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Higher Social Choice

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Abstract. The problem of finding a social choice function in a given space of preferences has been dominated by Arrow's and Chichilnisky's Impossibility Theorems. Based on previous work by Carrasquel, Lupton and Oprea, in this paper, we use tools from Algebraic Topology to introduce a notion of *higher social choice complexity* that determines the minimum number of local social choices that must be aggregated to cover all possible individual preferences. We prove that this invariant is bounded between two known topological invariants, the higher topological complexity and its symmetric version. This result shifts the focus onto the topological structure of the preference space when studying social choice processes.

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1. Introduction

1.1. Impossibility Theorems

Social Choice theory studies how to aggregate the preferences of various individuals in order to obtain a common group preference. The idea is to provide a rule for making collective decisions when individuals have different choices. It has applications in a large variety of branches as economics, political science, philosophy and mathematics [1]. Recently, it is being applied also in computer science and biology [2,3]. The best known result is Arrow's impossibility theorem [4], showing that under reasonable axioms (Pareto efficiency, non-dictatorship and independence of irrelevant alternatives), it is not possible to aggregate the individual preferences into a social one. Arrow's theorem and its proof are of a combinatorial nature.

Much later, G. Chichilnisky [5] introduced a topological approach to social choice. She considered the space of preferences as a topological space and imposed an analogous list of axioms (continuity, anonymity, unanimity) adapted to this setting. She proved that under these hypothesis, for usual preference spaces (namely, parafinite CW-complexes), an aggregation rule exists if and only if the space of preferences is contractible.

The social choice maps on a topological space had been studied before by G. Aumann [6], but with the name of "topological n-means". This was also the name used by B. Eckmann in 1954 [7] (see also [8]) to obtain independently the same result as Chichilnisky. However, Eckmann did not mention the social choice maps in his first paper. Later, S. Weinberger [9] proved that it is possible to find examples of non-finite non-contractible CW-complexes on which choices may be aggregated.

Y.M. Baryshnikov [10] unified the combinatorial and topological approaches, by proving that Arrow's theorem is a consequence of the non-contractibility of spheres. The work of S. Rajsbaum and A. Raventós [11] is a nice merge of combinatorial and algebraic topology that explores the relation between both arguments and deepens in the understanding of the impossibility theorem.

1.2. Topological Methods

The study of the social choice paradox from a topological point of view has been a subject of interest since its early beginnings. L. Lauwers gives in [12] a detailed survey where Arrow's and Chichilnisky's models are carefully analyzed. The requirements of the topological model have been examined in order to study its accuracy for the economic setting. For instance, in [13], the election of the chosen topology is discussed. Also, in the surveys by N. Baigent [14] and Lauwers [15], the convenience of the continuity demand is critically discussed. In the work of C.D. Horvath [16], it is shown that the structure of CW complex of the space of preferences is not necessary for the existence of social rules. Moreover, in this case, contractibility is not a requirement. For example, in the space $\mathbb{Q}$ of rational numbers, the arithmetic mean is a social choice rule.

There has been a lot of attempts to redefine the assumptions for an appropriate social choice rule. The topological approach to the question gives a new insight, because, as an alternative to analyze the conditions on the aggregation map, we can investigate the topological features of the space of preferences [17]. For a general class of spaces of preferences, contractibility is a necessary and sufficient condition for the existence of a social choice rule. Recently, it has been proposed to convert a non-contractible preference space into a contractible one by adding a null preference [18], the problem is that this new space is no longer a CW-complex. However, as Horvath said, "even in the absence of contractibility, the local structure of the space might be nice enough to allow for the existence of partial social choice functions" [16]. Then, instead of looking for a global social choice rule, it was proposed by J. Carrasquel, G. Lupton and J. Oprea in [19] to try to measure the minimal number of pieces in which we need to split the space of preferences so that for each piece it is possible to give a local aggregation rule.

1.3. Social Choice Complexity

Moreover, these authors pointed out a parallelism between social choice and the topological complexity introduced by M. Farber in [20]. They introduced in [19] the so-called *social choice complexity* for two individuals. Inequalities relating this social complexity with the topological complexity and the symmetric topological complexity are given in [19, Theorem 3.5]. They also sketched a theory of higher social choice complexity. However, they indicated that "there is not, at present, a suitable notion of 'symmetric higher topological complexity' that might be placed as an upper bound" of that higher social choice complexity.

In this work, we give (Theorem 4.4) a detailed proof of the result stated in [19, Theorem 3.7] that relates the social choice complexity with the topological complexity. Furthermore, we carefully establish the definition of *higher or sequential social choice complexity* and we study its properties. The relevance of this generalization comes from the fact that, in practice, we will be interested in aggregating the preferences of a whole society and not only of a pair of individuals. In Theorem 4.7, we show that there is a deep relation between the topology of the space of preferences and the minimum number of local social choice functions which are needed when there does not exist a global social choice. Specifically, we bound this minimum between the higher topological complexity and the *higher symmetric topological complexity* defined by Y.B. Rudyak [21] and later by I. Basabe et al. [22]. and M. Grant [26].

1.4. Contents

The contents of the paper are as follows: in Sect. 2, we explain the basics of Social Choice theory. Section 3 is about several topological invariants related with Social Choice, namely topological complexity, symmetrized topological complexity and their higher versions. In Sect. 4, we introduce the notion of social choice complexity for two agents and higher social choice complexity for three or more agents, and we prove their relationship with the topological invariants cited before.

Finally, following the advice of the referee and considering the potential audience of economists who may not be experts in topology, we have included an Appendix with a brief introduction to the essential definitions and results from homotopy theory needed to fully understand the paper.

