



A simple method for automatic recreation of railway horizontal alignments

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Abstract

This paper deals with the problem of recreating horizontal alignments of existing railway lines. The main objective is to propose a simple method for automatically obtaining optimized recreated alignments located as close as possible to an existing one. Based on a previously defined geometric model, two different constrained optimization problems are formulated. The first problem uses only the information provided by a set of points representing the track centerline while the second one also considers additional data about the existing alignment. The proposed methodology consists of a two-stage process in which both problems are solved consecutively using numerical techniques. The main results obtained applying this methodology are presented to show its performance and to prove its practical usefulness: an academic example used to compare with other methods, and a case study of a railway section located in Parga (Spain) in which the geometry of its horizontal alignment is successfully recovered.

Keywords Railway horizontal alignment · Geometrical model · Constrained optimization problem · Numerical resolution

List of symbols

A	Clothoid parameter.	f_{sp}	Function defining the piecewise cubic Hermite interpolating polynomial of the surveyed points.
$c_i^{\max} = (s_i^{\max}, \kappa_i^{\max})$	Local maxima of the curvature diagram.	J_1	Objective function of the first optimization problem.
$c_i^{\min} = (s_i^{\min}, \kappa_i^{\min})$	Local minima of the curvature diagram.	J_2	Objective function of the second optimization problem.
C_{i-}	Ingoing clothoid of the i th circular curve.	J_p	Penalty function used in the second optimization problem.
C_{i+}	Outgoing clothoid of the i th circular curve.	κ	Curvature.
CC_i	Circular curve i .	κ_{\min}	Minimum curvature used for estimating the number of curves.
d_{\min}	Minimum distance used for estimating the number of curves.	L	Length of the horizontal alignment.
		L_i^C	Length of the i th clothoid.
		$L_i^{C_0}$	Known length of the i th clothoid.
		$L_i^{C_{\min}}$	Minimum length of the clothoids.
		L_i^{CC}	Length of the i th circular curve.
		$L_i^{CC_0}$	Known length of the i th circular curve.
		L_{\min}^{CC}	Minimum length of the circular curves.
		L^E	Length of a geometrical element.
		L_j^T	Length of the j th tangent segment.
		$L_j^{T_0}$	Known length of the j th tangent segment.

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L_{\min}^T	Minimum length of the tangent segments.	\mathbf{x}_{S1}^N	Design vector obtained as solution of stage 1.
M_S	Number of multi-starts.	\mathbf{x}_{S2}^N	Design vector obtained as solution of stage 2.
N	Number of curves.	Y_i	Design variable defining the second coordinate of the j th vertex of the layout.
N_e	Estimated value of the number of curves.	α	Heading.
N_{\max}	Upper bound for the number of curves.	α^{sm}	Smoothed heading.
N_{\min}	Lower bound for the number of curves.	$\theta_i \in (0, \pi)$	Deflection angle of the i th curve.
n_f	Size of the kernel of the Gaussian filter.	$\theta_{\max} \in (0, \pi)$	Maximum allowable deflection angle for random generation.
p_i^C	Penalty parameter of the length of the i th clothoid.	σ	Parametrization of the horizontal alignment in terms of the arc length parameter s .
p_i^{CC}	Penalty parameter of the length of the i th circular curve.	σ_f	Standard deviation of the Gaussian distribution used to define the filter.
p_i^R	Penalty parameter of the radius of the i th circular curve.	ω_i	Angle of the i th circular curve.
p_j^T	Penalty parameter of the length of the j th tangent segment.		
$P_a, P_b \in \mathbb{R}^2$	Starting and ending points of the horizontal alignment.		
$P_k = (x_k, y_k) \in \mathbb{R}^2$	Set of points representing the current position of the railway centerline.		
q_i	Probability of number of curves equal to i .		
R_i	Radius of the i th circular curve.		
R_{\min}	Minimum radius.		
r	Parameter used to define N_{\max} and N_{\min} from N_e .		
s	Arc length parameter.		
s_i	Initial coordinate s of a geometrical element.		
s_f	Final coordinate s of a geometrical element.		
$\text{sgn}(\cdot)$	Sign function.		
T_j	Tangent j .		
T_i^-, T_i^+	Tangency points of the i th curve.		
$V_i \in \mathbb{R}^2$	The i th vertex of the polygonal chain defining the main axis of the horizontal alignment.		
X_{ad}^N	Set of admissible horizontal alignments.		
X_i	Design variable defining the first coordinate of the j th vertex of the layout.		
\mathbf{x}^N	Design vector of the horizontal alignment.		
$\mathbf{x}_{\text{inic}}^N$	Design vector of an initial horizontal alignment.		
$\mathbf{x}_{\text{opt}}^N$	Design vector of a local minimum obtained as solution of problem (6).		

1 Introduction

The recreation of horizontal alignments of existing railway lines is an essential task that must be carried out during periodic maintenance in order to prevent the process of track geometry degradation and assure the safety of train operations and the comfort of the passengers [1]. The interaction between the wheels and the railway track provokes over time deviations of its centerline with respect to its original position that must be corrected by means of a track calibration process before they reach unacceptable values. During calibration, the track centerline is adjusted by adopting a recreated horizontal alignment (RHA) defined as close as possible to the existing one taking into account the technical constraints that must be satisfied according to the operational requirements of the railway line (Fig. 1). Nowadays, the current position of the centerline is typically known by means of the coordinates of a set of points $P_k = (x_k, y_k) \in \mathbb{R}^2$ located along the railway line measured by track surveying or obtained from other sources.

The design of an optimal RHA is not only necessary for maintenance operations but also for renewal or reconstruction projects in order to improve the operational characteristics of existing railway lines. Another practical application related to the recreation of horizontal alignments is the possibility of recovering the geometrical parameters of older railway lines or road routes when that information is not available, missing or incomplete, which is usually very common.

The design of a RHA is a complicated constrained optimization problem [2] in which different types of constraints must be considered in order to obtain a geometrical and technical admissible alignment. Owing to its complexity, the

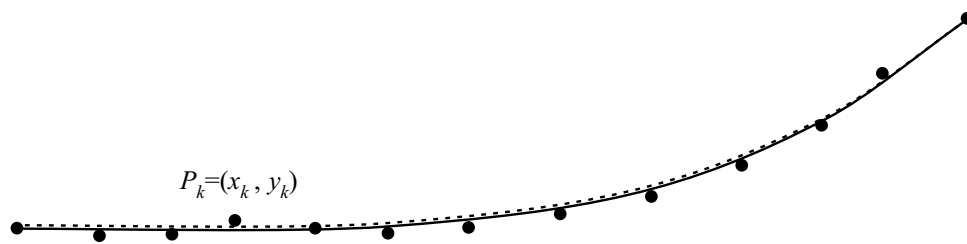


Fig. 1 Recreation of a horizontal alignment (gray solid line) obtained as close as possible to a set of measured points (black solid dots) representing the existing railway centerline. The recreated alignment may be different from the original design (black dashed line) of the railway line

search for an optimal RHA from the information provided by the measured points is generally conducted in two different stages: (1) identification of each potential geometrical element (tangents, circular curves, and transition curves) of the existing alignment and the boundaries between them, and (2) optimization of the parameters of the elements previously identified subjected to some technical constraints. The first one is clearly the most difficult part of the process, owing to the fact that it is not always easy to assign a certain type of geometrical element to each of the measured points of the existing alignment, being especially difficult to estimate the location of the boundaries between elements. This two stages: identification and optimization, were considered separately in many research papers, but in practice, both stages are unquestionably dependent on each other, which makes the problem of obtaining an optimal RHA even more complicated.

Several methods have been proposed so far for conducting the identification of railway or road horizontal alignments from geometrical features extracted from the measurement points. The traditional method based on curvature analysis is one of the most widely used, due to the fact that each of the geometrical elements of a horizontal alignment (tangents, circular curves, and transition curves) is characterized by a well-known curvature behavior. Some authors estimate the curvature of each measured point along the existing centerline by means of a local fitting using circular curves [3–5], while others propose to construct a spline curve passing through the measured points [6–8] and then compute the curvature from it instead of using directly the coordinates of the points. Subsequently, the identification of the geometrical elements can be conducted by analyzing the curvature obtained from those estimations using thresholds to assign some type of element to each measured point. In order to avoid some limitations associated with the curvature analysis, some authors [9–13] propose to use heading direction graphs or bearing graphs due to the fact that they are less noisy than curvature graphs. However, not only the curvature or the heading direction was considered for conducting this task. Other authors carried out this identification stage by using the values of versine measured along the track [14] or calculated from the coordinates of a set of known points [15], while others applied image

processing of satellite or photographic images [16–18] to identify the geometric elements of an existing alignment by using some artificial vision techniques, such as edge detectors and the Hough transform.

