Characterization of the constant sign of a class of Periodic and Neumann Green's functions via spectral theory

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Abstract In this paper we characterize the regions of constant sign of the Green's fucntions related to operator $T_n[p,M]\,u(t) = u^{(n)}(t) + p\,u^{(n-2)}(t) + M\,u(t)$, with n an even number, $n \geq 4$, and $p \leq 0$, coupled to periodic or Neumann boundary conditions. The results generalize the situation considered in [3] for the particular case of $p = 0$.

1 Introduction and Preliminaries

The study of Nonlinear Boundary Value Problems is closely related to the constant sign of the solutions related to the linear part of the studied equation. Such constant sign is fundamental to develop the method of lower and upper solutions [4], the monotone iterative techniques [5] or the existence of solutions defined in suitable cones [7]. Such property is equivalent to the constant sign of the related Green's function [8]. Due to the difficulty in obtaining the exact expression of such functions and that, in case of having such expression, it is very difficult to manage it, to develop a theory that allows us to know when the Green's function has constant sign in a direct way, without necessity of obtaining its expression, is of a great importance (see [6, 8, 9] and references therein for the Hill's equation). In this paper we consider an even order periodic equation and extend the results obtained by the authors in [3] to a more general situation. So, let $T > 0$, $p \leq 0$, $M \in \mathbb{R}$ and $n \in \mathbb{N}$ an even number,

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$n \geq 4$, be given. Consider the n-th order linear operator

$$T_n[p,M]\,u(t) := u^{(n)}(t) + p\,u^{(n-2)}(t) + M\,u(t), \quad \text{for all } t \in I := [0,T]. \tag{1}$$

Definition 1 Given a Banach space $X \subset C^n(I)$, operator $T_n[p,M]$ is said to be nonresonant in X if and only if the homogeneous equation

$$T_n[p,M]\,u(t) = 0 \quad \text{for all } \ t \in I, \qquad u \in X,$$

has only the trivial solution.

Definition 2 Given a Banach space $X \subset C^n(I)$ and $\bar{M} \in \mathbb{R}$, we say that $\bar{\lambda} \in \mathbb{R}$ is an eigenvalue of operator $T_n[p,\bar{M}]$ in X if and only if the homogeneous equation

$$T_n[p,\bar{M}+\bar{\lambda}]\,u(t) = 0 \quad \text{for all } \ t \in I, \qquad u \in X,$$

has non trivial solutions.

It is very well known that if operator $T_n[p,M]$ is nonresonant in X then, for any $\sigma \in C(I)$, the non homogeneous problem

$$T_n[p,M]\,u(t) = \sigma(t) \quad \text{for all } t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t) = \int_0^T G[p,M,T](t,s)\,\sigma(s)\,ds, \quad \text{for all } t \in I.$$

Function $G[p,M,T]$ is the so-called Green's function related to operator $T_n[p,M]$ on X. See [1] for details.

Now we introduce the concept of inverse positive and inverse negative operators.

Definition 3 Operator $T_n[p,M]$ is inverse positive (negative) in X if and only if the related Green's function $G[p,M,T] \geq 0$ ($G[p,M,T] \leq 0$) on $I \times I$.

Thus, the Banach space X for the periodic boundary conditions becomes

$$X^n_{P,T} = \left\{u \in C^n(I) : \ u^{(j)}(0) = u^{(j)}(T), \ j = 0, \dots, n-1\right\},$$

and the corresponding Green's function is denoted by $G_P[p,M,T]$.

As a direct consequence of [1, Theorems 1.8.5 and 1.8.9, Lemmas 1.8.25 and 1.8.33], since $M = 0$ is the main eigenvalue of $T_n[p,0]$ in $X^n_{P,T}$ (the corresponding eigenfunctions have constant sign in I), which implies that $G_P[p,0,T]$ does not exist, we deduce that for any $p \leq 0$ given, there are $M_1(p) \leq 0 \leq M_2(p)$ (or $M_1(p) = -\infty$ and/or $M_2(p) = \infty$) for which $M\,G_P[p,M,T] > 0$ in $I \times I$ if and only if $M \in (M_1(p), M_2(p))\backslash\{0\}$. Moreover, if $M_1(p) < 0$ ($M_2(p) > 0$), it is not an eigenvalue of operator $T_n[p,0]$ in $X^n_{P,T}$ and the nonpositive function $G_P[p,M_1(p),T]$ (nonnegative function $G_P[p,M_2(p),T]$) vanishes at some point of $I \times I$. Furthermore, function $G_P[p, M, T]$ is monotone decreasing with respect to M on $[M_1(p), 0)$ $((0, M_2(p)])$.