2. Basics of Social Choice Theory

The notion of a *social choice map* is well known (see [12,23,24]). There are two perspectives: the classical one is combinatorial, and leads to the famous impossibility theorem of Arrow [4]; the second one, developed by Eckmann [8] and, independently, by Chichilnisky [5], is topological. However, both are probably different aspects of a more general unifying theory [10].

We adopt here the topological approach. Let X be the space of possible individual choices, modeled as a topological space. The space X can be just a set of points —that is, a discrete space representing, for instance, political parties— or a continuous one —like for instance a highway or the spectrum of colors.

If our society has n members, then the input for the aggregation rule is an n-tuple formed by the preferences of each agent, and the output is a single value. So, a social choice is represented by a map $m\colon X^n \to X$ which is called a *social choice map*. That means that we must assign to each collection $(x_1, \dots, x_n)$ of individual choices a "social choice" or "mean", which is some point $m(x_1, \dots, x_n) \in X$.

The map m must verify the following conditions:

1. *Stability*. The map m must be continuous. A small change in the input should lead to a small change in the output.
2. *Unanimity*. If all agents agree in some option x, then the social option must be x, that is, $m(x, \dots, x) = x$.
3. *Anonymity*. None of the agents is more important than any other. Then, the choice must be a consequence of the elections $x_1, \dots, x_n$, but not of the agents that choose them. We can formalize this idea by saying that m does not depend on the permutation of its entries.

The classical result (proved by B. Eckmann [7] and independently by G. Chichilnisky [5]) is that such a map m does not exist, except when the topological space X is contractible.

Theorem 2.1 ([7], [5, Theorem 1], see also [8, Theorem 6]). *A finite CW complex X admits an n-mean for some $n \geq 2$ if and only if X is contractible (and thus all n-means exist).*

See [8] for a discussion of the history of this result.

Remark 2.2. Contractibility and homotopy can be interpreted in terms of *closeness*. Consequently, the idea of "close compromise" in [19] will express that the social choice must be close to each of the individual chosen preferences. We will say that the choices of two different agents are "close" if it is possible for each of them to evolve continuously from their original positions to the social choice, hence to the option of the other one. In [12], contractibility indicates that a single individual can play a dominant role in the determination of the social preference.

As a referee pointed out, Weinberger suggested that a close compromise might be achieved through negotiations among agents. Although we could not locate the specific reference, the concept of Solomonicity described in [9] appears to reflect this idea.

3. Topological Invariants

We recall several notions of applied algebraic topology that appeared in the last twenty years and will be useful in the context of social choice.

3.1. Topological Complexity

Let X be a topological space. In many applications, X is the configuration space of a mechanical device, like a drone or a robot. Here, we will interpret it as the space of preferences of a set of individuals.

Definition 3.1 (*Local motion planning algorithm*). A *local motion planning algorithm* is any rule defined on a subset $U \subset X \times X$ which assigns, in a continuous way, to each pair of points $(x, y) \in U$, a continuous path γ on X going from x to y. That is, $\gamma(0) = x$ and $\gamma(1) = y$.

Usually we shall ask U to be an *open subset* of $X \times X$, but in some situations more general types of subsets may appear (see Sect. 4.4).

We will assume that the path γ depends continuously on the initial and final positions x, y. For this continuity condition in Definition 3.1, we choose the *compact-open topology* in the space X^I of all paths on X. With this topology, the map $\pi\colon X^I \to X \times X$ which sends each path γ to its initial and final points $(\gamma(0), \gamma(1))$ is continuous. Then, a local motion planning can be seen as a local section of π, that is, a continuous map $s\colon U \subset X \times X \to X^I$ such that the composition $\pi \circ s$ is the inclusion, that is, $\pi s(x, y) = (x, y)$ for all $(x, y) \in U$.

It is not hard to see that a *global* (that is, with $U = X \times X$) motion planning exists if and only if the space X is contractible.

Theorem 3.2 ([20, Theorem 1]). *A continuous map $s\colon X \times X \to X^I$ such that $\pi s(x, y) = (x, y)$ for all $(x, y) \in X \times X$ exists if and only if the space X is contractible.*

In view of the latter result, it is natural to ask which is the minimum number of local solutions that we need to cover all contingencies when X is not contractible. Alternatively, we can describe this invariant as the minimal number of discontinuities of any global motion planning algorithm.

Definition 3.3 (*Topological complexity*, [20]). The *topological complexity* of X, denoted by $\mathrm{TC}(X)$, is the minimum number $k \geq 1$ of subsets U_j covering $X \times X = U_1 \cup \cdots \cup U_k$, such that on each set U_j, there is a local motion planning algorithm $s_j\colon U_j \to X^I$.

Then, $\mathrm{TC}(X) = 1$ means that there is a global motion planning algorithm.

3.2. Symmetric Topological Complexity

In [25], the following refinements to Definition 3.3 were suggested. First, if $x = y$, the path connecting x and y should be the constant path. Second, the solution for going from y to x should be the same as that for going from x to y, but run backwards. This leads us to the following definition.

Definition 3.4 (*Symmetric topological complexity*). The *symmetric topological complexity* of X, denoted by $\mathrm{TC}^S(X)$, is the least integer $k \geq 1$ such that $X \times X$ can be covered by k subsets $U_1 \cup \cdots \cup U_k = X \times X$, verifying

1. Each set U_j is symmetric, that is, $(x, y) \in U_j$ implies $(y, x) \in U_j$.

2. On each U_j, there is a local motion planning algorithm $s_j\colon U_j \to X^I$.
3. The algorithms s_j are equivariant, that is, $s_j(y,x)(t) = s_j(x,y)(1-t)$, for all $(x,y) \in U_j$ and $t \in I$.
4. If $(x,x) \in U_j$ then $s_j(x,x)$ is the constant path c_x.

