TOPOLOGY OF THE SPACE OF CONORMAL DISTRIBUTIONS

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ABSTRACT. Given a closed manifold M and a closed regular submanifold L, consider the corresponding locally convex space $I = I(M, L)$ of conormal distributions, with its natural topology, and the strong dual $I' = I'(M, L) = I(M, L; \Omega)'$ of the space of conormal densities. It is shown that I is a barreled, ultrabornological, webbed, Montel, acyclic LF-space, and I' is a complete Montel space, which is a projective limit of bornological barreled spaces. In the case of codimension one, similar properties and additional descriptions are proved for the subspace $K \subset I$ of conormal distributions supported in L and for its strong dual K'. We construct a locally convex Hausdoff space J and a continuous linear map $I \to J$ such that the sequence $0 \to K \to I \to J \to 0$ as well as the transpose sequence $0 \to J' \to I' \to K' \to 0$ are short exact sequences in the category of continuous linear maps between locally convex spaces. Finally, it is shown that $I \cap I' = C^\infty(M)$ in the space of distributions. In another publication, these results are applied to prove a Lefschetz trace formula for a simple foliated flow $\phi = \{\phi^t\}$ on a compact foliated manifold $(M, \mathcal{F})$. It describes a Lefschetz distribution $L_{\mathrm{dis}}(\phi)$ defined by the induced action $\phi^* = \{\phi^{t\,*}\}$ on the reduced cohomologies $\bar{H}^\bullet I(\mathcal{F})$ and $\bar{H}^\bullet I'(\mathcal{F})$ of the complexes of leafwise currents that are conormal and dual-conormal at the leaves preserved by ϕ.

Contents

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1. Introduction

Given a smooth manifold M and a closed regular submanifold $L \subset M$, the corresponding space $I = I(M, L)$ of conormal distributions was considered in [17, 13], [15, Section 18.2], [36, Chapters 3–5], [25, Chapters 4 and 6], [27, Chapters 3 and 9]. But its study was mainly oriented to the important role played in the analysis of pseudodifferential operators. This space was also used as a tool to get appropriate generalizations of those operators to manifolds with boundary or corners, stratified spaces, etc. For instance, the existence of asymptotic expansions of their symbols was well analyzed. However we are not aware of any publication with a deep study of its natural topology; actually, this project was begun in [25, Chapters 4 and 6], but that publication remains incomplete. The main objective of this paper is to fill in that gap of study.

Assume M is compact for the sake of simplicity. Let $\mathrm{Diff}(M, L) \subset \mathrm{Diff}(M)$ be the subalgebra of differential operators generated by the vector fields on M tangent to L, and let $H^s(M)$ be the Sobolev space of order $s \in \mathbb{R}$. The space $I^{(s)} = I^{(s)}(M, L)$ of conormal distributions of Sobolev order s consists of the distributions $u \in C^{-\infty}(M)$ satisfying $\mathrm{Diff}(M, L)u \subset H^s(M)$, endowed with the projective topology given by the maps $P : I^{(s)} \to H^s(M)$ ($P \in \mathrm{Diff}(M, L)$).

By definition, $I = \bigcup_s I^{(s)}$ with the corresponding locally convex inductive topology, which is continuously contained in $C^{-\infty}(M)$. There is another expression $I = \bigcup_m I^m$, using spaces $I^m = I^m(M, L)$ of conormal distributions with symbol order m, which can be locally described in terms of symbol spaces with a partial Fourier transform.

We show that every $I^{(s)}$ is a totally reflexive Fréchet space (Proposition 4.1), and the LF-space I is a barreled, ultrabornological, webbed, acyclic Montel space, and therefore complete, boundedly retractive and reflexive (Corollaries 4.2 and 4.7). As a first step, these properties are established for symbol spaces.

All notions and properties considered here have straightforward extensions for distributional sections of vector bundles. In particular, for the density bundle $\Omega = \Omega M$, the strong dual $I'(M, L) = I(M, L; \Omega)'$, simply denoted by I', is also continuously contained in $C^{-\infty}(M)$. We prove that I' is a complete Montel space and $I' = \varprojlim I'^{(s)}$, where $I'^{(s)} = I'^{(s)}(M, L) = I^{(-s)}(M, L; \Omega)'$ is bornological and barreled (Corollaries 5.1 to 5.3).

Now assume L is of codimension one. For simplicity reasons, consider the case where L is transversely oriented. Then cut M along L to obtain a compact manifold with boundary $\boldsymbol{M}$ and a projection $\boldsymbol{\pi} : \boldsymbol{M} \to M$. In this way, we can take advantage of the machinery developed in [15, 25] to study conormal distributions at the boundary; in particular, some notions from small b-calculus are used. For instance, with the terminology and notation of [15, 25], let $\mathcal{A}(\boldsymbol{M})$ (respectively, $\dot{\mathcal{A}}(\boldsymbol{M})$) be the locally convex space of extendable (respectively, supported) conormal distributions at the boundary. Then, via the push-down map $\boldsymbol{\pi}_*$, the image of $\dot{\mathcal{A}}(\boldsymbol{M})$ is I, and $\mathcal{A}(\boldsymbol{M})$ becomes isomorphic to another locally convex space $J = J(M, L)$. Let $K = K(M, L) \subset I$ be the subspace of conormal distributions supported in L. Like in the definition of I', consider also the strong dual spaces $K'(M, L) = K(M, L; \Omega)'$ and $J'(M, L) = J(M, L; \Omega)'$, simply denoted by K' and J'. It is proved that K is a limit subspace (Corollary 7.19), the spaces K, J, K' and J' satisfy the properties stated for I and I' (Corollaries 7.21, 7.23 and 8.1 to 8.3), and there are short exact sequences, $0 \to K \to I \to J \to 0$ and its transpose $0 \leftarrow K' \leftarrow I' \leftarrow J' \leftarrow 0$, in the category of continuous linear maps between locally convex spaces (Propositions 7.29 and 8.8). These sequences are relevant because J, K, J' and K' have better descriptions than I and I' (Corollaries 6.40 and 6.49 and Proposition 7.26). Finally, it is shown that $I \cap I' = C^\infty(M)$ (Theorem 8.11), extending a result of [15, 25] for the boundary case. Most of these properties are first established in the boundary case (Section 6).

Besides the extensions for distributional sections of vector bundles, some results are extended to non-compact manifolds. We also analyze the action of differential operators on these spaces, as well as the pull-back and push-forward homomorphisms induced by maps on these spaces (Sections 4, 5, 7 and 8).

Via the Schwartz kernel theorem, the spaces of pseudodifferential and differential operators can be described as $\Psi(M) \equiv I(M^2, \Delta)$ and $\mathrm{Diff}(M) \equiv K(M^2, \Delta)$, where Δ is the diagonal of M^2. Thus $\Psi(M)$ and $\mathrm{Diff}(M)$ become examples of locally convex spaces satisfying the above properties.

The wave front set of any $u \in I(M, L)$ satisfies $\mathrm{WF}(u) \subset N^*L \setminus 0_L$ (considering $N^*L \subset T^*M$) [14, Chapter VIII], [15, Chapter XVIII]; this is the reason of the term "conormal distribution." The larger space of all distributions whose wave front set is contained in any prescribed closed cone of $T^*M \setminus 0_M$, like $N^*L \setminus 0_L$, also has a natural topology which was studied in [6].

Our results for codimension one can be clearly extended to arbitrary codimension. We only consider codimension one for simplicity reasons. It is also clear that there are further extensions to manifolds with corners, stratified spaces, etc.

The case of codimension one is also enough for our application in a trace formula for simple foliated flows [4]. These are simple flows $\phi = \{\phi^t\}$ that preserve the leaves of a foliation $\mathcal{F}$ on M. C. Deninger conjectured the existence of a "Lefschetz distribution" $L_{\mathrm{dis}}(\phi)$ on $\mathbb{R}$ for the induced pull-back action $\phi^* = \{\phi^{t\,*}\}$ on the leafwise reduced cohomology $\bar{H}^\bullet(\mathcal{F})$, and predicted a formula for $L_{\mathrm{dis}}(\phi)$ involving data from the closed orbits and fixed points [8]. Here, $\bar{H}^\bullet(\mathcal{F})$ is the maximal Hausdorff quotient of the leafwise cohomology $H^\bullet(\mathcal{F})$, defined by the de Rham derivative of the leaves acting on leafwise differential forms smooth on M, equipped with the C^∞ topology. But we can not use leafwise forms smooth on M if there are leaves preserved by ϕ; they do not work well. Instead, we consider the spaces $I(\mathcal{F})$ and $I'(\mathcal{F})$ of distributional leafwise currents that are conormal and dual-conormal at the preserved leaves, giving rise to reduced cohomologies, $\bar{H}^\bullet I(\mathcal{F})$ and $\bar{H}^\bullet I'(\mathcal{F})$, with actions ϕ^*. The spaces $K(\mathcal{F})$, $J(\mathcal{F})$, $K'(\mathcal{F})$ and $J'(\mathcal{F})$ are similarly defined, obtaining short exact sequences, $0 \to \bar{H}^\bullet K(\mathcal{F}) \to \bar{H}^\bullet I(\mathcal{F}) \to \bar{H}^\bullet J(\mathcal{F}) \to 0$ and $0 \leftarrow \bar{H}^\bullet K'(\mathcal{F}) \leftarrow \bar{H}^\bullet I'(\mathcal{F}) \leftarrow \bar{H}^\bullet J'(\mathcal{F}) \leftarrow 0$. In this way, the definition of $L_{\mathrm{dis}}(\phi)$ for both $\bar{H}^\bullet I(\mathcal{F})$ and $\bar{H}^\bullet I'(\mathcal{F})$ together can be reduced to the cases of $\bar{H}^\bullet K(\mathcal{F})$, $\bar{H}^\bullet J(\mathcal{F})$, $\bar{H}^\bullet K'(\mathcal{F})$ and $\bar{H}^\bullet J'(\mathcal{F})$. This can be done by using the descriptions of $K(\mathcal{F})$, $J(\mathcal{F})$, $K'(\mathcal{F})$ and $J'(\mathcal{F})$, and some additional ingredients. Using these ideas, we define in [4] the Lefschetz distribution $L_{\mathrm{dis}}(\phi)$, which has the desired expression plus a zeta invariant produced by the use of the b-trace of R. Melrose [24]. However the ingredients can be chosen so that the zeta invariant vanishes [3], and the predicted formula becomes correct. We hope that this method will provide useful tools in future developments of Deninger's project.

2. Preliminaries

2.1. **Topological vector spaces.** The field of coefficients is $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For the general theory of topological vector spaces (TVSs), we follow the references [9, 16, 21, 31, 28], assuming the following conventions. We always consider locally convex spaces (LCSs), which are not assumed to be Hausdorff (contrary to the definition of [31]); the abbreviation LCHS is used in the Hausdorff case. Local convexity is preserved by all operations we use. For any inductive/projective system (or spectrum) of continuous linear maps between LCSs, we have its (locally convex) inductive/projective limit; in particular, when the inductive/projective spectrum consists of a sequence of continuous inclusions, their union/intersection is endowed with the inductive/projective limit topology. This applies to the locally convex direct sum and the topological product of LCSs. LF-spaces are not assumed to be strict. For any LCS X, its (continuous) dual X' is always endowed with the strong topology; i.e., we write $X' = X'_\beta$ with the usual notation.

Some homological theory of LCSs will be used (see [39] and references therein) For instance, for an inductive spectrum of LCSs of the form $(X_k) = (X_0 \subset X_1 \subset \cdots)$, the condition of being *acyclic* can be described as follows [39, Theorem 6.1]: for all k, there is some $k' \geq k$ such that, for all $k'' \geq k'$, the topologies of $X_{k'}$ and $X_{k''}$ coincide on some 0-neighborhood of X_k. In this case, $X := \bigcup_k X_k$ is Hausdorff if and only if all X_k are Hausdorff [39, Proposition 6.3]. It is said that (X_k) is *regular* if any bounded $B \subset X$ is contained and bounded in some step X_k. If moreover the topologies of X and X_k coincide on B, then (X_k) is said to be *boundedly retractive*. The conditions of being *compactly retractive* or *sequentially retractive* are similarly defined, using compact sets or convergent sequences.

If the steps X_k are Fréchet spaces, the above properties of (X_k) depend only on the LF-space X [39, Chapter 6, p. 111]; thus it may be said that they are properties of X. In this case, X is acyclic if and only if it is boundedly/compactly/sequentially retractive [39, Proposition 6.4]. As a consequence, acyclic LF-spaces are complete and regular [39, Corollary 6.5].

A topological vector subspace $Y \subset X$ is called a *limit subspace* if $Y \equiv \bigcup_k Y_k$, where $Y_k = X \cap Y_k$. This condition is satisfied if and only if the spectrum consisting of the spaces X_k/Y_k is acyclic [39, Chapter 6, p. 110].

Assume the steps X_k are LCHSs. It is said that (X_k) is *compact* if the inclusion maps are compact operators. In this case, (X_k) is clearly acyclic, and so X is Hausdorff. Moreover X is a complete bornological DF Montel space [18, Theorem 6'].

The above concepts and properties also apply to an inductive/projective spectrum consisting of continuous inclusions $X_r \subset X_{r'}$ for $r < r'$ in $\mathbb{R}$ because $\bigcap_r X_r = \bigcap_k X_{r_k}$ and $\bigcup_r X_r = \bigcup_k X_{s_k}$ for sequences $r_k \downarrow -\infty$ and $s_k \uparrow \infty$.

In the category of continuous linear maps between LCSs, the exactness of a sequence $0 \to X \to Y \to Z \to 0$ means that it is exact as a sequence of linear maps and consists of topological homomorphisms [39, Sections 2.1 and 2.2].

2.2. **Smooth functions on open subsets of $\mathbb{R}^n$.** For an open $U \subset \mathbb{R}^n$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), we will use the Fréchet space $C^\infty(U)$ of smooth ($\mathbb{K}$-valued) functions on U with the topology of uniform approximation of all partial derivatives on compact subsets, which is described by the semi-norms

$$\|u\|_{K,C^k} = \sup_{x \in K,\ |\alpha| \leq k} |\partial^\alpha u(x)| \,, \tag{2.1}$$

for any compact $K \subset U$, $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, using standard multi-index notation. (Sometimes the notation $C^\infty_{\mathrm{loc}}(U)$ is used for this space, and $C^\infty(U)$ is used for the uniform space denoted by $C^\infty_{\mathrm{ub}}(U)$ in this paper.) For any $S \subset U$, the notation $C^\infty_S(U)$ is used for the subspace of smooth functions supported in S (with the subspace topology). (The common notation $C^\infty(S) \equiv C^\infty_S(U)$ would be confusing when extended to other function spaces.) Recall also the strict LF-space of compactly supported functions,

$$C^\infty_{\mathrm{c}}(U) = \bigcup_K C^\infty_K(U) , \tag{2.2}$$

for compact subsets $K \subset U$ (an exhausting increasing sequence of compact subsets is enough).

The above definitions have straightforward generalizations to the case of functions with values in $\mathbb{K}^l$ ($l \in \mathbb{N}$), obtaining

$$C^\infty_{\cdot/\mathrm{c}}(U, \mathbb{K}^l) \equiv C^\infty_{\cdot/\mathrm{c}}(U) \otimes \mathbb{K}^l . \tag{2.3}$$

(The notation $C^\infty_{\cdot/\mathrm{c}}$ or $C^\infty_{\mathrm{c}/\cdot}$ refers to both C^∞ and C^∞_{c}.)

2.3. **Vector bundles.** The notation M will be used for a smooth manifold of dimension n, and E for a ($\mathbb{K}$-) vector bundle over M. The fibers of E are denoted by E_x ($x \in M$), the zero in every E_x by 0_x, and the image of the zero section by 0_M. Let $\Omega^a E$ ($a \in \mathbb{R}$) denote the line bundle of a-densities of E, let $\Omega E = \Omega^1 E$, and let $o(E)$ be the flat line bundle of orientations of E. We may use the notation $E_L = E|_L$ for the restriction of E to a submanifold $L \subset M$. As particular cases, we have the tangent and cotangent $\mathbb{R}$-vector bundles, TM and T^*M, and the associated $\mathbb{K}$-vector bundles $o(M) = o(TM)$, $\Omega^a M = \Omega^a TM$ and $\Omega M = \Omega TM$.

2.4. **Smooth and distributional sections.** Our notation for spaces of distributional sections mainly follows [25], with some minor changes to fit our application in [4]. Some notation from [14, 15] is also used.

Generalizing $C^\infty(U, \mathbb{K}^l)$, we have the Fréchet space $C^\infty(M; E)$ of smooth sections of E, whose topology is described by semi-norms $\|\cdot\|_{K,C^k}$ defined as in (2.1) via charts (U, x) of M and diffeomorphisms of triviality $E_U \equiv U \times \mathbb{K}^l$, with $K \subset U$. This procedure is standard and will be used again with other section spaces.

Redundant notation will be removed as usual. For instance, we write $C^\infty(M)$ (respectively, $C^\infty(M, \mathbb{K}^l)$) in the case of the trivial vector bundle of rank 1 (respectively, l). We also write $C^\infty(L, E) = C^\infty(L, E_L)$ and $C^\infty(M; \Omega^a) = C^\infty(M; \Omega^a M)$. We may write $C^\infty(E) = C^\infty(M; E)$ if M is fixed, but this may also mean the space of smooth functions on E. In particular, $\mathfrak{X}(M) = C^\infty(M; TM)$ is the Lie algebra of vector fields. The subspace $C^\infty_S(M; E)$ is defined like in Section 2.2. Similar notation will be used with any LCHS and $C^\infty(M)$-module continuously included in $C^\infty(M; E)$, or in the space $C^{-\infty}(M; E)$ defined below.

The notation $C^\infty(M; E)$, or $C^\infty(E)$, is also used with any smooth fiber bundle E, obtaining a completely metrizable topological space with the weak C^∞ topology.

The strict LF-space $C^\infty_{\mathrm{c}}(M; E)$ of compactly supported smooth sections is defined like in (2.2), using compact subsets $K \subset M$. There is a continuous inclusion $C^\infty_{\mathrm{c}}(M; E) \subset C^\infty(M; E)$. If M is a fiber bundle, the LCHS $C^\infty_{\mathrm{cv}}(M; E)$ of smooth sections with compact support in the vertical direction is similarly defined using (2.1) and (2.2) with closed subsets $K \subset M$ whose intersection with the fibers is compact (now an exhaustive increasing sequence of such subsets K is not enough).

The space of distributional sections with arbitrary/compact support is

$$C^{-\infty}_{\cdot/\mathrm{c}}(M;E) = C^{\infty}_{\mathrm{c}/\cdot}(M;E^*\otimes\Omega)' \;. \tag{2.4}$$

(In [14], these dual spaces are endowed with the weak topology, contrary to our convention.) Integration of smooth densities on M and the canonical pairing of E and E^* define a continuous dense inclusion $C^{\infty}_{\cdot/\mathrm{c}}(M;E) \subset C^{-\infty}_{\cdot/\mathrm{c}}(M;E)$. If $U \subset M$ is open, the extension by zero defines a TVS-embedding $C^{\pm\infty}_{\mathrm{c}}(U;E) \subset C^{\pm\infty}_{\mathrm{c}}(M;E)$.

The above spaces of distributional sections can be also described in terms of the corresponding spaces of distributions as the algebraic tensor product as $C^\infty(M)$-modules

$$C^{-\infty}_{\cdot/\mathrm{c}}(M;E) \equiv C^{-\infty}_{\cdot/\mathrm{c}}(M) \otimes_{C^\infty(M)} C^\infty(M;E) \;. \tag{2.5}$$

To show this identity, E can be realized as a vector subbundle of a trivial vector bundle $F = M \times \mathbb{K}^{l'}$ [12, Theorem 4.3.1]. Then, like in (2.3),

$$\begin{aligned} C^{-\infty}_{\cdot/\mathrm{c}}(M;F) &\equiv C^{-\infty}_{\cdot/\mathrm{c}}(M) \otimes \mathbb{K}^{l'} \equiv C^{-\infty}_{\cdot/\mathrm{c}}(M) \otimes_{C^\infty(M)} C^\infty(M) \otimes \mathbb{K}^{l'} \\ &\equiv C^{-\infty}_{\cdot/\mathrm{c}}(M) \otimes_{C^\infty(M)} C^\infty(M;F) \;, \end{aligned}$$

and the spaces of (2.5) clearly correspond by these identities. Expressions like (2.5) hold for most of the LCSs of distributional sections we will consider, which are also $C^\infty(M)$-modules. Thus, from now on, we will mostly define and study those spaces for the trivial line bundle or density bundles, and then the notation for arbitrary vector bundles will be used without further comment, and the properties have straightforward extensions.

Consider also the Fréchet space $C^k(M)$ ($k \in \mathbb{N}_0$) of C^k functions, with the semi-norms $\|\cdot\|_{K,C^k}$ given like in (2.1), the LF-space $C^k_{\mathrm{c}}(M)$ of C^k functions with compact support, defined like in (2.2), and the space $C'^{\,-k}_{\cdot/\mathrm{c}}(M)$ of distributions of order k with arbitrary/compact support, defined like in (2.4). (A prime is added to this notation to distinguish $C'^{\,0}_{\cdot/\mathrm{c}}(M)$ from $C^0_{\cdot/\mathrm{c}}(M)$.) There are continuous dense inclusions

$$C^{k'}_{\cdot/\mathrm{c}}(M) \subset C^{k}_{\cdot/\mathrm{c}}(M) \;, \quad C'^{\,-k'}_{\mathrm{c}/\cdot}(M) \supset C'^{\,-k}_{\mathrm{c}/\cdot}(M) \quad (k<k') \;, \tag{2.6}$$

with [28, Exercise 12.108]

$$\bigcap_k C^{k}_{\cdot/\mathrm{c}}(M) = C^{\infty}_{\cdot/\mathrm{c}}(M) \;, \quad \bigcup_k C'^{\,-k}_{\mathrm{c}}(M) = C^{-\infty}_{\mathrm{c}}(M) \;. \tag{2.7}$$

The space $\bigcup_k C'^{\,-k}(M)$ consists of the distributions with some order; it is $C^{-\infty}(M)$ just when M is compact.

Let us recall some properties of the spaces we have seen. In addition of the fact that $C^\infty(M)$ and $C^k(M)$ are Fréchet spaces [16, Example 2.9.3], $C^\infty_{\mathrm{c}}(M)$ and $C^k_{\mathrm{c}}(M)$ are complete and Hausdorff [16, Examples 2.12.6 and 2.12.8]. $C^{\infty}_{\cdot/\mathrm{c}}(M)$ and $C^{k}_{\cdot/\mathrm{c}}(M)$ are ultrabornological because this property is satisfied by Fréchet spaces and preserved by inductive limits [28, Example 13.2.8 (d) and Theorem 13.2.11], and therefore they are barreled [29, Observation 6.1.2 (b)]. $C^{\pm\infty}_{\cdot/\mathrm{c}}(M)$ is a Montel space (in particular, barreled) [16, Examples 3.9.3, 3.9.4 and 3.9.6 and Proposition 3.9.9], [9, Section 8.4.7, Theorem 8.4.11 and Application 8.4.12], [31, the paragraph before IV.5.9], and therefore reflexive [9, Section 8.4.7], [21, 6.27.2 (1)], [31, IV.5.8]. $C^{\infty}_{\cdot/\mathrm{c}}(M)$ is a Schwartz space [16, Examples 3.15.2 and 3.15.3], and therefore $C^{-\infty}_{\cdot/\mathrm{c}}(M)$ is ultrabornological [16, Exercise 3.15.9 (c)]. $C^\infty(M)$ is distinguished [16, Examples 3.16.1]. $C^{\pm\infty}_{\cdot/\mathrm{c}}(M)$ is webbed because this property is satisfied by LF-spaces and strong duals of strict inductive limits of sequences of metrizable LCSs [7, Proposition IV.4.6], [22, 7.35.1 (4) and 7.35.4 (8)], [28, Theorem 14.6.5].

2.5. **Linear operators on section spaces.** Let E and F be vector bundles over M, and let $A : C^\infty_{\mathrm{c}}(M;E) \to C^\infty(M;F)$ be a continuous linear map. Recall that the *transpose* of A is the continuous linear map

$$A^t : C^{-\infty}_{\mathrm{c}}(M;F^*\otimes\Omega) \to C^{-\infty}(M;E^*\otimes\Omega)\;,$$
$$\langle A^t v, u\rangle = \langle v, Au\rangle\;,\quad u\in C^\infty_{\mathrm{c}}(M;E)\;,\quad v\in C^{-\infty}_{\mathrm{c}}(M;F^*\otimes\Omega)\;.$$

For instance, the transpose of $C^\infty_{\mathrm{c}}(M;E^*\otimes\Omega)\subset C^\infty(M;E^*\otimes\Omega)$ is a continuous dense injection $C^{-\infty}_{\mathrm{c}}(M;E)\subset C^{-\infty}(M;E)$. If A^t restricts to a continuous linear map $C^\infty_{\mathrm{c}}(M;F^*\otimes\Omega)\to C^\infty(M;E^*\otimes\Omega)$, then $A^{tt} : C^{-\infty}_{\mathrm{c}}(M;E)\to C^{-\infty}(M;F)$ is a continuous extension of A, also denoted by A.

There are versions of the construction of A^t and A^{tt} when both the domain and codomain of A have compact support, or no support restriction. For example, for any open $U\subset M$, the transpose of the extension by zero $C^\infty_{\mathrm{c}}(U;E^*\otimes\Omega)\subset C^\infty_{\mathrm{c}}(M;E^*\otimes\Omega)$ is the restriction map $C^{-\infty}(M;E)\to C^{-\infty}(U,E)$, $u\mapsto u|_U$, and the transpose of the restriction map $C^\infty(M;E^*\otimes\Omega)\to C^\infty(U,E^*\otimes\Omega)$ is the extension by zero $C^{-\infty}_{\mathrm{c}}(U;E)\subset C^{-\infty}_{\mathrm{c}}(M;E)$. In the whole paper, inclusion maps may be denoted by ι and restriction maps by R, without further comment.