In the sequel we will prove that $M_1(p) < 0 < M_2(p)$, i.e., the set of values for which operator $T_n[p, M]$ is inverse positive (negative) in $X^n_{P,T}$ is not empty. Before this, we will enunciate the following very well know result (see [1, Section 1.9] and references therein):

Lemma 1 *The following properties are verified:*

1. *Operator* $T_2[0, M]$ *is inverse negative in* $X^2_{P,T}$ *if and only if* $M < 0$.
2. *Operator* $N_{A,B}\, u = u'' - 2\,A\,u' + (A^2 + B^2)\,u$ *is inverse positive in* $X^2_{P,T}$ *if and only if* $0 < B \leq \dfrac{\pi}{T}$.
3. *If the linear operators* L_1 *and* L_2 *are either both inverse positive or both inverse negative operators in* $X^n_{P,T}$ *and* $X^m_{P,T}$ *respectively, then* $L_1 \circ L_2$ *is inverse positive in* $X^{n+m}_{P,T}$.
4. *If the linear operators* L_1 *and* L_2 *are, respectively, inverse positive in* $X^n_{P,T}$ *and inverse negative in* $X^m_{P,T}$*, then* $L_1 \circ L_2$ *is inverse negative in* $X^{n+m}_{P,T}$.

Let us prove now the aforementioned result. The proof uses similar arguments to the ones used in ([1, Section 1.9] and references therein) for the particular case of $p = 0$. It consists on to rewrite the n-order operator as the composition of $n/2$ operators of second order.

Lemma 2 *Let* $n \geq 4$ *be an even number and* $p \leq 0$. *Then there exist* $M_1(p) < 0 < M_2(p)$ *such that* $T_n[p, M]$ *is inverse negative in* $X^n_{P,T}$ *if* $[M_1(p), 0)$ *and inverse positive in* $X^n_{P,T}$ *if* $(0, M_2(p)]$.

Proof. Let us consider the polynomial function $f_M(\lambda) := \lambda^n + p\,\lambda^{n-2} + M$. Since n is even, it is obvious that $f_M(\lambda) = 0$ if and only if $f_M(-\lambda) = 0$, i.e., its roots are symmetric. Let us study such roots for values of M near to 0:

- If $M = 0$ then $f_0(\lambda) = \lambda^{n-2}\,(\lambda^2 + p)$ has a root of multiplicity $n-2$ at $\lambda = 0$ and two simple roots $\lambda = \sqrt{-p}$ and $\lambda = -\sqrt{-p}$.
- If $M < 0$ then $f_M(\lambda)$ has two simple real roots $\lambda_1 > 0 > -\lambda_1$, and $n-2$ conjugated complex roots: $\alpha_j \pm i\,\beta_j$, $j = 1, \dots, (n-2)/2$.
- If $M > 0$ is next to 0, using the continuity of the roots of the polynomial, $f_M(\lambda)$ has four simple real roots $\lambda_1 > 0 > -\lambda_1$, $\lambda_2 > 0 > -\lambda_2$, and $n-4$ conjugated complex roots: $\tilde{\alpha}_j \pm i\,\tilde{\beta}_j$, $j = 1, \dots, (n-4)/2$.

As a consequence we have that if $M < 0$, then

$$f_M(\lambda) = (\lambda^2 - \lambda_1^2) \prod_{j=1}^{\frac{n-2}{2}} ((\lambda - \alpha_j)^2 + \beta_j^2), \quad \text{and so}$$

$$T_n[p, M] \equiv \overline{T}_1 \circ T_1 \circ \cdots \circ T_{\frac{n-2}{2}},$$

where $\overline{T}_1 u = u'' - \lambda_1^2\,u$ and $T_j u = u'' - 2\,\alpha_j\,u' + (\alpha_j^2 + \beta_j^2)\,u$.

