INTERNATIONAL DOCTORAL
SCHOOL OF THE USC

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PhD Thesis

De Rham Cohomology in Diffeological Spaces

Santiago de Compostela, 2024

Doctoral Programme in Mathematics

USC
UNIVERSIDADE
DE SANTIAGO
DE COMPOSTELA

DOCTORAL THESIS

De Rham Cohomology In Diffeological Spaces

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INTERNATIONAL PHD SCHOOL OF THE UNIVERSITY OF SANTIAGO DE COMPOSTELA

PHD PROGRAMME IN MATHEMATICS

SANTIAGO DE COMPOSTELA

YEAR 2024

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DE COMPOSTELA

Acknowledgement

This thesis would not have been possible without the guidance and the help of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of this study, it is a pleasure to thank those who made it a possibility.

I am indebted to my supervisors, Professors Enrique Macías Virgós and Antonio Gómez Tato, whose guidance, support and expertise have enabled me to complete my PhD within the time frame available to me as an international student.

And my biggest thanks to my lovely family for all the support you have shown me through this research, the culmination of four years of distance learning.

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Abstract

Diffeological spaces are a generalization of smooth manifolds. They were introduced around 1970 by the French mathematician and physicist Jean-Marie Souriau (1922-2012), in an attempt to formalize quantum mechanics and geometric quantization. This approach proved to be useful in several settings which involve objects that rarely are finite dimensional manifolds, like the space of smooth maps between two manifolds or the space of leaves of a foliation.

In this thesis we are mainly interested in the Cartan calculus on diffeological spaces. We clarify the properties of the De Rham cohomology groups, which are typical of the algebraic topology of manifolds but were less known in this setting. We introduce several new constructions like the Mayer-Vietoris sequence, the relative De Rham cohomology groups and the relative cup-product.

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Hypothesis

Diffeological spaces are a generalization of smooth manifolds. Essentially, a diffeology on a set X indicates for each integer number $m \geq 0$ and each open subset $U \subset \mathbb{R}^m$ which are the maps from U to X that are considered differentiable. These maps are called *plots* and are subject to three axioms: covering, locality, and compatibility. A *smooth map* between diffeological spaces is any map which sends plots to plots.

This approach proves to be useful in several settings which involve objects that rarely are (finite dimensional) manifolds. For instance, the space of smooth maps between two manifolds or the subductions of manifolds, like the space of leaves of a foliation [35].

In fact, *Fréchet manifolds* form a full subcategory of the category **Diff** of diffeological spaces [30]. This includes (infinite dimensional) Banach manifolds [15]. *Manifolds with borders* also can be viewed as diffeological spaces [14]. Finally, *orbifolds* (that is, spaces locally modeled on subductions of open subsets of $\mathbb{R}^n$ by finite group actions) are diffeological spaces [25].

Another advantage is that the definition of a diffeology on a set X is essentially categorical, as it is given as a presheaf on the category whose objects are the open subsets of Euclidean spaces and whose maps are the smooth maps. This framework facilitates natural contravariant constructions which are usually less complex than those within the scope of manifolds. In fact, the category **Diff** is closed under limits and colimits [2]. For example, every subset of a diffeological space inherits a diffeological structure with the good properties of a regularly immersed manifold.

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One negative aspect of the category of diffeological spaces is that it is so large that does not allow for strong geometric results that hold for all its objects [3]. We will comment this issue when speaking about the Mayer-Vietoris sequence. Furthermore, the covariant constructions like the tangent bundle or vector fields are more involved to define. A discussion on the best setting for developing Cartan calculus based on vector fields, Lie brackets and a tangent bundle has been carried out by Blohmann in [3]. This author introduced the so-called *elastic* diffeologies, but we will not follow this approach because limits and colimits do not work as well as in the general context.

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Objectives

Diffeological spaces were introduced around 1979 by the French mathematician and physicist Jean-Marie Souriau (1922-2012), in an attempt to formalize quantum mechanics and geometric quantization [38]. They were designed for dealing easily with infinite-dimensional groups of diffeomorphisms in symplectic geometry [21]. They are built on the model of Kuo-Tsai Chen's *differentiable spaces* [5], for which the structure is defined over convex Euclidean subsets instead of open Euclidean domains. But the latter setting makes diffeology more suitable to extending differential geometry. The formal definition of a diffeological space, without refering to group theory, is due to Souriau's disciples Paul Donato and Patrick Iglesias-Zemmour [8]. The main reference for this theory is now Iglesias'book [20]. A quick introduction to the main ideas of diffeology is given in [21].

Let us comment the relationship between diffeology and topology. Contrary to many constructions in differential geometry, a diffeology is defined just on a set without any preexisting structure, neither a topology nor anything else. Once the diffeology is given on X, a natural topology can be defined, the so-called $\mathcal{D}$-topology: a subset $A \subset X$ is $\mathcal{D}$-open iff $P^{-1}(A)$ is a (usual) open set in the domain $U \subset \mathbb{R}^n$ of any plot $P\colon U \to X$ of the diffeology. In other words, the $\mathcal{D}$-topology is the largest topology on X that makes continuous all the plots.

Even though the $\mathcal{D}$- topology facilitates the introduction of various topological and homotopical concepts within diffeological spaces (such as homotopy groups), we have refrained from employing it. Our belief is that it disrupts the intrinsic con-

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travariant nature of our framework. Additionally, the subspace diffeology is incompatible with $\mathcal{D}$-topologies. Also, this topology is determined by the smooth curves only, so that many different diffeologies induce the same topology [2].

Today, diffeology covers many traditional fields in Geometry and Topology, like higher homotopy theory, fiber bundles, Cartan-de Rham calculus, moment map and symplectic geometry.

In this thesis we are mainly interested in the *Cartan calculus* on diffeological spaces, and more specifically in the *De Rham cohomology*. As we shall see, differential calculus is one of the most developed domains in diffeology. Chen's spaces were introduced thinking about homology and cohomology in mind, and that is why he chose the convex subsets as domains for his plots. Anyway, the very definition of a diffeology leads naturally to define contravariant objects like differential forms.

Essentially, a differential form ω in the diffeological space X is defined by giving its pullbacks $\omega_P = P^*\omega$ for all the plots $P\colon U \to X$, submitted to a compatibility condition. That is, we give a compatible collection of usual differential forms on Euclidean domains. An important fact is that 0-forms on X can be identified with smooth functions $X \to \mathbb{R}$. The two main operations on the differential forms on a diffeological space are the pullback $(f^*\omega)_P = \omega_{f\circ P}$ for a smooth map $f\colon Y \to X$, and the exterior derivative $(\mathrm{d}\omega)_P = \mathrm{d}\omega_P$. Then, we have a De Rham complex $(\Omega(X), \mathrm{d})$, with $\mathrm{d}\circ\mathrm{d} = 0$, whose cohomology is denoted by $\mathrm{H}(X)$.

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Methodology

In this thesis we follow the usual way to do Mathematics. First, we guess what it is interesting to approach, in our case the study of De Rham's cohomology in the framework of diffeological spaces, in parallel to what is done for the differentiable manifolds of finite dimension or for the space of leaves of a foliation. Then we studied the relevant antecedents that exist in the literature. Finally we developed results that were published in two prestigious journals, as they are *Indagationes Mathematicae* and *MDPI Mathematics*. Finally, we prepared this memoir that contains the cited results plus an extensive work explaining the theory and the preliminary results, as well as some open questions.

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Introduction

De Rham cohomology in diffeological spaces has several interesting applications. For instance, Iglesias extended the notion of symplectic formalism and moment maps to diffeology in [23]. It was Souriau who introduced the notion of a moment map for the first time in the classical setting. Also, it has been proved that the cohomology of the space of leaves of a foliation equals the so-called *basic cohomology*, a very well known tool in foliation theory [18].

We now review several recent works in this field. First we have two papers where the notion of differential form in a diffeological space is applied to manifolds with boundary and corners [11] and to stratified spaces [12, 13].

An important issue in Cartan Calculus is the Mayer-Vietoris sequence. However, in the setting of diffeological spaces we encounter a difficulty to obtain partitions of unity and thus the Mayer-Vietoris is not an exact sequence in general. This problem was studied in [26, 27] when trying to obtain a De Rham theorem (that is, an isomorphism between the De Rham cohomology and the singular cohomology of the underlying $\mathcal{D}$-topological space). In those papers a notion of partition of unity and some kind of nice open covering for the $\mathcal{D}$-topology are introduced. These notions were also given in [16]. Even with these tools, the authors need a different notion of differential form, for obtaining the so-called *cubical* De Rham cohomology.

In [28], Kuribayashi compares the De Rham complex with another singular de Rham complex defined by a simplicial differential graded algebra. This author considers a singular De Rham complex that can be regarded as a diffeological variant

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of Sullivan's polynomial simplicial forms for a topological space [29]. Finally, the cochain complex of smooth singular simplicial set of $\mathcal{C}^\infty$ maps from the standard n-simplex into X defined by G. Hector in [19] is also considered.

Haraguchi's paper [16] shows that Mayer-Vietoris long exact sequences exist for the De Rham cohomology with compact support of diffeological subcartesian spaces (that is, a diffeological space locally diffeomorphic to some subspace of an Euclidean space), if one can find a subordinated partition of unity.

Iglesias has proposed a Čech-De Rham bicomplex in cohomology [22] which is connected with the De Rham cohomology, through a double-complex.

Recently, Minichiello [34] has obtained new exact sequences in all higher degrees for the Čech-De Rahm cohomology by using homotopy pull-backs. Diffeological Čech cohomology was introduced by Alireza Ahmadi in [1].

We will approach these problems in a very different way [32]. First we will replace the notion of open covering with that of *generating family*, which is much more intrinsic than $\mathcal{D}$-topology. Also, we will use systematically categorical notions as *pullback*, *push-out* and *join* to obtain the Mayer-Vietoris sequence. Hence our results do not depend on the $\mathcal{D}$-topology.

Our main contribution will be to define two different versions of a *relative* version of De Rham cohomology. The first one is inspired in Godbillon's definition of the De Rham cohomology of a pair formed by a manifold M and a submanifold $i\colon N \to M$ [10]; it is the cohomology of the complex $\ker i^*$ of differential forms on M which are null on N. The main problem with this cohomology is that there is not a long exact sequence of the pair (M, N) due to the lack of partitions of unity. Hence differential forms on N cannot be extend to the whole ambient manifold M.

The second one is a generalization of the cohomology of a pair defined by Bott-Tu [4] . In order to do that we have introduced the notion of f-horizontal forms for a smooth map f. In this way we can develop many of the usual properties of the relative cohomology groups.

For both cohomologies we define a *relative cup-product* and we prove its main properties, specially the skew-commutativity.

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Contents

We will explain in some details the contents of each chapter.

In the First Chapter we introduce some preliminaries about diffeological spaces, following Iglesias' book [20], which is the main reference in the field. We study the definition of diffeologies, smooth maps and then we show that a manifold is a diffeological space. Comparing diffeologies is necessary in the understanding of some definitions. The difeologies can be built starting with a generating family; this notion is one of the most used in this thesis. Any subset of a diffeological space inherits a diffeology. Pushforward and pullbak help us to describe the transposing of a diffeology along a map. Subductions are an important particular example of smooth maps in diffeology: they are surjective smooth maps characterized by the local factorization of the plots of the codomain through the diffeology of the domain.

In the Second Chapter we introduce some categorical concepts. They are of a more general nature that allow us to construct new diffeological spaces from old ones. Some of these constructions are familiar from topology and sets. It is worth note that the category of diffeological spaces and smooth maps allows any kind of limits and colimits. We will sudy in detail the pullback of two maps and then we will study pushouts. Also in this chapter we will show how to construct the join of two maps, and finally we will give the non-finite pullback.

The Third Chapter is dedicated to the study of Cartan calculus. Differential forms and De Rham cohomology can be generalized to the context of diffeological spaces. We study all concepts related to them in this chapter, like for instance the exterior product and the exterior derivative.

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In Chapter Four we introduce the notion of horizontal forms. This is another fundamental idea in this work and we will need all the results of this chapter in the following ones.

Chapter Five is devoted to the relative cohomology groups. Many classical constructions in Algebraic Topology and Cartan calculus can be generalized to diffeological spaces. Among them, the graded algebra $\Omega^\bullet(X)$ of differential forms, the exterior derivative d and the De Rham cohomology algebra $\mathrm{H}^\bullet(X) = \mathrm{H}(\Omega^\bullet(X), \mathrm{d})$. We will study two different types of relative cohomology groups, denoted with the letters H (Godbillon type) and $\mathcal{H}$ (Bott-Tu type) and then we will show the relationship between them.

In Chapter Six we will study the cohomological properties of generating families. When $\alpha_1, \ldots, \alpha_n$ is a generating family of the diffeology we will prove that

$$\mathrm{H}(X, \alpha_1 * \cdots * \alpha_n) = 0.$$

and

$$\mathcal{H}(X, \alpha_1 * \cdots * \alpha_n) = 0.$$

We also show that how the cohomology of X is determined by any generating family.

In Theorem 6.17, we will prove that if $\{(\alpha_1, U_1), \ldots, (\alpha_n, U_n)\}$ is a generating family of the diffeology $\mathcal{D}$ on X, then

$$\mathrm{H}(X, \mathcal{D}) \cong \mathrm{H}(U_1 * \cdots * U_n).$$

This result can be extended to the infinite case.

Chapter Seven is the biggest one in this thesis and it is devoted to study the Mayer-Vietoris Sequence.

Instead of a covering $\{U, V\}$ of X by two open sets of the so-called $\mathcal{D}$-topology, which would give the classical sequence for manifolds

$$0 \to \Omega(X) \to \Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \to 0,$$

our version of the Mayer-Vietoris result is based on a generating family $\{\alpha, \beta\}$ formed by two plots α, β. We get an exact sequence

$$0 \to \Omega(X) \to \Omega(\alpha) \oplus \Omega(\beta) \to \Omega(P),$$

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where $\Omega(\alpha)$ (resp. $\Omega(\beta)$) is the subcomplex of $\Omega(U)$ of α- (resp. β-horizontal) forms and P is the pullback of two plots (see 1.6). This idea is natural, because $U \cap V$ can be viewed as the pullback of the inclusions $U, V \subseteq X$.

Our objective is to understand how the De Rham complex of X is determined by α and β. This is achieved in Theorem 7.3. Our procedure is close to the diffeological gluing procedure in [36], which can be considered as a form of Mayer-Vietoris construction.

Also, we construct an improved Mayer-Vietoris sequence and get an exact sequence

$$0 \to \Omega(\alpha * \beta) \to \Omega(\alpha) \oplus \Omega(\beta) \to \Omega(p_X) \tag{0.1}$$

where $p_X = \alpha \circ p_U = \beta \circ p_V$.

In the last section we introduce split Mayer-Vietoris sequences, those which admit a global linear section.

In the last Chapter we will study cup product in relative cohomology.

In Section 8.2 we introduce a relative cup product by considering two arbitrary plots $\alpha\colon U \to X$ and $\beta\colon V \to X$ of the diffeology. In order to do that, we define a smooth map $\alpha * \beta\colon U * V \to X$, the join of α and β, which plays the role of the union of two open sets; actually this map is the categorical push-out of the pull-back of α and β. In such a way we have a well defined bilinear map

$$\mathrm{H}^r(X, \alpha) \otimes \mathrm{H}^s(X, \beta) \to \mathrm{H}^{r+s}(X, \alpha * \beta)$$

for the first type relative cohomology groups. In Section 8.3 we will define a cup product in relative cohomology of the second type when the Mayer-Vietoris is split and we show that it is skew-symmetric.

Unfortunately not every pair of plots is split due to the lack of partitions of unity.

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Conclusions

We have developed the theory of De Rham cohomology in diffeological spaces, in order to clarify the relationship between several invariants and constructions which are typical of the algebraic topology of manifolds but were less studied in this setting.

MV The existence of a Mayer-Vietoris sequence, which fails to be exact due to the inexistence of partitions of unity.

LESP The existence of a long exact sequence for a pair (X, U) of a diffeological space and a subspace, or more generally, of a smooth map $U \to X$.

CP The definition of a relative cup-product, which could be useful in several applications.

Our main tools are two different versions of a *relative De Rham cohomology*.

- One follows Godbillon's idea of considering the kernel of the induced *restriction map* $\Omega(X) \to \Omega(U)$. This version verifies CP but not LESP.
- The second one adapts a Bott-Tu construction by defining

$$\Omega^r(X, f) = \Omega^r(X) \oplus \Omega^{r-1}(U).$$

It verifies LESP, but not CP, excepting when an additional condition is assumed, namely that of being an "split" pair. Anyway, the proof is much more dificult.

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Another powerful tool is the notion of *generating families*. In fact, our general philosophy is to adapt the usual constructions for *open coverings* to the generating families. The main example is the Mayer-Vietoris sequence.

For instance, we have proved several results about how the definition of a differential form can be simplified with this notion.

Also, we have given a crucial role to the notion of being a *horizontal form*, a property that all the pullbacks of a form satisfy.

The behaviour of differential forms face to *subductions* (quotient maps) is an important step to generalize results of fiber bundles and foliation theory.

Finally, we have focused our attention to categorical constructions like the pullback, the push-out and more importantly the *join* of a family of maps. In fact we have proven that a diffeological space is the join of any generating family.

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Chapter 1

Diffeological Spaces

1.1 Introduction

Diffeological spaces were initiated by Jean-Marie Souriau in the 1980s under the name *espace différentiel* and later expanded by his students Paul Donato and Patrick Iglesias.
A related idea was introduced by Kuo-Tsai Chen in the 1970s, using convex sets instead of open sets for the domains of the plots.
Diffeological spaces are a generalization of differentiable manifolds, which provides a unified framework for non-classical objects in differential geometry like subductions of manifolds or spaces of smooth functions. Also, diffeology theory seems to be a convenient setting for many problems that appear in various areas of physics.

1.2 Diffeologies and diffeological spaces

The set of $\mathcal{C}^k$ mappings from U to V is denoted by $\mathcal{C}^k(U,V)$. For $k=\infty$, we say that f is *infinitely differentiable* or *smooth*. Let V be a domain, we call *smooth parametrization in* V every smooth (infinitely differentiable) map $f\colon U\to V$.
An open subset $U\subset\mathbb{R}^m$ of some Euclidean space, $m\in\mathbb{N}$, is called an m-domain.

A parametrization on the set X is any set map $\alpha\colon U\to X$, where $U\subset\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, $n\geq 0$. Often we shall denote it by (U,α).
Let X be a non-empty set. A diffeology on X is any set $\mathcal{D}$ of parametrizations such that the three following axioms are satisfied:

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D1. Covering: any constant parametrization on X belongs to $\mathcal{D}$;

D2. Locality: if $\alpha\colon U \subseteq \mathbb{R}^n \to X$ is a parametrization that locally belongs to $\mathcal{D}$ (i.e. for every $x \in U$ there exists a neighbourhood U_x such that $\alpha_{|U_x} \in \mathcal{D}$), then $\alpha \in D$;

D3. Smooth compatibility: if $\alpha \in \mathcal{D}$ and $h\colon V \to U$ is a change of coordinates, then $\alpha \circ h \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ will be called a diffeological space. Axiom 1 ensures that $\mathcal{D} \neq \emptyset$; from Axiom 2, parametrizations can be glued together; Axiom 3 means that we can change coordinates inside $\mathcal{D}$.

Example 1.1. If $U \subset \mathbb{R}^m$ is an m-domain, $m \geq 0$, the $\mathcal{C}^\infty$-differentiable maps $V \to U$ with codomain U form its usual diffeology. More generally, if M is a finite-dimentional differentiable manifold, the C^∞ maps $V \subset \mathbb{R}^m \to M$ with any m-domain, form the usual or standard diffeology on M. In this case, all the diffeological objects that we shall define correspond to the usual ones.

1.2.1 Smooth maps

A map $(X, \mathcal{D}) \xrightarrow{f} (X', \mathcal{D}')$ between diffeological spaces is smooth if $f \circ \alpha \in \mathcal{D}'$ for any $\alpha \in \mathcal{D}$.
The set of smooth maps from X to X' is denoted by $\mathcal{D}(X, X')$.