This differs from the definition of the symmetrized topological complexity condition $\mathrm{TC}^\Sigma(X)$ given in [22]. The difference lies in the *monoidal* Condition 4.

For the sake of future generalizations (see Sect. 3.4), Conditions 1 and 3 in Definition 3.4 can be formalized as follows. Let Σ_2 be the group of permutations of the set $\{1,2\}$, that is $\Sigma_2 = \{\mathrm{id}, \sigma\}$, where $\mathrm{id} = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ and $\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. The action of Σ_2 over $X \times X$ is given by $\sigma(x,y) = (y,x)$, while the action on the space of paths is $\sigma(\gamma)(t) = \gamma(1-t)$. In other words, the symmetric group acts on $X \times X$ by permuting coordinates and on X^I by reversing paths. Then Condition 1 can be written as $\sigma(U_j) = U_j$ while Condition 3 is $s_j\sigma = \sigma s_j$.

3.3. Higher Topological Complexity

Rudyak introduced the notion of higher topological complexity in [21]. Nowadays, the term *sequential* is commonly used instead of *higher*. This time, given n points $x_1, \ldots, x_n$ we need to find a path passing successively through each one. So, for $n = 2$, we shall recover the original topological complexity. For $n = 3$, given three points x, y, z, we should find a path γ such that $\gamma(0) = x$, $\gamma(1/2) = y$ and $\gamma(1) = z$.

For $n \geq 2$, we will consider the map $\pi_n\colon X^I \to X^n$ given by

$$\pi_n(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{n-1}\right), \ldots, \gamma\left(\frac{n-2}{n-1}\right), \gamma(1)\right).$$

A continuous motion planning algorithm will be a continuous section of π_n, that is, a map $s_j\colon U_j \subset X^n \to X^I$ such that $\pi_n \circ s_j = \iota_j\colon U_j \hookrightarrow X^n$ is the inclusion.

Definition 3.5 (*Sequential topological complexity*, [19, Definition 3.6]). Given X a path-connected space, the *sequential topological complexity of* X, denoted by $\mathrm{TC}'_n(X)$, is the minimum integer $k \geq 1$ such that X^n can be covered by k pieces, $U_1 \cup \cdots \cup U_k = X^n$, in such a way that on each U_j, there is a continuous motion planning algorithm $s_j\colon U_j \to X^I$.

We denote this invariant by TC'_n to distinguish it from the (equivalent) definition of TC_n that will be given in Definition 3.8. When $n = 2$, we have $\mathrm{TC}'_2(X) = \mathrm{TC}(X)$.

Since the parametrization of the paths is not relevant, the following technical tool was introduced in [21]; it will simplify several proofs, mainly in Sect. 3.4.

Definition 3.6. For $n \in \mathbb{N}$, let J_n be the union of n intervals $I_k = [0,1]$, $k = 1, \ldots, n$, by their common point 0 (see Fig. 1).

Usually this is called a "wedge" or a "bouquet" of intervals. Each element $t_k = (t,k)$ of $I_k \subset J_n$ is denoted by the parameter $t \in [0,1]$ and the label k indicating the arm to which the point belongs.

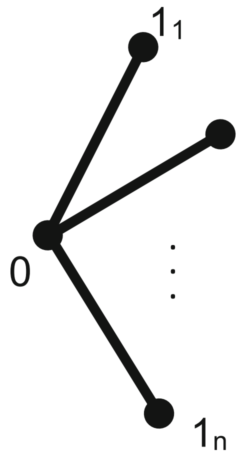

Figure 1. The wedge J_n

A continuous map $\alpha\colon J_n \to X$ will be called an *octopus of n arms* or an *n-multi-path*. The space of all such maps will be denoted by X^{J_n}. In this space, it is natural to define the map $e_n\colon X^{J_n} \to X^n$ given by

$$e_n(\alpha) = (\alpha(1_1), \dots, \alpha(1_n)).$$

It is not hard to translate the concepts of Sect. 3.2 to this new tool.

The precise relationship between π_n and e_n is as follows: first, for $n \geq 2$, we can define the map $A\colon I = [0,1] \to J_n$ given by

$$A(t) = \begin{cases} (-2(n-1)t + 2k + 1, k+1) & \text{if } \frac{k}{n-1} \leq t \leq \frac{2k+1}{2(n-1)}, \\ (2(n-1)t - (2k+1), k+2) & \text{if } \frac{2k+1}{2(n-1)} \leq t \leq \frac{k+1}{n-1}, \end{cases} \quad \text{for } 0 \leq k \leq n-2.$$

That means that each segment $\left[\frac{k}{n-1}, \frac{k+1}{n-1}\right] \subset I$ is sent onto the arms $k+1$ and $k+2$ of the octopus J_n. In particular, the image of the mid-point is

$$A\left(\frac{2k+1}{2(n-1)}\right) = (0, k+1) = (0, k+2),$$

while for the endpoints we have

$$A\left(\frac{k}{n-1}\right) = (1, k+1), \quad A\left(\frac{k+1}{n-1}\right) = (1, k+2).$$

Now, we will define a map $B\colon J_n \to I$.