Once the whole set of measured points has been divided in different subsets and classified into the different elements in the identification stage, the geometrical parameters of each element should be determined in the second stage according to some optimization process. Several methods have been proposed to accomplish this task. Easa and Wang [19] defined an optimization model for composite horizontal curves that simultaneously fits all its elements (circular curve, transition curves and tangents) by using the total least squares method. Gikas and Stratakos [11] proposed an algorithm based on the manipulation of the bearing diagram and its first (curvature) and second order derivative (rate of change of the curvature) to fit the centerline geometry of railways or roads. Liu et al. [20] used the Hough transform to improve the identification stage and then defined an optimization model which is solved by the Powell method. Skala-Szymanska et al. [21] proposed the use of the Nelder–Mead simplex method for fitting horizontal alignments. Camacho-Torregrosa et al. [12] introduced an analytical-heuristic method to obtain an optimal horizontal alignment by minimizing the square-mean error of headings, taking into account the restrictions of continuity required to heading and curvature. Li et al. [22] proposed a method for the reconstruction of existing railway alignments by using particle swarm optimization with a full direction search. Li et al. [13, 23] introduced a new methodology called swing iterations in which an iterative process of reclassification of the measured points is conducted for the identification of the alignment components, being their geometrical parameters optimized by means of genetic algorithm (GA) techniques. Song et al. [24] applied the Levenberg–Marquardt (LM) algorithm with an heuristic strategy to perform the fitting of transition curves. Pu et al. [2] developed an optimization model for railway reconstruction which is solved by means of a two-stage method. After applying an automatic segmenting method for separating existing alignments into reused and reconstructed sections, a multi-directional distance transform is proposed to searching redesigned alignments. Recently, Shi

et al. [15] proposed a smoothness optimization method based on orthogonal least squares and the theory of controlling the track smoothness for obtaining an optimal RHA.

In this paper, a simple automatic method for obtaining an RHA of an existing railway line is presented. It considers a geometrical model previously defined in [25] as a starting point for the subsequent formulation of two constrained optimization problems. The objective of both problems is the same: minimizing the distance between the existing alignment and the RHA, but the definition of the corresponding objective functions is different. In the first problem, its objective function only takes into account distances between the alignments, while, in the second problem, a penalty term is also included to consider during the optimization process some known data about the geometrical parameters of the existing alignment. If this kind of information is available, this second problem can be solved by using the optimal solution obtained from the first one as an initial solution. As regards to the constraints, the same geometrical and technical requirements are initially imposed on both problems. The first ones must be fulfilled by the solutions in order to assure an appropriate definition of their geometry, and the second ones are typical technical restrictions considered in such optimization problems: minimum radius of the circular curves and minimum length of each of the three different elements considered in the geometrical model: tangents, circular curves, and transition curves. However, if additional technical constraints should be taken into account for a specific case, they could be easily included. Once formulated, these optimization problems can be solved by gradient-type techniques as it has been proved in similar cases [26, 27].

This method has the following advantages that distinguish it from other previous research. Firstly, the formulation of the two constrained optimization problems makes unnecessary to conduct the identification stage to assign a potential geometrical element (tangent, circular curve or transition curve) to each one of the measured points. The initial alignments needed to start the process of optimization are obtained by using a previously developed algorithm for the automatic generation of random admissible horizontal alignments described in [28]. Therefore, the proposed method avoids all the difficulties related with the identification stage and especially the estimation of the boundaries between elements. Secondly, the second stage allows to consider during the optimization process some available information about the geometry of the existing alignment. Thirdly, unlike other methodologies previously proposed, the fact of starting the optimization process with an admissible horizontal alignment allows the whole RHA to be recreated automatically, instead of being composed by different recreated parts that should be properly connected later ensuring that the geometrical restrictions between them are fulfilled. Fourthly, all the geometrical elements (tangents, circular curves, and transition curves) of the RHA are optimized

simultaneously, according to the decision variables of the geometrical model defined in a continuous domain [25], without any of them being predefined in advance. Finally, the definition of the optimization function on both constrained problems avoids the need of spending a great deal of time on solving repeatedly the problem of computing the distance between a point and a transition curve during the optimization process.

The paper is organized as follows: the mathematical model is described in Sect. 2; the numerical results obtained in two different cases: an academic example previously proposed by Easa and Wang [19] selected to compare with another existing method, and a case study that consists of the recreation of an existing section of a railway line is presented and discussed in Sect. 3; finally, the conclusions of this research are summarized in Sect. 4.

2 Mathematical model

The geometrical model used in this work considers the horizontal alignment as the combination of tangent segments, joined by circular curves with transition curves (clothoids). Initially, the model considers exclusively symmetrical clothoids for each individual circular curve, but it can be easily modified in order to include the possibility of adopting different clothoid parameters on each side of the curves. A horizontal alignment composed by N curves joining two previously defined starting and ending points ($P_a = (x_a, y_a) \in \mathbb{R}^2$ and $P_b = (x_b, y_b) \in \mathbb{R}^2$, respectively) is determined by the N vertices of the main axis ($V_i = (X_i, Y_i) \in \mathbb{R}^2$), and the radii $R_i > 0$ and the angles $\omega_i \geq 0$ of its circular curves (Fig. 2). Therefore, the following vector of decision variables can be used to uniquely define the horizontal alignment:

$$\mathbf{x}^N = (X_1, Y_1, R_1, \omega_1, X_2, Y_2, R_2, \omega_2, \dots, X_N, Y_N, R_N, \omega_N) \in \mathbb{R}^{4N}. \quad (1)$$

From any decision vector \mathbf{x}^N , the length of the corresponding horizontal alignment $L(\mathbf{x}^N)$ and its parametrization in terms of the arc length parameter s ,

$$\begin{aligned} \sigma_{\mathbf{x}^N} : [0, L(\mathbf{x}^N)] &\mapsto \mathbb{R}^2 \\ s &\mapsto \sigma_{\mathbf{x}^N}(s) = (x(s), y(s)) \end{aligned} \quad (2)$$

can be computed by using the method proposed in [25].

2.1 Formulation of the optimization problem

The main goal of this work is to obtain a recreated horizontal alignment as close as possible to an existing one determined by the coordinates of a set of points $P_k = (x_k, y_k)$ measured along its horizontal axis. We assume that the first and the end of these surveyed points are, respectively, P_a and P_b . Additionally, to establish a rigorous formulation of the

