v_l > 0$ on I, $v_l \in C^n(I)$ such that

$$T_n[M]\,u(t) = v_1(t)\,\dots\,v_n(t)\,T_n\,u(t)\,,\; t \in I\,,$$

and, moreover, such a decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{aligned} T_{\sigma_1}\,u(a) = &\quad \cdots \quad = T_{\sigma_k}\,u(a) \quad = 0\,, \\ T_{\varepsilon_1}\,u(b) = &\quad \cdots \quad = T_{\varepsilon_{n-k}}\,u(b) = 0\,. \end{aligned}$$

Remark 8.1.4. *Note that decomposition* (8.1.6) *is not unique, it depends on the choice of v_k for $k = 1,\dots,n$. Throughout this chapter, for the operator $T_n[0]\,u(t) = u^{(n)}(t)$ we choose the following decomposition:*

$$T_0\,u(t) = u(t)\,, \quad T_k\,u(t) = \frac{d}{dt}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right)\,,\; k = 1,\dots,n\,,\;\; t \in I,$$

where $v_k \equiv 1$ on I. That is, $T_k\,u(t) = u^{(k)}(t)$, $t \in I$. In particular:

$$T_n[0]\,u(t) = v_1(t)\,\dots\,v_n(t)\,T_n\,u(t)\,,\; t \in I\,,$$

and, moreover, this decomposition satisfies, for every $u \in X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$:

$$\begin{aligned} T_{\sigma_1}\,u(a) = u^{(\sigma_1)}(a) = 0\,, &\quad \dots\,, \quad T_{\sigma_k}\,u(a) \quad = u^{(\sigma_k)}(a) \quad = 0\,, \\ T_{\varepsilon_1}\,u(b) = u^{(\varepsilon_1)}(b) = 0\,, &\quad \dots\,, \quad T_{\varepsilon_{n-k}}\,u(b) = u^{(\varepsilon_{n-k})}(b) = 0\,, \end{aligned}$$

or which is the same $T_n[0]$ satisfies property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Lemma 8.1.5. *[26, Lemma 3.8] Let $\bar{M} \in \mathbb{R}$ be such that $T_n[\bar{M}]$ satisfies the property (T_d) in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$. Then $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfy the property (N_a) if, and only if, $M = 0$ is not an eigenvalue of $T_n[\bar{M}]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

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Let us denote by G_M the Green's matrix associated with the equivalent n-dimensional first order system associated to (8.0.1)–(8.0.3), which is given by the expression

$$G_M(t,s) = \begin{pmatrix} g_1(t,s) & g_2(t,s) & \cdots & g_{n-1}(t,s) & g_M(t,s) \\ \frac{\partial}{\partial t} g_1(t,s) & \frac{\partial}{\partial t} g_2(t,s) & \cdots & \frac{\partial}{\partial t} g_{n-1}(t,s) & \frac{\partial}{\partial t} g_M(t,s) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}} g_1(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}} g_2(t,s) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}} g_{n-1}(t,s) & \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t,s) \end{pmatrix}, \tag{8.1.7}$$

where g_M is the scalar Green's function related to operator $T_n[M]$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ (see [6, Section 1.4] for details). Moreover, by the axiomatic definition of G_M, we have that for all $s \in (a,b)$, the following equality holds

$$\frac{\partial}{\partial t} G_M(t,s) = A\, G_M(t,s), \quad \text{for all } t \in I \setminus \{s\}, \tag{8.1.8}$$

where $A = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -M & 0 \end{array}\right)$, with I_{n-1} the $(n-1)-$dimensional identity matrix, denotes the matrix that defines the n-dimensional system equivalent to our problem (8.0.1)–(8.0.3) and

$$B\, G_M(a,s) + C\, G_M(b,s) = 0, \quad \text{for all } \ s \in (a,b), \tag{8.1.9}$$

with $B,\ C \in \mathcal{M}_{n\times n}$, defined by $b_{j,1+\sigma_j} = 1$ for $j = 1,\ldots,k$ and $c_{j+k,1+\varepsilon_j} = 1$ for $j = 1,\ldots,n-k$; otherwise, $b_{ij} = 0$ and $c_{ij} = 0$.

In particular, it is easy to see, using the expressions (8.1.7) and (8.1.8) (see [6, Section 1.4] for details), that $g_M \in C^{n-2}(I \times I)$. Moreover, it is a C^n function on the triangles $a \le s < t \le b$ and $a \le t < s \le b$, it satisfies, as a function of t for any $s \in (a,b)$ fixed, the two-point boundary value conditions (8.0.2)-(8.0.3) and solves equation (8.0.1) for all $t \in I \setminus \{s\}$.

Moreover, for any $t \in (a,b)$ it satisfies

$$\frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^+,t) = \frac{\partial^{n-1}}{\partial t^{n-1}} g_M(t^-,t) + 1. \tag{8.1.10}$$

Studying the matrix Green's function G_M, it is proved in [24, page 13] that, for this particular case, $g_{n-j}(t,s)$ may be expressed as a function of $g_M(t,s)$ for all $j = 1,\ldots,n-1$ as follows:

$$g_{n-j}(t,s) = (-1)^j \frac{\partial^j}{\partial s^j} g_M(t,s)\,. \tag{8.1.11}$$

The adjoint operator of $T_n[M]$ follows the expression

$$T_n^*[M]v(t) \equiv (-1)^n\, v^{(n)}(t) + M\, v(t)\,, \tag{8.1.12}$$

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defined on the domain

$$D(T_n^*[M]) = \Big\{ v \in C^n(I) \;\Big|\; \sum_{i=0}^{n-1} (-1)^{n-1-i}\, v^{(n-1-i)}(b)\, u^{(i)}(b) = \sum_{i=0}^{n-1} (-1)^{n-1-i}\, v^{(n-1-i)}(a)\, u^{(i)}(a) \Big\},$$

for all $u \in D(T_n[M])$.

We denote by $g_M^*(t,s)$ the Green's function related to operator $T_n^*[M]$. Furthermore, the following equality is satisfied (see [6, section 1.4])

$$g_M^*(t,s) = g_M(s,t), \quad (t,s) \in I \times I. \tag{8.1.13}$$

In such a case, if we define the following operator

$$\widehat{T}_n[(-1)^n\, M] := (-1)^n\, T_n^*[M]\,, \tag{8.1.14}$$

we have, according to the above equality, that

$$\widehat{g}_{(-1)^n\, M}(t,s) = (-1)^n\, g_M^*(t,s) = (-1)^n\, g_M(s,t)\,, \tag{8.1.15}$$

where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n[(-1)^n M]$ in $D\left(T_n^*[M]\right)$.

As a particular case of the obtained results in [26, section 4], we can deduce that

$$D(T_n^*[M]) \equiv X^{*\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}} = X^{\{\delta_1,\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}\}},$$

being $\{\delta_1,\ldots,\delta_k\}, \{\tau_1,\ldots,\tau_{n-k}\} \subset \{0,\ldots,n-1\}$, such that $\delta_i < \delta_{i+1}$ and $\tau_j < \tau_{j+1}$, for $i = 1,\ldots,k-1$ and $j = 1,\ldots,n-k-1$, satisfying:

$$\begin{aligned} \{\sigma_1,\ldots,\sigma_k, n-1-\tau_1,\ldots,\, n-1-\tau_{n-k}\} &= \{0,\ldots,n-1\}, \\ \{\varepsilon_1,\ldots,\varepsilon_{n-k}, n-1-\delta_1,\ldots,\, n-1-\delta_k\} &= \{0,\ldots,n-1\}. \end{aligned}$$

Notation 8.1.6. *Let us define η, γ in the following way:*

$$\eta = \min\left\{ i \in \{0,\ldots,n-1\} :\;\; i \notin \{\tau_1,\ldots,\tau_{n-k}\} \right\}, \tag{8.1.16}$$

$$\gamma = \min\left\{ i \in \{0,\ldots,n-1\} :\;\; i \notin \{\delta_1,\ldots,\delta_k\} \right\}. \tag{8.1.17}$$

Analogously, let us denote by $\widehat{G}(t,s)$ the Green's matrix associated with the equivalent $n-$dimensional problem of $\widehat{T}_n[(-1)^n\, M]\, v(t) = 0$, $v \in X^{\{\delta_1,\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}\}}$ given by

$$\widehat{G}(t,s) = \begin{pmatrix} \widehat{g}_1(t,s) & \cdots & \widehat{g}_{n-1}(t,s) & \widehat{g}_{(-1)^n\, M}(t,s) \\ \dfrac{\partial}{\partial t}\,\widehat{g}_1(t,s) & \cdots & \dfrac{\partial}{\partial t}\,\widehat{g}_{n-1}(t,s) & \dfrac{\partial}{\partial t}\,\widehat{g}_{(-1)^n\, M}(t,s) \\ \vdots & \cdots & \vdots & \vdots \\ \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_1(t,s) & \cdots & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{n-1}(t,s) & \dfrac{\partial^{n-1}}{\partial t^{n-1}}\,\widehat{g}_{(-1)^n\, M}(t,s) \end{pmatrix}, \tag{8.1.18}$$

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where $\widehat{g}_{(-1)^n M}(t,s)$ is the scalar Green's function related to operator $\widehat{T}_n[(-1)^n M]$ in $X^{\{\delta_1,\ldots,\delta_k\}}_{\{\tau_1,\ldots,\tau_{n-k}\}}$.

Similarly to (8.1.11) we can deduce that

$$\widehat{g}_{n-j}(t,s) = (-1)^j \frac{\partial^j}{\partial s^j}\, \widehat{g}_{(-1)^n M}(t,s)\,. \tag{8.1.19}$$

By definition, $\widehat{G}(t,s)$ satisfies the equality

$$\widehat{B}\,\widehat{G}(a,s) + \widehat{C}\,\widehat{G}(b,s) = 0, \quad \text{for all } \ s \in (a,b), \tag{8.1.20}$$

where $\widehat{B}, \widehat{C} \in \mathcal{M}_{n\times n}$, defined as $(\widehat{B})_{i,\,\tau_i+1} = 1$ for $i = 1,\ldots,n-k$ and $(\widehat{C})_{j+n-k,\,1+\delta_j} = 1$ for $j = 1,\ldots,k$; otherwise, $(\widehat{B})_{i\,j} = 0$ and $(\widehat{C})_{i\,j} = 0$.

Remark 8.1.7. *From the previous definitions, as it is pointed out in [26, Remarks 4.1 and 4.2] we have that* $\alpha = n-1-\tau_{n-k}$, $\beta = n-1-\delta_k$, $\eta = n-1-\sigma_k$ *and* $\gamma = n-1-\varepsilon_{n-k}$.

Example 8.1.8. *Let us consider operator* $T_4[M]$ *coupled with the boundary conditions*

$$u(0) = u'(0) = u''(0) = u''(1) = 0.$$

In this case, $\{\sigma_1,\sigma_2,\sigma_3\} = \{0,1,2\}, \{\varepsilon_1\} = \{2\}, \{\tau_1\} = \{0\}$ *and* $\{\delta_1,\delta_2,\delta_3\} = \{0,2,3\}$. *Thus, we deduce that* $X^{*\{2\}}_{\{0,1,2\}} = X^{\{0,2,3\}}_{\{0\}}$. *Moreover,* $\alpha = 3\,, \beta = 0,\ \eta = 1$ *and* $\gamma = 1$.

Definition 8.1.9. Operator $T_n[M]$ is said to be inverse positive (negative) on $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ if every $u \in X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ such that $T_n[M]\,u \geq 0$ on I, satisfies $u \geq 0$ ($u \leq 0$) on I.

The above definition is equivalent to the constant sign of Green's function g_M related to $T_n[M]$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ as it is shown in the following result whose proof is analogous the one made in [26, Theorem 2.9].

Theorem 8.1.10. *Operator* $T_n[M]$ *is inverse positive (negative) in* $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ *if, and only if, the Green's function related to problem* (8.0.1)–(8.0.3) *is nonnegative (nonpositive) on its square of definition.*

Definition 8.1.11. Operator $T_n[M]$ is said to be strongly inverse positive (negative) in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ if every $u \in X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ such that $T_n[M]\,u \gneqq 0$ on I, satisfies that $u > 0$ (< 0) on (a,b) and, moreover, $u^{(\alpha)}(a) > 0$ (< 0) and $(-1)^\beta u^{(\beta)}(b) > 0$ (< 0), where α and β are defined in (8.1.4) and (8.1.5), respectively.

Analogously to Theorem 8.1.10, the following characterization can be shown:

Theorem 8.1.12. *Operator* $T_n[M]$ *is strongly inverse positive (negative) in* $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ *if, and only if, the Green's function related to problem* (8.0.1)–(8.0.3) *satisfies the following properties:*

- $g_M(t,s) > 0\,(< 0)$ *a.e. on* (a,b).

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- $\frac{\partial^\alpha}{\partial t^\alpha}\, g_M(t,s) > 0\,(<0)$ *for a.e.* $s \in (a,b)$.
- $(-1)^\beta \frac{\partial^\beta}{\partial t^\beta}\, g_M(t,s) > 0\,(<0)$ *for a.e.* $s \in (a,b)$.

Let $u \in X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ be such that $T_n[M]\,u(t) = 0$, $t \in I$. Then, differentiating equation (8.0.1) with respect to t, as a direct consequence of equations (8.1.7) and (8.1.8) we deduce that, for any $s \in (a,b)$ given, the $q^{\text{th}}-$derivative $v_s^q[M](t) := \dfrac{\partial^q}{\partial t^q}\, g_M(t,s)$, with $1 \le q \le n-1$, satisfies the equation $T_n[M]\, v_s^q[M](t) = 0$, for all $t \in I \setminus \{s\}$, together with the new boundary conditions

$$(v_s^q)^{(\mu_1^q)}[M](a) = \cdots = (v_s^q)^{(\mu_k^q)}[M](a) \quad = 0\,, \tag{8.1.21}$$

$$(v_s^q)^{(\rho_1^q)}[M](b) = \cdots = (v_s^q)^{(\rho_{n-k}^q)}[M](b) = 0\,. \tag{8.1.22}$$

Here, the sets of indices $\{\mu_1^q,\dots,\mu_k^q\},\ \{\rho_1^q,\dots,\rho_{n-k}^q\} \subset \{0,\dots,n-1\}$, satisfy that

$$0 \le \mu_1^q < \mu_2^q < \cdots < \mu_k^q \le n-1\,, \quad 0 \le \rho_1^q < \rho_2^q < \cdots < \rho_{n-k}^q \le n-1\,,$$

and are defined in the following way:

- If $A_q = \{i \in \{1,\dots,k\} /\ \ \sigma_i \ge q\} \neq \varnothing$, then

$$\mu_1^q = \sigma_j - q,\ \mu_2^q = \sigma_{j+1} - q,\ \dots,\ \mu_{k-(j-1)}^q = \sigma_k - q,$$
$$\mu_{k-(j-2)}^q = \sigma_1 - q + n,\ \dots,\ \mu_k^q = \sigma_{j-1} - q + n\,,$$

  with

$$j = \min A_q \ge 1. \tag{8.1.23}$$

- If $A_q = \varnothing$, then

$$\mu_i^q = \sigma_i - q + n,\ \ i = 1,\dots,k.$$

- If $B_q = \{i \in \{1,\dots,n-k\} /\ \ \varepsilon_i \ge q\} \neq \varnothing$, then

$$\rho_1^q = \varepsilon_r - q,\ \rho_2^q = \varepsilon_{r+1} - q,\ \dots,\ \rho_{n-k-(r-1)}^q = \varepsilon_{n-k} - q,$$
$$\rho_{n-k-(r-2)}^q = \varepsilon_1 - q + n,\ \dots,\ \rho_{n-k}^q = \varepsilon_{r-1} - q + n,$$

  with

$$r = \min B_q \ge 1. \tag{8.1.24}$$

- If $B_q = \varnothing$, then

$$\rho_i^q = \varepsilon_i - q + n,\ \ i = 1,\dots,n-k.$$

Remark 8.1.13. *The above process of calculating the boundary conditions of* $v_s^q[M]$ *is valid for both* $M \neq 0$ *and* $M = 0$.

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Taking into account previous computations, we may define ϕ as the function which maps the space $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ into the space of the boundary conditions that satisfies the first derivative, that is, $X^{\{\rho_1^1,\dots,\rho_{n-k}^1\}}_{\{\mu_1^1,\dots,\mu_k^1\}}$. This way, we can extend this definition to the $q^{\text{th}}-$derivative as

$$\phi^q = \phi \circ \overset{q)}{\cdots} \circ \phi : X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} \longrightarrow X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}.$$

In particular, ϕ^n is the identity function.

