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# Lyapunov-type inequality for higher order left and right fractional p-Laplacian problems

Alberto Cabada[1] orcid.org/0000-0003-1488-935X
Rabah Khaldi[2] orcid.org/0000-0002-6462-4424

[1]Universidade de Santiago de Compostela, Dept. de Estatística, Análise Matemática e Optimización, Santiago de Compostela, Galicia, Spain.
alberto.cabada@usc.gal
[2]Badji Mokhtar Annaba University, Dept. of Mathematics, Annaba, Algeria.
rkhadi@yahoo.fr

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**Abstract:**

*In this paper, we consider a p-Laplacian eigenvalue boundary value problem involving both right Caputo and left Riemann-Liouville types fractional derivatives. To prove the existence of solutions, we apply the Schaefer's fixed point theorem. Furthermore, we present the Lyapunov inequality for the corresponding problem.*

**Keywords:** Fractional calculus; Lyapunov inequality; p-Laplacian operator; Eigenvalue problem.

**MSC (2020):** 74H20, 26D15, 34A08.

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## 1. Introduction

In this work, firstly, we prove the existence of solutions for an iterated fractional boundary value problem

$$(1.1)\qquad \begin{cases} -{}^{C}D_{1^-}^{\alpha}\left(\phi_p\left(D_{0^+}^{\beta}u\left(t\right)\right)\right)+\lambda f\left(t,u(t)\right)=0, 0<t<1 \\ u^{(i)}\left(0\right)=D_{0^+}^{\beta+i}u\left(1\right)=0\ , i=0,1,...,n-1, \end{cases}$$

then, we obtain Lyapunov type inequalities for the corresponding problem

$$(1.2)\qquad \begin{cases} -{}^{C}D_{b^-}^{\alpha}\left(\phi_p\left(D_{a^+}^{\beta}u\left(t\right)\right)\right)+\chi(t)\phi_p\left(u(t)\right)=0, a<t<b \\ u^{(i)}\left(a\right)=D_{a^+}^{\beta+i}u\left(b\right)=0\ , i=0,1,...,n-1, \end{cases}$$

where $n-1<\alpha,\beta\leq n$, $n\geq 2$, ${}^{C}D_{b^-}^{\alpha}$ and $D_{a^+}^{\beta}$ refer to the right Caputo derivative and the left Riemann–Liouville derivative respectevely, $\phi_p(s)=s\left|s\right|^{p-2}$, $p>2$, $\phi_p^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$ and $\chi:[a,b]\rightarrow\mathbf{R}$ is a continuous function.

The problems generated by equations involving both left and right fractional derivatives, arise in the study of Euler–Lagrange equations for fractional problems of calculus of variations, see [4,14,15]. Recently, this type of problems has been considered by many authors, see [1,2,3,5,6, 7,8,9,10,12,16].

Concerning the Lyapunov inequality and its generalizations, it has been shown to be very useful in different problems, such as oscillation, asymptotic theory, disconjugacy and eigenvalue problems. For results on Lyapunov inequality one may suggest the papers [7,10,11,17].

The paper is organized as follows. In Section 2, we briefly recall some essential definitions and lemmas on fractional calculus to be used later. Our results are formulated and proved in Sections 3 and 4. The main results are Theorem 2, which establishes existence of solution for the eigenvalue problem for left and right fractional differential equation 1.1, and Theorem t-lyapunov where, we obtain a new Lyapunov type inequality for problem 1.2.

## 2. Basic results

In this section, we recall some essential definitions and preliminary facts that will be used in the sequel and can be found in [13,18,19].

**Definition 1.** *The left and right fractional integrals of order $\alpha>0$ of a function $g\in L^{1}\left[a,b\right]$ are defined respectively by*

$$
\begin{aligned}
I_{a+}^{\alpha} g(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)\, ds, \\
I_{b-}^{\alpha} g(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s)\, ds.
\end{aligned}
$$

**Definition 2.** *The left and right Riemann-Liouville fractional derivatives of order $n-1<\alpha\leq n$, $n\geq 1$ of a function $g\in AC^n[a,b]$ are defined respectively by*

$$
\begin{aligned}
D_{a+}^{\alpha} g(t) &= \frac{d^n}{dt^n}\left(I_{a^+}^{n-\alpha} g\right)(t), \\
D_{b-}^{\alpha} g(t) &= (-1)^n \frac{d^n}{dt^n}\left(I_{a^+}^{n-\alpha} g\right)(t).
\end{aligned}
$$