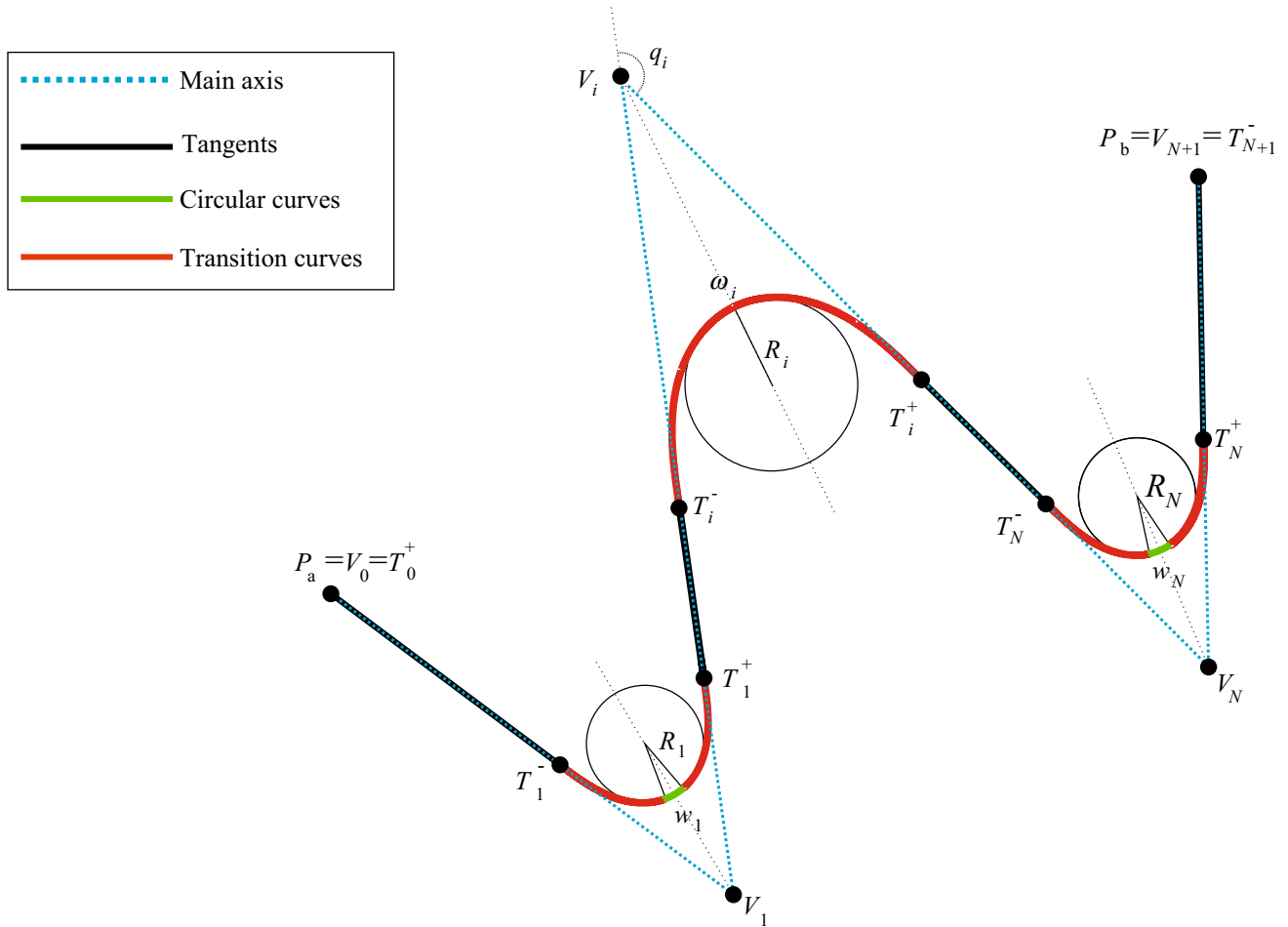


Fig. 2 Geometrical model of an horizontal alignment between to given points using a combination of $N + 1$ tangents connected by N circular curves with symmetrical transition curves (clothoids)

problem, we assume that: (H1) $x_k < x_{k+1}, \forall k$. This hypothesis is not very restrictive because, if there is not backtracking between surveyed points P_k , it can be just fulfilled through the rotation which achieves that $y_a = y_b$.

The first step to formulate the problem is to consider a smooth interpolation of the surveyed points. In this work, we use the piecewise cubic Hermite interpolating polynomial (PCHIP) because it has no overshoots and has low oscillation if the data is not smooth, connecting flat regions more accurately [29, 30]. We denote by $f_{sp}(x)$ the function defining the PCHIP of the surveyed points. If (H1) holds, it can be assumed that the graph of this function represents accurately enough the position of the existing alignment, and the quality of the fitting between the recreated alignment and the existing one can be easily quantified: we assume that backtracking is not allowed in the recreated alignment, specifically,

$$x_a \leq X_l \leq X_{l+1} \leq x_b, \quad l = 1, 2, \dots, N - 1, \quad (3)$$

which guarantees that $x(s) \in [x_a, x_b] \forall s \in [0, L(x^N)]$ and, consequently, that the function

$$J_1(x^N) = \int_0^{L(x^N)} (y(s) - f_{sp}(x(s)))^2 x'(s) ds \quad (4)$$

is well defined. This function gives the sum of the squares of the y -distance between both (existing and recreated) alignments and can be used to quantify the goodness of the fitting.

Remark 1 If there is backtracking along the surveyed points, for example due to the presence of complex reverse curves, a possible approach to overcome this difficulty will be to divide in a proper manner the set of points in different parts that could be optimized separately. For those parts presenting backtracking, an appropriate rotation should be applied to their surveyed points before solving the optimization problems presented below. Later the different parts should be properly joined assuring that the final solution represents an admissible horizontal alignment.

The optimization problem will consist of minimizing the function $J_1(\mathbf{x}^N)$ considering; in addition to the relation (3), some constraints over the decision vector \mathbf{x}^N related to the definition of the geometrical model or some technical restrictions that must be taken into account. Regarding the geometrical model, the following two constraints are required (Fig. 2):

1. The angles ω_i occupied by each circular curve cannot be negative ($\omega_i \geq 0$) or greater than its corresponding deflection angle θ_i .
2. A curve cannot begin before the preceding one ends, that is, $\text{sign}_j \geq 0$, being

$$\text{sign}_j = \text{sgn}\left((T_j^- - T_{j-1}^+) \cdot (V_j - V_{j-1})\right), \quad j = 1, 2, \dots, N + 1,$$

where $\text{sgn}(\cdot)$ is the sign function, $V_0 = T_0^+ = P_a$, $V_{N+1} = T_{N+1}^- = P_b$ and, for $i = 1, 2, \dots, N$, T_i^- and T_i^+ are the tangency points of the i th curve (see Fig. 2).

With respect to the technical constraints associated with the horizontal alignment, they mainly consist of establishing minimum values for the radii of its circular curves R_i and the length of its tangent segments $L_j^T = \text{sign}_j ||T_j^- - T_{j-1}^+||$, circular curves $L_i^{CC} = \omega_i R_i$, and clothoids $L_i^C = R_i(\theta_i - \omega_i)$. In order to fulfill these requirements, the following admissible set (X_{ad}^N) is defined:

$$X_{ad}^N = \left\{ \mathbf{x}^N \in \mathbb{R}^{4N} / \begin{array}{l} x_b \geq X_{l+1} \geq X_l \geq x_a \\ R_i \geq R_{\min} \quad i = 1, 2, \dots, N-1 \\ L_j^T(\mathbf{x}^N) \geq L_{\min}^T, \quad j = 1, 2, \dots, N \\ L_i^{CC}(\mathbf{x}^N) \geq L_{\min}^{CC} \quad j = 1, 2, \dots, N+1 \\ L_i^C(\mathbf{x}^N) \geq L_{\min}^C \end{array} \right\}, \quad (5)$$

with $R_{\min} > 0$ and $L_{\min}^T, L_{\min}^{CC}, L_{\min}^C \geq 0$ being the minimum values associated with radii and lengths of tangent segments, circular curves and clothoids, respectively.

Therefore, this optimization problem can be formulated as follows:

$$\min_{\mathbf{x}^N \in X_{ad}^N} J_1(\mathbf{x}^N). \quad (6)$$

Additionally, if there is available information about the values of some geometrical parameters of the existing alignment, it is possible to use them in order to penalize solutions with values far from those expected, forcing them to be as similar as possible. The following penalty function $J_p(\mathbf{x}^N)$ can be used for that purpose:

$$J_p(\mathbf{x}^N) = \sum_{i=1}^N p_i^R (R_i - R_i^o)^2 + p_i^{CC} (L_i^{CC} - L_i^{CC_o})^2 + p_i^C (L_i^C - L_i^{C_o})^2 + \sum_{j=1}^{N+1} p_j^T (L_j^T - L_j^{T_o})^2. \quad (7)$$

This penalty function is composed of the sum of the quadratic differences between all the obtained values of radii and lengths of each element and their corresponding known values: $R_i^o, L_i^{CC_o}, L_i^{C_o}$ and $L_j^{T_o}$. In addition, each of these quadratic differences is multiplied by an individual penalty parameter ($p_i^R, p_i^{CC}, p_i^C, p_j^T$) allowing us to select which geometrical parameters will be penalized and to weight the importance of each of them in the optimization process. However, the values of these penalty parameters cannot be set in advance, being necessary to carry out preliminary tests to select appropriate values that cause the desired effect. In order to take this penalization into account in the problem, a new objective function $J_2(\mathbf{x}^N)$ is defined as Eq. (8) by adding to the initial objective function $J_1(\mathbf{x}^N)$ the penalty function $J_p(\mathbf{x}^N)$:

$$J_2(\mathbf{x}^N) = J_1(\mathbf{x}^N) + J_p(\mathbf{x}^N). \quad (8)$$

By considering this new function $J_2(\mathbf{x}^N)$ as an objective to be minimized, a second optimization problem can be formulated as follows, taking into account the same set of admissible solutions X_{ad}^N previously defined for the initial optimization problem.

$$\min_{\mathbf{x}^N \in X_{ad}^N} J_2(\mathbf{x}^N). \quad (9)$$

Therefore, assuming that there is available information about some of its geometrical parameters, the method proposed in this research for obtaining a recreated alignment consists of conducting the two following stages:

- **Stage 1:** Solving the first optimization problem (6) with a random multi-start of the sequential quadratic programming (SQP) method (see [31]). As mentioned before, the initial alignments needed to start this stage are obtained by using an algorithm for the automatic generation of random admissible horizontal alignments previously developed by the authors [28].
- **Stage 2:** Using the optimal solution obtained in stage 1 as an initial solution for solving the second optimization problem (9). Of course, this stage can only be conducted if geometrical information about the existing alignment is known and therefore the penalty term of the objective function can be evaluated.

