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ON PHYSICAL LAWS AND COMPUTABILITY*

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§.1. Introduction

There is a wide spectrum of philosophical views about the relation between science and reality in general, and between physical theories and reality in particular. Realism in any of its varieties, conventionalism or the more neutral instrumentalist position are basic options. From a logical perspective, the spectrum of possibilities includes a defence of inductive procedures or some form of sophisticated falsificationism. No matter what is the epistemological route one opts for, scientific theories (and, in particular, physical theories) must, all of them, fulfil the following requirement: one must be capable of deriving empirical predictions from given initial conditions. The question whether this is *the* only requirement —the «only possibility», as someone sharing Cardinal Bellarmino's view on science may wish to emphasize— or just one of the necessary conditions on such theories is, after all, a philosophical issue. But what do the expressions «prediction» and «initial conditions» mean? This question is strictly related to another one: What is a *physical law*? These classical questions are analyzed in this paper from a particular perspective, that is, with the aim of exploring a possible connection between them and the theory of computability.

A physical law, from a logical standpoint, is generally expressed by means of a universal statement and, from a mathematical standpoint, by means of an ordinary or a partial differential equation. So-called predictions and initial conditions are expressed, in principle, by means of real numbers. Let us further distinguish between the interpretative aspect and the more formal aspects of a physical law. The signs appearing in an equation expressing a physical law represent (functions of) physical magnitudes. This interpreta-

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tive aspect is not explicitly examined in the present paper, which concentrates on a specific formal property of physical laws. Let us keep in mind, however, that a *sharp* separation between the two aspects is not really possible: the mathematical form of the equations is constrained by the fact that the latter are supposed to account for classes of phenomena in the physical world. Real numbers (values of predictions) are related by means of a physical law to real numbers (values of initial conditions). Is this a purely abstract, «platonic», correspondence or is it given by means of an effective procedure? This is the main question addressed in this paper. More specifically, we examine the following three issues:

1. What does it mean to assert that a physical law —more generally, a physical theory— is *computable* (or, as we shall also say, *mechanistic*)?
2. Are the usual physical laws (theories) computable?
3. Are there non-computable physical laws (theories)?

§.2. Computable laws: uniformity and physical constraints

Kreisel (1974) provides an answer to the first question, by means of a definition of computable (or mechanistic) theory:

> «The general idea is this. We consider theories, by which we mean such things as classical or quantum mechanics, and ask if every sequence of natural numbers or every real number which is well defined (observable) according to the theory must be recursive or, more generally, recursive in the data (which, according to the theory, determine the observations considered). Equivalently, we may ask whether any such sequence of numbers, etc., can also be generated by an ideal computing or Turing machine if the data are used as input. (This formulation explains our terminology 'mechanistic'.)». (Kreisel 1974, p. 11).

In connection with Kreisel's definition, we wish to emphasize the following points:

(i) Kreisel provides a *weak* definition, in the following sense. A theory is computable if for each real number x, which is observable according to the theory, there is a Turing machine which generates x. (For precise definitions of computable real number see, for instance, Pour-El and Richards [1989], p. 14). In other words, for every such real number there may be a *distinct* Turing machine. In this sense, Kreisel's definion does not impose a *uniformity* requirement.

(ii) The real numbers considered in the definition are those numbers that are well-defined (observable) according to the physical theory in question. In other words, the definition singles out those reals that for physical reasons, independent of their mathematical features, are possible values of observable magnitudes of the physical theory (or law). In this sense, Kreisel's definition introduces a *physical constraint*.

Let us consider what we have called the uniformity requirement. The question whether physical theories are computable has been addressed by various authors, even though without providing explicit definitions such as Kreisel's. For example, Penrose asserts:

> «Newtonian mechanics is, as we all know, deterministic, but is it computable? Suppose initial data for some physical situation is given in terms of computable numbers (all constants involved being also computable numbers), and we wait for a computable time. Is the state at that time computable from the initial data?» (Penrose 1989, p. 649).