**Definition 3.** *The left and right Caputo fractional derivatives of order $n-1<\alpha\leq n$, $n\geq 1$ of a function $g\in AC^n[a,b]$ are defined respectively by*

$$
\begin{aligned}
{}^{C}\mathcal{D}_{a+}^{\alpha} g(t) &= I_{d^-}^{n-\alpha} g^{(n)}(t), \\
{}^{C}\mathcal{D}_{b-}^{\alpha} g(t) &= (-1)^n I_{d^-}^{n-\alpha} g^{(n)}(t).
\end{aligned}
$$

With respect to the composition rule of fractional operators, we mention the following.

**Proposition 4.** *Let $\alpha\in(n-1,n]$ and $f\in L_1[a,b]$. Then*

$$
I_{a^+}^{\alpha C} D_{a+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k, \tag{2.1}
$$

$$
I_{b^-}^{\alpha C} D_{b-}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-t)^k. \tag{2.2}
$$

Next, we cite Schaefer's fixed point theorem:

**Theorem 5.** *[19]. Let $T$ be a continuous and compact mapping of a Banach space $X$ into itself, such that the set*

$\{$x$\in X : x=\mu T x$ *for some* $0\leq\mu\leq 1\}$ *is bounded. Then $T$ has a fixed point.*

## 3. Existence of solutions

In this section, we prove the existence of solutions for the eigenvalue problem 1.1.

First, we begin by solving the following linear fractional problem:

$$\left\{ \begin{array}{l} D_{0+}^{\beta} u(t) + y(t) = 0, 0 < t < 1 \\ u^{(i)}(0) = 0 \ , i = 0, 1, ..., n-1. \end{array} \right. \tag{3.1}$$

**Lemma 1.** *For $n-1 < \beta \leq n$, $n \geq 2$, the solution of problem 1.3 is given by*

$$u(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds. \tag{3.2}$$

**Proof.** We get (3.2) by applying (2.1) in (3.1) and using the boundary conditions $u^{(i)}(0) = 0$, $i = 0, 1, ..., n-1$. □

Now, we transform the boundary value problem to an integral equation.

**Lemma 2.** *A function $u$ is a solution of the linear boundary value problem 1.1 if and only if $u$ satisfies the integral equation*
$u(t) = \frac{1}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left( \int_s^1 (r-s)^{\alpha-1} \lambda f(r, u(r))\, dr \right) ds.$

**Proof.** Applying the operator $I_{1-}^{\alpha}$ on both sides of the differential equation in 1.1 and using (2.2) together with the boundary conditions $D_{0+}^{\beta-i} u(1) = 0$, $i = 0, 1, ..., n-1$, then the boundary value problem 1.1 reduces to the following problem

$$\left\{ \begin{array}{l} D_{0+}^{\beta} u(t) - \phi_q \left( \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} \lambda f(s, u(s))\, ds \right) = 0 \\ u^{(i)}(0) = 0 \ , i = 0, 1, ..., n-1. \end{array} \right.$$

Now, equation e-ID-1 yields
$\mathrm{u}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left( \frac{1}{\Gamma(\alpha)} \int_s^1 (r-s)^{\alpha-1} \lambda f(r, u(r))\, dr \right) ds.$
From which we obtain the required result. □

Define the integral operator $T : E \to E$ by

$\mathrm{Tu}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \phi_q \left( \frac{1}{\Gamma(\alpha)} \int_s^1 (r-s)^{\alpha-1} \lambda f(r, u(r))\, dr \right) ds,$
where the Banach space $E = C[0,1]$ is endowed with the norm

$\|u\| = t \in [0,1] \max |u(t)|$.

Now, we are in a position to establish the existence of nontrivial solution for the fractional boundary value problem 1.1.

