J. Nonlinear Var. Anal. 5 (2021), No. 4, pp. 561-572
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.5.2021.4.05

THE APPLICATION OF MEIR-KEELER CONDENSING OPERATORS TO A NEW CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING ψ-CAPUTO FRACTIONAL DERIVATIVE

ZIDANE BAITICHE[1], CHOUKRI DERBAZI[1], MOUFFAK BENCHOHRA[2,*], ALBERTO CABADA[3]

[1]*Laboratory of Mathematics And Applied Sciences, University of Ghardaia, Ghardaia, Algeria*
[2]*Laboratory of Mathematics, Djillali Liabes University of Sidi-Bel-Abbes, Sidi Bel-Abbes, Algeria*
[3]*Departamento de Estatistica, Análise Matemática e Optimización, Instituto de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, Spain*

Abstract. In this paper, we discuss the existence of solutions for a new class of nonlinear differential equations involving the ψ-Caputo fractional derivative with Dirichlet boundary conditions in Banach spaces. Our approach is based on a new fixed point theorem with respect to Meir-Keeler condensing operators. Two illustrative examples are also given in support of our existence results.
Keywords. ψ-Caputo fractional derivative; Dirichlet boundary conditions; Meir-Keeler condensing operators; Fixed point theorem; Measures of noncompactness.

1. INTRODUCTION

In the past few years, several researchers employed the fractional calculus as a way of describing natural phenomena in various areas, such as, mathematics, physics, biology, chemistry, finance, economics, and engineering; see, e.g., [1, 2, 3, 4, 5] and the references therein. Nowadays, this subject has become a matter of deep interest for many scholars both in mathematics and in applications. We refer the reader to [6, 7, 8, 9, 10]. In the same line, there are several definitions of fractional derivatives and fractional integrals, such as, Riemann-Liouville, Caputo, Caputo–Hadamard, Katugampola, and Hilfer. For the class of fractional operators, which is known as ψ-fractional operators, it has been shown that these operators unify a wide class of fractional derivatives. For some recent results involving ψ-fractional derivatives, one can refer to [11, 12, 13]. On the other hand, many authors studied the problems of time-fractional differential equations involving different kinds of fractional derivatives under various boundary conditions analytically and numerically. In particular, for the existence, the uniqueness, and the stability of solutions, we refer to [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein. In the same direction, in 2015, Aghajani and Mursaleen [25] introduced the notion of the Meir-Keleer condensing operator in Banach spaces, they established a new fixed

*Corresponding author.
E-mail addresses: baitichezidane19@gmail.com (Z. Baitiche), choukriedp@yahoo.com (C. Derbazi), benchohra@yahoo.com (M. benchohra), alberto.cabada@usc.gal (A. Cabada).
Received September 26, 2020; Accepted March 16, 2021.

point theorem with respect to the Meir-Keleer condensing operators, which generalize the famous Darbo's fixed point theorem. Consequently, the fixed point theorem for the Meir-Keleer condensing operators was extensively investigated for the existence of solutions for nonlinear differential equations; see, e.g., [26, 27, 28, 29] and the references therein.

Next, we list some results on Caputo fractional differential equations. One of the first results with boundary conditions is due to Zhang [30], who studied the following boundary value problem with Caputo fractional derivative of the form

$$\begin{cases} {}^{c}D^{q}u(t) = f(t,u(t)), & t \in [0,1], 1 < q \leq 2, \\ u(0) = a \neq 0, & u(1) = b \neq 0, \end{cases}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order q, and $f \in C(J \times \mathbb{R}, \mathbb{R})$. Zhang obtained the existence of solutions by means of Schauder's fixed point theorem.

In [16], Agarwal, Benchohra and Seba discussed the following Caputo FDEs with boundary conditions:

$$\begin{cases} {}^{c}D^{q}u(t) = f(t,u(t)), & t \in [0,T], 1 < q \leq 2, \\ u(0) = u_0, & u(T) = u_T, \end{cases}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $q, f : [0,T] \times E \longrightarrow E$ is a given function, T is a fixed positive constant, E is a Banach space with norm $\|\cdot\|$, and $u_0, u_T \in E$. They obtained the existence of solutions by employing Mönch fixed point theorem combined with the technique of measure of noncompactness.

Using the ideas of Meir-Keeler condensing operators and the measure of noncompactness, Mursaleen and Rizvi [27], in 2016, studied the solvability of the infinite systems of second-order differential equations in the sequence spaces c_0 and ℓ_1

$$\begin{cases} u_i''(t) = -f_i(t,u(t)), & t \in [0,T], \\ u_i(0) = u_i(T) = 0, & i = 1,2,\ldots. \end{cases} \tag{1.1}$$

Motivated by results mentioned above, in particular, the results presented in [16, 27, 30], we consider the existence of solutions for a new boundary value problems with ψ-Caputo differential equations in Banach spaces. More precisely, we will consider the following problem

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}u(t) = f(t,u(t)), & t \in \mathrm{J} := [a,b], \\ u(a) = \theta_a, \quad u(b) = u_b, \end{cases} \tag{1.2}$$

where ${}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (1,2]$, $f : [a,b] \times E \longrightarrow E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$ and $\theta_a, u_b \in E$.