Example 8.1.14. *Consider the operator $T_5[M]$ defined in $X^{\{0,2\}}_{\{0,2,4\}}$. Applying the above definition of the spaces of the derivatives we obtain the following sequence of spaces:*

$$X^{\{0,2\}}_{\{0,2,4\}} \xrightarrow{\phi} X^{\{1,4\}}_{\{1,3,4\}} \xrightarrow{\phi} X^{\{0,3\}}_{\{0,2,3\}} \xrightarrow{\phi} X^{\{2,4\}}_{\{1,2,4\}} \xrightarrow{\phi} X^{\{1,3\}}_{\{0,1,3\}} \xrightarrow{\phi} X^{\{0,2\}}_{\{0,2,4\}}.$$

Example 8.1.15. *Let us consider the operator $T_4[M]$ defined in $X^{\{0,2\}}_{\{0,2\}}$. In this case, the sequence of spaces of the derivatives is:*

$$X^{\{0,2\}}_{\{0,2\}} \xrightarrow{\phi} X^{\{1,3\}}_{\{1,3\}} \xrightarrow{\phi} X^{\{0,2\}}_{\{0,2\}} \xrightarrow{\phi} X^{\{1,3\}}_{\{1,3\}} \xrightarrow{\phi} X^{\{0,2\}}_{\{0,2\}}.$$

Analogously to the Notation 8.1.2, let us define α^q, β^q as follows:

$$\alpha^q = \min\left\{i \in \{0,\dots,n-1\} : \ \ i \notin \{\mu_1^q,\dots,\mu_k^q\}\right\}, \tag{8.1.25}$$

$$\beta^q = \min\left\{i \in \{0,\dots,n-1\} : \ \ i \notin \{\rho_1^q,\dots,\rho_{n-k}^q\}\right\}. \tag{8.1.26}$$

Remark 8.1.16. *It is important to point out that for any $q \geq 1$, the function $v_s^q[M]$ is not the Green's function related to the operator $T_n[M]$ in $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$. This is due to the fact that $v_s^q[M] \in C^{n-2-q}(I \times I)$.*

Remark 8.1.17. *The operator $T_n[0]$ satisfies the property (T_d) in $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$. From Remark 8.1.4, we know that operator $T_n[0]\,u(t) = u^{(n)}(t)$ satisfies the following decomposition:*

$$T_0\,u(t) = u(t)\,, \quad T_k\,u(t) = \frac{d}{dt}\left(\frac{T_{k-1}\,u(t)}{v_k(t)}\right),\ k = 1,\dots,n\,,$$

where $v_k \equiv 1$ on I and

$$T_n[0]\,u(t) = v_1(t)\,\dots\,v_n(t)\,T_n\,u(t)\,,\ t \in I\,,$$

that is, $T_k\,u(t) = u^{(k)}(t)$.

Moreover, this decomposition satisfies, for every $u \in X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$:

$$\begin{aligned} T_{\mu_1^q}\,u(a) = u^{(\mu_1^q)}(a) = 0\,, \quad \dots\,, \quad T_{\mu_k^q}\,u(a) \quad = u^{(\mu_k^q)}(a) \quad = 0\,, \\ T_{\rho_1^q}\,u(b) = u^{(\rho_1^q)}(b) = 0\,, \quad \dots\,, \quad T_{\rho_{n-k}^q}\,u(b) = u^{(\rho_{n-k}^q)}(b) = 0\,. \end{aligned}$$

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We consider the general expression of the solution of equation

$$T_n[0]\,u(t)+\lambda\,u(t)\equiv u^{(n)}(t)+\lambda\,u(t)=0,\quad t\in I,$$

to obtain the different eigenvalues.

Next we present a result concerning the spectrum of the spaces $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$.

Lemma 8.1.18. *$\bar\lambda\neq 0$ is an eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ if and only if $\bar\lambda$ is an eigenvalue of $T_n[0]$ in $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$ for all $q\in\{1,\ldots,n-1\}$.*

Proof. Suppose that $\bar\lambda\neq 0$ is an eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$. So, there is $u\not\equiv 0$ in I such that $u^{(n)}(t)+\bar\lambda\,u(t)=0$, $t\in I$ and $u\in X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$. Since $u\not\equiv 0$ in I we have that $u^{(n)}\not\equiv 0$ in I. Therefore, $u^{(j)}\not\equiv 0$ in I for all $j\in\{1,\ldots,n-1\}$.

Taking into account that $T_n[0]\,u^{(j)}(t)=u^{(j+n)}(t)=-\bar\lambda\,u^{(j)}(t)$ for all $t\in I$ and $j\in\mathbb{N}$, we have that $u^{(j)}$ is an eigenvector related to the eigenvalue $\bar\lambda$ such that $u^{(j)}\in X^{\{\rho_1^j,\ldots,\rho_{n-k}^j\}}_{\{\mu_1^j,\ldots,\mu_k^j\}}$ for all $j\in\{1,\ldots,n-1\}$.

For $j=n$, we have that $\mu_i^n=\sigma_i$ for all $i\in\{1,\ldots,k\}$ and $\rho_l^n=\varepsilon_l$ for all $l\in\{1,\ldots,n-k\}$. Therefore, the result holds. □

Remark 8.1.19. *As a direct consequence of Remark 8.1.4, we have, from Lemma 8.1.5, that $\{\sigma_1,\ldots,\sigma_k\}-\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}$ satisfies condition (N_a) if and only if $M=0$ is not an eigenvalue of $u^{(n)}$ in $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$.*

Remark 8.1.20. *Note that if $\lambda=0$ is not an eigenvalue of $T_n[0]$ in $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$, then the spaces $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$ and $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$ have the same spectrum. However, it is very important to point out that $\lambda=0$ can appear as an eigenvalue in some space $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$, as it is shown in the following example.*

Example 8.1.21. *Let us consider again the Example 8.1.14. As we know, the following sequence of spaces is fulfilled:*

$$X^{\{0,2\}}_{\{0,2,4\}}\xrightarrow{\phi}X^{\{1,4\}}_{\{1,3,4\}}\xrightarrow{\phi}X^{\{0,3\}}_{\{0,2,3\}}\xrightarrow{\phi}X^{\{2,4\}}_{\{1,2,4\}}\xrightarrow{\phi}X^{\{1,3\}}_{\{0,1,3\}}\xrightarrow{\phi}X^{\{0,2\}}_{\{0,2,4\}}.$$

In this case, we have that $\lambda=0$ is not an eigenvalue of $T_5[0]$ in $X^{\{0,2\}}_{\{0,2,4\}}$, $X^{\{0,3\}}_{\{0,2,3\}}$ and $X^{\{1,3\}}_{\{0,1,3\}}$, while $\lambda=0$ is an eigenvalue of $T_5[0]$ in $X^{\{1,4\}}_{\{1,3,4\}}$ and $X^{\{2,4\}}_{\{1,2,4\}}$.

Now, we will show how to determine the adjoint space of $X^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}_{\{\mu_1^q,\ldots,\mu_k^q\}}$ from the adjoint space of $X^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}_{\{\sigma_1,\ldots,\sigma_k\}}$. We arrive to the next result.

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Theorem 8.1.22. *The adjoint space of* $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ *is given by* $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$, *where the indices* $\{\tau_1^q,\dots,\tau_{n-k}^q\},\{\delta_1^q,\dots,\delta_k^q\}\subset\{0,\dots,n-1\}$ *satisfy*

$$0\leq\tau_1^q<\tau_2^q<\cdots<\tau_{n-k}^q\leq n-1\,,\quad 0\leq\delta_1^q<\delta_2^q<\cdots<\delta_k^q\leq n-1\,,$$

and are defined as follows:

1) If $C_q:=\{i\in\{1,\dots,n-k\}\,/\;\tau_i+q\leq n-1\}\neq\varnothing$, *then*

$$\begin{aligned}&\tau_1^q=\tau_{l+1}+q-n,\,\tau_2^q=\tau_{l+2}+q-n,\,\dots,\tau_{n-k-l}^q=\tau_{n-k}+q-n,\\&\tau_{n-k-l+1}^q=\tau_1+q,\,\dots,\,\tau_{n-k}^q=\tau_l+q,\end{aligned}$$

with

$$l=\max C_q. \tag{8.1.27}$$

2) If $C_q=\varnothing$, *then*

$$\tau_i^q=\tau_i+q-n,\;\;i=1,\dots,n-k.$$

3) If $D_q:=\{i\in\{1,\dots,k\}\,/\;\delta_i+q\leq n-1\}\neq\varnothing$, *then*

$$\begin{aligned}&\delta_1^q=\delta_{p+1}+q-n,\,\delta_2^q=\delta_{p+2}+q-n,\,\dots,\delta_{k-p}^q=\delta_k+q-n,\\&\delta_{k-p+1}^q=\delta_1+q\,,\dots,\,\delta_k^q=\delta_p+q,\end{aligned}$$

with

$$p=\max D_q. \tag{8.1.28}$$

4) If $D_q=\varnothing$, *then*

$$\delta_i^q=\delta_d+q-n,\;\;i=1,\dots,k.$$

As a consequence, the following diagram

$$\begin{array}{ccc}X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} & \xrightarrow{\;\phi^q\;} & X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}\\ \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *}\\ X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}} & \xleftarrow[\;\phi^q\;]{} & X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}\end{array} \tag{8.1.29}$$

is commutative.

Proof. We do the proof only for τ_i^q. For δ_i^q, the proof is done in a similar way. To prove it, we start from the definition of τ_i, we have that $\tau_i<\tau_{i+1}$ for all $i\in\{1,\dots,n-k-1\}$ and

$$\{\sigma_1,\dots,\sigma_k,n-1-\tau_{n-k},\dots,\,n-1-\tau_1\}\equiv\{0,\dots,n-1\}.$$

Taking into account the definition of $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ we have that

$$\{\sigma_1,\dots,\sigma_k,n-1-\tau_{n-k},\dots,\,n-1-\tau_1\}\xrightarrow{\;\phi^q\;}\{\mu_1^q,\dots,\mu_k^q,n-1-\tau_{n-k}^q,\dots,\,n-1-\tau_1^q\},$$

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with $\{\mu_1^q, \dots, \mu_k^q, n-1-\tau_{n-k}^q, \dots, n-1-\tau_1^q\} \equiv \{0, \dots, n-1\}$ where τ_i^q has to fulfill that

$$0 \leq \tau_1^q < \tau_2^q < \cdots < \tau_{n-k}^q \leq n-1\,.$$

To calculate them, we distinguish two cases:

1) If $C_q \neq \varnothing$, then:

   - If $n-1-\tau_i \geq q$ (i.e, $i \leq l$), $n-1-\tau_i$ becomes $n-1-\tau_i-q$ by applying ϕ^q.
   - If $n-1-\tau_i < q$ (i.e, $i > l$), $n-1-\tau_i$ becomes $n-1-\tau_i-q+n$ by applying ϕ^q.

   Reordering the elements, we obtain that:

$$\begin{aligned} 0 \leq n-1-\tau_l-q < n-1-\tau_{l-1}-q < \cdots < n-1-\tau_1-q \\ < n-1-\tau_{n-k}-q+n < \cdots < n-1-\tau_{l+1}-q+n \leq n-1, \end{aligned}$$

   or, equivalently,

$$0 \leq \tau_{l+1}+q-n < \tau_{l+2}+q-n < \cdots < \tau_{n-k}+q-n < \tau_1+q < \cdots < \tau_l+q \leq n-1\,.$$

   Taking

$$\begin{aligned} &\tau_1^q = \tau_{l+1}+q-n,\ \tau_2^q = \tau_{l+2}+q-n,\ \dots, \tau_{n-k-l}^q = \tau_{n-k}+q-n, \\ &\tau_{n-k-l+1}^q = \tau_1+q, \dots,\ \tau_{n-k}^q = \tau_l+q\,, \end{aligned}$$

   the first statement yields.

2) If $C_q = \varnothing$, then $n-1-\tau_i$ becomes $n-1-\tau_i-q+n$ by applying ϕ^q and we obtain that

$$0 \leq n-1-\tau_{n-k}-q+n < \cdots < n-1-\tau_2-q+n < n-1-\tau_1-q+n \leq n-1\,,$$

   and, reordering, we infer that

$$0 \leq \tau_1+q-n < \tau_2+q-n < \cdots < \tau_{n-k}+q-n \leq n-1\,.$$

   Taking $\tau_i^q = \tau_i+q-n,\ \ i=1,\dots,n-k$, the second statement yields.

Therefore, $X^{*\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}} = X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$. □

Remark 8.1.23. *Note that, unlike the direct arrows between the leading spaces, the arrows between the adjoint spaces are in the inverse direction.*

Example 8.1.24. *Consider the operator $T_4[M]$ coupled with the boundary conditions*

$$u(a) = u''(a) = u'(b) = u''(b).$$

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In this case, $u \in X_{\{0,2\}}^{\{1,2\}}$ *and* $X^{*\{1,2\}}_{\{0,2\}} = X^{\{0,3\}}_{\{0,2\}}$. *Using the diagram* (8.1.29)*, we get that*

$$\begin{array}{ccccccc} X^{\{1,2\}}_{\{0,2\}} & \xrightarrow{\phi} & X^{\{0,1\}}_{\{1,3\}} & \xrightarrow{\phi} & X^{\{0,3\}}_{\{0,2\}} & \xrightarrow{\phi} & X^{\{2,3\}}_{\{1,3\}} \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} & & \downarrow{\scriptstyle *} \\ X^{\{0,3\}}_{\{0,2\}} & \xleftarrow[\phi]{} & X^{\{0,1\}}_{\{1,3\}} & \xleftarrow[\phi]{} & X^{\{1,2\}}_{\{0,2\}} & \xleftarrow[\phi]{} & X^{\{2,3\}}_{\{1,3\}} \end{array}$$

Using the fact that the adjoint operators have the same spectrum as the original one, from Lemma 8.1.18, we arrive at the following result.

Lemma 8.1.25. *$\lambda^* \neq 0$ is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$ if and only if λ^* is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$ for all $q \in \{0,\dots,n-1\}$. Moreover, for each fixed $q \in \{0,\dots,n-1\}$, $\bar{\lambda} \in \mathbb{R}$ is an eigenvalue of $T_n[0]$ in $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_n[0]$ in $X^{\{\delta_1^q,\dots,\delta_k^q\}}_{\{\tau_1^q,\dots,\tau_{n-k}^q\}}$.*

Following the line of Definition 8.1.11 and the characterization of Theorem 8.1.12, we introduce the following concept for $v_s^q[M]$.

Definition 8.1.26. Let $q \in \{0,\dots,n-1\}$ be fixed. We said that the function $v_s^q[M]$ is strongly positive (strongly negative) on $I \times I$ if it satisfies the following properties:

- $v_s^q[M] > 0\,(<0)$ a.e. on (a,b).
- $(v_s^q)^{(\alpha^q)}[M](a) > 0\,(<0)$ for a.e. $s \in (a,b)$, with α^q defined in (8.1.25).
- $(-1)^{\beta^q}(v_s^q)^{(\beta^q)}[M](b) > 0\,(<0)$ for a.e. $s \in (a,b)$, with β^q defined in (8.1.26).

As a particular case of [26, Theorem 5.1], the following result is attained.

Theorem 8.1.27. *Suppose that $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a). Then the following properties are fulfilled:*

- *If $n-k$ is even, then $T_n[0]$ is strongly inverse positive in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and, moreover, the related Green's function, $g_0(t,s)$, satisfies (P_g).*
- *If $n-k$ is odd, then $T_n[0]$ is strongly inverse negative in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and, moreover, the related Green's function, $g_0(t,s)$, satisfies (N_g).*

Next, for any $q \in \{1,\dots,n-1\}$ we determine the positive (negative) sign of $v_s^q(t) := v_s^q[0](t)$ on the interval I.

Let us define $\{\sigma_1^q,\dots,\sigma_{c_q}^q\}, \{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\} \subset \{0,\dots,n-1\}$, in the following way:

$$\{\sigma_1^q,\dots,\sigma_{c_q}^q\} := \{\sigma_1 - q,\dots,\sigma_k - q\} \cap \{0,\dots,n-q-1\}, \tag{8.1.30}$$

and

$$\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\} := \{\varepsilon_1 - q,\dots,\varepsilon_{n-k} - q\} \cap \{0,\dots,n-q-1\}. \tag{8.1.31}$$

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If $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies the condition (N_a) then $c_q+d_q\leq n-q$. Moreover, if $c_q+d_q<n-q$ then $\{\sigma_1^q,\dots,\sigma_{c_q}^q\}-\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}$ does not satisfy (N_a), and in this case $\lambda=0$ is an eigenvalue of $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$ (we cannot study in this case the constant sign of its corresponding derivative $v_s^q[M]$).

