Multiplicity results for integral boundary value problems of fractional order with parametric dependence

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Abstract

This paper is devoted to the study of nonlinear fractional differential equation with parameter dependence and integral boundary value conditions. In the paper various existence and multiplicity results for positive solutions are derived depending of different values of the parameter. Some illustrative examples are also discussed.

Keywords: Fractional differential equations; Integral boundary conditions; Parameter dependence; Positive solution; Green's function; Krasnoselskii's fixed point theorem.

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines. In the monographs [3, 11, 14, 15, 16, 18] one can find several examples related to the mathematical modeling of systems and processes in, among others, the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, and polymer rheology.

Since only positive solutions are useful for many applications, in this paper we discuss the existence and multiplicity of positive solutions of a nonlinear fractional differential equation with integral boundary conditions and parameter dependence. More precisely, in Section 3, we consider the following problem

$$\begin{cases} D^{\alpha} u\,(t) + \mu\, g(t)\, f(u(t)) = 0, & 0 \leq t \leq 1 \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s)ds, & 0 < \lambda < \alpha \end{cases} \tag{1}$$

depending on the real parameter $\mu > 0$. Here D^{α} denotes the Rieman-Liouville fractional derivative of order $\alpha \in (2, 3]$ and f and g are appropriate functions to be specified later.

Integral boundary conditions constitute a very interesting and important class of problem. They arises in different areas of applied mathematics and physics as, for instance (see [1, 12, 13, 19] for details), heat conduction, chemical engineering, underground water flow, and plasma physics can be reduced to nonlocal problems with integral boundary conditions.

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In the last years, several papers which deal with the existence of solutions for nonlinear fractional differential equations coupled with integral boundary conditions have appeared. In many of them the main results follow by means of suitable fixed point theorems [2, 4, 5, 9, 10, 17]. In [5], Benchohra and Ouaar investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition:

$$\begin{cases} {}^cD^{\alpha}u\,(t) &= f(t,u(t)), \quad 0 \le t \le T,\ \alpha \in (0,1], \\ u(T) &= u(0) + \lambda \int_0^T u(s)ds, \end{cases} \tag{2}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$.

In 2012, by the use of the Guo-Krasnosellskii fixed-point theorem, Cabada and Wang [9] obtained existence results of positive solution for the following nonlinear fractional differential equation with integral boundary condition:

$$\begin{cases} {}^CD^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1,\ \alpha \in (2,3) \\ \\ u(0) = u''(0) = 0,\ u(1) = \lambda \int_0^1 u(s)ds, \end{cases}$$

where $0 < \lambda < 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

In section 2 we show the main properties, proved by the authors in [8], of the Green's function related to the linear problem of fractional order

$$\begin{cases} D^{\alpha}u\,(t) + y(t) = 0, & 0 \le t \le 1, \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s)ds, & 0 < \lambda < \alpha. \end{cases} \tag{3}$$

Section 3 is devoted to the construction of a suitable cone in the space of the nonnegative continuous functions and so, by means of the Krasnoselskii's fixed point theorem, we give sufficient conditions for the existence of nonnegative solutions and, under additional conditions on the behavior of function f at 0 and at ∞, we prove our main results for existence and multiplicity of positive solutions for the problem (1). We shall present in Section 4 an example that illustrates our main results.

In some sense, the results given in this work follow similar steps to the ones obtained in [6] and [7] for second order differential and difference equations with periodic boundary value conditions. The main difference in our approach is that in this paper the Green's function vanishes at $t = 0$ and $s = 0, 1$, which makes the main properties of the Green's function used in those references not valid here.

2 Preliminaries

In this section we recompile the fundamental properties of the Green's function related to problem (3). Such properties has been proven in [8].

Theorem 2.1. *[8, Theorem 2.1] Let $2 < \alpha \le 3$ and $\lambda \neq \alpha$. Assume $y \in C[0,1]$, then problem (3) has a unique solution $u \in C^1[0,1]$, given by the expression*

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0\le s\le t\le 1,\\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \tag{4}$$

In the next result, are showed the fundamental properties of the Green's function G, that will be used to prove our main results.