Fix a base point $\mathcal{O}$ in I, for instance, the mid-point $\mathcal{O} = \frac{1}{2(n-1)}$ of the first interval of the subdivision

$$x_1 = 0, x_1 = \frac{1}{n-1}, \dots, x_n = 1.$$

Each arm k of J_n, for $1 \leq k \leq n$, will be sent by B onto the segment connecting the points $\mathcal{O}$ and $x_k = \frac{k-1}{n-1}$. Namely,

$$B(t,k) = (1-t)\frac{1}{2(n-1)} + t\frac{k-1}{n-1}.$$

Excepting the case $n = 2$, the map A is not a homeomorphism. In fact, the map B is a homotopic inverse of A, because both J_n and $I = [0, 1]$ are contractible spaces.

Definition 3.7. 1. If $\alpha \in X^{J_n}$ is a multi-path, we define the path $\alpha^\flat \in X^I$ as $\alpha^\flat = \alpha \circ A$.
2. If $\gamma \in X^I$ is a path, we define the multi-path $\gamma^\sharp = \gamma \circ B$.

With these notations, for $\sigma \in \Sigma_n$ a permutation, the octopus

$$\gamma^\sharp(t, \sigma(k)), \quad 1 \leq k \leq n,$$

determines a path $\sigma \circ \gamma$ with permuted variables. This action would be much more difficult to define directly on subdivided paths instead of multi-paths.

We can give an alternative definition of higher topological complexity. It was first introduced by Rudyak [21, Definition 3.1], in a formulation that can be paraphrased as follows.

Definition 3.8 (*Sequential topological complexity, second definition*). The *sequential topological complexity* $\mathrm{TC}_n(X)$ of the space X is the minimum integer $k \geq 1$ such that X^n can be covered by k pieces $U_1 \cup \cdots \cup U_k = X^n$ so that on each piece U_j, there is a local homotopic section of e_n, that is, a continuous map $s_j : U_j \to X^{J_n}$ such that $e_n \circ s_j$ is homotopic to the inclusion, $e_n \circ s_j \simeq \iota_j$.

This is equivalent to Definition 3.5, as proved in [21, Remark 3.2.5]. Note that this follows from the relations $\pi_n \circ \flat \simeq e_n$ and $e_n \circ \sharp \simeq \pi_n$.

Proposition 3.9 ([21], Proposition 3.3). *If the space X is path connected, then* $\mathrm{TC}_n(X) \leq \mathrm{TC}_{n+1}(X)$ *for $n \geq 2$.*

The invariant $\mathrm{TC}_n(X)$ is usually rather high. For instance, if X is a connected finite CW-complex, then either X is contractible or $\mathrm{TC}_n(X) \geq n$ [21, Proposition 3.5].

3.4. Higher Symmetrized Monoidal Topological Complexity

The higher analog of the symmetric version of the topological complexity was first introduced by Rudyak in [21]. A few years later, Basabe et al. described in [22], a new higher symmetric topological complexity $\mathrm{TC}_n^\Sigma(X)$, that is a true homotopy invariant (see [22, Proposition 4.7]). In the final section of [21], M. Grant points out that there is a sequential version of the symmetrized monoidal topological complexity, which he denotes by $\mathrm{TC}_n^{M,\Sigma}$. Here we introduce a slight variant of it.

Definition 3.10 (*Higher symmetrized monoidal topological complexity*). The *higher symmetrized monoidal topological complexity* of the space X, denoted by $\left(\mathrm{TC}_n^{M,\Sigma}\right)'(X)$, is the least integer $k \geq 1$ such that there exists a covering $U_1 \cup \cdots \cup U_k = X^n$ such that, for each $j = 1, \ldots, k$,

1. $\sigma(U_j) = U_j$ for every $\sigma \in \Sigma_n$,
2. there is an equivariant continuous homotopic section of e_n, $s_j : U_j \to X^{J_n}$, that is, $e_n \circ s_j \simeq \iota_j$ with $\sigma s_j = s_j \sigma$;

3. moreover, $s_j(x, \ldots, x) = c_x$ where c_x is the constant map at x.

In [22], it is implicitly proved that the existence of a homotopic section $e_n \circ s_j \simeq \iota_j$ is equivalent to the existence of an actual section $e_n \circ S_j = \iota_j$, because the maps involved are fibrations. However, the equivariance of s_j does not imply that of S_j.

It is not hard to prove that $\mathrm{TC}_n^{\Sigma}(X) \leq (\mathrm{TC}_n^{M,\Sigma})'(X) \leq \mathrm{TC}_n^{M,\Sigma}(X)$. By Theorem 7.3 in [26] we have that, for X a paracompact ENR, the three versions coincide.

4. Social Choice Complexity

For two individuals that want to reach a common social choice, we know that it is not possible to provide a global social choice map, with the exception of very particular situations (cf. Theorem 2.1). So it is natural to seek for the minimal number of pieces in which we need to split the n-product of the space of preferences to ensure that we have local social choice maps in each piece. Carrasquel, Lupton and Oprea established this concept in [19].

4.1. Social Choice Complexity

Following the ideas developed in [19], we will consider *local social choice maps* $m_j \colon U_j \subset X \times X \to X$. These are social choice maps (continuous, unanimous and anonymous) defined only on a subset $U_j \subset X \times X$. We require these maps to be local homotopic sections of the diagonal $\Delta \colon X \to X^2$. The meaning of this condition will be explained later, but the idea behind it is that in the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle c}\downarrow & \searrow^{\Delta} & \\ X^I & \xrightarrow[\pi_2]{} & X^2 \end{array}$$

where $c(x) = c_x$ is the constant path at x, the path fibration π_2 is the fibrational replacement for the diagonal inclusion (see the Appendix).