Remark 8.1.28. *Taking into account that function $v_s^q(t)=\frac{\partial^q}{\partial t^q}\,g_0(t,s)$ is the Green's function related to operator $T_{n-q}[0]$ in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$, then operator $T_{n-q}[0]$ is strongly inverse positive (negative) in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$ if and only if it satisfies the Definition 8.1.11 with the same α^q and β^q since*

$$\mu_1^q=\sigma_1^q,\ \mu_2^q=\sigma_2^q,\ \dots,\mu_{k-(j-1)}^q=\sigma_{c_q}^q,$$

and

$$\rho_1^q=\varepsilon_1^q,\ \rho_2^q=\varepsilon_2^q,\ \dots,\rho_{n-k-(r-1)}^q=\varepsilon_{d_q}^q.$$

Theorem 8.1.29. *Let $q\in\{1,\dots,n-1\}$. Suppose that $\{\sigma_1,\dots,\sigma_k\}-\{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a) and $c_q+d_q=n-q$. The following properties are fulfilled:*

(a) If $c_q\geq 1$ and $d_q\geq 1$, then v_s^q is strongly positive on $I\times I$ if $n-q-c_q$ is even and v_s^q is strongly negative on $I\times I$ if $n-q-c_q$ is odd.

(b) If $c_q=n-q$ and $d_q=0$, then $v_s^q(t)>0,\dots,v_s^{n-1}(t)>0$ in $(s,b]$, $v_s^q(t)=\cdots=v_s^{n-2}(t)=0$ in $[a,s]$ and $v_s^{n-1}(t)=0$ in $[a,s)$.

(c) If $c_q=0$ and $d_q=n-q$, then $v_s^q(t)>0$ in $[a,s)$ if $q=n-l$ with l even and $v_s^q(t)<0$ in $[a,s)$ if $q=n-l$ with l odd. Moreover, $v_s^q(t)=\cdots=v_s^{n-2}(t)=0$ in $[s,b]$ and $v_s^{n-1}(t)=0$ in $(s,b]$.

Proof. (a) In this case, it easy to verify that the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$ satisfies the condition (N_a) since

$$\sum_{\sigma_j-q<h}1+\sum_{\varepsilon_j-q<h}1\geq h\,,\quad\forall h\in\{1,\dots,n-q-1\}\,.$$

So, taking into account the fact that v_s^q is the Green's function related to operator $T_{n-q}[0]$ in the space $X_{\{\sigma_1^q,\dots,\sigma_{c_q}^q\}}^{\{\varepsilon_1^q,\dots,\varepsilon_{d_q}^q\}}$, we apply Theorem 8.1.27, and we have proved the first statement.

(b) Since $v_s^{(n)}(t)=0$ for all $t\neq 0$, we have, by (8.1.10) and $v_s^{(n-1)}(a)=0$, that $\frac{\partial^{n-1}}{\partial t^{n-1}}\,g_0(t,s)=0$ if $t<s$ and $\frac{\partial^{n-1}}{\partial t^{n-1}}\,g_0(t,s)=1$ if $t>s$.
Since $v_s^h(a)=0$ for all $q\leq h\leq n-2$, and $g_0\in C^{n-2}(I\times I)$, we deduce that $\frac{\partial^h}{\partial t^h}\,g_0(t,s)=0$ if $t<s$ and $\frac{\partial^h}{\partial t^h}\,g_0(t,s)>0$ if $t>s$. So, we conclude the proof of second claim.

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(c) In this case arguing as in the previous case, since $v_s^{(n)}(b) = 0$, we have that $\frac{\partial^{n-1}}{\partial t^{n-1}} g_0(t,s) = 0$ if $t > s$ and $\frac{\partial^{n-1}}{\partial t^{n-1}} g_0(t,s) = -1$ if $t < s$.
As consequence, since $v_s^h(b) = 0$ for all $q \leq h \leq n-2$, and $g_0 \in C^{n-2}(I \times I)$, we deduce, by recurrence, that $\frac{\partial^h}{\partial t^h} g_0(t,s) = 0$ for all $t > s$ and $(-1)^{n-h} \frac{\partial^h}{\partial t^h} g_0(t,s) > 0$ for all $t < s$. Thus, the third assertion holds.

□

Let $q \in \{0, \ldots, n-1\}$ be fixed. Suppose that $(-1)^{n-q-c_q}\, v_s^q$ is strongly positive on $I \times I$. Then, for each $s \in (a,b)$, we obtain the following limits:

$$\ell_1^q(s) := \lim_{t \to a^+} \frac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}} = \frac{(-1)^{n-q-c_q}\,(v_s^q)^{(\alpha^q)}(a)}{\alpha^q!\,(b-a)^{\beta^q}}\,,$$

$$\ell_2^q(s) := \lim_{t \to b^-} \frac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}} = \frac{(-1)^{n-q-c_q-\beta^q}\,(v_s^q)^{(\beta^q)}(b)}{\beta^q!\,(b-a)^{\alpha^q}}.$$

For each $s \in (a,b)$, let us consider the following function defined on I by

$$\tilde{u}_s^q(t) = \begin{cases} \ell_1^q(s)\,, & t = a, \\ \dfrac{(-1)^{n-q-c_q}\, v_s^q(t)}{(t-a)^{\alpha^q}\,(b-t)^{\beta^q}}\,, & t \in (a,b), \\ \ell_2^q(s)\,, & t = b. \end{cases}$$

It is clear that $\tilde{u}_s^q > 0$ on $[a,b]$ for all $s \in (a,b)$.

Since $g_0(t,s) \in C^{n-2}(I \times I)$, $\frac{\partial^{n-1}}{\partial t^{n-1}} g_0 \in C^\infty\left((I \times I) \setminus \{(t,t) \,/\, t \in I\}\right)$ and there exists $\lim_{s \to t^\pm} \frac{\partial^{n-1}}{\partial t^{n-1}} g_0(t,s) \in \mathbb{R}$, we deduce that there exists $K^q > 0$ such that $\tilde{u}_s^q(t) \leq K^q$ for every $(t,s) \in I \times I$ and $q \in \{1, \ldots, n-1\}$. Therefore, the following functions

$$\tilde{k}_1^q(s) = \min_{t \in I} \tilde{u}_s^q(t)\,, \quad s \in I\,,$$

$$\tilde{k}_2^q(s) = \max_{t \in I} \tilde{u}_s^q(t)\,, \quad s \in I\,,$$

are continuous on I and positive in (a,b).

Taking $\phi(t) = (t-a)^{\alpha^q}\,(b-t)^{\beta^q} > 0$ on (a,b), the function $v_s^q(t)$ satisfies the condition (P_g) if $n-q-c_q$ is even with $k_1(s) = \tilde{k}_1^q(s)$ and $k_2(s) = \tilde{k}_2^q(s)$ and condition (N_g) if $n-q-c_q$ is odd with $k_1(s) = -\tilde{k}_2^q(s)$ and $k_2(s) = -\tilde{k}_1^q(s)$.

As for the set of parameters of M in which the function $v_s^q[M]$, $q \in \{1, \ldots, n-1\}$ maintains constant sign and monotony with respect to M, we have the following results:

Lemma 8.1.30. *Let $q \in \{1, \ldots, n-1\}$. Suppose that $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ satisfies condition (N_a). If there exists $M \in \mathbb{R}$ be such that $\frac{\partial^q}{\partial t^q} g_M$ has constant sign, then g_M has constant sign.*

Proof. If $v_s^q[M](t) = \frac{\partial^q}{\partial t^q} g_M(t,s)$ has constant sign, then $v_s^{q-1}[M](t) = \frac{\partial^{q-1}}{\partial t^{q-1}} g_M(t,s)$ is a monotone and continuous function. Thus, $v_s^{q-1}[M]$ has at most one zero on I.

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By recurrence, we have that $v_s^{q-l}[M](t) = \frac{\partial^{q-l}}{\partial t^{q-l}} g_M(t,s)$ has at most l zeros on I. In particular, $g_M(\cdot,s)$ has at most q zeros on I.

Now, since condition (N_a) implies that $g_M(\cdot,s)$ has at least q zeros of order smaller or equal to q on the boundary, we deduce that $g_M(\cdot,s)$ cannot have any zero on (a,b). Thus, g_M has constant sign. □

Moreover, we recall the following facts:

- If $n-k$ is even, then from Theorem 8.1.27 and Lemma 1.2.4 we have that $g_M(t,s) > 0$ for all $(t,s) \in (a,b) \times (a,b)$ if and only if $M \in (\lambda_1, +\infty)$ or $M \in (\lambda_1, \lambda_2]$, and it is monotone decreasing with respect to M on such interval, where $\lambda_1 < 0$ is the first eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\lambda_2 > 0$ (if the interval is bounded) is not an eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. The value of λ_2 is characterized in [26, Sect.7 and Sect.8].

- If $n-k$ is odd, then from Theorem 8.1.27 and Lemma 1.2.3 we have that $g_M(t,s) < 0$ for all $(t,s) \in (a,b) \times (a,b)$ if and only if $M \in (-\infty, \lambda_1)$ or $M \in [\overline{\lambda_2}, \lambda_1)$ and it is monotone decreasing with respect to M on such interval, where $\lambda_1 > 0$ is the first eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ and $\overline{\lambda_2} < 0$ (if the interval is bounded) is not an eigenvalue of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. The value of $\overline{\lambda_2}$ is characterized in [26, Sect. 7 and Sect. 8].

Therefore, as a direct consequence of Theorem 8.1.27, Lemma 8.1.30 and [16, Theorem 3.6], the following result is deduced.

Lemma 8.1.31. *The following properties hold:*

- *If $n-k$ is even and $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ on $(a,b) \times (a,b)$ for some $M \in \mathbb{R}$, then $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ if and only if $M \in (\lambda_1, M_q]$, with $M_q \leq \lambda_2$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone decreasing with respect to $M \in (\lambda_1, M_q]$.*

- *If $n-k$ is even and $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ on $(a,b) \times (a,b)$ for some $M \in \mathbb{R}$, then $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ if and only if $M \in (\lambda_1, M_q]$, with $M_q \leq \lambda_2$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone increasing with respect to $M \in (\lambda_1, M_q]$.*

- *If $n-k$ is odd and $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ on $(a,b) \times (a,b)$ for some $M \in \mathbb{R}$, then $\frac{\partial^q}{\partial t^q} g_M(t,s) > 0$ if and only if $M \in [M_q, \lambda_1)$, with $M_q \geq \overline{\lambda_2}$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone increasing with respect to $M \in [M_q, \lambda_1)$.*

- *If $n-k$ is odd and $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ on $(a,b) \times (a,b)$ for some $M \in \mathbb{R}$, then $\frac{\partial^q}{\partial t^q} g_M(t,s) < 0$ if and only if $M \in [M_q, \lambda_1)$, with $M_q \geq \overline{\lambda_2}$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone decreasing with respect to $M \in [M_q, \lambda_1)$.*

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8.2 Study of the Constant Sign of the Derivatives of the Green's Function

In this section for any $q \in \{1, \dots, n-1\}$ be fixed, we will give a proof of the main result that characterizes the constant sign of the partial derivative with respect to t of order q of the Green's function related to operator $T_n[M]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

We distinguish three cases:

(a) $c_q \geq 1$ and $d_q \geq 1$,

(b) $c_q = n-q$ and $d_q = 0$,

(c) $c_q = 0$, and $d_q = n-q$,

where c_q and d_q are defined in (8.1.30) and (8.1.31) and satisfy $c_q + d_q = n - q$.

8.2.1 Case (a): $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$

Lemma 8.2.1. *Let $q \in \{1, \dots, n-1\}$ be fixed. If $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$, then the indices $z = k - (j-1)$ and $h = n - k - (r-1)$ are such that*

$$\mu_z^q + q + \eta = n - 1, \quad \text{and} \quad \rho_h^q + q + \gamma = n - 1,$$

with j and r defined in (8.1.23) *and* (8.1.24)*, η and γ defined in* (8.1.16) *and* (8.1.17)*.*

Proof. By the definition of $\mu_z^q = \mu_{k-(j-1)}^q$ and $\rho_h^q = \rho_{n-k-(r-1)}^q$ we have that since $c_q \geq 1$, $\mu_z^q = \sigma_k - q \geq 0$ and since $d_q \geq 1$, $\rho_h^q = \varepsilon_{n-k} - q \geq 0$. Using Remark 8.1.7, we deduce that

$$\mu_z^q + q + \eta = \sigma_k - q + q + n - 1 - \sigma_k = n - 1,$$

and

$$\rho_h^q + q + \gamma = \varepsilon_{n-k} - q + q + n - 1 - \varepsilon_{n-k} = n - 1.$$

□

Lemma 8.2.2. *Let $q \in \{1, \dots, n-1\}$ be fixed. If $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$, then $z = p$ and $h = l$ with z and h defined in Lemma 8.2.1, p and l defined in* (8.1.28) *and* (8.1.27)*.*

Proof. First, note that $z = |A_q|$ is the cardinal of the set $A_q = \{i \in \{1, \dots, k\} / \ \sigma_i \geq q\}$ and $p = |D_q|$ is the cardinal of the set $D_q = \{i \in \{1, \dots, k\} / \ \delta_i + q \leq n-1\}$. Moreover, by (8.1.30), it is clear that c_q corresponds with the number of σ's that are equal or greater than q, that is, $z = c_q$.

On the other hand, we have that

$$\{\varepsilon_1, \dots, \varepsilon_{n-k}, n-1-\delta_k, \dots, n-1-\delta_1\} \equiv \{0, \dots, n-1\}.$$

Among these n elements (all the numbers between 0 and $n-1$) there are $n-q$ elements that are equal or greater than q. By (8.1.31), we have that the number of ε's that are equal or

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greater than q is d_q. Therefore, the number of δ's such that $n-1-\delta_i \geq q$ is $n-q-d_q$, that is, $p = n-q-d_q$. Since, $c_q + d_q = n-q$, we have that $z = p$. In an analogous way, we can prove that $h = l$. □

Remark 8.2.3. *It should be noted that in the previous proof, since j and r depend on q, we have that z and h also depend on q, but we will omit such dependence in the notation for the sake of simplicity.*

Remark 8.2.4. *Indices z and h depend on the space $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ and are unique. Indeed, for the rest of the components of space $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$ we have that $(\mu_i^q + q + \eta) \bmod n < n-1$ for $i \neq z$ with $i \in \{1,\dots,k\}$ and $(\rho_l^q + q + \gamma) \bmod n < n-1$ for $l \neq h$ with $l \in \{1,\dots,n-k\}$.*

Now, we present some results that provide sufficient conditions to ensure the constant sign of solutions of (8.0.1) in the spaces $X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}$. Using Lemma 8.1.17, these results can be proved analogously to [26, Proposition 6.7] and [26, Proposition 6.9], respectively, and we omit them.

Let us consider the following spaces

$$X_2 := \begin{cases} X^{\{\rho_1^q,\dots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q,\dots,\mu_{z-1}^q,\mu_{z+1}^q,\dots,\mu_k^q\}}, & \text{if } z \neq 1, \\ X^{\{\rho_1^q,\dots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_2^q,\dots,\mu_k^q\}}, & \text{if } z = 1, \end{cases} \tag{8.2.32}$$

$$X_3 := \begin{cases} X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_{z-1}^q,\mu_{z+1}^q,\dots,\mu_k^q|\alpha^q\}}, & \text{if } z \neq 1, \\ X^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}_{\{\mu_2^q,\dots,\mu_k^q|\alpha^q\}}, & \text{if } z = 1, \end{cases} \tag{8.2.33}$$

$$X_4 := \begin{cases} X^{\{\rho_1^q,\dots,\rho_{h-1}^q,\rho_{h+1}^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q|\alpha^q\}}, & \text{if } h \neq 1, \\ X^{\{\rho_2^q,\dots,\rho_{n-k}^q\}}_{\{\mu_1^q,\dots,\mu_k^q|\alpha^q\}}, & \text{if } h = 1, \end{cases} \tag{8.2.34}$$

and

$$X_5 := \begin{cases} X^{\{\rho_1^q,\dots,\rho_{h-1}^q,\rho_{h+1}^q,\dots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}, & \text{if } h \neq 1, \\ X^{\{\rho_2^q,\dots,\rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q,\dots,\mu_k^q\}}, & \text{if } h = 1, \end{cases} \tag{8.2.35}$$

where z and h are defined in Lemma 8.2.1.