Lemma 2.2. *[8, Lemmas 2.2 and 2.3] Let G be the Green's function related to problem (3), which is given by the expression (4). Then, for all $\alpha\in(2,3]$ and $\lambda\ge 0$, the following properties are fulfilled:*

(i) $G(0,s)=G(t,1)=0$ *for all* $t,\ s\in[0,1]$ *and* $\lambda\neq\alpha$.

(ii) $G(1,s)=0$ *for all* $s\in(0,1)$ *if and only if* $\lambda=0$.

(iii) $(\alpha-\lambda)\,G(1,s)>0$ *for all* $s\in(0,1)$ *if and only if* $\lambda\neq\alpha$.

(iv) $G(t,0)=0$ *for all* $t\in[0,1]$.

(v) $G(t,s)\le\frac{1}{(\alpha-\lambda)\Gamma(\alpha-1)}$ *for all* $t,s\in[0,1]$ *and* $\lambda\in[0,\alpha)$.

(vi) $G(t,s)$ *is a continuous function for all* $t,s\in[0,1]$ *and* $\lambda\neq\alpha$.

(vii) $t^{\alpha-1}\,G(1,s)\le G(t,s)\le\frac{\alpha}{\lambda}\,G(1,s),\qquad$ *for all* $t,\ s\in(0,1)$.

As a corollary of the previous result and Lemma 2.2, we deduce the following:

Corollary 2.3. *Let G be the Green's function related to problem (3), which is given by the expression (4). Then, for all $\alpha\in(2,3]$ and $\lambda\ge 0$, the following property holds:*

$$G(t,s)>0 \text{ for all } t,s\in(0,1) \text{ and all } \lambda\in[0,\alpha).$$

3 Main result

In this section we consider the nonlinear problem with parameter dependence (1). We will prove some existence, nonexistence and multiplicity results for some suitable values of the positive real parameter μ. In order to do this, we use the well known Guo-Krasnoselskii fixed point Theorem [11] in cones. We recall that a cone $\mathcal{P}$ is a subset of a Banach space which is nonempty, nontrivial, convex and closed. Moreover $\rho\mathcal{P}\subset\mathcal{P}$ for all $\rho>0$ and $\mathcal{P}\cap(-\mathcal{P})=\{0\}$.

Theorem 3.1. *Let E be a Banach space, and let $\mathcal{P}\subset E$ be a cone. Assume that Ω_1,Ω_2 are open and bounded subsets of E with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that*

(i) $\|Tu\|\ge\|u\|,\ u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\le\|u\|,\ u\in\mathcal{P}\cap\partial\Omega_2$; *or*

(ii) $\|Tu\|\le\|u\|,\ u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\ge\|u\|,\ u\in\mathcal{P}\cap\partial\Omega_2$.

Then operator T has at least one fixed point in $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Let $E = C[0,1]$ be the Banach space endowed with the usual supremum norm $\|\cdot\|$. And we assume the following hypotheses:

(H1) $\mu > 0$,

(H2) $g \in L^1[0,1]$, $g(t) \geq 0$ for a.e. $t \in [0,1]$ and $\int_{1/2}^1 g(s)\, ds > 0$,

(H3) $f : [0,\infty) \to [0,\infty)$ is continuous. Moreover $f(x) > 0$ for all $x > 0$.

In view of Theorem 2.1, we define the operator $T : C[0,1] \to C[0,1]$ as follows,

$$T_\mu u(t) := \mu \int_0^1 G(t,s)\, g(s)\, f(u(s))\, ds, \tag{5}$$

with G defined in (4).

It is clear that the fixed points of operator T_μ coincide with the solutions of problem (1).

Define now the cone $\mathcal{P} \subset E$ as follows,

$$\mathcal{P} = \left\{ u \in E,\ u(t) \geq 0 \text{ for all } t \in [0,1],\ u(t) \geq \frac{t^{\alpha-1}\lambda}{\alpha}\|u\|, \text{ for all } t \in \left[\frac{1}{2}, 1\right] \right\}. \tag{6}$$

Let's see that $T_\mu(\mathcal{P}) \subset \mathcal{P}$.

It is obvious that for $T_\mu(u) \geq 0$ for all $u \in \mathcal{P}$ and $\mu > 0$.