The main novelties of the present paper are as the following.

(1) We present a new class of fractional differential equations involving the ψ-Caputo fractional derivative in Banach spaces.
(2) From the choice of $\psi(t) = t$ and $\psi(t) = \ln t$, we have the problems with their solutions for the Caputo and Caputo-Hadamard fractional derivatives, respectively.
(3) The main tools used in our study are the Hausdorff's measure of noncompactness and Meir-Keleer condensing operators.
(4) The conditions imposed on the BVP (1.2) are weak.

(5) The results obtained in this paper are generalizations and partial continuation of some results obtained in [16, 27, 30].

The outline of this paper is as follows. In Section 2, we mainly present the measure of non-compactness and the Meir-Keleer condensing operator. In Section 3, we present the existence of solutions to the BVP problem (1.2). In Section 4, two examples are given to demonstrate the theoretical results. Finally, in Section 5, we wrap up this paper by a concluding remark.

2. Preliminaries

We begin this section by introducing some necessary definitions and basic results.

Let $\mathrm{J} := [a,b]$ $(0 < a < b < \infty)$ be a finite interval, and let $\psi : \mathrm{J} \to \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$. Let $C(\mathrm{J},E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm $\|u\|_\infty = \sup\{\|u(t)\|, t \in \mathrm{J}\}$. By $L^1(\mathrm{J},E)$ we denote the space of Bochner-integrable functions $u : \mathrm{J} \to E$ with the norm $\|u\|_1 = \int_a^b \|u(t)\|\mathrm{dt}$.

Next, we give some results and properties from the theory of fractional calculus. We begin by defining ψ-Riemann-Liouville fractional integrals and derivatives.

Definition 2.1 ([9, 11])**.** For $\alpha > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order α for an integrable function $u\colon \mathrm{J} \longrightarrow \mathbb{R}$ with respect to another function $\psi\colon \mathrm{J} \longrightarrow \mathbb{R}$, which is an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$ is defined as follows

$$\mathscr{I}_{a^+}^{\alpha;\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s)\mathrm{ds}, \tag{2.1}$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\mathrm{dt}, \quad \alpha > 0.$

Definition 2.2 ([11])**.** Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J},\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$. The left-sided ψ-Riemann-Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned} \mathscr{D}_{a^+}^{\alpha;\psi} u(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathscr{I}_{a^+}^{n-\alpha;\psi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} u(s)\mathrm{ds}, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3 ([11])**.** Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(\mathrm{J},\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of u of order α is defined by

$${}^c\mathscr{D}_{a^+}^{\alpha;\psi} u(t) = \mathscr{I}_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, and $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify the notation, we will use the following symbol

$$u_\psi^{[n]} u(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t). \tag{2.2}$$

From the definition, it is clear that

$$ {}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}u(t) = \begin{cases} \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} u_\psi^{[n]}(s)\mathrm{d}s, & \text{if } \alpha \notin \mathbb{N}, \\ u_\psi^{[n]}(t), & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.3} $$

(2.3) yields the Caputo fractional derivative operator when $\psi(t) = t$. Moreover, for $\psi(t) = \ln t$, it gives the Caputo-Hadamard fractional derivative. We note that if $u \in C^n(\mathrm{J}, \mathbb{R})$, then the ψ-Caputo fractional derivative of order α of u is determined as

$$ {}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}u(t) = \mathscr{D}_{a^+}^{\alpha;\psi}\left[u(t) - \sum_{k=0}^{n-1}\frac{u_\psi^{[k]}u(a)}{k!}(\psi(t)-\psi(a))^k\right]. $$

(see, e.g., [11, Theorem 3]).

Lemma 2.1 ([12])**.** *Let $\alpha, \beta > 0$, and $u \in L^1(\mathrm{J}, \mathbb{R})$. Then $\mathscr{I}_{a^+}^{\alpha;\psi}\mathscr{I}_{a^+}^{\beta;\psi}u(t) = \mathscr{I}_{a^+}^{\alpha+\beta;\psi}u(t)$, a.e. $t \in \mathrm{J}$. In particular, if $u \in C(\mathrm{J}, \mathbb{R})$, then $\mathscr{I}_{a^+}^{\alpha;\psi}\mathscr{I}_{a^+}^{\beta;\psi}u(t) = \mathscr{I}_{a^+}^{\alpha+\beta;\psi}u(t)$, $t \in \mathrm{J}$.*

Next, we recall the property describing the composition rules for fractional ψ-integrals and ψ-derivatives.