Proposition 8.2.5. *Let $q \in \{1,\dots,n-1\}$. Suppose that $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n-q$, $c_q \geq 1$ and $d_q \geq 1$. If $u \in C^n(I)$ is a solution of* (8.0.1) *on (a,b), satisfying the boundary conditions:*

$$u^{(\mu_1^q)}(a) = \cdots = u^{(\mu_{z-1}^q)}(a) = u^{(\mu_{z+1}^q)}(a) = \cdots = u^{(\mu_k^q)}(a) \quad = 0\,, \tag{8.2.36}$$

$$u^{(\rho_1^q)}(b) = \cdots = u^{(\rho_{n-k}^q)}(b) = 0\,, \tag{8.2.37}$$

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then it does not have any zero on (a,b) *provided that one of the following assertions is satisfied:*

- *Let* $n-k$ *be even:*
  - *If* $k>1$, $\mu_z^q \neq z-1$ *and* $M \in [\lambda_3^q, \lambda_2^q]$, *where:*
    - $\lambda_3^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_3.
    - $\lambda_2^q > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* X_2.
  - *If* $k>1$, $\mu_z^q = z-1$ *and* $M \in [\lambda_1, \lambda_2^q]$, *where:*
    - $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.
    - $\lambda_2^q > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* X_2.
  - *If* $k=1$, $\mu_1^q \neq 0$ *and* $M \in [\lambda_3^q, +\infty)$, *where:*
    - $\lambda_3^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* $X_{\{\alpha^q\}}^{\{\rho_1^q,\ldots,\rho_{n-1}^q\}}$, *where* $\alpha^q = 0$.
  - *If* $k=1$, $\mu_1^q = 0$ *and* $M \in [\lambda_1, +\infty)$, *where:*
    - $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-1}\}}$.
- *Let* $n-k$ *be odd:*
  - *If* $k>1$, $\mu_z^q \neq z-1$ *and* $M \in [\lambda_2^q, \lambda_3^q]$, *where:*
    - $\lambda_3^q > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* X_3.
    - $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.
  - *If* $k>1$, $\mu_z^q = z-1$ *and* $M \in [\lambda_2^q, \lambda_1]$, *where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.
    - $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.
  - *If* $k=1$, $\mu_1^q \neq 0$ *and* $M \in (-\infty, \lambda_3^q]$, *where:*
    - $\lambda_3^q > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\alpha^q\}}^{\{\rho_1^q,\ldots,\rho_{n-1}^q\}}$, *where* $\alpha^q = 0$.
  - *If* $k=1$, $\mu_1^q = 0$ *and* $M \in (-\infty, \lambda_1]$, *where:*
    - $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-1}\}}$.

Proposition 8.2.6. *Let* $q \in \{1,\ldots,n-1\}$. *Suppose that* $\{\sigma_1,\ldots,\sigma_k\} - \{\varepsilon_1,\ldots,\varepsilon_{n-k}\}$ *satisfies condition* (N_a), $c_q + d_q = n-q$, $c_q \geq 1$ *and* $d_q \geq 1$. *If* $u \in C^n(I)$ *is a solution of* (8.0.1) *on* (a,b) *satisfying the boundary conditions:*

$$u^{(\mu_1^q)}(a) = \cdots = u^{(\mu_k^q)}(a) \quad = 0\,, \tag{8.2.38}$$

$$u^{(\rho_1^q)}(b) = \cdots = u^{(\rho_{h-1}^q)}(b) = u^{(\rho_{h+1}^q)}(b) = \cdots = u^{(\rho_{n-k}^q)}(b) = 0\,, \tag{8.2.39}$$

then it does not have any zero on (a,b) *provided that one of the following assertions is satisfied:*

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- *Let $n-k$ be even:*
  - *If $\rho_h^q \neq h-1$ and $M \in [\lambda_5^q, \lambda_4^q]$, where:*
    - *$\lambda_5^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_5.*
    - *$\lambda_4^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_4.*
  - *If $\rho_h^q = h-1$ and $M \in [\lambda_1, \lambda_4^q]$, where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_4^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_4.*
- *Let $n-k$ be odd:*
  - *If $k < n-1$, $\rho_h^q \neq h-1$ and $M \in [\lambda_4^q, \lambda_5^q]$, where:*
    - *$\lambda_5^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_5.*
    - *$\lambda_4^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_4.*
  - *If $k < n-1$, $\rho_h^q = h-1$ and $M \in [\lambda_4^q, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_4^q < 0$ is the biggest negative eigenvalue of $T_n[0]$ in X_4.*
  - *If $k = n-1$, $\rho_1^q \neq 0$ and $M \in (-\infty, \lambda_5^q]$, where:*
    - *$\lambda_5^q > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\mu_1^q,\ldots,\mu_{n-1}^q\}}^{\{\beta^q\}}$, where $\beta^q = 0$.*
  - *If $k = n-1$, $\rho_1^q = 0$ and $M \in (-\infty, \lambda_1]$, where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_{n-1}\}}^{\{\varepsilon_1\}}$.*

Example 8.2.7. *Let us consider the operator $T_4[M]$ with the boundary conditions:*

$$u'(0) = u'''(0) = u(1) = u'(1) = 0.$$

The hypotheses of propositions 8.2.5 and 8.2.6 are satisfy only for the value $q = 1$. Moreover, $z = 2$, $h = 1$ and $v_s^q[M]$ satisfies the conditions of space $X_{\{0,2\}}^{\{0,3\}}$.

From Proposition 8.2.5, we can affirm that any nontrivial solution to problem

$$T_4[M] = 0,\ t \in [0,1],\ \ u(0) = u(1) = u'''(1) = 0\,, \tag{8.2.40}$$

does not have any zero on $(0,1)$ for $M \in [\lambda_3^1, \lambda_2^1]$, where $\lambda_3^1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$ and $\lambda_2^1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{1\}}^{\{0,1,3\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$ are given by $-\lambda^4$, where λ is a positive solution of $\sin(\lambda) = 0$. The small positive solution of this equation is π, so $\lambda_3^1 = -\pi^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$.

Similarly, the eigenvalues of $T_4[0]$ in $X_{\{1\}}^{\{0,1,3\}}$ are given by λ^4, where λ is a positive solution of

$$\left(-1 + e^{\sqrt{2}\,m}\right)\cos\left(\frac{m}{\sqrt{2}}\right) + \left(1 + e^{\sqrt{2}\,m}\right)\sin\left(\frac{m}{\sqrt{2}}\right) = 0.$$

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Denoting by m_2 the smallest positive solution of this equation, we deduce that $\lambda_2^1 = m_2^4 \approx 3.34^4$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0\}}^{\{0,1,3\}}$.

It is easy to verify that the solutions of problem (8.2.40) *are given as multiples of the following expression:*

$$\begin{cases} \cosh(m-m\,t)\,\sin(m)-\sin(m\,t)-\cosh(m)\,\sin(m-m\,t) \\ -\cos(m-m\,t)\,\sinh(m)+\sinh(m\,t)+\cos(m)\,\sinh(m-m\,t)\,, \\ \text{if } M=-m^4<0\,, \\ t^2-t\,,\ \text{ if } M=0\,, \\ e^{-\frac{m\,t}{\sqrt{2}}}\left(e^{\sqrt{2}\,m}\left(-1+e^{\sqrt{2}\,m\,t}\right)\cos\left(\dfrac{m(-2+t)}{\sqrt{2}}\right)-e^{\sqrt{2}\,m}\left(-1+e^{\sqrt{2}\,m}\right)\cos\left(\dfrac{m\,t}{\sqrt{2}}\right)\right. \\ \left.+\left(-1+e^{\sqrt{2}\,m}\right)\left(-e^{\sqrt{2}\,m}+e^{\sqrt{2}\,m\,t}\right)\sin\left(\dfrac{m\,t}{\sqrt{2}}\right)\right)\,, \\ \text{if } M=m^4>0. \end{cases}$$

Analogously, from Proposition 8.2.6, we conclude that any solution to problem

$$T_4[M]\,u(t)=0,\ t\in[0,1],\ \ u(0)=u''(0)=u'''(1)=0, \tag{8.2.41}$$

does not have any zero on $(0,1)$ for $M\in[\lambda_1,\lambda_4^1]$ where $\lambda_1<0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{1,3\}}^{\{0,1\}}$ and $\lambda_4^1>0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$.

The eigenvalues of $T_4[0]$ in $X_{\{1,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is a positive solution of

$$\sin(\lambda)+\cos(\lambda)\,\tanh(\lambda)=0\,.$$

Denote m_1 the smallest positive solution of this equation. Then, $\lambda_1=-m_1^4\approx-2.36^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{0,3\}}$.

On the other hand, the eigenvalues of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$ are given by λ^4, where λ is a positive solution of $\cos\left(\frac{m}{\sqrt{2}}\right)=0$. The small positive solution of this equation is $\frac{\sqrt{2}}{2}\pi$, so $\lambda_4^1=\left(\frac{\sqrt{2}}{2}\pi\right)^4$ is the least postive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{3\}}$.

It is immediate to show that solutions of problem (8.2.41) *are given as multiples of:*

$$\begin{cases} \sin(m\,t)+\cos(m)\,\operatorname{sech}(m)\sinh(m\,t)\,,\ \text{ if } M=-m^4<0\,, \\ t\,,\ \text{ if } M=0\,, \\ e^{-\frac{m\,t}{\sqrt{2}}}\left(\left(-1+e^{\sqrt{2}\,m\,(1+t)}\right)\cos\left(\dfrac{m(-1+t)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}\,m}+e^{\sqrt{2}\,m\,t}\right)\cos\left(\dfrac{m(1+t)}{\sqrt{2}}\right)\right. \\ \left.+\left(1+e^{\sqrt{2}\,m\,(1+t)}\right)\sin\left(\dfrac{m(-1+t)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}\,m}+e^{\sqrt{2}\,m\,t}\right)\sin\left(\dfrac{m(1+t)}{\sqrt{2}}\right)\right)\,, \\ \text{if } M=m^4>0. \end{cases}$$

Remark 8.2.8. *It should be noted that in the above propositions we have used the fact proven in Lemma 8.1.18 that the first nonzero eigenvalues of the spaces $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$ and $X_{\{\mu_1^q,\dots,\mu_k^q\}}^{\{\rho_1^q,\dots,\rho_{n-k}^q\}}$ coincide.*

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Let us define the following spaces

$$X_2^* := \begin{cases} X^{\{\delta_1, \ldots, \delta_{z-1}, \delta_{z+1}, \ldots, \delta_k\}}_{\{\tau_1, \ldots, \tau_{n-k}|\eta\}}, & \text{if } z \neq 1, \\ \\ X^{\{\delta_2, \ldots, \delta_k\}}_{\{\tau_1, \ldots, \tau_{n-k}|\eta\}}, & \text{if } z = 1, \end{cases} \tag{8.2.42}$$

and

$$X_4^* := \begin{cases} X^{\{\delta_1, \ldots, \delta_k|\gamma\}}_{\{\tau_1, \ldots, \tau_{h-1}, \tau_{h+1}, \ldots, \tau_{n-k}\}}, & \text{if } h \neq 1, \\ \\ X^{\{\delta_1, \ldots, \delta_k|\gamma\}}_{\{\tau_2, \ldots, \tau_{n-k}\}}, & \text{if } h = 1. \end{cases} \tag{8.2.43}$$

Lemma 8.2.9. *$\lambda \neq 0$ is an eigenvalue of $T_n[0]$ in X_2^* defined in* (8.2.42) *if and only if λ is an eigenvalue of $T_n[0]$ in X_2. In particular, $(\lambda_2^*)^q = \lambda_2^q$ where $(\lambda_2^*)^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_2^* and $\lambda_2^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_2.*

Proof. Suppose that $z \neq 1$ (the case $z = 1$ is proved similarly). By the definition of adjoint space of $X^{\{\rho_1^q, \ldots, \rho_{n-k}^q\}}_{\{\mu_1^q, \ldots, \mu_k^q\}}$ we have that

$$\{\mu_1^q, \ldots, \mu_{z-1}^q, \mu_z^q, \mu_{z+1}^q, \ldots, \mu_k^q, n-1-\tau_{n-k}^q, \ldots, n-1-\tau_1^q\} = \{0, \ldots, n-1\},$$

and

$$\{\rho_1^q, \ldots, \rho_{n-k}^q, n-1-\delta_k^q, \ldots, n-1-\delta_1^q\} = \{0, \ldots, n-1\}.$$

Since, $\mu_z^q = \sigma_k - q$ and $\beta^q = n-1-\delta_k^q$ we deduce that

$$X^{\{\rho_1^q, \ldots, \rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q, \ldots, \mu_{z-1}^q, \mu_{z+1}^q, \ldots, \mu_k^q\}} \xrightarrow{*} X^{\{\delta_1^q, \ldots, \delta_{k-1}^q\}}_{\{\tau_1^q, \ldots, \tau_{n-k}^q|n-1-\sigma_k+q\}}. \tag{8.2.44}$$

Using that $\eta = n-1-\sigma_k$, the definition of $X^{\{\delta_1^q, \ldots, \delta_k^q\}}_{\{\tau_1^q, \ldots, \tau_{n-k}^q\}}$ and the diagram (8.1.29) we infer that

$$X^{\{\delta_1^q, \ldots, \delta_{k-1}^q\}}_{\{\tau_1^q, \ldots, \tau_{n-k}^q|n-1-\sigma_k+q\}} \xrightarrow{\phi^q} X^{\{\delta_1, \ldots, \delta_{p-1}, \delta_{p+1}, \ldots, \delta_k\}}_{\{\tau_1, \ldots, \tau_{n-k}|\eta\}}.$$

From Lemma 8.2.2 we know that $p = z$ and therefore

$$X^{\{\delta_1^q, \ldots, \delta_{k-1}^q\}}_{\{\tau_1^q, \ldots, \tau_{n-k}^q|n-1-\sigma_k+q\}} \xrightarrow{\phi^q} X^{\{\delta_1, \ldots, \delta_{z-1}, \delta_{z+1}, \ldots, \delta_k\}}_{\{\tau_1, \ldots, \tau_{n-k}|\eta\}}. \tag{8.2.45}$$

Considering Lemma 8.1.18 and the fact that the adjoint spaces have the same eigenvalues, from (8.2.44) and (8.2.45), we obtain that the spaces $X^{\{\rho_1^q, \ldots, \rho_{n-k}^q|\beta^q\}}_{\{\mu_1^q, \ldots, \mu_{z-1}^q, \mu_{z+1}^q, \ldots, \mu_k^q\}}$ and $X^{\{\delta_1, \ldots, \delta_{z-1}, \delta_{z+1}, \ldots, \delta_k\}}_{\{\tau_1, \ldots, \tau_{n-k}|\eta\}}$ have the same eigenvalues. In particular, $(\lambda_2^*)^q = \lambda_2^q$. □

Using similar argument to the previous result, we arrive to the next one.

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Lemma 8.2.10. *$\bar{\lambda} \neq 0$ is an eigenvalue of $T_n[0]$ in X_4^* defined in* (8.2.43) *if and only if $\bar{\lambda}$ is an eigenvalue of $T_n[0]$ in X_4. In particular, $(\lambda_4^*)^q = \lambda_4^q$ where $(\lambda_4^*)^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_4^* and $\lambda_4^q \neq 0$ is the least positive (biggest negative) eigenvalue of $T_n[0]$ in X_4.*

Next, we state and prove the main result, which gives the characterization of the set of parameter M where $v_s^q[M]$ is of constant sign.

Theorem 8.2.11. *Let $q \in \{1, \ldots, n-1\}$. Suppose that $\{\sigma_1, \ldots, \sigma_k\} - \{\varepsilon_1, \ldots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q + d_q = n - q$, $c_q \geq 1$ and $d_q \geq 1$. The following properties are fulfilled:*

- *Let $2 \leq k \leq n-2$ and $n-k$ even:*
  - *If $n - q - c_q$ is even, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in (\lambda_1, \lambda^q]$, and if $n - q - c_q$ is odd, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in (\lambda_1, \lambda^q]$ where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda^q > 0$ is the minimum between:*
      - *$\lambda_2^q > 0$, the least positive eigenvalue of $T_n[0]$ in X_2 defined in* (8.2.32).
      - *$\lambda_4^q > 0$, the least positive eigenvalue of $T_n[0]$ in X_4 defined in* (8.2.34).
- *Let $2 \leq k \leq n-2$ and $n-k$ odd:*
  - *If $n - q - c_q$ is even, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in [\lambda^q, \lambda_1)$, and if $n - q - c_q$ is odd, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in [\lambda^q, \lambda_1)$ where:*
    - *$\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda^q < 0$ is the maximum between:*
      - *$\lambda_2^q < 0$, the biggest negative eigenvalue of $T_n[0]$ in X_2.*
      - *$\lambda_4^q < 0$, the biggest negative eigenvalue of $T_n[0]$ in X_4.*
- *Let $k = 1$ and $n > 2$ odd:*
  - *If $n - q - 1$ is even, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in (\lambda_1, \lambda_2^q]$, and if $n - q - 1$ is odd, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in (\lambda_1, \lambda_2^q]$ where:*
    - *$\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-1}\}}$.*
    - *$\lambda_2^q > 0$ is the least positive eigenvalue of $T_n[0]$ in X_2.*
- *Let $k = 1$ and $n > 2$ even:*
  - *If $n - q - 1$ is even, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in [\lambda_2^q, \lambda_1)$, and if $n - q - 1$ is odd, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in [\lambda_2^q, \lambda_1)$, where:*

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  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X^{\{\varepsilon_1,\dots,\varepsilon_{n-1}\}}_{\{\sigma_1\}}$.
  * $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.