Other related concepts and results, like singular support, Schwartz kernel and the Schwartz kernel theorem, can be seen e.g. in [24].

2.6. **Pull-back and push-forward of distributional sections.** Recall that any smooth map $\phi : M'\to M$ induces the continuous linear pull-back map

$$\phi^* : C^\infty(M;E)\to C^\infty(M';\phi^*E)\;. \tag{2.8}$$

Suppose that moreover ϕ is a submersion. Then it also induces the continuous linear push-forward map

$$\phi_* : C^\infty_{\mathrm{c}}(M';\phi^*E\otimes\Omega_{\mathrm{fiber}})\to C^\infty_{\mathrm{c}}(M;E)\;, \tag{2.9}$$

where $\Omega_{\mathrm{fiber}}=\Omega_{\mathrm{fiber}}M'=\Omega\mathcal{V}$ for the vertical subbundle $\mathcal{V}=\ker\phi_*\subset TM'$. Since $\phi^*\Omega M\equiv\Omega(TM/\mathcal{V})\equiv\Omega^{-1}_{\mathrm{fiber}}\otimes\Omega M'$, the transposes of the versions of (2.8) and (2.9) with $E^*\otimes\Omega M$ are continuous extensions of (2.9) and (2.8) [14, Theorem 6.1.2],

$$\phi_* : C^{-\infty}_{\mathrm{c}}(M';\phi^*E\otimes\Omega_{\mathrm{fiber}})\to C^{-\infty}_{\mathrm{c}}(M;E)\;, \tag{2.10}$$

$$\phi^* : C^{-\infty}(M;E)\to C^{-\infty}(M';\phi^*E)\;, \tag{2.11}$$

also called push-forward and pull-back maps. The term integration along the fibers is also used for ϕ_*.

If $\phi : M'\to M$ is a proper local diffeomorphism, then we can omit Ω_{fiber} and the compact support condition in (2.9) and (2.10), and therefore the compositions $\phi_*\phi^*$ and $\phi^*\phi_*$ are defined on smooth/distributional sections.

The space $C^\infty(M';\phi^*E)$ becomes a $C^\infty(M)$-module via the algebra homomorphism $\phi^* : C^\infty(M)\to C^\infty(M')$, and we have

$$C^{\pm\infty}_{\cdot/\mathrm{c}}(M';\phi^*E)=C^{\pm\infty}_{\cdot/\mathrm{c}}(M')\otimes_{C^\infty(M)}C^\infty(M;E)\;. \tag{2.12}$$

Using (2.5) and (2.12), we can describe (2.8)–(2.11) as the $C^\infty(M)$-tensor products of their trivial-line-bundle versions with the identity map on $C^\infty(M;E)$. Thus, from now on, only pull-back and push-forward of distributions will be considered.

2.7. **Differential operators.** Let $\operatorname{Diff}(M)$ be the filtered algebra and $C^\infty(M)$-module of differential operators, filtered by the order. Every $\operatorname{Diff}^m(M)$ ($m \in \mathbb{N}_0$) is spanned as $C^\infty(M)$-module by all compositions of up to m elements of $\mathfrak{X}(M)$, considered as the Lie algebra of derivations of $C^\infty_{\cdot/\mathrm{c}}(M)$. In particular, $\operatorname{Diff}^0(M) \equiv C^\infty(M)$.

For vector bundles E and F over M, the above concepts can be extended by taking the $C^\infty(M)$-tensor product with $C^\infty(M;F\otimes E^*)$, obtaining $\operatorname{Diff}^m(M;E,F)$ ($\operatorname{Diff}^m(M;E)$ being obtained if $E=F$); here, redundant notation is simplified like in the case of $C^{\pm\infty}(M;E)$ (Section 2.4). If E is a line bundle, then

$$\begin{aligned}\operatorname{Diff}^m(M;E) &\equiv \operatorname{Diff}^m(M)\otimes_{C^\infty(M)} C^\infty(M;E\otimes E^*)\\ &\equiv \operatorname{Diff}^m(M)\otimes_{C^\infty(M)} C^\infty(M) \equiv \operatorname{Diff}^m(M)\;.\end{aligned} \tag{2.13}$$

Any $A\in\operatorname{Diff}^m(M;E)$ defines a continuous linear endomorphism A of $C^\infty_{\cdot/\mathrm{c}}(M;E)$. We get $A^t \in \operatorname{Diff}^m(M;E^*\otimes\Omega)$ using integration by parts. So A has continuous extensions to a continuous endomorphism A of $C^{-\infty}_{\cdot/\mathrm{c}}(M;E)$ (Section 2.5). A similar map is defined when $A\in\operatorname{Diff}^m(M;E,F)$.

Other related concepts like symbols and ellipticity can be seen e.g. in [24].

2.8. L^2 **sections.** Recall that the Hilbert space $L^2(M;\Omega^{1/2})$ of square-integrable half-densities is the completion of $C^\infty_{\mathrm{c}}(M;\Omega^{1/2})$ with the scalar product $\langle u,v\rangle = \int_M u\bar v$. The induced norm is denoted by $\|\cdot\|$.

If M is compact, the space $L^2(M;E)$ of square-integrable sections of E can be described as the $C^\infty(M)$-tensor product of $L^2(M;\Omega^{1/2})$ and $C^\infty(M;\Omega^{-1/2}\otimes E)$. It becomes a Hilbert space with the scalar product $\langle u,v\rangle = \int_M (u,v)\,\omega$ determined by the choice of a Euclidean/Hermitian structure $(\cdot,\cdot)$ on E and a non-vanishing $\omega\in C^\infty(M;\Omega)$. The equivalence class of its norm $\|\cdot\|$ is independent of those choices; in this sense, $L^2(M;E)$ is called a *Hilbertian space* if no norm is distinguished.

When M is not assumed to be compact, any choice of $(\cdot,\cdot)$ and ω can be used to define $L^2(M;E)$ and $\langle\cdot,\cdot\rangle$. Now $L^2(M;E)$ and the equivalence class of $\|\cdot\|$ depends on the choices involved. The independence still holds for sections supported in any compact $K\subset M$, obtaining the Hilbertian space $L^2_K(M;E)$. Then the strict LF-space $L^2_{\mathrm{c}}(M;E)$ is defined like in (2.2). On the other hand, let

$$L^2_{\mathrm{loc}}(M;E) = \{\, u\in C^{-\infty}(M;E) \mid C^\infty_{\mathrm{c}}(M)\,u \subset L^2_{\mathrm{c}}(M;E)\,\}\;, \tag{2.14}$$

which is a Fréchet space with the semi-norms $u\mapsto\|f_k u\|$, for a countable partition of unity $\{f_k\}\subset C^\infty_{\mathrm{c}}(M)$. If M is compact, then $L^2_{\mathrm{loc/c}}(M;E)\equiv L^2(M;E)$ as TVSs. The spaces $L^2_{\mathrm{loc/c}}(M;E)$ satisfy the obvious version of (2.4).

Any $A\in\operatorname{Diff}^m(M;E)$ can be considered as a densely defined operator in $L^2(M;E)$. Integration by parts shows that the adjoint A^* is defined by an element $A^*\in\operatorname{Diff}^m(M;E)$ (the *formal adjoint* of A).

2.9. L^∞ **sections.** A Euclidean/Hermitian structure can be also used to define the Banach space $L^\infty(M;E)$ of its essentially bounded sections, with the norm $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x\in M}|u(x)|$. There is a continuous injection $L^\infty(M;E)\subset L^2_{\mathrm{loc}}(M;E)$. If M is compact, then the equivalence class of $\|\cdot\|_{L^\infty}$ is independent of $(\cdot,\cdot)$.

2.10. **Sobolev spaces.**

2.10.1. *Local and compactly supported versions.* Recall that the Fourier transform, $f \mapsto \hat{f}$, defines a TVS-automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, which extends to a TVS-automorphism of the space $\mathcal{S}(\mathbb{R}^n)'$ of tempered distributions [14, Section 7.1]. In turn, for every $s \in \mathbb{R}$, this automorphism of $\mathcal{S}(\mathbb{R}^n)'$ restricts to unitary isomorphism

$$H^s(\mathbb{R}^n) \xrightarrow{\cong} L^2(\mathbb{R}^n, (1+|\xi|^2)^s\, d\xi)\;, \quad f \mapsto \hat{f}\;, \tag{2.15}$$

for some Hilbert space $H^s(\mathbb{R}^n)$, called the Sobolev space of order s of $\mathbb{R}^n$. There is a canonical continuous inclusion $H^s(\mathbb{R}^n) \subset C^{-\infty}(\mathbb{R}^n)$.

For any compact $K \subset \mathbb{R}^n$, we have the Hilbert subspace $H^s_K(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ of elements supported in K. Then the LCHSs $H^s_{\mathrm{c/loc}}(U)$ are defined like in (2.4) and (2.14), using the spaces $H^s_K(\mathbb{R}^n)$ for compact subsets $K \subset U$. They are continuously included in $C^{-\infty}_{\mathrm{c/\cdot}}(U)$.

For a manifold M, the definition of the LCHSs $H^s_{\mathrm{c/loc}}(M)$ can be extended in a standard way, by using a locally finite atlas and a partition of unity consisting of compactly supported smooth functions. These are the compactly supported and local versions of the Sobolev space of order s of M. They are continuously included in $C^{-\infty}_{\mathrm{c/\cdot}}(M)$.

2.10.2. *Case of compact manifolds.* Suppose for a while that M is compact. Then $H^s(M) := H^s_{\mathrm{loc}}(M) = H^s_{\mathrm{c}}(M)$ is a Hilbertian space called the *Sobolev space* of order s of M. We have

$$H^{-s}(M) = H^s(M;\Omega)'\;, \tag{2.16}$$

given by (2.4). Moreover there are continuous dense inclusions,

$$H^s(M) \subset H^{s'}(M)\;, \tag{2.17}$$

for $s' < s$, and

$$H^s(M) \subset C^k(M) \subset H^k(M)\;, \tag{2.18}$$

$$H^{-s}(M) \supset C'^{\,-k}(M) \supset H^{-k}(M)\;, \tag{2.19}$$

for $s > k + n/2$. The first inclusion of (2.18) is the Sobolev embedding theorem, and (2.19) is the transpose of the version of (2.18) with ΩM. Moreover the inclusions (2.17) are compact (Rellich theorem). So the spaces $H^s(M)$ form a compact spectrum with

$$C^\infty(M) = \bigcap_s H^s(M) \qquad C^{-\infty}(M) = \bigcup_s H^s(M)\;. \tag{2.20}$$

Any $A \in \mathrm{Diff}^m(M;E)$ defines a bounded operator $A : H^{s+m}(M;E) \to H^s(M;E)$. It can be considered as a densely defined operator in $H^s(M;E)$, which is closable because, after fixing a scalar product in $H^s(M;E)$, the adjoint of A in $H^s(M;E)$ is densely defined since it is induced by $\bar{A}^t \in \mathrm{Diff}^m(M;\bar{E}^* \otimes \Omega)$ via the identity $H^s(M;E) \equiv H^s(M;\bar{E})' = H^{-s}(M;\bar{E}^* \otimes \Omega)$, where the bar stands for the complex conjugate. In the case $s = 0$, the adjoint of A is induced by the formal adjoint $A^* \in \mathrm{Diff}^m(M;E)$.

By the elliptic estimate, a scalar product on $H^s(M)$ can be defined by $\langle u, v\rangle_s = \langle (1+P)^s u, v\rangle$, for any choice of a nonnegative symmetric elliptic $P \in \mathrm{Diff}^2(M)$, where $\langle\cdot,\cdot\rangle$ is defined like in Section 2.8 and $(1+P)^s$ is given by the spectral theorem for all $s\in\mathbb{R}$. The corresponding norm $\|\cdot\|_s$ is independent of the choice of P. For a vector bundle E, a precise scalar product on $H^s(M;E)$ can be defined as above, using any choice of a Euclidean/Hermitian structure $(\cdot,\cdot)$ on E and a non-vanishing $\omega\in C^\infty(M;\Omega)$ (Section 2.8), besides a nonnegative symmetric elliptic $P\in\operatorname{Diff}^2(M;E)$. If $E=\Omega^{1/2}M$, then $\langle\cdot,\cdot\rangle_s$ can be defined independently of $(\cdot,\cdot)$ and ω (Section 2.8).

If $s\in\mathbb{N}_0$, we can also describe

$$H^s(M)=\{\,u\in C^{-\infty}(M)\mid \operatorname{Diff}^s(M)\,u\subset L^2(M)\,\}\,, \tag{2.21}$$

$$H^{-s}(M)=\operatorname{Diff}^s(M)\,L^2(M)\,, \tag{2.22}$$

with the respective projective and injective topologies given by the maps $A: H^s(M)\to L^2(M)$ and $A:L^2(M)\to H^{-s}(M)$ ($A\in\operatorname{Diff}^s(M)$).

2.10.3. *Extension to non-compact manifolds.* If M is not assumed to be compact, then $H^s(M;E)$ can be defined as the completion of $C^\infty_{\mathrm{c}}(M;E)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_s$ defined by the above choices of $(\cdot,\cdot)$, ω and P; in this case, $H^s(M;E)$ and the equivalence class of $\|\cdot\|_s$ depends on the choices involved. For instance, in (2.15), $H^s(\mathbb{R}^n)$ can be also described with the Laplacian of $\mathbb{R}^n$ and the standard density and Euclidean/Hermitian structure. The version of (2.16) with E can be used to define $H^{-s}(M;E)$. With this generality, the versions of (2.21), (2.22), the right-hand side inclusions of (2.18) and (2.19), and the inclusions "$\subset$" of (2.20) are wrong, but the versions of (2.17), the left-hand side continuous inclusions of (2.18) and (2.19), and the continuous inclusion "$\supset$" of (2.20) are true. Thus the intersection and union of (2.20) define new LCHSs $H^{\pm\infty}(M)$, which are continuously included in $C^{\pm\infty}(M)$. Any $A\in\operatorname{Diff}^m(M;E)$ defines continuous linear maps $A:H^s_{\mathrm{c/loc}}(M;E)\to H^{s-m}_{\mathrm{c/loc}}(M;F)$.

2.11. **Weighted spaces.**

2.11.1. *Case of compact manifolds.* Assume first that M is compact. Take any $h\in C^\infty(M)$ which is positive almost everywhere; for instance, $\{h=0\}$ could be any countable union of submanifolds of positive codimension. Then the *weighted Sobolev space* $hH^s(M;E)$ is a Hilbertian space; a scalar product $\langle\cdot,\cdot\rangle_{hH^s}$ is given by $\langle u,v\rangle_{hH^s}=\langle h^{-1}u,h^{-1}v\rangle_s$, depending on the choice of a scalar product $\langle\cdot,\cdot\rangle_s$ on $H^s(M;E)$ like in Section 2.10. The corresponding norm is denoted by $\|\cdot\|_{hH^s}$. In particular, we get the *weighted L^2 space* $hL^2(M;E)$. We have $h>0$ just when $hH^m(M;E)=H^m(M;E)$; in this case, $\langle\cdot,\cdot\rangle_{hH^s}$ can be described like $\langle\cdot,\cdot\rangle_s$ using $h^{-2}\omega$ instead of ω. Thus the notation $hH^m(M;E)$ for $h>0$ is used when changing the density; e.g., if it is different from a distinguished choice, say a Riemanian volume.

2.11.2. *Extension to non-compact manifolds.* If M is not compact, $hH^s(M;E)$ and $\langle u,v\rangle_{hH^s}$ depend on h and the chosen definitions of $H^s(M;E)$ and $\langle u,v\rangle_s$ (Section 2.10). We also get the weighted spaces $hH^s_{\mathrm{c/loc}}(M;E)$, and the weighted Banach space $hL^\infty(M;E)$ with the norm $\|u\|_{hL^\infty}=\|h^{-1}u\|_{L^\infty}$. There is a continuous injection $hL^\infty(M;E)\subset hL^2_{\mathrm{loc}}(M;E)$.

2.12. **Bounded geometry.** Concerning this topic, we follow [10, 30, 35, 32, 33]; see also [2] for the way we present it and examples.

2.12.1. *Manifolds and vector bundles of bounded geometry.* The concepts recalled here become relevant when M is not compact. Equip M with a Riemannian metric g, and let ∇ denote its Levi-Civita connection, R its curvature and $\operatorname{inj}_M \geq 0$ its injectivity radius (the infimum of the injectivity radius at all points). If M is connected, we have an induced distance function d. If M is not connected, we can also define d taking $d(p,q)=\infty$ if p and q belong to different connected components. Observe that M is complete if $\operatorname{inj}_M > 0$. For $r>0$ and $p \in M$, let $B(p,r)$ and $\overline{B}(p,r)$ denote the open and closed r-balls centered at p.

Recall that M is said to be of *bounded geometry* if $\operatorname{inj}_M > 0$ and $\sup|\nabla^m R| < \infty$ for every $m \in \mathbb{N}_0$. This concept has the following chart description.

Theorem 2.1 (Eichhorn [10]; see also [30, 32, 33])**.** *M is of bounded geometry if and only if, for some open ball $B \subset \mathbb{R}^n$ centered at 0, there are normal coordinates at every $p \in M$ defining a diffeomorphism $y_p : V_p \to B$ such that the corresponding Christoffel symbols Γ^i_{jk}, as a family of functions on B parametrized by i, j, k and p, lie in a bounded set of the Fréchet space $C^\infty(B)$. This equivalence holds as well replacing the Cristoffel symbols with the metric coefficients g_{ij}.*

From now on in this subsection, assume M is of bounded geometry and consider the charts $y_p : V_p \to B$ given by Theorem 2.1. The radius of B is denoted by r_0.

Proposition 2.2 (Schick [32, Theorem A.22], [33, Proposition 3.3])**.** *For every $\alpha \in \mathbb{N}_0^n$, the function $|\partial^\alpha(y_q y_p^{-1})|$ is bounded on $y_p(V_p \cap V_q)$, uniformly on $p,q \in M$.*

Proposition 2.3 (Shubin [35, Appendix A1.1, Lemma 1.2])**.** *For any $0 < 2r \leq r_0$, there is a subset $\{p_k\} \subset M$ and some $N \in \mathbb{N}$ such that the balls $B(p_k, r)$ cover M, and every intersection of $N+1$ sets $B(p_k, 2r)$ is empty.*

A vector bundle E of rank l over M is said to be of *bounded geometry* when it is equipped with a family of local trivializations over the charts (V_p, y_p), for small enough r_0, with corresponding defining cocycle $a_{pq} : V_p \cap V_q \to \mathrm{GL}(l,\mathbb{K}) \subset \mathbb{K}^{l\times l}$, such that, for all $\alpha \in \mathbb{N}_0^n$, the function $|\partial^\alpha(a_{pq} y_p^{-1})|$ is bounded on $y_p(V_p \cap V_q)$, uniformly on $p,q \in M$. When referring to local trivializations of a vector bundle of bounded geometry, we always mean that they satisfy this condition. If the corresponding defining cocycle is valued in the orthogonal/unitary group, then E is said to be of *bounded geometry* as a Euclidean/Hermitian vector bundle.

2.12.2. *Uniform spaces.* For every $m \in \mathbb{N}_0$, a function $u \in C^m(M)$ is said to be C^m-*uniformy bounded* if there is some $C_m \geq 0$ with $|\nabla^{m'} u| \leq C_m$ on M for all $m' \leq m$. These functions form the *uniform C^m space* $C^m_{\mathrm{ub}}(M)$, which is a Banach space with the norm $\|\cdot\|_{C^m_{\mathrm{ub}}}$ defined by the best constant C_m. Equivalently, we may take the norm $\|\cdot\|'_{C^m_{\mathrm{ub}}}$ defined by the best constant $C'_m \geq 0$ such that $|\partial^\alpha(u y_p^{-1})| \leq C'_m$ on B for all $p \in M$ and $|\alpha| \leq m$; in fact, it is enough to consider any subset of points p so that $\{V_p\}$ covers M [32, Theorem A.22], [33, Proposition 3.3]. The *uniform C^∞ space* is $C^\infty_{\mathrm{ub}}(M) = \bigcap_m C^m_{\mathrm{ub}}(M)$. This is a Fréchet space with the semi-norms $\|\cdot\|_{C^m_{\mathrm{ub}}}$ or $\|\cdot\|'_{C^m_{\mathrm{ub}}}$. It consists of the functions $u \in C^\infty(M)$ such that all functions $u y_p^{-1}$ lie in a bounded set of $C^\infty(B)$, which are said to be C^∞-*uniformy bounded.*

The same definitions apply to functions with values in $\mathbb{C}^l$. Moreover the definition of uniform spaces with covariant derivative can be also considered for non-complete Riemannian manifolds.

Proposition 2.4 (Shubin [35, Appendix A1.1, Lemma 1.3]; see also [33, Proposition 3.2])**.** *Given r, $\{p_k\}$ and N like in Proposition 2.3, there is a partition of unity $\{f_k\}$ subordinated to the open covering $\{B(p_k,r)\}$, which is bounded in the Fréchet space $C^\infty_{\rm ub}(M)$.*

For a Euclidean/Hermitian vector bundle E of bounded geometry over M, the *uniform C^m space* $C^m_{\rm ub}(M;E)$, of *C^m-uniformly bounded* sections, can be defined by introducing $\|\cdot\|'_{C^m_{\rm ub}}$ like the case of functions, using local trivializations of E to consider every uy_p^{-1} in $C^m(B,\mathbb{C}^l)$ for all $u\in C^m(M;E)$. Then, as above, we get the *uniform C^∞ space* $C^\infty_{\rm ub}(M;E)$ of *C^∞-uniformly bounded* sections, which are the sections $u\in C^\infty(M;E)$ such that all functions uy_p^{-1} define a bounded set of $C^\infty_{\rm ub}(B;\mathbb{C}^l)$. In particular, $\mathfrak{X}_{\rm ub}(M):=C^\infty_{\rm ub}(M;TM)$ is a $C^\infty_{\rm ub}(M)$-submodule and Lie subalgebra of $\mathfrak{X}(M)$.

2.12.3. *Differential operators of bounded geometry.* Like in Section 2.7, by using $\mathfrak{X}_{\rm ub}(M)$ and $C^\infty_{\rm ub}(M)$ instead of $\mathfrak{X}(M)$ and $C^\infty(M)$, we get the filtered subalgebra and $C^\infty_{\rm ub}(M)$-submodule $\operatorname{Diff}_{\rm ub}(M)\subset\operatorname{Diff}(M)$ of differential operators of *bounded geometry*. Observe that

$$C^m_{\rm ub}(M)=\{\,u\in C^m(M)\mid \operatorname{Diff}_{\rm ub}(M)\,u\subset L^\infty(M)\ \forall m'\le m\,\}\ . \tag{2.23}$$

For vector bundles of bounded geometry E and F over M, by taking the $C^\infty_{\rm ub}(M)$-tensor product of $\operatorname{Diff}_{\rm ub}(M)$ and $C^\infty_{\rm ub}(M;F\otimes E^*)$, we obtain the filtered $C^\infty_{\rm ub}(M)$-submodule $\operatorname{Diff}_{\rm ub}(M;E,F)\subset\operatorname{Diff}(M;E,F)$ (or $\operatorname{Diff}_{\rm ub}(M;E)$ if $E=F$). Bounded geometry of differential operators is preserved by compositions and by taking transposes, and by taking formal adjoints in the case of Hermitian vector bundles of bounded geometry; in particular, $\operatorname{Diff}_{\rm ub}(M;E)$ is a filtered subalgebra of $\operatorname{Diff}(M;E)$. Like in (2.13), if E is a line bundle of bounded geometry, then

$$\operatorname{Diff}^m_{\rm ub}(M;E)\equiv\operatorname{Diff}^m_{\rm ub}(M)\ . \tag{2.24}$$

Every $A\in\operatorname{Diff}^m_{\rm ub}(M;E)$ defines continuous linear maps $A:C^{m+k}_{\rm ub}(M;E)\to C^k_{\rm ub}(M;E)$ ($k\in\mathbb{N}_0$), which induce a continuous endomorphism A of $C^\infty_{\rm ub}(M;E)$. It is said that A is *uniformly elliptic* if there is some $C\ge1$ such that, for all $p\in M$ and $\xi\in T^*_pM$, its leading symbol $\sigma_m(A)$ satisfies

$$C^{-1}|\xi|^m\le|\sigma_m(A)(p,\xi)|\le C|\xi|^m\ .$$

This condition is independent of the choice of the Hermitian metric of bounded geometry on E. Any $A\in\operatorname{Diff}^m_{\rm ub}(M;E,F)$ satisfies the second inequality. The case where $A\in\operatorname{Diff}^m_{\rm ub}(M;E,F)$ is similar.

2.12.4. *Sobolev spaces of manifolds of bounded geometry.* For any Hermitian vector bundle E of bounded geometry over M, any nonnegative symmetric uniformly elliptic $P\in\operatorname{Diff}^2_{\rm ub}(M;E)$ can be used to define the Sobolev space $H^s(M;E)$ ($s\in\mathbb{R}$) with a scalar product $\langle\cdot,\cdot\rangle_s$ (Section 2.10). Any choice of P defines the same Hilbertian space $H^s(M;E)$, which is a $C^\infty_{\rm ub}(M)$-module. In particular, $L^2(M;E)$ is the $C^\infty_{\rm ub}(M)$-tensor product of $L^2(M;\Omega^{1/2})$ and $C^\infty_{\rm ub}(M;E\otimes\Omega^{1/2})$, and $H^s(M;E)$ is the $C^\infty_{\rm ub}(M)$-tensor product of $H^s(M)$ and $C^\infty_{\rm ub}(M;E)$. For instance, we may take $P=\nabla^*\nabla$ for any unitary connection ∇ of bounded geometry on E. For $s\in\mathbb{N}_0$, the Sobolev space $H^s(M)$ can be also described with the scalar product

$$\langle u,v\rangle'_s=\sum_k\sum_{|\alpha|\le s}\int_B f_k^2(x)\cdot\partial^\alpha(uy_{p_k}^{-1})(x)\cdot\overline{\partial^\alpha(vy_{p_k}^{-1})(x)}\,dx\ ,$$

using the partition of unity $\{f_k\}$ given by Proposition 2.4 [32, Theorem A.22], [33, Propositions 3.2 and 3.3], [35, Appendices A1.2 and A1.3]. A similar scalar product $\langle\cdot,\cdot\rangle'_s$ can be defined for $H^s(M;E)$ with the help of local trivializations defining the bounded geometry of E. Every $A \in \mathrm{Diff}^m_{\mathrm{ub}}(M;E)$ defines bounded operators $A : H^{m+s}(M;E) \to H^s(M;E)$ $(s \in \mathbb{R})$, which induce a continuous endomorphism A of $H^{\pm\infty}(M;E)$. For any almost everywhere positive $h \in C^\infty(M)$, we have $hH^m(M;E) = H^m(M;E)$ if and only if $h > 0$ and $h^{\pm 1} \in C^\infty_{\mathrm{ub}}(M)$.