Theorem 1.2. *The composition of two smooth maps is a smooth map.*

Proof. Let X, X', X'' be three diffeological spaces whose diffeologies are denoted by $\mathcal{D}, \mathcal{D}', \mathcal{D}''$. Let $f\colon X \to X'$ and $g\colon X' \to X''$ be two differentiable maps. that is $f \circ \mathcal{D} \subset \mathcal{D}'$ and $g \circ \mathcal{D}' \subset \mathcal{D}''$, then $(g \circ f) \circ \mathcal{D} = g \circ (f \circ \mathcal{D}) \subset g \circ \mathcal{D}' \subset \mathcal{D}''$.
Therefore, $g \circ f$ is a smooth map. □

Theorem 1.3. *The plots $U \to X$ of $\mathcal{D}$ are smooth maps for the usual diffeology on U.*

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Let $f : U \to X$ be a smooth parametrization. Thanks to Axiom (D3) , the composite $f \circ i_u$, where i_u is the identity of U, is a plot of X. Thus, $\mathcal{D}(U, X) \subset \mathcal{D}(U)$. Now, let $\alpha \colon U \to X$ be a plot. Thanks again to the the axiom (D3), for every smooth parametrization $\mathcal{F}$ in U, the paramerization $\alpha \circ f$ belongs to $\mathcal{D}$. Hence, α is smooth. Thus, $\mathcal{D}(U) \subset \mathcal{D}(U, X)$.

1.3 A manifold is a diffeological space

Let M be a finite dimensional smooth manifold. The plots of the "standard" diffeology are the maps $\alpha \colon U \to M$ which are smooth in the usual sense of differential geometry. We will briefly check if the three axioms of a diffeology are satisfied. (D1) clearly constant maps are smooth. (D2) let α be the smallest extension of the α_i for any $x \in U$, there is an $i \in I$ such that $x \in U_i$ and hence $\alpha|_{U_i} = \alpha_i$ is smooth. Now, smoothness is local condition, so α is smooth on all of U_α. (D3) composition of smooth maps are smooth.

Lemma 1.4. *Given two smooth manifolds M and N, the usual smooth maps coincide with the smooth maps as a diffeological space.*

Proof. If $\alpha \colon U \to M$ is a smooth map and $f \in \mathcal{C}^\infty(M, N)$, then the composition $f \circ \alpha$ is again smooth. Conversely, let f be a morphism between M, N. We have to show that f is smooth, so let $\varphi \colon U \to \mathbb{R}^n$ be a chart in a given atlas A of M.
By choice of f, the map $f \circ \varphi^{-1}$ is a plot for N, and therefore a smooth map. This is true for every chart in A. □

Remark 1.5. In the terminology of category theory, the above lemma is expressed by saying the category of smooth manifolds is a full subcategory of the diffeological category.

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1.4 Comparing Diffeologies

The inclusion of diffeologies, regarded as subsets of the power set of all the parametrizations of a set, is a partial ordering.

1.4.1 Smallness of diffeologies

We shall avoid the words *finer* and *coarser* when comparing diffeologies on the same set X. If $\mathcal{D} \subseteq \mathcal{D}'$ we say that $\mathcal{D}'$ is *larger* than $\mathcal{D}$, equivalently $\mathcal{D}$ is *smaller* than $\mathcal{D}'$.

Remark 1.6. Let us denote by X_1 and X_2 the same set X equipped with two diffeologies $\mathcal{D}_1$ and $\mathcal{D}_2$. The diffeology $\mathcal{D}_1$ is smaller than $\mathcal{D}_2$ if and only if the identity map $1_X \colon X_2 \to X_1$ is smooth.

1.4.2 Discrete diffeology

Let X be a set. The *locally constant parametrizations* of X are defined as follows:
A parametrization $\alpha \colon U \to X$ is said to be *locally constant* if for all $r \in U$ there exists an open neighborhood V of r such that $\alpha|_V$ is constant.
The locally constant parametrizations in X form a diffeology called the *discrete diffeology*

1.4.3 Indiscrete diffeology.

Let X be a set. The set of all the parametrizations in X is a diffeology. This diffeology, larger than every other one, is called the *indiscrete diffeology*.

Theorem 1.7. *Let X be any set. Let $\mathbf{D} = \{\mathcal{D}_i\}_{i\in\mathcal{J}}$ be any family of diffeologies of X, indexed by some set $\mathcal{J}$. The intersection of the elements of the family, denoted by*

$$\mathcal{D} = \bigcap_{i\in\mathcal{J}} \mathcal{D}_i,$$

is still a diffeology of X, smaller than each element of the family $\mathbf{D}$.

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Proof. Let us first check the three axioms of diffeology.
Axiom D1: Since every diffeology contains the constant parametrizations, they are contained in each element $\mathcal{D}_i$ of $\mathbf{D}$. Hence, the constant parametrizations also are contained in the intersection $\bigcap_{i\in\mathcal{J}} \mathcal{D}_i$.
Axiom D2: Let $\alpha\colon V \to X$ be a parametrization in X such that for every point r in U there exists an open neighborhood V of r for which $\alpha|_V$ is a parametrization belonging to the intersection $\bigcap_{i\in\mathcal{J}} \mathcal{D}_i$, that is, for each i in $\mathcal{J}$, $\alpha|_V \in \mathcal{D}_i$. Thus, by application of the axiom D2 to the diffeology $\mathcal{D}_i$, α is a plot for $\mathcal{D}_i$. Therefore, for all $\alpha \in \mathcal{D}_i$, $i \in \mathcal{J}$, that is, $\alpha \in \bigcap_{i\in\mathcal{J}} \mathcal{D}_i$.
Axiom D3: Let $\alpha \in \bigcap_{i\in\mathcal{J}} \mathcal{D}_i$ and let F be a smooth parametrization in the domain of α. For each i in $\mathcal{J}$, $\alpha \in \mathcal{D}_i$. Since each $\mathcal{D}_i$ is a diffeology, $\alpha \circ F$ belongs to $\mathcal{D}_i$, hence $\alpha \circ F$ belongs to the intersection $\bigcap_{i\in\mathcal{J}} \mathcal{D}_i$. Therefore, $\mathcal{D} = \bigcap_{i\in\mathcal{J}} \mathcal{D}_i$ is a diffeology of X. □

1.5 Generating families

A diffeology can be obtained from a generating family. Any collection of parametrizations of a set generates a diffeology. This way of building diffeologies is very convenient since it can simplify the analysis of the properties of a diffeological space to a subset of its plots, most times smaller than the total diffeology.

1.5.1 Generating diffeology

Let $\mathcal{F} = \{\alpha_j\}$ be an arbitrary family of parametrizations $\alpha_j\colon U_j \to X$ and let $\mathcal{D} = \langle\mathcal{F}\rangle$ be the intersection of all the diffeologies on X containing $\mathcal{F}$. We say that $\mathcal{F}$ is a generating family for $\mathcal{D}$.

The diffeology $\mathcal{D} = \langle F\rangle$ generated by the family $\mathcal{F}$ is then the smallest diffeology containing it. The intersection is not void because $\mathcal{F}$ is always contained in the indiscrete diffeology. It is always possible to add to a generating family any family of parameterizations that generate the discrete diffeology, without altering the generated diffeology; consequently, we shall always assume that the family $\mathcal{F}$ contains all the

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locally constant parametrizations.

Example 1.8. Let X be a set, and let us consider the empty family $\mathscr{F} = \emptyset$. The diffeology of X generated by the empty family is the discrete diffeology, that is, the intersection of all diffeologies of X. $\langle\emptyset\rangle = \mathcal{D}_\circ(X)$.

Example 1.9. Any atlas on a manifold M is a generating family of the manifold diffeology.

Theorem 1.10. *The parametrization (U, α) is a plot in $\langle\mathcal{F}\rangle$ if and only if for all $x \in U$ there exists a neighborhood V of x in U such that there exist a plot $f\colon W \to X$ in the family $\mathcal{F}$ and a change of coordinates $h\colon V \to W$ such that $\alpha_{|V} = f \circ h$.*

1.6 Pullbacks of diffeologies

In this Section we describe the transport of a diffeology by pullback. Given any set X and any map f from X to a diffeological space X', the set X acquire a natural diffeology by pullback of the diffeology of X'.

Let X be a set. Let X' be a diffeological space and $\mathcal{D}'$ its diffeology. Let $f\colon X \to X'$ be a set map.There exists a largest diffeology on X such that the map f is smooth. This diffeology is called the pullback of the diffeology $\mathcal{D}'$ by f, and it is denoted by $f^*(\mathcal{D}')$.

Theorem 1.11. *A parametrization α in X belongs to $f^*(\mathcal{D}')$ if and only if $f \circ \alpha$ belongs to $\mathcal{D}'$.*

Proof. Let us consider the family $\mathbf{D}$ of all the diffeologies of X such that f is smooth. A diffeology of X belongs to $\mathbf{D}$ if and only if the left composition of each of its elements by f is an element of $\mathcal{D}'$. The family $\mathbf{D}$ has a supremum. It is the intersection of all the diffeologies of X containing the set of all the parametrizations α of X such that $f \circ \alpha$ belongs to $\mathcal{D}'$. But this set, let us call it $\mathcal{D}$, is already a diffeology, let us check it.

D1. The composition of every constant map by any function f is constant. Hence the

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constant parametrizations belong to $\mathcal{D}$.
D2. Let $\alpha\colon U \to X$ be a parametrization such that, locally at each point r of U, $f \circ \alpha \in \mathcal{D}'$. Since $\mathcal{D}'$ is a diffeology, $f \circ \alpha$ is a plot for $\mathcal{D}'$. Hence, α belongs to $\mathcal{D}$.
D3. Let α be a parametrization in X. If $f \circ \alpha$ is a plot of X', for every smooth parametrization F in the domain of α, $f \circ (\alpha \circ F) = (f \circ \alpha) \circ F$ is a plot of X'. Hence, $\alpha \circ F$ belongs to $\mathcal{D}$.
Therefore, $\mathcal{D}$ is the supremum of $\mathbf{D}$, that is, the largest diffeology such that f is smooth. □

1.7 Subspaces and Diffeological Space

A nice characteristic of diffeological spaces is that any subset inherits a diffeology, making the theory much more clean than the usual theory of differentiable manifolds and submanifolds.

1.7.1 Subspaces

If $(X, \mathcal{D}_X)$ is a diffeological space and $Y \subseteq X$ is any subset, we define the induced diffeology or subspace diffeology $\mathcal{D}_Y$ on Y as the family of plots (U, α) in $\mathcal{D}_X$ whose image is contained in Y.
The inclusion $i_Y\colon Y \to X$ is smooth. In fact, a map $F\colon Z \to Y$ is smooth if and only if the composition $i_Y \circ F\colon Z \to X$ is smooth.

Example 1.12. Let $N \subset M$ be any subset of the differentiable manifold M. We endow N with the subspace diffeology formed by plots $\alpha\colon U \to M$ of the manifold diffeology $\mathcal{D}_M$ such that $\alpha(U) \subset N$.

Theorem 1.13. *$\mathcal{D}_Y$ is a diffeology on Y.*

Proof. Constant plots certainly are in $\mathcal{D}_Y$. Now take a compatible family $(U_i, \alpha_i) \in \mathcal{D}_Y$ and $\alpha|_{U_i} = \alpha_i$ because $\alpha_i \in \mathcal{D}_Y$ so $\alpha|_{U_i} \in \mathcal{D}_Y$ therefore $\alpha|_{U_i} \in \mathcal{D}_X$ and $\operatorname{Im}\alpha\,|_{U_i} \subseteq Y$ then $\alpha \in \mathcal{D}_Y$.

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To verify Axiom 3, let α and $h\colon V \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$ be composable where $\alpha \in \mathcal{D}_Y$ and h is smooth. We have $\alpha \in \mathcal{D}_X$, $\operatorname{Im}\alpha \subseteq Y$ and $\alpha \circ h \in \mathcal{D}_X$. We know $\operatorname{Im}\alpha \circ h \subseteq Im\,\alpha$ so $\operatorname{Im}\alpha \circ h \subseteq Y$ therefore $\alpha \circ h \in \mathcal{D}_Y$. □

1.8 Pushforward of diffeologies

In this section we describe the transport of a diffeology by pushforward. Given a diffeological space X and a map f from X to some set X', the set X' inherits a natural diffeology by pushing forward the diffeology of X to X'. Let X be a diffeological space, and let X' be a set. Let $f\colon X \to X'$ be a map. There exists a smallest diffeology of X' such that the map f is smooth. This diffeology is called the *pushforward*, by f of the diffeology $\mathcal{D}$ of X, and will be denoted by $f_*(\mathcal{D})$.

Theorem 1.14. *A parametrization $\alpha\colon U \to X'$ is a plot for $f_*(\mathcal{D})$ if and only if it satisfies the following condition:*

> *For every $r \in U$, there exists an open neighborhood V of r such that, either $\alpha|_V$ is a constant parametrization, or there exists a plot $\beta\colon V \to X$ such that $\alpha|_V = f \circ \beta$.*

In the case where $\alpha|_V = f \circ \beta$, we say that α lifts locally, at the point r, along the map f.

Proof. The set $\mathcal{D}$ of all the diffeologies of X' such that f is smooth is nonempty, since it contains the indescrete diffeology. Now, this family $\mathcal{D}$ has an infimum, let us denote it by $\mathcal{D}'$, it is the intersection of all the elements of $\mathcal{D}$. We have now to check that this infimum is actually a minimum, that is, f is smooth for the diffeology $\mathcal{D}'$. But by the definition of smooth maps, for any plot α belonging to $\mathcal{D}$, $f \circ \alpha$ belongs to every element of $\mathcal{D}$. Thus, $\alpha \circ f$ belongs to their intersection $\mathcal{D}'$. Next, let us check that the set of parametrizations satisfying condition is a diffeology. The axioms $D1$ and $D2$ are satisfied by definition. Let us consider then a parametrization $\alpha\colon U \to X'$ satisfying condition, and let $F\colon U' \to U$ be a smooth parametrization.

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Let $\alpha' = \alpha \circ F$, $r' \in U'$, $r = F(r')$, $V' = F^{-1}(V)$. Since F is a continuous, V' is a domain, and by construction an open neighborhood of r'. Then, let $\beta' = \beta \circ F$, then $\alpha|_V = f \circ \beta$ implies $\alpha' \,|_{V'} = f \circ \beta'$. Therefore, the axiom $D3$ is satisfied. Now, let $\mathcal{D}'$ be a diffeology such that the map f is smooth, that is, for every plot β of X, $f \circ \beta$ is a plot of X'. Hence, the diffeology $\mathcal{D}'$ contains the set of all the parametrizations of the type $f \circ \beta$. By restriction to any subdomain of the plots β (allowed by the axiom D3), $\mathcal{D}'$ contains the elements of the diffeology described by condition. It also contains the locally constant parametrizations. Therefore, this diffeology is the smallest diffeology such that f is smooth. □

Theorem 1.15. *Let X be a diffeological space and let X' and X'' be two sets. Let $f\colon X \to X'$ and $g\colon X' \to X''$ be two maps, and*

$$(X, \mathcal{D}) \xrightarrow{f} X' \xrightarrow{g} X''$$

we have $g_*(f_*(\mathcal{D})) = (g \circ f)_*(\mathcal{D})$. In other words, the pushforward of diffeologies is associative.

1.9 Subductions

Subductions are surjections between diffeological spaces, where the diffeology of the codomain is determined by the pushforward of the domain. Let X, X' be two diffeological spaces, and let $f\colon X \to X'$ be some map. The map f is said to be a *subduction* if it satisfies the following conditions:

1. The map f is surjective.

2. The diffeology D' of X' is the pushforward of the diffeology $\mathcal{D}$ of X.

Theorem 1.16. *Let X, X' be two diffeological spaces, and let $f\colon X \to X'$ be some map. The map f is a subduction if and only if the following conditions are satisfied:*

1. *The map f is a smooth surjection.*

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2. *For every plot $\alpha\colon U \to X'$, for every $r \in U$, there exist an open neighborhood V of r and a plot $\beta\colon V \to X$ such that $\alpha|_V = f \circ \beta$.*

Put differently, the map f is a subduction if and only if f is smooth and every plot of X' lifts locally along f, at each point of its domain.

Proof. The first condition is equivalent to $f_*(\mathcal{D}) \subset \mathcal{D}'$. The second condition is the reduction of the condition $\mathcal{D}' \subset f_*(\mathcal{D})$, in the case of a surjection. □

Theorem 1.17. *Let X, X' be two diffeological spaces, and let $f\colon X \to X'$ be a subduction. If f is injective, then f is a diffeomorphism. Conversely, diffeomorphisms are injective subductions.*

Proof. A subduction is, by definition, smooth and surjective. If moreover, f is injective, then f is a smooth bijection. Now, let $\alpha\colon U \to X'$ be a plot, for every $r \in U$ there exist an open neighborhood V of r and a plot β of X such that $f \circ \beta = \alpha|_V$. But since f is bijective, β is unique and $\beta = f^{-1} \circ (\alpha|_V)$, that is, $f^{-1} \circ \alpha$ is locally a plot of X at each point r of U. Hence, $f^{-1} \circ \alpha$ is a plot of X (axiom D2) and f^{-1} is smooth. Thus f is a diffeomorphism. Conversely, let $\mathcal{D}$ and $\mathcal{D}'$ denote the diffeologies of X and X'. If f is a diffeomorphism, then it is smooth and $f_*(\mathcal{D}) \subset \mathcal{D}'$. Now, its inverse also is smooth and $f_*^{-1}(\mathcal{D}') \subset \mathcal{D}$. But, this is equivalent to $\mathcal{D}' \subset f_*(\mathcal{D})$. Therefore, $f_*(\mathcal{D}) = \mathcal{D}'$. □

1.10 Quotients by an equivalence relation

Any quotient of a diffeological space acquires the subduction diffeology defined as the pushforward of the diffeology of the source space.

Let X be a diffeological space, and let $\mathcal{D}$ be its diffeology. Pick an equivalence relation $\mathcal{R}$ on X. The quotient set $X/\mathcal{R}$ carries a natural diffeology $\mathcal{D}/\mathcal{R}$, the pushforward of the diffeology $\mathcal{D}$ by the natural projection map $X \to X/\mathcal{R}$. This diffeology is called the *subduction diffeology*. For the subduction diffeology, the projection map is a subduction. The set $X/\mathcal{R}$ equipped with the subduction diffeology

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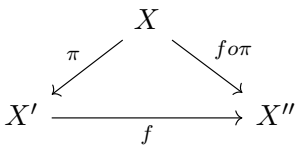
is called the *quotient space* or the *diffeological quotient* of X by the relation $\mathcal{R}$. For example, let β be a diffeological space. Every surjection $\pi\colon X \to \beta$ defines a natural equivalence relation $\mathcal{R}$, that is, $x\mathcal{R}x'$ if $\pi(x) = \pi(x')$.

Example 1.18. Let $\mathcal{R}$ be any equivalence relation on the manifold M and let $\pi\colon M \to M/\mathcal{R}$ be the quotient map. We endow $M/\mathcal{R}$ with the subduction diffeology $\mathcal{D}_M/\mathcal{R}$ where $\alpha\colon U \to M/\mathcal{R}$ belongs to $\mathcal{D}_M/\mathcal{R}$ if it locally factors through some plots of the manifold diffeology $\mathcal{D}_M$ on M.

Theorem 1.19. *Let X, X' and X'' be three diffeological spaces. Let $\pi\colon X \to X'$ be a subduction. A map $f\colon X' \to X''$ is smooth if and only if $f \circ \pi$ is smooth.*



(1.1)

Proof. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be the diffeologies of X, X' and X''. The map $f \circ \pi$ is smooth if and only if $(f \circ \pi)_*(\mathcal{D}) \subset \mathcal{D}''$. But $(f \circ \pi)_*(\mathcal{D}) = f_*(\pi_*(\mathcal{D}))$. Since π is a subduction, $\pi_*(\mathcal{D}) = \mathcal{D}'$ and $f_*(\pi_*(\mathcal{D})) = f_*(\mathcal{D}')$. Hence, $(f \circ \pi)_*(\mathcal{D}) \subset \mathcal{D}''$ is equivalent to $f_*(\mathcal{D}') \subset \mathcal{D}''$ which means that f is smooth. Note that if $f \circ \pi$ is surjective, then f is surjective. Next, replacing above the inclusion by an equality proves the second assertion. □

Corollary 1.20. *A subduction map $\alpha\colon U \to X$ is a diffeomorphism if and only if it is bijective.*

Theorem 1.21. *Let X be a diffeological space and $\mathcal{R}$ be a equivalence relation on X. Let $X/\mathcal{R}$ be the diffeological quotient. Let Q be a set and $\alpha\colon X \to Q$ be a surjection such that $\alpha(x) = \alpha(x')$ if and only if $x\mathcal{R}x'$. Then, the map $f\colon [x] \mapsto \alpha(x)$, is a diffeomorphism from $X/\mathcal{R}$ to Q, where Q is equipped with the pushforward*

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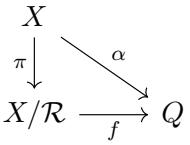
diffeology of X.



(1.2)

We say that the space Q is a representation of the subduction $X/\mathcal{R}$.