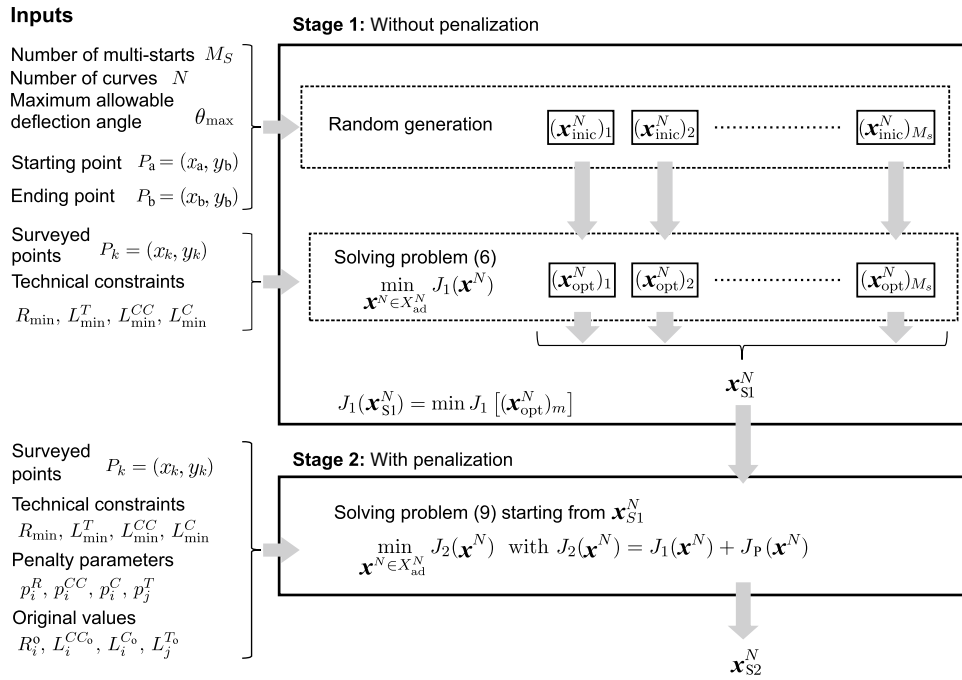


Fig. 3 Flowchart of the two-stage method according to Algorithm 1

The pseudocode of this two-stage method is described in Algorithm 1 and illustrated by means of a flowchart presented in Fig. 3 to make easier to understand the logical relationship between stage 1 (without penalization) and stage 2 (with penalization). As it can be observed (Fig. 3), this flowchart provides detailed information about the different steps conducted during the stage 1 (random generation, solving problem (6) and selecting the best local minimum), the specific inputs used in each step of the algorithm, and the outputs obtaining as a result of each stage: the decision vectors x_{S1}^N and x_{S2}^N for stage 1 and stage 2, respectively.

An important point to consider is the fact that stage 2 uses as initial solution a previously optimized alignment obtained from the stage 1. Therefore, the second optimization problem including the penalty term is solved starting from a solution located near to the set of measured points. Trying to solve directly the second optimization problem (stage 2) from initial alignments randomly generated can lead to solutions with geometrical data very close to the original one, but worse fitting with the set of measured points.

Algorithm 1 Two-stage method for automatic recreation of horizontal alignments

Input: Surveyed points ($P_k \in \mathbb{R}^2$), technical constraints ($R_{min}, L_{min}^T, L_{min}^{CC}, L_{min}^C$), available data ($N, R_i^o, L_i^{CCo}, L_i^{Co}, L_j^{To}$), penalty parameters ($p_i^R, p_i^{CC}, p_i^C, p_j^T$), maximum allowable deflection angle for random generation ($\theta_{max} \in (0, \pi)$), and number of multi-starts (M_S).

Output: Decision vector x_{S2}^N given the optimal RHA.

Stage 1

- 1: **for** $m \leftarrow 1$ to M_S **do**
- 2: **Compute** $(x_{inic}^N)_m$ by running the Algorithm 2 detailed in [28];
- 3: **Compute** a local minimum $(x_{opt}^N)_m$ of problem (6) with the SQP method described in [31], starting from $(x_{inic}^N)_m$
- 4: **end for**
- 5: **Take** x_{S1}^N as the best local minimum, verifying $J_1(x_{S1}^N) = \min J_1[(x_{opt}^N)_m]$

Stage 2

- 1: **Compute** x_{S2}^N by solving the problem (9) with the SQP method described in [31], starting from x_{S1}^N

Algorithm 2 Method proposed for solving problem (10)

Input: Surveyed points ($P_k \in \mathbb{R}^2$), technical constraints ($R_{\min}, L_{\min}^T, L_{\min}^{CC}, L_{\min}^C$), upper bound (N_{\max}), lower bound (N_{\min}), and probability estimates ($q_i \in [0, 1]$) for the number of curves, maximum allowable deflection angle for random generation ($\theta_{\max} \in (0, \pi)$), number of multi-starts ($M_S \gg N_{\max} - N_{\min} + 1$).

Output: Decision vector \mathbf{x}_{S1}^N given the optimal RHA.

Stage 1

- 1: **for** $m \leftarrow 1$ to M_S **do**
- 2: **Take** $N_m \in \{N_{\min}, \dots, N_{\max}\}$ randomly, with probabilities $p(\{N_m = i\}) = q_i$
- 3: **Compute** $\mathbf{x}_{\text{inic}}^{N_m}$ by running the Algorithm 2 detailed in [28];
- 4: **Compute** a local minimum of problem (6) with the SQP method described in [31], starting from $\mathbf{x}_{\text{inic}}^{N_m}$
- 5: **end for**
- 6: **Take** \mathbf{x}_{S1}^N as the best local minimum obtained for problem (6)

Remark 2 This two-stage method can be only applied if there is available information about the values of some geometrical parameters, particularly, if the number of curves (N) is known in advance. Otherwise, if geometrical data are not available and N is unknown, the problem (9) cannot be defined and N must be considered as other decision variable, to be determined during the optimization process. In this situation, the optimal RHA is given by the following slight modification of problem (6):

$$\begin{aligned} & \min_{\substack{N \in \{N_{\min}, \dots, N_{\max}\} \\ \mathbf{x}^N \in X_{\text{ad}}^N}} J_1(\mathbf{x}^N), \end{aligned} \quad (10)$$

where N_{\min} and $N_{\max} \in \mathbb{N}$ are, respectively, the lower and upper bounds of the number of curves. This new problem (10) is more complex than problem (6), because it is formulated in the framework of mixed integer nonlinear programming (MINLP). However, if the number of possible values of N is not too large, the solution to problem (10) can be reached by solving the problem (6) $N_{\max} - N_{\min} + 1$ times (corresponding to $N = N_{\min} - N_{\max}$). From a computational point of view, if there are only rough estimates of the probability that the number of curves is i (values $q_i \in [0, 1]$ verifying $\sum_{i=N_{\min}}^{N_{\max}} q_i = 1$), the solution to problem (10), which provides the optimal RHA in this situation, can be obtained by the following method (Algorithm 2), which the authors have already used successfully in a similar problem (see [28]).