Proposition 2.5 (Roe [30, Proposition 2.8]). *If $m' > m+n/2$, then $H^{m'}(M;E) \subset C^m_{\mathrm{ub}}(M;E)$, continuously. Thus $H^\infty(M;E) \subset C^\infty_{\mathrm{ub}}(M;E)$, continuously.*

3. Symbols

The canonical coordinates of $\mathbb{R}^n \times \mathbb{R}^l$ $(n,l \in \mathbb{N}_0)$ are denoted by $(x,\xi) = (x^1,\dots,x^n,\xi^1,\dots,\xi^l)$, and let $dx = dx^1 \wedge \cdots \wedge dx^n$ and $d\xi = d\xi^1 \wedge \cdots \wedge d\xi^l$. Recall that a *symbol* of *order* at most $m \in \mathbb{R}$ on $U \times \mathbb{R}^l$, or simply on U, is a function $a \in C^\infty(U \times \mathbb{R}^l)$ such that, for any compact $K \subset U$, and multi-indices $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^l$,

$$\|a\|_{K,\alpha,\beta,m} := \sup_{x\in K,\ \xi\in\mathbb{R}^l} \frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{(1+|\xi|)^{m-|\beta|}} < \infty\ . \tag{3.1}$$

The set of symbols of order at most m, $S^m(U \times \mathbb{R}^l)$, becomes a Fréchet space with the semi-norms $\|\cdot\|_{K,\alpha,\beta,m}$ given by (3.1). There are continuous inclusions

$$S^m(U \times \mathbb{R}^l) \subset S^{m'}(U \times \mathbb{R}^l) \quad (m < m')\ , \tag{3.2}$$

giving rise to the LCSs

$$S^\infty(U \times \mathbb{R}^l) = \bigcup_m S^m(U \times \mathbb{R}^l)\ , \quad S^{-\infty}(U \times \mathbb{R}^l) = \bigcap_m S^m(U \times \mathbb{R}^l)\ .$$

The LF-space $S^\infty(U \times \mathbb{R}^l)$ is a filtered algebra and $C^\infty(U)$-module with the pointwise multiplication. The Fréchet space $S^{-\infty}(U \times \mathbb{R}^l)$ is a filtered ideal and $C^\infty(U)$-submodule of $S^\infty(U \times \mathbb{R}^l)$. The homogeneous components of the corresponding graded algebra are

$$S^{(m)}(U \times \mathbb{R}^l) = S^m(U \times \mathbb{R}^l)/S^{m-1}(U \times \mathbb{R}^l)\ .$$

When $U = \mathbb{R}^0 = \{0\}$, the notation $S^m(\mathbb{R}^l)$, $S^{\pm\infty}(\mathbb{R}^l)$ and $S^{(m)}(\mathbb{R}^l)$ is used, and the subscripts K and α are omitted from the notation of the semi-norms in (3.1).

Since $S^\infty(U \times \mathbb{R}^l)$ is an LF-space, we get the following (see Section 2.4).

Proposition 3.1. *$S^\infty(U \times \mathbb{R}^l)$ is barreled, ultrabornological and webbed.*

There are continuous inclusions (see Section 2.4 for the definition of $C^\infty_{\mathrm{cv}}(U \times \mathbb{R}^l)$)

$$C^\infty_{\mathrm{cv}}(U \times \mathbb{R}^l) \subset S^{-\infty}(U \times \mathbb{R}^l)\ , \quad S^\infty(U \times \mathbb{R}^l) \subset C^\infty(U \times \mathbb{R}^l)\ ; \tag{3.3}$$

in particular, $S^\infty(U \times \mathbb{R}^l)$ is Hausdorff. According to (2.1) and (3.3), we get continuous semi-norms $\|\cdot\|_{Q,C^k}$ on $S^\infty(U \times \mathbb{R}^l)$, for any compact $Q \subset U \times \mathbb{R}^l$ and $k \in \mathbb{N}_0$, given by

$$\|a\|_{Q,C^k} = \sup_{(x,\xi)\in Q,\ |\alpha|+|\beta|\le k} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|\ . \tag{3.4}$$

With the notation of (3.1), consider also the continuous semi-norms $\|\cdot\|'_{K,\alpha,\beta,m}$ on $S^m(U\times\mathbb{R}^l)$ given by

$$\|a\|'_{K,\alpha,\beta,m}=\sup_{x\in K}\limsup_{|\xi|\to\infty}\frac{\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|}{|\xi|^{m-|\beta|}}\;. \tag{3.5}$$

In the case of $S^m(\mathbb{R}^l)$, the subscripts K and α are omitted from the notation of the semi-norms (3.5).

Proposition 3.2. *The semi-norms* (3.4) *and* (3.5) *together describe the topology of* $S^m(U\times\mathbb{R}^l)$.

Proof. Let $S'^{\,m}(U\times\mathbb{R}^l)$ denote the LCHS defined by endowing the vector space $S^m(U\times\mathbb{R}^l)$ with the topology induced by the semi-norms (3.4) and (3.5) together; in fact, countably many semi-norms of these types are enough to describe its topology (taking exhausting increasing sequences of compact sets), and therefore $S'^{\,m}(U\times\mathbb{R}^l)$ is metrizable. Let $\widehat{S}'^{\,m}(U\times\mathbb{R}^l)$ denote its completion, where the stated semi-norms have continuous extensions. There is a continuous inclusion $S'^{\,m}(U\times\mathbb{R}^l)\subset C^\infty(U\times\mathbb{R}^l)$, which can be extended to a continuous map $\phi:\widehat{S}'^{\,m}(U\times\mathbb{R}^l)\to C^\infty(U\times\mathbb{R}^l)$ because $C^\infty(U\times\mathbb{R}^l)$ is complete. For any $a\in\widehat{S}'^{\,m}(U\times\mathbb{R}^l)$, and K, α and β like in (3.5), since $\|\phi(a)\|'_{K,\alpha,\beta,m}=\|a\|'_{K,\alpha,\beta,m}<\infty$, there are $C,R>0$ so that, if $x\in K$ and $|\xi|\ge R$, then

$$\frac{|\partial_x^\alpha\partial_\xi^\beta\phi(a)(x,\xi)|}{(1+|\xi|)^{m-|\beta|}}\le C\;.$$

Let $B_R\subset\mathbb{R}^l$ denote the open ball of center 0 and radius R. For $Q=K\times\overline{B_R}\subset U\times\mathbb{R}^l$ and $k=|\alpha|+|\beta|$, since $\|\phi(a)\|_{Q,C^k}=\|a\|_{Q,C^k}<\infty$, there is some $C'>0$ such that $|\partial_x^\alpha\partial_\xi^\beta\phi(a)(x,\xi)|<C'$ for $(x,\xi)\in Q$, yielding

$$\frac{|\partial_x^\alpha\partial_\xi^\beta\phi(a)(x,\xi)|}{(1+|\xi|)^{m-|\beta|}}\le\begin{cases}C' & \text{if } |\beta|\le m\\ C'(1+R)^{|\beta|-m} & \text{if } |\beta|\ge m\;.\end{cases}$$

This shows that $\|\phi(a)\|_{K,\alpha,\beta,m}<\infty$, obtaining that $a\equiv\phi(a)\in S^m(U\times\mathbb{R}^l)$. Hence $S'^{\,m}(U\times\mathbb{R}^l)$ is complete, and therefore it is a Fréchet space. Thus the identity map $S^m(U\times\mathbb{R}^l)\to S'^{\,m}(U\times\mathbb{R}^l)$ is a continuous linear isomorphism between Fréchet spaces, obtaining that it is indeed a homeomorphism by a version of the open mapping theorem [21, Section 15.12], [31, Theorem II.2.1], [28, Theorem 14.4.6]. □

Proposition 3.3. *For* $m,m'\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^n$, $\beta\in\mathbb{N}_0^l$ *and any compact* $K\subset U$, *if* $m<m'$, *then* $\|\cdot\|'_{K,\alpha,\beta,m'}=0$ *on* $S^m(U\times\mathbb{R}^l)$.

Proof. According to (3.5), for all $a\in S^m(U\times\mathbb{R}^l)$,

$$\|a\|'_{K,\alpha,\beta,m'}=\|a\|'_{K,\alpha,\beta,m}\lim_{|\xi|\to\infty}|\xi|^{m-m'}=0\;.$$ □

Corollary 3.4. *For* $m<m'$, *the topologies of* $S^{m'}(U\times\mathbb{R}^l)$ *and* $C^\infty(U\times\mathbb{R}^l)$ *coincide on* $S^m(U\times\mathbb{R}^l)$. *Therefore the topologies of* $S^\infty(U\times\mathbb{R}^l)$ *and* $C^\infty(U\times\mathbb{R}^l)$ *coincide on* $S^m(U\times\mathbb{R}^l)$.

Proof. The first assertion is a consequence of Propositions 3.2 and 3.3.

To prove the second assertion, by (3.3), it is enough to show that the topology of $S^\infty(U\times\mathbb{R}^l)$ is finer or equal than the topology of $C^\infty(U\times\mathbb{R}^l)$ on $S^m(U\times\mathbb{R}^l)$.

For every open $O \subset S^{\infty}(U \times \mathbb{R}^l)$ and $m' > m$, since $O \cap S^{m'}(U \times \mathbb{R}^l)$ is open in $S^{m'}(U \times \mathbb{R}^l)$, it follows from the first assertion that there is some open $P \subset C^{\infty}(U \times \mathbb{R}^l)$ such that $O \cap S^m(U \times \mathbb{R}^l) = P \cap S^m(U \times \mathbb{R}^l)$. □

Corollary 3.5. *For $m < m'$, $C^{\infty}_{\mathrm{c}}(U \times \mathbb{R}^l)$ is dense in $S^m(U \times \mathbb{R}^l)$ with the topology of $S^{m'}(U \times \mathbb{R}^l)$. Therefore $C^{\infty}_{\mathrm{c}}(U \times \mathbb{R}^l)$ is dense in $S^{\infty}(U \times \mathbb{R}^l)$.*

Proof. The first assertion is given by Corollary 3.4 and the density of $C^{\infty}_{\mathrm{c}}(U \times \mathbb{R}^l)$ in $C^{\infty}(U \times \mathbb{R}^l)$.

To prove the second assertion, take any open $O \neq \emptyset$ in $S^{\infty}(U \times \mathbb{R}^l)$. We have $O \cap S^m(U \times \mathbb{R}^l) \neq \emptyset$ for some m. This intersection is open in $S^m(U \times \mathbb{R}^l)$ with the topology of any $S^{m'}(U \times \mathbb{R}^l)$ for all $m' \geq m$. So $O \cap C^{\infty}_{\mathrm{c}}(U \times \mathbb{R}^l) \neq \emptyset$ by the first assertion. □

Corollary 3.6. *$S^{\infty}(U \times \mathbb{R}^l)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

Proof. Corollary 3.4 gives the property of being acyclic, and therefore complete and boundedly retractive (Section 2.1). Since $S^{\infty}(U \times \mathbb{R}^l)$ is barreled (Proposition 3.1) and every Montel space is reflexive [21, 6.27.2 (1)], [9, Section 8.4.7], [31, after the examples of IV.5.8], it only remains to prove that $S^{\infty}(U \times \mathbb{R}^l)$ is semi-Montel.

Take any closed bounded subset $B \subset S^{\infty}(U \times \mathbb{R}^l)$; in particular, B is complete because $S^{\infty}(U \times \mathbb{R}^l)$ is complete. Since $S^{\infty}(U \times \mathbb{R}^l)$ is boundedly retractive, B is contained and bounded in some $S^m(U \times \mathbb{R}^l)$, and the topologies of $S^{\infty}(U \times \mathbb{R}^l)$ and $S^m(U \times \mathbb{R}^l)$ coincide on B. By Corollary 3.4, it follows that B is a complete bounded subspace of $C^{\infty}(U \times \mathbb{R}^l)$, and therefore closed because $C^{\infty}(U \times \mathbb{R}^l)$ is complete. So B is compact because $C^{\infty}(U \times \mathbb{R}^l)$ is a Montel space. □

Remark 3.7. Another proof of Corollary 3.5 could be given like in Proposition 6.10.

Remark 3.8. Despite of Corollary 3.4, the following argument shows that the second inclusion of (3.3) is not a TVS-embedding. Let $a_m \in S^{\infty}(U \times \mathbb{R}^l)$ ($m \in \mathbb{N}_0$) such that $a_m(x, \xi) = 0$ if $|\xi^1| \leq m$, and $a_m(x, \xi) = (\xi^1 - m)^m$ if $|\xi^1| \geq m + 1$. Then $a_m \in S^m(U \times \mathbb{R}^l) \setminus S^{m-1}(U \times \mathbb{R}^l)$ and $a_m \to 0$ in $C^{\infty}(U \times \mathbb{R}^l)$ as $m \uparrow \infty$. However $a_m \not\to 0$ in $S^{\infty}(U \times \mathbb{R}^l)$; otherwise, since $S^{\infty}(U \times \mathbb{R}^l)$ is sequentially retractive (Corollary 3.6), all a_m would lie in some step $S^{m_0}(U \times \mathbb{R}^l)$, a contradiction.

With more generality, a symbol of order m on a vector bundle E over M is a smooth function on E satisfying (3.1) via charts of M and local trivializations of E, with K contained in the domains of charts where E is trivial. As above, they form a Fréchet space $S^m(E)$ with the topology described by the semi-norms given by this version of (3.1). The version of (3.2) in this setting is true, obtaining the corresponding inductive and projective limits $S^{\pm\infty}(E)$, and quotient spaces $S^{(m)}(E)$. We can similarly define the norms (3.4) and (3.5) on $S^m(E)$, and Propositions 3.2 and 3.3 and Corollaries 3.4 to 3.6 can be directly extended to this setting.

Given another vector bundle F over M, we can further take the $C^{\infty}(M)$-tensor product of these spaces with $C^{\infty}(M; F)$, obtaining the spaces $S^m(E; F)$, $S^{\pm\infty}(E; F)$ and $S^{(m)}(E; F)$, satisfying analogous properties and results. Now (3.3) becomes

$$C^{\infty}_{\mathrm{cv}}(E; \pi^*F) \subset S^{-\infty}(E; F)\,, \quad S^{\infty}(E; F) \subset C^{\infty}(E; \pi^*F)\,,$$

where $\pi : E \to M$ is the vector bundle projection.

4. Conormal distributions

4.1. **Differential operators tangent to a submanifold.** Let L is a regular submanifold of M of codimension n' and dimension n'', which is a closed subset. Let $\mathfrak{X}(M,L) \subset \mathfrak{X}(M)$ be the Lie subalgebra and $C^\infty(M)$-submodule of vector fields tangent to L. Using $\mathfrak{X}(M,L)$ instead of $\mathfrak{X}(M)$, we can define the filtered subalgebra and $C^\infty(M)$-submodule $\operatorname{Diff}(M,L) \subset \operatorname{Diff}(M)$ like in Section 2.7. We have

$$A \in \operatorname{Diff}(M,L) \Rightarrow A^t \in \operatorname{Diff}(M,L;\Omega) \ . \tag{4.1}$$

By the conditions on L, every $\operatorname{Diff}^m(M,L)$ ($m \in \mathbb{N}_0$) is locally finitely $C^\infty(M)$-generated, and therefore $\operatorname{Diff}(M,L)$ is countably $C^\infty(M)$-generated. The surjective restriction map $\mathfrak{X}(M,L) \to \mathfrak{X}(L)$, $X \mapsto X|_L$, induces a surjective linear restriction map of filtered algebras and $C^\infty(M)$-modules,

$$\operatorname{Diff}(M,L) \to \operatorname{Diff}(L) \ , \quad A \mapsto A|_L \ . \tag{4.2}$$

Let (U,x) be a chart of M adapted to L; i.e., for open subsets $U' \subset \mathbb{R}^{n'}$ and $U'' \subset \mathbb{R}^{n''}$,

$$x = (x^1,\dots,x^n) \equiv (x',x'') : U \to U' \times U'' \ ,$$
$$x' = (x'^1,\dots,x'^{n'}) \ , \quad x'' = (x''^1,\dots,x''^{n''}) \ , \quad L_0 := L \cap U = \{x'=0\} \ .$$

If L is of codimension one, then we will use the notation (x,y) instead of (x',x''). For every $m \in \mathbb{N}_0$, $\operatorname{Diff}^m(U,L_0)$ is $C^\infty(U)$-spanned by the operators $x'^\alpha \partial_{x'}^\beta \partial_{x''}^\gamma$ with $|\beta|+|\gamma| \le m$ and $|\alpha| = |\beta|$; we may use the generators $\partial_{x'}^\beta \partial_{x''}^\gamma x'^\alpha$ as well, with the same conditions on the multi-indices.

4.2. Conormal distributions filtered by Sobolev order.

4.2.1. *Case of compact manifolds.* Suppose first that M is compact. Then the space of *conormal distributions* at L of *Sobolev order* at most $s \in \mathbb{R}$ is the LCS and $C^\infty(M)$-module

$$I^{(s)}(M,L) = \{\, u \in C^{-\infty}(M) \mid \operatorname{Diff}(M,L)\, u \subset H^s(M) \,\} \ , \tag{4.3}$$

with the projective topology given by the maps $P : I^{(s)}(M,L) \to H^s(M)$ ($P \in \operatorname{Diff}(M,L)$).

Proposition 4.1. *$I^{(s)}(M,L)$ is a totally reflexive Fréchet space.*

Proof. For any countable $C^\infty(M)$-spanning set $\{P_j \mid j \in \mathbb{N}_0\}$ of $\operatorname{Diff}(M,L)$, the space $I^{(s)}(M,L)$ has the projective topology given by the maps $P_j : I^{(s)}(M,L) \to H^s(M)$. Let

$$I^{(s)}_k(M,L) = \{\, u \in C^{-\infty}(M) \mid P_j u \subset H^s(M),\ j = 0,\dots,k \,\} \ ,$$

with the projective topology given by the maps $P_j : I^{(s)}(M,L) \to H^s(M)$ ($j = 0,\dots,k$). We can assume $P_0 = 1$, and therefore $I^{(s)}_0(M,L) = H^s(M)$. Every $I^{(s)}_k(M,L)$ is a Hilbert space with the scalar product

$$\langle u,v\rangle_{s,k} = \sum_{j=0}^k \langle P_j u, P_j v\rangle_s \ ,$$

there are continuous inclusions $I^{(s)}_{k'}(M,L) \subset I^{(s)}_k(M,L)$ ($k < k'$), and $I^{(s)}(M,L) = \bigcap_k I^{(s)}_k(M,L)$. So $I^{(s)}(M,L)$ is a totally reflexive Fréchet space [38, Theorem 4]. $\square$

We have continuous inclusions

$$I^{(s)}(M,L) \subset I^{(s')}(M,L) \quad (s' < s) \;, \tag{4.4}$$

and consider the LCSs and $C^\infty(M)$-modules

$$I(M,L) = \bigcup_s I^{(s)}(M,L) \;, \quad I^{(\infty)}(M,L) = \bigcap_s I^{(s)}(M,L) \;.$$

$I(M,L)$ is an LF-space, and $I^{(\infty)}(M,L)$ is a Fréchet space and submodule of $I(M,L)$. The elements of $I(M,L)$ are called *conormal distributions* of M at L (or of (M,L)). The spaces $I^{(s)}(M,L)$ form the *Sobolev order filtration* of $I(M,L)$. From (4.3), it follows that there are canonical continuous inclusions,

$$C^\infty(M) \subset I^{(\infty)}(M,L) \;, \quad I(M,L) \subset C^{-\infty}(M) \;; \tag{4.5}$$

in particular, $I(M,L)$ is Hausdorff.

Since every $I^{(s)}(M,L)$ is a Fréchet space (Proposition 4.1), the following analog of Proposition 3.1 holds true by the same reason.

Corollary 4.2. *$I(M,L)$ is barreled, ultrabornological and webbed.*

4.2.2. *Extension to non-compact manifolds.* If M is not assumed to be compact, we can similarly define the LCHS $I^{(s)}_{\cdot/\mathrm{c}}(M,L)$ by using $C^{-\infty}_{\cdot/\mathrm{c}}(M)$ and $H^s_{\mathrm{loc}/\mathrm{c}}(M)$. Every $I^{(s)}(M,L)$ is a Fréchet space, as follows like in the proof of Proposition 4.1, using the Fréchet spaces $H^s_{\mathrm{loc}}(M)$. We can describe $I^{(s)}_{\mathrm{c}}(M,L) = \bigcup_K I^{(s)}_K(M,L)$ like in (2.2), which is a strict LF-space because every $I^{(s)}_K(M,L)$ satisfies an analog of Proposition 4.1. Therefore $I_{\mathrm{c}}(M,L) = \bigcup_s I^{(s)}_{\mathrm{c}}(M,L)$ is an LF-space [28, Exercise 12.108]; moreover $I_{\mathrm{c}}(M,L) = \bigcup_K I_K(M,L)$. We also have the LCHS $I^{(\infty)}_{\mathrm{c}}(M,L) = \bigcap_s I^{(s)}_{\mathrm{c}}(M,L)$. All of these spaces are modules over $C^\infty(M)$; $I_{\mathrm{c}}(M,L)$ is a filtered module and $I^{(\infty)}_{\mathrm{c}}(M,L)$ a submodule. The extension by zero defines a continuous inclusion $I_{\mathrm{c}}(U, L\cap U) \subset I_{\mathrm{c}}(M,L)$ for any open $U \subset M$. We also define the space $I^{(\infty)}(M,L)$ like in the compact case, as well as the space $\bigcup_s I^{(s)}(M,L)$, which consists of the conormal distributions with a Sobolev order. But now let (cf. [15, Definition 18.2.6])

$$I(M,L) = \{\, u \in C^{-\infty}(M) \mid C^\infty_{\mathrm{c}}(M)\, u \subset I_{\mathrm{c}}(M,L) \,\} \;, \tag{4.6}$$

which is a LCS with the projective topology given by the (multiplication) maps $f_j : I(M,L) \to I_{\mathrm{c}}(M,L)$, for a countable partition of unity $\{f_j\} \subset C^\infty_{\mathrm{c}}(M)$. We have $I(M,L) = \bigcup_s I^{(s)}(M,L)$ if and only if L is compact; thus the spaces $I^{(s)}(M,L)$ form a filtration of $I(M,L)$ just when L is compact. There is an extension of (4.5) for non-compact M, taking arbitrary/compact support; in particular, $I_{\cdot/\mathrm{c}}(M,L)$ is Hausdorff.

4.3. **Filtration of $I(M,L)$ by the symbol order.**

4.3.1. *Local description of conormal distributions with symbols.* Consider the notation of Section 4.1 for a chart $(U, x = (x', x''))$ of M adapted to L. We use the identity $U'' \times \mathbb{R}^{n'} \equiv N^*U''$, and the symbol spaces $S^m(U'' \times \mathbb{R}^{n'}) \equiv S^m(N^*U'')$ (Section 3). Define

$$C^\infty_{\mathrm{cv}}(N^*U'') \to C^\infty(U) \;, \quad a \mapsto u \;, \tag{4.7}$$

$$C^\infty_{\mathrm{c}}(U) \to C^\infty(N^*U'') \;, \quad u \mapsto a \;, \tag{4.8}$$

by the following partial inverse Fourier transform and partial Fourier transform:

$$u(x) = (2\pi)^{-n'} \int_{\mathbb{R}^{n'}} e^{i\langle x',\xi\rangle} a(x'',\xi)\, d\xi \ ,$$
$$a(x'',\xi) = \int_{\mathbb{R}^{n'}} e^{-i\langle x',\xi\rangle} u(x',x'')\, dx' \ .$$

Proposition 4.3 ([15, Theorem 18.2.8], [25, Proposition 6.1.1], [27, Lemma 9.33])**.** *If $s < -\bar{m} - n'/2$, then (4.7) has a continuous extension $S^{\bar{m}}(N^*U'') \to I^{(s)}(U, L_0)$. If $\bar{m} > -s - n'/2$, then (4.8) induces a continuous linear map $I^{(s)}_{\mathrm{c}}(U, L_0) \to S^{\bar{m}}(N^*U'')$.*

Remark 4.4. The continuity of the maps of Proposition 4.3 is not stated in [15, Theorem 18.2.8], [25, Proposition 6.1.1], [27, Lemma 9.33], but it follows easily from their proofs.