Lemma 2.2 ([12])**.** *Let $\alpha > 0$. The following assertions hold.*
If $u \in C(\mathrm{J}, \mathbb{R})$, then ${}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}\mathscr{I}_{a^+}^{\alpha;\psi}u(t) = u(t)$, $t \in \mathrm{J}$.
If $u \in C^n(\mathrm{J}, \mathbb{R})$, $n-1 < \alpha < n$, then

$$ \mathscr{I}_{a^+}^{\alpha;\psi}\,{}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{u_\psi^{[k]}(a)}{k!}\left[\psi(t)-\psi(a)\right]^k, $$

for all $t \in \mathrm{J}$.

Lemma 2.3 ([9, 12])**.** *Let $t > a$, $\alpha \geq 0$, and $\beta > 0$. Then*

- $\mathscr{I}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$,
- ${}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}$,
- ${}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = 0$, *for all $k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.*

Remark 2.1. Note that for an abstract function $u : \mathrm{J} \longrightarrow E$, the integrals which appear in the previous definitions are taken in Bochner's sense. (see, for instance, [31]).

Now, we define the Hausdorff measure of noncompactness and give some of its important properties.

Definition 2.4 ([32])**.** Let E be a Banach space and B a bounded subsets of E. Then the Hausdorff measure of non-compactness of B is defined by

$$ \chi(B) = \inf\left\{ \begin{array}{c} \varepsilon > 0 : B \text{ can be covered by} \\ \text{finitely many balls with radius } < \varepsilon \end{array} \right\}. $$

Lemma 2.4 ([32])**.** *Let $A, B \subset E$ be bounded. Then Hausdorff measure of non-compactness has the following properties*

(1) $\chi(A) = 0 \Longleftrightarrow A$ *is relatively compact,*
(2) $A \subset B \Longrightarrow \chi(A) \leq \chi(B)$*,*
(3) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$*,*

(4) $\chi(A)=\chi(\overline{A})=\chi(conv(A))$, *where* $\overline{A}$ *and* $convA$ *represent the closure and the convex hull of* A *, respectively,*
(5) $\chi(A+B)\leq\chi(A)+\chi(B)$, *where* $A+B=\{x+y: x\in A, y\in B\}$,
(6) $\chi(\lambda A)\leq|\lambda|\chi(A)$, *for any* $\lambda\in\mathbb{R}$.

In 1969, Meir and Keeler introduced a new contractive mapping, which is now called Meir-Keeler contraction.

Definition 2.5 ([33])**.** Let (X,d) be a metric space. A mapping $\mathscr{T}$ on X is said to be a Meir-Keeler contraction (MKC, for short) if, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\varepsilon\leq d(x,y)<\varepsilon+\delta\Rightarrow d(\mathscr{T}x,\mathscr{T}y)<\varepsilon,\quad\forall x,y\in X.$$

In [25], Aghajani and Mursaleen further defined the concept of Mei-keeler condensing operators on a Banach space.

Definition 2.6 ([25])**.** Let C be a nonempty subset of a Banach space E, and let μ be a measure of noncompactness on E. An operator $\mathscr{T}: C\to C$ is said to be a Meir-Keeler condensing operator if, for any $\varepsilon>0$, there exists $\delta>0$ such that $\varepsilon\leq\mu(\Omega)<\varepsilon+\delta\Rightarrow\mu(\mathscr{T}\Omega)<\varepsilon$, for any bounded subset Ω of C.

The following fixed point theorem with respect to the Meir-Keeler condensing operator was proved by Aghajani and Mursaleen [25] plays a key role in the proof of our main results.

Theorem 2.1 ([25])**.** *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* E*. Let* μ *be an arbitrary measure of noncompactness on* E*. If* $\mathscr{T}:\Omega\to\Omega$ *is a continuous Meir-Keeler condensing operator, then* $\mathscr{T}$ *has at least one fixed point and the set of all fixed points of* $\mathscr{T}$ *in* Ω *is compact.*

In addition, we also need the following lemmas.

Lemma 2.5 ([34])**.** *Let* E *be a Banach space, and let* $B\subset C(\mathrm{J},E)$ *be bounded and equicontinuous. Then* $\chi(B(t))$ *is continuous on* $[a,b]$*, and* $\chi_C(B)=\max_{t\in[a,b]}\chi(B(t))$.

Lemma 2.6 ([35])**.** *Let* E *be a Banach space and let* $B\subset E$ *be bounded. Then, for each* ε*, there is a sequence* $\{u_n\}_{n=1}^{\infty}\subset B$*, such that* $\chi(B)\leq 2\chi(\{u_n\}_{n=1}^{\infty})+\varepsilon$.

We call $B\subset L^1(\mathrm{J},E)$ uniformly integrable if there exists $\eta\in L^1(\mathrm{J},\mathbb{R}^+)$ such that $\|u(s)\|\leq\eta(s)$, for all $u\in B$ and a.e. $s\in\mathrm{J}$.