If $u \in \mathcal{P}$ then, for all $t \in [0,1]$, by using Lemma 2.3 and knowing that $G(t,s)$ is continuous, the following inequalities are satisfied for all $t \in [0,1]$:

$$\begin{aligned} T_\mu u(t) &= \mu \int_0^1 G(t,s)\, g(s)\ f(u(s))\, ds \\ &\geq \mu\, t^{\alpha-1} \int_0^1 G(1,s)\, g(s)\ f(u(s))\, ds \\ &\geq \mu \frac{t^{\alpha-1}\lambda}{\alpha} \int_0^1 \max_{t\in[0,1]} \{G(t,s)\}\, g(s)\ f(u(s))\, ds \\ &\geq \frac{t^{\alpha-1}\lambda}{\alpha} \max_{t\in[0,1]} \left\{ \mu \int_0^1 G(t,s)\, g(s)\ f(u(s))\, ds \right\} \\ &= \frac{t^{\alpha-1}\lambda}{\alpha} \|T_\mu u\|. \end{aligned}$$

It is not difficult to show that $T_\mu : \mathcal{P} \to \mathcal{P}$ is completely continuous. Then, in order to apply the Guo-Krasnoselskii's fixed point theorem, showed in Theorem 3.1, we introduce a series of preliminary results depending on the value of the positive parameter μ.

Before this, we denote

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x} \text{ and } f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}.$$

Lemma 3.2. *Assume that hypothesis (H1), (H2) and (H3) hold. Then, for every $R > 0$, there exists $\mu_0(R) > 0$ such that for every $0 < \mu \leq \mu_0(R)$, we have*

$$\|T_\mu u\| \leq \|u\|,\ \textit{for all } u \in \mathcal{P} \textit{ with } \|u\| = R.$$

Proof. Fix $R>0,$ and let $u \in \mathcal{P}$ with $\|u\| = R$. Let $\mu > 0$ be such that

$$0<\mu \leq \mu_0(R) = \frac{R(\alpha-\lambda)\Gamma(\alpha-1)}{\max_{u\in[0,R]}\{f(u)\}\int_0^1 g(s)\,ds}. \tag{7}$$

Notice that from $(H3)$ we know that $\max_{u\in[0,R]}\{f(u)\} > 0$ for all $R>0$. Moreover $(H2)$ implies that $\int_0^1 g(s)\,ds > 0$.

Then, for all $t\in[0,1]$ the following inequalities hold:

$$\begin{aligned} T_\mu u(t) &= \mu\int_0^1 G(t,s)\,g(s)\,f(u(s))\,ds \\ &\leq \frac{\mu}{(\alpha-\lambda)\Gamma(\alpha-1)}\int_0^1 g(s)\,f(u(s))\,ds \\ &\leq \frac{\mu}{(\alpha-\lambda)\Gamma(\alpha-1)}\max_{u\in[0,R]}\{f(u)\}\int_0^1 g(s)\,ds \\ &\leq R = \|u\|, \end{aligned}$$

and so $\|T_\mu u\| \leq \|u\|$. □

Lemma 3.3. *Assume that hypothesis (H1), (H2) and (H3) hold. Then, for each $r>0$, there exists $\mu_0(r)>0$ such that for every $\mu \geq \mu_0(r)$, we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u\in\mathcal{P} \text{ with } \|u\| = r.$$

Proof. Fix $r>0$ and let $u\in\mathcal{P}$ with $\|u\|=r$. Let

$$\mu_0(r) := \frac{r}{\min_{u\in\left[\frac{\lambda}{\alpha 2^{\alpha-1}}r,r\right]}\{f(u)\}\int_{\frac{1}{2}}^1 G(1,s)\,g(s)\,ds} > 0. \tag{8}$$

We point out that conditions $(H2)$ and $(H3)$ warrant that $\mu_0(r)$ is well defined.