When applying Proposition 4.3 to M via (U, x), it will be convenient to use

$$a\,|d\xi| \in S^{\bar{m}}(N^*U''; \Omega N^*U'') \equiv S^{\bar{m}}(N^*L_0; \Omega N^*L_0) \ .$$

4.3.2. *Case of compact manifolds.* Assume first that M is compact. Take a finite cover of L by relatively compact charts (U_j, x_j) of M adapted to L, and write $L_j = L \cap U_j$. Let $\{h, f_j\}$ be a C^∞ partition of unity of M subordinated to the open covering $\{M \setminus L, U_j\}$. Then $I(M, L)$ consists of the distributions $u \in C^{-\infty}(M)$ such that $hu \in C^\infty(M \setminus L)$ and $f_j u \in I_{\mathrm{c}}(U_j, L_j)$ for all j. According to Proposition 4.3, every $f_j u$ is given by some $a_j \in S^\infty(N^*L_j; \Omega N^*L_j)$. For

$$\bar{m} = m + n/4 - n'/2 \ , \tag{4.9}$$

the condition $a_j \in S^{\bar{m}}(N^*L_j; \Omega N^*L_j)$ describes the elements u of a $C^\infty(M)$-submodule $I^m(M, L) \subset I(M, L)$, which is independent of the choices involved [27, Proposition 9.33] (see also [25, Definition 6.2.19] and [36, Definition 4.3.9]). Moreover, applying the versions of semi-norms (2.1) on $C^\infty(M \setminus L)$ to hu and versions of semi-norms (3.1) on $S^{\bar{m}}(N^*L_j; \Omega N^*L_j)$ to every a_j, we get semi-norms on $I^m(M, L)$, which becomes a Fréchet space [25, Sections 6.2 and 6.10]. In other words, the following map is required to be a TVS-embedding:

$$I^m(M, L) \to C^\infty(M \setminus L) \oplus \prod_j S^{\bar{m}}(N^*L_j; \Omega N^*L_j) \ , \quad u \mapsto (hu, (a_j)) \ . \tag{4.10}$$

The version of (3.2) for the spaces $S^{\bar{m}}(N^*L_j; \Omega N^*L_j)$ gives continuous inclusions

$$I^m(M, L) \subset I^{m'}(M, L) \quad (m < m') \ . \tag{4.11}$$

The element $\sigma_m(u) \in S^{(\bar{m})}(N^*L; \Omega N^*L)$ represented by $\sum_j a_j \in S^{\bar{m}}(N^*L; \Omega N^*L)$ is called the *principal symbol* of u. This defines the exact sequence

$$0 \to I^{m-1}(M, L) \hookrightarrow I^m(M, L) \xrightarrow{\sigma_m} S^{(\bar{m})}(N^*L; \Omega N^*L) \to 0 \ .$$

From Proposition 4.3 and (4.9), we also get continuous inclusions

$$I^{(-m-n/4+\epsilon)}(M, L) \subset I^m(M, L) \subset I^{(-m-n/4-\epsilon)}(M, L) \ , \tag{4.12}$$

for all $m \in \mathbb{R}$ and $\epsilon > 0$ (cf. [25, Eq. (6.2.5)], [27, Eq. (9.35)]). So

$$I(M, L) = \bigcup_m I^m(M, L) \ , \quad I^{(\infty)}(M, L) = I^{-\infty}(M, L) := \bigcap_m I^m(M, L) \ .$$

The spaces $I^m(M,L)$ form the *symbol order filtration* of $I(M,L)$. The maps (4.10) induce a TVS-embedding

$$I(M,L) \to C^\infty(M \setminus L) \oplus \prod_j S^\infty(N^*L_j; \Omega N^*L_j)\ . \tag{4.13}$$

Corollary 4.5. *For $m < m', m''$, the topologies of $I^{m'}(M,L)$ and $I^{m''}(M,L)$ coincide on $I^m(M,L)$.*

Proof. Use Corollary 3.4 and the TVS-embeddings (4.10). □

Corollary 4.6 ([25, Eq. (6.2.12)])**.** *For $m < m'$, $C^\infty(M)$ is dense in $I^m(M,L)$ with the topology of $I^{m'}(M,L)$. Therefore $C^\infty(M)$ is dense in $I(M,L)$.*

Proof. $C^\infty(M)$ is contained in the stated spaces by (4.5).

Let us prove the first density, and the second one follows like in Corollary 3.5. Given $u \in I^m(M,L)$, let $a_j \in S^{\bar m}(N^*L_j; \Omega N^*L_j)$ be the symbol corresponding to $f_j u$ by Proposition 4.3, like in (4.10). By Corollary 3.5, there is a sequence $b_{j,k} \in C^\infty_{\rm c}(N^*L_j; \Omega N^*L_j)$ converging to a_j in $S^{\bar m'}(N^*L_j; \Omega N^*L_j)$ ($\bar m' = m' + n/4 - n'/2$). Let $v_{j,k}$ be the sequence in $C^\infty(U)$ that corresponds to $b_{j,k}$ via (4.7); it converges to $f_j u$ in $I^{m'}(U,L)$ as $k \to \infty$ by Proposition 4.3. Take functions $\tilde f_j \in C^\infty_{\rm c}(U_j)$ with $\tilde f_j = 1$ on $\operatorname{supp} f_j$. Then $\tilde f_j v_{j,k} \to f_j u$ in $I^{m'}_{\rm c}(U,L)$, and therefore $hu + \sum_j \tilde f_j v_{j,k} \in C^\infty(M)$ is convergent to u in $I^{m'}(M,L)$. □

Corollary 4.7. *$I(M,L)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

Proof. Like in Corollary 3.6, by Corollaries 4.2 and 4.5, it is enough to prove that $I(M,L)$ is semi-Montel. The TVS-embedding (4.13) is closed because $I(M,L)$ is complete. Then $I(M,L)$ is semi-Montel because $C^\infty(M \setminus L)$ and $S^\infty(N^*L_j; \Omega N^*L_j)$ are Montel spaces (Corollary 3.6), and this property is inherited by closed subspaces and products [16, Propositions 3.9.3 and 3.9.4], [28, Exercise 12.203 (c)]. □

Remark 4.8. The reflexivity of $I(M,L)$ is also a consequence of the reflexivity of $I^{(s)}(M,L)$ (Proposition 4.1) and the regularity of $I(M,L)$ (Corollary 4.7) [23].

4.3.3. *Extension to non-compact manifolds.* When M is not assumed to be compact, the definition of $I^m(M,L)$ can be immediately extended assuming $\{U_j\}$ is a locally finite cover of L, obtaining an analog of (4.10). We can similarly define $I^m_K(M,L)$ for all compact $K \subset M$, and take $I^m_{\rm c}(M,L) = \bigcup_K I^m_K(M,L)$ like in (2.2). The space of conormal distributions with a symbol order is $\bigcup_m I^m(M,L)$, and let $I^{-\infty}_{\cdot/{\rm c}}(M,L) = \bigcap_m I^m_{\cdot/{\rm c}}(M,L)$. There are extensions of (4.10)–(4.13) and Corollaries 4.5 and 4.6, with arbitrary/compact support (using direct sums instead of products in the case of compact support). So $\bigcup_m I^m(M,L) = \bigcup_s I^{(s)}(M,L)$, $I_{\rm c}(M,L) = \bigcup_m I^m_{\rm c}(M,L)$ and $I^{(\infty)}_{\cdot/{\rm c}}(M,L) = I^{-\infty}_{\cdot/{\rm c}}(M,L)$. Corollary 4.7 has extensions for $\bigcup_m I^m(M,L)$ and $I_{\cdot/{\rm c}}(M,L)$, except acyclicity in the case of $I(M,L)$.

4.4. **Dirac sections at submanifolds.** Let NL and N^*L denote the normal and conormal bundles of L. We have $\Omega NL \otimes \Omega L \equiv \Omega_L M$. The transpose of the restriction map $C^\infty_{{\rm c}/\cdot}(M; E^* \otimes \Omega M) \to C^\infty_{{\rm c}/\cdot}(L; E^* \otimes \Omega_L M)$ is a continuous inclusion

$$C^{-\infty}_{\cdot/\mathrm{c}}(L; E \otimes \Omega^{-1}NL) \subset C^{-\infty}_{\cdot/\mathrm{c}}(M; E) \; , \tag{4.14}$$
$$u \mapsto \delta^u_L \; , \quad \langle \delta^u_L, v \rangle = \langle u, v|_L \rangle \; , \quad v \in C^\infty_{\mathrm{c}/\cdot}(M; E^* \otimes \Omega) \; .$$

By restriction of (4.14), we get a continuous inclusion [11, p. 310],

$$C^{\infty}_{\cdot/\mathrm{c}}(L; E \otimes \Omega^{-1}NL) \subset C^{-\infty}_{\cdot/\mathrm{c}}(M; E) \; ; \tag{4.15}$$

in this case, we can write $\langle \delta^u_L, v \rangle = \int_L u\, v|_L$. This is the subspace of *δ-sections* or *Dirac sections* at L. Actually, the following sharpening of (4.15) is true.

Proposition 4.9. *The inclusion* (4.15) *induces a continuous injection*

$$C^{\infty}_{\cdot/\mathrm{c}}(L; E \otimes \Omega^{-1}NL) \subset H^s_{\mathrm{loc/c}}(M; E) \quad (s < -n'/2)$$

with

$$C^{\infty}_{\cdot/\mathrm{c}}(L; E \otimes \Omega^{-1}NL) \cap H^{-n'/2}_{\mathrm{loc/c}}(M; E) = 0 \; .$$

Proof. First, take $M = \mathbb{R}^n$, $L = \mathbb{R}^{n''} \times \{0\} \equiv \mathbb{R}^{n''}$ and $E = M \times \mathbb{C}$ (the trivial line bundle). Let δ_0 be the Dirac mass at 0 in $\mathbb{R}^{n'}$. For any $\phi \in \mathcal{S}(\mathbb{R}^{n''})$, consider the tensor product distribution $\phi \otimes \delta_0 \in \mathcal{S}(\mathbb{R}^n)'$ [14, Section 5.1]. Its Fourier transform is $\hat{\phi} \otimes \hat{\delta}_0 = \hat{\phi} \otimes 1$. If $\phi \neq 0$, then $\hat{\phi} \otimes 1 \in L^2(\mathbb{R}^n, (1 + |\xi|^2)^s \, d\xi)$ if and only if $1 \in L^2(\mathbb{R}^{n'}, (1 + |\xi|^2)^s \, d\xi)$, which holds just when $s < -n'/2$. Moreover the map

$$\mathcal{S}(\mathbb{R}^{n''}) \to L^2(\mathbb{R}^n, (1 + |\xi|^2)^s \, d\xi) \; , \quad \phi \mapsto \hat{\phi} \otimes 1 \; ,$$

is continuous if $s < -n'/2$. Thus (2.15) yields versions of the stated properties using $\mathcal{S}(\mathbb{R}^{n''})$ and $H^s(R^n)$.

For arbitrary M, L and E, the result follows from the previous case by using a locally finite atlas, a subordinated partition of unity, and diffeomorphisms of triviality of E. □

For instance, for any $p \in M$ and $u \in E_p \otimes \Omega^{-1}_p M$, we get $\delta^u_p \in H^s_{\mathrm{c}}(M; E)$ if $s < -n/2$, with $\langle \delta^u_p, v \rangle = u \cdot v(p)$ for $v \in C^\infty(M; E^* \otimes \Omega)$, obtaining a continuous map

$$M \times C^\infty(M; E \otimes \Omega^{-1}) \to H^s_{\mathrm{c}}(M; E) \; , \quad (p, u) \mapsto \delta^{u(p)}_p \; .$$

As a particular case, the Dirac mass at any $p \in \mathbb{R}^n$ is $\delta_p = \delta_p^{1 \otimes |dx|^{-1}} \in H^s_{\mathrm{c}}(\mathbb{R}^n)$.

4.5. **Differential operators on conormal distributional sections.** Any $A \in \mathrm{Diff}^k(M; E)$ induces continuous linear maps [25, Lemma 6.1.1]

$$A : I^{(s)}_{\cdot/\mathrm{c}}(M, L; E) \to I^{(s-k)}_{\cdot/\mathrm{c}}(M, L; E) \; , \tag{4.16}$$

which induce a continuous endomorphism A of $I_{\cdot/\mathrm{c}}(M, L; E)$. If $A \in \mathrm{Diff}(M, L; E)$, then it clearly induces a continuous endomorphism A of every $I^{(s)}_{\cdot/\mathrm{c}}(M, L; E)$.

According to (4.14), for $A \in \mathrm{Diff}(M, L; E)$ and $u \in C^\infty_{\cdot/\mathrm{c}}(L; E \otimes \Omega^{-1}NL)$,

$$A\delta^u_L = \delta^{A'u}_L \; , \quad A' = ((A^t)|_L)^t \in \mathrm{Diff}(L; E \otimes \Omega^{-1}NL) \; , \tag{4.17}$$

where $A^t \in \mathrm{Diff}(M, L; E^* \otimes \Omega)$ and $(A^t)|_L \in \mathrm{Diff}(L, E^* \otimes \Omega_L M)$ using the vector bundle versions of (4.1) and (4.2). In fact, for $v \in C^\infty_{\mathrm{c}/\cdot}(M; E^* \otimes \Omega)$,

$$\langle A\delta^u_L, v \rangle = \langle \delta^u_L, A^t v \rangle = \langle u, (A^t v)|_L \rangle = \langle u, (A^t)|_L(v|_L) \rangle = \langle A'u, v|_L \rangle = \langle \delta^{A'u}_L, v \rangle \; .$$

By (4.17), $\mathrm{Diff}(M,L;E)$ preserves the subspace of Dirac sections given by (4.15). Thus the continuous inclusion of Proposition 4.9 induces a continuous inclusion

$$C^{\infty}_{\cdot/\mathrm{c}}(L;E\otimes\Omega^{-1}NL)\subset I^{(s)}_{\cdot/\mathrm{c}}(M,L;E)\quad(s<-n'/2)\ . \tag{4.18}$$

4.6. **Pull-back of conormal distributions.** If a smooth map $\phi:M'\to M$ is transverse to a regular submanifold $L\subset M$, then $L':=\phi^{-1}(L)\subset M'$ is a regular submanifold and (the trivial-line-bundle version of) (2.8) has continuous extensions

$$\phi^*:I^m(M,L)\to I^{m+k/4}(M',L')\quad(m\in\mathbb{R})\ , \tag{4.19}$$

where $k=\dim M-\dim M'$ [36, Theorem 5.3.8], [25, Proposition 6.6.1]. Taking inductive limits and using (4.12), we get a continuous linear map

$$\phi^*:I(M,L)\to I(M',L')\ . \tag{4.20}$$

If ϕ is a submersion, then (4.20) is a restriction of (2.11). If ϕ is a local diffeomorphism, then (4.20) is compatible with the Sobolev and symbol order filtrations in the sense that it restricts to continuous maps between the spaces defining those filtrations.

A more general pull-back of distributional sections can be defined under conditions on the wave front set [14, Theorem 8.2.4], but we will not use it.

4.7. **Push-forward of conormal distributional sections.** Now let $\phi:M'\to M$ be a smooth submersion, and let $L\subset M$ and $L'\subset M'$ be regular submanifolds such that $\phi(L')\subset L$ and the restriction $\phi:L'\to L$ is also a smooth submersion. Then (2.9) has continuous extensions

$$\phi_*:I^m_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}})\to I^{m+l/2-k/4}_{\mathrm{c}}(M,L)\quad(m\in\mathbb{R})\ , \tag{4.21}$$

where $k=\dim M'-\dim M$ and $l=\dim L'-\dim L$ [36, Theorem 5.3.6], [25, Proposition 6.7.2]. Taking inductive limits, we get a continuous linear map

$$\phi_*:I_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}})\to I_{\mathrm{c}}(M,L)\ , \tag{4.22}$$

which is a restriction of (2.10). If ϕ is a local diffeomorphism, then (4.22) is compatible with the Sobolev and symbol order filtrations.

4.8. **Pseudodifferential operators.** This type of operators is the main application of conormal distributions (see e.g. [37, 15, 26, 36]).

4.8.1. *Case of compact manifolds.* Suppose first that M is compact. The filtered algebra and $C^\infty(M^2)$-module of pseudodifferential operators, $\Psi(M)$, consists of the continuous endomorphisms A of $C^\infty(M)$ with Schwartz kernel $K_A\in I(M^2,\Delta)$, where Δ is the diagonal. In fact, by the Schwartz kernel theorem, we may consider $\Psi(M)\equiv I(M^2,\Delta)$. It is filtered by the symbol order, $\Psi^m(M)\equiv I^m(M^2,\Delta)$ ($m\in\mathbb{R}$), and $\Psi^{-\infty}(M)\equiv I^{-\infty}(M^2,\Delta)$ consists of the smoothing operators. The analogs of (2.21) and (2.22) hold true using $\Psi^s(M)$ instead of $\mathrm{Diff}^s(M)$ for any $s\in\mathbb{R}$. In this way, $\Psi(M)$ also becomes a LCHS satisfying the properties indicated in Sections 4.2.1 and 4.3.2.

Taking the $C^\infty(M^2)$-tensor product of $\Psi(M)$ with $C^\infty(M;F\boxtimes E^*)$, we get $\Psi(M;E,F)$ (or $\Psi(M;E)$ if $E=F$) as in Section 2.7, satisfying the analog of (2.13). In this case, we have $\bar m=m$ in (4.9), and the *symbol* of any $A\in\Psi^m(M;E,F)$ can be given by $\sigma_m(A)\equiv\sigma_m(K_A)$; this symbol is used to extend the concept of *ellipticity* to pseudodifferential operators (see e.g. [24]).

$\Psi(M;E)$ is preserved by taking transposes, and therefore any $A \in \Psi(M;E)$ defines a continuous endomorphism A of $C^{-\infty}(M;E)$ (Section 2.5), and $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$ for all $u \in C^{-\infty}(M;E)$ (*pseudolocality*).

If $A \in \Psi^m(M;E)$, it defines a bounded operator $A : H^{s+m}(M;E) \to H^s(M;E)$. This can be considered as a closable densely defined operator in $H^s(M;E)$, like in the case of differential operators (Section 2.10). In the case $s = 0$, the adjoint of A is induced by the formal adjoint $A^* \in \Psi^m(M;E)$. The symbol map on $\Psi(M;E)$ is multiplicative and compatible with transposition and taking adjoints.

The class of pseudodifferential operators is preserved by transposition. So any $A \in \Psi^m(M)$ defines a continuous endomorphism A of $C^{-\infty}(M)$ (Section 2.5), and $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$ for all $u \in C^{-\infty}_{\mathrm{c}}(M)$ (*pseudolocality*).

4.8.2. *Extension to non-compact manifolds.* If M is not assumed to be compact, $\Psi(M)$ is similarly defined with the change that any $A \in \Psi^m(M)$ defines continuous linear maps $A : C^{\pm\infty}_{\mathrm{c}}(M) \to C^{\pm\infty}(M)$ and $A : H^{s+m}_{\mathrm{c}}(M) \to H^s_{\mathrm{loc}}(M)$. Thus $\Psi(M)$ is not an algebra in this case. However, if $A \in \Psi^m(M)$ is properly supported (both factor projections $M^2 \to M$ restrict to proper maps $\operatorname{supp} K_A \to M$), then it defines a continuous endomorphism A of $C^{-\infty}_{\mathrm{c}}(M;E)$; in this sense, properly supported pseudodifferential operators can be composed. Pseudodifferential operators are properly supported modulo $\Psi^{-\infty}(M)$. Like in the compact case, $\Psi(M) \equiv I(M^2,\Delta)$ becomes a filtered $C^\infty(M^2)$-module and LCHS satisfying the properties indicated in Sections 4.2.2 and 4.3.3.

In the setting of bounded geometry (Section 2.12.3), properly supported pseudodifferential operators with uniformly bounded symbols, and their uniform ellipticity, were studied in [19, 20].

5. Dual-conormal distributions

5.1. **Dual-conormal distributions.** Consider the notation of Sections 4.2 and 4.3.

5.1.1. *Case of compact manifolds.* Assume first that M is compact. The space of *dual-conormal distributions* of M at L (or of (M,L)) is [25, Chapter 6]

$$I'(M,L) = I(M,L;\Omega)' \,. \tag{5.1}$$

Corollary 5.1. *$I'(M,L)$ is a complete Montel space.*

Proof. Since $I(M,L;\Omega)$ is bornological (the version Corollary 4.2 with ΩM), $I'(M,L)$ is complete [31, IV.6.1], [29, Corollary 6.1.18], [28, Theorem 13.2.13].

Since $I(M,L;\Omega)$ is a Montel space (the version Corollary 4.7 with ΩM), $I'(M,L)$ is a Montel space [16, Proposition 3.9.9], [21, 6.27.2 (2)], [31, IV.5.9]. □

Let also

$$I'^{\,(s)}(M,L) = I^{(-s)}(M,L;\Omega)' \,, \quad I'^{\,m}(M,L) = I^{-m}(M,L;\Omega)' \,. \tag{5.2}$$

Corollary 5.2. *$I'^{\,(s)}(M,L)$ is bornological and barreled.*

Proof. Since $I^{(-s)}(M,L;\Omega)$ is a reflexive Fréchet space (the version of Proposition 4.1 with ΩM), $I'^{\,(s)}(M,L)$ is bornological [31, Corollary 1 of IV.6.6], and therefore barreled [31, IV.6.6]. □

Transposing the versions of (4.4) and (4.11) with ΩM, we get continuous restrictions, for $s' < s$ and $m < m'$,

$$I'^{(s)}(M,L) \to I'^{(s')}(M,L)\ , \quad I'^{m}(M,L) \to I'^{m'}(M,L)\ .$$

These maps form projective systems, giving rise to $\varprojlim I'^{(s)}(M,L)$ as $s \uparrow \infty$ and $\varprojlim I'^{m}(M,L)$ as $m \downarrow -\infty$. Transposing the versions of (4.5) and (4.12) with ΩM, we get continuous inclusions

$$C^{-\infty}(M) \supset I'(M,L) \supset C^{\infty}(M)\ , \tag{5.3}$$

and, for all $m \in \mathbb{R}$ and $\epsilon > 0$, continuous restrictions

$$I'^{(-m+n/4-\epsilon)}(M,L) \leftarrow I'^{m}(M,L) \leftarrow I'^{(-m+n/4+\epsilon)}(M,L)\ . \tag{5.4}$$

Thus

$$\varprojlim I'^{(s)}(M,L) \equiv \varprojlim I'^{m}(M,L)\ . \tag{5.5}$$

Corollary 5.3. $I'(M,L) \equiv \varprojlim I'^{(s)}(M,L)$.

Proof. This holds because $I(M,L)$ is regular (Corollary 4.7) [23, Lemma 1].

Alternatively, the following argument can be used. $I'(M,L)$ is a Montel space (Corollary 5.1); in particular, it is barreled, and therefore a Mackey space [31, IV.3.4]. On the other hand, every $I'^{(s)}(M,L)$ is bornological (Corollary 5.2), and therefore a Mackey space [16, Proposition 3.7.2], [31, IV.3.4], [28, Theorem 13.2.10]. So the result follows applying [31, Remark of IV.4.5]. □

5.1.2. *Extension to non-compact manifolds.* If M is not supposed to be compact, we can similarly define the space $I'_K(M,L)$ of dual-conormal distributions supported in any compact $K \subset M$. Then define the LCHSs, $I'_{\mathrm{c}}(M,L) = \bigcup_K I'_K(M,L)$ like in (2.2), and $I'(M,L)$ like in (4.6) using $I'_{\mathrm{c}}(M,L)$ instead of $I_{\mathrm{c}}(M,L)$. These spaces satisfy a version of (5.1), interchanging arbitrary/compact support like in (2.4). Given a smooth partition of unity $\{f_j\}$ so that every $K_j := \operatorname{supp} f_j$ is compact, the multiplication by the functions f_j defines closed TVS-embeddings

$$I'(M,L) \to \prod_j I'_{K_j}(M,L)\ , \quad I'_{\mathrm{c}}(M,L) \to \bigoplus_j I'_{K_j}(M,L)\ . \tag{5.6}$$

By the extension of Corollary 4.7 for $I_{\mathrm{c}}(M,L;\Omega)$, the obvious extension of Corollary 5.2 for every $I'_{K_j}(M,L)$, the indicated extension of (5.1) and the properties of (5.6), we get an extension of Corollary 5.1.

Similarly, we can define the spaces $I'^{(s)}_{\cdot/\mathrm{c}}(M,L)$ and $I'^{m}_{\cdot/\mathrm{c}}(M,L)$. They satisfy (5.2) interchanging the support condition, and also obvious versions of (5.3)–(5.5). Since $I_{\mathrm{c}}(M,L)$ is an acyclic Montel space (Section 4.3.3), there is an extension of Corollary 5.3 for $I'(M,L)$.

5.2. **Differential operators on dual-conormal distributional sections.** For any $A \in \operatorname{Diff}(M;E)$, consider $A^t \in \operatorname{Diff}(M;E^* \otimes \Omega)$. The transpose of A^t on $I_{\mathrm{c}/\cdot}(M,L;E^* \otimes \Omega)$ is a continuous endomorphism A of $I'_{\cdot/\mathrm{c}}(M,L;E)$, which is a restriction of the map A on $C^{-\infty}(M;E)$ (Section 2.7). By (4.16), if $A \in \operatorname{Diff}^m(M;E)$, we get induced continuous linear maps

$$A : I'^{(s)}_{\cdot/\mathrm{c}}(M,L;E) \to I'^{(s-m)}_{\cdot/\mathrm{c}}(M,L;E)\ , \tag{5.7}$$

If $A \in \operatorname{Diff}(M, L; E)$, the transpose of A^t of $I^{(-s)}_{\mathrm{c}/\cdot}(M, L; E^* \otimes \Omega)$ is a continuous endomorphism A of $I'^{(s)}_{\cdot/\mathrm{c}}(M, L; E)$.

5.3. **Pull-back of dual-conormal distributions.** If the conditions of Section 4.7 hold, transposing the versions of (4.21) and (4.22) with ΩM and $-m$, we get continuous linear pull-back maps

$$\phi^* : I'^{\,m}(M, L) \to I'^{\,m+l/2-k/4}(M', L') \quad (m \in \mathbb{R}) , \tag{5.8}$$

$$\phi^* : I'(M, L) \to I'(M', L') . \tag{5.9}$$

The map (5.9) is an extension of (2.8), a restriction of (2.11) and the projective limit of the maps (5.8). If ϕ is a local diffeomorphism, then (5.9) is compatible with the Sobolev and symbol order filtrations.

5.4. **Push-forward of dual-conormal distributions.** With the notation of Section 4.6, if ϕ is a submersion, transposing the versions of (4.19) and (4.20) with ΩM and $-m$, we get continuous linear push-forward maps

$$\phi_* : I'^{\,m}_{\mathrm{c}}(M', L' \otimes \Omega_{\mathrm{fiber}}) \to I'^{\,m-k/4}_{\mathrm{c}}(M, L) \quad (m \in \mathbb{R}) , \tag{5.10}$$

$$\phi_* : I'_{\mathrm{c}}(M', L'; \Omega_{\mathrm{fiber}}) \to I'_{\mathrm{c}}(M, L) . \tag{5.11}$$

The map (5.11) is an extension of (2.9), a restriction of (2.10) and the projective limit of the maps (5.10). If ϕ is a local diffeomorphism, then (5.11) is compatible with the Sobolev and symbol order filtrations.