To be able to apply the Algorithm 2, the lower bound (N_{\min}), the upper bound (N_{\max}), and the probability estimates ($q_i \in [0, 1]$) must be defined. To determine appropriate values for N_{\min} and N_{\max} , a method for obtaining an estimate of the number of curves N_e was implemented. This method, described in Algorithm 3, is based on the analysis of a curvature diagram derived from a previously smoothed heading diagram obtained by applying a Gaussian filter to the initial heading values estimated from the surveyed points $P_k = (x_k, y_k)$. The following equations (see [12] for more details) were used, respectively, to compute the heading (α), the distance between consecutive points (Δs), and the curvature (κ), with $\Delta y_k = y_{k+1} - y_k$ and $\Delta x_k = x_{k+1} - x_k$.

$$\alpha_k = \begin{cases} \arctan\left(\frac{\Delta y_k}{\Delta x_k}\right); & \Delta x_k \neq 0 \\ \pm \frac{\pi}{2} & \Delta x_k = 0 \end{cases}, \quad (11)$$

$$\Delta s_k = \sqrt{\Delta y_k^2 + \Delta x_k^2}, \quad (12)$$

$$\kappa_k = \frac{\alpha_{k+1} - \alpha_k}{\Delta s_k}. \quad (13)$$

Algorithm 3 Method for estimating the number of curves.

Input: Surveyed points ($P_k \in \mathbb{R}^2$), size of the kernel of the gaussian filter (n_f), standard deviation of the gaussian distribution (σ_f), minimum value of curvature ($\kappa_{\min} > 0$), and minimum distance between circular curves ($d_{\min} > 0$).

Output: Estimation of the number of curves N_e .

- 1: **Compute** the initial heading diagram (s, α) using Eq. (11).
- 2: **Compute** the smoothed heading diagram (s, α^{sm}) by applying a gaussian filter with kernel size (n_f) and standard deviation (σ_f).
- 3: **Compute** the curvature diagram (s, κ) from the smoothed heading diagram by using Eqs. (12) and (13).
- 4: **Identify** the local maxima $c_i^{\text{max}} = (s_i^{\text{max}}, \kappa_i^{\text{max}})$ and local minima $c_i^{\text{min}} = (s_i^{\text{min}}, \kappa_i^{\text{min}})$ of the curvature diagram.
- 5: **Take** N_e as the number of local extrema (maxima and minima) verifying: $\kappa_i^{\text{max}} \geq \kappa_{\min}$ or $\kappa_i^{\text{min}} \leq -\kappa_{\min}$, and separated all of them with a distance in the s -axis $\geq d_{\min}$.

From the value of N_e obtained by Algorithm 3, the lower and upper bounds can be defined, respectively, as follows: $N_{\min} = N_e - r$ and $N_{\max} = N_e + r$, with $r \geq 0$ being a parameter chosen to consider in the optimization process other alternative values of N close to N_e . The selection of this parameter should be made taking into account the visual information provided by the heading and curvature diagrams.

Finally, the probability estimates for the number of curves $N \in \{N_{\min}, \dots, N_{\max}\}$ can be defined by using a constant value $q_i = 1/(N_{\max} - N_{\min} + 1)$ for all $i = N_{\min}, \dots, N_{\max}$ if all of them are considered equiprobable, or by assigning decreasing probabilities according to their distance to N_e , for example, using again a Gaussian distribution for this purpose.

3 Numerical results

Several academic tests were conducted to assess the performance of the method under different scenarios. These tests considered hypothetical alignments with different numbers of circular curves (N) and different values of their design variables, and the results obtained are very satisfactory. After these tests, the behavior of the method was compared with an academic example previously used by Easa and Wang [19]. Despite the fact of its simplicity (only one circular curve), it was selected because all the information needed to simulate the test was available. Moreover, no other academic case was found to test the method proposed in this paper by comparing the results obtained with the same set of surveyed points. Finally, the practical usefulness of the method was tested in a case study consisting of the

recreation of an existing section of a railway line located in the Northwest of Spain. In this section, a comparison with the academic example of Easa and Wang [19] and the results obtained for the case study are presented and discussed.

3.1 Comparison with the academic example of Easa and Wang

The academic example proposed by Easa and Wang [19] consists of a hypothetical alignment composed by a single circular curve ($R = 300$ m) with symmetrical clothoids ($L^C = 100$ m) inserted between two known tangents with a deflection angle of 40° [19]. The first tangent of the alignment was the axis $Y = 0$, and the vertex was located at the origin of coordinates. Easa and Wang [19] used a set of 42 points randomly selected along the alignment to show the ability of their methodology to fit its geometry and to recover the exact parameters of radius and clothoid length. The coordinates of this set of points expressed in centimeters were published by the authors, being therefore available in order to simulate the experiment with the method proposed in this work. However, owing to the accuracy of the data, those points cannot be considered to belong to the exact alignment but to represent the position of a slightly distorted geometry.

In this simple illustrative case, only the first stage of the method described in the previous section (solving problem (6)) was conducted by considering $N = 1$ and the same technical constraints of the original experiment: $R_{\min} = 150$ m, $L_{\min}^T = 0$ m, $L_{\min}^{CC} = 10$ m, and $L_{\min}^C = 20$ m, except those related to the maximum values of radius or length which are not taking into account here. A multi-start with 10 initial alignments (seeds) obtained with the aforementioned algorithm for the automatic generation of random

admissible horizontal alignments was used. The optimal solution obtained was: $X_1 = -0.011$ m, $Y_1 = -0.008$ m, $R_1 = 301.020$ m, and $\omega_1 = 0.3618$ rad, with a value of the objective function J_1 equal to $8.9216 \times 10^{-3} \text{ m}^3$. The mean y -distance between the RHA and the spline constructed along the points was equal to 0.004 m, with the length of the clothoids and the circular curve obtained being 101.235 and 108.916 m, respectively. The set of points and the optimal RHA obtained by applying the method are shown in Fig. 4. It can be observed the high quality of the fitting with the set of points and the location of the tangency points (boundaries between elements) provided by the RHA.

3.2 Case study

After verifying the good behavior of the method with academic tests and the academic example of Easa and Wang [19], it is also tested in a case study in which a railway line in the Northwest of Spain crossing the village of Parga (Lugo) is considered (Fig. 5). The method was applied to an approximately 10 km long section of the existing single-track railway between the locations of San Alberte and Guitiriz. The recreation of this horizontal alignment was an initial task conducted by the authors in order to be able to design later an optimal railway bypass avoiding the urban zone of Parga [32]. The objective is to identify its geometrical elements and recover its parameters as accurately as possible in order to use some of them as candidates for bypass connections.

In the absence of measured data along the railway track, the Universal Transverse Mercator (UTM) coordinates of a set of points corresponding to its centerline were obtained from the database of the National Geographic Institute (IGN). This set contains only 253 points distributed non-uniformly along the entire length of the railway line section. The distances between consecutive points vary between a minimum of 1.37 m and a maximum of 137.06 m, making very difficult to obtain a good estimation of some geometrical elements by means of identification methods based on curvature or heading. According to the information provided by the IGN regarding the positional accuracy of the data, it can be expected errors are up to 1.00 m, and the recreation results that can be obtained from this set of points are significantly affected by this fact. In addition, informative boards (Fig. 6) with geometrical data about the horizontal alignment are located at the proximity of the boundaries between elements, and hence, the two-stage method proposed in Sect. 2 can be applied to this specific case. Considering all the information available about the railway line section and after applying the method for estimating the number of curves (Algorithm 3) for further verification, a number of eleven circular curves ($N = 11$) was assumed for the application of the two-stage method. For this specific case, considering the technical standard pertaining to this railway line [33] and assuming a maximum velocity equals to 100 km/h, the following technical constraints were considered: the minimum radius of circular curves: $R_{\min} = 495$ m; the minimum length of circular curves: L_{\min}^{CC}