Definition 4.1 (*Social choice complexity*, [19, Definition 3.3]). The *social choice complexity* of a space X, denoted by $\mathrm{sc}(X)$, is the least integer $k \geq 1$ such that $X \times X$ can be covered by k pieces $U_1 \cup \cdots \cup U_k = X \times X$ verifying:

1. Each U_j is symmetric, that is, $\sigma(U_j) = U_j$ for any permutation $\sigma \in \Sigma_2$.
2. For each U_j, there is a local social choice map $m_j \colon U_j \to X$. That is, m_j is a continuous map such that $m_j(x, x) = x$ for all $(x, x) \in U_j$ and $m_j(y, x) = m_j(x, y)$ for all $(x, y) \in U_j$. The latter condition can also be written as $m_j \circ \sigma = m_j$, for any $\sigma \in \Sigma_2$.
3. Each m_j is a local homotopic section of the diagonal map $\Delta \colon X \to X^2$, $\Delta(x) = (x, x)$, that is, $\Delta \circ m_j \simeq \iota_j$.

Remark 4.2. Regarding Condition 3, it is important to note that when we have a social choice m in a contractible space X, then the following additional property holds: the composition $\Delta \circ m\colon X^2 \to X^2$ is homotopic to the identity, because X^2 is contractible and then any two maps are homotopic.

Remark 4.3. Notice that in Condition 3, we have the notion of "homotopic section", which is different from a true section. The latter means $\Delta \circ m_j = \iota_j$, which would imply that if the "mean" of (x, y) is $m = m_j(x, y)$, then $(x, y) = (m, m)$, which is impossible when $x \neq y$.

Instead, if m_j is a homotopic section, we have a homotopy $H\colon \Delta \circ m_j \simeq \iota_j$. Then, for each pair (x, y), there is a path H_t between (x, y) and (m, m). In other words, the mean position m is "reachable" from both x and y. Of course this is only possible if x and y are in the same path-component of X.

Condition 3 is called "close compromise" in [19], so the social choice should be "close" to each of the individual preferences. This means that an individual could gradually change his preference to the result of the social choice. It could be said that both individuals could come to accept the "average" of their preferences. A simple example would be that of a married couple who have to decide a color to paint their house. One of them chooses color "yellow", and the other color "red". So a solution to their problem could be the color "orange", which would be the mixture of both colors.

The notions of social complexity and topological complexity are deeply related. The next result establishes the connection between these concepts and with the symmetrized topological complexity. We provide here a complete proof.

Theorem 4.4 ([19], Theorem 3.5). *Let X be a topological space, then*

$$\mathrm{TC}(X) \leq \mathrm{sc}(X) \leq \mathrm{TC}^{\Sigma}(X).$$

Proof. Assume $\mathrm{TC}^{\Sigma}(X) = \ell$ and let $U_1 \cup \cdots \cup U_\ell = X \times X$ be a covering with $s_j\colon U_j \to X^I$ a local symmetric motion planning algorithm in U_j. Take $r\colon X^I \to X$ the map that associates to each path γ its mid-point $\gamma(1/2)$. We can define on each U_j the local social choice map $m_j = r \circ s_j$. By taking the homotopy

$$G(t, x, y) = (s_j(x, y)(t/2), s_j(x, y)(1 - t/2))$$

we have $\Delta \circ m_j \simeq \iota_j$. Then $\mathrm{sc}(X) \leq \ell$.

For the remaining inequality, if we suppose $\mathrm{sc}(X) = \ell$, we have $X \times X = V_1 \cup \ldots \cup V_\ell$ where each V_j is symmetric and there exist local social choice maps $m_j\colon V_j \to X$ verifying $\Delta \circ m_j \simeq \iota_j$. Let $H_j\colon I \times V_j \to X \times X$ be the homotopy such that $(H_j)_1(x, y) = (x, y)$ and $(H_j)_0(x, y) = (m, m)$ where $m = m_j(x, y)$. Then, for each V_j we can, take the 2-multi-path $\alpha_j\colon V_j \to X^{J_2}$ given by

$$\alpha_j(x, y)(t_k) = p_k((H_j)_t(x, y)), \quad k = 1, 2,$$

where $p_k\colon X \times X \to X$ is the projection $p_k(x_1, x_2) = x_k$. We have that α_j is a local homotopic section of e_2, and it satisfies $\alpha_j(1, 1) = x$, $\alpha_j(1, 2) = y$ and $\alpha_j(0, k) = m$. Then $\mathrm{TC}(X) \leq \ell$. □

Example 4.5. This is a classical example in Social Choice theory [27]. Think of two people who want to have a picnic on the beach that surrounds a lake. They can have it at any point of the beach, so their space of preferences is an annulus which has the homotopy type of a circle $\mathbb{S}^1$. On the one hand, $\mathrm{TC}(\mathbb{S}^1) = 2$ [20]. On the other hand, even if $\mathrm{TC}^\Sigma(\mathbb{S}^1)$ is unknown, in [22, Example 4.5], it is shown that it is bounded above by 3. Now, we will give three local social choice maps. Whether two are enough or not is an open issue.