Lemma 2.7 ([36])**.** *If* $\{u_n\}_{n=1}^{\infty}\subset L^1(\mathrm{J},E)$ *is uniformly integrable, then* $t\mapsto\chi(\{u_n(t)\}_{n=1}^{\infty})$ *is measurable, and*

$$\chi\left(\left\{\int_a^t u_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right)\leq 2\int_a^t\chi(\{u_n(s)\}_{n=1}^{\infty})\mathrm{d}s.$$

Definition 2.7 ([37])**.** A function $f:[a,b]\times E\longrightarrow E$ is said to satisfy the Carathéodory conditions if

(i) $f(t,u)$ is measurable with respect to t for $u\in E$,
(ii) $f(t,u)$ is continuous with respect to $u\in E$ a.e. $t\in\mathrm{J}$.

3. Main Results

First, we define the solution to boundary value problem (1.2).

Definition 3.1. A function $u \in AC^1(\mathrm{J}, E)$ is said to be a solution of Eq. (1.2) if u satisfies the equation ${}^c\mathscr{D}_{a^+}^{\alpha;\psi} u(t) = f(t, u(t))$, a.e. on J, and the conditions $u(a) = \theta_a, \quad u(b) = u_b$.

For the existence of solutions to problem (1.2), we need the following lemma.

Lemma 3.1. *For a given $h \in L^1(\mathrm{J}, \mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^c\mathscr{D}_{a^+}^{\alpha;\psi} = h(t),\ 1 < \alpha \leq 2\,, t \in \mathrm{J} := [a,b], \\ u(a) = \theta_a, \quad u(b) = u_b, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} u(t) &= \theta_a + \mathscr{I}_{a^+}^{\alpha;\psi} h(t) + \frac{(u_b - \theta_a)(\psi(t) - \psi(a))}{(\psi(b) - \psi(a))} - \frac{(\psi(t) - \psi(a))}{(\psi(b) - \psi(a))} I_{a^+}^{\alpha;\psi} h(b) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s) \mathrm{ds} \\ &- \frac{(\psi(t) - \psi(a))}{\Gamma(\alpha)(\psi(b) - \psi(a))} \int_a^b \psi'(s)(\psi(b) - \psi(s))^{\alpha-1} h(s) \mathrm{ds} \\ &+ \theta_a + \frac{(u_b - \theta_a)(\psi(t) - \psi(a))}{(\psi(b) - \psi(a))}. \end{aligned} \tag{3.2}$$

Proof. Taking the ψ-Riemann-Liouville fractional integral of order α to the first equation of (3.1), we get

$$u(t) = \mathscr{I}_{a^+}^{\alpha;\psi} h(t) + c_0 + c_1(\psi(t) - \psi(a)), \quad c_0, c_1 \in \mathbb{R}. \tag{3.3}$$

Substituting $t = a$ in (3.3) and applying the first boundary condition of (3.1), we arrive at

$$c_0 = \theta_a.$$

Letting $t = b$ in (3.3) and using the second boundary condition of (3.1), we have

$$u(b) = u_b = \mathscr{I}_{a^+}^{\alpha;\psi} h(b) + \theta_a + c_1(\psi(b) - \psi(a)). \tag{3.4}$$

Solving (3.4), we find that

$$c_1 = \frac{u_b - \theta_a}{(\psi(b) - \psi(a))} - \frac{1}{(\psi(b) - \psi(a))} \mathscr{I}_{a^+}^{\alpha;\psi} h(b). \tag{3.5}$$

Substituting the values of c_0 and c_1 into (3.3), we get integral equation (3.2). This completes the proof. □

Now, we present our main result concerning the existence of solutions of problem (1.2). Let us introduce the following hypotheses first.

(H1) The function $f : [a,b] \times E \longrightarrow E$ satisfies Carathéodory conditions.

(H2) There exist a continuous function $p_f \colon \mathrm{J} \to \mathbb{R}_+$ and a continuous nondecreasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\|f(t,u)\| \leq p_f(t)\phi(\|u\|), \quad \text{for a.e. } t \in \mathrm{J}, \text{ and each } u \in E.$$

(H3) For each bounded set $B \subset E$, and each $t \in \mathrm{J}$, the following inequality holds,

$$\chi(f(t,B)) \leq p_f(t)\chi(B).$$

(H4) There exists a constant $r > 0$ such that

$$r \geq 2\|\theta_a\| + \|u_b\| + \phi(r)\mathscr{M}_\psi, \tag{3.6}$$

Now, we prove the following theorem concerning the existence of solutions of problem (1.2).

Theorem 3.1. *Assume that hypotheses (H1)–(H4) are satisfied. If*

$$4\mathscr{M}_\psi < 1, \tag{3.7}$$

where $\mathscr{M}_\psi = \frac{2p_f^*(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}$ *and* $p_f^* := \sup_{t\in \mathrm{J}} p_f(t)$, *then the problem* (1.2) *has at least one solution defined on* J.