6. Conormal distributions at the boundary

For the sake of simplicity, in this section and in Sections 7 and 8, we only consider the case of compact manifolds unless otherwise stated. But the concepts, notation and some of the results, can be extended to the non-compact case like in Sections 4.2.2, 4.3.3 and 5.1.2, using arbitrary/compact support conditions. Such extensions to non-compact manifolds may be used without further comment.

6.1. **Some notions of b-geometry.** R. Melrose introduced b-calculus, a way to extend calculus to manifolds with boundary [24, 25]. We will only use a part of it called small b-calculus. Let M be a compact (smooth) n-manifold with boundary; its interior is denoted by $\mathring{M}$. There exists a function $x \in C^\infty(M)$ so that $x \geq 0$, $\partial M = \{x = 0\}$ (i.e., $x^{-1}(0)$) and $dx \neq 0$ on ∂M, which is called a *boundary defining function*. Let ${}_+N\partial M \subset N\partial M$ be the inward-pointing subbundle of the normal bundle to the boundary. There is a unique trivialization $\nu \in C^\infty(\partial M; {}_+N\partial M)$ of ${}_+N\partial M$ so that $dx(\nu) = 1$. Take a collar neighborhood $T \equiv [0, \epsilon_0)_x \times \partial M$ of ∂M, whose projection $\varpi : T \to \partial M$ is the second factor projection. (In a product expression, every factor projection may be indicated as subscript of the corresponding factor.) Given coordinates $y = (y^1, \dots, y^{n-1})$ on some open $V \subset \partial M$, we get via ϖ coordinates $(x, y) = (x, y^1, \dots, y^{n-1})$ adapted (to ∂M) on the open subset $U \equiv [0, \epsilon_0) \times V \subset M$. There are vector bundles over M, ${}^{\mathrm{b}}TM$ and ${}^{\mathrm{b}}T^*M$, called *b-tangent* and *b-cotangent* bundles, which have the same restrictions as TM and T^*M to $\mathring{M}$, and such that $x\partial_x, \partial_{y^1}, \dots, \partial_{y^{n-1}}$ and $x^{-1}dx, dy^1, \dots, dy^{n-1}$ extend to smooth local frames around boundary points. This gives rise to versions of induced vector bundles, like ${}^{\mathrm{b}}\Omega^s M := \Omega^s({}^{\mathrm{b}}TM)$ ($s \in \mathbb{R}$) and ${}^{\mathrm{b}}\Omega M := {}^{\mathrm{b}}\Omega^1 M$. Clearly,

$$C^\infty(M;\Omega^s) \equiv x^s C^\infty(M;{}^{\mathrm{b}}\Omega^s)\ . \tag{6.1}$$

Thus the integration operator $\int_M$ is defined on $xC^\infty(M;{}^{\mathrm{b}}\Omega)$, and induces a pairing between $C^\infty(M)$ and $xC^\infty(M;{}^{\mathrm{b}}\Omega)$.

At the points of ∂M, the local section $x\partial_x$ is independent of the choice of adapted local coordinates, spanning a trivial line subbundle ${}^{\mathrm{b}}N\partial M \subset {}^{\mathrm{b}}T_{\partial M}M$ with $T\partial M = {}^{\mathrm{b}}T_{\partial M}M/{}^{\mathrm{b}}N\partial M$. So ${}^{\mathrm{b}}\Omega^s_{\partial M}M \equiv \Omega^s\partial M \otimes \Omega^s({}^{\mathrm{b}}N\partial M)$, and a restriction map $C^\infty(M;{}^{\mathrm{b}}\Omega^s) \to C^\infty(\partial M;\Omega^s)$ is locally given by

$$u = a(x,y)\left|\frac{dx}{x}dy\right|^s \mapsto u|_{\partial M} = a(0,y)\,|dy|^s\ .$$

A Euclidean structure g on ${}^{\mathrm{b}}TM$ is called a *b-metric*. Locally,

$$g = a_0\Big(\frac{dx}{x}\Big)^2 + 2\sum_{j=1}^{n-1} a_{0j}\,\frac{dx}{x}\,dy^j + \sum_{j,k=1}^{n-1} a_{jk}\,dy^j\,dy^k\ ,$$

where a_0, a_{0j} and a_{jk} are C^∞ functions, on condition that g is positive definite. If moreover $a_0 = 1 + O(x^2)$ and $a_{0j} = O(x)$ as $x \downarrow 0$, then g is called *exact*. In this case, the restriction of g to $\mathring{T} \equiv (0,\epsilon_0)\times\partial M$ is asymptotically cylindrical, and therefore $g|_{\mathring{M}}$ is complete. This restriction is of bounded geometry if it is cylindrical around the boundary; i.e., $g = (\frac{dx}{x})^2 + h$ on $\mathring{T}$ for (the pull-back via ϖ of) some Riemannian metric h on ∂M, taking ϵ_0 small enough; i.e., $a_0 = 1$ and $a_{0j} = 0$ using adapted local coordinates.

6.2. **Supported and extendible functions.** Let $\breve{M}$ be any closed manifold of dimension n which contains M as submanifold (for instance, $\breve{M}$ could be the double of M), and let $M' = \breve{M}\setminus\mathring{M}$, which is another compact n-submanifold with boundary of $\breve{M}$, with dimension n and $\partial M' = M \cap M' = \partial M$.

The concepts, notation and conventions of Section 2.4 have straightforward extensions to manifolds with boundary, like the Fréchet space $C^\infty(M)$. Its elements are called *extendible functions* because the continuous linear restriction map

$$R : C^\infty(\breve{M}) \to C^\infty(M) \tag{6.2}$$

is surjective; in fact, there is a continuous linear extension map $E : C^\infty(M) \to C^\infty(\breve{M})$ [34]. Since $C^\infty(\breve{M})$ and $C^\infty(M)$ are Fréchet spaces, the map (6.2) is open by the open mapping theorem, and therefore it is a surjective topological homomorphism. Its null space is $C^\infty_{M'}(\breve{M})$.

The Fréchet space of *supported* functions is the closed subspace of the smooth functions on M that vanish to all orders at the points of ∂M,

$$\dot{C}^\infty(M) = \bigcap_{m\ge 0} x^m C^\infty(M) \subset C^\infty(M)\ . \tag{6.3}$$

The extension by zero realizes $\dot{C}^\infty(M)$ as the closed subspace of functions on $\breve{M}$ supported in M,

$$\dot{C}^\infty(M) \equiv C^\infty_M(\breve{M}) \subset C^\infty(\breve{M})\ . \tag{6.4}$$

By (6.3),

$$x^m\dot{C}^\infty(M) = \dot{C}^\infty(M) \quad (m \in \mathbb{R})\ , \tag{6.5}$$

and therefore, by (6.1),

$$\dot{C}^\infty(M;{}^{\mathrm{b}}\Omega^s) \equiv \dot{C}^\infty(M;\Omega^s) \quad (s\in\mathbb{R}) . \tag{6.6}$$

We can similarly define Banach spaces $C^k(M)$ and $\dot{C}^k(M)$ ($k\in\mathbb{N}_0$) satisfying the analogs of (6.2)–(6.4), which in turn yield analogs of the first inclusions of (2.6), obtaining $C^\infty(M)=\bigcap_k C^k(M)$ and $\dot{C}^\infty(M)=\bigcap_k \dot{C}^k(M)$.

6.3. **Supported and extendible distributions.** The spaces of *supported* and *extendible* distributions on M are

$$\dot{C}^{-\infty}(M)=C^\infty(M;\Omega)' , \quad C^{-\infty}(M)=\dot{C}^\infty(M;\Omega)' .$$

Transposing the version of (6.2) with ΩM, we get [25, Proposition 3.2.1]

$$\dot{C}^{-\infty}(M)\equiv C^{-\infty}_M(\breve{M})\subset C^{-\infty}(\breve{M}) . \tag{6.7}$$

Similarly, (6.4) and (6.3) give rise to continuous linear restriction maps

$$R: C^{-\infty}(\breve{M})\to C^{-\infty}(M) , \tag{6.8}$$

$$R: \dot{C}^{-\infty}(M)\to C^{-\infty}(M) , \tag{6.9}$$

which are surjective by the Hahn-Banach theorem. Their null spaces are $C^{-\infty}_{M'}(\breve{M})=\dot{C}^{-\infty}(M')$ and $\dot{C}^{-\infty}_{\partial M}(M)$ [25, Proposition 3.3.1], respectively. According to (6.7), the map (6.9) is a restriction of (6.8). As a consequence of (6.7), there are continuous dense inclusions [25, Lemma 3.2.1]

$$C^\infty_{\mathrm{c}}(\mathring{M})\subset \dot{C}^\infty(M)\subset C^\infty(M)\subset \dot{C}^{-\infty}(M) , \tag{6.10}$$

the last one given by the integration pairing between $C^\infty(M)$ and $C^\infty(M;\Omega)$. The restriction of this pairing to $\dot{C}^\infty(M;\Omega)$ induces a continuous dense inclusion

$$C^\infty(M)\subset C^{-\infty}(M) . \tag{6.11}$$

Moreover (6.9) is the identity map on $C^\infty(M)$.

As before, from (6.5) and (6.6), we get

$$x^m C^{-\infty}(M)=C^{-\infty}(M) \quad (m\in\mathbb{R}) , \tag{6.12}$$

$$C^{-\infty}(M;{}^{\mathrm{b}}\Omega^s)\equiv C^{-\infty}(M;\Omega^s) \quad (s\in\mathbb{R}) . \tag{6.13}$$

The Banach spaces $C'^{-k}(M)$ and $\dot{C}'^{-k}(M)$ ($k\in\mathbb{N}_0$) are similarly defined and satisfy the analogs of (6.7)–(6.13). These spaces satisfy the analogs of the second inclusions of (2.6), obtaining $\bigcup_k C'^{-k}(M)=C^{-\infty}(M)$ and $\bigcup_k \dot{C}'^{-k}(M)=\dot{C}^{-\infty}(M)$.

6.4. **Supported and extendible Sobolev spaces.** The *supported* Sobolev space of order $s\in\mathbb{R}$ is the closed subspace of the elements supported in M,

$$\dot{H}^s(M)=H^s_M(\breve{M})\subset H^s(\breve{M}) . \tag{6.14}$$

On the other hand, using the map (6.9), the *extendible* Sobolev space of order s is $H^s(M)=R(H^s(\breve{M}))$ with the inductive topology given by

$$R: H^s(\breve{M})\to H^s(M) ; \tag{6.15}$$

i.e., this is a surjective topological homomorphism. Its null space is $H^s_{M'}(\breve{M})$. The analogs of (2.17)–(2.20) hold true in this setting using $\dot{C}^{\pm\infty}(M)$ and $C^{\pm\infty}(M)$. Furthermore the analogs of (2.17) are also compact operators because (6.14) is a closed embedding and (6.15) a surjective topological homomorphism.

The following properties are satisfied [25, Proposition 3.5.1]. $C^\infty(M)$ is dense in $H^s(M)$, we have

$$\dot H^s(M) \equiv H^{-s}(M;\Omega)' , \quad H^s(M) \equiv \dot H^{-s}(M;\Omega)' , \tag{6.16}$$

and the map (6.9) has a continuous restriction

$$R : \dot H^s(M) \to H^s(M) , \tag{6.17}$$

which is surjective if $s \le 1/2$, and injective if $s \ge -1/2$. In particular, $\dot H^0(M) \equiv H^0(M) \equiv L^2(M)$. The null space of (6.17) is $\dot H^s_{\partial M}(M)$.

Since $\dot H^s(M)$ and $H^s(M)$ form compact spectra of Hilbertian spaces, we get the following result.

Proposition 6.1. *$\dot C^{-\infty}(M)$ and $C^{-\infty}(M)$ are barreled, ultrabornological, webbed, acyclic DF Montel spaces, and therefore complete, boundedly retractive and reflexive.*

Proposition 6.2. *The maps* (6.8) *and* (6.9) *are surjective topological homomorphisms.*

Proof. We already know that these maps are linear, continuous and surjective. Since $C^{-\infty}(\breve M)$ is webbed, and $\dot C^{-\infty}(M)$ and $C^{-\infty}(M)$ are webbed and ultrabornological (Proposition 6.1), the stated maps are also open by the open mapping theorem [22, 7.35.3 (1)], [28, Exercise 14.202 (a)], [7, Section IV.5], [5]. □

6.5. **The space $\dot C^{-\infty}_{\partial M}(M)$.** Consider the sequences

$$0 \to \dot C^{-\infty}(M') \xrightarrow{\iota} C^{-\infty}(\breve M) \xrightarrow{R} C^{-\infty}(M) \to 0 , \tag{6.18}$$

$$0 \to \dot C^{-\infty}_{\partial M}(M) \xrightarrow{\iota} \dot C^{-\infty}(M) \xrightarrow{R} C^{-\infty}(M) \to 0 . \tag{6.19}$$

Proposition 6.2 has the following direct consequence.

Corollary 6.3. *The sequences* (6.18) *and* (6.19) *are exact sequences in the category of continuous linear maps between LCSs.*

From (6.7), we get

$$\dot C^{-\infty}_{\partial M}(M) \equiv C^{-\infty}_{\partial M}(\breve M) \subset C^{-\infty}(\breve M) . \tag{6.20}$$

The analogs of the second inclusion of (2.6), (2.17) and (2.19) for the spaces $\dot C'^{\,-k}(M)$ and $\dot H^s(M)$ yield corresponding analogs for the spaces $\dot C'^{\,-k}_{\partial M}(M)$ and $\dot H^s_{\partial M}(M)$. Thus the spaces $\dot C'^{\,-k}_{\partial M}(M)$ and $\dot H^s_{\partial M}(M)$ form spectra with the same union; the spectrum of spaces $\dot H^s_{\partial M}(M)$ is compact.

Proposition 6.4. *$\dot C^{-\infty}_{\partial M}(M)$ is a limit subspace of the LF-space $\dot C^{-\infty}(M)$.*

Proof. By Propositions 6.1 and 6.2, $\dot C^{-\infty}(M)/\dot C^{-\infty}_{\partial M}(M) \equiv C^{-\infty}(M)$ is acyclic, which is equivalent to the statement. □

The following analog of Proposition 6.1 hold true with the same arguments, applying Proposition 6.4 and using that the Hilbertian spaces $\dot H^s_{\partial M}(M)$ form a compact spectrum.

Corollary 6.5. *$\dot C^{-\infty}_{\partial M}(M)$ is barreled, ultrabornological, webbed acyclic DF Montel space, and therefore complete, boundedly retractive and reflexive.*

A description of $\dot C^{-\infty}_{\partial M}(M)$ will be indicated in Remark 7.9.

6.6. **Differential operators acting on $C^{-\infty}(M)$ and $\dot C^{-\infty}(M)$.** The notions of Section 2.7 also have straightforward extensions to manifolds with boundary. The action of any $A \in \operatorname{Diff}(M)$ on $C^\infty(M)$ preserves $\dot C^\infty(M)$. Taking the version of this property with ΩM, we get that A^t acts on $\dot C^\infty(M;\Omega)$ and $C^\infty(M;\Omega)$. Using the transpose again, we get extended continuous actions of A on $C^{-\infty}(M)$ and $\dot C^{-\infty}(M)$. They fit into commutative diagrams

$$
\begin{array}{ccc}
\dot C^{-\infty}(M) & \xrightarrow{A} & \dot C^{-\infty}(M) \\
{\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\
C^{-\infty}(M) & \xrightarrow{A} & C^{-\infty}(M)
\end{array}
\qquad
\begin{array}{ccc}
C^{-\infty}(M) & \xrightarrow{A} & C^{-\infty}(M) \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
C^{\infty}(M) & \xrightarrow{A} & C^{\infty}(M)\,.
\end{array}
\tag{6.21}
$$

However the analogous diagram

$$
\begin{array}{ccc}
\dot C^{-\infty}(M) & \xrightarrow{A} & \dot C^{-\infty}(M) \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
C^{\infty}(M) & \xrightarrow{A} & C^{\infty}(M)
\end{array}
\tag{6.22}
$$

may not be commutative. Let us use the notation $u \mapsto u_{\rm c}$ for the injection $C^\infty(M) \subset \dot C^{-\infty}(M)$ (see (6.10)). (Following Melrose, the subscript "c" stands for "cutoff at the boundary.") We have $A(u_{\rm c}) - (Au)_{\rm c} \in C^{-\infty}_{\partial M}(M)$ for all $u \in C^\infty(M)$ [25, Eq. (3.4.8)]. For instance, if $M = [x_0, x_1]$, where $x_0 < x_1$ in $\mathbb{R}$, and $A = \partial_x$, integration by parts gives

$$
\partial_x(u_{\rm c}) - (\partial_x u)_{\rm c} = u(x_1)\,\delta_{x_1} - u(x_0)\,\delta_{x_0}
$$

for all $u \in C^\infty([x_0, x_1])$, using the Dirac mass at x_j ($j = 0, 1$).

Using (6.2) and its version for vector fields, we get a surjective restriction map

$$
\operatorname{Diff}(\breve M) \to \operatorname{Diff}(M)\,, \quad \breve A \mapsto \breve A|_M\,. \tag{6.23}
$$

For any $\breve A \in \operatorname{Diff}(\breve M)$ with $\breve A|_M = A$, we have the commutative diagrams

$$
\begin{array}{ccc}
C^{-\infty}(\breve M) & \xrightarrow{\breve A} & C^{-\infty}(\breve M) \\
{\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\
C^{-\infty}(M) & \xrightarrow{A} & C^{-\infty}(M)\,,
\end{array}
\qquad
\begin{array}{ccc}
C^{-\infty}(\breve M) & \xrightarrow{\breve A} & C^{-\infty}(\breve M) \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
\dot C^{-\infty}(M) & \xrightarrow{A} & \dot C^{-\infty}(M)\,,
\end{array}
\tag{6.24}
$$

where the left-hand side square extends the left-hand side square of (6.21).

If $A \in \operatorname{Diff}^m(M)$ ($m \in \mathbb{N}_0$), its actions on $\dot C^{-\infty}(M)$ and $C^{-\infty}(M)$ define continuous linear maps,

$$
A : \dot H^s(M) \to \dot H^{s-m}(M)\,, \quad A : H^s(M) \to H^{s-m}(M)\,. \tag{6.25}
$$

The maps (6.17) and (6.25) fit into a commutative diagram given by the left-hand side square of (6.21).

6.7. **Differential operators tangent to the boundary.** The concepts of Section 4 can be generalized to the case with boundary when $L = \partial M$ [25, Chapter 6] (see also [24, Section 4.9]), giving rise to the Lie subalgebra and $C^\infty(M)$-submodule $\mathfrak{X}_{\rm b}(M) \subset \mathfrak{X}(M)$ of vector fields tangent to ∂M, called *b-vector fields*. There is a canonical identity $\mathfrak{X}_{\rm b}(M) \equiv C^\infty(M; {}^{\rm b}TM)$. Using $\mathfrak{X}_{\rm b}(M)$ like in Section 2.7, we get the filtered $C^\infty(M)$-submodule and filtered subalgebra $\operatorname{Diff}_{\rm b}(M) \subset \operatorname{Diff}(M)$ of *b-differential operators*. It consists of the operators $A \in \operatorname{Diff}(M)$ such that (6.22) is commutative [25, Exercise 3.4.20]. The extension of $\operatorname{Diff}_{\mathrm{b}}(M)$ to arbitrary vector bundles is closed by taking transposes and formal adjoints.

Clearly, the restriction map (6.23) satisfies

$$\operatorname{Diff}(\breve{M}, \partial M)|_M = \operatorname{Diff}_{\mathrm{b}}(M) . \tag{6.26}$$

For all $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have [25, Eqs. (4.2.7) and (4.2.8)]

$$\operatorname{Diff}^k_{\mathrm{b}}(M)\, x^a = x^a \operatorname{Diff}^k_{\mathrm{b}}(M) . \tag{6.27}$$

Since $\operatorname{Diff}(M)$ is spanned by ∂_x and $\operatorname{Diff}_{\mathrm{b}}(M)$ as algebra, it follows that

$$\operatorname{Diff}^k(M)\, x^a \subset x^{a-k} \operatorname{Diff}^k(M) . \tag{6.28}$$

6.8. **Conormal distributions at the boundary.** The spaces of *supported* and *extendible* conormal distributions at the boundary of Sobolev order $s \in \mathbb{R}$ are the $C^\infty(M)$-modules and LCSs

$$\dot{\mathcal{A}}^{(s)}(M) = \{\, u \in \dot{C}^{-\infty}(M) \mid \operatorname{Diff}_{\mathrm{b}}(M)\, u \subset \dot{H}^s(M)\,\} ,$$
$$\mathcal{A}^{(s)}(M) = \{\, u \in C^{-\infty}(M) \mid \operatorname{Diff}_{\mathrm{b}}(M)\, u \subset H^s(M)\,\} ,$$

with the projective topologies given by the maps $P : \dot{\mathcal{A}}^{(s)}(M) \to H^s(M)$ and $P : \mathcal{A}^{(s)}(M) \to \dot{H}^s(M)$ ($P \in \operatorname{Diff}_{\mathrm{b}}(M)$). They satisfy the analogs of the continuous inclusions (4.4), giving rise to the filtered $C^\infty(M)$-modules and LF-spaces of *supported* and *extendible* conormal distributions at the boundary,

$$\dot{\mathcal{A}}(M) = \bigcup_s \dot{\mathcal{A}}^{(s)}(M) , \quad \mathcal{A}(M) = \bigcup_s \mathcal{A}^{(s)}(M) . \tag{6.29}$$

By definition, there are continuous inclusions

$$\dot{\mathcal{A}}(M) \subset \dot{C}^{-\infty}(M) , \quad \mathcal{A}(M) \subset C^{-\infty}(M) . \tag{6.30}$$

Thus $\dot{\mathcal{A}}(M)$ and $\mathcal{A}(M)$ are Hausdorff.

The following analogs of Propositions 3.1 and 4.1 hold true with formally the same proofs.

Proposition 6.6. *$\dot{\mathcal{A}}^{(s)}(M)$ and $\mathcal{A}^{(s)}(M)$ are totally reflexive Fréchet spaces.*

Corollary 6.7. *$\dot{\mathcal{A}}(M)$ and $\mathcal{A}(M)$ are barreled, ultrabornological and webbed.*

We have

$$\bigcap_s \dot{\mathcal{A}}^{(s)}(M) = \dot{C}^\infty(M) , \quad \bigcap_s \mathcal{A}^{(s)}(M) = C^\infty(M) , \tag{6.31}$$

obtaining continuous dense inclusions [25, Proposition 4.1.1 and Lemma 4.6.1]

$$\dot{C}^\infty(M) \subset \dot{\mathcal{A}}(M) , \quad C^\infty(M) \subset \mathcal{A}(M) . \tag{6.32}$$

By elliptic regularity, we also get continuous inclusions [25, Eq. (4.1.4)]

$$\dot{\mathcal{A}}(M)|_{\mathring{M}}, \mathcal{A}(M) \subset C^\infty(\mathring{M}) . \tag{6.33}$$

Using (6.9), (6.15) and the commutativity of the left-hand side square of (6.21), we get the continuous linear restriction maps

$$R : \dot{\mathcal{A}}^{(s)}(M) \to \mathcal{A}^{(s)}(M) , \tag{6.34}$$

which are surjective for $s \leq 1/2$ and injective for $s \geq -1/2$ because this is true for the maps (6.17). From (6.3), (6.32) and (6.34) for $s = 0$, we get a continuous inclusion $C^\infty(M) \subset \dot{\mathcal{A}}^{(0)}(M)$, yielding a continuous inclusion [25, Proposition 4.1.1]

$$C^\infty(M) \subset \dot{\mathcal{A}}(M) \; . \tag{6.35}$$

The maps (6.34) induce a surjective continuous linear restriction map

$$R : \dot{\mathcal{A}}(M) \to \mathcal{A}(M) \; , \tag{6.36}$$

with $R = 1$ on $C^\infty(M)$ [25, Proposition 4.1.1]. The surjectivity of (6.36) is a consequence of the existence of enough partial extension maps [25, Section 4.4]; the precise statement is recalled in Proposition 6.29 for later use. The maps (6.34) and (6.36) are restrictions of (6.9).

The following analog of Proposition 6.2 holds true with formally the same proof, using that $\dot{\mathcal{A}}(M)$ is webbed and $\mathcal{A}(M)$ ultrabornological (Corollary 6.7).