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Chapter 2

Categorical Constructions

2.1 Introduction

In this chapter, we introduce some categorical concepts. If we have a finite family $\{\alpha_1, \ldots, \alpha_n\}$ of plots $\alpha_i \colon U_i \to X$, we can compute its limit by taking the limit $L = L(\alpha_1, \ldots, \alpha_n)$ of the underlying sets and endowing it with the initial diffeology for the natural maps $p_i \colon L \to U_i$. This is easily seen to generalize the pull-back construction, because the product diffeology and the subspace diffeology are initial diffeologies for the projections and the inclusion, respectively.

2.2 Initial and final Diffeology

Given a family of diffeological spaces $\{(Y_i, \mathcal{D}_{Y_i})\}_{i \in I}$ and maps $f_i \colon X \to Y_i$ from a set X, we define the initial diffeology on X with respect to the maps f_i to be the diffeology consisting of all plots $P \colon U \to X$ such that $f_i \circ P \in \mathcal{D}_{Y_i}$ for all $i \in I$. This is the largest diffeology making all the f_i smooth.
Given a family of diffeological spaces $\{(X_i, \mathcal{D}_{X_i})\}_{i \in I}$ and maps $f_i \colon X_i \to Y$ to a set Y, we define the final diffeology on Y with respect to the maps f_i to be the diffeology generated by the maps $f_i \circ P$ for all $i \in I$ and all plots $P \in \mathcal{D}_X$. It is the smallest diffeology making all the f_i smooth.

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2.3 Product Diffeology and Disjoint Union

Given a family of diffeological spaces $\{(X_i, \mathcal{D}_{X_i})\}_{i\in I}$, we can form the direct product of the sets X_i in the category of sets and maps. We equip the direct product $\prod_i X_i$ with the initial diffeology with respect to the projections π_i. Then projections are subductions.
Note that for two spaces X, Y the product diffeology is generated by plots of the form (P_1, P_2) where $P_1 \in \mathcal{D}_X$ and $P_2 \in \mathcal{D}_Y$.

Given a family of diffeological spaces $\{(X_i, \mathcal{D}_{X_i})\}_{i\in I}$, we can form the disjoint union of the sets X_i in the category of sets and maps. We equip the disjoint union $\bigsqcup_i X_i$ with the final diffeology with respect to the injections $L_i : X_i \to \bigsqcup_i X_i$. Then the injections are subspaces.

2.4 Pullbacks

The category of diffeological spaces and smooth maps allows any kind of categorical limit. In particular, we shall need the following development of the pullback (or fibered product) of two maps (in particular, of two plots).

Let (U, α) and (V, β) be two maps into the diffeological space $(X, \mathcal{D})$.
We consider the subspace P (endowed with the induced diffeology) of the product $U \times V$ (endowed with the usual diffeology) given by

$$P = \{(u, v) \in U \times V : \alpha(u) = \beta(v)\};$$

also we consider the smooth maps

$$p_U : P \to U \quad , \quad p_U(u, v) = u,$$

and

$$p_V : P \to V \quad , \quad p_V(u, v) = v,$$

induced by projections. They verify

$$\alpha \circ p_U = \beta \circ p_V.$$

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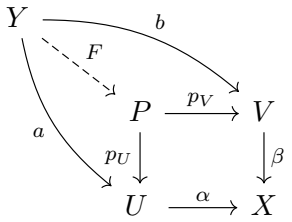
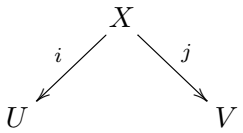
Proposition 2.1. *P is a pullback in the category of diffeological spaces. That means that it verifies the following universal property: given any diffeological space Y and two smooth maps $a\colon Y \to U$ and $b\colon Y \to V$ such that $\alpha \circ a = \beta \circ b$, then there exists a unique smooth map $F\colon Y \to P$ such that $p_U \circ F = a$ and $p_V \circ F = b$,*



(2.1)

2.5 Pushouts

Given a diagram of sets and maps like this:



(2.2)

We consider Q as a quotient of the coproduct $U \sqcup V$ given by

$$U \sqcup V/_{i(x)\sim j(x)};$$

also we consider smooth maps

$$q_U\colon U \to Q \quad , \quad q_U(u) = [u],$$

and

$$q_V\colon V \to Q \quad , \quad q_V(v) = [v],$$

They verify $q_U \circ i = q_V \circ j$.

Proposition 2.2. *Q is a pushout in the category of diffeological spaces, that is, it verifies the following property:*

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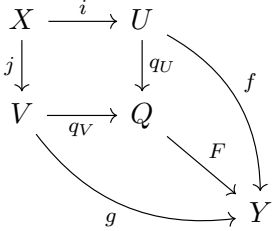
given any set Y and two smooth maps $q_U\colon U \to Q$ and $q_V\colon V \to Q$ such that $q_U \circ i = q_V \circ j$ where $i\colon X \to U$ and $j\colon X \to V$, then there exists a unique smooth map $F\colon Q \to Y$ such that $F \circ Q_U = f$ and $F \circ Q_V = g$.



(2.3)

2.6 Joins

The category of diffeological spaces is stable under limits and colimits ([2]). This allows to introduce the *join* of two maps (in particular, two plots). Let $\alpha\colon U \to X$ and $\beta\colon V \to X$ be two maps into $(X, \mathcal{D})$. The *join* of α and β is a diffeological space $U * V$ endowed with a smooth map $\alpha * \beta\colon U * V \to X$, defined as follows.

We take the disjoint union $U \sqcup V$ with the coproduct diffeology generated by the manifold diffeologies $\mathcal{D}_U$ and $\mathcal{D}_V$.

We introduce the equivalence relation in $U \sqcup V$ given by

$$u \sim v \quad \text{if f} \quad u \in U, v \in V, \alpha(u) = \beta(v),$$

and we endow the quotient space $U * V = (U \sqcup V)/\!\sim$ with the subduction diffeology.

Finally we define the smooth map $\alpha * \beta\colon U * V \to X$ as

$$(\alpha * \beta)([w]) = \begin{cases} \alpha(u), & \text{if } w = u \in U, \\ \beta(v), & \text{if } w = u \in V. \end{cases}$$

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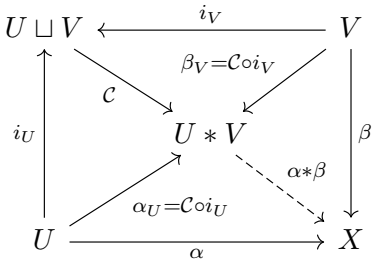
Then we have a commutative diagram



(2.4)

Remark 2.3. This join of two maps is in fact *the push-out* of the pull-back of the two plots, as it is clear from the following constructions.

First, one can consider the *pullback* P of the maps α and β. It is defined as the subspace $P \subset U \times V$ of pairs $(u, v) \in U \times V$ such that $\alpha(u) = \beta(v)$. Then the projections induce the maps p_U, p_V in the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{p_V} & V \\ {\scriptstyle p_U}\downarrow & & \downarrow{\scriptstyle \beta} \\ U & \xrightarrow{\alpha} & X. \end{array} \tag{2.5}$$

Now we consider the *push-out* of p_U, p_V. It is given as the space Q obtained by introducing in $U \sqcup V$ the equivalence relation $u \sim v$ iff $u \in U$, $v \in V$, and there exists $w \in P$ such that $p_U(w) = u$ and $p_V(w) = v$. New maps α_U, β_V appear, as in the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{p_V} & V \\ {\scriptstyle p_U}\downarrow & & \downarrow{\scriptstyle \beta_V} \\ U & \xrightarrow{\alpha_U} & Q. \end{array} \tag{2.6}$$

It is easy to check that $Q = U * V$, and that $\alpha * \beta \colon U * V \to X$ is the map obtained by the universal property of the push-out.

We refer to [31] for more information about the join construction in other settings.

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2.7 Infinite families

It is possible to give non-finite versions of all those constructions. For instance, the non-finite pull-back would be defined as follows.

Definition 2.4. Let $\mathcal{F} = \{\alpha_\lambda \colon U_\lambda \to X\}_{\lambda\in\Lambda}$ be an arbitrary family of smooth maps with codomain the diffeological space $(X, \mathcal{D})$. The *pull-back* or *limit* of the family $\mathcal{F}$ is the diffeological space

$$P = \{(x_\lambda) \in \prod_{\lambda\in\Lambda} U_\lambda \colon \alpha_\lambda(x_\lambda) = \alpha_\mu(x_\mu) \quad \forall\lambda, \mu \in \Lambda\}.$$

We endow $\prod_{\lambda\in\Lambda} U_\lambda$ with the product diffeology (the smallest diffeology making smooth all the projections) and P with the subspace diffeology. The projections $\pi_\lambda \colon \prod X_\lambda \to X_\lambda$ induce smooth maps $p_\lambda \colon P \to U_\lambda$ such that

$$\alpha_\lambda \circ p_\lambda = \alpha_\mu \circ p_\mu \quad \forall\lambda, \mu \in \Lambda.$$

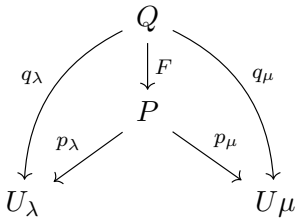
Then the usual universal property of the pull-back holds, that is, if Q is a diffeological space and $q_\lambda \colon Q \to U_\lambda$ are smooth maps such that

$$\alpha_\lambda \circ q_\lambda = \alpha_\mu \circ q_\mu \quad \forall\lambda, \mu \in \Lambda,$$

then there exists a unique smooth map $f \colon Q \to P$ such that

$$p_\lambda \circ f = q_\lambda \quad \forall\lambda \in \Lambda.$$

(see Diagram (2.7))



(2.7)

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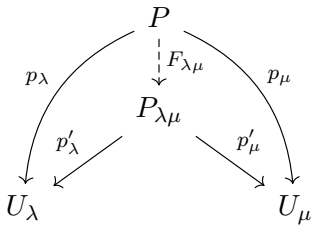
Lemma 2.5. *Let $\lambda, \mu \in \Lambda$ and let $P_{\lambda\mu}$ be the pull-back of α_λ and α_μ. Then there exists a unique map $F_{\lambda\mu}\colon P \to P_{\lambda\mu}$ such that*

$$p_\lambda = p'_\lambda \circ F_{\lambda\mu},$$
$$p_\mu = p'_\mu \circ F_{\lambda\mu}$$

(see Diagram (2.8)*). Notice that $F_{\lambda\mu}$ is the projection onto the coordinates λ and μ.*



(2.8)

Remark 2.6. Analogously we can define the *join* of the family $\mathcal{F}$, that will be denoted by $\lambda \in \Lambda\alpha_\lambda$.

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Chapter 3

Cartan Calculus

3.1 Introduction

The general philosophy of diffeology is to transport, functorially, geometrical objects defined on real domains to diffeological spaces. Since a diffeology on a set X is defined by parametrizations in X, geometrical contravariant objects fit well to this method (differential forms are one of the main examples).

In the second section of this chapter, the bases of differential calculus on manifolds are recalled.

In the third section the differential calculus in diffeology is developed. There we begin to state the main results relative to differential forms in diffeology.

3.2 Differential forms and the exterior differentation

Let M be a differentiable manifold. We can construct the *exterior r-bundle*

$$\Lambda^r(M) = \cup_{p\in M}\Lambda^r(T_p^*M)$$

and the *exterior algebra bundle*

$$\Lambda(M) = \cup_{p\in M}\Lambda(T_p^*M)$$

over M. Under a coordinate chart (U, x^i) of M, the natural basis of $\Lambda^r(T_p^*M)$ for $p \in U$ is given by

$$\{dx_{i_1} \wedge \cdots \wedge dx^{i_r} : 1 \leq i_1 < \cdots < i_r \leq n\}$$

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evaluated at p. A section of the exterior r-bundle $\Lambda^r(M)$ is called an r-form on M. A *differential form on* M is a section of the exterior algebra bundle $\Lambda(M)$ which is *smooth*, in the sense that it varies differentiably with regard to the point and the variables. The space of r-forms (differential forms, respectively) is denoted by $\Omega^r(M)$ ($\Omega(M)$, respectively).

By definition, $\Omega^0(M)$ is simply the space $C^\infty(M)$ of smooth functions on M. There is a canonical exterior product structure on the real vector space $\Omega(M)$ induced by the exterior product "$\wedge$" on each fiber point. It turns out that $\Omega(M)$ is an infinite dimensional (graded) algebra over $\mathbb{R}$. A differential form w can be uniquely written as

$$w = w^0 + w^1 + \cdots + w^n,$$

where each w^r is an r-form in $\Omega^r(M)$, and $n = \dim M$.
By the definition of the exterior algebra, an r-form ω can be regarded as a $(0, r)$-type (antisymmetric) tensor field. The form ω acts on the space $\mathfrak{X}(M)$ of smooth vector fields on M as an alternating r-linear map

$$\omega\colon \mathfrak{X}(M) \times \cdots \mathfrak{X}(M) \to C^\infty(M).$$

Under a coordinate chart (U, x^i), an r-form w can be expressed as

$$w = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Analogously to functions, differential forms can be "derived" via the exterior differentiation.

Theorem 3.1. *There exists a unique linear operator*

$$\mathrm{d}\colon \Omega(M) \to \Omega(M)$$

such that:

(1) For any $r \ge 0$,

$$\mathrm{d}(\Omega^r(M)) \subset \Omega^{r+1}(M)$$

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(2) If w_1 is an r-form, then for any $w_2 \in \Omega(M)$,

$$\mathrm{d}(w_1 \wedge w_2) = \mathrm{d}w_1 \wedge w_2 + (-1)^r w_1 \wedge \mathrm{d}w_2.$$

(3) If $f \in C^\infty(M)$, then $\mathrm{d}f$ is the differential of f.

(4) $\mathrm{d}^2 = 0$.

Assume that $F\colon M \to N$ is a C^∞ map between manifolds. We define the *pull-back of differential forms by F* to be the linear map $F^*\colon \Omega^r(N) \to \Omega^r(M)$ given by

$$(F^* w)_p(v_1, \ldots, v_r) = w_{F(p)}(F_{p*}(v_1), \ldots, F_{p*}(v_r))$$

Proposition 3.2. *F^* commutes with d. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \Omega^r(N) & \xrightarrow{F^*} & \Omega^r(M) \\ \mathrm{d}\downarrow & & \mathrm{d}\downarrow \\ \Omega^{r+1}(N) & \xrightarrow{F^*} & \Omega^{r+1}(M). \end{array} \tag{3.1}$$

It is immediate to prove that if $F\colon M \to N$ and $G\colon N \to P$ are C^∞ maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^*\colon \Omega(P) \to \Omega(M). \tag{3.2}$$

3.2.1 The de Rham cohomology groups

We will be mainly focused on the study of a very important algebraic topological invariant over differentiable manifolds: the de Rham cohomology. This is a fundamental object carrying topological information of the manifold.

Let M be a manifold. An r-form ω on M is called *closed* if $\mathrm{d}\omega = 0$; it is called *exact* if $\mathrm{d}\tau = \omega$ for some $(r-1)$-form τ. The space of closed (exact, respectively) r-forms on M is denoted by $Z^r(M)$ ($B^r(M)$, respectively). It is easy to see that $Z^r(M)$ and $B^r(M)$ are (usually infinite dimensional) real vector spaces. Since $\mathrm{d}^2 = 0$, it is immediate that $B^r(M)$ is a subspace of $Z^r(M)$ for all $r \geq 0$.

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3.2.2 De Rham cohomology groups of a manifold

The graded vector space $\Omega^\bullet(M) = (\Omega^r(M))_{r\geq 0}$ of differential forms together with the exterior derivative d is called the *de Rham complex* of M. The quotient space

$$\frac{Z^r(M)}{B^r(M)} = \frac{\ker(\mathrm{d}\colon \Omega^r(M) \to \Omega^{r+1}(M))}{\mathrm{im}(\mathrm{d}\colon \Omega^{r-1} \to \Omega^r(M))}$$

as a real vector space is called the r-th *de Rham cohomology group* of M, and it is denoted by $\mathrm{H}^r(M)$. Elements in $\mathrm{H}^r(M)$ are called *cohomology classes* of degree r.

By definition, a closed r-form ω defines a cohomology class $[w]$ of degree r. If $[\omega] = [\omega']$, then there exists some $(r-1)$-form τ such that $\omega' - \omega = \mathrm{d}\tau$. $\mathrm{H}^0(M)$ is just the space of C^∞ functions f with $\mathrm{d}f = 0$, which is equivalent to the fact that f is locally constant (that is, constant on each connected component). Therefore, if M is connected, $\mathrm{H}^0(M)$ is isomorphic to $\mathbb{R}$. In general, $\mathrm{H}^0(M)$ is isomorphic to $\mathbb{R}^N$, where N is the cardinality of the set of connected components of M (at most countable since the topology of M is second countable). In particular, if M is compact (so there will be only finitely many connected components of M since the set of connected components from an open cover of M), then $\mathrm{H}^0(M)$ is finite dimentional.

3.3 Differential Forms on Diffeological Spaces

Differential forms on diffeological spaces are defined by their evaluations on the plots, which are the pullbacks of the forms by the plots. These pullbacks are ordinary smooth forms. Hence, a differential form of a diffeological space is known as soon as we know all its pullbacks by all the plots. There is an additional condition on the pullbacks wich expresses the condition of compatibility under changes of variables.

Definition 3.3. Let X be a diffeological space. A *differential k-form* on X is a map ω which associates, to every plot $\alpha\colon U \to X$, a smooth k-form $\omega(\alpha)$ or ω_α defined on the domain U of α such that, for every smooth parametrization $F\colon V \to U$ in the

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domain of the plot α, we have

$$\omega(\alpha \circ F) = F^*(\omega_\alpha). \tag{3.3}$$

The condition $\omega(\alpha \circ F) = F^*(\omega_\alpha)$ is the *smooth compatibility* condition, and we shall say that $\omega(\alpha)$ *represents* the differential form ω in the plot α.

We shall denote by $\Omega^k(X, \mathcal{D})$ the vector space of k-forms on $(X, \mathcal{D})$. Its linear structure is inherited from the linear structure of classical forms, that is,

$$(t\omega + \omega')_\alpha = t\omega_\alpha + \omega'_\alpha, \quad \forall \omega, \omega' \in \Omega^k(X, \mathcal{D}), t \in \mathbb{R}, \alpha \in \mathcal{D}.$$

Theorem 3.4. *The zero forms on a diffeological space $(X, \mathcal{D})$ are identified with the smooth maps $X \to \mathbb{R}$, for the usual diffeology on $\mathbb{R}$.*

Proof. If $f\colon X \to \mathbb{R}$ is a smooth map, we define the form $\omega(f) \in \Omega^0(X)$ as

$$\omega(f)_\alpha = f \circ \alpha \in \Omega^0(U) = \mathcal{C}^\infty(U, \mathbb{R})$$

for any plot (U, α) on X.

Conversely, if $\omega \in \Omega^0(X)$, we define the function

$$f(\omega)\colon X \to \mathbb{R}, \quad f(\omega)(x) = \omega_{c_x},$$

where for each $x \in X$ we take the constant plot $c_x\colon \mathbb{R}^0 = \{0\} \to X$ with $c_x(0) = x$.

The function $f = f(\omega)$ is differentiable because for any plot (U, α) on X, we have

$$(f \circ \alpha)(u) = \omega_{c_{\alpha(u)}} = \omega_{\alpha \circ c_u} = c_u^*(\omega_\alpha) = \omega_\alpha(u),$$

where $c_u\colon \mathbb{R}^0 \to U$ is the constant map u, and $\omega_\alpha \in \mathcal{C}^\infty(U, \mathbb{R})$. □

3.4 Pullback of a form

Definition 3.5. Let $(X, \mathcal{D})$ be a diffeological space. A differential k-form on X, $k \geq 0$, is any collection $w = \{w_\alpha\}$, where for each plot $\alpha\colon U \to X$ in $\mathcal{D}$ we

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have a usual differential form $w_\alpha \in \Omega^k(U)$. Moreover, it must verify the following compatibility condition: for any $\mathcal{C}^\infty$ change of coordinates $h\colon V \to U$ we must have

$$w_{\alpha\circ h} = h^* w_\alpha,$$

where $h^*\omega_\alpha \in \Omega^k(V)$ is the usual pullback of a differential form.

Let X be a diffeological space and $\omega \in \Omega^k(X)$. Let $\alpha\colon U \to X$ be a plot and let the domain U be regarded as a smooth diffeological space. The plot α is a smooth map from U to X, so $\alpha^*(\omega) \in \Omega^k(U)$. The k-form $\alpha^*(\omega)$ is characterized by its values on the plot 1_U, that is, $\alpha^*(\omega)(1_U) = \omega(\alpha \circ 1_U) = \omega(\alpha)$. Therefore, the values of ω on the plots, which define the form ω, represent just the pullbacks of ω by the plots. In other words, the differential form ω is just defined by its pullbacks by the plots of X, and the condition of compatibility just writes $(\alpha \circ F)^*(\omega) = F^*(\alpha^*(\omega))$, for any plot α of X and any smooth parametrization F in the domain of α.