Thus

$$\|T_\mu u\| \geq T_\mu u(1) = \mu\int_0^1 G(1,s)\,g(s)\,f(u(s))\,ds \geq \mu\int_{\frac{1}{2}}^1 G(1,s)\,g(s)\,f(u(s))\,ds.$$

Now, since $u\in\mathcal{P}$ and $s\in[1/2,1]$, we have that the last expression is bigger than or equals to

$$\mu\min_{u\in\left[\frac{\lambda}{\alpha 2^{\alpha-1}}r,r\right]}\{f(u)\}\int_{\frac{1}{2}}^1 G(1,s)\,g(s)\,ds \geq r = \|u\|,$$

i.e., $\|T_\mu u\| \geq \|u\|$. □

Lemma 3.4. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if $f_0=\infty$, there exists $r_0(\mu)>0$ such that for every $0<r<r_0(\mu)$, we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u\in\mathcal{P} \text{ with } \|u\| = r.$$

Proof. Since $f_0 = \infty$, then there exists a constant $r_0(\mu) > 0$ such that $f(u) \geq Lu$ for all $0 < u \leq r_0(\mu)$, where $L = L(\mu) > 0$ is defined as (condition $(H2)$ says us that it is well defined):

$$L = L(\mu) = \frac{\alpha}{\mu\lambda \int_{\frac{1}{2}}^{1} G(1,s)\, g(s)\ s^{\alpha-1} ds}.$$

Fix $0 < r < r_0(\mu)$, and let $u \in \mathcal{P}$ be such that $\|u\| = r$. Then, for all $t \in [0,1]$ we have that

$$T_\mu u(t) \geq t^{\alpha-1} \mu \int_0^1 G(1,s)\, g(s)\ f(u(s))\, ds.$$

As a consequence

$$\begin{aligned}
\|T_\mu u\| &\geq \max_{t\in[0,1]} \left\{ t^{\alpha-1}\mu \int_0^1 G(1,s)\, g(s)\ f(u(s)) ds \right\} \\
&= \mu \int_0^1 G(1,s)\, g(s)\ f(u(s))\, ds \\
&\geq \mu L \int_0^1 G(1,s)\, g(s)\ u(s) ds \\
&\geq \mu L \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\ u(s) ds \\
&\geq \mu L \frac{\lambda}{\alpha} \|u\| \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\ s^{\alpha-1} ds = \|u\|.
\end{aligned}$$

□

Lemma 3.5. *Assume that hypothesis (H1), (H2) and (H3) hold. Then, if $f_0 = 0$, there exists $r_0(\mu) > 0$ such that for every $0 < r < r_0(\mu)$ we have*

$$\|T_\mu u\| \leq \|u\|, \ \textit{for } u \in \mathcal{P} \textit{ with } \|u\| = r.$$

Proof. Since $f_0 = 0$, for

$$\epsilon = \epsilon(\mu) = \frac{(\alpha-\lambda)\,\Gamma(\alpha-1)}{\mu \int_0^1 g(s)\, ds} > 0,$$

there exists $r_0(\mu) > 0$ such that $f(u) \leq \epsilon u$ for each $0 < r \leq r_0(\mu)$.

Fix $0 < r < r_0(\mu)$, and let $u \in \mathcal{P}$ with $\|u\| = r$. Then, for every $t \in [0,1]$, it is fulfilled

$$\begin{aligned}
T_\mu u(t) &= \mu \int_0^1 G(t,s)\, g(s)\ f(u(s))\, ds \\
&\leq \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \int_0^1 g(s)\ f(u(s))\, ds \\
&\leq \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \int_0^1 g(s)\ \epsilon u(s)\, ds \\
&\leq \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \epsilon \|u\| \int_0^1 g(s)\, ds \\
&= \|u\|.
\end{aligned}$$

As consequence, we conclude that $\|T_\mu u\| \leq \|u\|$. $\square$

Lemma 3.6. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if* $f_\infty = \infty$, *there exists* $R_0(\mu) > 0$ *such that for every* $R \geq R_0(\mu)$, *we have*

$$\|T_\mu u\| \geq \|u\|, \text{ for } u \in \mathcal{P} \text{ with } \|u\| = R.$$

Proof. Since $f_\infty = \infty$, for

$$L = L(\mu) = \frac{\alpha}{\mu\lambda \int_{\frac{1}{2}}^{1} G(1,s)\, g(s)\; s^{\alpha-1} ds} > 0$$

there exists $R_1(\mu) > 0$ such that $f(u) \geq Lu$ for each $u \geq R_1(\mu)$.

Define now $R_0(\mu) := \frac{2^{\alpha-1}\alpha}{\lambda} R_1(\mu) > R_1(\mu)$.