Fix a point in the circle, for instance $x_0 = (1, 0)$. Then each point of $\mathbb{S}^1 \setminus \{x_0\}$ can be parametrized by its argument, $0 < \alpha < 2\pi$. Consider the set $U \subset \mathbb{S}^1 \times \mathbb{S}^1$ of pairs of points both different from x_0. Hence, U is homeomorphic to $(0, 2\pi) \times (0, 2\pi)$, so it is contractible. The arithmetic mean of the arguments is a social choice $m\colon U \to \mathbb{S}^1$ on U. Clearly, the first two conditions in Definition 4.1 are verified. To check the third one, consider the homotopy $H\colon [0, 1] \times U \to \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$H(t, \alpha_1, \alpha_2) = ((1 - t)\alpha_1 + tm, (1 - t)\alpha_2 + tm),$$

where $m = (1/2)(\alpha_1 + \alpha_2)$. Then $H_0 = (\alpha_1, \alpha_2)$ and $H_1 = (m, m)$, as stated.

The second piece will be the subset V of $(\{x_0\} \times \mathbb{S}^1) \cup (\mathbb{S}^1 \times \{x_0\})$ formed by the pairs such that exactly one of the coordinates equals x_0. This time the social choice will be the constant map $x_0\colon V \to \mathbb{S}^1$. Condition 3 in Definition 4.1 is ensured by the continuous homotopy

$$H(t, \alpha_1, \alpha_2) = ((1 - t)\alpha_1, (1 - t)\alpha_2)$$

if we give the argument $\alpha_0 = 0$ to the point x_0. The other conditions are obvious.

Finally, we take $W = \{(x_0, x_0)\}$ as the third piece.

Note that this is a classical example in the discrete case. For instance, if we have n people choosing one of the k camping sites located along a circular road, if the number of sites is large enough, $k > 2n$, there is not any global social choice function (unanimous, anonymous and continuous) (see [10, Section 2]).

4.2. Higher Social Choice Complexity

In order to aggregate the preferences of a whole society and not only of a couple of individuals, we will study the number of sets in which we need to split X^n to construct continuous social choice maps. The following definition was suggested at the end of the paper [19].

Definition 4.6 (*Higher social choice complexity*). Let X be a space of preferences and denote the symmetric group on n letters by Σ_n. The *higher social choice complexity*, $\mathrm{sc}_n(X)$, is the least integer $k \geq 1$ such that X^n can be covered by k pieces $U_1, \ldots, U_k$ verifying:

1. $\sigma(U_j) = U_j$ for any permutation $\sigma \in \Sigma_n$, that is, U_j is symmetric.
2. For each U_j, there is a local social choice map $m_j\colon U_j \to X$.
3. Each m_j is a homotopic section of the diagonal map $\Delta\colon X \to X^n$, $\Delta(x) = (x, \ldots, x)$, that is, $\Delta \circ m_j \simeq \iota_j$.

4.3. Relation Between Higher Social Complexity and Topological Complexity

In [19], it was pointed out that "there is not, at present, a suitable notion of symmetric higher topological complexity that might be placed as an upper bound on $\mathrm{sc}_n(X)$ as there was in [Theorem 4.4]". We will fill that gap.

Theorem 4.7. *If the space of preferences X is a topological space, then*

$$\mathrm{TC}_n(X) \leq \mathrm{sc}_n(X) \leq \left(\mathrm{TC}_n^{M,\Sigma}\right)'(X).$$

Proof. Let us begin with the left inequality. Assume that $\mathrm{sc}_n(X) = \ell$. Then there exist symmetric sets $U_1, \ldots, U_\ell$ which cover X^n, such that on each of them there is a local social choice map $m_j \colon U_j \to X$. By Property 3 in Definition 4.6, we have that $\Delta \circ m_j \simeq \iota_j$, that is, there exists a homotopy defined on each U_j, say $H_j \colon I \times U_j \to X^n$, verifying:

$$(H_j)_0(x_1, \ldots, x_n) = \Delta \circ m_j(x_1, \ldots, x_n) = (m, \ldots, m),$$

where we denote $m = m_j(x_1, \ldots, x_n)$, and

$$(H_j)_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n).$$

For each j, we define the section $s_j \colon U_j \to X^{J_n}$ as

$$s_j(x_1, \ldots, x_n)(t, k) = p_k[(H_j)_t(x_1, \ldots, x_n)],$$

where $p_k \colon X^n \to X$, $k = 1, \ldots, n$, are the projections. Let us check that it verifies the conditions in Definition 3.8:

1. It is well defined, because, for each $(x_1, \ldots, x_n) \in U_j$, $s_j(x_1, \ldots, x_n)$ is a continuous map from J_n into X. In fact, it is the combined map of the restrictions s_j^k to each arm $k \in \{1, \ldots, n\}$ of J_n, where the maps $s_j^k \colon I \to X$ are given by

   $$s_j^k(x_1, \ldots, x_n)(t) = p_k\left[(H_j)_t(x_1, \ldots, x_n)\right].$$

   They are continuous because they are the compositions of the projection p_k with the map $H_j \circ \iota$, where $\iota \colon I \to I \times U_j$ is given by

   $$\iota(t) = (t, x_1, \ldots, x_n).$$

2. Each s_j is continuous on U_j because it can be written as a composition of continuous maps, namely

   $$(p_k)_* \circ (\sharp) \circ H_j^*,$$

   where

   $$(p_k)_* \colon (X^n)^{J_n} \to X^{J_n}$$

   is given by

   $$(p_k)_*(\alpha) = p_k \circ \alpha$$

   for α a multi-path on X^n; the map

   $$\sharp : (X^n)^I \to (X^n)^{J_n}$$

   transforms paths into multi-paths, as in Definition 3.7; and, finally,

   $$H_j^* \colon U_j \to (X^n)^I$$

   is the map associated to $H_j \colon I \times U_j \to X^n$, by the exponential law.

3. Furthermore, since

$$s_j(x_1,\dots,x_n)(1,k) = p_k\left[(H_j)_1(x_1,\dots,x_n)\right] = x_k,$$

we have $e_n \circ s_j = \iota_j$.