3.5 Exterior Product and exterior derivative

3.5.1 $\mathscr{C}^\infty$ differential forms on $\mathbb{R}^n$

Let $x_1, \cdots, x_n$ be the linear coordinates on $\mathbb{R}^n$. We define $\Omega(\mathbb{R}^n)$ to be the algebra over $\mathbb{R}$ generated by $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$ with the relations

$$\begin{cases} (\mathrm{d}x_i)^2 = 0, \\ \mathrm{d}x_i\,\mathrm{d}x_j = -\mathrm{d}x_j\,\mathrm{d}x_i,\ i \neq j. \end{cases}$$

As a vector space over $\mathbb{R}$, $\Omega(\mathbb{R}^n)$ has the basis

$$1, \mathrm{d}x_i, \mathrm{d}x_i\mathrm{d}x_j, \mathrm{d}x_i\mathrm{d}x_j\mathrm{d}x_k, \ldots, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \quad 1 \leq i < j < k \leq n.$$

The $\mathscr{C}^\infty$ differential forms on $\mathbb{R}^n$ are then elements of

$$\mathscr{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega(\mathbb{R}^n)$$

Thus, if ω is such a form, it can be uniqely written as $\sum f_{i_1\cdots i_q}\mathrm{d}x_{i_1}\cdots\mathrm{d}x_{i_q}$ where the coefficients $f_{i_1\cdots i_q}$ are $\mathscr{C}^\infty$ functions. We also write $\omega = \sum f_I dx_I$. The algebra $\Omega^\bullet(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is naturally graded.

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3.5.2 The exterior derivative

There is a differential operator

$$\mathrm{d}\colon \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$$

defined as follows:

i) if $f \in \Omega^0(\mathbb{R}^n)$, then $\mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\mathrm{d}x_i$,

ii) if $\omega = \sum f_i \mathrm{d}x_i$, then $\mathrm{d}\omega = \sum \mathrm{d}f_i \mathrm{d}x_i$.

The exterior product of two differential forms, written $\tau \wedge \omega$, is defined as follows: if $\tau = \sum f_I dx_I$ and $\omega = \sum g_I dx_J$, then

$$\tau \wedge \omega = \sum f_I \, g_J \, dx_I \, dx_J.$$

Note that $\tau \wedge \omega = (-1)^{\deg \tau \, \deg \omega} \omega \wedge \tau$.

Proposition 3.6. d *is an antiderivation. In other words:*

$$d(\tau \wedge \omega) = \mathrm{d}\tau \wedge \omega + (-1)^{\deg \tau} \tau \wedge \mathrm{d}\omega$$

Proposition 3.7. $\mathrm{d}^2 = 0$

3.5.3 Exterior derivative of forms

Now we translate the preceding constructions to diffeological spaces.

Definition 3.8. The exterior derivative $\mathrm{d}_X\colon \Omega^k(X, \mathcal{D}) \to \Omega^{k+1}(X, \mathcal{D})$ is defined, for any plot $(U, \alpha) \in \mathcal{D}$, by

$$(\mathrm{d}_X \omega)_\alpha = \mathrm{d}_U(\omega_\alpha),$$

where $\mathrm{d}_U\colon \Omega^k(U) \to \Omega^{k+1}(U)$ is the usual exterior derivative.

That $\mathrm{d}_X\omega$ is a well-defined form follows from the commutativity of pullbacks with the classical exterior derivative.

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Theorem 3.9. *Let X' be another diffeological space, and let $f\colon X \to X'$ be a smooth map. Then, for all differential forms ω' on X' we have*

(i) $\mathrm{d}_X(f^*(\omega')) = f^*(\mathrm{d}_{X'}\omega')$,

(ii) $\mathrm{d} \circ \mathrm{d} = 0$,

(iii) d *is* $\mathbb{R}$*-linear.*

3.5.4 Exterior product of differential forms.

Let X be a diffeological space, let $\omega \in \Omega^k(X)$ and $\omega' \in \Omega^l(X)$. The *exterior product* $\omega \wedge \omega'$ is the differential $(k+l)$-form defined on X by

$$(\omega \wedge \omega')(\alpha) = \omega(\alpha) \wedge \omega'(\alpha),$$

for all plots α of X.

Theorem 3.10. *The exterior product is bilinear. The following properties of the exterior product of smooth forms pass naturally to the exterior product of differential forms on diffeological spaces,*

1. $\omega \wedge \omega' = (-1)^{kl}\omega' \wedge \omega$,
2. $f^*(\omega \wedge \omega') = f^*(\omega) \wedge f^*(\omega')$, *where* $f\colon X' \to X$ *is a smooth map.*

Proof. First of all, the exterior product is well defined. Indeed, let $\alpha\colon U \to X$ be a plot, and $F \in \mathscr{C}^\infty(V, U)$ be a smooth parametrization, then

$$\begin{aligned}(\omega \wedge \omega')(\alpha \circ F) &= \omega(\alpha \circ F) \wedge \omega'(\alpha \circ F)\\ &= F^*(\omega(\alpha)) \wedge F^*(\omega'(\alpha))\\ &= F^*(\alpha^*(\omega) \wedge \alpha^*(\omega'))\\ &= F^*((\omega \wedge \omega')(\alpha).\end{aligned}$$

Hence, the map $\alpha \mapsto \alpha^*(\omega \wedge \omega')$ defines a differential form $\omega \wedge \omega' \in \Omega^{k+l}(X)$. □

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3.6 De Rham Cohomology

In this section we introduce the De Rham cohomology of diffeological spaces, according to the definition of differential forms and the exterior derivative. Let X be a diffeological space. The exterior derivative satisfies the coboundary condition

$$\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X),\ k \geq 0, \quad \mathrm{d}\circ\mathrm{d} = 0.$$

As it is usual in cohomology theories, when we have a chain complex of real vector spaces $\Omega^\bullet(X) = \{\Omega^k(X)\}_{k=0}^\infty$ with a coboundary operator d, the space of k-cocycles is defined as the kernel in $\Omega^k(X)$ of the operator d, and the space of k-coboundaries is defined as the image in $\Omega^k(X)$ of the operator d. They will be denoted by

$$\begin{aligned} Z^k_{dR}(X) &= \ker \mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X), \\ B^k_{\mathrm{dR}}(X) &= \mathrm{d}(\Omega^{k-1}(X)) \subset Z^k_{\mathrm{dR}}(X). \end{aligned}$$

The De Rham cohomology groups of X are then defined as the quotients of the spaces of cocycles by the spaces of coboundaries.We denote them by

$$\mathrm{H}^k_{\mathrm{dR}}(X) = \mathrm{Z}^k_{\mathrm{dR}}(X)/\mathrm{B}^k_{\mathrm{dR}}(X).$$

Example 3.11. It was proved in [18] that the cohomology of the leaf space $M/\mathcal{F}$ of a foliated manifold $(M, \mathcal{F})$, endowed with the quotient diffeology, equals the so-called basic cohomology of the foliation.

Each differentiable map $F\colon (X, \mathcal{D}) \to (X', \mathcal{D}')$ induces a morphism

$$F^*\colon \mathrm{H}^k(X', \mathcal{D}') \to \mathrm{H}^k(X, \mathcal{D}), \quad F^*([\omega]) = [F^*\omega],$$

and the usual properties hold,

$$(G\circ F)^* = F^*\circ G^*, \quad \mathrm{id}^*_X = \mathrm{id}_{\mathrm{H}(X)}.$$

Proposition 3.12. *Let X be a diffeological space. Let $\{X_\lambda\}$ be a collection of subspaces of X such that $X = \sqcup_{\lambda\in\Lambda} X_\lambda$. Then $\mathrm{H}^k_{\mathrm{dR}}(X)$ and $\Pi_{\lambda\in\Lambda}\mathrm{H}^k_{\mathrm{dR}}(X_\lambda)$ are isomorphic for each k.*

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Proof. We define a homomorphism $\Pi i_\lambda^* \colon \Omega^k(X) \to \Pi_{\lambda\in\Lambda}\Omega^k(X_\lambda)$ by $(\Pi i_\lambda^*)(\omega) = (i_\lambda^*(\omega))_{\lambda\in\Lambda}$ for any $\omega \in \Omega^k(X)$, where $i_\lambda^* \colon \Omega^k(X) \to \Omega^k(X_\lambda)$ is induced by the inclusion $i_\lambda^* \colon X_\lambda \to X$. Let ω be an element of $\ker(\Pi i_\lambda^*)$. Since $(\Pi i_\lambda^*)(\omega) = (i_\lambda^*(\omega))_{\lambda\in\Lambda} = 0$, we have $\omega = 0$. Let $(\tau_\lambda)_{\lambda\in\Lambda}$ be an element of $\Pi_{\lambda\in\Lambda}\Omega^k(X_\lambda)$. We define $\tau \in \Omega^k(X)$ by any $\lambda \in \Lambda$, $\tau|X_\lambda = \tau_\lambda$. Then we have $(\Pi i_\lambda^*)(\tau) = (\tau_\lambda)_{\lambda\in\Lambda}$ since $X_\lambda \cap X'_\lambda$ is empty for each $\lambda \neq \lambda'$. □

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Chapter 4

Horizontal Forms

4.1 Introduction

The concept of horizontal forms was first introduced by Iglesias in [20].
We will use the following property for the definition of horizontal forms:
If $h, h' \colon V \to U$ are two changes of coordinates such that $\alpha \circ h = \alpha \circ h'$, then

$$h^* \omega_\alpha = (h')^* \omega_\alpha$$

where $\alpha \colon U \to X$ is a plot in $\mathcal{D}$ and $\omega_\alpha \in \Omega^k(U)$.

4.2 Horizontal forms

If (U, α) is a plot on X, we define the subspace $\Omega^k(\alpha) \subset \Omega^k(U)$ of α-horizontal forms as those forms $\mu \in \Omega^k(U)$ verifying: if $h, h' \colon V \to U$ are changes of coordinates such that $\alpha \circ h = \alpha \circ h'$, then $h^* \mu = (h')^* \mu$. They form a subcomplex of $\Omega(U)$ because if $\omega \in \Omega^k(\alpha)$ and $\alpha \circ h = \alpha \circ h'$ then

$$h^*(\mathrm{d}_U \omega) = d_V h^* \omega = \mathrm{d}_V (h')^* \omega = (h')^* \mathrm{d}_U \omega,$$

hence $\mathrm{d}_U \omega \in \Omega^{k+1}(\alpha)$.

Proposition 4.1. *Let $f \colon X \to Y$ be a smooth map between diffeological spaces and let $\omega \in \Omega(Y)$ then $f^*\omega$ is f-horizontal.*

Proof. If $fh = fh'$ then $h^*(f^*\omega) = (fh)^*\omega = (fh')^*\omega = (h')^*(f^*\omega)$. □

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Lemma 4.2. *If $P\colon X \to Y$ is a subduction map between diffeological spaces, then $P^*\colon \Omega(P) \to \Omega(X)$ is an isomorphism.*

Proof. We shall make the proof in several steps:
Step 1. The map is well-defined by Proposition 4.1). That is $P^*(\Omega(Y)) \subset \Omega(X)$.
Step 2. The map is injective.
Let $\omega \in \Omega(Y)$ such that $P^*(\omega) = 0$. We must show that $\omega_\gamma = 0$ for any plot $\gamma\colon W \to Y$. For each $p \in W$ there exist a neighborhood W of p such that γ on W factor through as $\gamma = P\tilde{\gamma}$ where $\tilde{\gamma}\colon W \to X$, then $\omega_\gamma = \gamma^*\omega = (P\tilde{\gamma})^*\omega = (\tilde{\gamma})^*(P^*\omega) = (\gamma)^*(0) = 0$. This proves that $\omega = 0$.
Step 3. The map is surjective.
Let $\xi \in \Omega(p)$ that is $\xi \in \Omega(X)$ is P-horizontal. Let us define $\omega \in \Omega(Y)$ such that $P^*\omega = \xi$ if $\gamma\colon W \to Y$ is a plot, we need to define $\omega_\gamma$in a neighborhood of $p \in W$. Since $\gamma = P\tilde{\gamma}$, we define $\omega_{\gamma|_{\omega_P}} \in \tilde{\gamma}^*(\xi)$ in order to prove that ω is well-defined if $P\tilde{\gamma}' = \gamma$ for another $\tilde{\gamma}'$, then we have $P\tilde{\gamma}' = \gamma = P\gamma$. Then, since ξ is P-horizontal, we have $\gamma'^*\xi = \tilde{\gamma}^*\xi$. Finally we have $P^*\omega = \xi$ because for any plot $\tilde{\gamma}\colon W \to X$, we have $(P^*\omega)_{\tilde{\gamma}} = \tilde{\gamma}^*P^*\omega = (P\tilde{\gamma})^*\omega = \xi$. □

Lemma 4.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth maps between diffeological spaces. If $\rho \in \Omega(Y)$ is a g-horizontal form, then $f^*\rho \in \Omega(X)$ is $g \circ f$-horizontal.*

Proof. Let $h, h'\colon W \to X$ be changes of coordinates such that $(g\circ f)\circ h = (g\circ f)\circ h'$, then $(fh)^*\rho = (fh')^*\rho$ because ρ is a horizontal. So, $h^*(f^*\rho) = h'^*(f^*\rho)$. □

Lemma 4.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth maps between diffeological spaces. If $\omega \in \Omega(X)$ is a $g \circ f$-horizontal, then ω is f-horizontal.*

Proof. Let $h, h'\colon W \to X$ be change of coordinates such that $f \circ h = f \circ h'$, then $(g \circ f) \circ h = (g \circ f) \circ h'$. So $h^*\omega = (h')^*\omega$. □

Lemma 4.5. *Let $X \xrightarrow{P} Y \xrightarrow{g} Z$ are smooth maps between diffeological spaces with P a subduction map, if $\omega \in \Omega(X)$ is $(g \circ P)$-horizontal and $\omega = P^*\xi$, for some $\xi \in \Omega(Y)$, then ξ is g-horizontal.*

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Proof. If ω is $g \circ P$-horizontal, then it is P-horizontal (lemma 4.4) also we know $\Omega(Y) \overset{P^*}{\cong} \Omega(P)$ (lemma 4.2), then there exist an unique ξ such that $\omega = P^*\xi$. We need to prove that ξ is g-horizontal. Let $h, h' \colon W \to Y$ be change of coordinates such that $g \circ h = g \circ h'$. Locally there are $H, H' \colon W_p \to X$ (where W_P is an open neighborhood of p) such that $PH = h'$, $PH' = h'$, then $g \circ P \circ H = g \circ P \circ H'$, and then $H^*\omega = (H')^*\omega$ because ω is $g \circ P$-horizontal, then $H^*P^*\xi = (H)'^*P^*\xi$. So $(PH)^*\xi = (PH')^*\xi$ then $h^*\xi = (h')^*\xi$. □

Proposition 4.6. *Let $f \colon X \to Y$ be a smooth map. The r-form $\omega \in \Omega^r(X)$ is f-horizontal if and only if for any diffeological space Z and smooth maps $h, h' \colon Z \to X$ such that $f \circ h = f \circ h'$, we have $h^*\omega = (h')^*\omega$.*

Proof. If $h, h' \colon Z \to X$ with $fh = fh'$. We must show that $(h^*\omega)_\gamma = ((h')^*\omega)_\gamma$ for any plot $\gamma \colon W \to Z$.
Since $h\gamma, h'\gamma \colon W \to X$ verifies $fh\gamma = fh'\gamma$, then $(h\gamma)^*\omega = (h'\gamma)^*\omega$. So $\gamma^*h^*\omega = (\gamma')^*(h')^*\omega$. We have $(h^*\omega)_\gamma = ((h')^*\omega)_\gamma$ for any plot γ. □

Theorem 4.7. *If $\pi \colon U \to X$ is a subduction then the induced morphism $\pi^* \colon \Omega(X) \to \Omega(U)$ is injective.*

Corollary 4.8. *For a subduction $\pi \colon U \to X$, the complex $\Omega(X)$ is isomorphic to the complex $\Omega(\pi) \subset \Omega(U)$ of π-horizontal forms.*

Proof. Let $\theta \in \Omega^k(\pi) \subset \Omega^k(U)$ be a π-horizontal form. We define a form $\omega \in \Omega^k(X)$ as follows: if $\gamma \colon W \to X$ is a plot, and $p \in W$, then γ locally factors though π, say $\gamma = \pi \circ h$ for some $h \colon W_p \to U$. We define $\omega_\gamma = h^*\theta$ in the neighbourhood W_p.

It is easy to check that this definition does not depend on h (because θ is π-horizontal), that the form ω is well defined (that is, it verifies the compatibility with the changes of coordinates), and that $\pi^*\omega = \theta$.

This proves that the image of π^* is $\Omega(\pi)$. Hence $\Omega(X) \cong \Omega(\pi)$ by Theorem 4.7. □

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Chapter 5

Relative cohomology groups

5.1 Introduction

In this chapter we study two different versions of the *relative cohomology*.

In section 5.4 we explain the relationship between the two types of relative cohomologies.

Finally, in the last section we have some examples.

5.2 A first type relative De Rham complex

In the setting of topological spaces, the relative cohomology groups are defined (see [17, Section 3.1] or [39, Example 5.4.5]) for any topological pair (X, A), that is, $A \subset X$ is a subspace.

In the diffeological setting, the relative De Rham cohomology group $\mathrm{H}(X, \alpha)$ of any smooth map $\alpha\colon U \to X$ was defined, in our previous paper [32], as the cohomology of the complex $\Omega(X, \alpha) = \ker \alpha^*$, the kernel of the induced morphism $\alpha^*\colon \Omega(X) \to \Omega(U)$. However, this relative complex $\Omega(X, \alpha)$ has not an associated long exact sequence in cohomology.

5.3 A second type relative De Rham complex

The problem is that, in general, the morphism α^* is not surjective, because an argument based on partitions of unity is missing in the diffeological setting. In order to overcome this inconvenience, we will introduce in this section a second type relative

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De Rham complex, denoted by $\Omega(X, \alpha)$. This complex mimics the analogous one given by Bott-Tu for differentiable manifolds in [4, Prop. 6.49].

Definition 5.1. Let $\alpha\colon U \to X$ be a smooth map between diffeological spaces. The *relative De Rham complex* $\Omega(X, \alpha)$ is defined as

$$\Omega^r(X, \alpha) = \Omega^r(X) \oplus \Omega^{r-1}(\alpha), \quad r \geq 0,$$

with differential $d\colon \Omega^r(X, \alpha) \to \Omega^{r+1}(X, \alpha)$ given by

$$d(\omega, \theta) = (\mathrm{d}\omega, \alpha^*\omega - \mathrm{d}\theta).$$

The differential is well defined because $\omega \in \Omega(X)$ implies $\alpha^*\omega \in \Omega(\alpha)$. It is easy to check that $d \circ d = 0$. We denote by $\mathcal{H}^r(X, \alpha)$ the cohomology groups of $(\Omega^\bullet(X, \alpha), d)$. A cohomology class is represented by a closed form ω in X which becomes exact when pulled back to U.

There is the short exact sequence of complexes

$$0 \longrightarrow \Omega^{r-1}(\alpha) \xrightarrow{i} \Omega^r(X, \alpha) \xrightarrow{\pi} \Omega^r(X) \longrightarrow 0, \tag{5.1}$$

where

$$i(\theta) = (0, \theta), \quad \pi(\omega, \theta) = \omega,$$

and the associated long exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^{r-1}(\alpha) \xrightarrow{i^*} \mathcal{H}^r(X, \alpha) \xrightarrow{\pi^*} \mathrm{H}^r(X) \xrightarrow{\alpha^*} \mathrm{H}^r(\alpha) \longrightarrow \cdots \tag{5.2}$$

Notice that i anti-commutes with the derivatives. It is easy to check that the connecting morphism in (5.2) is just α^*.

Remark 5.2. For a differentiable manifold X and an open subet $i_U\colon U \hookrightarrow X$, the complexes $\Omega(X, U)$ and $\Omega(X, i_U)$ are not isomorphic but have isomorphic cohomology groups, by a partition of unity argument. As far as we now, the diffeological De Rham complexes $\Omega(X, \alpha)$ and $\Omega(X, \alpha)$ are not related in this way.

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5.4 Relationship between the two type of relative cohomologies

Let $\alpha\colon U \to X$ be a smooth map between diffeological spaces. Let $\alpha^*\colon \Omega(X) \to \Omega(U)$ be the induced map in De Rham complexes.