Fix $R \geq R_0(\mu)$, and let $u \in \mathcal{P}$ with $\|u\| = R$. Then, for all $t \in [0,1]$ the following inequalities hold:

$$\begin{aligned}
\|T_\mu u\| &\geq \max_{t\in[0,1]} \left\{ t^{\alpha-1}\mu \int_0^1 G(1,s)\, g(s)\; f(u(s))ds \right\} \\
&= \mu \int_0^1 G(1,s)\, g(s)\; f(u(s))\, ds \\
&\geq \mu \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\; f(u(s))ds.
\end{aligned}$$

Note that, for all $s \in \left[\frac{1}{2}, 1\right]$ we have

$$u(s) \geq s^{\alpha-1}\frac{\lambda}{\alpha}\|u\| \geq \left(\frac{1}{2}\right)^{\alpha-1}\frac{\lambda}{\alpha}\|u\| = \left(\frac{1}{2}\right)^{\alpha-1}\frac{\lambda}{\alpha}R \geq \left(\frac{1}{2}\right)^{\alpha-1}\frac{\lambda}{\alpha}R_0 = R_1,$$

and so

$$\begin{aligned}
\|T_\mu u\| &\geq \mu L \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\; u(s)ds \\
&\geq \mu L \frac{\lambda}{\alpha}\|u\| \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\; s^{\alpha-1}ds = \|u\|.
\end{aligned}$$

$\square$

Lemma 3.7. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then if* $f_\infty = 0$ *then, there exists* $R_0(\mu) > 0$ *such that for every* $R \geq R_0(\mu)$, *we have*

$$\|T_\mu u\| \leq \|u\|, \text{ for } u \in \mathcal{P} \text{ with } \|u\| = R.$$

Proof. Since $f_\infty = 0$, for

$$\epsilon(\mu) = \frac{(\alpha-\lambda)\,\Gamma(\alpha-1)}{2\mu\int_0^1 g(s)\,ds}$$

there exists $R_1(\mu) > 0$ such that $f(u) \le \epsilon u$ for each $u \ge R_1(\mu)$.

Let $R_0(\mu) := \max\left\{\frac{2^{\alpha-1}\alpha}{\lambda} R_1(\mu), 2\gamma\right\}$, where

$$\gamma = \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \max_{u\in[0,R_1]}\{f(u)\}\int_0^1 g(s)\,ds$$

Note that $R_0(\mu) > R_1(\mu)$.

Fix $R \ge R_0(\mu)$, and let $u \in \mathcal{P}$ with $\|u\| = R$. Then, for any $t \in [0,1]$ we obtain

$$\begin{aligned}
T_\mu u(t) &= \mu\int_0^1 G(t,s)\,g(s)\,f(u(s))\,ds \\
&\le \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \max_{u\in[0,R]}\{f(u)\}\int_0^1 g(s)ds \\
&\le \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \left(\max_{u\in[0,R_1(\mu)]}\{f(u)\} + \max_{u\in[R_1(\mu),R]}\{f(u)\}\right)\int_0^1 g(s)ds \\
&\le \frac{\mu}{(\alpha-\lambda)\,\Gamma(\alpha-1)} \left(\max_{u\in[0,R_1(\mu)]}\{f(u)\} + \epsilon\, R\right)\int_0^1 g(s)ds \\
&\le \gamma + \frac{R}{2} < \frac{R}{2} + \frac{R}{2} = R = \|u\|,
\end{aligned}$$

and thus $\|T_\mu u\| \le \|u\|$. □

Now, after these technical lemmas, we are in a position to prove the following existence, non existence and multiplicity result for problem (1). We refer to a positive solution of problem (1) as any solution u of such problem such that $u(t) > 0$ for all $t \in (0,1]$.