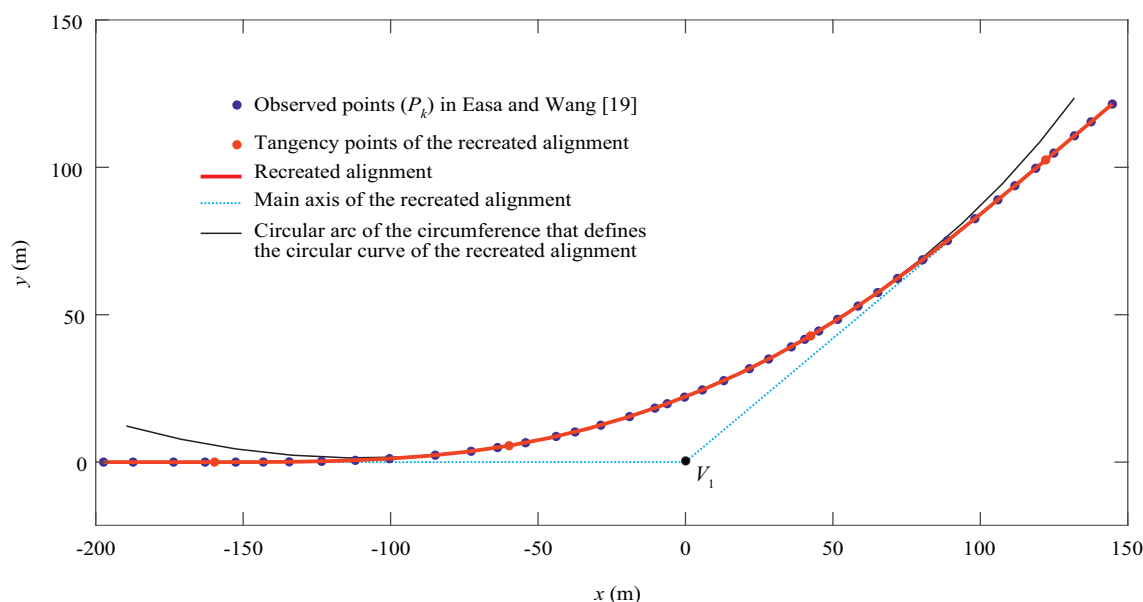


Fig. 4 Optimal RHA obtained for the academic example

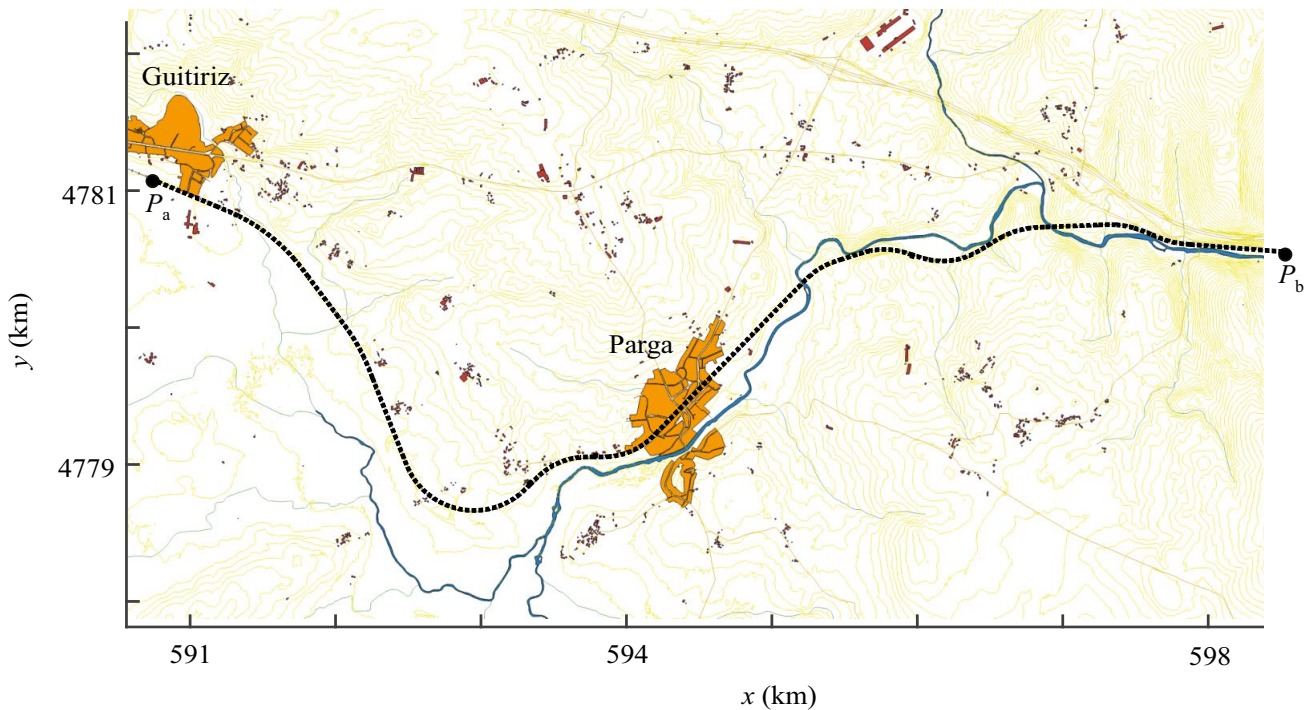


Fig. 5 Segment of railway line used as case study



Fig. 6 Informative board located along the railway track with geometrical information about the horizontal alignment elements

= 33.33 m; and the minimum length of clothoids: $L_{\min}^C = 95$ m. Regarding the minimum length of tangents, the corresponding value according to the maximum velocity was $L_{\min}^T = 33.33$ m. However, the technical standard states

that if it is not possible to achieve this value for reverse curves, it is preferable to connect the transition curves without an intermediate tangent section. Considering this fact and the presence of reverse curves in the railway line in which it is likely that this recommendation should be taken into account, a value of $L_{\min}^T = 0$ m was finally consider. Finally, the UTM coordinates of the starting point and the ending point selected for the RHA were: $P_a = (590,562.500, 4,781,157.300)$ m and $P_b = (598,709.200, 4,780,626.100)$ m.

The first stage of the method (solving problem (6)) was carried out by using a multi-start with 200 initial alignments (seeds) obtained with the algorithm for the automatic generation of random admissible horizontal alignments. The optimal solution obtained after solving the optimization problem with all of them is presented in Table 1 in terms of its decision variables: the coordinates of its vertices (X_i, Y_i) and the radii R_i and angles ω_i of its circular curves. The minimum value of the objective function J_1 corresponding to this optimal solution was equal to $2,335.709 \text{ m}^3$.

As mentioned before, in the second stage (solving problem (9)), the optimal solution obtained from the first stage is used as the initial solution. Regarding the penalty function, only the original radius values R_i^0 provided by the informative boards

Table 1 Optimal solution obtained in stage 1 (x_{S1}^N)

i	X_i (m)	Y_i (m)	R_i (m)	ω_i (rad)
1	591,458.874	4,780,782.897	1,194.134	0.2609
2	592,081.477	4,779,946.497	1,146.321	0.1034
3	592,706.596	4,778,342.170	518.006	1.7186
4	593,505.012	4,779,142.511	497.760	0.5805
5	594,043.301	4,779,140.346	501.947	0.5853
6	595,376.846	4,780,517.370	524.139	0.2570
7	595,854.003	4,780,667.431	502.121	0.4125
8	596,328.141	4,780,508.600	496.450	0.6582
9	596,798.364	4,780,793.441	526.664	0.2728
10	597,579.975	4,780,825.623	586.293	0.2180
11	597,922.560	4,780,686.093	617.175	0.1257

Table 2 Optimal solution obtained in stage 2 (x_{S2}^N)

i	X_i (m)	Y_i (m)	R_i (m)	ω_i (rad)
1	591,457.587	4,780,783.589	1,273.009	0.3606
2	592,081.399	4,779,947.431	1,300.030	0.1626
3	592,707.497	4,778,338.049	519.863	1.7503
4	593,502.823	4,779,141.725	499.999	0.5856
5	594,043.893	4,779,141.113	497.497	0.5706
6	595,376.484	4,780,516.927	511.998	0.2511
7	595,854.120	4,780,667.198	499.996	0.4120
8	596,332.603	4,780,507.806	499.992	0.6596
9	596,795.473	4,780,794.276	600.001	0.3546
10	597,580.974	4,780,825.301	601.998	0.2334
11	597,922.999	4,780,686.363	499.999	0.0777

were considered as the geometrical parameters to be penalized, assuming in this specific case that all values of radius should have the same importance. Therefore, the penalty parameters applied to the length of each element were set to zero: $p_i^{CC} = 0$, $p_i^C = 0$ for $i = 1, 2, \dots, 11$; and $p_j^T = 0$ for $j = 1, 2, \dots, 12$. In order to select an appropriate value for the penalty parameters of the radii, a sensitivity analysis was performed based on the results of preliminary tests consisting of solving the second optimization problem (9) with different values: $p_i^R = 1, 10, 100$ and 1000 m for $i = 1, 2, \dots, 11$. After confirming that the desired effect on the radii was achieved, taking into account the differences obtained between the results and the known values (R_i^o), the last value ($p_i^R = 1000$ m for $i = 1, 2, \dots, 11$) was finally selected for this case study.