Definition 5.3. The "relative cohomology" $\mathrm{H}(X,\alpha)$ of the first type is the cohomology of the complex $\Omega(X,\alpha) = \mathrm{ker}\alpha^*$.

Note that if $\alpha^*\omega = 0$, then $\alpha^* d\omega = d\alpha^*\omega = 0$, hence $\mathrm{ker}\alpha^*$ is a subcomplex of $\Omega(X)$; hence we have an exact sequence

$$0 \longrightarrow \Omega(X,\alpha) \longrightarrow \Omega(X) \longrightarrow \Omega(U). \tag{5.3}$$

Unfortunately the map α^* is not surjective in general. The cohomology of second type is defined as the cohomology of the complex.

$$\Omega^r(X,\alpha) = \Omega^r(X) \bigoplus \Omega^{r-1}(\alpha)$$

where $\Omega(\alpha)$ is the complex of α-horizontal forms. The differential is given as

$$d(\omega,\theta) = (d\omega, \alpha^*\omega - d\theta)$$

.

Proposition 5.4. *We have a morphism of complexes $F\colon \Omega(X,\alpha) \to \underline{\Omega}(X,\alpha)$ where $F(\omega) = (\omega, 0)$.*

Proof. We have $dF(\omega) = d(\omega,0) = (d\omega, \alpha^*\omega) = (d\omega, 0)$ because $\omega \in ker\alpha^*$ hence $dF = Fd$. □

Let $F^*\colon \mathrm{H}(X,\alpha) \to \mathcal{H}(X,\alpha)$ be the morphism induced in cohomology, that is $F^*([\omega]) = [(\omega,0)]$, $d\omega = 0$ and $\alpha^*\omega = 0$.

Proposition 5.5. *If α^* is surjective then F^* is an isomorphism.*

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Proof. Firstly we show that F^* is injective.

Let $F^*([\omega]) = 0$ that is $[(\omega, 0)] = 0$, hence if

$$(\omega, 0) = d(\omega_1, \theta_1) = (d\omega_1, \alpha^*\omega_1 - d\theta_1),$$

then $\omega = d\omega_1$. The problem is that ω_1 is not α-horizontal. Let $\tilde{\theta_1} \in \Omega(X)$ such that $\alpha^*\tilde{\theta_1} = \theta_1$. Then $\omega = d(\omega_1 - d\tilde{\theta_1})$; but

$$\alpha^*(\omega_1 - d\tilde{\theta_1}) = \alpha^*\omega_1 - \alpha^* d\tilde{\theta_1} = \alpha^*\omega_1 - d\alpha^*\tilde{\theta_1} = \alpha^*(\omega_1 - d\theta_1) = 0$$

because $\alpha^*\omega_1 = d\theta_1$. Hence $\omega = d(\alpha - \text{horizontal form})$, so $[\omega] = 0$.

Now, we show that F^* is surjective.

Let $[(\omega, \theta)] \in \mathcal{H}(X, \alpha)$ that is $d\omega = 0$, $\alpha^*\omega = d\theta$. Let $\tilde{\theta} \in \Omega(X)$ such that $\alpha^*\tilde{\theta} = \theta$. Then

$$(\omega - d\tilde{\theta}, 0) + d(\tilde{\theta}, 0) = (\omega - d\tilde{\theta}, 0) + (\mathrm{d}\tilde{\theta}, \alpha^*\tilde{\theta}) = (\omega, \theta),$$

hence

$$[(\omega, \theta)] = [(\omega - d\tilde{\theta}, 0)] = F^*([\omega - d\tilde{\theta}]).$$

Notice that

$$\alpha^*(\omega - d\tilde{\theta}) = \alpha^*\omega - \alpha^* d\tilde{\theta} = \alpha^*\omega - d\theta = 0.$$

□

5.5 Examples

Example 5.6 (The real line)**.** The cohomology of the real line $\mathbb{R}$, with its diffeology as a manifold, is the usual De Rham cohomology ([18]), that is, $\mathrm{H}^0(X) = 0$, $\mathrm{H}^1(X) = 0$.

Example 5.7 (The irrational torus)**.** Let a be an irrational number. The irrational torus $X = \mathbb{T}_a$ ([8], cf. [20, Exercise 1.4]) is the quotient of $U = \mathbb{R}$ by the equivalence relation $s \sim t$ iff $s - t = m + na$ for some integers $m, n \in \mathbb{Z}$.

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This diffeological space X is the space of leaves of the flow $\mathcal{F}$ on the torus $\mathbb{T}^2$ by lines of constant slope a. This is an example of a foliation with dense leaves, which is defined by the closed form $\mathrm{d}y - a\mathrm{d}x$.

It is well known that the *basic cohomology* $\mathrm{H}(\mathbb{T}^2/\mathcal{F})$ ([9, Prop. II.1]) of the foliation $\mathcal{F}$ is the cohomology of the abelian Lie algebra $\mathbb{R}$, that is, $\mathrm{H}^0 = \langle 1 \rangle$, $\mathrm{H}^1 = 0$.

On the other hand it has been proved that the basic cohomology $\mathrm{H}(M/\mathcal{F})$ of any foliation equals the De Rham cohomology of its space of leaves $M/\mathcal{F}$ as a diffeological space ([18]), hence $\mathrm{H}(\mathbb{T}_a) \cong \mathrm{H}(\mathbb{T}^2/\mathcal{F})$.

Example 5.8 (Subductions)**.** Let $\pi\colon U \to X$ be a subduction. We know that

1. The induced morphism $\pi^*\colon \Omega(X) \to \Omega(U)$ is injective (Theorem 4.7).
2. $\pi^*\colon \Omega(X) \cong \Omega(\pi)$ is an isomorphism with the complex of π-horizontal forms (Corollary 4.8).

Then the first type relative cohomology group is $\mathrm{H}(X, \pi) = 0$ because $\ker \pi^* = 0$.

On the other hand, the second type complex is isomorphic by π^* with the complex

$$\cdots \to \Omega^r(\pi) \oplus \Omega^{r-1}(\pi) \xrightarrow{\mathrm{d}} \Omega^{r+1}(\pi) \oplus \Omega^r(\pi) \to \cdots \tag{5.4}$$

with differential

$$\mathrm{d}(\omega, \mu) = (\mathrm{d}\omega, \omega - \mathrm{d}\mu).$$

Example 5.9 (A subduction map)**.** Let $X = \mathbb{T}_a$ be the irrational torus of Example 5.7 and let $\pi\colon \mathbb{R} \to X$ be the subduction map for the equivalence relation defining X. Since π is a subduction, the forms in $\Omega(X)$ are the π-horizontal forms in $M = \mathbb{R}$.

First, the horizontal 0-forms are the constant functions. In fact, let f be a 0-form, that is, a real function. We consider the maps $h, h'\colon U = \mathbb{R} \to \mathbb{R}$ given by

$$h(t) = t, \quad h'(t) = t + m + n\alpha.$$

Clearly $\pi \circ h = \pi \circ h'$. The condition $h^* f = (h')^* f$ means $f \circ h = f \circ h'$, that is,

$$f(t) = f(t + m + n\alpha) \quad \forall t.$$

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Since $m, n \in \mathbb{Z}$ are arbitrary and the set

$$\Delta(\alpha) = \{m + n\alpha \colon m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$, we obtain that f is a constant function.

Analogously, the π-horizontal 1-forms are $\omega = c\,dt$ where $c \in \mathbb{R}$ is a constant.

Then we can write $\Omega^0(\pi) = \langle 1 \rangle_{\mathbb{R}} \cong \mathbb{R}$ and $\Omega^1(\pi) = \langle dt \rangle_{\mathbb{R}} \cong \mathbb{R}$, and the differential $d \colon \Omega^0 \to \Omega^1$ is $d = 0$.

We can now compute the two types of relative cohomology groups for π.

The first type relative groups are $\mathrm{H}(X, \pi) = 0$ because they are the cohomology of $\ker \pi^* = 0$.

For the second type groups $\mathcal{H}(\pi)$ we consider the complex (5.4):

$$\cdots \to \Omega^0(\pi) \oplus 0 \xrightarrow{\mathrm{d}_0} \Omega^1(\pi) \oplus \Omega^0(\pi) \xrightarrow{\mathrm{d}_1} 0 \oplus \Omega^1(\pi) \to \cdots$$

with differential

$$\mathrm{d}_0(c, 0) = (0, c), \quad \mathrm{d}_1(a\,dt, b) = (0, a\,dt).$$

Then

$$\mathcal{H}^0(X, \pi) = \ker \mathrm{d}_0 = 0$$

and

$$\mathcal{H}^1(X, \pi) = \ker \mathrm{d}_1 / \operatorname{im} \mathrm{d}_0 = 0.$$

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Chapter 6

Cohomology and generating families

6.1 Introduction

This chapter is devoted to cohomological properties of generating families. This is again an indication that generating families play the role of open coverings in the diffeological setting, as we pointed out in [32].

The results of this chapter were published in our paper [33].

Definition 6.1 ([20, Art. 1.66])**.** A collection $\mathcal{F} = \{(\alpha_\lambda, U_\lambda)\}_{\lambda\in\Lambda}$ of plots is a *generating family* of the diffeology $\mathcal{D}$ if any plot of $\mathcal{D}$ locally factors through some element of the family $\mathcal{F}$.

Lemma 6.2. *If $\{(\alpha_1, U_1), \ldots, (\alpha_n, U_n)\}$ is a generating family of the diffeology $\mathcal{D}$ on X, then the universal map*

$$\alpha = \alpha_1 * \cdots * \alpha_n \colon U = U_1 * \cdots * U_n \to X$$

is a subduction.

Proof. Let $\gamma \colon W \to X$ be a plot on X, and let $p \in W$. Then, by definition of generating family, there is a neighbourhood W_p and a smooth change of coordinates $h \colon W_p \to U_k$, for some $k \in \{1, \ldots, n\}$, such that $\gamma = \alpha_k \circ h$ on W_p. Then $\gamma_{|W_p}$ factors through the plot $\alpha \circ \mathcal{C} \circ i_k \colon U_k \to U$. □

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Theorem 6.3. *Let $(X, \mathcal{D})$ be a diffeological space. A family $\mathcal{F} = \{\alpha_i \colon U_i \to X\}$ of plots is a generating family for $\mathcal{D}$ if and only if there is a diffeomorphism $\alpha = \alpha_1 * \cdots * \alpha_n \colon U = U_1 * \cdots * U_n \to X$ commuting with the maps α_i and the natural maps $j_i \colon U_i \to U$, that is, $\alpha \circ j_i = \alpha_i$ for all i.*

Proof. Let $\mathcal{F}$ be a generating family. The map α is surjective because generating families include all constant plots.

It is injective because $\alpha([u_i]) = \alpha([u_j])$ means that either $u_i = u_j$ or $u_i \in U_i$, $u_j \in U_j$ and $\alpha_i(u_i) = \alpha_j(u_j)$, that is, $[u_i] = [u_j]$. Then, α is bijective and Corollary 1.20 applies. Hence, α is a diffeomorphism.

The converse statement follows from Lemma 6.2. □

Example 6.4. Let X be the disjoint union of a line $X_1 = \mathbb{R}$ and a point $X_0 = \{0\}$, endowed with the diffeology generated by the constant plot $\alpha_0 \colon \mathbb{R}^0 \to X$, $\alpha_0(0) = 0$, and the identity plot $\alpha_1 \colon \mathbb{R}^1 \to X_1 \subset X$, $\alpha_1(t) = t$, respectively. Clearly, $X \cong U_0 * U_1 = U_0 \sqcup U_1$.

Example 6.5. Let X be the set

$$X = \{(x, y) \in \mathbb{R}^2 \colon xy = 0\}.$$

We consider the diffeology on X generated by the two plots

$$\alpha_1 \colon U_1 = \mathbb{R} \to X, \quad \alpha_1(t) = (t, 0),$$
$$\alpha_2 \colon U_2 = \mathbb{R} \to X, \quad \alpha_2(t) = (0, t).$$

Then, $X \cong U_1 * U_2$.

As we pointed out in [32], this is not the *cross* diffeology on X induced by the manifold diffeology of $\mathbb{R}^2$.

Example 6.6. The manifold diffeology on a manifold M is the join of any atlas. More precisely, let $\{\alpha_i \subset \mathbb{R}^m \colon U_i \to M\}$ be an atlas on the manifold M^m. Then

$$\bigcup_i \alpha_i(U_i) = M \cong *_i U_i,$$

where the open subsets $U_i \cong \alpha(U_i) \subset M$ are glued by inclusions.

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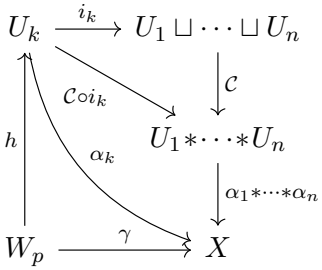
Theorem 6.7. *If $\alpha_1, \dots, \alpha_n$ is a generating family of $\mathcal{D}$, then*

$$\mathrm{H}(X, \alpha_1 * \cdots * \alpha_n) = 0.$$

Proof. Let $[\omega] \in \mathrm{H}^r(X, \alpha_1 * \cdots * \alpha_n)$ be a relative cohomology class, that is, $\omega \in \Omega^r(X)$ is a differential form such that $\mathrm{d}_X\omega = 0$ and $(\alpha_1 * \cdots * \alpha_n)^*\omega = 0$. We need to prove that $\omega = 0$, that is, $\omega_\gamma = 0$ for any plot (W, γ) on X. Since $W \subset \mathbb{R}^n$ is an Euclidean domain, we can proceed locally. Let $p \in W$. By definition of generating family, there is a neighbourhood $W_p \subset W$ such that γ factors though some α_k, $1 \leq k \leq n$, say

$$\gamma_{|W_p} = \alpha_k \circ h,$$

for some $\mathcal{C}^\infty$-map $h\colon W_p \to U_k$ (see Diagram (6.1))



(6.1)

So, in a neighbourhood of $p \in W$, we have

$$\gamma = (\alpha_1 * \cdots * \alpha_n) \circ \mathcal{C} \circ i_k$$

hence

$$\omega_\gamma = \gamma^*\omega = (\mathcal{C} \circ i_k)^*(\alpha_1 * \cdots * \alpha_n)^*\omega = 0. \quad \square$$

Remark 6.8. The two latter results easily extend to any infinite generating family.

We show in the next results how the cohomology of X is determined by any generating family.

Lemma 6.9. *Let $\{(\alpha_\lambda, U_\lambda)\}_{\lambda \in \Lambda}$ be a generating family of plots for the diffeological space X. If $\omega \in \Omega(X)$ is a form such that $\alpha_\lambda^*\omega = 0$ for all $\lambda \in \Lambda$, then $\omega = 0$.*

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Proof. Any plot $\gamma\colon W \to X$ locally factors through some α_λ, say $\gamma = \alpha_\lambda \circ h$ for some change of coordinates $h\colon W \to U_\lambda$. Then, locally,

$$\omega_\gamma = \gamma^*\omega = h^*\alpha_\lambda^*\omega = h^*0 = 0. \quad \square$$

Proposition 6.10. *Let $\{(\alpha_\lambda, U_\lambda)\}_{\lambda\in\Lambda}$ be a generating familiy of the diffeological space X. Let P be the pullback of those plots, with projections $p_\lambda\colon P \to U_\lambda$ as in Diagram (2.5). If $\theta_\lambda \in \Omega(U_\lambda)$ are differential forms of the same degree r, such that $p_\lambda^*\theta_\lambda = p_\mu^*\theta_\mu$ for all $\lambda, \mu \in \Lambda$, then there is an unique form $\theta \in \Omega(X)$ such that $\alpha_\lambda^*\theta = \theta_\lambda$ for any $\lambda \in \Lambda$.*

In other words, any form ω in X is completely determined by the pull-backs $\alpha_\lambda^*\omega$.

Proof. The form θ will be defined by giving its value θ_γ for any plot $\gamma\colon W \to X$. Let $p \in W$, then γ locally factors through some $\alpha_\lambda\colon U_\lambda \to X$, say $\gamma_{|W_p} = \alpha_\lambda \circ h_\lambda$ for some $h_\lambda\colon W_p \to U_\lambda$. Then we define $\theta_\gamma = h_\lambda^*\theta_\lambda$ on W_p.

To prove that it is well defined, assume that there are $\lambda, \mu \in \Lambda$ such that

$$\alpha_\lambda \circ h_\lambda = \gamma_{|W_p} = \alpha_\mu \circ h_\mu.$$

Then there exists $F\colon W_p \to P_{\lambda\mu}$ such that $p'_\lambda \circ F = h_\lambda$ and $p'_\mu \circ F = h_\mu$, where $P_{\lambda\mu}$ is the pull-back of α_λ and α_μ. By Lemma 2.5 there exists $F_{\lambda\mu}\colon P \to P_{\lambda\mu}$ such that $p_\lambda = p'_\lambda \circ F_{\lambda\mu}$ and $p_\mu = p'_\mu \circ F_{\lambda\mu}$. By hypothesis, we have $p_\lambda^*\theta_\lambda = p_\mu^*\theta_\mu$, hence

$$F_{\lambda\mu}^*(p'_\lambda)^*\theta_\lambda = F_{\lambda\mu}^*(p'_\mu)^*\theta_\mu.$$

But $F_{\lambda\mu}$ is a subduction map, so the induced morphism $F_{\lambda\mu}^*$ is injective, by Theorem 4.7. That means that $(p'_\lambda)^*\theta_\lambda = (p'_\mu)^*\theta_\mu$, so

$$\begin{aligned} h_\lambda^*\theta_\lambda &= (p'_\lambda \circ F)^*\theta_\lambda = F^*(p'_\lambda)^*\theta_\lambda \\ &= F^*(p'_\mu)^*\theta_\mu = (p'_\mu \circ F)^*\theta_\mu = h_\mu^*\theta_\mu. \end{aligned}$$ $\square$

Corollary 6.11. *For the disjoint union $U \sqcup V$, a form $\theta \in \Omega(U \sqcup V)$ is completely determined by giving the forms $i_U^*\theta \in \Omega(U)$ and $i_V^*\theta \in \Omega(V)$.*

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Proof. The pullback of i_U and i_V is $P = \emptyset$. □

Lemma 6.12. *Let $\{(\alpha_\lambda, U_\lambda)\}_{\lambda\in\Lambda}$ be a generating family of plots for the diffeology $\mathcal{D}$ on X. Denote $U = *_{\lambda\in\Lambda}U_\lambda$ and $\alpha = *_{\lambda\in\Lambda}\alpha_\lambda \colon U \to X$. If $h, h' \colon W \to U$ are plots such that $\alpha \circ h = \alpha \circ h'$, then $h = h'$.*

As a consequence, every form in U is α-horizontal, that is, $\Omega(\alpha) = \Omega(U)$.

Proof. Let $w \in W$. If, $h(w) = [u_\lambda]$ for some $\lambda \in \Lambda$, then $(\alpha \circ h)(w) = \alpha_\lambda(u_\lambda)$. Analogously, $(\alpha \circ h')(w) = \alpha_\mu(u_\mu)$ for some μ. By hypothesis, $u_\lambda \sim u_\mu$, that is, $[u_\lambda] = [u_\mu]$, hence $h(w) = h'(w)$. □

Proposition 6.13. *Let $G \colon A \to B$ be a subduction. If $\{\alpha_i\}_{i\in\mathcal{J}}$ is a generating family for the diffeology in A then $\{G \circ \alpha_i\}_{i\in\mathcal{J}}$ is a generating family for the diffeology in B.*

Proof. Let $\beta \colon W \to B$ be a plot in B, if $p \in W$ there exist $\gamma \colon W_p \to A$ such that $G \circ \gamma = \beta$. But since $\{\alpha_i\}_{i\in\mathcal{J}}$ generates the diffeology in A, there exist $\gamma' \colon W'_p \to U_i$ where $W'_P \subset W_P \subset W$ for some plot (α_j, U_j) such that $\gamma = \alpha_j \circ \gamma'$. then $\beta = G \circ \gamma = G\alpha_j \circ \gamma'$ on W'_p.

This proves that $\{G \circ \alpha_i\}_{i\in\mathcal{J}}$ is a generating family. □

Corollary 6.14. *Let $U \sqcup V$ be the disjoint union of the domains $\alpha \colon U \to X$, $\beta \colon V \to X$ of two plots, then $i_U \colon U \to U \sqcup V$, $i_V \colon U \to U \sqcup V$ are a generating family. Let $G \colon U \sqcup V \to U * V$ be the quotient map, then $J_U \colon U \to U * V$, $J_V \colon V \to U * V$ are generating families (with $J_U = G \circ\ i_U$, $J_V = G \circ\ i_V$).*

Let $\alpha_1 \colon U_1 \to X$, $\alpha_2 \colon U_2 \to X$ be a generating family for diffeological space on X, If $\omega \in \Omega^r(X)$ is a differential form on X, We have the forms $\alpha_1^*\omega \in \Omega^r(U_1)$, $\alpha_2^*\omega \in \Omega^r(U_2)$, they verify that the restriction to the Pull-back in the same:

$p_1^*(\alpha_1^*\omega) = (\alpha_1 p_1)^*\omega = (\alpha_2 p_2)^*\omega = p_2^*(\alpha_2^*\omega)$.