Now, we shall prove the inequality on the right side. Assume that $\left(\mathrm{TC}_n^{M,\Sigma}\right)'(X) = \ell$. Hence, there is a decomposition $U_1 \cup \dots \cup U_\ell = X^n$ by ℓ symmetric sets

such that on each of them there is a homotopic section $s_j\colon U_j \to X^{J_n}$ of e_n verifying $\sigma s_j = s_j \sigma$ for any $\sigma \in \Sigma_n$.

Consider the continuous maps $r\colon X^{J_n} \to X$ with $r(\alpha) = \alpha(0)$ and $m_j = r \circ s_j\colon U_j \to X$, that is, $m_j(x_1,\dots,x_n) = s_j(x_1,\dots,x_n)(0)$.

Let us check that the map m_j is a social choice map:

a) If $x \in X$ and $c_x\colon J_n \to X$ is the constant multi-path such that $c_x(t,k) = x$ for all $k \in \{1,\dots,n\}$ and $t \in [0,1]$, then, by Definition 3.10 (3),

$$m_j(x,\dots,x) = s_j(x,\dots,x)(0) = x.$$

b) For any $(x_1,\dots,x_n) \in U_j \subset X^n$, let $\alpha = s_j(x_1,\dots,x_n) \in X^{J_n}$

We have $m_j(x_1,\dots,x_n) = r(\alpha) = \alpha(0)$ and $s_j(x_1,\dots,x_n)(0_k) = m$ for all k. So, for $\sigma \in \Sigma_n$,

$$m_j(x_1,\dots,x_n) = \alpha(0) = (\sigma\alpha)(0) = r(\sigma\alpha) = m_j(\sigma(x_1,\dots,x_n)),$$

where the final equality comes from Definition 3.10 (2).

c) Last, we will show that $\Delta \circ m_j \simeq \iota_j$. By the homotopy lifting property of the fibration e_n (see the Appendix) there exists a homotopy $\tilde{G}\colon U_j \times I \to X^{J_n}$ which lifts the homotopy G that exists between $e_n \circ s_j$ and ι_j, with the initial condition $(\tilde{G})_0 = s_j$.

Then s_j is homotopic to the continuous map $S_j{:=}(\tilde{G})_1\colon U_j \to X^{J_n}$, which is a true local section of e_n because $e_n \circ S_j = e_n \circ (\tilde{G})_1 = G_1 = \iota_j$.

Now, let $\Delta\colon X \to X^n$ be the diagonal map. For any $(x_1,\dots,x_n) \in U_j \subset X^n$, let $\beta = S_j(x_1,\dots,x_n) \in X^{J_n}$ and $M_j = r \circ S_j\colon U_j \to X$, that is, $M_j(x_1,\dots,x_n) = S_j(x_1,\dots,x_n)(0)$. Then

$$\Delta \circ M_j(x_1,\dots,x_n) = \Delta(r(\beta)) = \Delta(\beta(0)) = (\beta(0),\dots,\beta(0)).$$

Hence, we have $\Delta \circ M_j \simeq \iota_j$ with the homotopy defined by

$$H(x_1,\dots,x_n,t) = (\beta(t,1),\dots,\beta(t,n)).$$

Finally, $\Delta \circ r \circ \tilde{G}$ is a homotopy between $\Delta \circ m_j$ and $\Delta \circ M_j$, so we obtain the homotopies

$$\Delta \circ m_j \simeq \Delta \circ M_j \simeq \iota_j,$$

as stated.

Notice that H is a continuous map because for the compact-open topology the following *exponential law* $Y^{I\times X} {\equiv} (Y^I)^X$ (bijection) is always satisfied for X, Y arbitrary topological spaces. Namely, the map $H\colon I \times X \to Y$ is continuous if and only if the map $H^*\colon X \to Y^I$, given by

$$H^*(x)(t) = H(t, x)$$

is continuous [28, Theorem 1]. So in order to prove that the k-component of H is continuous, we only need to prove that the composition of S_j with the map $X^{J_n} \to X^{I_k}$ induced by the inclusion $I_k \subset J_n$ is continuous, which is an exercise. □

Remark 4.8. In the preceding proof, we do not require S_j to be equivariant. Grant's definition of $TC^{M,\Sigma}$ requires s_j to be a true section, but we are able to work around this using only a homotopic section.

4.4. Decompositions and Pieces

In this article, we have defined several topological and homotopic invariants that require decomposing a given space X into "pieces". We have purposely avoided being too specific on what kind of pieces we are allowed to take. For example, topological complexity can be defined using *open* subsets. This has the advantage of avoiding many tedious continuity checks. However, *closed* sets are sometimes used, which allows to easily define combined maps. The equivalences between both definitions require that the ambient space meets some technical conditions, such as being a *normal* space.

Sometimes more general decompositions are allowed, using so-called ENRs (Euclidean neighborhood retracts) [29]. It is even possible to take decompositions using completely arbitrary subsets in the context of ANRs (Absolute neighborhood retracts) [30, 31]. We will avoid this issue since the interested reader can always consult the bibliography to establish the necessary specific hypotheses.

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Declarations

Conflict of interest The authors declare no conflict of interest.

5. Appendix

We assume the reader is familiar with the basic concepts of topological space and continuity. As a general reference, we recommend Hatcher's book [32].

5.1. Homotopy

A *homeomorphism* is a topological equivalence between two topological spaces, that is, a bijective continuous map with a continuous inverse.