From Lemma 1 we have that $\overline{T}_1$ is inverse negative in $X^2_{P,T}$. Moreover, if M is near to 0, by the continuity of the roots of the polynomial, we have that $0 < \beta_j \leq \frac{\pi}{T}$ and so T_j is inverse positive in $X^2_{P,T}$ for all $j = 1, \ldots, \frac{n-2}{2}$. Thus, using Lemma 1 again, we conclude that $T_n[p, M]$ is inverse negative in $X^n_{P,T}$ for $M < 0$ next to 0.

On the other hand, if $M > 0$ is next to 0 then

$$f_M(\lambda) = (\lambda^2 - \lambda_1^2)\,(\lambda^2 - \lambda_2^2) \prod_{j=1}^{\frac{n-4}{2}} ((\lambda - \tilde{\alpha}_j)^2 + \tilde{\beta}_j^2), \quad \text{and so}$$

$$T_n[p, M] \equiv \overline{T}_1 \circ \overline{T}_2 \circ \tilde{T}_1 \circ \cdots \circ \tilde{T}_{\frac{n-4}{2}},$$

where $\overline{T}_1 u = u'' - \lambda_1^2\, u$, $\overline{T}_2 u = u'' - \lambda_2^2\, u$ and $\tilde{T}_j u = u'' - 2\,\tilde{\alpha}_j\, u' + (\tilde{\alpha}_j^2 + \tilde{\beta}_j^2)\, u$.

From Lemma 1, both $\overline{T}_1$ and $\overline{T}_2$ are inverse negative in $X^2_{P,T}$. Moreover, if M is near to 0 it holds that $0 < \tilde{\beta}_j \leq \frac{\pi}{T}$ and so $\tilde{T}_j$ is inverse positive in $X^2_{P,T}$ for all $j = 1, \ldots, \frac{n-4}{2}$ and, as a consequence, Lemma 1 implies that $T_n[p, M]$ is inverse positive in $X^n_{P,T}$ for $M > 0$ next to 0. □

Since all the coefficients in operator $T_n[p, M]$ are constant, we are in conditions to apply the following result (see [1, Section 1.4] and references therein), that ensures that $G_P[p, M, T]$ is constant over the straight lines of slope equals to 1.

Lemma 3 *The Green's function $G_P[p, M, T]$ related to the operator $T_n[p, M]$ in $X^n_{P,T}$ is given by the following expression:*

$$G_P[p, M, T](t, s) = \begin{cases} G_P[p, M, T](t - s, 0), & 0 \leq s \leq t \leq T, \\ G_P[p, M, T](T + t - s, 0), & 0 \leq t \leq s \leq T. \end{cases}$$

Moreover, $r_M(t) := G_P[p, M, T](t, 0)$ is the unique solution of the following problem:

$$\begin{cases} T_n[p, M]\, r_M(t) = 0,\ t \in I, \\ r_M^{(i)}(0) - r_M^{(i)}(T) = 0,\ i = 0, \ldots, n-2, \\ r_M^{(n-1)}(0) - r_M^{(n-1)}(T) = 1. \end{cases} \tag{2}$$

As it is stated on the proof of [1, Corollary 1.4.12] for a more general situation, it is immediate to verify that if $n = 2\,k$ is even, then $r_M(t) = r_M(T - t)$ for all $t \in I$. Notice that, as a direct consequence, we deduce that

$$r_M^{(j)}(t) = (-1)^j\, r_M^{(j)}(T - t) \qquad \text{for all } t \in I \text{ and } j \in \{0, 1, \ldots, 2\,k\}. \tag{3}$$

In particular,

$$r_M^{(2\,j+1)}(T/2) = 0 \qquad \text{for all } \ j \in \{0, 1, \ldots, k-1\}, \tag{4}$$

$$r_M^{(2\,j+1)}(0) = r_M^{(2\,j+1)}(T) = 0, \qquad j \in \{0, 1, \ldots, k-2\}, \tag{5}$$

$$r_M^{(2\,k-1)}(0) = 1/2 \qquad \text{and} \qquad r_M^{(2\,k-1)}(T) = -1/2. \tag{6}$$