**Theorem 3.** *Suppose that $f(t,0) \neq 0$ and there exist nonnegative functions $h \in L^1[0,1]$ and $\psi \in C[0,+\infty)$, with $\psi$ nondecreasing, such that*
$|f(t,x)| \leq h(t)\psi(|x|)$ *.Then problem 1.1 has at least one nontrivial solution for all $\lambda > 0$.*

**Proof.** We are going to use the Schaefer's fixed point Theorem to prove the result. So fix $\lambda > 0$. The following properties hold
(i) $T$ maps bounded sets into bounded sets. Let $u \in B_\delta = \{u \in E, \|u\| < \delta\}$, $\delta > 0$, then we have

$$\begin{aligned}
|Tu(t)| &\leq \frac{1}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\left(\int_s^1 (r-s)^{\alpha-1}|\lambda f(r,u(r))|\,dr\right)ds \\
&\leq \frac{\lambda^{q-1}}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)}\left(\int_0^1 h(r)\psi(|u(r)|)\,dr\right)^{q-1} \\
&\leq \frac{\left(\lambda\psi(\delta)\|h\|_{L^1[0,1]}\right)^{q-1}}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)} < +\infty.
\end{aligned}$$

Hence, $T(B_\delta)$ is uniformly bounded.
(ii) $T(B_\delta)$ is equicontinuous. For $0 \leq$
$t_1 < t_2 \leq 1$, we have

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
\leq\ & \frac{1}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)}\left(\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)\phi_q\right) \\
&\left(\int_s^1 (r-s)^{\alpha-1}|\lambda f(r,u(r))|\,dr\right)ds \\
&+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\phi_q\left(\int_s^1 (r-s)^{\alpha-1}|\lambda f(r,u(r))|\,dr\right)ds\Bigg) \\
\leq\ & \frac{\left(\lambda\psi(\delta)\|h\|_{L^1[0,1]}\right)^{q-1}}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)}\left(\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)ds\right. \\
&\left.+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\right) \to 0, \text{ as } t_1 \to t_2.
\end{aligned}$$

From Arzelà–Ascoli Theorem, we deduce that the operator $T$ is compact.

(iii) Let $\Omega = B_\delta, \delta > 0$, then for $u \in \Omega$, such that $u = \mu T u, 0 < \mu < 1$, we obtain
$|u(t)| \leq |Tu(t)| \leq \frac{\left(\lambda\psi(\delta)\|h\|_{L^1[0,1]}\right)^{q-1}}{(\Gamma(\alpha))^{q-1}\Gamma(\beta)}$, which implies that the set of solutions is uniformly bounded.

So, all assumptions of Schaefer's Theorem are fulfilled, consequently $T$ has a fixed point in $\overline{\Omega}$ which, since $f(t,0) \not\equiv 0$, is a nontrivial solution of problem 1.1 in $E$. □

## 4. Lyapunov inequality

In this section, we are going to establish the Lyapunov inequality for the fractional problem 1.2

**Theorem 1.** *(Lyapunov type inequality). Let $u$ be a nontrivial solution of problem 1.2, then the inequality $\int_a^b |\chi(r)|\, dr \geq \frac{\Gamma(\alpha+1)(\Gamma(\beta))^{p-1}}{(b-a)^{(p-1)\beta+\alpha-1}} \left(\frac{\beta(p-1)-1}{p-2}\right)^{p-2}$, holds, where $p > 2$ and $(p-1)\beta \geq 1$.*

**Proof.** From the properties of the Caputo derivative showed in Proposition 4 and the boundary conditions $D_{a^+}^{\beta+i} u(b) = 0, i = 0, 1, ..., n-1$, we obtain

$$I_{b^-}^{\alpha C} D_{b^-}^{\alpha} \left(\phi_p \left(D_{0^+}^{\beta} u(t)\right)\right) = \phi_p \left(D_{0^+}^{\beta} u(t)\right). \tag{4.1}$$

Now, by applying the operator $I_{b^-}^{\alpha}$ on both sides of the differential equation in 1.2 and using (4.1), we get
$\phi_p \left(D_{0^+}^{\beta} u(t)\right) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} \chi(s) \phi_p(u(s))\, ds$, thus, we have

$$\left|\left(D_{0^+}^{\beta} u(t)\right)\right|^{p-1} \leq \frac{1}{\Gamma(\alpha)} \int_t^b |\chi(s)| \, |u(s)|^{p-1} (s-t)^{\alpha-1} ds. \tag{4.2}$$