Theorem 3.8. *Assume that conditions (H1), (H2) and (H3) are satisfied. Then, the following results hold:*

1. *If $f_0 > 0$ and $f_\infty > 0$ then there exists $\mu_0 > 0$ such that problem (1) has no positive solutions for all $\mu > \mu_0$.*
2. *If $f_0 = \infty$ or $f_\infty = \infty$, then there exists $\mu_0 > 0$ such that problem (1) has a positive solution for every $0 < \mu < \mu_0$.*
3. *If $f_\infty = 0$, then there exists $\mu_0 > 0$ such that problem (1) has a positive solution for every $\mu > \mu_0$.*
4. *If $f_0 = \infty$ and $f_\infty = 0$, then problem (1) has a positive solution for every $\mu > 0$.*
5. *If $f_0 = \infty$ and $f_\infty = \infty$, then there exists $\mu_0 > 0$ such that problem (1) has two positive solutions for every $0 < \mu < \mu_0$.*
6. *If $f_0 = 0$ and $f_\infty = \infty$, then problem (1) has a positive solution for every $\mu > 0$.*

7. *If* $f_0 = 0$ *and* $f_\infty = 0$, *then there exists* $\mu_0 > 0$ *such that problem (1) has two positive solutions for every* $\mu > \mu_0$.

Proof. First, note that any solution u of problem (1) satisfies, from Lemma 2.2 (vii), the following inequalities for all $t \in [0,1]$:

$$\begin{aligned} u(t) = T_\mu\, u(t) &= \mu \int_0^1 G(t,s)\, g\,(s)\; f(u(s))\, ds \\ &\geq \mu\, t^{\alpha-1} \int_0^1 G(1,s)\, g\,(s)\; f(u(s))\, ds = t^{\alpha-1}\, T_\mu\, u(1) = t^{\alpha-1}\, u(1). \end{aligned}$$

On the other hand, if $u \in \mathcal{P}$, then $u(1) \geq \lambda\, \|u\|/\alpha$.

In particular, any solution $u \in \mathcal{P}$ of problem (1) satisfies

$$u(t) \geq t^{\alpha-1}\, \lambda\, \|u\|/\alpha, \qquad \text{for all } t \in [0,1],$$

and, if it is nontrivial on $[0,1]$, then is strictly positive on $(0,1]$.

As a consequence, since in all the considered situations in the enunciate, the solutions of problem (1) are obtained as nontrivial fixed points of operator T_μ in the cone $\mathcal{P}$, we deduce that all of them are strictly positive on $(0,1]$.

Let's prove the first assertion of the result.

Since $f_0 > 0$ and $f_\infty > 0$ there exists $L > 0$ such that $f(u) \geq L\, u$ for all $u > 0$.

Define

$$\bar{\mu} := \frac{\alpha}{L\, \lambda\, \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\, s^{\alpha-1}\, ds}. \tag{9}$$

Consider $\mu > \mu_0$. If there is a nontrivial solution $u \in \mathcal{P}$ of problem (1) we have that

$$\begin{aligned} \|u\| = \|T_\mu u\| &= \max_{t\in[0,1]} \left\{ \mu \int_0^1 G(t,s)\, g(s)\, f(u(s))\, ds \right\} \\ &\geq \max_{t\in[0,1]} \left\{ \mu\, t^{\alpha-1} \int_0^1 G(1,s)\, g(s)\, f(u(s))\, ds \right\} \\ &= \mu \int_0^1 G(1,s)\, g(s)\, f(u(s))\, ds \\ &\geq \mu\, L \int_0^1 G(1,s)\, g(s)\, u(s)\, ds \\ &\geq \mu\, L \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\, u(s)\, ds \\ &\geq \mu\, L\, \frac{\lambda}{\alpha} \int_{\frac{1}{2}}^1 G(1,s)\, g(s)\, s^{\alpha-1}\, \|u\|\, ds > \|u\|, \end{aligned}$$

which is a contradiction.

The rest of the cases follow by combining Lemmas 3.2 - 3.7 together with Theorem 3.1 one or twice. The proofs use similar arguments to the ones given in [6, Theorem 2.14] and we omit them. $\square$

Remark 3.1. *It is immediate to verify that, instead of condition* $(H2)$*, we can assume the weaker one:*

(H2') $g \in L^1[0,1]$, $g(t) \geq 0$ *for a.e.* $t \in [0,1]$ *and there is* $a \in (0,1)$ *such that* $\int_a^1 g(s)\,ds > 0$.