Proof. Consider the operator $\mathscr{T} : C(\mathrm{J},E) \longrightarrow C(\mathrm{J},E)$ defined by:

$$\begin{aligned}\mathscr{T}u(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,u(s))\mathrm{ds}\\ &-\frac{(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-1}f(s,u(s))\mathrm{ds}\\ &+\theta_a+\frac{(u_b-\theta_a)(\psi(t)-\psi(a))}{(\psi(b)-\psi(a))}.\end{aligned} \tag{3.8}$$

It is obvious that $\mathscr{T}$ is well defined due to (H1) and (H2). Then, fractional integral equation (3.2) can be written as the following operator equation

$$u = \mathscr{T}u. \tag{3.9}$$

Thus, the existence of a solution to (1.2) is equivalent to the existence of a fixed point to operator $\mathscr{T}$ which satisfies operator equation (3.9). Define the set

$$\Omega_r = \{w \in C(\mathrm{J},E) : \|w\|_\infty \leq r\}.$$

Notice that Ω_r is closed, convex and bounded subset of the Banach space $C(\mathrm{J},E)$. We show that operator $\mathscr{T}$ satisfies all the assumptions of Theorem 2.1. We split the proof into four steps.

Step 1. $\mathscr{T}$ maps Ω_r into itself.

From assumption (H2), we have

$$\begin{aligned}
\|\mathscr{T}u(t)\| \leq & \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,u(s))\|\mathrm{ds} \\
& + \frac{(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-1}\|f(s,u(s))\|\mathrm{ds} \\
& + \|\theta_a\| + \frac{(\|u_b\|+\|\theta_a\|)(\psi(t)-\psi(a))}{(\psi(b)-\psi(a))} \\
\leq & \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}p_f(s)\phi(\|u(s)\|)\mathrm{ds} \\
& + \frac{(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-1}p_f(s)\phi(\|u(s)\|)\mathrm{ds} \\
& + \|\theta_a\| + \frac{(\|u_b\|+\|\theta_a\|)(\psi(t)-\psi(a))}{(\psi(b)-\psi(a))} \\
\leq & 2\|\theta_a\| + \|u_b\| + \phi(r)\mathscr{M}_\psi \\
\leq & r.
\end{aligned}$$

Thus $\|\mathscr{T}u\| \leq r$. This proves that $\mathscr{T}$ transforms the ball Ω_r into itself.

Step 2. Show that $\mathscr{T}$ is continuous.

Suppose that $\{u_n\}$ is a sequence such that $u_n \to u$ in Ω_r as $n \to \infty$. It is easy to see that $f(s,u_n(s)) \to f(s,u(s))$, as $n \to +\infty$, due to the Carathéodory continuity of f. On the other hand, taking (H2) into consideration, we get the following relations:

$$\begin{aligned}
\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,u_n(s))-f(s,u(s))\| \leq 2p_f(s)\phi(r)\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}, \\
\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}\|f(s,u_n(s))-f(s,u(s))\| \leq 2p_f(s)\phi(r)\psi'(s)(\psi(b)-\psi(s))^{\alpha-1},
\end{aligned}$$

which imply that each term on the left is integrable. By the Lebesgue dominated convergent theorem, we obtain

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,u_n(s))-f(s,u(s))\|\mathrm{ds} \to 0 \text{ as } n \to +\infty, \\
\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha-1}\|f(s,u_n(s))-f(s,u(s))\|\mathrm{ds} \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

It follows that $\|\mathscr{T}u_n - \mathscr{T}u\| \to 0$ as $n \to +\infty$. This implies the continuity of the operator $\mathscr{T}$.

Step 3. $\mathscr{T}(\Omega_r)$ is equicontinuous.

For any $a < t_1 < t_2 < b$ and $u \in \Omega_r$, we get

$$
\begin{aligned}
&\|\mathscr{T}(u)(t_2)-\mathscr{T}(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi'(s)\left[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}\right]\|f(s,u(s))\|\mathrm{ds} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\|f(s,u(s))\|\mathrm{ds} \\
&\quad+\frac{(\psi(t_2)-\psi(t_1))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b\psi'(s)(\psi(b)-\psi(s))^{\alpha-1}\|f(s,u(s))\|\mathrm{ds} \\
&\quad+\frac{\|u_b-\theta_a\|(\psi(t_2)-\psi(t_1))}{(\psi(b)-\psi(a))} \\
&\leq \frac{\|p_f\|_{L^\infty}\phi(r)}{\Gamma(\alpha+1)}[(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha} \\
&\quad+(\psi(t_2)-\psi(t_1))(\psi(b)-\psi(a))^{\alpha-1}\big]+\frac{\|u_b-\theta_a\|(\psi(t_2)-\psi(t_1))}{(\psi(b)-\psi(a))}.
\end{aligned}
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero independently of $u \in \Omega_r$. Hence, we conclude that $\mathscr{T}(\Omega_r) \subseteq C(\mathrm{J},E)$ is bounded and equicontinuous.

Step 4. Show that $\mathscr{T}: \Omega_r \to \Omega_r$ is a Meir-Keeler condensing operator.