Now, for any even natural number $n = 2\,k$, we will denote by $G_N\,[p, M, T]$ the Green's function related to the operator $T_n\,[p, M]$ coupled to the so-called Neumann boundary conditions:

$$X^n_{N,T} = \left\{u \in C^n(I) : \; u^{(2j+1)}(0) = u^{(2j+1)}(T) = 0, \; j = 0, \dots, k-1\right\}.$$

As it is shown in [2, Theorem 3], in case of constant coefficients, the regions of constant sign of the Green's functions related to periodic and Neumann conditions coincide in intervals of double length. The result is the following:

Theorem 1 *The following property is fulfilled:*
$G_P[p, M, 2T] \leq 0$ *(*$G_P[p, M, 2T] \geq 0$*) on* $[0, 2T] \times [0, 2T]$ *if and only if* $G_N\,[p, M, T] \leq 0$ *(*$G_N\,[p, M, T] \leq 0$*) on* $I \times I$*.*

2 Characterization of constant sign of Periodic and Neumann Green's functions

In this section we will extend Theorems 2 and 4 in [3] to the case $p < 0$. In particular, we will consider problems of order $n = 2\,k \geq 4$. The case $n = 2$ is considered in [3] but such situation makes no sense in the context of this paper because in such a case $n - 2 = 0$ and so the problem would be reduced to the one studied in [3]. The obtained result is the following.

Theorem 2 *Let* $n = 2\,k$ *with* $k \in \mathbb{N}$, $k \geq 2$*, and* $p \leq 0$*. Then the following properties hold:*
1.- The Green's function related to operator $T_n[p, M]$ *on* $X^n_{P,T}$ *is nonnegative on* $I \times I$ *(and strictly positive on* $I \times I$ *if* M *is on the interior of the intervals) if and only if the following conditions are fulfilled:*

1. $k = 2\,l + 1$ *for some* $l \geq 1$*, and* $M \in (0, M_2(p)]$*, being* $M_2(p)$ *the least positive eigenvalue of problem*

$$\begin{cases} r^{(n)}(t) + p\,r^{(n-2)}(t) = 0, \; t \in [0, T/2], \\ r(0) = 0, \\ r^{(2\,j+1)}(0) = 0, \; j \in \{0, 1, \dots, k-2\}, \\ r^{(2\,j+1)}(T/2) = 0, \; j \in \{0, 1, \dots, k-1\}. \end{cases} \tag{7}$$

2. $k = 2\,l$ *for some* $l \geq 1$*, and* $M \in (0, M_2(p)]$*, being* $M_2(p)$ *the least positive eigenvalue of problem*

$$\begin{cases} r^{(n)}(t) + p\,r^{(n-2)}(t) = 0, \; t \in [0, T/2], \\ r(T/2) = 0, \\ r^{(2\,j+1)}(0) = 0, \; j \in \{0, 1, \dots, k-2\}, \\ r^{(2\,j+1)}(T/2) = 0 = 0, \; j \in \{0, 1, \dots, k-1\}. \end{cases} \tag{8}$$

2.- The Green's function related to operator $T_n[p,M]$ on $X^n_{P,T}$ is nonpositive on $I \times I$ (and strictly negative on $I \times I$ if M is on the interior of the intervals) if and only if the following conditions are fulfilled:

1. *$k = 2\,l+1$ for some $l \geq 1$, and $M \in [M_1(p), 0)$, being $M_1(p)$ the biggest negative eigenvalue of Problem* (8)*.*
2. *$k = 2\,l$ for some $l \geq 1$, and $M \in [M_1(p), 0)$, being $M_1(p)$ the biggest negative eigenvalue of Problem* (7)*.*

Proof. Let $p \leq 0$ be fixed. From Lemma 3, it is clear that the sign of the Green's function on $I \times I$ is characterized by the sign of the function r_M on I. Moreover, from Lemma 2 we know that there exist $M_1(p) < 0 < M_2(p)$ for which $M\, r_M > 0$ in I if and only if $M \in (M_1(p), M_2(p))\backslash\{0\}$. In particular, since r_M is monotone decreasing with respect to $M \in [M_1(p), 0) \cup (0, M_2(p)]$, we have that our problem is reduced to find the exact values of $M_1(p) < 0 < M_2(p)$, that satisfy that they are the unique real constants for which r_M has constant sign on I and vanishes at some point in I.