The optimal solution obtained in this stage is presented in Table 2, with the value of the objective function J_2 being equal to $2,555.246 \text{ m}^3$, slightly greater than the value obtained for J_1 in the precedent stage. This increment can be explained in part by the additional contribution of the penalty function to J_2 . However, it is also possible that some parts of the recreated alignment were forced to get more separation from the existing alignment in order to be able to fit the value of the radius on each circular curve. The effect of the penalization on circular curves can be observed in Table 3 where the values of the original radius and those obtained in both stages of the method can be compared. The optimal solution of the first stage provides values of radius similar to the original ones for the majority of the circular curves but some of them are clearly far from those expected. On the contrary, in the second stage, the penalty term forces the solution to adopt practically the same values as the original ones with very small deviations.

The final solution for the RHA obtained by means of the proposed two-stage optimization method is shown in Fig. 7. It can be observed the different geometrical elements (tangents, circular curves, and transition curves) and the position of the vertices of the main axis. The mean of the y-distances computed with the whole set of points was equal to 0.442 m, clearly lower than the positional accuracy that could be expected from the data.

Detailed information about the whole RHA is provided in Table 4, including the most relevant geometrical parameters of each element and the position (coordinate s) of the tangency points (boundaries between them). Of course, the geometrical model provides the coordinates (x, y) of these points, but this information was omitted to avoid overloading the table with an excessive number of parameters. It can be easily verified that the geometrical parameters of the obtained RHA actually satisfy the technical constraints (R_{\min} , L_{\min}^{CC} , L_{\min}^C , and L_{\min}^T) required for this specific case to ensure operational safety.

Table 3 Comparison between the original radii and the values obtained in the two-stage method (m)

i	R_i^o	$R_i(x_{S1}^N)$	$R_i(x_{S2}^N)$
1	1,273.0	1,194.134	1,273.009
2	1,300.0	1,146.321	1,300.030
3	520.0	518.006	519.863
4	500.0	497.760	499.999
5	497.5	501.947	497.497
6	512.0	524.139	511.998
7	500.0	502.121	499.996
8	500.0	496.450	499.992
9	600.0	526.664	600.001
10	602.0	586.293	601.998
11	500.0	617.175	499.999

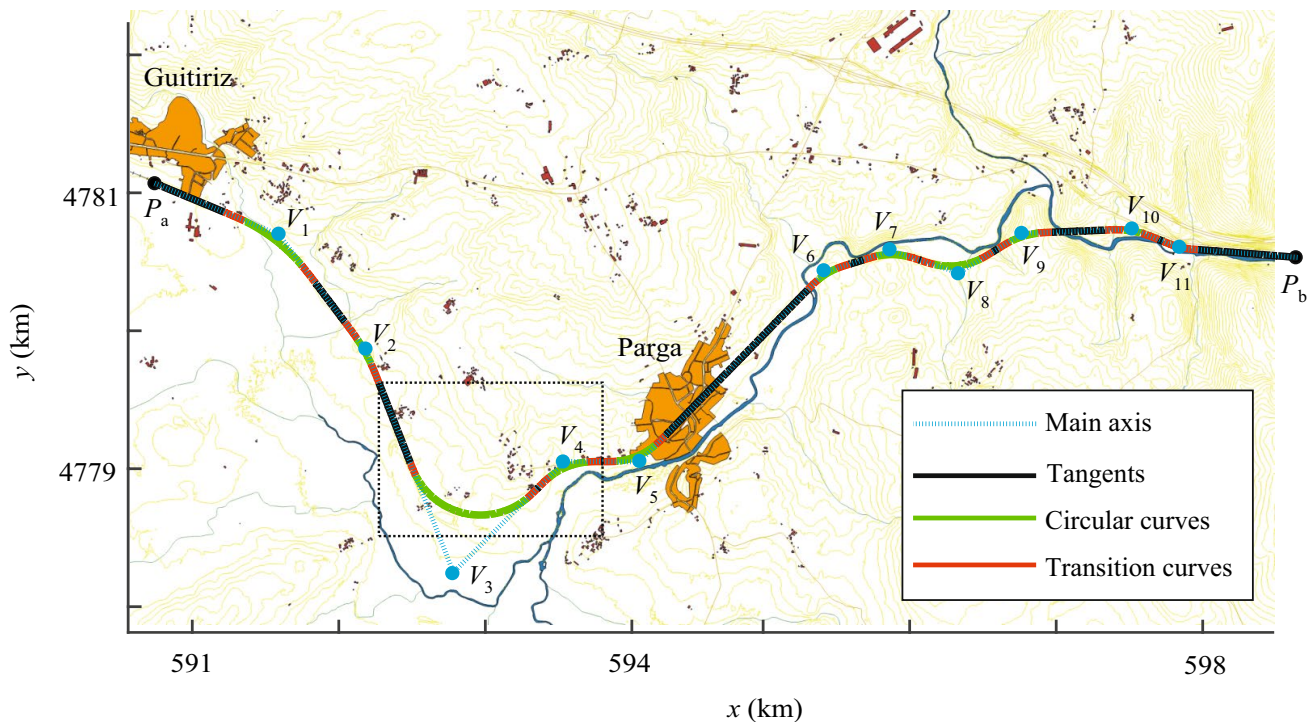


Fig. 7 Optimal horizontal alignment obtained with the proposed two-stage method

In order to appreciate the quality of the fitting with the existing alignment, the zone of the RHA identified with the dashed line box in Fig. 7 was selected as an example. This zone was enlarged and presented with additional information in Fig. 8, which includes the following geometrical elements: T_3 , C_{3-} , CC_3 , C_{3+} , T_4 , C_{4-} , CC_4 , C_{4+} , and T_5 , according to the names assigned in Table 4. The upper part of the figure shows the RHA obtained (red line) and the available points (blue crosses) on each geometrical element, while the bottom part of the figure provides the values of y -distance computed from the RHA to each point. It can be seen that the number of points finally assigned to each geometric element varies as well as the consecutive distance between them. For example, the first tangent T_3 has only five points associated with it. The results of y -distance obtained for this specific element, among which is

the highest value of this entire zone, show the difficulty of adjusting properly a straight line (tangent) to these points, showing that the recreation is clearly limited by the number and the accuracy of the available data. The average y -distance calculated for these nine geometrical elements was 0.337 m, slightly lower than the value obtained for the whole RHA (0.442 m). The values of y -distance obtained in other zones of the RHA can be considered similar without significant differences.

Taking into account the main objective of this case study mentioned before and despite the lack of more precise data measured along the railway centerline, the optimal RHA obtained as solution of applying the two-stage method recreates the existing alignment accurately enough within the limitations imposed by the data, allowing us to recover the geometric parameters of its elements.