Proposition 6.8. *The map* (6.36) *is a surjective topological homomorphism.*

Corollary 6.9. *The inclusion* (6.35) *is dense.*

6.9. **The spaces $x^m L^\infty(M)$.** For $m \in \mathbb{R}$, consider the weighted space $x^m L^\infty(M)$ (Section 2.11). From (6.12) and since $L^\infty(M) \subset C^{-\infty}(M)$, it follows that there is a continuous inclusion

$$x^m L^\infty(M) \subset C^{-\infty}(M) \; .$$

Moreover, for $m' < m$, from $x^{m-m'} \in L^\infty(M)$, we easily get a continuous inclusion

$$x^m L^\infty(M) \subset x^{m'} L^\infty(M) \; . \tag{6.37}$$

Proposition 6.10. *For $m' < m$, $C^\infty_{\mathrm{c}}(\mathring{M})$ is dense in $x^m L^\infty(M)$ with the topology of $x^{m'} L^\infty(M)$.*

Proof. Given $u \in x^m L^\infty(M)$ and $\epsilon > 0$, let B be the ball in $x^{m'} L^\infty(M)$ of center u and radius ϵ. Let $S = \sup_M x^{m-m'} > 0$. Since $C^\infty(M)$ is dense in $L^\infty(M)$, there is some $f \in C^\infty(M)$ so that $|f - x^{-m} u| < \min\{\epsilon/2, \epsilon/S\}$ (Lebesgue-) almost everywhere. (Recall that the sets of Lebesgue measure zero are well-defined in any C^1 manifold [12, Lemma 3.1.1].) There is some $0 < \delta < 1$ so that $\delta x^{-m}|u| < \epsilon/4$ almost everywhere. Take some $\lambda \in C^\infty(\mathbb{R})$ so that $\lambda \geq 0$, $\lambda(r) \leq r^m$ if $r > 0$, $\lambda(r) = 0$ if $r^{m-m'} \leq \delta/2$, and $\lambda(r) = r^m$ if $r^{m-m'} \geq \delta$. Let $h = \lambda(x) f \in C^\infty_{\mathrm{c}}(\mathring{M})$. If $x^{m-m'} \leq \delta$, then, almost everywhere,

$$\begin{aligned} x^{-m'}|h - u| &\leq \delta x^{-m}(|h| + |u|) \leq \delta(|f| + x^{-m}|u|) \\ &\leq \delta(|f - x^{-m}u| + 2x^{-m}|u|) < \delta\Big(\frac{\epsilon}{2} + \frac{\epsilon}{2\delta}\Big) < \epsilon \; . \end{aligned}$$

If $x^{m-m'} \geq \delta$, then, almost everywhere,

$$x^{-m'}|h - u| = x^{m-m'}|x^{-m}\lambda(x) f - x^{-m}u| \leq S|f - x^{-m}u| < \epsilon \; .$$

Thus $h \in B \cap C^\infty_{\mathrm{c}}(\mathring{M})$. $\square$

6.10. **Filtration of $\mathcal{A}(M)$ by bounds.** For every $m \in \mathbb{R}$, let

$$\mathcal{A}^m(M) = \{\, u \in C^{-\infty}(M) \mid \operatorname{Diff}_{\mathrm{b}}(M)\, u \subset x^m L^\infty(M) \,\} .$$

This is another $C^\infty(M)$-module and LCS, with the projective topology given by the maps $P : \mathcal{A}^m(M) \to x^m L^\infty(M)$ ($P \in \operatorname{Diff}_{\mathrm{b}}(M)$).

Example 6.11 ([25, Exercises 4.2.23 and 4.2.24])**.** Via the injection of $\mathbb{R}^l$ into its stereographic compactification $\mathbb{S}^l_+ = \{\, x \in \mathbb{S}^l \mid x^{l+1} \geq 0 \,\}$, the space $\mathcal{A}^{-m}(\mathbb{S}^l_+)$ corresponds to the symbol space $S^m(\mathbb{R}^l)$ (Section 3).

Note that (6.37) yields a continuous inclusion

$$\mathcal{A}^m(M) \subset \mathcal{A}^{m'}(M) \quad (m' < m) . \tag{6.38}$$

Moreover there are continuous inclusions [25, Proof of Proposition 4.2.1]

$$\mathcal{A}^{(s)}(M) \subset \mathcal{A}^m(M) \subset \mathcal{A}^{(\min\{m,0\})}(M) \quad (m < s - n/2 - 1) . \tag{6.39}$$

Hence

$$\mathcal{A}(M) = \bigcup_m \mathcal{A}^m(M) . \tag{6.40}$$

Despite of defining the same LF-space, the filtrations of $\mathcal{A}(M)$ defined by the spaces $\mathcal{A}^{(s)}(M)$ and $\mathcal{A}^m(M)$ are not equivalent because, in contrast with (6.31),

$$\dot{C}^\infty(M) = \bigcap_m \mathcal{A}^m(M) .$$

Let $\{\, P_j \mid j \in \mathbb{N}_0 \,\}$ be a countable $C^\infty(M)$-spanning set of $\operatorname{Diff}_{\mathrm{b}}(M)$. The topology of $\mathcal{A}^m(M)$ can be described by the semi-norms $\|{\cdot}\|_{k,m}$ ($k \in \mathbb{N}_0$) given by

$$\|u\|_{k,m} = \|P_k u\|_{x^m L^\infty} = \operatorname*{ess\,sup}_M \left|x^{-m} P_k u\right| = \sup_{\mathring{M}} \left|x^{-m} P_k u\right| , \tag{6.41}$$

using (6.33) in the last expression. From (2.1) and (6.33), we also get the continuous semi-norms $\|{\cdot}\|_{K,k,m}$ (for any compact $K \subset \mathring{M}$ and $k \in \mathbb{N}_0$) on $\mathcal{A}^m(M)$ given by

$$\|u\|_{K,k,m} = \sup_K |P_k u| . \tag{6.42}$$

Other continuous semi-norms $\|{\cdot}\|'_{k,m}$ ($k \in \mathbb{N}_0$) on $\mathcal{A}^m(M)$ are defined by

$$\|u\|'_{k,m} = \lim_{\epsilon \downarrow 0} \sup_{\{0<x<\epsilon\}} \left|x^{-m} P_k u\right| . \tag{6.43}$$

The proofs of the following results are similar to the proofs of Propositions 3.2 and 3.3 and Corollaries 3.4 to 3.6, using (6.33).

Proposition 6.12. *The semi-norms* (6.42) *and* (6.43) *together describe the topology of $\mathcal{A}^m(M)$.*

Proposition 6.13. *For $m, m', k \in \mathbb{N}_0$, if $m' < m$, then $\|{\cdot}\|'_{k,m'} = 0$ on $\mathcal{A}^m(M)$.*

Corollary 6.14. *If $m' < m$, then the topologies of $\mathcal{A}^{m'}(M)$ and $C^\infty(\mathring{M})$ coincide on $\mathcal{A}^m(M)$. Therefore the topologies of $\mathcal{A}(M)$ and $C^\infty(\mathring{M})$ coincide on $\mathcal{A}^m(M)$.*

Corollary 6.15. *For $m' < m$, $C^\infty_{\mathrm{c}}(\mathring{M})$ is dense in $\mathcal{A}^m(M)$ with the topology of $\mathcal{A}^{m'}(M)$. Therefore $C^\infty_{\mathrm{c}}(\mathring{M})$ is dense in $\mathcal{A}(M)$.*

Corollary 6.16. *$\mathcal{A}(M)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

Remark 6.17. Proposition 6.10 provides an alternative direct proof of Corollary 6.15. Actually, it will be shown that $C^\infty_{\mathrm{c}}(\mathring{M})$ is dense in every $\mathcal{A}^m(M)$ with its own topology (Corollary 6.39 and Remark 6.41), reconfirming Corollary 6.9.

The obvious analog of Remark 3.8 makes sense for (6.33) and Corollary 6.14.

6.11. **$\dot{\mathcal{A}}(M)$ and $\mathcal{A}(M)$ vs $I(\breve{M},\partial M)$.** Using (6.8), (6.15), (6.26) and the commutativity of the left-hand side square of (6.24), we get continuous linear restriction maps

$$R : I^{(s)}(\breve{M},\partial M) \to \mathcal{A}^{(s)}(M) ,$$

which induce a continuous linear restriction map

$$R : I(\breve{M},\partial M) \to \mathcal{A}(M) . \tag{6.44}$$

By (6.14), (6.26) and the commutativity of the right-hand side square of (6.24), we get the TVS-identities

$$\dot{\mathcal{A}}^{(s)}(M) \equiv I^{(s)}_M(\breve{M},\partial M) , \tag{6.45}$$

inducing a continuous linear isomorphism

$$\dot{\mathcal{A}}(M) \xrightarrow{\cong} I_M(\breve{M},\partial M) . \tag{6.46}$$

By (6.46) and Proposition 6.8, the map (6.44) is also surjective. Then the following analog of Proposition 6.2 follows with formally the same proof, using that $I(\breve{M},\partial M)$ is webbed (Corollary 4.2) and $\mathcal{A}(M)$ ultrabornological (Corollary 6.7).

Proposition 6.18. *The map* (6.44) *is a surjective topological homomorphism.*

The null space of (6.44) is $I_{M'}(\breve{M},\partial M)$. The following analog of Proposition 6.4 follows with formally the same proof, using Proposition 6.18 and Corollary 6.16.

Proposition 6.19. *$I_M(\breve{M},\partial M)$ is a limit subspace of the LF-space $I(\breve{M},\partial M)$.*

Corollary 6.20. *The map* (6.46) *is a TVS-isomorphism.*

Proof. Apply (6.29), (6.45) and Proposition 6.19. □

6.12. **Filtration of $\dot{\mathcal{A}}(M)$ by the symbol order.** Inspired by (6.45), let

$$\dot{\mathcal{A}}^m(M) = I^m_M(\breve{M},\partial M) \subset I^m(\breve{M},\partial M) \quad (m \in \mathbb{R}) , \tag{6.47}$$

which are closed subspaces satisfying the analogs of (4.11) and (4.12). Thus

$$\dot{\mathcal{A}}(M) = \bigcup_m \dot{\mathcal{A}}^m(M) , \quad \dot{C}^\infty(M) = \bigcap_m \dot{\mathcal{A}}^m(M) ,$$

and the TVS-isomorphism (6.46) is also compatible with the symbol filtration.

The following is a consequence of Corollary 4.5 applied to $(\breve{M},\partial M)$.

Corollary 6.21. *For $m < m', m''$, the topologies of $\dot{\mathcal{A}}^{m'}(M)$ and $\dot{\mathcal{A}}^{m''}(M)$ coincide on $\dot{\mathcal{A}}^m(M)$.*

The following result follows like Corollary 3.6, applying Corollary 6.21 and using that $\dot{\mathcal{A}}(M)$ is barreled (Corollary 6.7) and a closed subspace of the Montel space $I(\breve{M},\partial M)$ (Corollary 4.7).

Corollary 6.22. *$\dot{\mathcal{A}}(M)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

6.13. **The space $\mathcal{K}(M)$.** Using the condition of being supported in ∂M, define the LCHSs and $C^\infty(M)$-modules

$$\mathcal{K}^{(s)}(M) = \dot{\mathcal{A}}^{(s)}_{\partial M}(M)\ , \quad \mathcal{K}^m(M) = \dot{\mathcal{A}}^m_{\partial M}(M)\ , \quad \mathcal{K}(M) = \dot{\mathcal{A}}_{\partial M}(M)\ .$$

These are closed subspaces of $\dot{\mathcal{A}}^{(s)}(M)$, $\dot{\mathcal{A}}^m(M)$ and $\dot{\mathcal{A}}(M)$, respectively; more precisely, they are the null spaces of the corresponding restrictions of the map (6.36). They satisfy the analogs of (4.4), (4.11) and (4.12). So

$$\bigcup_s \mathcal{K}^{(s)}(M) = \bigcup_m \mathcal{K}^m(M)\ .$$

The term *conormal sequence at the boundary* of M will be used for

$$0 \to \mathcal{K}(M) \xrightarrow{\iota} \dot{\mathcal{A}}(M) \xrightarrow{R} \mathcal{A}(M) \to 0\ . \tag{6.48}$$

Proposition 6.8 has the following direct consequence.

Corollary 6.23. *The conormal sequence at the boundary of M is exact in the category of continuous linear maps between LCSs.*

The following analog of Proposition 6.4 holds true with formally the same proof, using Proposition 6.8 and Corollary 6.16.

Proposition 6.24. *$\mathcal{K}(M)$ is a limit subspace of the LF-space $\dot{\mathcal{A}}(M)$.*

From the definition of $\dot{\mathcal{A}}^{(s)}(M)$ (Section 6.8), we get

$$\mathcal{K}^{(s)}(M) = \{\, u \in \dot{C}^{-\infty}_{\partial M}(M) \mid \mathrm{Diff}_{\mathrm{b}}(M)\, u \subset \dot{H}^s_{\partial M}(M)\,\}\ ,$$

with the projective topology given by the maps $P : \mathcal{K}^{(s)}(M) \to \dot{H}^s_{\partial M}(M)$ ($P \in \mathrm{Diff}_{\mathrm{b}}(M)$). Hence the following analogs of Propositions 3.1 and 4.1 hold true with formally the same proofs.

Proposition 6.25. *$\mathcal{K}^{(s)}(M)$ is a totally reflexive Fréchet space.*

Corollary 6.26. *$\mathcal{K}(M)$ is barreled, ultrabornological and webbed.*

Now the following analogs of Corollaries 6.21 and 6.22 hold true with formally the same proofs, using Corollaries 6.21, 6.22 and 6.26.

Corollary 6.27. *For $m < m', m''$, the topologies of $\mathcal{K}^{m'}(M)$ and $\mathcal{K}^{m''}(M)$ coincide on $\mathcal{K}^m(M)$.*

Corollary 6.28. *$\mathcal{K}(M)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

By Corollary 6.20,

$$\mathcal{K}(M) \equiv I_{\partial M}(\breve{M}, \partial M)\ , \tag{6.49}$$

which restricts to identities between the spaces defining the Sobolev and symbol order filtrations, according to (6.45) and (6.47).

A description of $\mathcal{K}^{(s)}(M)$ and $\mathcal{K}(M)$ will be indicated in Remark 7.28.

6.14. **Action of** $\operatorname{Diff}(M)$ **on** $\dot{\mathcal{A}}(M)$**,** $\mathcal{A}(M)$ **and** $\mathcal{K}(M)$**.** According to Section 4.5, and using (6.26), (6.45), Proposition 6.18 and locality, any $A \in \operatorname{Diff}(M)$ defines continuous endomorphisms A of $\dot{\mathcal{A}}(M)$, $\mathcal{A}(M)$ and $\mathcal{K}(M)$. If $A \in \operatorname{Diff}^k(M)$, these maps also satisfy the analogs of (4.16). If $A \in \operatorname{Diff}_{\mathrm{b}}(M)$, it clearly defines continuous endomorphisms of $\dot{\mathcal{A}}^{(s)}(M)$, $\mathcal{A}^{(s)}(M)$, $\mathcal{A}^m(M)$ and $\mathcal{K}^{(s)}(M)$.

According to Section 6.6, (6.30), (6.32) and (6.33), the maps of this subsection are restrictions of the endomorphisms A of $\dot{C}^{-\infty}(M)$, $C^{-\infty}(M)$ and $C^\infty(\mathring{M})$, and extensions of the endomorphisms A of $\dot{C}^\infty(M)$ and $C^\infty(M)$.

6.15. **Partial extension maps.** Given linear subspaces, $X \subset \mathcal{A}(M)$ and $Y \subset \dot{\mathcal{A}}(M)$, a map $E : X \to Y$ is called a *partial extension map* if $R(Y) \subset X$ and $RE = 1$ on X.

Proposition 6.29 (Cf. [25, Section 4.4])**.** *For all $m \in \mathbb{R}$, there is a continuous linear partial extension map $E_m : \mathcal{A}^m(M) \to \dot{\mathcal{A}}^{(s)}(M)$, where $s = 0$ if $m \geq 0$, and $m > s \in \mathbb{Z}^-$ if $m < 0$. For $m \geq 0$, $E_m : \mathcal{A}^m(M) \to \dot{\mathcal{A}}^{(0)}(M)$ is a continuous inclusion map.*

Remark 6.30. By (6.39) and Proposition 6.29, for any $A \in \operatorname{Diff}^k(M)$, the endomorphism A of $\mathcal{A}(M)$ (Section 6.14) is induced by the continuous linear compositions

$$\mathcal{A}^m(M) \xrightarrow{E_m} \dot{\mathcal{A}}^{(s)}(M) \xrightarrow{A} \dot{\mathcal{A}}^{(s-k)}(M) \xrightarrow{R} \mathcal{A}^{(s-k)}(M) \subset \mathcal{A}^{m'-k}(M) ,$$

where $m' = s - n/2 - 1$ for m and s satisfying the conditions of Proposition 6.29.

6.16. L^2 **half-b-densities.** By (6.1),

$$\begin{aligned} C^\infty(M; \Omega^{-\frac{1}{2}} \otimes {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) &\equiv C^\infty(M; \Omega^{-\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty(M; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \\ &\equiv C^\infty(M; \Omega^{-\frac{1}{2}}) \otimes_{C^\infty(M)} x^{-\frac{1}{2}} C^\infty(M; \Omega^{\frac{1}{2}}) \\ &\equiv x^{-\frac{1}{2}} C^\infty(M; \Omega^{-\frac{1}{2}} \otimes \Omega^{\frac{1}{2}}) \equiv x^{-\frac{1}{2}} C^\infty(M) . \end{aligned}$$

So

$$\begin{aligned} L^2(M; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) &\equiv L^2(M; \Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty(M; \Omega^{-\frac{1}{2}} \otimes {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \\ &\equiv L^2(M; \Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} x^{-\frac{1}{2}} C^\infty(M) \equiv x^{-\frac{1}{2}} L^2(M; \Omega^{\frac{1}{2}}) . \end{aligned} \tag{6.50}$$

This is an identity of Hilbert spaces, using the weighted L^2 space structure of $x^{-1/2} L^2(M; \Omega^{1/2})$ (Section 2.11) and the Hilbert space structure on $L^2(M; {}^{\mathrm{b}}\Omega^{1/2})$ induced by the canonical identity

$$L^2(\mathring{M}; \Omega^{\frac{1}{2}}) \equiv L^2(M; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) . \tag{6.51}$$

6.17. L^∞ **half-b-densities.** Like in (6.50), we get

$$L^\infty(M; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \equiv x^{-\frac{1}{2}} L^\infty(M; \Omega^{\frac{1}{2}}) , \tag{6.52}$$

as LCSs endowed with a family of equivalent Banach space norms.

Equip M with a b-metric g (Section 6.1), and endow $\mathring{M}$ with the restriction of g, also denoted by g. With the corresponding Euclidean/Hermitean structures on $\Omega^{1/2}\mathring{M}$ and ${}^{\mathrm{b}}\Omega^{1/2}M$, we have the identity of Banach spaces

$$L^\infty(\mathring{M}; \Omega^{\frac{1}{2}}) \equiv L^\infty(M; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}) . \tag{6.53}$$

6.18. **b-Sobolev spaces.** For $m \in \mathbb{N}_0$, the *b-Sobolev spaces* of *order* $\pm m$ are defined by the following analogs of (2.16), (2.21) and (2.22):

$$H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) = \{\, u \in L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \mid \mathrm{Diff}_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}})\, u \subset L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \,\}\ ,$$
$$\mathrm{Diff}_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}})\, L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) = H_{\mathrm{b}}^{-m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) = H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}})'\ .$$

These are $C^\infty(M)$-modules and Hilbertian spaces with no canonical choice of a scalar product in general; we can use any finite set of $C^\infty(M)$-generators of $\mathrm{Diff}_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{1/2})$ to define a scalar product on $H_{\mathrm{b}}^{\pm m}(M;{}^{\mathrm{b}}\Omega^{1/2})$. The intersections and unions of the b-Sobolev spaces are denoted by $H_{\mathrm{b}}^{\pm\infty}(M;{}^{\mathrm{b}}\Omega^{1/2})$. In particular, $H_{\mathrm{b}}^{\infty}(M;{}^{\mathrm{b}}\Omega^{1/2}) = \mathcal{A}^{(0)}(M;{}^{\mathrm{b}}\Omega^{1/2})$.

6.19. **Weighted b-Sobolev spaces.** We will also use the *weighted b-Sobolev space* $x^a H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{1/2})$ ($a \in \mathbb{R}$), which is another Hilbertian space with no canonical choice of a scalar product; given a scalar product on $H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{1/2})$ with norm $\|\cdot\|_{H_{\mathrm{b}}^{m}}$, we get a scalar product on $x^a H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{1/2})$ with norm $\|\cdot\|_{x^a H_{\mathrm{b}}^{m}}$, like in Section 2.11. Observe that

$$\bigcap_{a,m} x^a H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) = \dot{C}^\infty(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}})\ .$$

6.20. **Action of $\mathrm{Diff}_{\mathrm{b}}^{m}(M)$ on weighted b-Sobolev spaces.** Like in (2.13),

$$\mathrm{Diff}_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \equiv \mathrm{Diff}_{\mathrm{b}}^{m}(M) \equiv \mathrm{Diff}_{\mathrm{b}}^{m}(M;\Omega^{\frac{1}{2}})\ . \tag{6.54}$$

By (6.27), for all $k \in \mathbb{N}_0$, $m \in \mathbb{Z}$ and $a \in \mathbb{R}$, any $A \in \mathrm{Diff}_{\mathrm{b}}^{k}(M;{}^{\mathrm{b}}\Omega^{1/2})$ defines a continuous linear map [24, Lemma 5.14]

$$A : x^a H_{\mathrm{b}}^{m}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \to x^a H_{\mathrm{b}}^{m-k}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}})\ .$$

Thus it induces a continuous endomorphism A of $x^a H_{\mathrm{b}}^{\pm\infty}(M;{}^{\mathrm{b}}\Omega^{1/2})$.

6.21. **A description of $\mathcal{A}(M)$.** In this subsection, unless the contrary is indicated, assume the following properties:

(A) $\mathring{M}$ is of bounded geometry with g.
(B) The collar neighborhood T of ∂M can be chosen so that:
   (a) every $A \in \mathfrak{X}(\partial M)$ has an extension $A' \in \mathfrak{X}_{\mathrm{b}}(T)$ such that A' is ϖ-projectable to A, and $A'|_{\mathring{T}}$ is orthogonal to the ϖ-fibers; and
   (b) $\mathfrak{X}_{\mathrm{ub}}(\mathring{M})|_{\mathring{T}}$ is $C_{\mathrm{ub}}^\infty(\mathring{M})|_{\mathring{T}}$-generated by $x\partial_x$ and the restrictions $A'|_{\mathring{T}}$ of the vector fields A' of (a), for $A \in \mathfrak{X}(\partial M)$.

For instance, these properties hold if $\mathring{T}$ is cylindrical with g (Section 6.1).

Lemma 6.31. *$\mathfrak{X}_{\mathrm{b}}(M)|_T$ is $C^\infty(M)|_T$-generated by $x\partial_x$ and the vector fields A' of* (B)*, for $A \in \mathfrak{X}(\partial M)$.*

Proof. For every $A \in \mathfrak{X}(\partial M)$, there is a unique $A'' \in \mathfrak{X}(T)$ such that A'' is ϖ-projectable to A and $dx(A'') = 0$. Since $A' - A''$ is tangent to the ϖ-fibers and vanishes on ∂M, we have $A' - A'' = fx\partial_x$ for some $f \in C^\infty(T)$. Then the result follows because $\mathfrak{X}_{\mathrm{b}}(M)|_T$ is $C^\infty(M)|_T$-spanned by $x\partial_x$ and the vector fields A''. □

Consider the notation of Sections 2.12.2 to 2.12.4 for $\mathring{M}$ with g.

Corollary 6.32. *The restriction to $\mathring{M}$ defines a continuous injection $C^\infty(M) \subset C_{\mathrm{ub}}^\infty(\mathring{M})$; in particular, $C_{\mathrm{ub}}^\infty(\mathring{M})$ becomes a $C^\infty(M)$-module.*

Proof. It is enough to work on a collar neighborhood T of the boundary satisfying (A) and (B). But, by (2.23), (b) and Lemma 6.31, the restriction to $\mathring{T}$ defines an injection of $C^\infty(M)|_T$ into $C^\infty_{\mathrm{ub}}(\mathring{M})|_{\mathring{T}}$. □

Proposition 6.33. *There is a canonical identity of $C^\infty_{\mathrm{ub}}(\mathring{M})$-modules,*

$$\mathrm{Diff}^m_{\mathrm{ub}}(\mathring{M}) \equiv \mathrm{Diff}^m_{\mathrm{b}}(M) \otimes_{C^\infty(M)} C^\infty_{\mathrm{ub}}(\mathring{M}) \ .$$

Proof. We have to prove that $\mathrm{Diff}^m_{\mathrm{ub}}(\mathring{M})$ is $C^\infty_{\mathrm{ub}}(\mathring{M})$-spanned by $\mathrm{Diff}^m_{\mathrm{b}}(M)$. It is enough to consider the case $m = 1$ because the filtered algebra $\mathrm{Diff}_{\mathrm{ub}}(\mathring{M})$ (respectively, $\mathrm{Diff}_{\mathrm{b}}(M)$) is spanned by $\mathrm{Diff}^1_{\mathrm{ub}}(\mathring{M})$ (respectively, $\mathrm{Diff}^1_{\mathrm{b}}(M)$). Moreover it is clearly enough to work on a collar neighborhood T of the boundary satisfying (A) and (B). By (b), Lemma 6.31 and Corollary 6.32, the restriction to $\mathring{T}$ defines an injection of $\mathrm{Diff}^1_{\mathrm{b}}(M)|_T$ as a $C^\infty_{\mathrm{ub}}(\mathring{M})|_{\mathring{T}}$-spanning subset of $\mathrm{Diff}^1_{\mathrm{ub}}(\mathring{M})|_{\mathring{T}}$. □

Corollary 6.34. *There is a canonical identity of $C^\infty_{\mathrm{ub}}(\mathring{M})$-modules,*

$$\mathrm{Diff}^m_{\mathrm{ub}}(\mathring{M};\Omega^{\frac{1}{2}}) \equiv \mathrm{Diff}^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty_{\mathrm{ub}}(\mathring{M}) \ .$$

Proof. This follows from (2.24) for $\mathring{M}$, (6.54) and Proposition 6.33. □

Corollary 6.35. $H^m(\mathring{M};\Omega^{1/2}) \equiv H^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$ *($m \in \mathbb{Z}$) as $C^\infty(M)$-modules and Hilbertian spaces, and therefore $H^{\pm\infty}(\mathring{M};\Omega^{1/2}) \equiv H^{\pm\infty}_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$.*

Proof. We show the case where $m \geq 0$, and the case where $m < 0$ follows by taking dual spaces. For any $m \in \mathbb{N}_0$, let $\{P_k\}$ be a finite $C^\infty(M)$-spanning set of $\mathrm{Diff}^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$, which is also a $C^\infty_{\mathrm{ub}}(\mathring{M})$-spanning set of $\mathrm{Diff}^m_{\mathrm{ub}}(\mathring{M};\Omega^{1/2})$ by Corollary 6.34. Then, by (6.51),

$$\begin{aligned} H^m(\mathring{M};\Omega^{\frac{1}{2}}) &= \{\, u \in L^2(\mathring{M};\Omega^{\frac{1}{2}}) \mid P_k u \in L^2(\mathring{M};\Omega^{\frac{1}{2}}) \ \forall k \,\} \\ &\equiv \{\, u \in L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \mid P_k u \in L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ \forall k \,\} = H^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ . \end{aligned}$$

Moreover $\{P_k\}$ can be used to define scalar products on both $H^m(\mathring{M};\Omega^{1/2})$ and $H^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$, obtaining that the above identity is unitary. □

Proposition 6.36. $\mathcal{A}^m(M;\Omega^{1/2}) \equiv x^{m+1/2} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$ $(m \in \mathbb{R})$.