Sometimes, a more general notion of "deformation" is needed. We shall denote by I the closed interval $[0, 1]$ with the usual Euclidean topology.

- A *path* in the topological space X is any continuous map $\gamma\colon I \to X$. The *path-connected component* of $x_0 \in X$ is the set of points $x \in X$ that can be connected to x_0 by a path γ such that the initial point $\gamma(0) = x_0$ and the final point $\gamma(1) = x$. A space is *path-connected* if it has a single path-connected component.
- A *homotopy* between two continuous maps $f, g\colon X \to Y$ is any continuous map $H\colon X \times I \to X$ such that $H(-, 0) = f$ and $H(-, 1) = g$.

Then $H(-, t)$ can be understood as a "path" between f and g in the space of continuous maps from X to Y. When such a homotopy exists, we will write $f \simeq g$. In this case, composing with a third map (on the left or on the right), preserves the relation, $h \circ f \simeq h \circ g$ and $f \circ h \simeq g \circ h$.

Notice that in a path-connected space Y any two arbitrary constant maps $c_{y_0}, c_{y_1}\colon Y \to Y$ are homotopic.

- Two spaces X, Y are said *homotopically equivalent* if there exists a map $f\colon X \to Y$ with a *homotopic inverse*, that is, a map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. In this case, we also say that X and Y have the same homotopy type, denoted by $X \simeq Y$.

Many topological invariants (by homeomorphisms) are in fact homotopic ones (by homotopy equivalences).

Example 1. The Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic (otherwise, deleting a point would result in homeomorphic subspaces, which is not the case due to the number of resulting path-components). However, they are homotopically equivalent, because the inclusion $f\colon \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 0)$, and the projection $g\colon \mathbb{R}^2 \to \mathbb{R}, g(x, y) = x$, are homotopic inverses.

- The space X is *contractible* if it is homotopically equivalent to a point.

Example 2. All Euclidean spaces $\mathbb{R}^n, n \geq 0$, are contractible. However, the spheres S^n, $n \geq 1$ are not contractible, a fact that can be checked by computing its *fundamental group.*

From the very definition, being contractible is equivalent to the existence of a homotopy (called a contraction) $\mathrm{id}_X \simeq c_{x_0}$ between the identity of X and some constant map $c_{x_0}\colon X \to X$. As a consequence, any arbitrary map $X \to Y$ with contractible codomain Y is homotopic to a constant map (just compose the map with the contraction).

5.2. Fibrations

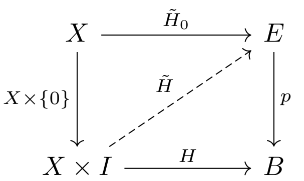
A particular class of continuous maps, the so-called *fibrations*, plays a central role in algebraic topology.

Fibrations, though abstract, may have interesting applications in economic analysis. In this paper, they help model decision structures at various levels of aggregation. We guess that fibrations could also be useful in models addressing complex networks, growth, development, and game theory, by organizing variables or relationships across layers of a system.

- A continuous map $p\colon E \to B$ satisfies the *homotopy lifting property* for a space X if
  - for every homotopy $H\colon X \times I \to B$ and
  - for every continuous map $\tilde{H}_0\colon X \to E$ lifting $H_0 = H_{|X \times 0}$ (i.e., $H_0 = p \circ \tilde{H}_0$)

  there exists a (not necessarily unique) homotopy $\tilde{H}\colon X \times I \to E$ lifting H (i.e., $H = p \circ \tilde{H}$) with $\tilde{H}_0 = \tilde{H}_{|X \times 0}$.

  That is, any partial lift of $H_0 = H_{|X \times 0}$ can be extended to a lift of the entire homotopy H, as in the following diagram



- A *fibration* (also called Hurewicz fibration) is a continuous map satisfying the homotopy lifting property for all spaces X.

The space E (respectively B) is called the *total space* (respectively the *base space*) of the fibration. If $b \in B$ is a base point, then $p^{-1}(b) \subset E$ is the *fiber* over b.

Example 3.

- The map $f: \mathbb{R} \to S^1$, $f(t) = (\cos 2\pi t, \sin 2\pi t)$, that wraps the real line around the circle repeatedly is a special type of fibration called a *covering map.*
- Let X^I be the space of paths on X. The map $\pi: X^I \to X \times X$ that sends each path γ to its initial and final points $(\gamma(0), \gamma(1))$ is called the *path fibration.*

Fibrations have nice properties:

- When the base space B is path connected, any two fibers have the same homotopy type.
- If the base B is contractible, then the total space is homotopically equivalent to a product $E \simeq B \times F$, where F is any fiber, in a compatible way with the projections.

In homotopy theory, spaces are often studied up to homotopy equivalence, and fibrations are particularly useful for this purpose, as they preserve properties that depend on the homotopy types of the spaces involved.

Example 4. For a continuous map $f: X \to Y$, a *section*, that is a map $s: Y \to X$ satisfying $f \circ s = \mathrm{id}_Y$, is not the same as a *homotopy section*, where $f \circ s \simeq \mathrm{id}_Y$. However, for fibrations, the existence of one implies the other, as shown by the homotopy lifting property.

Finally, arbitrary continuous maps can be homotopically turned into fibrations. More precisely,

- Let $f: X \to Y$ be a continuous map. Then there exists a fibration $p: E \to Y$ and a homotopy equivalence $h: E \to X$ such that $f \circ h = p$.

Example 5. The fibrational replacement of the diagonal map $\Delta: X \to X \times X$, $\Delta(x) = (x, x)$, is the path fibration $\pi_2: X^I \to X \times X$ of Example 3.

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