Proposition 6.15. *let α_1, α_2 be a generating family on X, Let $\omega_1 \in \Omega^r(U_1)$, $\omega_2 \in \Omega^r(U_2)$ are two r-forms such that $p_1^*\omega_1 = p_2^*\omega_2$ on the Pull-back P, Then exists an unique form $\omega \in \Omega^r(X)$ such that $\omega_1 = \alpha_1^*\omega$ and $\omega_2 = \alpha_2^*\omega$.*

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Proof. In order to define $\omega \in \Omega(x)$, we need to define $\gamma^*\omega = \omega_\gamma$ for any plot $\gamma\colon W \to X$. Assume that $p \in W_p$ and let γ factors through α_1, that is, there exists $\gamma_1\colon W_p \to U_1$ with $\gamma = \alpha_1\gamma_1$ on W_p. Then we define $\omega_\gamma = \gamma_1^*\alpha_1^*\omega = \gamma_1^*\omega_1$.

We need to prove that this form is well defined:

Assume thatγ factors through α_1 and α_2, then there exists ω_p'' such that $\gamma = \alpha_1\gamma_1 = \alpha_2\gamma_2$ on W_p'' . By definition of pull-back, there exists $F\colon W_p'' \to U_1$ such that $p_1F = \gamma_1$, $p_2F = \gamma_2$, then $\gamma_1^*\omega_1 = F^*(p_1^*\omega_1) = F^*(p_2^*\omega_2) = \gamma_2^*\omega_1$.

Finally, we check that we have defined two forms:

If $h\colon W' \to W$ is a change of coordinates in W_p' (by community) h_γ factors through the same α_i thatγ, hence $\omega_h\gamma = (\alpha_i h)^*\omega_i = h^*(\alpha_i^*\omega_i) = h^*\omega_\gamma$. □

Corollary 6.16. *For the disjoint union , the pull-back of the plots $i_U\colon U \to U \sqcup V$, $i_V\colon U \to U \sqcup V$ is the empty set, then $\Omega(U \sqcup V) = \Omega(U) \bigoplus \Omega(V)$. That is, a form on $U \sqcup V$ is an arbitrary pair of forms $\omega_1 \in \Omega(U), \omega_2 \in \Omega(V)$.*

By applying Corollary 4.8 and Theorem 6.2, we have

Theorem 6.17. *If $\{(\alpha_1, U_1), \ldots, (\alpha_n, U_n)\}$ is a generating family of the diffeology $\mathcal{D}$ on X, then*

$$\mathrm{H}(X, \mathcal{D}) \cong \mathrm{H}(U_1 * \cdots * U_n).$$

Clearly this result and the next one extend to the infinite case.

The following result is the second version of Theorem 6.7.

Corollary 6.18. *If $\alpha_1, \ldots, \alpha_n$ is a generating family of $\mathcal{D}$, then*

$$\mathcal{H}(X, \alpha_1 * \cdots * \alpha_n) = 0.$$

Proof. Let $\alpha = \alpha_1 * \cdots * \alpha_n$. It follows from Theorem 6.17 and the long exact sequence (5.2), that $0 = \ker \alpha^* = \operatorname{im} \pi^*$. Also, $i^* = 0$ because $\operatorname{im} \alpha^* = \mathrm{H}(\alpha) = \ker i^*$. Hence, $\mathcal{H}(X, \alpha) = 0$. □

Actually we have proved

Theorem 6.19. *If π is a subduction then $\mathrm{H}(X, \pi) = 0 = \mathcal{H}(X, \pi)$.*

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Proof. We give a direct proof. Since $\pi^*\colon \Omega(X) \cong \Omega(\alpha)$ is an isomorphism (Theorem 4.7), we have $\Omega(X,\pi) = \ker \pi^* = 0$, hence $\mathrm{H}(X,\pi) = 0$.

Denote $C^r = \Omega^r(X,\pi)$ and let

$$h^r\colon C^r \to C^{r-1}, \quad h(\omega,\theta) = ((\pi^*)^{-1}\theta, 0).$$

It holds

$$d\circ h + h\circ d = \mathrm{id}_C,$$

hence id_C is homotopic to the zero map, that is, $\mathcal{H}(X,\pi) = 0$. □

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Chapter 7

Mayer-Vietoris Sequence

7.1 Introduction

In the classical De Rham theory, the Mayer-Vietoris sequence is considered the main basic tool for computing the cohomology groups of a manifold that can be written as the union of two open subspaces. In diffeological spaces, versions of this Mayer-Vietoris result have been given. For instance, Iwase and Izumida [26] introduced a new version of differential forms (cubical differential forms) in order to obtain partitions of unity and an exact Mayer-Vietoris sequence. Also, Haraguchi [16] considered the Mayer-Vietoris sequence for general differential forms and gave sufficient conditions for the existence of partitions of unity. In [29], Kuribayashi developed cohomological methods in diffeological spaces and used a Mayer-Vietoris argument for different cochain complexes.

In all these cases, the classical theory is imitated by considering two open sets of the so-called $\mathcal{D}$-topology. Those are the sets $W \subseteq X$ such that $\alpha^{-1}(W)$ is open in U for any plot $\alpha\colon U \to X$ of the diffeology. However, it is our opinion that introducing the $\mathcal{D}$-topology fits poorly with the general philosophy of diffeological spaces, where topology is often irrelevant and differential objects are defined intrinsically: we can give a space a differentiable structure without first giving it a topology. See [40] for a discussion on this issue. In other words, smoothness is a property that is not based on continuity: for instance, the $\mathcal{D}$-topology is determined by the smooth curves, while diffeologies are not [6, Theorem 3.7].

There is another notion more intimately linked to the foundations of the theory, namely that of a *generating family* (1.5.1). Many diffeologies are constructed by only

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giving a generating family of plots, often much smaller than the whole diffeology. This mode of construction of diffeologies is very useful because it reduces the study of a diffeological space to a subset of its plots.

In section 7.2 we recall Mayer-Vietoris Sequence in a manifold. Section 7.3 is devoted to introducing Mayer-Vietoris sequence for finite generating families in diffeological spaces. In section 7.5 we will study generalization to a countable generating family. Section 7.6 is dedicated to an improved Mayer-Vietoris sequence.

Finally, in the last section we will have split Mayer-Vietoris sequences.

The results of this chapter were published in our paper [32]

7.2 Mayer-Vietoris Sequence in manifolds

The classical Mayer-Vietoris sequence allows one to compute the cohomology of the union of two open sets. Suppose $M = U \cup V$ with U, V open. Then there is a sequence of inclusions

$$M \leftarrow U \sqcup V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

where $U \sqcup V$ is the disjoint of U and V and ∂_0 and ∂_1 are the inclusions of $U \cap V$ in V and in U respectively. Applying the contravariant functor Ω^*, we get a sequence of restrictions of forms

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \Omega^*(U \cap V)$$

where by the restriction of a form to a submanifold we mean its pullback by the inclusion. By taking the difference of the last two maps, we obtain the Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

Proposition 7.1. *Mayer-Vietoris sequence is exact.*

Proof. Let ω be a form on $U \cap V$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$.

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Note that $\rho_V\omega$ is a form on U, that is, to get a *differentiable* form on an open set we must multiply by the partition function of the other open set. Since

$$(\rho_U\omega) - (-\rho_V\omega) = \omega$$

we see that for a general manifold M, if $\omega \in \Omega(U \cap V)$, then $(-\rho_V\omega, \rho_U\omega)$ in $\Omega(U) \oplus \Omega^k(V)$ maps onto ω. □

7.3 Mayer-Vietoris sequence for generating families in diffeological spaces

We shall consider a generating family of $(X, \mathcal{D})$ formed by *two* plots (U, α) and (V, β). We want to understand to what extent the cohomology $\mathrm{H}(X, \mathcal{D})$ is determined by these two plots.

7.3.1 Restriction and difference morphisms

Let (P, p_U, p_V) be the pullback defined in Section 2.4 and Proposition 2.1:

$$\begin{array}{ccc} P & \xrightarrow{p_V} & V \\ \downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle \beta} \\ U & \xrightarrow{\alpha} & X \end{array} \tag{7.1}$$

We have the sequence of complexes

$$0 \to \Omega^k(X) \xrightarrow{r} \Omega^k(\alpha) \oplus \Omega^k(\beta) \xrightarrow{\delta} \Omega^k(P) \tag{7.2}$$

where we call r a *restriction morphism* and δ the *difference morphism*. They are defined by

$$r(\omega) = (\alpha^*\omega, \beta^*\omega)$$

and

$$\delta(\mu, \nu) = p_U^*\mu - p_V^*\nu.$$

The complexes $\Omega(\alpha)$ and $\Omega(\beta)$ of horizontal forms were considered in Section 4.2.

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Lemma 7.2. *r and δ are well defined morphisms of complexes.*

Proof. If $\omega \in \Omega^k(X)$, and $h, h' \colon V \to U$ are two changes of coordinates such that $\alpha \circ h = \alpha \circ h'$, then

$$h^*(\alpha^*\omega) = (\alpha \circ h)^*\omega = (h')^*(\alpha^*\omega).$$

Then $\alpha^*\omega$ is a horizontal form. Analogously for β. This proves that r is well defined.

The morphism r commutes with the differentials, $(\mathrm{d}_U \oplus \mathrm{d}_V) \circ r = r \circ \mathrm{d}_X$, because

$$\begin{aligned}(\mathrm{d}_U \oplus \mathrm{d}_V) r(\omega) =& (\mathrm{d}_U \oplus \mathrm{d}_V)(\alpha^*\omega, \beta^*\omega) = (\mathrm{d}_U \alpha^*\omega, \mathrm{d}_V \beta^*\omega) \\ =& (\alpha^* \mathrm{d}_X \omega, \beta^* \mathrm{d}_X \omega) = r(\mathrm{d}_X(\omega)).\end{aligned}$$

That δ is well defined follows from the definition of the pull-back of forms (section 3.3). Finally, δ is a morphism of complexes because

$$\begin{aligned}(\mathrm{d}_P \circ \delta)(\mu, \nu) =& \mathrm{d}_P(p_U^*\mu - p_V^*\nu) = \mathrm{d}_P p_U^*\mu - \mathrm{d}_P p_V^*\nu \\ =& p_U^* \mathrm{d}_U \mu - p_V^* \mathrm{d}_V \nu = \delta(\mathrm{d}_U \mu, \mathrm{d}_V \nu).\end{aligned}$$

This proves $\mathrm{d}_P \circ \delta = \delta \circ (\mathrm{d}_U \oplus \mathrm{d}_V)$. □

Theorem 7.3. *The sequence* (7.2) *is exact.*

Proof. We shall make the proof in several steps.

Step 1. The morphism r is injective.

Let $\omega \in \Omega^k(X)$ such that $r(\omega) = 0$. We must show that $w_\gamma = 0$ for any plot $\gamma \colon W \to X$. Since $\{\alpha, \beta\}$ is a generating family, for each $p \in W$ there exists a neighborhood W of p such that either $\gamma_{|W}$ is constant, in which case γ factors through $c_0 \colon W \to \mathbb{R}^0 = \{*\}$ and we have $\omega_\gamma = c_0^* 0 = 0$ on W, or there is a change of coordinates $h \colon W \to U$ with $\gamma_{|W} = \alpha \circ h$ (analogously $\gamma_{|W} = \beta \circ h'$ for some $h' \colon W \to V$). Then, on W we have

$$\omega_\gamma = \gamma^*\omega = (\alpha \circ h)^*\omega = h^*\alpha^*\omega = 0.$$

This proves that $\omega = 0$.

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Step 2. $\operatorname{im} r \subseteq \ker \delta$.

We show $\delta \circ r = 0$:

$$p_U^* \alpha^* \omega - p_V^* \beta^* \omega = (\alpha \circ p_U)^* \omega - (\beta \circ p_V)^* \omega = 0,$$

because Diagram (7.1) is commutative.

Step 3. $\ker \delta \subseteq \operatorname{im} r$.

Let $(\mu, \nu) \in \Omega^k(\alpha) \oplus \Omega^k(\beta)$ such that $p_U^* \mu = p_V^* \nu$. We need to define a form $\omega \in \Omega^k(X)$ such that $\alpha^* \omega = \mu$ and $\beta^* \omega = \nu$. For that, we need to define $\omega_\gamma \in \Omega^k(W)$ for any plot (W, γ) on X.

Obviously, for $\gamma = \alpha$ we take $\omega_\alpha = \mu$. If $h \colon W \to U$ is a change of coordinates, we put $\omega_{\alpha \circ h} = h^* \mu$. This form does not depend on h due to the definition of horizontal form, that is, if $\alpha \circ h = \alpha \circ h'$ then $h^* \mu = (h')^* \mu$. Analogously, for $\gamma = \beta$ we take $\omega_\beta = \nu$, and for a change of coordinates $h' \colon W \to V$ we put $\omega_{\beta \circ h'} = (h')^* \nu$. It may happen that $\alpha \circ h = \beta \circ h'$, for some changes of coordinates $h \colon W \to U$ and $h' \colon W \to V$. In this case, by the pullback property of Proposition 2.1, there exists a differentiable map $F \colon W \to P$ such that $p_U \circ F = h$ and $p_V \circ F = h'$. Hence

$$(h')^* \nu = (p_V \circ F)^* \nu = F^* p_V^* \nu = F^* p_U^* \mu = (p_U \circ F)^* \mu = h^* \mu.$$

This proves that the preceding definitions are consistent.

Finally, let $\gamma \colon W \to X$ be an arbitrary plot. For each point $p \in W$ there is a neighbourhood W_p such that the restriction of γ to W_p either is constant, in which case we define $\omega_\gamma = 0$ on the neighbourhood W_p, or $\gamma_{|W_p} = \alpha \circ h$ for some $h \colon W_p \to U$, in which case we define $\omega_\gamma = h^* \omega_\alpha$, or $\gamma_{|W_p} = \beta \circ h'$, in which case we state ω_γ to be $(h')^* \omega_\beta$ on W_p.

The compatibility condition can be checked easily. This proves that we have a well defined k-form ω on X such that $r(\omega) = (\mu, \nu)$. □

The latter Theorem shows that the complex $\Omega(X)$ is isomorphic to the subcomplex $\ker \delta$ of $\Omega(\alpha) \oplus \Omega(\beta)$.

Note that we do not prove the surjectiveness of δ. This would require the existence of partitions of unity, a practically impossible objective in such a wide context.

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Example 7.4. In order to emphasize the analogy between a generating family and an open cover, let us consider $X = S^1$, with the usual diffeology as a differentiable manifold, and the open subsets $U = S^1 \setminus \{\mathrm{NP}\}$ and $V = S^1 \setminus \{\mathrm{SP}\}$, where NP (SP) stands for "North pole" (respectively "South pole"). Accordingly to Example 1.9, since $X = U \cup V$, the inclusions $\alpha = i_U \colon U \hookrightarrow X$ and $\beta = i_V \hookrightarrow X$ form a generating family. Since those plots are injective, the horizontal forms are $\Omega(\alpha) = \Omega(U)$ and $\Omega(\beta) = \Omega(V)$. Finally, the pull-back of i_U and i_V is the intersection $U \cap V$. Hence, one recovers the usual Mayer-Vietoris sequence.

7.4 An Example

We shall compute the De Rham cohomology of the following Example as an application of our Theorem 7.3.

Let X be the set obtained as the union of the two coordinate axis in $\mathbb{R}^2$, that is,

$$X = \{(x, y) \in \mathbb{R}^2 \colon xy = 0\}.$$

We take on X the diffeology $\mathcal{D}$ generated by the two axis inclusions, that is, by the plots

$$\alpha \colon U = \mathbb{R} \to X, \quad \alpha(s) = (s, 0),$$

and

$$\beta \colon V = \mathbb{R} \to X, \quad \beta(t) = (0, t).$$

The pullback of these two plots is $P = \{(0, 0)\}$, because $\alpha(s) = \beta(t)$ if and only if $s = t = 0$. Then $\Omega^0(P) = \mathbb{R}$ and $\Omega^K(P) = 0$ for $k \geq 1$.

Remark 7.5. Notice that this diffeology $\mathcal{D}$ is strictly smaller than the subspace diffeology $\mathcal{D}'$ of X as a subset of $\mathbb{R}^2$. In fact, the 1-plot $\alpha \colon \mathbb{R} \to X \subseteq \mathbb{R}^2$ given by the $\mathcal{C}^\infty$ map

$$\alpha(t) = \begin{cases} (e^{1/t}, 0) & \text{if } t < 0, \\ (0, 0) & \text{if } t = 0, \\ (0, e^{-1/t}) & \text{if } t > 0, \end{cases}$$

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does not belong to the diffeology, by Theorem 1.9. Hence the computation of the 1-differential forms for $(X, \mathcal{D}')$ (the so-called *cross*) in [24] does not apply here.

We compute the horizontal forms for $\mathcal{D}$: since $p_1\alpha = i\mathrm{d}_{\mathbb{R}}$, for the first projection $p_1 \colon \mathbb{R}^2 \to \mathbb{R}$, the condition $\alpha \circ h = \alpha \circ h'$ for $h, h' \colon W \to U = \mathbb{R}$ means in fact $h = h'$. Hence $\Omega(\alpha) = \Omega(U)$. Analogously $\Omega(\beta) = \Omega(V)$.

Now, we know that $\Omega^k(X) \cong \operatorname{im} r = \ker \delta \subseteq \Omega^k(\alpha) \oplus \Omega^k(\beta)$. For $k \geq 2$ that means $\Omega^k(X) = 0$. For $k = 0$ we have that $(F, G) \in \Omega^0(\mathbb{R}) \oplus \Omega^0(\mathbb{R})$ is a pair of functions $F, G \colon \mathbb{R} \to \mathbb{R}$, and being in the kernel of δ means $F(0) = G(0)$. For $k = 1$, we have a pair of 1-forms $(\mu, \nu) \in \Omega^1(\mathbb{R}) \oplus \Omega^1(\mathbb{R})$, say $\mu = f(t)\mathrm{d}t$ and $\nu = g(t)\mathrm{d}t$. The pair of functions (f, g) is arbitrary because $p_U^*\mu = 0 = p_V^*\nu$ in $\Omega^1(P) = 0$.

We consider the resulting De Rham complex of X, say,

$$0 \to \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \to 0 \to \cdots$$

which can be written as

$$0 \to \mathcal{C}^\infty(\mathbb{R}) \oplus_0 \mathcal{C}^\infty(\mathbb{R}) \xrightarrow{\mathrm{d}} \mathcal{C}^\infty(\mathbb{R}) \oplus \mathcal{C}^\infty(\mathbb{R}) \to 0 \to \cdots$$

where

$$\mathrm{d}(F, G) = (F', G').$$

The kernel of d is a pair of constant functions, which must be equal by the condition $F(0) = G(0)$, hence

$$\mathrm{H}^0(X, \mathcal{D}) = \mathbb{R}.$$

We compute the image of d. Since any $f \in \mathcal{C}^\infty(\mathbb{R})$ is the derivative of some F, unique up to a constant, we can always write $f = \mathrm{d}F$, $g = \mathrm{d}G$ and we can assume that $F(0) = G(0)$, by taking $G - G(0) + F(0)$. Hence d is surjective and

$$\mathrm{H}^1(X, \mathcal{D}) = 0.$$

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7.5 Generalization to a countable generating family

Instead of a generating family with only two plots, we shall now consider an ordered countable generating family $\{\alpha_1, \ldots, \alpha_n, \ldots\}$ of plots $\alpha_i \colon U_i \to X$. Of course the family may be finite.

7.5.1 Generalized Mayer-Vietoris sequence

We shall follow as closely as possible the notations in Bott-Tu's book [4, Definition 8.2] for the classical generalized Mayer-Vietoris sequence. Denote the finite limit $L(\alpha_1, \ldots, \alpha_n)$ by $L_{\alpha_1 \ldots \alpha_n}$. In particular, $L_{\alpha_i} = U_i$, the domain of α_i.