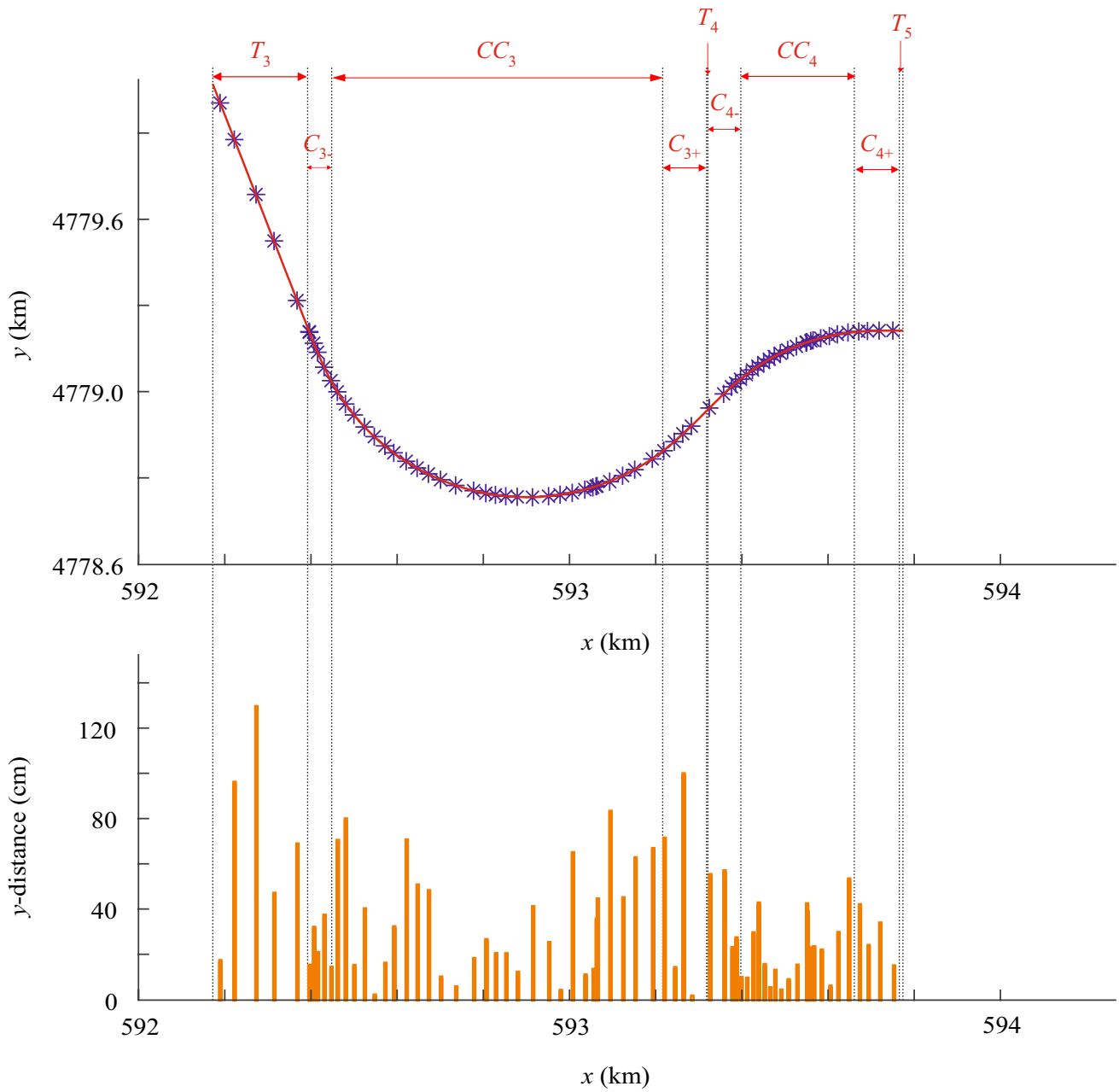


Fig. 8 Optimal horizontal alignment obtained and values of y -distances computed in the zone highlighted (dashed line box) in Fig. 7

Table 4 Geometrical information of each geometrical element of the optimal RHA

Element	s_i (m)	s_f (m)	L^E (m)	R (m)	A (m)
T_1	0+000.000	0+510.595	510.595	–	–
C_{1-}	0+510.595	0+731.693	221.097	–	530,527
CC_1	0+731.693	1+190.788	459.096	1273.009	–
C_{1+}	1+190.788	1+411.886	221.097	–	530,527
T_2	1+411.886	1+749.328	337,443	–	–
C_{2-}	1+749.328	1+888.885	139.557	–	425,944
CC_2	1+888.885	2+100.273	211.388	1300.030	–
C_{2+}	2+100.273	2+239.830	139.557	–	425,944
T_3	2+239.830	2+854.948	615.118	–	–
C_{3-}	2+854.948	2+979.749	124.801	–	254,715
CC_3	2+979.749	3+889.684	909.935	519.863	–
C_{3+}	3+889.684	4+014.485	124.801	–	254,715
T_4	4+014.485	4+018.977	4.491	–	–
C_{4-}	4+018.977	4+122.029	103.053	–	226.994
CC_4	4+122.029	4+414,851	292.822	499.999	–
C_{4+}	4+414,851	4+517.904	103.053	–	226.994
T_5	4+517.904	4+528.971	11.067	–	–
C_{5-}	4+528.971	4+644.332	115.361	–	239.566
CC_5	4+644.332	4+928,206	283.874	497.497	–
C_{5+}	4+928,206	5+043.568	115.361	–	239.566
T_6	5+043.568	6+496.847	1453.280	–	–
C_{6-}	6+496.847	6+622,540	125.691	–	253.680
CC_6	6+622,540	6+751,078	128.539	511.998	–
C_{6+}	6+751,078	6+876.769	125.691	–	253.680
T_7	6+876.769	6+968,774	92.005	–	–
C_{7-}	6+968,774	7+075.965	107.191	–	231.506
CC_7	7+075.965	7+281.956	205.991	499.996	–
C_{7+}	7+281.956	7+389.148	107.191	–	231.506
T_8	7+389.148	7+389.148	0.001	–	–
C_{8-}	7+389.148	7+497.221	108.072	–	232.455
CC_8	7+497.221	7+827.021	329.800	499.992	–
C_{8+}	7+827.021	7+935.093	108.072	–	232.455
T_9	7+935.093	7+984,784	49.691	–	–
C_{9-}	7+984,784	8+080.877	96.093	–	240.116
CC_9	8+080.877	8+293.619	212.743	600.001	–
C_{9+}	8+293.619	8+389.712	96.093	–	240.116
T_{10}	8+389.712	8+781.781	392.070	–	–
C_{10-}	8+781.781	8+897.306	115.525	–	263.715
CC_{10}	8+897.306	9+037.833	140.527	601.998	–
C_{10+}	9+037.833	9+153.358	115.525	–	263.715
T_{11}	9+153.358	9+198.579	45.221	–	–
C_{11-}	9+198.579	9+314.407	115.828	–	240.653
CC_{11}	9+314.407	9+353.258	38.851	499.999	–
C_{11+}	9+353.258	9+469.086	115.828	–	240.653
T_{12}	9+469.086	10+121,575	652,489	–	–

4 Conclusions

In this paper, a simple method for automatically recreating horizontal alignments of existing railway lines is proposed. On the basis of a previous geometrical model developed to easily obtain a continuous parametrization of the centerline of a horizontal alignment composed of tangents, circular curves, and clothoids, the formulation of two constrained optimization problems is defined. In both of them, the main objective is optimizing a recreated alignment by minimizing the distance from an existing one, but in the second one, an additional penalty function is included in order to force the solutions to adjust their geometrical parameters to some known values. The method consists of two stages in which both optimization problems are solved consecutively by means of numerical techniques using the optimal solution of the first one as initial alignment for starting the resolution of the second one. However, if there is no information about the geometric parameters of the existing alignment, then, the first stage of the method can still be applied. After being tested with an academic example in order to assess its performance, the method was successfully applied to a railway line section located at Parga (Spain), proving to be a very useful tool for recovering the geometry of existing horizontal alignments. The main advantages of this simple method can be summarized as follows:

- It is not necessary to conduct the identification stage in order to assign a potential geometrical element, avoiding all the inherent difficulties related to this task.
- The second stage allows to consider during the optimization process some available information about the geometry of the existing alignment.
- The whole horizontal alignment can be recreated automatically, instead of obtaining different separate parts that should be properly connected later.
- All the geometrical elements (tangents, circular curves, and transition curves) are optimized simultaneously, without any of them being predefined in advance.
- The definition of the optimization functions avoids the need of spending time on solving over and over the problem of computing the distance between a point and a clothoid curve.

On the contrary, the proposed method presents the following limitations and drawbacks that should be taken into account in further research in order to improve its performance or being able to apply it to other scenarios.

- Some difficulties during the generation process were observed in the recreation of the case study, owing to the different spatial distribution of the curves along the railway section. As it can be observed in Fig. 7, the distance between the vertices of the RHA is greater in the western zone than in the eastern zone. This fact, not considered at the moment, should be studied in order to improve the random generation of initial alignments.
- The mathematical model considers exclusively symmetrical clothoids (as mentioned in Sect. 2), being of great interest to implement in the future the possibility of considering non-symmetrical transition curves.
- The mathematical model considers only one combination of geometrical elements: tangent segments, joined by circular curves with symmetrical clothoids. It would be very useful to adapt the model to other combinations, such as compound curves, very common in older railway lines

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Data availability statement The data that support the findings of this research are available from the corresponding author, upon reasonable request.

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