It is important to point out that if $M \in [M_1(p), M_2(p)]\backslash\{0\}$ then M is not an eigenvalue of operator $u^{(n)} + p\, u^{(n-2)}$ on $X^n_{P,T}$. As a direct consequence, identities (4) and (5) implies that r_M satisfies the two last sets of boundary conditions imposed in Problems (7) and (8) for all M in such intervals.

Let us define $v(t) := r''_M(t) + p\, r_M(t)$, $t \in I$. Taking into account (3), it is immediate to verify that v satisfies the same property, that is:

$$v^{(j)}(t) = (-1)^j\, v^{(j)}(T-t) \qquad \text{for all } t \in I \text{ and } j \in \{0, 1, \ldots, 2\,k\}. \tag{9}$$

In particular,

$$v^{(2\,j+1)}(T/2) = 0 \qquad \text{for all } \; j \in \{0, 1, \ldots, k-1\}. \tag{10}$$

Moreover, from (5) and (6), we deduce that

$$v^{(2\,j+1)}(0) = v^{(2\,j+1)}(T) = 0, \qquad j \in \{0, 1, \ldots, k-3\} \cup \{k-1\} \tag{11}$$

and

$$v^{(2\,k-3)}(0) = 1/2 \qquad \text{and} \qquad v^{(2\,k-3)}(T) = -1/2. \tag{12}$$

Finally, by integration, we reach to

$$\int_0^1 v(s)\, ds = \int_0^1 r''_M(s)\, ds + p \int_0^1 r_M(s)\, ds = p \int_0^1 r_M(s)\, ds. \tag{13}$$

Now, if $M \in (M_1(p), M_2(p))\backslash\{0\}$, since $M\, r_M(t) > 0$ for all $t \in I$, we have that $v^{(2\,k-2)}(t) = r^{(2\,k)}_M(t) + p\, r^{(2\,k-2)}_M(t) = -M\, r_M(t) < 0$ for all $t \in I$. Thus, we deduce that $v^{(2\,k-3)}$ is strictly decreasing on I. Moreover, as a direct consequence of (10) and (12), we conclude that

$$v^{(2\,k-3)}(t) > 0 \quad \text{for all } t \in (0, T/2) \quad \text{and} \quad v^{(2\,k-3)}(t) < 0 \quad \text{for all } t \in (T/2, T).$$

Therefore, $v^{(2k-4)}$ is strictly increasing on $(0,T/2)$ and strictly decreasing in $(T/2,T)$.

Now, if $\mathbf{k=2}$ (i.e. $2k-4=0$), we shall distinguish two different cases:
Case 1: $\mathbf{M} \in (\mathbf{0}, \mathbf{M_2(p)}]$.
In this case, it is clear that r_M is nonnegative on I and so, from (13), we know that either v is nonpositive or changes its sign on I.

Now, let $t_0 \in I$ be such that $r_{M_2(p)}(t_0) = 0$ (by symmetry, we may assume that $t_0 \in [0,T/2]$). Since $r_{M_2(p)}$ is nonnegative on I, t_0 is a minimum of $r_{M_2(p)}$ and so $r'_{M_2(p)}(t_0) = 0$ and $r''_{M_2(p)}(t_0) \geq 0$. In such a case, $v(t_0) = p\, r''_{M_2(p)}(t_0) \geq 0$. Therefore, if $r_{M_2(p)}$ vanishes at some point t_0, v must change its sign on I, that is, there exists $t_1 \in (0,T/2)$ such that

$$v(t) > 0 \text{ for all } t \in (t_1, T-t_1) \text{ and } v(t) < 0 \text{ for all } t \in [0,t_1) \cup (T-t_1, T],$$

with $t_0 \in [t_1, T-t_1)$.