Then we have a sequence of differentiable maps

$$\cdots \quad \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \quad \bigsqcup_{i_1<i_2<i_3} L_{\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}} \quad \substack{\xrightarrow{\partial_1} \\ \xrightarrow{\partial_2} \\ \xrightarrow{\partial_3}} \quad \bigsqcup_{i_1<i_2} L_{\alpha_{i_1}\alpha_{i_2}} \quad \substack{\xrightarrow{\partial_1} \\ \xrightarrow{\partial_2}} \quad \bigsqcup_{\alpha_1} L_{\alpha_1} \xrightarrow{\partial_1} X$$

where ∂_k is the map which "ignores" α_{i_k}. For example, $\partial_1 \colon L_{\alpha_1\alpha_2\alpha_3} \to L_{\alpha_2\alpha_3}$ is the arrow given by the universal property of the limit $L_{\alpha_2\alpha_3}$ with respect to the arrows $L_{\alpha_1\alpha_2\alpha_3} \to U_i$, for $i = 2, 3$.

This sequence of maps induces a sequence of restriction morphisms between horizontal forms

$$\Omega(X) \xrightarrow{\delta_1} \bigoplus_{\alpha_1} \Omega(\alpha_1) \quad \substack{\xrightarrow{\delta_1} \\ \xrightarrow{\delta_2}} \quad \bigoplus_{i_1<i_2} \Omega(\alpha_{i_1}, \alpha_{i_2}) \quad \substack{\xrightarrow{\delta_1} \\ \xrightarrow{\delta_2} \\ \xrightarrow{\delta_3}} \quad \bigoplus_{i_1<i_2<i_3} \Omega(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) \quad \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

Clearly the morphisms are well defined. Now we define a "difference morphism"

$$\delta \colon \bigoplus \Omega(\alpha_{i_1}, \ldots, \alpha_{i_n}) \to \bigoplus \Omega(\alpha_{i_1}, \ldots, \alpha_{i_{n+1}})$$

whose components are the alternating sum

$$(\delta\omega)_{\alpha_1 \ldots \alpha_n} = \sum_{i=1}^{n+1} (-1)^{i+1} \omega_{\alpha_1 \ldots \hat{\alpha_i} \ldots \alpha_{n+1}},$$

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where we have suppressed the restriction operator on the right side. The hat denotes omission, as usual.

It is not hard to check that $\delta \circ \delta = 0$. Moreover we have the following generalization of Theorem 7.3.

Theorem 7.6. *The sequence*

$$\Omega(X) \xrightarrow{\delta} \bigoplus_{\alpha_1} \Omega(\alpha_{i_1}) \xrightarrow{\delta} \bigoplus_{i_1<i_2} \Omega(\alpha_{i_1}, \alpha_{i_2}) \xrightarrow{\delta} \bigoplus_{i_1<i_2<i_3} \Omega(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) \xrightarrow{\delta} \dots$$

is exact.

We omit the proof since it is completely analogous to the case $n = 2$.

7.6 Improved Mayer-Vietoris sequence

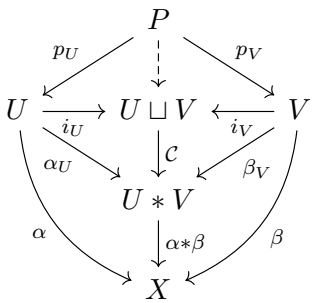
In order to obtain new results in 7.3, we can replace $\Omega(P)$ with $\Omega(p_X)$. In this Section, we will construct a Mayer-Vietoris sequence

$$0 \longrightarrow \Omega(\alpha * \beta) \xrightarrow{J} \Omega(\alpha) \oplus \Omega(\beta) \xrightarrow{\Pi} \Omega(p_X) \tag{7.3}$$

where $\Omega(p_X) \subset \Omega(P)$ is the space of p_X-horizontal forms, for

$$p_X = \alpha \circ p_U = \beta \circ p_V \colon P \to X$$

(see Diagram (7.4))



(7.4)

The morphisms J, Π in (7.3) are defined as

$$J(\rho) = (\alpha_U^* \rho, \beta_V^* \rho), \quad \Pi(\theta, \tau) = p_U^* \theta - p_V^* \tau.$$

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Proposition 7.7. *The sequence* (7.3) *is well defined and it is exact.*

Proof. Step 1: The morphism J is well defined: If ρ is $(\alpha * \beta)$-horizontal then $(\mathcal{C} \circ i_U)^* \rho$ is α-horizontal, as follows from Lemma 4.3. Analogously, $(\mathcal{C} \circ i_V)^* \rho$ is β-horizontal.

Step 2. That the morphism Π is well defined follows again from Lemma 4.3.

Step 3. It is an exercise to check that J and Π are morphisms of complexes (i.e. they commute with the differentials).

Step 4. That J is injective follows from Lemma 6.9, because $\{\mathcal{C} \circ i_U, \mathcal{C} \circ i_V\}$ is a generating family of plots for $U * V$.

Step 5. $\operatorname{im} J \subset \ker \Pi$ because $\Pi \circ J = 0$; on the other side, $\ker \Pi \subset \operatorname{im} J$ because given $(\theta, \tau) \in \Omega(\alpha) \oplus \Omega(\beta)$ such that $p_U^* \theta = p_V^* \tau$, there exists $\rho \in \Omega(U * V)$ such that $\alpha_U^* \rho = \theta$ and $\beta_V^* \rho = \tau$, by Lemma 6.10. Here we are using that $\{\mathcal{C} \circ i_U, \mathcal{C} \circ i_V\}$ is a generating family for $U * V$. Moreover, the form ρ is $(\alpha * \beta)$-horizontal. □

7.7 Split Mayer-Vietoris sequence

The formulae in this Section are inspired in those of Macías-Virgós in [31]. In the previous section we constructed a Mayer-Vietoris sequence

$$0 \longrightarrow \Omega^r(\alpha * \beta) \xrightarrow{J} \Omega^r(\alpha) \oplus \Omega^r(\beta) \xrightarrow{\Pi} \Omega^r(p_X) \tag{7.5}$$

where $\Omega(p_X) \subset \Omega(P)$ is the space of p_X-horizontal forms, for

$$p_X = \alpha \circ p_U = \beta \circ p_V \colon P \to X,$$

and

$$J(\rho) = (\alpha_U^* \rho, \beta_V^* \rho), \quad \Pi(\theta, \theta') = p_U^* \theta - p_V^* \theta'.$$

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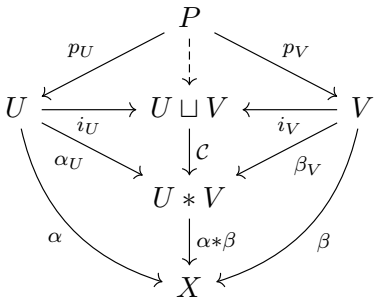
(see Diagram (7.6))



(7.6)

In general, the morphism Π will not be surjective.

Definition 7.8. When Π is surjective , we will say that the pair (α, β) is "SPLIT".

In this case we can take a (vector space) linear section S,

$$S\colon \Omega^r(p_X) \to \Omega^r(\alpha) \oplus \Omega^r(\beta)$$

where $\Pi \circ S = \mathrm{id}$.

In general the section S is not a morphism of complexes. Hence we consider $\Delta = \mathrm{d}S - S\mathrm{d}$,

$$\begin{array}{ccc} \Omega^r(\alpha) \oplus \Omega^r(\beta) & \xleftarrow{S} & \Omega^r(p_X) \\ \downarrow{\scriptstyle d} & \swarrow{\scriptstyle \Delta} & \downarrow{\scriptstyle d} \\ \Omega^{r+1}(\alpha) \oplus \Omega^{r+1}(\beta) & \xleftarrow[S]{} & \Omega^{r+1}(p_X) \end{array} \tag{7.7}$$

which is a morphism of complexes (up to sign)

$$\Delta\colon \Omega^r(p_X) \to \ \Omega^{r+1}(\alpha) \oplus \Omega^{r+1}(\beta)$$

because

$$\mathrm{d}\Delta = \mathrm{d}(\mathrm{d}S - S\mathrm{d}) = \mathrm{d}^2 - \mathrm{d}S\mathrm{d} = -\mathrm{d}S\mathrm{d}$$

$$\Delta\mathrm{d} = (\mathrm{d}S - S\mathrm{d})\mathrm{d} = \mathrm{d}S\mathrm{d} - S\mathrm{d}^2 = \mathrm{d}S\mathrm{d}.$$

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Then it passes to cohomology. But notice that $\Pi\Delta = 0$ because

$$\Pi(\mathrm{d}S - S\mathrm{d}) = \Pi\mathrm{d}S - \Pi S\mathrm{d} = \mathrm{d}\Pi S - \mathrm{d} = \mathrm{d} - \mathrm{d} = 0$$

since $\Pi S = \mathrm{id}$, hence $\mathrm{im}\Delta \subset \Omega^{r+1}(\alpha * \beta)$.

So it is more appropriate to write

$$J\Delta = \mathrm{d}S - S\mathrm{d}. \tag{7.8}$$

Proposition 7.9. *If Π is surjective, we have a long Mayer-Vietoris exact sequence in cohomology,*

$$\cdots \longrightarrow \mathrm{H}^r(\alpha * \beta) \longrightarrow \mathrm{H}^r(\alpha) \oplus \mathrm{H}^r(\beta) \longrightarrow \mathrm{H}^r(p_X) \xrightarrow{\Delta^*} \mathrm{H}^{r+1}(\alpha * \beta) \longrightarrow \cdots \tag{7.9}$$

where the connecting morphism is Δ^*.

Example 7.10. In a finite dimensional manifold M with the natural diffeology, the Mayer-Vietoris sequences for the inclusions of two open sets $U \subset M$ and $V \subset M$ are split. See proposiition 7.1.

Example 7.11. In [26], Iwase and Izumida considered a *nice covering* of a diffeological space as a pair U, V of open subsets of the $\mathcal{D}$-topology for which there exists a partition of unity ρ_U, ρ_V ([26, Definition 2.1]). Then the corresponding Mayer-Vietoris sequence is split ([26, Definition 2.1])

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Chapter 8

Cup product in relative cohomology

8.1 Introduction

In section 8.2 we introduce a relative cup product

$$\mathrm{H}^r(X,\alpha)\otimes \mathrm{H}^s(X,\beta)\to \mathrm{H}^{r+s}(X,\alpha*\beta)$$

by considering two arbitrary plots (α,U), (β,V) of the diffeology.

Also, in this chapter we define a relative cup product

$$\mathcal{H}^r(X,\alpha)\otimes \mathcal{H}^s(X,\beta)\to \mathcal{H}^{r+s}(X,\alpha*\beta)$$

and we check that it verifies the usual properties.

8.2 Relative cup product

We can define a structure of algebra in the De Rham complex $\Omega^*(X)=\oplus_{r\geq 0}\Omega^r(X)$ by means of the exterior product of forms. Since the exterior derivation d verifies

$$d(\omega\wedge\mu)=(d\omega)\wedge\mu+(-1)^r\omega\wedge(d\mu),\qquad \omega\in\Omega^r(X),\ \mu\in\Omega^s(X).$$

the exterior product induces a structure of algebra in the De Rham cohomology $H(X,\mathcal{D})$. The morphism

$$\smile\colon \mathrm{H}^{\mathrm{r}}(X)\otimes \mathrm{H}^{\mathrm{s}}(X)\to H^{r+s}(X),$$

$$[\omega]\smile[\mu]=[\omega\wedge\mu].$$

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is usually called the absolute cup product.

Our aim is to define a *relative cup product*

$$\smile\colon \mathrm{H}^r(X,\alpha)\otimes \mathrm{H}^s(X,\beta)\to \mathrm{H}^{r+s}(X,\alpha{*}\beta),$$

where (α,U), (β,V) are plots and $(\alpha{*}\beta, U{*}V)$ is the join.

Definition 8.1. The relative cup product is defined as

$$[\omega]\smile[\mu]=[\omega\wedge\mu].$$

Proposition 8.2. *The relative cup product is well defined.*

Proof. We have differential forms $\omega\in\Omega^r(X)$ with $\mathrm{d}_X\omega=0$ and $\alpha^*\omega=0$, analogously $\mu\in\Omega^s(X)$, with $\mathrm{d}_X\mu=0$ and $\beta^*\mu=0$.

Clearly $\omega\wedge\mu\in\Omega^{r+s}(X)$ and $\mathrm{d}_X(\omega\wedge\mu)=0$

Denote

$$\rho:=(\alpha{*}\beta)^*(\omega\wedge\mu)\in\Omega(U{*}V).$$

It only remains to show that $\rho=0$.

Let $\gamma\colon W\to U{*}V$ be a plot in $U{*}V$. We have to show that $(\rho_\gamma)_p=0$, for any $p\in W$. By definition of subduction diffeology, there is some neighbourhood $W_p\subset W$ such that $\gamma_{|W_p}$ factors through $U\sqcup V$, say

$$\gamma_{|W_p}=\mathcal{C}\circ\tilde{\gamma}$$

where $C\colon U\sqcup V\to U{*}V$ and $\tilde{\gamma}\colon W_p\to U\sqcup V$.

Assume for instance that $\tilde{\gamma}(p)\in U$. Then, by definition of coproduct diffeology, there is some neigbourhood $W_p'\subset W_p$ such that γ factors through U, that is (see Diagram (2.4)),

$$\gamma_{|W_p'}=\mathcal{C}\circ i_U\circ\bar{\gamma},$$

for some map $\bar{\gamma}\colon W_p'\to U$.

Then in a neighbourhood W_p' of $p\in W$ we have

$$(\alpha{*}\beta)\circ\gamma=(\alpha{*}\beta)\circ\mathcal{C}\circ i_u\circ\bar{\gamma}=\alpha\circ\bar{\gamma},$$

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hence

$$\begin{aligned}\rho_\gamma = \gamma^*\rho = \gamma^*(\alpha * \beta)^*(\omega \wedge \mu) = ((\alpha * \beta) \circ \gamma)^*(\omega \wedge \mu) \\ = (\alpha \circ \bar{\gamma})^*(\omega \wedge \mu) = (\bar{\gamma})^*\alpha^*(\omega \wedge \mu) = 0,\end{aligned}$$

because $\alpha^*(\omega \wedge \mu) = \alpha^*\omega \wedge \alpha^*\mu$ and $\alpha^*\omega = 0$.

The proof is analogous when $\tilde{\gamma}(p) \in V$. □

8.3 Cup product

If the Mayer-Vietoris sequence is split in Chapter 7, then we have the map

$$K = \mathrm{id} - S\Pi \colon\ \Omega^r(\alpha) \oplus \Omega^r(\beta) \to\ \Omega^r(\alpha) \oplus \Omega^r(\beta)$$

with

$$\Pi K = \Pi - \Pi S \Pi = \Pi - \Pi = 0$$

because $\Pi S = \mathrm{id}$. Hence

$$K \colon\ \Omega^r(\alpha) \oplus \Omega^r(\beta) \to \Omega^r(\alpha * \beta).$$

Then it is more appropriate to write

$$JK = \mathrm{id} - S\Pi. \tag{8.1}$$

Moreover

$$KJ = (\mathrm{id} - S\Pi)J = J - S\Pi J = J.$$

Finally K is not a morphism of complexes, in fact

$$\mathrm{d}K - K\mathrm{d} = -\Delta\Pi$$

because

$$\begin{aligned}\mathrm{d}K - K\mathrm{d} =& \mathrm{d}(\mathrm{id} - S\Pi) - (\mathrm{id} - S\Pi)\mathrm{d} \\ =& \mathrm{d} - \mathrm{d}S\Pi - \mathrm{d} + S\Pi\mathrm{d} \\ =& (-\mathrm{d}S + S\mathrm{d})\Pi \\ =& -\Delta\Pi.\end{aligned}$$

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Definition 8.3. We define a relative cup product,

$$\smile\colon \mathcal{H}^r(X,\alpha)\otimes\mathcal{H}^s(X,\beta)\to\mathcal{H}^{r+s}(X,\alpha*\beta),$$

for differentiable maps $\alpha\colon U\to X$ and $\beta\colon V\to X$, as

$$[(\omega,\theta)]\smile[(\omega',\theta')]=[(\omega\wedge\omega',\xi)], \tag{8.2}$$

where

$$\xi=K(\theta\wedge\alpha^*\omega',(-1)^\omega\beta^*\omega\wedge\theta')+(-1)^\omega\Delta(p_U^*\theta\wedge p_V^*\theta').$$

In what follows we will verify that this product is well defined.

Proposition 8.4. *The product in* (8.2) *is well defined.*

Proof. First, $\mathrm{d}(\omega\wedge\omega')=0$ because

$$\mathrm{d}(\omega\wedge\omega')=\mathrm{d}\omega\wedge\omega'+(-1)^r\omega\wedge\mathrm{d}\omega',$$

with $\mathrm{d}\omega=0$ and $\mathrm{d}\omega'=0$.

Now we will prove that $(\alpha*\beta)^*(\omega\wedge\omega')=\mathrm{d}\xi$.

We have

$$\alpha_U^*(\alpha*\beta)^*(\omega\wedge\omega')=\alpha^*(\omega\wedge\omega')=\alpha^*\omega\wedge\alpha^*\omega'=\mathrm{d}\theta\wedge\alpha^*\omega'=\mathrm{d}(\theta\wedge\alpha^*\omega')$$

and

$$\beta_V^*(\alpha*\beta)^*(\omega\wedge\omega')=\beta^*(\omega\wedge\omega')=\beta^*\omega\wedge\beta^*\omega'=\beta^*\omega\wedge\mathrm{d}\theta'=(-1)^\omega\mathrm{d}(\beta^*\omega\wedge\theta'),$$

because $\mathrm{d}\beta^*\omega=0$ and

$$\mathrm{d}(\beta^*\omega\wedge\theta')=(-1)^\omega\beta^*\omega\wedge\mathrm{d}\theta'.$$

Hence

$$J(\alpha*\beta)^*(\omega\wedge\omega')=\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^\omega\beta^*\omega\wedge\theta'). \tag{8.3}$$

Now we compute $J\mathrm{d}\xi$. We have

$$J\mathrm{d}\xi=\mathrm{d}J\xi=\mathrm{d}(JK(\theta\wedge\alpha^*\omega',(-1)^\omega\beta^*\omega\wedge\theta'))$$

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$$
\begin{aligned}
&+(-1)^{\omega}\mathrm{d}J\Delta(p_U^*\theta\wedge p_V^*\theta')\\
=&\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^{\omega}\beta^*\omega\wedge\theta')-\mathrm{d}S\Pi(\theta\wedge\alpha^*\omega',(-1)^{\omega}\beta^*\omega\wedge\theta')\\
&+(-1)^{\omega}\mathrm{d}(\mathrm{d}S-S\mathrm{d})(p_U^*\theta\wedge p_V^*\theta')\\
=&\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^{\omega}\beta^*\omega\wedge\theta')-\mathrm{d}S(p_U^*\theta\wedge p_U^*\alpha^*\omega'-(-1)^{\omega}p_V^*\beta^*\omega\wedge p_V^*\theta')\\
&-(-1)^{\omega}\mathrm{d}S(\mathrm{d}p_U^*\theta\wedge p_V^*\theta'+(-1)^{\theta}p_U^*\theta\wedge\mathrm{d}p_V^*\theta')\\
=&\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^{\omega}\beta^*\omega\wedge\theta')-\mathrm{d}S(p_U^*\theta\wedge p_U^*\alpha^*\omega'-(-1)^{\omega}p_V^*\beta^*\omega\wedge p_V^*\theta')\\
&-(-1)^{\omega}\mathrm{d}S(p_U^*\mathrm{d}\theta\wedge p_V^*\theta'+(-1)^{\theta}p_U^*\theta\wedge p_V^*\mathrm{d}\theta')\\
=&\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^{\omega}\wedge\theta')-\mathrm{d}S(p_U^*\theta\wedge p_U^*\alpha^*\omega'-(-1)^{\omega}p_V^*\beta^*\omega\wedge p_V^*\theta')\\
&-(-1)^{\omega}\mathrm{d}S(p_U^*\alpha^*\omega\wedge p_V^*\theta'+(-1)^{\theta}p_U^*\theta\wedge p_V^*\beta^*\omega').
\end{aligned}
$$

Now, since $\alpha\circ p_U=p_X=\beta\circ p_V$, it is $p_U^*\alpha^*=p_V^*\beta^*$, so we have

$$J\mathrm{d}\xi=\mathrm{d}(\theta\wedge\alpha^*\omega',(-1)^{\omega}\beta^*\omega\wedge\theta')$$

like in Equation (8.3) because the term in

$$p_U^*\theta\wedge p_U^*\alpha^*\omega'=p_U^*\theta\wedge p_V^*\beta^*\omega'=p_U^*\theta\wedge p_V^*\mathrm{d}\theta'$$

has coefficient

$$(-1)-(-1)^{\omega}(-1)^{\theta}=(-1)-(-1)^{\omega+\omega-1}=0,$$

while the term

$$p_V^*\beta^*\omega\wedge p_V^*\theta'=p_U^*\alpha^*\omega\wedge p_V^*\theta'=p_U^*\mathrm{d}\theta\wedge p_V^*\theta'$$

has coefficient

$$(-1)^{\omega}-(-1)^{\omega}=0.$$

□

Theorem 8.5. *The cup product is skew-symmetric, that is*

$$[(\omega',\theta')]\smile[(\omega,\theta)]=(-1)^{\omega\omega'}[(\omega,\theta)]\smile[(\omega',\theta')]. \tag{8.4}$$

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Proof. First we will carefully recall the notation we are using. Given two smooth maps $\alpha\colon U \to X$ and $\beta\colon V \to X$ we consider the pullback P of α and β and the map $p_X = \alpha p_U = \beta p_V$.