- *If* $k = n-1$, $n > 2$ *and* $n - q - c_q = 1$, *then* $v_s^q[M]$ *is strongly negative on* $I \times I$ *if, and only if,* $M \in [\lambda_2^q, \lambda_1)$, *where*

  * $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X^{\{\varepsilon_1\}}_{\{\sigma_1,\dots,\sigma_{n-1}\}}$.
  * $\lambda_2^q < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* X_2.

Proof. First, it has been shown in Theorem 8.1.29 that v_s^q satisfies the property (P_g) if $n-q-c_q$ is even and the property (N_g) if $n-q-c_q$ is odd. In addition, from Lemma 8.1.31 we know that if $v_s^q[M]$ has constant sign then the first eigenvalue λ_1 of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$ is one of the extremes of the interval where $v_s^q[M]$ holds that sign.

Again, taking into account Lemma 8.1.31 on the monotony of $v_s^q[M]$ with respect to M, the constant sign starts/ends at the first eigenvalue λ_1 of $T_n[0]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$. Thus, from Theorems 8.1.27 and 8.1.29 we conclude that:

- If $n-k$ and $n-q-c_q$ are even, and $M \leq 0$, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$.
- If $n-k$ is even, $n-q-c_q$ is odd and $M \leq 0$, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$.
- If $n-k$ is odd, $n-q-c_q$ is even and $M \geq 0$, then $v_s^q[M]$ is strongly positive on $I \times I$ if, and only if, $M \in [0, \lambda_1)$.
- If $n-k$ and $n-q-c_q$ are odd, and $M \geq 0$, then $v_s^q[M]$ is strongly negative on $I \times I$ if, and only if, $M \in [0, \lambda_1)$.

We now need to determine the other extreme of the constant sign interval of $v_s^q[M]$ using Definition 8.1.26, Propositions 8.2.5 and 8.2.6. We divide the proof into five steps assuming that $n-k$ and $n-q-c_q$ are even, and $2 \leq k \leq n-2$. For the rest of cases the proof is done in an analogous way.

Step 1. Study of $v_s^q[M]$ at $s = a$.

Step 2. Study of $v_s^q[M]$ at $s = b$.

Step 3. Study of $v_s^q[M]$ at $t = a$.

Step 4. Study of $v_s^q[M]$ at $t = b$.

Step 5. Study of $v_s^q[M]$ on $(a,b) \times (a,b)$.

Let us denote

$$g_M(t,s) = \begin{cases} g_M^1(t,s)\,, & a \leq s \leq t \leq b\,, \\ \\ g_M^2(t,s)\,, & a < t < s < b\,, \end{cases}$$

as the related Green's function of $T_n[M]$ in $X^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}}$.

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Step 1. Study of the function $v_s^q[M]$ at $s = a$.

Let us consider the function

$$w_{M,q}(t) = \frac{\partial^\eta}{\partial s^\eta}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a},$$

where η has been defined in (8.1.16).

If $\eta > 0$, since $\{0,\dots,\eta-1\} \subset \{\tau_1,\dots,\tau_{n-k}\}$ using the equality (8.1.13) we obtain that

$$g_M^1(t,a) = \frac{\partial}{\partial s} g_M^1(t,a) = \cdots = \frac{\partial^{\eta-1}}{\partial s^{\eta-1}} g_M^1(t,a) = 0\,, \quad \text{for all } t \in (a,b].$$

Therefore, for all $t \in (a,b]$ and $q \in \{1,\dots,n-1\}$ we infer that

$$\frac{\partial^q}{\partial t^q} g_M^1(t,a) = \frac{\partial^q}{\partial t^q}\Big(\frac{\partial}{\partial s} g_M^1(t,s)\Big)_{|s=a} = \cdots = \frac{\partial^q}{\partial t^q}\Big(\frac{\partial^{\eta-1}}{\partial s^{\eta-1}} g_M^1(t,s)\Big)_{|s=a} = 0\,,$$

equivalently,

$$\frac{\partial^q}{\partial t^q} g_M^1(t,a) = \frac{\partial}{\partial s}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a} = \cdots = \frac{\partial^{\eta-1}}{\partial s^{\eta-1}}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a} = 0\,.$$

Note that it is necessary that $w_{M,q} > 0$ to guarantee the strongly positive sign of the function $v_s^q[M]$. Indeed, if there exists $t^* \in [a,b]$, such that $w_{M,q}(t^*) < 0$, then there exists $\rho(t^*) > a$ such that $v_s^q[M](t^*) < 0$ for all $s \in (a,\rho(t^*))$, which contradicts the strongly positive sign.

Since the function $v_s^q[M]$ is a solution of $T_n[M]\, v_s^q[M](t) = 0$ for $t \neq s$, we have that

$$\frac{\partial^\eta}{\partial s^\eta}\left(T_n[M]\, v_s^q[M](t)\right)_{|s=a} = T_n[M]\, w_{M,q}(t) = 0\,, \quad t \in (a,b]\,,$$

that is, the function $w_{M,q}$ solves the equation $T_n[M]\, u(t) = 0$, $t \in (a,b]$.

It remains to obtain the boundary conditions satisfied by the function $w_{M,q}$.

Using the matrix of the Green's function given in (8.1.7), the equality (8.1.9), the expression of g_{n-j}, $j = 1,\dots,n-1$ given in (8.1.11) and the following identity

$$\frac{\partial}{\partial t}\Big(\frac{\partial^q}{\partial t^q} G_M(t,s)\Big) = A\,\frac{\partial^q}{\partial t^q} G_M(t,s), \quad \text{for all } t \in I\backslash\{s\}\,,\ q \in \{1,\dots,n-1\},$$

it follows that

$$\begin{cases} \dfrac{\partial^{\mu_1^q+q}}{\partial t^{\mu_1^q+q}} g_M^2(t,s)_{|t=a} = 0\,, \\ -\dfrac{\partial^{\mu_1^q+q+1}}{\partial t^{\mu_1^q+q}\partial s} g_M^2(t,s)_{|t=a} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^2(t,s)_{|t=a} = 0\,. \end{cases}$$

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Since $(\eta+\mu_1^q+q) \mod n < n-1$, then we are not in an element of the diagonal of $\dfrac{\partial^q}{\partial t^q}G_M(t,s)$, and taking in account that the above system is satisfied for $s=a$, by continuity we deduce that

$$\begin{cases} \dfrac{\partial^{\mu_1^q+q}}{\partial t^{\mu_1^q+q}} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\ -\dfrac{\partial^{\mu_1^q+q+1}}{\partial t^{\mu_1^q+q}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,. \end{cases}$$

Therefore,

$$w_{M,q}^{(\mu_1^q)}(a) = \frac{\partial^{\mu_1^q+q+\eta}}{\partial t^{\mu_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,.$$

Making a similar reasoning we deduce that

$$w_{M,q}^{(\mu_2^q)}(a) = \cdots = w_{M,q}^{(\mu_{z-1}^q)}(a) = w_{M,q}^{(\mu_{z+1}^q)}(a) = \cdots = w_{M,q}^{(\mu_k^q)}(a) = 0\,.$$

For μ_z^q-derivative, we have the following system

$$\begin{cases} \dfrac{\partial^{\mu_z^q+q}}{\partial t^{\mu_z^q+q}} g_M^2(t,s)_{|t=a} = 0\,, \\ -\dfrac{\partial^{\mu_z^q+q+1}}{\partial t^{\mu_z^q+q}\partial s} g_M^2(t,s)_{|t=a} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^2(t,s)_{|t=a} = 0\,. \end{cases}$$

Since $\eta+\mu_z^q+q = n-1$, then we are in an element of the diagonal of $\dfrac{\partial^q}{\partial t^q}G_M(t,s)$. Again, taking into account again that the previous system is satisfied for $s=a$ and that

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$\frac{\partial^q}{\partial t^q} G_M(t,s)$ has a jump equal to $+1$ on the diagonal we infer that

$$\begin{cases} \dfrac{\partial^{\mu_z^q+q}}{\partial t^{\mu_z^q+q}} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\ -\dfrac{\partial^{\mu_z^q+q+1}}{\partial t^{\mu_z^q+q}\partial s} g_M^1(t,s)_{|(t,s)=(a,a)} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = 1\,. \end{cases}$$

Therefore,

$$w_{M,q}^{(\mu_z^q)}(a) = \frac{\partial^{\mu_z^q+q+\eta}}{\partial t^{\mu_z^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(a,a)} = (-1)^\eta\,.$$

Now, let us determine the conditions of $w_{M,q}$ at $t = b$. Similarly, reasoning as the previous argument we obtain that

$$\begin{cases} \dfrac{\partial^{\rho_1^q+q}}{\partial t^{\rho_1^q+q}} g_M^1(t,s)_{|t=b} = 0\,, \\ -\dfrac{\partial^{\rho_1^q+q+1}}{\partial t^{\rho_1^q+q}\partial s} g_M^1(t,s)_{|t=b} = 0\,, \\ \vdots \\ (-1)^\eta \dfrac{\partial^{\rho_1^q+q+\eta}}{\partial t^{\rho_1^q+q}\partial s^\eta} g_M^1(t,s)_{|t=b} = 0\,. \end{cases}$$

The previous system is satisfied at $s = a$ and since $b \neq a$ we have that

$$w_{M,q}^{(\rho_1^q)}(b) = \frac{\partial^{\rho_1^q+q+\eta}}{\partial t^{\rho_1^q+q}\partial s^\eta} g_M^1(t,s)_{|(t,s)=(b,a)} = 0\,.$$

Analogously we obtain that

$$w_{M,q}^{(\rho_2^q)}(b) = \cdots = w_{M,q}^{(\rho_{n-k}^q)}(b) = 0\,.$$

If $\eta = 0$, then it follows from the matrix argument we made earlier that the function $w_{M,q}(t) = \frac{\partial^q}{\partial t^q} g_M^1(t,a)$ satisfies the same boundary conditions (8.2.36)–(8.2.37).

Then from Proposition 8.2.5 and Theorem 8.1.29 we conclude that $w_{M,q} > 0$ on (a,b) for all $M \in [0, \lambda_2^q]$.

Now, let us see that $v_s^q[M]$ cannot be positive for $M > \lambda_2^q$.

Suppose that there exists $\widehat{M} > \lambda_2^q$ such that $v_s^q[\widehat{M}]$ is positive. Then, by monotony of $v_s^q[M]$ we have for every $M \in [\lambda_2^q, \widehat{M}]$ that $w_{\widehat{M},q} \leq w_{M,q} \leq w_{\lambda_2^q,q}$.

In particular, $w_{\widehat{M},q}^{(\beta^q)}(b) \geq w_{M,q}^{(\beta^q)}(b) \geq w_{\lambda_2^q,q}^{(\beta^q)}(b) = 0$ if β^q is even and $w_{\widehat{M},q}^{(\beta^q)}(b) \leq w_{M,q}^{(\beta^q)}(b) \leq w_{\lambda_2^q,q}^{(\beta^q)}(b) = 0$ if β^q is odd.

If $w_{\widehat{M},q}^{(\beta^q)}(b) \neq 0$, then there exists $\rho > 0$ such that $w_{\widehat{M},q}(t) < 0$ for all $t \in (b-\rho, b)$, which contradicts our assumption. Hence,

$$0 = w_{\widehat{M},q}^{(\beta^q)}(b) = w_{M,q}^{(\beta^q)}(b) = w_{\lambda_2^q,q}^{(\beta^q)}(b)\,, \quad \forall M \in [\lambda_2^q, \widehat{M}],$$

and this fact contradicts the discrete character of the spectrum.

Thus, we conclude that, if $M \in [0, \lambda_2^q]$, then

$$\forall t \in (a,b)\,, \quad \exists \rho^q(t) > 0 \;\mid\; v_s^q[M](t) > 0 \; \forall s \in (a, a+\rho^q(t))\,.$$

Moreover, if $M > \lambda_2^q$, then $v_s^q[M]$ is not positive.

Step 2. Study of the function $v_s^q[M]$ at $s = b$.

We study the following function by applying similar reasoning to Step 1. In this case we consider

$$y_{M,q}(t) = \frac{\partial^\gamma}{\partial s^\gamma}\Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s)\Big)_{|s=b}\,,$$

where γ has been defined in (8.1.17).

In this case, using arguments analogous to Step 1 we obtain that if $\gamma > 0$, then

$$\frac{\partial^q}{\partial t^q} g_M^2(t,b) = \frac{\partial}{\partial s} g_M^2(t,s)_{|s=b} = \cdots = \frac{\partial^{\gamma-1}}{\partial s^{\gamma-1}}\Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s)\Big)_{|s=b} = 0\,.$$

Moreover, if γ is even, then $y_{0,q} \geq 0$ and if γ is odd, then $y_{0,q} \leq 0$.

Again, we have that

$$T_n[M]\, y_{M,q}(t) = 0\,, \quad \forall\, t \in [a,b)\,.$$

In the same way than above, studying $\dfrac{\partial^q}{\partial t^q} G_M(t,s)$ to determine the boundary conditions of $y_{M,q}$ we infer that

$$\begin{aligned} y_{M,q}^{(\mu_1^q)}(a) = \cdots = y_{M,q}^{(\mu_k^q)}(a) &= 0\,, \\ y_{M,q}^{(\rho_1^q)}(b) = \cdots = y_{M,q}^{(\rho_{h-1}^q)}(b) = y_{M,q}^{(\rho_{h+1}^q)}(b) = \cdots = y_{M,q}^{(\rho_{n-k}^q)}(b) &= 0\,, \\ y_{M,q}^{(\rho_h^q)}(b) &= (-1)^{\gamma+1}\,. \end{aligned}$$

Analogously, if $\gamma = 0$, then $y_{M,q}(t) = \frac{\partial^q}{\partial t^q} g_M^2(t,b)$ satisfies the conditions (8.2.38)–(8.2.39) by an argument similar to the case $\gamma > 0$.

Then from Proposition 8.2.6 and Theorem 8.1.29 we conclude that $y_{M,q} > 0$ if γ is even and $y_{M,q} < 0$ if γ is odd on (a,b) for all $M \in [0, \lambda_4^q]$.

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As in the previous step it can be seen that for $M > \lambda_4^q$, the function $v_s^q[M]$ is never positive. Therefore, we conclude from this step that, if $M \in [0, \lambda_4^q]$, then

$$\forall t \in (a,b)\,, \quad \exists \rho^q(t) > 0 \;\mid\; v_s^q[M](t) > 0 \;\forall s \in (b-\rho^q(t), b)\,.$$

Moreover, if $M > \lambda_4^q$, then $v_s^q[M]$ is cannot be positive.

Step 3. Study of the function $v_s^q[M]$ at $t = a$.

Let us denote

$$\widehat{g}_{(-1)^n M}(t,s) = \begin{cases} \widehat{g}^1_{(-1)^n M}(t,s)\,, & a \le s \le t \le b\,, \\ \\ \widehat{g}^2_{(-1)^n M}(t,s)\,, & a < t < s < b\,, \end{cases}$$

as the related Green's function of $\widehat{T}_n[(-1)^n M]$ in $X^{*\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}_{\{\sigma_1,\dots,\sigma_k\}} = X^{\{\delta_1,\dots,\delta_k\}}_{\{\tau_1,\dots,\tau_{n-k}\}}$.