Moreover, for $t \in (t_1, T-t_1)$, it occurs that

$$r''_{M_2(p)}(t) \geq r''_{M_2(p)}(t) + p\, r_{M_2(p)}(t) = v(t) > 0$$

and so $r'_{M_2(p)}$ is strictly increasing in $(t_1, T-t_1)$. Since $r'_{M_2(p)}(T/2) = r'_{M_2(p)}(t_0) = 0$, it must occur that $t_0 = T/2$. Thus, $M_2(p)$ is the least positive eigenvalue of problem

$$r^{(4)}(t) + p\, r''(t) = 0,\ t \in [0,T/2], \quad r(T/2) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

Case 2: $\mathbf{M} \in [\mathbf{M_1(p)}, \mathbf{0})$.
Similarly, in this case it occurs that r_M is nonpositive on I and so (13) implies that either v is nonnegative os changes sign on I.

Taking again $t_0 \in I$ such that $r_{M_1(p)}(t_0) = 0$ and reasoning as in previous case, it can be proved that $v(t_0) \leq 0$ and, consequently, v must change its sign on I. Thus, there exists $t_1 \in (0,T/2)$ such that

$$v(t) > 0 \text{ for all } t \in (t_1, T-t_1) \text{ and } v(t) < 0 \text{ for all } t \in [0,t_1) \cup (T-t_1, T],$$

with $t_0 \in [0, t_1]$.

Now, for $t \in [0, t_1)$,

$$r''_{M_1(p)}(t) \leq r''_{M_1(p)}(t) + p\, r_{M_1(p)}(t) = v(t) < 0$$

and $r'_{M_1(p)}$ is strictly decreasing in $[0, t_1)$. Since $r'_{M_1(p)}(0) = r'_{M_1(p)}(t_0) = 0$, it must occur that $t_0 = 0$. Thus, $M_1(p)$ is the biggest negative eigenvalue of problem

$$r^{(4)}(t) + p\, r''(t) = 0,\ t \in [0,T/2], \quad r(0) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

On the other hand, if $\mathbf{k>2}$, (11) together to the monotonicity properties previously proved for $v^{(2k-4)}$, implies that $v^{(2k-4)}$ must change its sign on I and so, we deduce that there exists $t_1 \in (0,T/2)$ such that

$v^{(2k-4)}(t) > 0$ for all $t \in (t_1, T-t_1)$ and $v^{(2k-4)}(t) < 0$ for all $t \in [0,t_1) \cup (T-t_1, T]$.

This, together with (9), (10) and (11) implies that

$$v^{(2k-5)}(t) < 0 \text{ for all } t \in (0, T/2) \text{ and } v^{(2k-5)}(t) > 0 \text{ for all } t \in (T/2, T),$$

and, as a consequence, $v^{(2k-6)}$ is strictly decreasing on $(0, T/2)$ and strictly increasing in $(T/2, T)$.

Now, if $\mathbf{k = 3}$, we shall distinguish two cases and reasoning similarly to the case $k = 2$, we arrive at the following results:

Case 1: $\mathbf{M} \in (\mathbf{0}, \mathbf{M_2(p)}]$.

Reasoning as in Case 1 for $k = 2$, we deduce that if t_0 is a minimum of $r_{M_2(p)}$ ($r_{M_2(p)}(t_0) = 0$) then v must change its sign on I. In particular, there exists $t_1 \in (0, T/2)$ such that

$$v(t) < 0 \text{ for all } t \in (t_1, T-t_1) \text{ and } v(t) > 0 \text{ for all } t \in [0, t_1) \cup (T-t_1, T],$$

with $t_0 \in [0, t_1)$. Also, it occurs that $r''_{M_2(p)}(t) > 0$ for $t \in [0, t_1)$ and so $r'_{M_2(p)}$ is strictly increasing in such interval and, since $r'_{M_2(p)}(0) = r'_{M_2(p)}(t_0) = 0$, necessarily $t_0 = 0$. This way, we conclude that $M_2(p)$ is the least positive eigenvalue of problem

$$\begin{cases} r^{(6)}(t) + p\, r^{(4)}(t) = 0, \ t \in [0, T/2], \\ r(0) = r'(0) = r'''(0) = r'(T/2) = r'''(T/2) = r^{(5)}(T/2) = 0. \end{cases}$$

Case 2: $\mathbf{M} \in [\mathbf{M_1(p)}, \mathbf{0})$.