To do this, let $\varepsilon > 0$ be given. We prove that there exists $\delta > 0$ such that

$$
\varepsilon \leq \chi_C(B) < \varepsilon+\delta \Rightarrow \chi_C(\mathscr{T}B) < \varepsilon, \quad \text{for any } B \subset \Omega_r.
$$

For every bounded subset $B \subset \Omega_r$ and $\varepsilon' > 0$, using Lemma 2.6 and the properties of χ, we find that there exists sequences $\{u_n\}_{n=1}^{\infty} \subset B$ such that

$$
\begin{aligned}
\chi(\mathscr{T}(B)(t)) &\leq 2\chi\left\{\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f\big(s,\{u_n(s)\}_{n=1}^{\infty}\big)\mathrm{ds}\right\} \\
&+2\chi\left\{\frac{(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f\big(s,\{u_n(s)\}_{n=1}^{\infty}\big)\mathrm{ds}\right\}+\varepsilon'.
\end{aligned}
$$

Using Lemma 2.7 and (H3), we have

$$
\begin{aligned}
\chi(\mathscr{T}(B)(t)) &\leq \frac{4}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\chi\big(f(s,\{u_n(s)\}_{n=1}^{\infty})\big)\mathrm{ds} \\
&+\frac{4(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\chi\big(f(s,\{u_n(s)\}_{n=1}^{\infty})\big)\mathrm{ds}+\varepsilon' \\
&\leq \frac{4}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}p_f(s)\chi\big(\{u_n(s)\}_{n=1}^{\infty}\big)\mathrm{ds} \\
&+\frac{4(\psi(t)-\psi(a))}{\Gamma(\alpha)(\psi(b)-\psi(a))}\int_a^b\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}p_f(s)\chi\big(\{u_n(s)\}_{n=1}^{\infty}\big)\mathrm{ds}+\varepsilon' \\
&\leq 4\frac{2p_f^*(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\chi_C(B)+\varepsilon' \\
&= 4\mathscr{M}_{\psi}\chi_C(B)+\varepsilon'.
\end{aligned}
$$

As the last inequality is true, for every $\varepsilon' > 0$, we have $\chi(\mathscr{T}(B)(t)) \leq 4\mathscr{M}_\psi \chi_C(B)$. Since $\mathscr{T}(B) \subset \Omega_r$ is bounded and equicontinuous, we know from Lemma 2.6 that

$$\chi_C(\mathscr{T}(B)) = \max_{t\in \mathrm{J}} \chi(\mathscr{T}(B)(t)).$$

Therefore, $\chi_C(\mathscr{T}(B)) \leq 4\mathscr{M}_\psi \chi_C(B)$. Observe that from the last estimates

$$\chi_C(\mathscr{T}(B)) \leq 4\mathscr{M}_\psi \chi_C(B) < \varepsilon \Rightarrow \chi_C(B) < \frac{1}{4\mathscr{M}_\psi}\varepsilon.$$

Consequently, for given $\varepsilon > 0$, and taking $\delta = \frac{1-4\mathscr{M}_\psi}{4\mathscr{M}_\psi}\varepsilon$, we get the following implication:

$$\varepsilon \leq \chi_C(B) < \varepsilon + \delta \Rightarrow \chi_C(\mathscr{T}B) < \varepsilon, \quad \text{for any } B \subset \Omega_r,$$

which means that $\mathscr{T} : \Omega_r \to \Omega_r$ is a Meir-Keeler condensing operator. It follows from Theorem 2.1 that the operator $\mathscr{T}$ defined by (3.8) has at least one fixed point $u \in \Omega_r$, which is just the solution of boundary value problem (1.2). This completes the proof of Theorem 3.1. □

4. EXAMPLES

In this section we give two examples to illustrate our main result. Let

$$E = c_0 = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0 \ (n \to \infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|u\|_\infty = \sup_{n\geq 1} |u_n|.$$

Example 4.1. Consider the following boundary value problem of a fractional differential posed in c_0:

$$\begin{cases} {}^{CH}\mathscr{D}^{1.8}_{1^+} u(t) = f(t, u(t)), \ t \in \mathrm{J} := [1, e], \\ u(1) = (0, 0, \ldots, 0, \ldots), \ u(1) = (0, 0, \ldots, 0, \ldots). \end{cases} \tag{4.1}$$

Note that, this problem is a particular case of BVP (1.2) with

$$\alpha = 1.8, a = 1, b = e, \psi(t) = \ln(t),$$

and $f : J \times c_0 \longrightarrow c_0$ given by

$$f(t,u) = \left\{ \frac{1}{e^t + 3} \left(\frac{1}{n^2} + \arctan(|u_n|) \right) \right\}_{n\geq 1}, \quad \text{for } t \in \mathrm{J}, u = \{u_n\}_{n\geq 1} \in c_0.$$