See the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{p_V} & V \\ {\scriptstyle p_U}\downarrow & & \downarrow{\scriptstyle \beta} \\ U & \xrightarrow{\alpha} & X. \end{array} \tag{8.5}$$

If we consider the pullback P' of plots β and α, there is an obvious isomorphism $\phi\colon P \cong P'$ induced by the *flip* map

$$\phi\colon U \times V \to V \times U, \quad \phi(u,v) = (v,u).$$

Notice that $p'_U = p_U \circ \phi, p'_V = p_V \circ \phi$ and $p'_X = p_X$. The induced isomorphism in cohomology

$$\phi^*\colon \Omega(p_X) \cong \Omega(p'_X)$$

sends $\Sigma\xi_i^U \otimes \xi_i^V$ into $\Sigma\xi_i^V \otimes \xi_i^U$. Then, in order to simplify notation, we will consider that $\phi^* = \mathrm{id}$.

Analogously, there is an obvious isomorphism between the joins $\alpha * \beta \cong \beta * \alpha$ induced by $U \sqcup V \cong V \sqcup U$. Again we shall consider that

$$\Omega(\alpha * \beta) \cong \Omega(\beta * \alpha)$$

is the identity, for the sake of simplicity.

Now remember that the pair (α, β) is "SPLIT" if the map Π in the Mayer-Vietoris sequence is surjective,

$$0 \longrightarrow \Omega^r(\alpha * \beta) \xrightarrow{J} \Omega^r(\alpha) \oplus \Omega^r(\beta) \xrightarrow{\Pi} \Omega^r(p_X) \tag{8.6}$$

where $\Pi(\theta, \theta') = p_U^*\theta - p_V^*\theta'$. In this case there is a linear section S,

$$S\colon \Omega^r(p_X) \to \Omega^r(\alpha) \oplus \Omega^r(\beta)$$

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where $\Pi \circ S = \mathrm{id}$.

Now we have

$$\begin{array}{ccc} \Omega^r(\alpha) \oplus \Omega^r(\beta) & \xrightarrow{\phi} & \Omega^r(\beta) \oplus \Omega^r(\alpha) \\ \downarrow{\scriptstyle \Pi} & & {\scriptstyle \Pi'}\downarrow \\ \Omega^r(p_X) & = & \Omega^r(p'_X). \end{array} \tag{8.7}$$

where $\phi(\theta, \theta') = (\theta', \theta)$ is the flip morphism. Since

$$\Pi'(\theta', \theta) = P_V'^* \theta' - P_U'^* \theta = \phi^* P_V \theta' - \phi^* P_U \theta = -\Pi(\theta, \theta'),$$

it is $\Pi' \phi = -\Pi$, then the Mayer-Vietoris sequence of (β, α) admits the linear section

$$S' = -\phi S.$$

In fact

$$\Pi' S' = -\Pi' \phi S = \Pi S = \mathrm{id}.$$

The linear right section S induces a left section

$$K\colon\ \Omega^r(\alpha) \oplus \Omega^r(\beta) \to\ \Omega^r(\alpha * \beta)$$

where $JK = \mathrm{id} - S\Pi$.

Notice that $JKJ = J - S\Pi J = J$, that is, $KJ = \mathrm{id}$.

Analogously we have

$$K'\colon\ \Omega^r(\beta) \oplus \Omega^r(\alpha) \to\ \Omega^r(\beta * \alpha)$$

given by $J'K' = \mathrm{id} - S'\Pi'$. In fact we have

$$K' = K\phi,$$

because

$$\phi J K' = \mathrm{id} - (-\phi S)(-\Pi\phi) = \mathrm{id} - \phi S \Pi \phi$$

hence

$$JK' = \phi - S\Pi\phi = (\mathrm{id} - S\Pi)\phi = JK\phi,$$

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so $K' = K\phi$.

Finally we had defined

$$\Delta\colon \Omega^r(p_X) \to \ \Omega^{r+1}(\alpha * \beta)$$

by $J\Delta = \mathrm{d}S - S\mathrm{d}$.

Analogously,

$$\Delta'\colon \Omega^r(p'_X) \to \ \Omega^{r+1}(\alpha) \oplus \Omega^{r+1}(\beta)$$

by $J'\Delta' = \mathrm{d}'S' - S'\mathrm{d}$ where $\mathrm{d}' = \phi\mathrm{d}\phi$. Notice that the differential of $\Omega(p'_X)$ is that of $\Omega(p_X)$. See the following diagram:

$$\begin{array}{ccc} \Omega^r(\alpha) \oplus \Omega^r(\beta) & \xrightarrow{\ \phi\ } & \Omega^r(\beta) \oplus \Omega^r(\alpha) \\ {\scriptstyle \mathrm{d}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}'} \\ \Omega^{r+1}(\alpha) \oplus \Omega^{r+1}(\beta) & \xrightarrow{\ \phi\ } & \Omega^{r+1}(\beta) \oplus \Omega^{r+1}(\alpha). \end{array} \tag{8.8}$$

Hence $\Delta' = -\Delta$ because

$$\begin{aligned} \phi J\Delta' =& J'\Delta' = \mathrm{d}'S' - S'\mathrm{d} = \phi\mathrm{d}\phi(-\phi S) - (-\phi S)\mathrm{d} \\ =& \phi(-\mathrm{d}S + S\mathrm{d}) = -\phi(J\Delta) \end{aligned}$$

Now, since

$$J\xi = JK(\theta \wedge \alpha^*\omega', (-1)^{\omega}\beta^*\omega \wedge \theta') + (-1)^{\omega} J\Delta(p_U^*\theta \wedge p_V^*\theta') \tag{8.9}$$

we have

$$J'\xi' = J'K'(\theta' \wedge \beta^*\omega, (-1)^{\omega'}\alpha^*\omega' \wedge \theta) + (-1)^{\omega'} J'\Delta'(p_V^*\theta' \wedge p_U^*\theta)$$

so

$$\begin{aligned} \phi J\xi' =& \phi JK\phi(\theta' \wedge \beta^*\omega, (-1)^{\omega'}\alpha^*\omega' \wedge \theta) + (-1)^{\omega'}\phi J(-\Delta)(p_V^*\theta' \wedge p_U^*\theta) \\ =& \phi JK((-1)^{\omega'}\alpha^*\omega' \wedge \theta, \theta' \wedge \beta^*\omega) - (-1)^{\omega'}\phi J(\Delta)(p_V^*\theta' \wedge p_U^*\theta) \\ =& JK((-1)^{\omega'}(-1)^{\theta\omega'}\theta \wedge \alpha^*\omega', (-1)^{\theta'\omega}\beta^*\omega \wedge \theta') \end{aligned}$$

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$$-(-1)^{\omega'}\phi J\Delta((-1)^{\theta\theta'}p_U^*\theta\wedge p_V^*\theta')$$

So composing with $(-1)^{\omega\omega'}\phi$ we have

$$(-1)^{\omega\omega'}J\xi' = JK((-1)^{\omega\omega'}(-1)^{\omega'}(-1)^{\omega'\theta}\theta\wedge\alpha^*\omega', (-1)^{\omega\omega'}(-1)^{\theta'\omega}\beta^*\omega\wedge\theta')$$

which equals $J\xi$ in Equation (8.9), after checking the signs:

- $\omega\omega'+\omega'+\omega'\theta=\omega'(\omega+1+\omega-1)=2\omega\omega'$
  hence
  $$(-1)^{\omega\omega'}(-1)^{\omega'}(-1)^{\omega'\theta}=1.$$
- $\omega\omega'+\theta'\omega=\omega(\omega'+\omega'-1)=2\omega\omega'-\omega$
  hence
  $$(-1)^{\omega\omega'}(-1)^{\theta'\omega}=(-1)^{\omega}.$$
- $1+\omega\omega'+\omega'+\theta\theta'=1+\omega\omega'+\omega'+(\omega-1)(\omega'-1)=2+2\omega\omega'-\omega$
  hence
  $$(-1)(-1)^{\omega\omega'}(-1)^{\omega'}(-1)^{\theta\theta'}=(-1)^{\omega}.$$

□

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Resumo

Hipóteses

Os espazos difeolóxicos son unha xeneralización das variedades diferenciabeis. Esencialmente, unha difeoloxía sobre un conxunto X indica para cada número enteiro $m \geq 0$ e cada subconxunto aberto $U \subset \mathbb{R}^m$ qué aplicacións de U en X considéranse diferenciabeis. Estas aplicacións chámanse *plots* e están suxeitas a tres axiomas: cobertura, localidade e compatibilidade. Unha *aplicación diferenciable* entre espazos difeolóxicos é calquera aplicación que envía plots en plots.

Este enfoque demostra ser útil en varios escenarios que implican obxectos que raramente son variedades (de dimension finita). Por exemplo, o espazo de aplicacións diferenciabeiss entre dúas variedades ou os cocientes de variedades, como o espazo das follas dunha foliación [35].

En particular, as *variedades de Fréchet* forman unha subcategoría completa da categoría **Diff** de espazos difeolóxicos [30]. Isto inclúe as variedades de Banach (é decir, espazos modelados por espazos de Banach) [15]. *As variedades con borde* tamén se poden ver como espazos difeolóxicos [14]. Finalmente, as *orbifolds* (é decir, espazos modelados localmente en cocientes de subconxuntos abertos de $\mathbb{R}^n$ pola acción dun grupo finito) son espazos difeolóxicos [25].

Outra vantaxe da definición dunha difeoloxía nun conxunto X é ser esencialmente categórica, xa que se dá como un *prefeixe* na categoría cuxos obxectos son os subconxuntos abertos de espazos euclidianos e cuxas flechas son as aplicacións

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diferenciabeis usuais. Este marco facilita construcións naturais contravariantes que adoitan ser menos complexas que aquelas dentro do ámbito das variedades. De feito, a categoría **Diff** é pechada baixo limites e colimites [2]. Por exemplo, cada subconxunto dun espazo difeolóxico herda a estrutura difeolóxica coas boas propiedades dunha variedade regularmente inmersa.

Obxectivos

Os espazos difeolóxicos foron introducidos arredor de 1979 polo físico e matemático francés Jean-Marie Souriau (1922-2012), nun intento de formalizar a mecánica cuántica e a cuantización xeométrica [38]. Foron deseñados para tratar con facilidade grupos infinito-dimensionais de difeomorfismos en xeometría simpléctica [21]. Están construídos sobre o modelo dos *espazos diferenciabeis* de Kuo-Tsai Chen [5], para os que a estrutura defínese sobre subconxuntos euclidianos convexos en lugar de dominios euclidianos abertos. Pero esta última configuración fai que a difeoloxía sexa máis adecuada para estender os resultados clásicos da xeometría diferencial. A definición formal dun espazo difeolóxico, sen referirse á teoría de grupos, débese aos discípulos de Souriau Paul Donato e Patrick Iglesias-Zemmour [8]. A principal referencia desta teoría é agora o libro de Iglesias [20]. Unha introdución rápida ás ideas principais de difeoloxía dáse en [21].

Comentemos a relación entré difeoloxía e topoloxía. Ao contrario de moitas construcións en xeometría diferencial, unha difeoloxía defínese nun conxunto sen ningunha estructura preexistente, nen topoloxía nen outra cousa. Unha vez dada a difeoloxía en X, pódese definir unha topoloxía natural, a chamada topoloxía $\mathcal{D}$: o subconxunto $A \subset X$ e $\mathcal{D}$-aberto se $P^{-1}(A)$ é un conxunto aberto (habitual) no dominio $U \subset \mathbb{R}^n$ de calquer plot $P\colon U \to X$ da difeoloxía. Noutras palabras, a topoloxía $\mathcal{D}$ é a máis grande topoloxía en X que fai continuos todos os plots.

Aínda que a topoloxía $\mathcal{D}$ facilita a introdución de varios conceptos topolóxicos e homotópicos no contexto dod espazos difeolóxicos (como os grupos de homotopía). abstivémonos de empregala. A nosa crenza é que perturba a natureza contravariante

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intrínseca do noso marco. Ademáis, a difeoloxía de subespazo é incompatible coa $\mathcal{D}$-topoloxía. Tamén, esta topoloxía está determinada só polas curvas suaves, de xeito que moitas difeoloxías diferentes inducen a mesma topoloxía [2].

Hoxe, a difeoloxía abrangue moitos campos tradicionais en xeometría e topoloxía, como teoría de homotopía superior, febrados, cálculo de Cartan-de Rham, aplicación momento e xeometría simpléctica.

Nesta tese interésanos principalmente o cálculo de Cartan sobre os espazos difeolóxicos, e máis concretamente a cohomoloxía de De Rham. Como veremos, o cálculo diferencial é un dos dominios máis desenvolvidos da difeoloxía. Os espazos de Chen foran introducidos tendo en conta a homoloxía, e por iso el escolleu os subconxuntos convexos como dominios dos plots. En fin, a propia definición de difeoloxía leva naturalmente a definir obxectos contravariantes como formas diferenciais.

Esencialmente, unha forma diferencial ω no espazo difeolóxico X defínese dando os seus pull-backs $\omega_P = P^*\omega$ para todos os plots $P\colon U \to X$, sometidos a unha condición de compatibilidade. É dicir, damos unha colección compatible de formas diferenciais habituais en dominios euclidianos. Un feito importante é que se poden identificar as 0-formas en X coas funcións diferenciabeis $X \to \mathbb{R}$. As dúas operacións principais sobre as formas diferenciais en espazos difeolóxicos son o pull-back $(f^*\omega)_P = \omega_{f\circ P}$ para unha aplicación diferenciable $f\colon Y \to X$, e a derivada exterior $(\mathrm{d}\omega)_P = \mathrm{d}\omega_P$. Así, temos un complexo de De Rham $(\Omega(X), \mathrm{d})$, con $\mathrm{d}\circ\mathrm{d} = 0$, cuxa cohomoloxía denótase $\mathrm{H}(X)$.

Metodoloxía

Nesta tese seguimos o método usual para facer Matemáticas. Primeiro, conxecturamos que é interesante abordar o estudo da cohomoloxía de De Rham no marco dos espazos difeolóxicos, en paralelo ao que se fai para as variedades diferenciabeis de dimensión finita ou para os espazos de follas dunha foliación. Despois estudamos os antecedentes relevantes que existan na literatura. Finalmente desenrolamos resultados que foron publicados en dúas revistas de prestixio, como son *Indagationes*

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Mathematicae e *MDPI Mathematics*. Un terceiro artigo depositado en arXiv está pendente de publicación.

Finalmente elaboramos esta memoria que contén os resultados citados máis un extenso traballo de explicar a teoría e os resultados preliminares, así como algunhas cuestións abertas.

Introdución

A cohomoloxía de De Rham en espazos difeolóxicos ten varias aplicacións interesantes. Por exemplo, Iglesias estendeu a noción de formalismo simpléctico e aplicacións de momentos á difeoloxía en [23]. Foi Souriau quen introduciu por primeira vez a noción dunha aplicación de momentos no marco clásico. Ademais, comprobouse que a cohomoloxía do espazo de follas dunha foliación é igual á chamada cohomoloxía básica, unha ferramenta moi coñecida na teoría da foliacións [18].

Repasamos agora varios traballos recentes neste campo. Primeiro temos dous artigos onde a noción de forma diferencial nun espazo difeolóxico aplícase a variedades con bozde e esquinas [11] e a espazos estratificados [12, 13].

Unha cuestión importante no Cálculo de Cartan é a sucesión de Mayer-Vietoris. Non obstante, na configuración de espazos difeolóxicos atopamos unha dificultade para obter particións da unidade e, polo tanto, a sucesión de Mayer-Vietoris non é exacta en xeral. Este problema xa foi estudado en [26, 27] ao tentar obter un teorema de De Rham (é dicir isomorfismo entre a cohomoloxía de De Rham e a cohomoloxía singular do espazo $\mathcal{D}$-topolóxico subxacente). Neses traballos introdúcese unha noción de partición da unidade e algún tipo de recubremento aberto adecuado para a topoloxía $\mathcal{D}$. Estas nocións tamén foron dadas en [16]. Aínda con estas ferramentas, os autores necesitan unha noción diferente de forma diferencial, para a obtención da chamada cohomóloxía *cúbica* de De Rham.

En [28], Kuribayashi compara o complexo de De Rham con outro complexo singular de Rham que pode ser considerado como unha variante difeolóxica das formas simpliciais polinómicas de Sullivan para un espazo topolóxico [29]. Finalmente, o

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complexo de cocadeas do conxunto de aplicacións $\mathcal{C}^\infty$ do n-simplex estándar en X definido por G. Hector considérase en [19].

O artigo de Haraguchi [16] mostra que as sucesións exactas longas de Mayer-Vietoris existen para a cohomoloxía de De Rham con soporte compacto de espazos subcartesianos difeolóxicos (é dicir, un espazo difeolóxico localmente difeomorfo a algún subespazo difeolóxico dun espazo euclidiano), se se pode atopar unha partición subordinada da unidade.

Iglesias propuxo un bicomplexo de Čech-De Rham en cohomoloxía [22] que está conectado coa cohomoloxía de De Rham, a través dun dobre complexo.

Recentemente, Minichiello [34] obtivo novas sucesións exactas en todos os graos superiores para a cohomoloxía Čech-De Rahm mediante o uso de pull-backs homotópicos. A cohomoloxía difeolóxica de Čech foi introducida por Alireza Ahmadi en [1].

Abordaremos estes problemas dun xeito moi diferente [32]. Primeiro substituiremos a noción de recubremento aberto pola de *familia xeradora*, que é moito máis intrínseca que a topoloxía $\mathcal{D}$. Ademáis, utilizaremos sistemáticamente nocións categóricas como pullback, push-out e *join* para obter a sucesión de Mayer-Vietoris. De aí que os nosos resultados non dependan da topoloxía $\mathcal{D}$.

A nosa principal contribución será definir dúas versións diferentes dunha cohomoloxía *relativa* de De Rham. A primeira está inspirada na definición de Godbillon da cohomoloxía de De Rham dun par formado por unha variedade M e unha subvariedade $i\colon N \to M$ [10]; é a cohomoloxía do complexo $\ker i^*$ de formas diferenciais sobre M que son nulas en N. O principal problema con esta cohomoloxía é que non hai unha sucesión exacta longa do par (M, N), debido á falta de particións da unidade. Por iso as formas diferenciais en N non se poden estender a toda a variedade ambiental M.

A segunda versión é unha xeneralización da cohomoloxía dun par definida por Bott-Tu [4]. Para iso introducimos a noción de formas f-horizontais para unha aplicación diferenciable f. Deste xeito podemos desenvolver moitas das propiedades habituais dos grupos de cohomoloxía relativa, e en particular unha noción de cup

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producto relativo.

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Publicaciones

- **Título:** Mayer-Vietoris sequence for generating families in diffeological spaces

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  Autores: Enrique Macías Virgós, Reihaneh Mehrabi

  Referencias: Indagationes mathematicae

  Índices de calidad:

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  Cuartil: Q3.

  Autorización de la revista: en la página siguiente.

  Presencia en la Tesis: Chapter 7

  Contribución de la doctoranda: Las contribuciones de la doctoranda fueron esenciales. Desarrolló la teoria sobre cohomología de De Rham más allá de lo que era conocido. Demostró los resultados y escribió el texto da memoria. La revisó y corrigió.

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Diffeological spaces are a generalization of smooth manifolds. They were introduced around 1979 by the French mathematician and physicist Jean-Marie Souriau (1922–2012) in an attempt to formalize quantum mechanics and geometric quantization. This approach has proven useful in various settings that involve objects which are rarely (finite-dimensional) manifolds, such as the space of smooth maps between two manifolds or the space of leaves of a foliation.

In this thesis, we are primarily interested in the Cartan calculus on diffeological spaces. We clarify the properties of De Rham cohomology groups, which are typical in the algebraic topology of manifolds but were less understood in this context. Additionally, we study several constructions, such as the Mayer-Vietoris sequence and the relative cup product."