Proof. By (6.52), (6.53), Proposition 2.5 and Corollary 6.35, we get the following identities and continuous inclusions:

$$\begin{aligned} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) &\equiv H^\infty(\mathring{M};\Omega^{\frac{1}{2}}) \subset C^\infty_{\mathrm{ub}}(\mathring{M};\Omega^{\frac{1}{2}}) \\ &\subset L^\infty(\mathring{M};\Omega^{\frac{1}{2}}) \equiv L^\infty(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \equiv x^{-\frac{1}{2}} L^\infty(M;\Omega^{\frac{1}{2}}) \ . \end{aligned}$$

So, according to Section 6.20, every $A \in \mathrm{Diff}_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2})$ induces a continuous linear map

$$x^{m+\frac{1}{2}} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{1/2}) \xrightarrow{A} x^{m+\frac{1}{2}} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \subset x^m L^\infty(M;\Omega^{\frac{1}{2}}) \ .$$

Hence there is a continuous inclusion

$$x^{m+\frac{1}{2}} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \subset \mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \ .$$

On the other hand, by (6.13) and the version of (6.12) with $\Omega^{1/2}M$, for all $a \in \mathbb{R}$,

$$x^a \mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \subset x^a C^{-\infty}(M;\Omega^{\frac{1}{2}}) = C^{-\infty}(M;\Omega^{\frac{1}{2}}) \equiv C^{-\infty}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ .$$

Moreover, by (6.50) and (6.54), every $A \in \operatorname{Diff}_{\mathrm{b}}(M; {}^{\mathrm{b}}\Omega^{1/2})$ induces a continuous linear map

$$\mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \xrightarrow{A} x^m L^\infty(M;\Omega^{\frac{1}{2}}) \subset x^m L^2(M;\Omega^{\frac{1}{2}}) \equiv x^{m+\frac{1}{2}} L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ .$$

Hence, by (6.27) and (6.54), A induces a continuous linear map

$$A : x^{-m-\frac{1}{2}}\mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \to L^2(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ .$$

It follows that there is a continuous inclusion

$$x^{-m-\frac{1}{2}}\mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \subset H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ ,$$

or, equivalently, there is a continuous inclusion

$$\mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \subset x^{m+\frac{1}{2}} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \ . \qquad \square$$

Corollary 6.37. *$H^m(\mathring{M}) = x^{-1/2}H^m_{\mathrm{b}}(M)$ ($m \in \mathbb{Z}$) as $C^\infty(M)$-modules and Hilbertian spaces, and therefore $H^{\pm\infty}(\mathring{M}) = x^{-1/2}H^{\pm\infty}_{\mathrm{b}}(M)$.*

Proof. By (6.1) and Corollary 6.35,

$$\begin{aligned} H^m(\mathring{M}) &\equiv H^m(\mathring{M};\Omega^{\frac{1}{2}}) \otimes_{C^\infty_{\mathrm{ub}}(\mathring{M})} C^\infty_{\mathrm{ub}}(\mathring{M};\Omega^{-\frac{1}{2}}) \\ &\equiv H^m(\mathring{M};\Omega^{\frac{1}{2}}) \otimes_{C^\infty_{\mathrm{ub}}(\mathring{M})} \big(C^\infty(M;\Omega^{-\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty_{\mathrm{ub}}(\mathring{M})\big) \\ &\equiv H^m(\mathring{M};\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty(M;\Omega^{-\frac{1}{2}}) \\ &\equiv H^m_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} x^{-\frac{1}{2}}C^\infty(M;{}^{\mathrm{b}}\Omega^{-\frac{1}{2}}) \equiv x^{-\frac{1}{2}}H^m_{\mathrm{b}}(M) \ . \qquad \square \end{aligned}$$

Corollary 6.38. *$\mathcal{A}^m(M) \equiv x^m H^\infty_{\mathrm{b}}(M) \equiv x^{m+1/2}H^\infty(\mathring{M})$ ($m \in \mathbb{R}$).*

Proof. The second identity is given by Corollary 6.37. By Proposition 6.36 and (6.1),

$$\begin{aligned} x^m H^\infty_{\mathrm{b}}(M) &\equiv x^{m+\frac{1}{2}} H^\infty_{\mathrm{b}}(M;{}^{\mathrm{b}}\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} x^{-\frac{1}{2}}C^\infty(M;{}^{\mathrm{b}}\Omega^{-\frac{1}{2}}) \\ &\equiv \mathcal{A}^m(M;\Omega^{\frac{1}{2}}) \otimes_{C^\infty(M)} C^\infty(M;\Omega^{-\frac{1}{2}}) \equiv \mathcal{A}^m(M) \ . \qquad \square \end{aligned}$$

By (6.39) and (6.40), we get the following consequences of Corollary 6.38.

Corollary 6.39. *$C^\infty_{\mathrm{c}}(\mathring{M})$ is dense in every $\mathcal{A}^m(M)$ and $\mathcal{A}^{(s)}(M)$.*

Corollary 6.40. *$\mathcal{A}(M) \equiv \bigcup_m x^m H^\infty_{\mathrm{b}}(M) = \bigcup_m x^m H^\infty(\mathring{M})$.*

Remark 6.41. Corollary 6.39 and the first identities of Corollaries 6.38 and 6.40 are independent of g. So they hold true without the assumptions (A) and (B). Observe that Corollary 6.39 is stronger than Corollary 6.15.

6.22. **Dual-conormal distributions at the boundary.** Consider the LCHSs [15, Section 18.3], [25, Chapter 4]

$$\mathcal{K}'(M) = \mathcal{K}(M;\Omega)' \ , \quad \mathcal{A}'(M) = \dot{\mathcal{A}}(M;\Omega)' \ , \quad \dot{\mathcal{A}}'(M) = \mathcal{A}(M;\Omega)' \ .$$

The elements of $\mathcal{A}'(M)$ (respectively, $\dot{\mathcal{A}}'(M)$) will be called *extendible* (respectively, *supported*) *dual-conormal distributions* at the boundary. The following analog of Corollary 5.1 holds true with formally the same proof, using the versions with ΩM of Corollaries 6.7, 6.16, 6.22, 6.26 and 6.28

Proposition 6.42. *$\mathcal{K}'(M)$, $\mathcal{A}'(M)$ and $\dot{\mathcal{A}}'(M)$ are complete Montel spaces.*

We also define the LCHSs

$$\mathcal{K}'^{(s)}(M) = \mathcal{K}^{(-s)}(M;\Omega)' , \quad \mathcal{K}'^{m}(M) = \mathcal{K}^{-m}(M;\Omega)' ,$$
$$\mathcal{A}'^{(s)}(M) = \dot{\mathcal{A}}^{(-s)}(M;\Omega)' , \quad \mathcal{A}'^{m}(M) = \dot{\mathcal{A}}^{-m}(M;\Omega)' ,$$
$$\dot{\mathcal{A}}'^{(s)}(M) = \mathcal{A}^{(-s)}(M;\Omega)' , \quad \dot{\mathcal{A}}'^{m}(M) = \mathcal{A}^{-m}(M;\Omega)' .$$

Transposing the analogs of (4.4) and (4.11) for the spaces $\mathcal{K}^{(s)}(M;\Omega)$, $\mathcal{K}^{m}(M;\Omega)$, $\dot{\mathcal{A}}^{(s)}(M;\Omega)$ and $\dot{\mathcal{A}}^{m}(M;\Omega)$, we get continuous linear restriction maps

$$\mathcal{K}'^{(s')}(M) \to \mathcal{K}'^{(s)}(M) , \quad \mathcal{K}'^{m}(M) \to \mathcal{K}'^{m'}(M) ,$$
$$\mathcal{A}'^{(s')}(M) \to \mathcal{A}'^{(s)}(M) , \quad \mathcal{A}'^{m}(M) \to \mathcal{A}'^{m'}(M) ,$$

for $s < s'$ and $m < m'$. These maps form projective spectra, giving rise to projective limits. The spaces $\mathcal{K}'^{(s)}(M)$, $\mathcal{K}'^{m}(M)$, $\mathcal{A}'^{(s)}(M)$ and $\mathcal{A}'^{m}(M)$ satisfy the analogs of (5.4). So the corresponding projective limits satisfy the analogs of (5.5).

Similarly, transposing the analog of (4.4) for the spaces $\mathcal{A}^{(s)}(M;\Omega)$ and the version of (6.38) with ΩM, by Corollary 6.39 and Remark 6.41, we get continuous inclusions

$$\dot{\mathcal{A}}'^{(s')}(M) \subset \dot{\mathcal{A}}'^{(s)}(M) , \quad \dot{\mathcal{A}}'^{m'}(M) \subset \dot{\mathcal{A}}'^{m}(M) ,$$

for $s < s'$ and $m < m'$. The version of (6.39) with ΩM yields continuous inclusions

$$\dot{\mathcal{A}}'^{(s)}(M) \supset \dot{\mathcal{A}}'^{m}(M) \supset \dot{\mathcal{A}}'^{(\max\{m,0\})}(M) \quad (m > s + n/2 + 1) . \tag{6.55}$$

Therefore

$$\bigcap_s \dot{\mathcal{A}}'^{(s)}(M) = \bigcap_m \dot{\mathcal{A}}'^{m}(M) . \tag{6.56}$$

The following analogs of Corollaries 5.2 and 5.3 hold true with formally the same proofs, using the versions with ΩM of Propositions 6.6 and 6.25 and Corollaries 6.16, 6.22 and 6.28; or use Proposition 6.42 for an alternative proof.

Corollary 6.43. *$\mathcal{K}'^{(s)}(M)$, $\mathcal{A}'^{(s)}(M)$ and $\dot{\mathcal{A}}'^{(s)}(M)$ are bornological and barreled.*

Corollary 6.44. *We have*

$$\mathcal{K}'(M) \equiv \varprojlim \mathcal{K}'^{(s)}(M) , \quad \mathcal{A}'(M) \equiv \varprojlim \mathcal{A}'^{(s)}(M) , \quad \dot{\mathcal{A}}'(M) = \bigcap_s \dot{\mathcal{A}}'^{(s)}(M) .$$

Transposing the versions of (6.3), (6.30), (6.32), (6.35) and Corollary 6.9 with ΩM, we get continuous inclusions [25, Section 4.6]

$$C^\infty(M) \subset \mathcal{A}'(M) \subset C^{-\infty}(M), \dot{C}^{-\infty}(M) , \tag{6.57}$$
$$\dot{C}^\infty(M) \subset \dot{\mathcal{A}}'(M) \subset \dot{C}^{-\infty}(M), C^{-\infty}(M) , \tag{6.58}$$

and $R : \dot{C}^{-\infty}(M) \to C^{-\infty}(M)$ restricts to the identity map on $\mathcal{A}'(M)$ and $\dot{\mathcal{A}}'(M)$.

6.23. **Dual-conormal sequence at the boundary.** Transposing maps in the version of (6.48) with ΩM, we get the sequence

$$0 \leftarrow \mathcal{K}'(M) \xleftarrow{\iota^t} \mathcal{A}'(M) \xleftarrow{R^t} \dot{\mathcal{A}}'(M) \leftarrow 0 ,$$

which will called the *dual-conormal sequence at the boundary* of M.

Proposition 6.45. *The dual-conormal sequence at the boundary of M is exact in the category of continuous linear maps between LCSs.*

Proof. By Proposition 6.8 and [39, Lemma 7.6], it is enough to prove that the map (6.36) satisfies the following condition of "topological lifting of bounded sets."

Claim 6.46. For all bounded subset $A \subset \mathcal{A}(M)$, there is some bounded subset $B \subset \dot{\mathcal{A}}(M)$ such that, for all 0-neighborhood $U \subset \dot{\mathcal{A}}(M)$, there is a 0-neighborhood $V \subset \mathcal{A}(M)$ so that $A \cap V \subset R(B \cap U)$.

Since $\mathcal{A}(M)$ is boundedly retractive (Corollary 6.16), A is contained and bounded in some step $\mathcal{A}^m(M)$. For any $m' > m$, let $E_{m'} : \mathcal{A}^m(M) \to \dot{\mathcal{A}}^{(s)}(M)$ be the partial extension map given by Proposition 6.29. Then $B := E_{m'}(A)$ is bounded in $\dot{\mathcal{A}}^{(s)}(M)$, and therefore in $\dot{\mathcal{A}}(M)$. Moreover, given any 0-neighborhood $U \subset \dot{\mathcal{A}}(M)$, there is some 0-neighborhood $W \subset \mathcal{A}^{m'}(M)$ so that $E_{m'}(W) \subset U \cap \dot{\mathcal{A}}^{(s)}(M)$. By Corollary 6.14, there is some 0-neighborhood $V \subset \mathcal{A}(M)$ such that $V \cap \mathcal{A}^m(M) = W \cap \mathcal{A}^m(M)$. Hence $E_{m'}(V \cap \mathcal{A}^m(M)) \subset U \cap \dot{\mathcal{A}}^{(s)}(M)$, yielding

$$A \cap V = R(E_{m'}(A \cap V)) \subset R(E_{m'}(A) \cap E_{m'}(V \cap \mathcal{A}^m(M))) \subset R(B \cap U) . \quad \square$$

Remark 6.47. Proposition 6.45 does not agree with [25, Proposition 4.6.2], which seems to be a minor error of that book project.

6.24. $\dot{\mathcal{A}}(M)$ **and** $\mathcal{A}(M)$ **vs** $\mathcal{A}'(M)$**.** Using (6.30), (6.32) and (6.57), we have [15, Proposition 18.3.24], [25, Theorem 4.6.1]

$$\dot{\mathcal{A}}(M) \cap \mathcal{A}'(M) = C^\infty(M) . \tag{6.59}$$

(Actually, the a priori weaker equality $\mathcal{A}(M) \cap \mathcal{A}'(M) = C^\infty(M)$ is proved in [25, Theorem 4.6.1], but it is equivalent to (6.59) because $R = 1$ on $\mathcal{A}'(M)$.)

6.25. **A description of** $\dot{\mathcal{A}}'(M)$**.** In this subsection, assume again (A) and (B).

Corollary 6.48. $\dot{\mathcal{A}}'^{\,m}(M) \equiv x^m H_{\mathrm{b}}^{-\infty}(M) = x^{m-\frac{1}{2}} H^{-\infty}(\mathring{M})$ $(m \in \mathbb{R})$.

Proof. Apply the version of Corollary 6.38 with ΩM. $\square$

Corollary 6.49. $\dot{\mathcal{A}}'(M) \equiv \bigcap_m x^m H_{\mathrm{b}}^{-\infty}(M) = \bigcap_m x^m H^{-\infty}(\mathring{M})$.

Proof. Apply (6.56) and Corollaries 6.44 and 6.48. $\square$

Corollary 6.50. $C_{\mathrm{c}}^\infty(\mathring{M})$ *is dense in every* $\dot{\mathcal{A}}'^{\,m}(M)$ *and in* $\dot{\mathcal{A}}'(M)$*. Therefore the first inclusion of* (6.58) *is also dense.*

Proof. Since $C_{\mathrm{c}}^\infty(\mathring{M})$ is dense in $H^{-\infty}(\mathring{M})$, we get that $C_{\mathrm{c}}^\infty(\mathring{M}) = x^m C_{\mathrm{c}}^\infty(\mathring{M})$ is dense in every $x^m H^{-\infty}(\mathring{M}) \equiv \dot{\mathcal{A}}'^{\,m}(M)$ (Corollary 6.48), and therefore in $\dot{\mathcal{A}}'(M)$ (Corollary 6.49). $\square$

Remark 6.51. Like in Remark 6.41, Corollary 6.50 and the first identities of Corollary 6.49 are independent of g, and hold true without the assumptions (A) and (B).

6.26. **Action of** $\mathrm{Diff}(M)$ **on** $\mathcal{A}'(M)$**,** $\dot{\mathcal{A}}'(M)$ **and** $\mathcal{K}'(M)$**.** According to Sections 2.5 and 6.14, any $A \in \mathrm{Diff}(M)$ induces continuous linear endomorphisms A of $\mathcal{A}'(M)$, $\dot{\mathcal{A}}'(M)$ and $\mathcal{K}'(M)$ [25, Proposition 4.6.1], which are the transposes of A^t on $\dot{\mathcal{A}}(M;\Omega)$, $\mathcal{A}(M;\Omega)$ and $\mathcal{K}(M;\Omega)$. If $A \in \mathrm{Diff}^k(M)$, these maps satisfy the analogs of (5.7). If $A \in \mathrm{Diff}_{\mathrm{b}}(M)$, it induces continuous endomorphisms of $\mathcal{A}'^{\,(s)}(M)$, $\mathcal{A}'^{\,m}(M)$, $\dot{\mathcal{A}}'^{\,(s)}(M)$ and $\mathcal{K}'^{\,(s)}(M)$.

7. Conormal sequence

7.1. **Cutting along a submanifold.** Let M be a closed connected manifold, and $L \subset M$ be a regular closed submanifold of codimension one. L may not be connected, and therefore $M \setminus L$ may have several connected components. First assume also that L is transversely oriented. Then, like in the boundary case Section 6.1, there is some real-valued smooth function x on some tubular neighborhood T of L in M, with projection $\varpi : T \to L$, so that $L = \{x = 0\}$ and $dx \neq 0$ on L. Any function x satisfying these conditions is called a *defining function* of L on T. We can suppose $T \equiv (-\epsilon, \epsilon)_x \times L$, for some $\epsilon > 0$, so that $\varpi : T \to L$ is the second factor projection. For any atlas $\{V_j, y_j\}$ of L, we get an atlas of T of the form $\{U_j \equiv (-\epsilon, \epsilon)_x \times V_j, (x, y)\}$, whose charts are adapted to L. The corresponding local vector fields $\partial_x \in \mathfrak{X}(U_j)$ can be combined to define a vector field $\partial_x \in \mathfrak{X}(T)$; we can consider ∂_x as the derivative operator on $C^\infty(T) \equiv C^\infty((-\epsilon, \epsilon), C^\infty(L))$. For every j, $\mathrm{Diff}(U_j, L \cap U_j)$ is spanned by $x\partial_x, \partial_j^1, \dots, \partial_j^{n-1}$ using the operations of $C^\infty(U_j)$-module and algebra, where $\partial_j^\alpha = \partial/\partial y_j^\alpha$. Using $T \equiv (-\epsilon, \epsilon)_x \times L$, any $A \in \mathrm{Diff}(L)$ induces an operator $1 \otimes A \in \mathrm{Diff}(T, L)$, such that $(1 \otimes A)(u(x)v(y)) = u(x)\,(Av)(y)$ for $u \in C^\infty(-\epsilon, \epsilon)$ and $v \in C^\infty(L)$. This defines a canonical injection $\mathrm{Diff}(L) \equiv 1 \otimes \mathrm{Diff}(L) \subset \mathrm{Diff}(T, L)$ so that $(1 \otimes A)|_L = A$. (This also shows the surjectivity of (4.2) in this case.) Moreover $\mathrm{Diff}(T)$ (respectively, $\mathrm{Diff}(T, L)$) is spanned by ∂_x (respectively, $x\partial_x$) and $1 \otimes \mathrm{Diff}(L)$ using the operations of $C^\infty(T)$-module and algebra. Clearly,

$$[\partial_x, 1 \otimes \mathrm{Diff}(L)] = 0\ , \quad [\partial_x, x\partial_x] = \partial_x\ , \tag{7.1}$$

yielding

$$[\partial_x, \mathrm{Diff}^k(T, L)] \subset \mathrm{Diff}^k(T, L) + \mathrm{Diff}^{k-1}(T, L)\,\partial_x\ . \tag{7.2}$$

$\mathrm{Diff}^k(T, L)$ and $\mathrm{Diff}^k(T)$ satisfy the obvious versions of (6.27) and (6.28).

For a vector bundle E over M, there is an identity $E_T \equiv (-\epsilon, \epsilon) \times E_L$ over $T \equiv (-\epsilon, \epsilon) \times L$, which can be used to define $\partial_x \in \mathrm{Diff}^1(T; E)$ using the above charts. With this interpretation of ∂_x and using tensor products like in (2.5), the vector bundle versions of the properties and spaces of distributions of this section are straightforward.

Let $\boldsymbol{M}$ be the smooth manifold with boundary defined by "cutting" M along L; i.e., modifying M only on the tubular neighborhood $T \equiv (-\epsilon, \epsilon) \times L$, which is replaced with $\boldsymbol{T} = ((-\epsilon, 0] \sqcup [0, \epsilon)) \times L$ in the obvious way. ($\boldsymbol{M}$ is the blowing-up $[M, L]$ of M along L [25, Chapter 5].) Thus $\partial \boldsymbol{M} \equiv L \sqcup L$ because L is transversely oriented, and $\mathring{\boldsymbol{M}} \equiv M \setminus L$. A canonical projection $\boldsymbol{\pi} : \boldsymbol{M} \to M$ is defined as the combination of the identity map $\mathring{\boldsymbol{M}} \to M \setminus L$ and the map $\boldsymbol{T} \to T$ given by the product of the canonical projection $(-\epsilon, 0] \sqcup [0, \epsilon) \to (-\epsilon, \epsilon)$ and id_L. This projection realizes M as a quotient space of $\boldsymbol{M}$ by the equivalence relation defined by the homeomorphism $h \equiv h_0 \times \mathrm{id}$ of $\partial \boldsymbol{M} \equiv \partial \boldsymbol{T} = (\{0\} \sqcup \{0\}) \times L$, where h_0 switches the two points of $\{0\} \sqcup \{0\}$. Moreover $\boldsymbol{\pi} : \boldsymbol{M} \to M$ is a local embedding of a compact manifold with boundary to a closed manifold of the same dimension.

Like in Section 2.6, we have the continuous linear pull-back map

$$\boldsymbol{\pi}^* : C^\infty(M) \to C^\infty(\boldsymbol{M})\ , \tag{7.3}$$

which is clearly injective. Then the transpose of the version of (7.3) with ΩM and $\Omega\boldsymbol{M} \equiv \boldsymbol{\pi}^*\Omega M$ is the continuous linear push-forward map

$$\boldsymbol{\pi}_* : \dot{C}^{-\infty}(\boldsymbol{M}) \to C^{-\infty}(M) , \tag{7.4}$$

which is surjective by a consequence of the Hahn-Banach theorem [31, Theorem II.4.2].

After distinguishing a connected component L_0 of L, let $\widetilde{M}$ and $\widetilde{L}$ be the quotients of $\boldsymbol{M} \sqcup \boldsymbol{M} \equiv \boldsymbol{M} \times \mathbb{Z}_2$ and $\partial\boldsymbol{M} \sqcup \partial\boldsymbol{M} \equiv \partial\boldsymbol{M} \times \mathbb{Z}_2$ by the equivalence relation generated by $(p, a) \sim (h(p), a)$ if $\boldsymbol{\pi}(p) \in L \setminus L_0$ and $(p, a) \sim (h(p), a + 1)$ if $\boldsymbol{\pi}(p) \in L_0$ ($p \in \boldsymbol{\pi}^{-1}(L) = \partial\boldsymbol{M}$ in both cases). Let us remark that $\widetilde{M}$ may not be homeomorphic to the double of $\boldsymbol{M}$, which is the quotient of $\boldsymbol{M} \times \mathbb{Z}_2$ by the equivalence relation generated by $(p, 0) \sim (p, 1)$, for $p \in \partial\boldsymbol{M}$. Note that $\widetilde{M}$ is a closed connected manifold and $\widetilde{L}$ is a closed regular submanifold. Moreover the quotient $\widetilde{T}$ of $\boldsymbol{T} \sqcup \boldsymbol{T}$ becomes a tubular neighborhood of $\widetilde{L}$ in $\widetilde{M}$. The combination $\boldsymbol{\pi} \sqcup \boldsymbol{\pi} : \boldsymbol{M} \sqcup \boldsymbol{M} \to M$ induces a two-fold covering map $\tilde{\pi} : \widetilde{M} \to M$, whose restrictions to $\widetilde{L}$ and $\widetilde{T}$ are trivial two-fold coverings of L and T, respectively; i.e., $\widetilde{L} \equiv L \sqcup L$ and $\widetilde{T} \equiv T \sqcup T$. The group of deck transformations of $\tilde{\pi} : \widetilde{M} \to M$ is $\{\mathrm{id}, \sigma\}$, where $\sigma : \widetilde{M} \to \widetilde{M}$ is induced by the map $\sigma_0 : \boldsymbol{M} \times \mathbb{Z}_2 \to \boldsymbol{M} \times \mathbb{Z}_2$ defined by switching the elements of $\mathbb{Z}_2$. The composition of the injection $\boldsymbol{M} \to \boldsymbol{M} \times \mathbb{Z}_2$, $p \mapsto (p, 0)$, with the quotient map $\boldsymbol{M} \sqcup \boldsymbol{M} \to \widetilde{M}$ is a smooth embedding $\boldsymbol{M} \to \widetilde{M}$. This will be considered as an inclusion map of a regular submanifold with boundary, obtaining $\partial\boldsymbol{M} \equiv \widetilde{L}$.