It is clear that condition (H1) hold, and

$$\|f(t,u)\| \leq \frac{1}{e^t + 3}(1 + \|u\|) = p_f(t)\phi(\|u\|).$$

Therefore, the assumption (H2) of Theorem 3.1 is satisfied with $p_f(t) = \frac{1}{e^t+3}, t \in \mathrm{J}$ and $\phi(x) = 1 + x$, $x \in [0, \infty)$. On the other hand, for any bounded set $B \subset c_0$, we have

$$\chi(f(t,B)) \leq p_f(t)\chi(B), \text{ a.e. } t \in \mathrm{J}.$$

Hence (H3) is satisfied. Now, we check that condition (3.7) is satisfied. Indeed $4\mathscr{M}_\psi = 0.8345 < 1$, and $(1 + r)\mathscr{M}_\psi \leq r$. Thus

$$r \geq \frac{\mathscr{M}_\psi}{1 - \mathscr{M}_\psi} = 5.0421,$$

Then r can be chosen as $r = 5.5$. Consequently, all the hypotheses of Theorem 3.1 are satisfied and we conclude that the BVP (4.1) has at least one solution $u \in C(\mathrm{J}, c_0)$.

Let

$$E = \ell^1 = \left\{ x = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Example 4.2. Consider the following boundary value problem of a fractional differential posed in ℓ^1:

$$\begin{cases} {}^c\mathscr{D}_{0^+}^{\frac{3}{2}} u(t) = f(t, u(t)), t \in \mathrm{J} := [0,1], \\ u(0) = (0.5, 0, \ldots, 0, \ldots), \ u(1) = (1, 0, \ldots, 0, \ldots). \end{cases} \tag{4.2}$$

Note that, this problem is a particular case of BVP (1.2) with

$$\alpha = \frac{3}{2}, a = 0, b = 1, \psi(t) = t$$

and $f : J \times \ell^1 \longrightarrow \ell^1$ given by

$$f(t,u) = \left\{ \frac{1}{(t+3)^2} \left(\frac{t}{2^n} + \frac{u_n}{\|u\|+1} \right) \right\}_{n \geq 1}, \quad \text{for } t \in \mathrm{J}, u = \{u_n\}_{n \geq 1} \in \ell^1.$$

It is clear that condition (H1) hold, and

$$\|f(t,u)\| \leq \frac{1}{(t+3)^2} (1 + \|u\|) = p_f(t) \phi(\|u\|).$$

Therefore, the assumption (H2) of Theorem 3.1 is satisfied with $p_f(t) = \frac{1}{(t+3)^2}, t \in \mathrm{J}$ and $\phi(x) = 1 + x,\ x \in [0, \infty)$. On the other hand, for any bounded set $B \subset c_0$, we have

$$\chi(f(t,B)) \leq \frac{1}{(t+3)^2} \chi(B), \text{ a.e. } t \in \mathrm{J}.$$

Hence (H3) is satisfied. Now, we check that condition (3.7) is satisfied. Indeed $4.\mathscr{M}_\psi = 0.6687 < 1$, and $2 + (1+r)\mathscr{M}_\psi \leq r$. Thus

$$r \geq \frac{2 + \mathscr{M}_\psi}{1 - \mathscr{M}_\psi} = 8.0544.$$

Then r can be chosen as $r = 8.5$. Consequently, all the hypotheses of Theorem 3.1 are satisfied and we conclude that the BVP (4.2) has at least one solution $u \in C(\mathrm{J}, \ell^1)$.

5. THE CONCLUSION

The main purpose of this paper was to study the existence of solutions for the boundary value problem involving the ψ-Caputo fractional derivative with Dirichlet boundary conditions in Banach spaces. The main results are essentially based on the Hausdorff's measure of noncompactness and Meir-Keleer condensing operators. Two examples are also provided to demonstrate the main results presented in this paper. Furthermore, it is of interest to intend to generalize the results presented in this paper by considering the existence, the uniqueness, and the stability of solutions to some fractional differential equations involving Hilfer derivative with respect to function ψ in the future.

Acknowledgments

The fourth author was partially supported by Xunta de Galicia (Spain), project EM2014/032 the Agencia Estatal de Investigación (AEI) of Spain under grant MTM2016-75140-P, co-financed by the European Community fund FEDER.