Using similar arguments to previous cases, we deduce that $M_1(p)$ is the biggest negative eigenvalue of problem

$$\begin{cases} r^{(6)}(t) + p\, r^{(4)}(t) = 0, \ t \in [0, T/2], \\ r'(0) = r'''(0) = r(T/2) = r'(T/2) = r'''(T/2) = r^{(5)}(T/2) = 0. \end{cases}$$

For $\mathbf{k > 3}$, by (11), $v^{(2k-6)}$ must change its sign on I and, reasoning as before, we may deduce that

$$v^{(2k-7)}(t) > 0 \text{ for all } t \in (0, T/2) \text{ and } v^{(2k-7)}(t) < 0 \text{ for all } t \in (T/2, T).$$

In this case, we are in the same situation as $v^{(2k-3)}$ and so the result holds by recurrence. □

To obtain a numerical approach of the eigenvalues is very simple. It consists on look for the zeros of the corresponding Wronskians. On [3, Section 5] it is explained in detail how to do it. In figures 1 and 2 are plotted the values of $M_1(p)$ and $M_2(p)$ for $p \in [-20, 0]$ and $T = 1$.

Remark 1 It is immediate to verify that v is a solution of $T_n[p, M]\, v = 0$ on $[a, b]$, together the boundary conditions (7) or (8) by replacing 0 and T by a and b respectively, if and only if $u(t) := v((b-a)t/T + a))$, $t \in I$, is a solution of

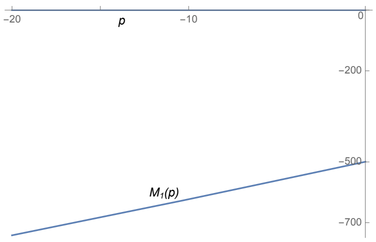

Fig. 1 Values of $M_1(p)$ for $T = 1$.

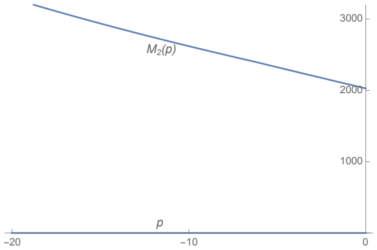

Fig. 2 Values of $M_2(p)$ for $T = 1$.

$T_n[p\,(T/(b-a))^2, M(T/(b-a))^n]\,u = 0$ on I. So, as a direct consequence, we obtain that the eigenvalues of such problems $\lambda_j(p,a,b)$ (with obvious notation) satisfy the equality

$$\lambda_j(p,a,b) = \lambda_j(p\,(T/(b-a))^2, 0, T)\,(T/(b-a))^n, \quad j \in \mathbb{N}.$$

As it has been done in [3], as a direct consequence of Theorem 1 we can rewrite the results for the periodic problem obtained in Theorem 2 to the Neumann boundary value problem. In this case, the result is the following one.

Theorem 3 *Let $n = 2\,k$ with $k \in \mathbb{N}$, $k \geq 2$, and $p \leq 0$. Then the following properties are fulfilled:*
1.- The Green's function related to operator $T_n[p,M]$ on the space $X^n_{N,T}$, of functions satisfying T–Neumann boundary conditions, is nonnegative on $I \times I$ (and strictly positive on $I \times I$ if M is on the interior of the intervals) if and only if $M \in (0, M_2^N(p,0,T)]$, being $M_2^N(p,0,T)$ the least positive eigenvalue of either Problem (7) *or* (8)*, as appropriate, by replacing $T/2$ instead of T.*
2.- The Green's function related to operator $T_n[p,M]$ on the space $X^n_{N,T}$, of functions satisfying T–Neumann boundary conditions, is nonpositive on $I \times I$ (and strictly negative on $I \times I$ if M is on the interior of the intervals) if and only if $M \in [M_1^N(p,0,T), 0)$, being $M_1^N(p,0,T)$ the biggest negative eigenvalue of either Problem (7) *or* (8)*, as appropriate, by replacing $T/2$ instead of T .*

Arguing as in Remark 1, we deduce that (with obvious notation)

$$M_i^N(p,0,T) = M_i(p/4,0,T)/2^n, \quad i = 1,2.$$

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