Furthermore, Geroch and Hartle (1986, p. 534) distinguish between the formulation of a physical theory and its possible implementations by means of algorithms: if there is an algorithm for implementing a physical theory then the theory is computable. Both Penrose's and Geroch and Hartle's observations seem to suggest the existence of an *implicitly accepted, intuitive* notion of computable theory in physics. However, while Penrose's remark seems to be perfectly tuned with Kreisel's definition, Geroch and Hartle's observation seems to suggest a stronger notion of computable theory. In fact, they seem to require the existence of *one* algorithm to implement the theory, whereas Kreisel's definition only requires that for each real number x, which is observable according to the theory, there must be a Turing machine for computing x (and therefore the choice of the Turing machine may depend on the particular value of the observable in question). These observations naturally suggest the question whether one can give a stronger definition of computable theory than the one advanced by Kreisel and possibly closer to the intuitive notion. By imposing, in addition to the physical constraint, what we have called uniformity condition, one might arrive at such a definition, i. e., one that satisfies, within the context of a suitable mathematical theory, the following minimal requirement:

A physical law (theory) is computable if it provides a computational procedure that enables one to derive in a uniform way the physical predictions of the law (theory), expressed as computable real numbers, from initial data expressed as computable real numbers.

This is an epistemological requirement of a rather general character. Indeed, one may attempt to follow a variety of approaches to express the uniformity condition. (Think, for example, of the notion of computable function of real variables introduced by Grzegorczyk [1955, 1957], given in terms of recursive functionals, or the equivalent definition of Pour-El and Caldwell [1975], or the notion of computability on the reals introduced by L. Blum [1990].)

There is another point we wish to emphasize in connection with the above epistemological requirement on a definition of uniformly computable physical law (theory). Such a law (theory) is in practice used to determine a correspondence between rationals rather than between reals. In fact, both initial conditions and predictions are, in any specific case, expressed by finite initial segments of reals. Thus, in practice, our uniformity requirement concerns an algorithm operating on inputs representable as finite strings and providing finite strings as outputs. (Let us recall that only a computable real can be approximated, with any degree of precision, by a single algorithm.)

In connection with the form in practice assumed by initial conditions and predictions (rational numbers), one has to observe that Kreisel's definition might be interpreted as a definition that implies some form of weak uniformity. In fact, Kreisel requires that «every *sequence* of natural numbers or every real number which is well defined (observable) according to the theory must be recursive or, more generally, recursive in the data.» But this requirement should be better specified. Indeed, such condition does not seem to imply that every finite approximation of every well-defined (observable) real number can be uniformly generated by the same Turing machine. A more precise formulation of the uniformity condition implicit in Kreisel's definition seems to depend on a deeper analysis of the notion of sequence of natural numbers that are observable within the framework of a physical theory.

§.3. Present-day physical laws

After this quick overview it is time to ask again our second question:

«Is each and every law (theory) of present-day physics computable?»

One must admit from the start, that there is no rigorous and straightforward answer to this question, although many elements support the belief that one should answer «Yes» (at least, inside the realm of purely intuitive requirements or, even, of the more specific ones advanced by Kreisel and reported at the beginning of the paper).

A central supporting element is the fact that first order differential equations —satisfying conditions assuring the existence and uniqueness of their solutions— yield solutions which are computable on computable initial data; moreover, higher-order differential equations are sometimes reducible to systems of first order differential equations. This fact was stressed by Earman (1986) (see pp. 117-120) who also states that «for partial differential equations the story is both more complicated and more interesting» (p. 118). Pour-El & Richards (1981) have, in fact, shown that there exist solutions of the wave equation which are not computable for given computable input data. One must, however, observe —as Penrose did (Penrose 1989, p. 247)— that such input data admit no reasonable physical interpretation. Penrose himself concludes, then, that «... it is hard to see that in *any* of the physical theories that I have been discussing so far [i. e., classical theories] there can be any significant 'non-computable' elements» (Penrose 1989, p. 216). Geroch & Hartle (1986), finally, glance at the possibility that a specific approach to quantum gravity could constitute a counterexample to the generally assumed hypothesis of the computability of physical theories (laws).

In this approach, in fact, a magnitude which is, in principle, observable, is expressed by means of an infinite summation S. The elements of this sum are to be selected among the members of a certain set *I*. However, this set *I* is such that the problem of correctly selecting the elements of this sum S from the members of *I* is provably undecidable. That is, one can show that there exist no Turing machine which, for any element belonging to *I*, given a suitable input, halts on a given symbol S_s, if the element must be added up and halts on a symbol S_n otherwise.[1]

It is obvious that should we be obliged to give up the computability of laws (or, more generally, theories) in some domain of physical (or, in general, scientific) investigation the same notion of scientific forecast would be shaken at its root. Coming back to physics, if we were really confronted with the actual presence of non-computable physical laws (theories) one could hardly speak of predictions *effectively* provided by these laws. Moreover, modifying our conceptual schemes would not be a straightforward matter. A non-computable law, in fact, although referring to a correspondence between the real values of certain observables and the real value of another observable (or a function of it), would not provide an algorithm to determine finite initial segments of reals (which, according to the law, express values assumed by observables) starting from finite initial segments of reals (expressing initial conditions).