In this case, we study the following function by applying similar reasoning to Step 1:

$$\widehat{w}_{M,q}(t) = (-1)^n \frac{\partial^{\alpha^q+q}}{\partial\, s^{\alpha^q+q}} \widehat{g}^1_{(-1)^n M}(t,s)_{|s=a}\,,$$

which is equivalent, using equality (8.1.15), to study the function

$$\widehat{w}_{M,q}(s) = \frac{\partial^{\alpha^q+q}}{\partial\, t^{\alpha^q+q}} g^2_M(t,s)_{|t=a} = \frac{\partial^{\alpha^q}}{\partial\, t^{\alpha^q}} \Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s)\Big)_{|t=a}\,, \quad s \in I\,.$$

In this case, if $\alpha^q > 0$ using that $\{0,\dots,\alpha^q-1\} \subset \{\mu_1^q,\dots,\mu_k^q\}$ we have

$$\frac{\partial^q}{\partial t^q} g^2_M(a,s) = \frac{\partial}{\partial t}\Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s)\Big)_{|t=a} = \cdots = \frac{\partial^{\alpha^q-1}}{\partial t^{\alpha^q-1}}\Big(\frac{\partial^q}{\partial t^q} g^2_M(t,s)\Big)_{|t=a} = 0\,,\ \forall s \in (a,b).$$

As in the Steps 1 and 2 we can deduce that

$$\widehat{T}_n[(-1)^n M]\, \widehat{w}_{M,q}(t) = 0, \;\; t \in (a,b].$$

Using the arguments of Step 1, we can affirm that if there exists $t^* \in (a,b)$ such that $\widehat{w}_{M,q}(t^*) < 0$, then $v_s^q[M]$ is not positive.

Next, we determine the conditions that $\widehat{w}_{M,q}$ saisfies at $t = a$. By the first row of (8.1.20) and (8.1.19) we deduce that:

$$\begin{cases} \dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,, \\ \\ -\dfrac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,, \\ \\ \vdots & \\ \\ (-1)^{\alpha^q+q} \dfrac{\partial^{\tau_1+\alpha^q+q}}{\partial t^{\tau_1}\partial s^{\alpha^q+q}} \widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,. \end{cases}$$

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Since $(\tau_1+\alpha^q+q) \mod n < n-1$, we do not reach any diagonal element, hence the previous system is satisfied for $s=a$, and we obtain:

$$\begin{cases} \dfrac{\partial^{\tau_1}}{\partial t^{\tau_1}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 0\,, \\ -\dfrac{\partial^{\tau_1+1}}{\partial t^{\tau_1}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 0\,, \\ \vdots & \\ (-1)^{\alpha^q+q}\dfrac{\partial^{\tau_1+\alpha^q+q}}{\partial t^{\tau_1}\partial s^{\alpha^q+q}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 0\,. \end{cases}$$

So, we conclude that $\widehat{w}^{(\tau_1)}_{M,q}(a)=0$.
Analogously we obtain that

$$\widehat{w}^{(\tau_2)}_{M,q}(a)=\cdots=\widehat{w}^{(\tau_{h-1})}_{M,q}(a)=\widehat{w}^{(\tau_{h+1})}_{M,q}(a)=\cdots=\widehat{w}^{(\tau_{n-k})}_{M,q}(a)=0.$$

For τ_h-derivative, we have the following system

$$\begin{cases} \dfrac{\partial^{\tau_h}}{\partial t^{\tau_h}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,, \\ -\dfrac{\partial^{\tau_h+1}}{\partial t^{\tau_h}\partial s}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,, \\ \vdots & \\ (-1)^{\alpha^q+q}\dfrac{\partial^{\tau_h+\alpha^q+q}}{\partial t^{\tau_h}\partial s^{\alpha^q+q}}\widehat{g}^2_{(-1)^n M}(t,s)_{|t=a} &= 0\,. \end{cases}$$

In this case, since $\tau_h+\alpha^q+q=n-1$, we reach a diagonal element of $\widehat{G}(t,s)$ given in (8.1.18), and as consequence we obtain the following system for $s=a$:

$$\begin{cases} \dfrac{\partial^{\tau_h}}{\partial t^{\tau_h}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 0\,, \\ -\dfrac{\partial^{\tau_h+1}}{\partial t^{\tau_h}\partial s}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 0\,, \\ \vdots & \\ (-1)^{\alpha^q+q}\dfrac{\partial^{\tau_h+\alpha^q+q}}{\partial t^{\tau_h}\partial s^{\alpha^q+q}}\widehat{g}^1_{(-1)^n M}(t,s)_{|(t,s)=(a,a)} &= 1\,. \end{cases}$$

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Therefore, $\widehat{w}_{M,q}^{(\tau_h)}(a) = (-1)^{n-\alpha^q-q}$.

Now, let us study the behavior of $\widehat{w}_{M,q}$ at $t = b$. Studying the $(n-k+1)^{\text{th}}$ row of (8.1.20), we have for all $s \in (a,b)$:

$$\begin{cases} \dfrac{\partial^{\delta_1}}{\partial t^{\delta_1}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} &= 0\,, \\ -\dfrac{\partial^{\delta_1+1}}{\partial t^{\delta_1}\partial s} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} &= 0\,, \\ \vdots & \\ (-1)^{\alpha^q+q} \dfrac{\partial^{\delta_1+\alpha^q+q}}{\partial t^{\delta_1}\partial s^{\alpha^q+q}} \widehat{g}^1_{(-1)^n M}(t,s)_{|t=b} &= 0\,. \end{cases}$$

The previous system is satisfied at $s = a$ and since $b \neq a$ we have that $\widehat{w}_{M,q}^{(\delta_1)}(b) = 0$.
Analogously we obtain that

$$\widehat{w}_{M,q}^{(\delta_2)}(b) = \cdots = \widehat{w}_{M,q}^{(\delta_k)}(b) = 0.$$

Note that if $\alpha^q = 0$, then from the same matrix argument above we deduce that the function $\widehat{w}_{M,q}(s) = \frac{\partial^q}{\partial t^q} g^2_M(a,s)$ verifies the same boundary conditions:

$$\begin{aligned} \widehat{w}_{M,q}^{(\tau_2)}(a) = \cdots = \widehat{w}_{M,q}^{(\tau_{h-1})}(a) = \widehat{w}_{M,q}^{(\tau_{h+1})}(a) = \quad \cdots \quad &= \widehat{w}_{M,q}^{(\tau_{n-k})}(a) = 0\,, \\ \widehat{w}_{M,q}^{(\delta_2)}(b) = \quad \cdots \quad &= \widehat{w}_{M,q}^{(\delta_k)}(b) \quad = 0\,. \end{aligned}$$

From (8.1.14) and (8.1.12) we deduce that

$$\widehat{T}_n[0]\, v(t) := (-1)^n\, T_n^*[0]\, v(t) = T_n[0]\, v(t).$$

Hence, using arguments similar to Step 1, we conclude that $\widehat{w}_{M,q} > 0$ on (a,b) for all $M \in [0, (\lambda_4^*)^q]$, where $(\lambda_4^*)^q > 0$ is the least positive eigenvalue of $\widehat{T}_n[0] \equiv T_n[0]$ in X_4^* defined in (8.2.43).

From Lemma 8.2.10, we know that $(\lambda_4^*)^q = \lambda_4^q$. Thus, from this step we have that, if $M \in [0, \lambda_4^q]$, then

$$\forall s \in (a,b)\,, \quad \exists \rho^q(s) > 0 \;\mid\; v_s^q[M](t) > 0 \;\forall t \in (a, a+\rho^q(s))\,.$$

Step 4. Study of the function $v_s^q[M]$ at $t = b$.

In this case, we study the following function by applying similar reasoning to Step 3

$$\widehat{y}_{M,q}(t) = (-1)^n \frac{\partial^{\beta^q+q}}{\partial\, s^{\beta^q+q}} \widehat{g}^2_{(-1)^n M}(t,s)_{|s=b}\,,$$

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which is equivalent using equality (8.1.15) to studying the function

$$\widehat{y}_{M,q}(s) = \frac{\partial^{\beta^q+q}}{\partial\, t^{\beta^q+q}} g_M^1(t,s)_{|t=b} = \frac{\partial^{\beta^q}}{\partial\, t^{\beta^q}} \Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s) \Big)_{|t=b}.$$

Since $\{0,\dots,\beta^q\} \subset \{\rho_1^q,\dots,\rho_{n-k}\}$ if $\beta^q > 0$ we obtain that

$$\frac{\partial^q}{\partial t^q} g_M(b,t) = \frac{\partial}{\partial t} \Big(\frac{\partial^q}{\partial t^q} g_M(t,s) \Big)_{|t=b} = \cdots = \frac{\partial^{\beta^q-1}}{\partial t^{\beta^q-1}} \Big(\frac{\partial^q}{\partial t^q} g_M(t,s) \Big)_{|t=b} = 0\,.$$

Moreover, using arguments analogous to Step 2 we obtain that if β^q is even, then $\widehat{y}_{0,q} \geq 0$ and if β^q is odd, then $\widehat{y}_{0,q} \leq 0$.

Again, we have that

$$\widehat{T}_n[(-1)^n M]\, \widehat{y}_{M,q}(t) = 0\,, \quad \forall\, t \in [a,b)\,.$$

A study analogous to Step 3 leads us to the fact that $\widehat{y}_{M,q}$ satisfies the boundary conditions

$$\begin{aligned} \widehat{y}_{M,q}^{(\tau_1)}(a) = \cdots = \widehat{y}_{M,q}^{(\tau_{n-k})}(a) = 0\,, \\ \widehat{y}_{M,q}^{(\delta_1)}(b) = \cdots = \widehat{y}_{M,q}^{(\delta_{z-1})}(b) = \widehat{y}_{M,q}^{(\delta_{z+1})}(b) = \cdots = \widehat{y}_{M,q}^{(\delta_k)}(b) = 0\,, \\ y_{M,q}^{(\delta_z)}(b) = (-1)^{n-\beta^q-q+1}\,. \end{aligned} \tag{8.2.46}$$

Analogously, if $\beta^q = 0$, then $\widehat{y}_{M,q}(s) = \frac{\partial^q}{\partial t^q} g_M^2(b,s)$ satisfies the conditions (8.2.46) by an argument similar to the case $\beta^q > 0$.

Thus, we deduce that $\widehat{y}_{M,q} > 0$ on (a,b) if β^q is even and $\widehat{y}_{M,q} < 0$ on (a,b) if β^q is odd for all $M \in [0,(\lambda_2^*)^q]$, where $(\lambda_2^*)^q > 0$ is the least positive eigenvalue of $\widehat{T}_n[0] \equiv T_n[0]$ in X_2^* defined in (8.2.42).

Again, from Lemma 8.2.9 we have that $(\lambda_2^*)^q = \lambda_2^q$. Therefore, we conclude from this step that, if $M \in [0,\lambda_2^q]$, then

$$\forall s \in (a,b)\,, \quad \exists \rho^q(s) > 0 \;\mid\; v_s^q[M](t) > 0\; \forall t \in (b-\rho^q(t),b)\,.$$

Step 5. Study of the function $v_s^q[M]$ on $(a,b)\times(a,b)$.

In this step, we verify that the function $v_s^q[M](t) > 0$ for all $(t,s) \in I \times I$ if M belongs to the given intervals. Let us denote $u_M^s(t) = g_M(t,s)$, for all $s \in (a,b)$.

The function $v_s^q[M](t)$ is in $C^{n-2-q}(I)$ and it satisfies the boundary conditions (8.1.21)-(8.1.22).

Using the definition of $v_s^q[M]$, the following equality holds for all $s \in (a,b)$:

$$T_{n-q}[0]\, v_s^q[M](t) = -M\, u_M^s(t)\,, \quad \forall\, t \neq s\,,\ t \in I\,. \tag{8.2.47}$$

Let us see that for the values of the parameter M for which $v_s^q[M](t)$ has constant sign in I, it cannot have a double zero in (a,b), and this implies that the change of the sign must be at $t=a$ or $t=b$, and the result is proven.

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We do the proof for $n-k$ even (the case $n-k$ odd is similarly proved). In this case, we study the behavior of $v_s^q[M](t)$ for $M>0$ and $u_M^s \geq 0$. From (8.2.47), we have that

$$\frac{d^{n-q}}{dt^{n-q}}\, v_s^q[M](t) = \frac{\partial^n}{\partial t^n}\, g_M(t,s) \leq 0.$$

Therefore, using the condition (T_d), since $v_1 = \cdots = v_n = 1$, we have that

$$\frac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t) = \frac{\partial^{n-1}}{\partial t^{n-1}}\, g_M(t,s)$$

is a decreasing function, with two continuous components and a jump $+1$ at $t=s$. Then, it has at most two zeros on I (see Figure 8.2.1).

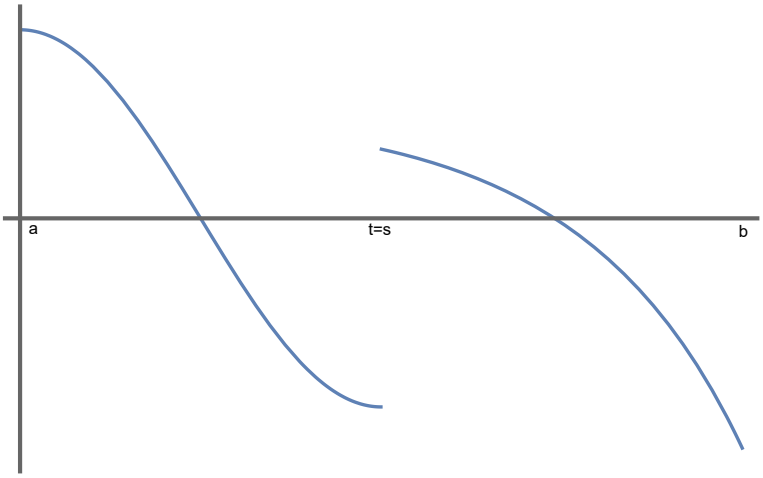

Figure 8.2.1: $\dfrac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t)$, maximal oscillation with $t \in I$.

Now, we have that

$$\frac{d^{n-q-2}}{dt^{n-q-2}}\, v_s^q[M](t) = \frac{\partial^{n-2}}{\partial t^{n-2}}\, g_M(t,s)$$

is a continuous function with at most four zeros on I (See Figure 8.2.2).

By recurrence, we have that

$$\frac{d^{n-q-l}}{dt^{n-q-l}}\, v_s^q[M](t) = \frac{\partial^{n-l}}{\partial t^{n-l}}\, g_M(t,s)$$

has at most $l+2$ zeros on I. In particular, $v_s^q[M]$ has at most $n-q+2$ zeros on I. Moreover, since

$$\begin{aligned}(v_s^q)^{(\sigma_1^q)}[M](a) &= \cdots = (v_s^q)^{(\sigma_{c_q}^q)}[M](a) = 0\,,\\ (v_s^q)^{(\varepsilon_1^q)}[M](b) &= \cdots = (v_s^q)^{(\varepsilon_{d_q}^q)}[M](b) = 0\,,\end{aligned}$$

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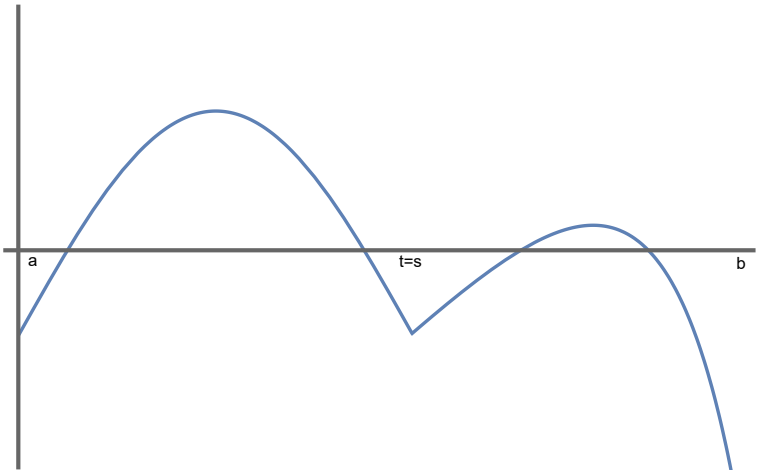

Figure 8.2.2: $\dfrac{d^{n-q-1}}{dt^{n-q-1}}\, v_s^q[M](t)$, maximal oscillation with $t \in I$.

$v_s^q[M]$ has $c_q + d_q = n - q$ zeros on the boundary. Thus, $v_s^q[M]$ has at most two zeros on (a, b).