The proofs follow by direct adaptation of the ones given here. In this case the solutions belong to the cone:

$$\mathcal{P}' = \left\{ u \in E,\ u(t) \geq 0 \text{ for all } t \in [0,1],\ u(t) \geq \frac{t^{\alpha-1}\lambda}{\alpha}\|u\|, \text{ for all } t \in [a,1] \right\}.$$

4 Examples

We now give three examples where we apply Theorem 3.8 and that illustrate our results. On them it is pointed out that we are able to get some estimates of the values μ_0 mentioned in Theorem 3.8. In all the cases it is immediate to verify that conditions $(H1)$, $(H2)$ and $(H3)$ are fulfilled.

Example 4.1. *Let consider the following integral boundary value problem*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu g(t) f(u(t)) = 0 & 0 \leq t \leq 1 \\ u(0) = u'(0) = 0, u(1) = 2\int_0^1 u(s)ds, & \end{cases} \tag{10}$$

where

$$f(x) = \sqrt{x} + \log\left(x^2+1\right) \quad \text{and} \quad g(t) = t^2.$$

By direct calculation, we obtain $f_0 = \infty$ *and* $f_\infty = 0$*. From Theorem 3.8, part 4, we get that problem (10) has a positive solution for every* $\mu > 0$.

Example 4.2. *Consider the fractional differential equation (10) with* $f(x) = e^{-1/x}\left(\sqrt{x} + \sin x\right)$ *and* $g(t) = 10\sqrt{t}$.

It is not difficult to verify that $f_0 = 0$ *and* $f_\infty = 0$*. In consequence, Theorem 3.8, part 7, is applicable and we can ensure the existence of two positive solutions of this problem for* μ *large enough.*

Moreover, we can deduce sharper results, in the sense that it is possible to get estimations on the values of the parameters for which the multiplicity result holds. To this end, let $r_1 = 1$ *and* $r_2 = 2$*. By pasting such values on (8) we obtain that* $\mu_0(1) \approx 38.1972$ *and* $\mu_0(2) \approx 4.10508$*. Now, Lemma 3.3 tell us that*

$$\|T_\mu u\| \geq \|u\|, \qquad \text{if } \|u\| = r_i \qquad \text{for all } \mu \geq \mu_0(r_i),\ i = 1,2.$$

Now, from Lemma 3.5 we have that for any $\mu > 0$ *given, there is* $r_0(\mu) > 0$ *for which*

$$\|T_\mu u\| \leq \|u\|, \qquad \text{if } \|u\| = r \qquad \text{for all } r \leq r_0(\mu).$$

In particular, from Theorem 3.1, we have that for any $\mu \geq 38.1972$ *there is a positive solution* u_1 *of problem (10) that satisfies* $0 < \|u_1\| \leq 1$

Using now Lemma 3.7 we deduce that if $\mu \geq 4.10508$ *then problem (10) has a positive solution* u_2 *such that* $2 \leq \|u_2\|$.

Obviously we can ensure the existence of the two solutions for $\mu \geq 38.1972$.

Example 4.3. *Consider the integral boundary value problem (10) with* $f(x) = e^x$ *and* $g(t) = \frac{t^2}{2}$*. Clearly,* $f_0 = \infty$ *and* $f_\infty = \infty$*.*

As consequence, part 1 of Theorem 3.8 ensures that for μ *large enough there is no solution of the considered problem. Moreover Theorem 3.8, part 4 implies that this problem has two positive solutions for positive* μ *small enough.*

Since $f(x)/x \geq e$ *for all* $x > 0$*, we obtain that the expresion* (9) *remains* $\bar{\mu} \approx 25.3927$*. As a consequence, we have that for* $\mu > 25.3927$ *problem (10) has no positive solution.*

On the other hand, let $R_1 = 1$ *and* $R_2 = 2$*. So* $\mu_0(1) \approx 0.978074$ *and* $\mu_0(2) \approx 0.719627$ *in (7). Thus, from Theorem 3.1 and Lemma 3.4, we deduce, for all* $\mu \leq \mu_0(1)$*, the existence of a positive solution* u_1 *such that* $0 < \|u_1\| \leq 1$*.*

Analogously, from Lemma 3.6 we have that if $\mu \leq \mu_0(2)$ *then problem (10) has a positive solution* u_2 *satisfying* $2 \leq \|u_2\|$*.*

As consequence, we can ensure the existence of the two positive solutions u_1 *and* u_2 *for all* $\mu \leq 0.719627$

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