Also, from the boundary conditions $u^{(i)}(a) = 0, i = 0, 1, ..., n-1$, yields $\mathrm{u}(t) = I_{a^+}^{\beta} D_{a^+}^{\beta} u(t)$, so, we have $|u(t)| \leq \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left|D_{0^+}^{\beta} u(s)\right| ds$.By using Hölder inequality on the integral of the right-hand side of the above inequality, we obtain

$$\begin{aligned} |u(t)| &\leq \frac{1}{\Gamma(\beta)} \left(\int_a^t (t-s)^{\frac{(\beta-1)(p-1)}{p-2}} ds\right)^{\frac{p-2}{p-1}} \left(\int_a^t \left|D_{a^+}^{\beta} u(s)\right|^{p-1} ds\right)^{\frac{1}{p-1}} \\ &= \frac{1}{\Gamma(\beta)} \left(\frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}}\right)^{\frac{p-2}{p-1}} \left(\int_a^t \left|D_{a^+}^{\beta} u(s)\right|^{p-1} ds\right)^{\frac{1}{p-1}}, \end{aligned}$$

then,

$$|u(t)|^{p-1} \leq \frac{1}{(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2} \int_a^t \left| D_{a^+}^{\beta} u(s) \right|^{p-1} ds. \tag{4.3}$$

Substituting (4.2) in (4.3) yields
$|u(t)|^{p-1} \leq \frac{1}{\Gamma(\alpha)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2}$
$\int_a^t \int_s^b |\chi(r)| \, |u(r)|^{p-1} (r-s)^{\alpha-1} dr ds$,
then, by interchanging the order of integration, we get

$$\begin{aligned}
|u(t)|^{p-1} &\leq \frac{1}{\Gamma(\alpha)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2} \\
&\quad \left( \int_a^t |\chi(r)| \, |u(r)|^{p-1} \int_a^r (r-s)^{\alpha-1} ds dr \right. \\
&\quad \left. + \int_t^b |\chi(r)| \, |u(r)|^{p-1} \int_a^t (r-s)^{\alpha-1} ds dr \right) \\
&= \frac{1}{\Gamma(\alpha)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2} \\
&\quad \int_a^t |\chi(r)| \, |u(r)|^{p-1} \frac{(r-a)^{\alpha}}{\alpha} dr \\
&\quad \left. + \int_t^b |\chi(r)| \, |u(r)|^{p-1} \frac{(r-a)^{\alpha} - (r-t)^{\alpha}}{\alpha} dr \right) \\
&\leq \frac{1}{\Gamma(\alpha+1)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (t-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2} \\
&\quad \int_a^b |\chi(r)| \, |u(r)|^{p-1} (r-a)^{\alpha} dr.
\end{aligned}$$

Therefore, we obtain $\|u\|^{p-1} \leq \frac{1}{\Gamma(\alpha+1)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} (b-a)^{\frac{(p-1)\beta-1}{p-2}} \right)^{p-2}$
$\int_a^b |\chi(r)| \, \|u\|^{p-1} (b-a)^{\alpha} dr$. Thus
$1 \leq \frac{(b-a)^{(p-1)\beta+\alpha-1}}{\Gamma(\alpha+1)(\Gamma(\beta))^{p-1}} \left( \frac{p-2}{\beta(p-1)-1} \right)^{p-2} \int_a^b |\chi(r)| \, dr$, which yields the required result. □

**Example 2.** *Consider the following problem*

$$\begin{cases} -{}^{C}D_{1^-}^{\frac{3}{2}} \left( \phi_p \left( D_{0^+}^{\frac{7}{4}} u(t) \right) \right) + \lambda f(t, u(t)) = 0, 0 < t < 1 \\ u(0) = u'(0) = D_{0^+}^{\frac{7}{4}} u(1) = D_{0^+}^{\frac{11}{4}} u(1) = 0, \end{cases}$$

*where* $f(t,x) = \frac{1+\arctan x}{\sqrt{1+t}}$.

*Obviosly, we have* $f(t,0) = \frac{1}{\sqrt{1+t}} = 0$ *and*
$|f(t,x)| \leq h(t)\psi(|x|)$, *where* $h(t) = \frac{1}{\sqrt{1+t}} \in L^1[0,1]$ *and* $\psi(x) = 1 + \arctan x$.

*From Theorem 8 we conclude that for all* $\lambda > 0$*, this problem admits at least one solution.*

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