Now, making an argument similar to the one made in Theorem 8.1.27, we deduce that the maximal oscillation (that is, the maximal number of zeros) is possible if and only if

$$\begin{cases} (v_s^q)^{(\alpha^q)}[M](a) \leq 0\,, & \text{if } n-q-c_q \text{ is even,} \\ \\ (v_s^q)^{(\alpha^q)}[M](a) \geq 0\,, & \text{if } n-q-c_q \text{ is odd.} \end{cases}$$

Since we are in the case $n - q - c_q$ even, the maximal oscillation is only possible if $(v_s^q)^{(\alpha^q)}[M](a) \leq 0$. However, by definition, $(v_s^q)^{(\alpha^q)}[M](a) \neq 0$ and, since $v_s^q[M]$ is nonnegative and α^q denotes the order of the smallest derivative at $t = a$ which is different from zero, necessarily $(v_s^q)^{(\alpha^q)}[M](a) > 0$. Therefore the maximal oscillation can not happen (that is, $v_s^q[M]$ cannot have two zero on (a, b)). Thus, $v_s^q[M]$ has at most one simple zero on (a, b), which is not possible due to the constant sign of $v_s^q[M]$, and we conclude that $v_s^q[M]$ does not have any zero on (a, b).

From this step we deduce that if $M > 0$ and $u_M^s \geq 0$, then $v_s^q[M] > 0$ on (a, b).

In conclusion, from the previous steps the result is proved. □

In the sequel, we present some examples of application of the previous result.

Example 8.2.12. *Consider again the fourth order operator $T_4[M]$ coupled with the boundary conditions $X^{\{2\}}_{\{0,1,2\}}$ defined in Example 8.1.8.*

The functions $\frac{\partial}{\partial t} g_M$ and $\frac{\partial^2}{\partial t^2} g_M$ satisfy the conditions of spaces $X^{\{1\}}_{\{0,1,3\}}$ and $X^{\{0\}}_{\{0,2,3\}}$, respectively.

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In this case $k = 3$ and $n - k = 1$ is odd. For $q = 1$, we have that $c_q = 2$, $d_q = 1$, $z = 2 \neq 1$ and $\beta^1 = 0$. By applying Theorem 8.2.11 it follows that $\frac{\partial}{\partial t} g_M$ is strongly negative on $[0,1] \times [0,1]$ if, and only if $M \in [\lambda_2^1, \lambda_1)$, where

* *$\lambda_1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$.*

* *$\lambda_2^1 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$.*

The eigenvalues of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$ are given by λ^4, where λ is a positive solution of

$$\tanh\left(\lambda\sqrt{2}\right) + \tan\left(\frac{\lambda}{\sqrt{2}}\right) = 0.$$

Let us denote m_1 as the smallest positive solution of this equation. Then, $\lambda_1 = m_1^4$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$.

In the same way, the eigenvalues of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is a positive solution of $\sin(\lambda) = 0$.

In this case, $\lambda_{min}^1 = \pi$ is the smallest positive solution of this equation. Then, $\lambda_2^1 = -(\lambda_{min}^1)^4 = -\pi^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{0,3\}}^{\{0,1\}}$.

By numerical approach, it can be seen that $\lambda_1 = m_1^4 \approx 3.34^4$.

For $q = 2$, we have that $c_q = n - q - 1 = 1$, $d_q = 1$, $z = 1$ and $\beta^2 = 1$. Again, by applying Theorem 8.2.11 it follows that $\frac{\partial^2}{\partial t^2} g_M$ is strongly negative on $[0,1] \times [0,1]$ if, and only if $M \in [\lambda_2^2, \lambda_1)$, where

* *$\lambda_1 > 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1,2\}}^{\{2\}}$.*

* *$\lambda_2^2 < 0$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$.*

The eigenvalues of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$ are given by $-\lambda^4$, where λ is a least positive solution of

$$2e^{2\lambda} + (e^{2\lambda} + 1)\cos(\lambda) = 0.$$

By denoting λ_{min}^2 as the smallest positive solution of this equation, then $\lambda_2^2 = -(\lambda_{min}^2)^4$ is the biggest negative eigenvalue of $T_4[0]$ in $X_{\{2,3\}}^{\{0,1\}}$.

In this case, we have that $\lambda_2^2 = -(\lambda_{min}^2)^4 \approx -1.87^4$.

Example 8.2.13. *We consider the fourth order operator $T_4[M]$ defined in the space of the boundary conditions $X_{\{0,1\}}^{\{1,3\}}$.*

In this case $k = 2$ and $n - k$ is even. For $q = 1$ we have that $c_q = 1$, $d_q = n - q - c_q = 2$, $z = 1$, $h = 2$ and $\alpha^1 = \beta^1 = 1$. Moreover, the function $\frac{\partial}{\partial t} g_M$ satisfies the conditions of space $X_{\{0,3\}}^{\{0,2\}}$.

By applying Theorem 8.2.11 we obtain that $\frac{\partial}{\partial t} g_M$ is strongly positive on $[0,1] \times [0,1]$ if, and only if $M \in (\lambda_1, \lambda^1]$, where

* *$\lambda_1 < 0$ is the least positive eigenvalue of $T_4[0]$ in $X_{\{0,1\}}^{\{1,3\}}$.*

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* $\lambda^1 > 0$ *is the minimum between:*

  ⋆ $\lambda_2^1 > 0$*, the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{3\}}^{\{0,1,2\}}$.

  ⋆ $\lambda_4^1 > 0$ *is the least positive eigenvalue of* $T_4[0]$ *in* $X_{\{0,1,3\}}^{\{0\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -2.36^4$*,* $\lambda_2^1 \approx 2.22^4$ *and* $\lambda_4^1 \approx 4.44^4$*. Therefore,* $\frac{\partial}{\partial t} g_M$ *is strongly positive on* $[0,1]\times[0,1]$ *if, and only if,* $M \in (-2.36^4, 2.22^4]$.

As a consequence of the Theorem 8.2.11 we arrive at the following result.

Corollary 8.2.14. *Let* $q \in \{1,\dots,n-1\}$*. Suppose that* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfies condition* (N_a)*,* $c_q + d_q = n - q$*,* $c_q \geq 1$ *and* $d_q \geq 1$*. If either* $\sigma_k = q + c_q - 1$ *or* $\varepsilon_{n-k} = q + d_q - 1$*, we have the following properties:*

- *If* $n - q - c_q$ *is even, then there is not any* $M \in \mathbb{R}$ *such that* $v_s^q[M]$ *is strongly negative on* $I \times I$*.*
- *If* $n - q - c_q$ *is odd, then there is not any* $M \in \mathbb{R}$ *such that* $v_s^q[M]$ *is strongly positive on* $I \times I$*.*

Proof. If $\sigma_k = q + c_q - 1$, then $\eta = n - 1 - \sigma_k = n - q - c_q$.

We consider

$$w_{M,q}(t) = \frac{\partial^\eta}{\partial s^\eta}\Big(\frac{\partial^q}{\partial t^q} g_M^1(t,s)\Big)_{|s=a},$$

defined in Step 1 of the proof of Theorem 8.2.11.

We know that this function satisfies the following boundary conditions:

$$\begin{aligned} w_{M,q}^{(\mu_1^q)}(a) = \cdots = w_{M,q}^{(\mu_{z-1}^q)}(a) = w_{M,q}^{(\mu_{z+1}^q)}(a) = \cdots = w_{M,q}^{(\mu_k^q)}(a) &= 0\,, \\ w_{M,q}^{(\rho_1^q)}(b) = \cdots = w_{M,q}^{(\rho_{n-k}^q)}(b) &= 0\,, \\ w_{M,q}^{(\mu_z^q)}(a) &= (-1)^\eta = (-1)^{n-q-c_q}\,. \end{aligned}$$

Hence, if $n - q - c_q$ is even, then there exists $\rho > 0$, such that $w_{M,q}(t) > 0$ for all $t \in (a, a+\rho)$. So, $v_s^q[M]$ cannot be negative for any real M.

Now, if $n - q - c_q$ is odd, then there exists $\rho > 0$, such that $w_{M,q}(t) < 0$ for all $t \in (a, a+\rho)$. Thus, $v_s^q[M]$ cannot be positive for any $M \in \mathbb{R}$.

Analogously, if $\varepsilon_{n-k} = q + d_q - 1$, then $\gamma = n - \varepsilon_{n-k} - 1 = n - q - d_q = c_q$. In this case, we consider the function

$$y_{M,q}(t) = \frac{\partial^\gamma}{\partial s^\gamma}\Big(\frac{\partial^q}{\partial t^q} g_M^2(t,s)\Big)_{|s=b},$$

defined in Step 2 of the proof of Theorem 8.2.11.

We know that for all $M \in \mathbb{R}$, the function $y_{M,q}$ satisfies the following boundary conditions:

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$$y_{M,q}^{(\mu_1^q)}(a) = \cdots = y_{M,q}^{(\mu_k^q)}(a) = 0\,,$$
$$y_{M,q}^{(\rho_1^q)}(b) = \cdots = y_{M,q}^{(\rho_{h-1}^q)}(b) = y_{M,q}^{(\rho_{h+1}^q)}(b) = \cdots = y_{M,q}^{(\rho_{n-k}^q)}(b) = 0\,,$$
$$y_{M,q}^{(\rho_h^q)}(b) = (-1)^{c_q+1}\,.$$

Hence, if $n-q-c_q$ and c_q are even, then there exists $\rho>0$, such that $y_{M,q}(t)>0$ for all $t\in(b-\rho,b)$. So, $v_s^q[M]$ cannot be negative for any real M.

Moreover, if $n-q-c_q$ is even and c_q odd, then there exists $\rho>0$, such that $y_{M,q}(t)<0$ for all $t\in(b-\rho,b)$. So, $v_s^q[M]$ cannot be positive for any real M.

Now, if $n-q-c_q$ is odd and c_q even, then there exists $\rho>0$, such that $y_{M,q}(t)<0$ for all $t\in(b-\rho,b)$. So, $v_s^q[M]$ cannot be positive for any real M.

Finally, if $n-q-c_q$ and c_q are odd, then there exists $\rho>0$, such that $y_{M,q}(t)>0$ for all $t\in(b-\rho,b)$.

As consequence, $v_s^q[M]$ cannot be negative for any real M. □

Remark 8.2.15. *It is important to note that in the process of obtaining the boundary conditions of $w_{M,q}$ and $y_{M,q}$ we have not used the property (T_d) of the operator $T_n[0]$ in $X_{\{\mu_1^q,\ldots,\mu_k^q\}}^{\{\rho_1^q,\ldots,\rho_{n-k}^q\}}$.*

Example 8.2.16. *Consider the operator $T_4[M]$ coupled with the boundary conditions defined in Example 8.1.8.*

For $q=1$ and $q=2$, we have that $n-q-c_q$ is odd. Since $\sigma_3=q+c_q-1$ for $q=1,2$, from Corollary 8.2.14 we have that there is not any $M\in\mathbb{R}$ such that $\frac{\partial}{\partial t}g_M$ and $\frac{\partial^2}{\partial t^2}g_M$ are strongly positive on $[0,1]\times[0,1]$.

Now, we give a necessary condition of the nonpositive (nonnegative) sign of $v_s^q[M]$. This characterization will be an interval in which $v_s^q[M]$ has a nonpositive (nonnegative) sign whose infimum or supremum is given by the first eigenvalue λ_1 of $T_n[M]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.

Theorem 8.2.17. *Let $q\in\{1,\ldots,n-1\}$. Suppose that $\{\sigma_1,\ldots,\sigma_k\}-\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q+d_q=n-q$, $c_q\geq 1$ and $d_q\geq 1$. If $\mu_z^q\neq z-1$ and $\rho_h^q\neq h-1$, then the following properties are fulfilled:*

- *Let $n-k$ be even:*
  - *If $n-q-c_q$ is even and $v_s^q[M]$ is nonpositive on $I\times I$, then $M\in[\lambda_*^q,\lambda_1)$, and if $n-q-c_q$ is odd and $v_s^q[M]$ is nonnegative on $I\times I$, then $M\in[\lambda_*^q,\lambda_1)$, where*
    - *$\lambda_1<0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\ldots,\sigma_k\}}^{\{\varepsilon_1,\ldots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_*^q<0$ is the maximum between:*
      - *$\lambda_3^q<0$ is the biggest negative eigenvalue of $T_n[0]$ in X_3 defined in* (8.2.33).
      - *$\lambda_5^q<0$, the biggest negative eigenvalue of $T_n[0]$ in X_5 defined in* (8.2.35).

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- *Let $n-k$ be odd:*
  - *If $n-q-c_q$ is even and $v_s^q[M]$ is nonpositive on $I\times I$, then $M\in(\lambda_1,\lambda_*^q]$, and if $n-q-c_q$ is odd and $v_s^q[M]$ is nonnegative on $I\times I$, then $M\in(\lambda_1,\lambda_*^q]$ where*
    - *$\lambda_1>0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
    - *$\lambda_*^q>0$ is the minimum between:*
      - *$\lambda_3^q>0$, the biggest negative eigenvalue of $T_n[0]$ in X_3.*
      - *$\lambda_5^q>0$, the biggest negative eigenvalue of $T_n[0]$ in X_5.*

Proof. Since $\mu_z^q\neq z-1$ and $\rho_h^q\neq h-1$ we have from Propositions 8.2.5 and 8.2.6 that there are eigenvalues λ_3^q and λ_5^q with corresponding related eigenfunctions of constant sign. We suppose that $n-k$ and $n-q-c_q$ are even. For the other cases the proof is done in an analogous way.

Let us assume that there exists $M^*\notin[\lambda_*^q,\lambda_1)$, such that $v_s^q[M^*]$ is nonpositive on $I\times I$. From Theorem 8.2.11, we can affirm that $M^*<\lambda_1$. So, $M^*<\lambda_*^q$ and for all $M\in[M^*,\lambda_1)$ the function $v_s^q[M]$ is nonpositive on $I\times I$ because the set where its takes nonpositive values is an interval. By the monotone decreasing character of $v_s^q[M]$ we infer that

$$v_s^q[\lambda_*^q](t)\leq v_s^q[M](t)\leq v_s^q[M^*](t)\leq 0.$$

So, in particular

$$w_{\lambda_*^q,q}(t)\leq w_{M,q}(t)\leq w_{M^*,q}(t)\leq 0\,,$$

and

$$\begin{cases} y_{\lambda_*^q,q}(t)\leq y_{M,q}(t)\leq y_{M^*,q}(t)\leq 0\,, & \text{if } \gamma \text{ is even,}\\[2ex] 0\leq y_{M^*,q}(t)\leq y_{M,q}(t)\leq y_{\lambda_*^q,q}(t)\,, & \text{if } \gamma \text{ is odd.}\end{cases}$$

If $\lambda_*^q=\lambda_3^q$, then $w_{\lambda_*^q,q}^{(\alpha^q)}(a)=0$. So, we conclude that, for all $M\in[M^*,\lambda_*^q)$, $w_{M,q}^{(\alpha^q)}(a)=0$, which contradicts the discrete character of the spectrum of $T_n[0]$ in X_3.

If $\lambda_*^q=\lambda_5^q$, then $y_{\lambda_*^q,q}^{(\beta^q)}(b)=0$. So, we conclude that, for all $M\in[M^*,\lambda_*^q)$, $y_{M,q}^{(\beta^q)}(b)=0$, which contradicts the discrete character of the spectrum of $T_n[0]$ in X_5.

So, we arrive to a contradiction, thus the result is proved. □

Next, we present an example to illustrate the previous result.

Example 8.2.18. *Consider the operator $T_5[M]$ defined in the space of the boundary conditions $X_{\{0,2,3\}}^{\{1,3\}}$.*

For $q=1$ we have that $c_q=2$, $n-q-c_q=d_q=2$ is even, $z=h=2$, $\alpha^1=0$ and $\beta^1=1$. Moreover, the function $\frac{\partial}{\partial t}g_M$ satisfies the conditions of space $X_{\{1,2,4\}}^{\{0,2\}}$ that fulfills the assumptions of Theorem 8.2.17. Therefore, if $\frac{\partial}{\partial t}g_M$ is nonpositive on $[0,1]\times[0,1]$, then $M\in[\lambda_*^1,\lambda^1)$, where

- *$\lambda_1<0$ is the biggest negative eigenvalue of $T_5[0]$ in $X_{\{0,2,3\}}^{\{1,3\}}$.*

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* $\lambda_*^1 < 0$ *is the maximum between:*

  - $\lambda_3^1 < 0$*, the biggest negative eigenvalue of* $T_5[0]$ *in* $X_{\{0,1,4\}}^{\{0,2\}}$.
  - $\lambda_5^1 < 0$ *is the biggest negative eigenvalue of* $T_5[0]$ *in* $X_{\{1,2,4\}}^{\{0,1\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -2.23^5$, $\lambda_3^1 \approx -3.67^5$ *and* $\lambda_5^1 \approx -2.88^5$. *Thus, if* $\frac{\partial}{\partial t} g_M$ *is nonpositive on* $[0,1]\times[0,1]$, *then* $M \in [-2.88^5, -2.23^5)$.