REFERENCES

[1] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific, New Jersey, 2001.
[2] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, 2010.
[3] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus-Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.
[4] V.E. Tarasov, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Heidelberg & Higher Edscation Press, Beijing, 2010.
[5] J. Vanterler da C. Sousa, M.N.N. dos Santos, L.A. Magna, E. Capelas de Oliveira, Validation of a fractional model for erythrocyte sedimentation rate, Comput. Appl. Math. 37 (2018), 6903-6919.
[6] S. Abbas, M. Benchohra, G. M. N'Guŕékata, Topics in Fractional Differential Equations, Dev. Math, 27, Springer, New York 2012.
[7] S. Abbas, M. Benchohra, G. M. N'Guŕékata, Advanced Fractional Differential and Integral Equations, Nova Science Publisher, New York, 2014.
[8] S. Abbas, M. Benchohra, G. M. N'Guŕékata, J.R. Graef, J. Henderson, Implicit Fractional Differential and Integral Equations: Existence and Stability, De Gruyter, Berlin, 2018.
[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, vol. 204, Elsevier Science, Amsterdam 2006.
[10] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, Singapore, 2014.
[11] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul. 44 (2017), 460-481.
[12] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, Math. Meth. Appl. Sci. 41 (2018), 336-352.
[13] J. Vanterler da C. Sousa, E. Capelas de Oliveira, On the ψ-Hilfer fractional derivative, Commun. Nonlinear Sci. Numer. Simul. 60 (2018), 72-91.
[14] S. Abbas, M. Benchohra, N. Hamidi, J. Henderson, Caputo–Hadamard fractional differential equations in Banach spaces, Fract. Calc. Appl. Anal. 21 (2018), 1027-1045.
[15] M. S. Abdo, S. K. Panchal, A. M. Saeed, Fractional boundary value problem with ψ-Caputo fractional derivative, Proc. Indian Acad. Sci. Math. Sci. 129 (2019), 65.
[16] R.P. Agarwal, M. Benchohra, D. Seba, On the application of measure of noncompactness to the existence of solutions for fractional differential equations, Results Math. 55 (2009), 221-230.
[17] B. Ahmad, Y. Alruwaily, A. Alsaedi, S.K. Ntouyas, Riemann-Stieltjes Integral boundary value problems involving mixed Riemann-Liouville and Caputo fractional derivatives, J. Nonlinear Funct. Anal. 2021 (2021), Article ID 11.
[18] A. Aghajani, E. Pourhadi, J.J. Trujillo, Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces, Fract. Calc. Appl. Anal. 16 (2013), 962-977.
[19] M. Benchohra, J. Henderson, D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, Commun. Appl. Anal. 12 (2008), 419-428.
[20] J.P. Kharade, K.D. Kucche, On the impulsive implicit Ψ-Hilfer fractional differential equations with delay, Math. Methods Appl. Sci. 43 (2020), 1938-1952.
[21] K.D. Kucche, A.D. Mali, J.V.C. Sousa, On the nonlinear Ψ-Hilfer fractional differential equations, Comput. Appl. Math. 38 (2019), 73.

[22] K.D. Kucche, A.D. Mali, Initial time difference quasilinearization method for fractional differential equations involving generalized Hilfer fractional derivative, Comput. Appl. Math. 39 (2020), 31.

[23] J.V.C. Sousa, K.D. Kucche, E.C. de Oliveira, Stability of ψ-Hilfer impulsive fractional differential equations, Appl. Math. Lett. 88 (2019), 73-80.

[24] J.V.C. Sousa, F.G. Rodrigues, E. Capelas de Oliveira, Stability of the fractional Volterra integro-differential equation by means of ψ-Hilfer operator, Math. Methods Appl. Sci. 42 (2019), 3033-3043.

[25] A. Aghajani, M. Mursaleen, A. Shole Haghighi, Fixed point theorems for Meir–Keeler condensing operators via measure of noncompactness, Acta Math. Sci. Ser. B Engl. Ed. 35 (2015), 552-566.

[26] A. Das, B. Hazarika, M. Mursaleen, Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in ℓ_p $(1 < p < \infty)$, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. 113 (2019), 31-40.

[27] M. Mursaleen, S.M.H. Rizvi, Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators, Proc. Amer. Math. Soc. 144 (2016), 4279-4289.

[28] R. Saadati, E. Pourhadi, M. Mursaleen, Solvability of infinite systems of third-order differential equations in c_0 by Meir-Keeler condensing operators, J. Fixed Point Theory Appl. 21 (2019), 64.

[29] H.M. Srivastava, A. Das, B. Hazarika, S.A. Mohiuddine, Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces c_0 and ℓ_1 via the measure of noncompactness, Math. Methods Appl. Sci. 41 (2018), 3558-3569.

[30] S. Zhang, Existence of solutions for a boundary value problem of fractional order, Acta Math. Sci. 26 (2006), 220-228.

[31] S. Schwabik, Y. Guoju, Topics in Banach Spaces Integration, Series in Real Analysis 10, World Scientific, Singapore, 2005.

[32] J. Banaѕ̀, K. Goebel, Measures of Noncompactness in Banach Spaces, Marcel Dekker, New York, 1980.

[33] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326-329.

[34] J. P. Aubin, I. Ekeland, Applied Nonlinear Analysis, John Wiley & Sons, New York, 1984.

[35] D. Bothe, Multivalued perturbations of m-accretive differential inclusions, Israel J. Math. 108 (1998), 109-138.

[36] H. R. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Anal. 7 (1983), 1351-1371.

[37] E. Zeidler, Nonlinear Functional Analysis and its Applications, part II/B: Nonlinear Monotone Operators, Springer, New York, 1989.