It may be worthwhile to delve into the formidable conceptual and technical problems surrounding the notion of (non-)computable physical theory because the problem of the computability of scientific laws may become more and more relevant in various areas of scientific investigation. We just wish to indicate two fields inside which this kind of discussion has already started.

§.4. Concerning analog computers, digital computers, and the mind

Kreisel notices a relation between his notion of computable physical theory and the problem of simulating by Turing machines the behaviour of analog computers. Kreisel states that if «the particular theory of the behavior of the analogue computers considered happens to be mechanistic in the sense described above» then «their behavior may be simulated by Turing machines» (Kreisel 1974, p. 11). We wish to stress two points in relation to this statement. The first one has to do with what we have called «uniformity constraint». In accordance with his definition of computable theory, which does not impose a uniformity constraint, Kreisel affirms that the behaviour of an analog computer can be simulated by Turing machines, according to the theory T of the analog computer under consideration, if T is a computable

[1] See Geroch & Hartle (1986), p. 547 and the references therein.

theory. Thus, according to Kreisel's definition, the following relationship holds among an analog computer C, a physical theory T and a set of Turing machines M: if the behaviour of C is described by a physical theory T which is computable then, according to T, for each real number expressing the result of a computation of C, there exists a Turing machine in M which computes in the data that number. This relationship, due to the lack of a uniformity constraint on the theory, is a rather *weak* simulability relationship. If a physical theory T satisfies —through an adequate formal explicatum— also a uniformity constraint, then one could establish the following *strong* simulability relationship among an analog computer C, the physical theory T in question and a single Turing machines T_m: if the behaviour of C is described by a physical theory T which is computable and satisfies the uniformity constraint then, according to T, there exists a Turing machine T_m which computes in the data each real number expressing the result of a computation of C. Only strong simulability, which relates a single analog computer and a single Turing machine, seems to pose also practically interesting simulation problems. Of course, the simulability of analog computation raises the further problem of the levels of accuracy of both predictions and experimental results, a problem we referred to above, by observing that the predictions of a physical theory are, as a matter of fact, expressed by means of rational numbers.[2]

Kreisel's definition, as it does not impose a uniformity constraint, is more general than other definitions embodying such a costraint. Thus, if a physical theory T happens to be non-computable in the sense of Kreisel, then T will not be uniformly computable according to any reasonable definition whatsoever of uniform computability of a physical theory. For this reason, the proof that a particular theory T is non-computable according to Kreisel's definition shows that T is non-computable in a quite general sense. As is well known, Penrose relates the problem of the existence of non-computable physical theories to the mind-machine problem. He states that such theories will be an essential ingredient for explaining brain behaviour and its cognitive aspects which, in his view, cannot be satisfactorily explained in algorithmic terms. The discovery of a (present or future) physical theory relevant for the description of the behaviour of the brain, which is not computable in Kreisel's sense, could be seen, just for the generality of Kreisel's definition, as a first necessary step of Penrose's programme[3]. To this first necessary step, of course, many others should be added (but no indications have been provided by Penrose in this direction) in order to bridge the gap between physical laws and the intelligent behaviour of human beings.

The (possible violation of the) computability —in the sense of Kreisel— of a physical law is not very significant for computational approaches to the

[2] For an analytic discussion, raising a variety of research problems on Turing machines and analog computation —some of which are open today— see also Myhill (1966).

[3] For a discussion of this point, see also Guccione (1995).

study of mind. Indeed, such approaches aim at establishing a relationship between the intelligent behaviour exhibited by a single physical system (the brain with all his cognitive abilities) and a single (although complex at will) digital machine. But we have already pointed out that the relationship entailed by a computable law in the sense of Kreisel is between a single physical system and a set of Turing machines. Thus, a notion of computable physical law satisfying a uniformity constraint would be more relevant for (epistemological discussions on) computational approaches to the study of mind.

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