8.2.2 Case (b): $c_q = n - q$ and $d_q = 0$

In this subsection we deal with the constant sign of $v_s^q[M]$ for the case $c_q = n-q$ and $d_q = 0$. We arrive to the next theorem.

Theorem 8.2.19. *Let* $q \in \{1,\dots,n-1\}$. *Suppose that* $\{\sigma_1,\dots,\sigma_k\} - \{\varepsilon_1,\dots,\varepsilon_{n-k}\}$ *satisfies condition* (N_a), $c_q = n-q$ *and* $d_q = 0$. *The following properties are fulfilled:*

- *If* $n-k$ *is even, then* $v_s^q[M]$ *is nonnegative on* $I\times I$ *if, and only if,* $M \in (\lambda_1, 0]$, *where* $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.
- *If* $n-k$ *is odd, then* $v_s^q[M]$ *is nonnegative on* $I\times I$ *if, and only if,* $M \in [0, \lambda_1)$, *where* $\lambda_1 > 0$ *is the least positive eigenvalue of* $T_n[0]$ *in* $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.

Proof. Let us assume that $n-k$ is even. For the case $n-k$ odd the proof is done in an analogous way. We know from the Theorem 8.1.29 that v_s^q satisfies an initial problem and it is nonnegative on $[a,b]$. Moreover, $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) > 0$ if $s < t$ and $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) = 0$ if $s > t$.

Taking into account Lemma 8.1.31, we obtain that $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) > 0$ on $(a,b)\times(a,b)$ if and only if $M \in (\lambda_1, M_q]$, with $M_q > \lambda_1$. Moreover, $\frac{\partial^q}{\partial t^q} g_M$ is monotone decreasing with respect to $M \in (\lambda_1, M_q]$.

Let's see that $M_q = 0$. Using the monotone decreasing character of $v_s^q[M]$ with respect to $M \in (\lambda_1, M_q]$ and since $v_s^q(t) = \frac{\partial^q}{\partial t^q} g_0(t,s) = 0$ if $s > t$ we have that $v_s^q[M_q](t) < v_s^q(t) = 0$ if $M_q > 0$. On the other hand, if $s < t$ then there exists $(t^*, s^*) \in (a,b)\times(a,b)$ with $s^* < t^*$ such that $0 < v_{s^*}^q[M_q](t^*) < v_{s^*}^q(t^*)$ if $M_q > 0$. Thus, $v_s^q[M]$ changes sign for $M_q > 0$ and this completes the proof. □

Example 8.2.20. *Consider the operator* $T_4[M]$ *coupled with the boundary conditions in the space* $X_{\{2,3\}}^{\{0,1\}}$.

In this case $n-k$ *is even. For* $q = 2$ *we have that* $c_q = n - q = 2$ *and* $d_q = 0$. *Then, by Theorem 8.2.19 we have that* $\frac{\partial^2}{\partial t^2} g_M$ *is nonnegative on* $[0,1]\times[0,1]$ *if, and only if,* $M \in (\lambda_1, 0]$, *where* $\lambda_1 < 0$ *is the biggest negative eigenvalue of* $T_4[0]$ *in* $X_{\{2,3\}}^{\{0,1\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -1.8751^4$. *Thus,* $\frac{\partial^2}{\partial t^2} g_M$ *is nonnegative on* $[0,1]\times[0,1]$ *if, and only if,* $M \in (-1.8751^4, 0]$.

For $q = 3$ *we have that* $c_q = n - q = 1$ *and* $d_q = 0$. *Again, by Theorem 8.2.19 we have that* $\frac{\partial^3}{\partial t^3} g_M$ *is nonnegative on* $[0,1]\times[0,1]$ *if, and only if,* $M \in (-1.8751^4, 0]$.

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8.2.3 Case (c): $c_q = 0$ and $d_q = n - q$

In this section, arguing in a similar manner than in previous one, we can characterize the constant sign of $v_s^q[M]$ for the case $c_q = 0$ and $d_q = n - q$.

Using Theorem 8.1.29, Lemma 8.1.31, and reasoning in an analogous way to the proof of Theorem 8.2.19, we get to the next result.

Theorem 8.2.21. *Let $q \in \{1, \dots, n-1\}$. Suppose that $\{\sigma_1, \dots, \sigma_k\} - \{\varepsilon_1, \dots, \varepsilon_{n-k}\}$ satisfies condition (N_a), $c_q = 0$ and $d_q = n - q$. Then, the following properties are fulfilled:*

- *Let $n - k$ be even:*
  - *If $n - q$ is even, then $v_s^q[M]$ is nonnegative on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *If $n - q$ is odd, then $v_s^q[M]$ is nonpositive on $I \times I$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
- *Let $n - k$ be odd:*
  - *If $n - q$ is even, then $v_s^q[M]$ is nonnegative on $I \times I$ if, and only if, $M \in [0, \lambda_1)$, where $\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*
  - *If $n - q$ is odd, then $v_s^q[M]$ is nonpositive on $I \times I$ if, and only if, $M \in [0, \lambda_1)$, where $\lambda_1 > 0$ is the least positive eigenvalue of $T_n[0]$ in $X_{\{\sigma_1,\dots,\sigma_k\}}^{\{\varepsilon_1,\dots,\varepsilon_{n-k}\}}$.*

Example 8.2.22. *Consider the operator $T_3[M]$ coupled with the boundary conditions*

$$u'(0) = u(1) = u''(1) = 0,$$

that is, $u \in X_{\{1\}}^{\{0,2\}}$.

For $q = 2$ we have that $c_q = 0$ and $d_q = n - q = 1$ is odd. Hence, since $n - k$ is even, from Theorem 8.2.21 we conclude that $\frac{\partial^2}{\partial t^2} g_M$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_3[0]$ in $X_{\{1\}}^{\{0,2\}}$.

By numerical approach, it can be seen that $\lambda_1 \approx -1.85^3$. Thus, $\frac{\partial^2}{\partial t^2} g_M$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.85^3, 0]$.

Example 8.2.23. *Consider the operator $T_6[M]$ coupled with the boundary conditions in the space $X_{\{0,2\}}^{\{1,3,4,5\}}$.*

In this case $n - k = 4$ is even. For $q = 3$ we have that $c_q = 0$ and $d_q = n - q = 3$ is odd. Then, by Theorem 8.2.21 we have that $\frac{\partial^3}{\partial t^3} g_M$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (\lambda_1, 0]$, where $\lambda_1 < 0$ is the biggest negative eigenvalue of $T_6[0]$ in $X_{\{0,2\}}^{\{1,3,4,5\}}$.

By numerical approximation, we obtain that $\lambda_1 \approx -1.953^6$. Thus, $\frac{\partial^3}{\partial t^3} g_M$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.953^6, 0]$.

For $q = 4$ we have that $c_q = 0$ and $d_q = n - q = 2$ is even. Again, from Theorem 8.2.21 we have that $\frac{\partial^4}{\partial t^4} g_M$ is nonnegative on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.953^6, 0]$.

Finally, for $q = 5$ we have that $c_q = 0$ and $d_q = n - q = 1$ is odd. Then, $\frac{\partial^5}{\partial t^5} g_M$ is nonpositive on $[0,1] \times [0,1]$ if, and only if, $M \in (-1.953^6, 0]$.

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General Conclusions and Future Work

This thesis has come to an end and has been framed in the field of the qualitative study of differential equations.

First of all, we have started by giving the preliminary results that are needed to develop the rest of the syllabus. Most of the results are given in reference [6].

Chapters 2 to 4, the contents of which have been obtained in the articles [17–19], deal with the study of differential equations with arbitrary nonlocal boundary conditions and an application to Dirichlet and periodic problems of second order subjects to integral conditions.

Chapters 5 and 6 are a consequence of the work published in the articles [20, 22] in which have been studied the properties and relationships between the Green's functions related to the Hill's equation, subject to different two-point boundary conditions, and the characterization of its spectrum with applications to nonlinear problems.

Chapter 7, which was published in article [21], generalizes the ideas developed in Chapter 2 to nonlinear systems subject to arbitrary nonlocal boundary conditions, showing the interest and advantages it has over other results proved in the literature.

Finally, the results compiled in Chapter 8 are collected in Arxiv, reference [23], whose final objective has been the characterization of the constant sign of certain derivatives of the Green's function related to a differential equation of order n coupled to two-point boundary conditions that generalize the work [26] for the particular case of the considered ordinary differential equation.

During the process of carrying out the thesis, several open questions have arisen that would be interesting to investigate in the future, which we list below:

- Study of ordinary differential equations with functional dependence on the equation itself. Specifically, we will study differential equations with constant arguments and with involution by combining the techniques used in [12, 77] for differential equations with constant arguments and [28] for differential equations with involutions.

- It would be interesting to investigate nonlinear problems defined in unbounded intervals, subject to additional functional conditions. When the intervals are not bounded, the asymptotic behavior of the solutions to a given function will be addressed, especially in the case where the spectrum of the linear operator has several imaginary complex roots, which allows us to consider the situations described in [38] where bounded solutions cannot exist.

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- For nonlinear problems, we will try to apply other techniques to guarantee the existence, non-existence and multiplicity of solutions defined in both bounded and unbounded intervals.
- It would be interesting to carry out a comparison study of the Green's function relative to a general differential equation of order n defined in an interval $[a, b]$ with the Green's function of the same equation but defined in a subinterval $W \subset [a, b]$, under additional boundary conditions.
- It would be interesting to study the characterization of the sign of the derivatives of Green's function related to the general operator of order n with constant coefficients:

$$L_n[M] \equiv u^{(n)}(t) + a_1\, u^{(n-1)}(t) + \cdots + (a_n + M)\, u(t), \quad t \in I := [a, b],$$

where $M \in \mathbb{R}$ and $a_k \in \mathbb{R}$ for all $k = 1, \ldots, n$, with two-point boundary conditions. In this case, since the technique developed in Chapter 8 would not be valid for operators that are not of the form $u^{(n)} + M\, u$, we must look for a different one. For example, if we consider the following third order problem:

$$\begin{cases} u'''(t) + u'(t) + M\, u(t) = 0, & t \in [0, 1], \\ u(0) = u'(1) = u''(1) = 0. \end{cases}$$

In this case, $u \in X_{\{0\}}^{\{1,2\}}$ but we cannot compute the condition of the derivative function at $t = 0$. In addition, the condition (T_d) of disconjugacy for the operators $L_n[M]$ is not generally verified, which is a necessary hypothesis that we have used to carry out the study.

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Publications and Further Information

Articles and journals

In compliance with the rules of doctoral studies at Universidade de Santiago de Compostela in *Regulamento dos estudos de doutoramento na USC, DOG de 16 de setembro de 2020*, we provide some information regarding the articles on which this work is based and the journals that published those articles. In particular we give the names of the authors, the title of the journals, the year each article was published, the ISSN (or EISSN), the publisher, the DOI-type link, the Journal Impact Factor and the quartile from the Journal Citation Reports, the CiteScore rating and the quartile from Scopus, and some relevant information regarding copyright and use of the articles. The links have been checked on June 05, 2024.

Chapter 2. Article [17].

TITLE: Green's Functions Related to a n-th Order Linear Differential Equation Coupled to Arbitrary Linear Non-Local Boundary Conditions.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

JOURNAL: Mathematics.

YEAR: 2021.

EISSN: 2227-7390.

PUBLISHER: Multidisciplinary Digital Publishing Institute (MDPI)

LINK: https://doi.org/10.3390/math9161948

JOURNAL IMPACT FACTOR: The data from 2021: 2.529 [Q1 in MATHEMATICS-SCIE (21/333)].

CITESCORE: The data from 2021: 2.9 [Q1 in Mathematics-General Mathematics (53/391), Q2 in Engineering-Engineering (miscellaneous) (34/90) and in Computer Science-Computer Science (miscellaneous) (45/115)]

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Chapter 3. Article [18].

TITLE: Constant-Sign Green's Function of a Second-Order Perturbed Periodic Problem.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

JOURNAL: Axioms.

YEAR: 2022.

EISSN: 2075-1680.

PUBLISHER: Multidisciplinary Digital Publishing Institute (MDPI).

LINK: https://doi.org/10.3390/axioms11030139.

JOURNAL IMPACT FACTOR: The data from 2022: 2 [Q1 in MATHEMATICS, APPLIED-SCIE (68/267)].

CITESCORE: The data from 2022: 2.2 [Q1 in Mathematics-Algebra and Number Theory (23/117)and in Mathematics-Geometry and Topology (21/103), Q2 in Mathematics-Logic (11/41), in Mathematics Analysis (71/187) and in Mathematics-Mathematical Physics (37/81)].

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Chapter 4. Article [19].

TITLE: Existence of Positive Solutions of Nonlinear Second Order Dirichlet Problems Perturbed by Integral boundary conditions.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

JOURNAL: Recent Advances in Mathematical Analysis, Trends in Mathemtics.

YEAR: 2022.

LINK: https://doi.org/10.1007/978-3-031-20021-2.

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Chapter 5. Article [20].

TITLE: Relationship of the Green's functions related to the Hill's equation coupled to different boundary conditions.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

JOURNAL: Memoirs on Differential Equations and Mathematical Physics.

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YEAR: To appear.

ISSN: 1512-0015.

EISSN: 1512-0015.

PUBLISHER: Georgian National Academy of Sciences and Andrea Razmadze Mathematical Institute.

JOURNAL CITATION INDICATOR (JCI): The data from 2023 is still not available. The data from 2022: 0.21 [Q4 in MATHEMATICS-APPLIED-ESCI].

CITESCORE: The data from 2023 is still not available. The data from 2022: 0.5 [Q4 in Mathematics-Analysis (164/187), in Mathematics-Mathematical Physics (64/290), in Multidisciplinary (13/110) and in Engineering-Miscellaneous (72/81) and in Mathematics-Applied Mathematics (558/609)]

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Chapter 6. Article [22].

TITLE: Characterization of the spectra of the Hill's equation coupled to different boundary value conditions and application to nonlinear boundary problems.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

JOURNAL: Filomat.

YEAR: 2023.

ISSN: 0354-5180.

EISSN: 0354-5180.

PUBLISHER: Faculty of Sciences and Mathematics, University of Nis.

LINK: https://www.pmf.ni.ac.rs/filomat-content/2024/38-1/38-1-15-21198.pdf

JOURNAL IMPACT FACTOR: The data from 2023 is still not available. The data from 2022: 0.8 [Q3 in MATHEMATICS-SCIE (170/330) and Q4 in MATHEMATICS-APPLIED-SCIE (210/267)].

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Chapter 7. Article [21].

TITLE: Existence of solutions of nonlinear systems subject to arbitrary linear non-local boundary conditions.

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AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi

JOURNAL: Journal of Fixed Point Theory and Applications.

YEAR: 2023.

ISSN: 1661-7738.

EISSN: 1661-7738.

PUBLISHER: Springer Nature.

LINK: https://doi.org/10.1007/s11784-023-01083-7.

JOURNAL IMPACT FACTOR: The data from 2023 is still not available. The data from 2022: 1.8 [Q1 in MATHEMATICS-SCIE (43/330) and Q2 in MATHEMATICS-APPLIED-SCIE (87/267)].

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Chapter 8. Article [23].

TITLE: Spectral characterization of the constant sign derivatives of Green's function related to two point boundary value conditions.

AUTHORS: A. Cabada, L. López-Somoza and M. Yousfi.

DATE OF PUBLICATION: 3 June 2024.

PUBLISHER: arXiv repository.

LINK: http://arxiv.org/abs/2406.01509.

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Author contributions

The contributions of the Ph.D. candidate were essential in all the included articles. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscripts.

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Funding Information

The results made on this PhD thesis, have been partially supported by projects PID2020-113275GB-I00, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe" of the "European Union", and by Xunta de Galicia (Spain), project ED431C 2023/12.

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March 20, 2024

C e r t i f i c a t e

The Editorial Board of the journal "*Memoirs on Differential Equations and Mathematical Physics*" certifies that the paper "*Relationship of the Green's functions related to the Hill's equation coupled to different boundary value conditions*" written by Alberto Cabada, Lucía López-Somoza and Mouhcine Yousfi is accepted for publication and authorizes **Mr. Mouhcine Yousfi** to include the results of the paper, as well as the graphs that appear therein, in his doctoral thesis.

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Differential equations represent an important tool for solving many real problems. In this thesis we focus on the qualitative properties of the solutions of functional equations with nonlocal boundary conditions, focusing on the study of constant sign solutions.

We will study comparison results between the Green's functions related to the Hill's equation subject to different boundary conditions. The relationship between the respective spectra of the different problems considered will be also studied.

Finally, we will addressed the study of nonlinear systems subject to nonlocal linear boundary conditions and the spectral charaterization of the constant sign derivatives of the Green's function.