Since $\tilde{\pi}$ is a two-fold covering map, we have continuous linear maps (Section 2.6)

$$\begin{gathered}\tilde{\pi}_* : C^\infty(\widetilde{M}) \to C^\infty(M) , \quad \tilde{\pi}^* : C^\infty(M) \to C^\infty(\widetilde{M}) , \\ \tilde{\pi}^* : C^{-\infty}(M) \to C^{-\infty}(\widetilde{M}) , \quad \tilde{\pi}_* : C^{-\infty}(\widetilde{M}) \to C^{-\infty}(M) ,\end{gathered} \tag{7.5}$$

both pairs of maps satisfying

$$\tilde{\pi}_*\tilde{\pi}^* = 2 , \quad \tilde{\pi}^*\tilde{\pi}_* = A_\sigma , \tag{7.6}$$

where $A_\sigma : C^{\pm\infty}(\widetilde{M}) \to C^{\pm\infty}(\widetilde{M})$ is given by $A_\sigma u = u + \sigma_* u$. Using the continuous linear restriction and inclusion maps given by (6.2) and (6.7), we get the commutative diagrams

$$\begin{array}{ccc} C^\infty(\widetilde{M}) & \xrightarrow{R} & C^\infty(\boldsymbol{M}) \\ {\scriptstyle\tilde{\pi}^*}\uparrow & & \uparrow{\scriptstyle\boldsymbol{\pi}^*} \\ C^\infty(M) & = & C^\infty(M) , \end{array} \qquad \begin{array}{ccc} \dot{C}^{-\infty}(\boldsymbol{M}) & \xrightarrow{\iota} & C^{-\infty}(\widetilde{M}) \\ {\scriptstyle\boldsymbol{\pi}_*}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}_*} \\ C^{-\infty}(M) & = & C^{-\infty}(M) , \end{array} \tag{7.7}$$

the second one being the transpose of the density-bundles version of the first one.

7.2. **Lift of differential operators from M to $\widetilde{M}$.** For any $A \in \mathrm{Diff}(M)$, let $\widetilde{A} \in \mathrm{Diff}(\widetilde{M})$ denote its lift via the covering map $\tilde{\pi} : \widetilde{M} \to M$. The action of $\widetilde{A}$ on $C^{\pm\infty}(\widetilde{M})$ corresponds to the action of A on $C^{\pm\infty}(M)$ via $\tilde{\pi}^* : C^{\pm\infty}(M) \to C^{\pm\infty}(\widetilde{M})$ and $\tilde{\pi}_* : C^{\pm\infty}(\widetilde{M}) \to C^{\pm\infty}(M)$. According to (6.23), $\widetilde{A}|_{\boldsymbol{M}} \in \mathrm{Diff}(\boldsymbol{M})$ is the lift of A via the local embedding $\boldsymbol{\pi} : \boldsymbol{M} \to M$, sometimes also denoted by $\widetilde{A}$. The action of $\widetilde{A}$ on $C^\infty(\boldsymbol{M})$ (respectively, $C^{-\infty}(\boldsymbol{M})$) corresponds to the action of A on $C^\infty(M)$ (respectively, $C^{-\infty}(M)$) via $\boldsymbol{\pi}^* : C^\infty(M) \to C^\infty(\boldsymbol{M})$

(respectively, $\boldsymbol{\pi}_* : C^{-\infty}(\boldsymbol{M}) \to C^{-\infty}(M)$). If $A \in \operatorname{Diff}(M, L)$, then $\widetilde{A} \in \operatorname{Diff}(\widetilde{M}, \widetilde{L})$ and $\widetilde{A}|_{\boldsymbol{M}} \in \operatorname{Diff}_{\mathrm{b}}(\boldsymbol{M})$ by (6.26).

7.3. **The spaces $C^{\pm\infty}(M, L)$.** Consider the closed subspaces,

$$C^\infty(M, L) \subset C^\infty(M) , \quad C^k(M, L) \subset C^k(M) \quad (k \in \mathbb{N}_0) , \tag{7.8}$$

consisting of functions that vanish to all orders at the points of L in the first case, and that vanish up to order k at the points of L in the second case. Then let

$$C^{-\infty}(M, L) = C^\infty(M, L; \Omega)' , \quad C'^{\,-k}(M, L) = C^k(M, L; \Omega)' .$$

Note that (7.3) restricts to TVS-isomorphisms

$$\boldsymbol{\pi}^* : C^\infty(M, L) \xrightarrow{\cong} \dot{C}^\infty(\boldsymbol{M}) , \quad \boldsymbol{\pi}^* : C^k(M, L) \xrightarrow{\cong} \dot{C}^k(\boldsymbol{M}) . \tag{7.9}$$

Taking the transposes of its versions with density bundles, it follows that (7.4) restricts to TVS-isomorphisms

$$\boldsymbol{\pi}_* : C^{-\infty}(\boldsymbol{M}) \xrightarrow{\cong} C^{-\infty}(M, L) , \quad \boldsymbol{\pi}_* : C'^{\,-k}(\boldsymbol{M}) \xrightarrow{\cong} C'^{\,-k}(M, L) . \tag{7.10}$$

So the spaces $C^\infty(M, L)$, $C^k(M, L)$, $C^{-\infty}(M, L)$ and $C'^{\,-k}(M, L)$ satisfy the analogs of (2.6) and (2.7).

On the other hand, there are Hilbertian spaces $H^r(M, L)$ ($r > n/2$) and $H'^{\,s}(M, L)$ ($s \in \mathbb{R}$), continuously included in $C^0(M, L)$ and $C^{-\infty}(M, L)$, respectively, such that the second map of (7.9) for $k = 0$ and the first map of (7.10) restrict to a TVS-isomorphisms

$$\boldsymbol{\pi}^* : H^r(M, L) \xrightarrow{\cong} \dot{H}^r(\boldsymbol{M}) , \quad \boldsymbol{\pi}_* : H^s(\boldsymbol{M}) \xrightarrow{\cong} H'^{\,s}(M, L) . \tag{7.11}$$

By (6.16),

$$H'^{\,-r}(M, L) \equiv H^r(M, L; \Omega)' , \quad H^r(M, L) \equiv H'^{\,-r}(M, L; \Omega)' . \tag{7.12}$$

Now, the second identity of (7.12) can be used to extend the definition of $H^r(M, L)$ for all $r \in \mathbb{R}$.

Alternatively, we may also use trace theorems [1, Theorem 7.53 and 7.58] to define $H^m(M, L)$ for $m \in \mathbb{Z}^+$, and then use the first identity of (7.12) to define $H'^{\,-m}(M, L)$.

From (7.3), (7.4), (7.11) and the analogs of (2.18)–(2.20) mentioned in Section 6.4, we get

$$C^\infty(M, L) = \bigcap_r H^r(M, L) , \quad C^{-\infty}(M, L) = \bigcup_s H'^{\,s}(M, L) , \tag{7.13}$$

as well as a continuous inclusion and a continuous linear surjection,

$$C^\infty(M) \subset \bigcap_s H'^{\,s}(M, L) , \qquad C^{-\infty}(M) \leftarrow \bigcup_r H^r(M, L) . \tag{7.14}$$

By (7.12) and (7.13),

$$C^\infty(M, L) = C^{-\infty}(M, L; \Omega)' . \tag{7.15}$$

Proposition 6.1 and (7.10) have the following consequence.

Corollary 7.1. *$C^{-\infty}(M, L)$ is a barreled, ultrabornological, webbed, acyclic DF Montel space, and therefore complete, boundedly retractive and reflexive.*

The transpose of the version of the first inclusion of (7.8) with ΩM is a continuous linear restriction map

$$R : C^{-\infty}(M) \to C^{-\infty}(M, L) , \tag{7.16}$$

whose restriction to $C^{\infty}(M)$ is the identity. This map can be also described as the composition

$$C^{-\infty}(M) \xrightarrow{\tilde{\pi}^*} C^{-\infty}(\widetilde{M}) \xrightarrow{R} C^{-\infty}(\boldsymbol{M}) \xrightarrow{\boldsymbol{\pi}_*} C^{-\infty}(M, L) .$$

The canonical pairing between $C^{\infty}(M)$ and $C^{\infty}(M, L; \Omega)$ defines a continuous inclusion

$$C^{\infty}(M) \subset C^{-\infty}(M, L) \tag{7.17}$$

such that (7.16) is the identity on $C^{\infty}(M)$. We also get commutative diagrams

$$\begin{array}{ccccccc} C^{\infty}(\boldsymbol{M}) & \xleftarrow{\iota} & \dot{C}^{\infty}(\boldsymbol{M}) & & \dot{C}^{-\infty}(\boldsymbol{M}) & \xrightarrow{R} & C^{-\infty}(\boldsymbol{M}) \\ \boldsymbol{\pi}^*\uparrow & & \cong\uparrow\boldsymbol{\pi}^* & & \boldsymbol{\pi}_*\downarrow & & \cong\downarrow\boldsymbol{\pi}_* \\ C^{\infty}(M) & \xleftarrow{\iota} & C^{\infty}(M, L) , & & C^{-\infty}(M) & \xrightarrow{R} & C^{-\infty}(M, L) , \end{array} \tag{7.18}$$

the second one being the transpose of the density-bundles version of the first one.

7.4. **The space $C^{-\infty}_L(M)$.** The closed subspaces of elements supported in L,

$$C^{-\infty}_L(M) \subset C^{-\infty}(M) , \quad C'^{\,-k}_L(M) \subset C'^{\,-k}(M) , \quad H^s_L(M) \subset H^s(M) ,$$

are the null spaces of restrictions of (7.16). These spaces satisfy continuous inclusions analogous to (2.6), (2.17) and (2.19).

According to (6.20) and Section 7.1,

$$\begin{aligned} \dot{C}^{-\infty}_{\partial \boldsymbol{M}}(\boldsymbol{M}) &\equiv C^{-\infty}_{\widetilde{L}}(\widetilde{M}) \equiv C^{-\infty}_{\widetilde{L}}(\widetilde{T}) \equiv C^{-\infty}_L(T) \oplus C^{-\infty}_L(T) \\ &\equiv C^{-\infty}_L(M) \oplus C^{-\infty}_L(M) . \end{aligned} \tag{7.19}$$

The maps (7.4) and (7.5) have restrictions

$$\boldsymbol{\pi}_* = \tilde{\pi}_* : \dot{C}^{-\infty}_{\partial \boldsymbol{M}}(\boldsymbol{M}) \to C^{-\infty}_L(M) , \quad \tilde{\pi}^* : C^{-\infty}_L(M) \to \dot{C}^{-\infty}_{\partial \boldsymbol{M}}(\boldsymbol{M}) . \tag{7.20}$$

Using (7.19), these maps are given by $\boldsymbol{\pi}_*(u, v) = u + v$ and $\tilde{\pi}^* u = (u, u)$.

From (7.19), Proposition 6.4 and Corollary 6.5, we get the following.

Corollary 7.2. *$C^{-\infty}_L(M)$ is a limit subspace of the LF-space $C^{-\infty}(M)$.*

Corollary 7.3. *$C^{-\infty}_L(M)$ is a barreled, ultrabornological, webbed, acyclic DF Montel space, and therefore complete, boundedly retractive and reflexive.*

Moreover the right-hand side diagram of (7.18) can be completed to get the commutative diagram

$$\begin{array}{ccccc} 0 \to \dot{C}^{-\infty}_{\partial \boldsymbol{M}}(\boldsymbol{M}) & \xrightarrow{\iota} & \dot{C}^{-\infty}(\boldsymbol{M}) & \xrightarrow{R} & C^{-\infty}(\boldsymbol{M}) \to 0 \\ \boldsymbol{\pi}_*\downarrow & & \boldsymbol{\pi}_*\downarrow & & \cong\downarrow\boldsymbol{\pi}_* \\ 0 \to C^{-\infty}_L(M) & \xrightarrow{\iota} & C^{-\infty}(M) & \xrightarrow{R} & C^{-\infty}(M, L) \to 0 . \end{array} \tag{7.21}$$

Proposition 7.4. *The maps* (7.4) *and* (7.16) *are surjective topological homomorphisms.*

Proof. In (7.21), the top row is exact in the category of continuous linear maps between LCSs by Corollary 6.3, the left-hand side vertical map is onto by (7.6), and the right-hand side vertical map is a TVS-isomorphism. Then, by the commutativity of its right-hand side square, the map (7.16) is surjective, and therefore the bottom row of (7.21) is exact in the category of linear maps between vector spaces.

By the above properties, chasing (7.21), we get that (7.4) is surjective. Since $\dot{C}^{-\infty}(\boldsymbol{M})$ is webbed (Proposition 6.1) and $C^{-\infty}(M)$ ultrabornological, by the open mapping theorem, it also follows that (7.4) is a topological homomorphism.

To get that (7.16) is another surjective topological homomorphism, apply the commutativity of the right-hand side square of (7.21) and the above properties. □

Corollary 7.5. *The bottom row of* (7.21) *is exact in the category of continuous linear maps between LCSs.*

Corollary 7.6. *The inclusion* (7.17) *is dense.*

Proof. Apply Proposition 7.4 and the density of $C^\infty(M)$ in $C^{-\infty}(M)$. □

7.5. **A description of $C^{-\infty}_L(M)$.** According to Sections 2.7 and 7.1 and (4.14), we have the subspaces

$$\partial_x^m C^{-\infty}(L;\Omega^{-1}NL) \subset C^{-\infty}_L(M) \tag{7.22}$$

for $m \in \mathbb{N}_0$, and continuous linear isomorphisms

$$\partial_x^m : C^{-\infty}(L;\Omega^{-1}NL) \xrightarrow{\cong} \partial_x^m C^{-\infty}(L;\Omega^{-1}NL)\ . \tag{7.23}$$

They induce a continuous linear injection

$$\bigoplus_{m=0}^{\infty} C^0_m \to C^{-\infty}_L(M)\ , \tag{7.24}$$

where $C^0_m = C^{-\infty}(L;\Omega^{-1}NL)$ for all m.

Proposition 7.7. *The map* (7.24) *is a TVS-isomorphism, which restricts to TVS-isomorphisms*

$$\bigoplus_{m=0}^{k} C^{m-k}(L;\Omega^{-1}NL) \xrightarrow{\cong} C'^{-k}_L(M) \quad (k \in \mathbb{N}_0)\ . \tag{7.25}$$

Proof. In the case where $M = \mathbb{R}^n$ and L is a linear subspace, it is known that (7.25) is a linear isomorphism [14, Theorem 2.3.5 and Example 5.1.2], which is easily seen to be continuous. This can be easily extended to arbitrary M by using charts of M adapted to L. Then we get the continuous linear isomorphism (7.24) by taking the locally convex inductive limit of (7.25) as $k \uparrow \infty$. Since $\bigoplus_m C^0_m$ is webbed and $C^{-\infty}_L(M)$ ultrabornological (Corollary 7.3), the map (7.24) is a TVS-isomorphism by the open mapping theorem. □

Remark 7.8. Proposition 7.7 reconfirms Corollary 7.2.

Remark 7.9 (See [25, Exercise 3.3.18]). In Section 6.5, for any compact manifold with boundary M, the analog of Proposition 7.7 for $\dot{C}^{-\infty}_{\partial M}(M)$ follows from the application of Proposition 7.7 to $C^{-\infty}_{\partial M}(\breve{M})$.

Corollary 7.10. *Every map* (7.23) *is a TVS-isomorphism.*

7.6. **Action of** $\mathrm{Diff}(M)$ **on** $C^{-\infty}(M,L)$ **and** $C^{-\infty}_L(M)$**.** For every $A \in \mathrm{Diff}(M)$, A^t preserves $C^\infty(M,L;\Omega)$, and therefore A induces a continuous linear map $A = A^{tt}$ on $C^{-\infty}(M,L)$. By locality, it restricts to a continuous endomorphism A of $C^{-\infty}_L(M)$.

7.7. **The space** $J(M,L)$**.** According to Sections 6.8 and 7.3, there is a LCHS $J(M,L)$, continuously included in $C^{-\infty}(M,L)$, so that (7.10) restricts to a TVS-isomorphism

$$\boldsymbol{\pi}_* : \mathcal{A}(\boldsymbol{M}) \xrightarrow{\cong} J(M,L) \;, \tag{7.26}$$

where $\mathcal{A}(\boldsymbol{M})$ is defined in (6.29). By (6.33), there is a continuous inclusion

$$J(M,L) \subset C^\infty(M \setminus L) \;.$$

We also get spaces $J^{(s)}(M,L)$ and $J^m(M,L)$ ($s,m \in \mathbb{R}$) corresponding to $\mathcal{A}^{(s)}(\boldsymbol{M})$ and $\mathcal{A}^m(\boldsymbol{M})$ via (7.26). Extend $|x|$ to a function $\boldsymbol{x}$ on M that is positive and smooth on $M \setminus L$. Its lift $\boldsymbol{\pi}^*\boldsymbol{x}$ is a boundary defining function of $\boldsymbol{M}$, also denoted by $\boldsymbol{x}$. Using the first map of (7.10) and second map of (7.11), and according to Section 7.2, we can also describe

$$\begin{aligned} J^{(s)}(M,L) &= \{\, u \in C^{-\infty}(M,L) \mid \mathrm{Diff}(M,L)\, u \subset H'^{\,s}(M,L) \,\} \;,\\ J^m(M,L) &= \{\, u \in C^{-\infty}(M,L) \mid \mathrm{Diff}(M,L)\, u \subset \boldsymbol{x}^m L^\infty(M) \,\} \;, \end{aligned} \tag{7.27}$$

equipped with topologies like in Sections 6.8 and 6.10. These spaces satisfy the analogs of (4.4), (6.29) and (6.38)–(6.40). By (7.14) and (7.27), there are continuous inclusions,

$$C^\infty(M) \subset J^{(\infty)}(M,L) := \bigcap_s J^{(s)}(M,L) \;, \quad J(M,L) \subset C^{-\infty}(M,L) \;; \tag{7.28}$$

in particular, $J(M,L)$ is Hausdorff. Moreover the following analogs of Proposition 6.6 and Corollaries 6.7 and 6.14 to 6.16 hold true.

Corollary 7.11. *$J^{(s)}(M,L)$ is a totally reflexive Fréchet space.*

Corollary 7.12. *$J(M,L)$ is barreled, ultrabornological and webbed.*

Corollary 7.13. *If $m' < m$, then the topologies of $J^{m'}(M,L)$ and $C^\infty(M \setminus L)$ coincide on $J^m(M,L)$. Therefore the topologies of $J(M,L)$ and $C^\infty(M\setminus L)$ coincide on $J^m(M,L)$.*

Corollary 7.14. *For $m' < m$, $C^\infty_{\mathrm{c}}(M \setminus L)$ is dense in $J^m(M,L)$ with the topology of $J^{m'}(M,L)$. Therefore $C^\infty_{\mathrm{c}}(M \setminus L)$ is dense in $J(M,L)$.*

Corollary 7.15. *$J(M,L)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

The analog of Remark 6.17 makes sense for $J(M,L)$.

7.8. **A description of** $J(M,L)$**.** Take a b-metric $\boldsymbol{g}$ on $\boldsymbol{M}$ satisfying (A) and (B), and consider its restriction to $\mathring{\boldsymbol{M}}$. Consider also the boundary defining function $\boldsymbol{x}$ of $\boldsymbol{M}$ (Section 7.7). Corollaries 6.38 to 6.40 and (7.26) have the following direct consequences.

Corollary 7.16. *$J^m(M,L) \cong \boldsymbol{x}^m H^\infty_{\mathrm{b}}(\boldsymbol{M}) \equiv \boldsymbol{x}^{m+1/2} H^\infty(\mathring{\boldsymbol{M}})$ ($m \in \mathbb{R}$).*

Corollary 7.17. *$C^\infty_{\mathrm{c}}(M \setminus L)$ is dense in every $J^m(M,L)$ and $J^{(s)}(M,L)$.*

Corollary 7.18. *$J(M,L) \cong \bigcup_m \boldsymbol{x}^m H^\infty_{\mathrm{b}}(\boldsymbol{M}) = \bigcup_m \boldsymbol{x}^m H^\infty(\mathring{\boldsymbol{M}})$.*

The analog of Remark 6.41 makes sense for $J(M,L)$.

7.9. $I(M,L)$ **vs** $\dot{\mathcal{A}}(\boldsymbol{M})$ **and** $J(M,L)$**.** According to Sections 4.6 and 4.7, we have the continuous linear maps

$$\tilde\pi^* : I(M,L) \to I(\widetilde M,\widetilde L)\ , \quad \tilde\pi_* : I(\widetilde M,\widetilde L) \to I(M,L)\ , \tag{7.29}$$

which are restrictions of the maps (7.5), and therefore they satisfy (7.6). These maps are compatible with the Sobolev and symbol order filtrations because $\tilde\pi : \widetilde M \to M$ is a covering map (Sections 4.6 and 4.7).

Using the TVS-embedding $\dot{\mathcal{A}}(\boldsymbol{M}) \subset I(\widetilde M,\widetilde L)$ (Corollary 6.20), which is compatible with the Sobolev and symbol order filtrations, the map $\tilde\pi_*$ of (7.29) has the restriction

$$\boldsymbol{\pi}_* : \dot{\mathcal{A}}(\boldsymbol{M}) \to I(M,L)\ . \tag{7.30}$$

This map is compatible with the Sobolev and symbol order filtration by the above properties.

On the other hand, the map (7.16) restricts to a continuous linear map

$$R : I(M,L) \to J(M,L)\ , \tag{7.31}$$

which can be also described as the composition

$$I(M,L) \xrightarrow{\tilde\pi^*} I(\widetilde M,\widetilde L) \xrightarrow{R} \mathcal{A}(\boldsymbol{M}) \xrightarrow{\boldsymbol{\pi}_*} J(M,L)\ .$$

From the properties of (6.44), (7.26) and (7.29), it follows that (7.31) is compatible with the Sobolev order filtration. According to (4.5) and (7.28), the map (7.31) is the identity on $C^\infty(M)$.

7.10. **The space** $K(M,L)$**.** Like in Section 6.13, the condition of being supported in L defines the LCHSs and $C^\infty(M)$-modules

$$K^{(s)}(M,L) = I_L^{(s)}(M,L)\ , \quad K^m(M,L) = I_L^m(M,L)\ , \quad K(M,L) = I_L(M,L)\ .$$

These are closed subspaces of $I^{(s)}(M,L)$, $I_L^m(M,L)$ and $I(M,L)$, respectively; more precisely, they are the null spaces of the corresponding restrictions of the map (7.31). According to Corollary 6.20, the identity (7.19) restricts to a TVS-identity

$$\mathcal{K}(\boldsymbol{M}) \equiv K(M,L) \oplus K(M,L)\ . \tag{7.32}$$

Furthermore the maps (7.20) induce continuous linear maps

$$\boldsymbol{\pi}_* : \mathcal{K}(\boldsymbol{M}) \to K(M,L)\ , \quad \tilde\pi^* : K(M,L) \to \mathcal{K}(\boldsymbol{M})\ . \tag{7.33}$$

Using (7.32), these maps are given by $\boldsymbol{\pi}_*(u,v) = u+v$ and $\tilde\pi^*u = (u,u)$.

By (6.45) and (6.47), $K^{(s)}(M,L)$ and $K^m(M,L)$ satisfy analogs of (7.32), using $\mathcal{K}^{(s)}(\boldsymbol{M})$ and $\mathcal{K}^m(\boldsymbol{M})$. Thus we get the following consequences of Propositions 6.24 and 6.25 and Corollaries 6.26 to 6.28.

Corollary 7.19. *$K(M,L)$ is a limit subspace of the LF-space $I(M,L)$.*

Corollary 7.20. *$K^{(s)}(M,L)$ is a totally reflexive Fréchet space.*

Corollary 7.21. *$K^{(s)}(M,L)$ is barreled, ultrabornological and webbed, and therefore so is $K(M,L)$.*

Corollary 7.22. *For $m < m', m''$, the topologies of $K^{m'}(M,L)$ and $K^{m''}(M,L)$ coincide on $K^m(M,L)$.*

Corollary 7.23. *$K(M,L)$ is an acyclic Montel space, and therefore complete, boundedly retractive and reflexive.*

Example 7.24. With the notation of Section 4.8, $\operatorname{Diff}(M) \equiv K(M^2,\Delta)$ becomes a filtered $C^\infty(M^2)$-submodule of $\Psi(M)$, with the order filtration corresponding to the symbol order filtration. In this way, $\operatorname{Diff}(M)$ also becomes a LCHS satisfying the above properties. If M is compact, it is also a filtered subalgebra of $\Psi(M)$.

7.11. **A description of $K(M,L)$.** By (4.18) and (4.16),

$$\partial_x^m C^\infty(L;\Omega^{-1}NL) \subset K^{(s-m)}(M,L) \quad (s<-1/2)\ , \tag{7.34}$$

and (7.23) restricts to a continuous linear isomorphism

$$\partial_x^m : C^\infty(L;\Omega^{-1}NL) \xrightarrow{\cong} \partial_x^m C^\infty(L;\Omega^{-1}NL)\ . \tag{7.35}$$

Lemma 7.25. *For all $m\in\mathbb{N}_0$,*

$$\partial_x^m C^\infty(L;\Omega^{-1}NL) \cap K^{(-\frac12-m)}(M,L) = 0\ .$$

Proof. We proceed by induction on m. The case $m=0$ is given by Proposition 4.9. Now assume $m\ge 1$, and let $v\in C^\infty(L;\Omega^{-1}NL)$ with $u=\partial_x^m\delta_L^v\in K^{(-\frac12-m)}(M,L)$. Take any $A\in\operatorname{Diff}^2(L;\Omega NL)$ such that $-\partial_x^2+B\in\operatorname{Diff}^2(T)$ is elliptic, where $B=(1\otimes A^t)^t\in\operatorname{Diff}^2(T,L)$; for instance, given a Riemannian metric on M, A can be the Laplacian of the flat line bundle ΩNL. By (7.34), $u_0:=\partial_x^{m-1}\delta_L^v\in K^{(\frac12-m-\epsilon)}(M,L)$ for $0<\epsilon<1$. By (7.2), given any $B_0\in\operatorname{Diff}(M,L)$, there is some $B_1,B_2,B_3\in\operatorname{Diff}(M,L)$ such that $[\partial_x^2,B_0]=B_1+B_2\partial_x+B_3\partial_x^2$. So, according to Section 4.5, (4.17) and (7.2), for all $B_0\in\operatorname{Diff}(M,L)$,

$$\begin{aligned}(-\partial_x^2+B)B_0u_0 &= -B_0\partial_xu-B_1u_0-B_2u-B_3\partial_xu+\partial_x^{m-1}\delta_L^{B_0'Av}+[B,B_0]u_0\\ &\in K^{(-\frac32-m)}(M,L)+K^{(\frac12-m-\epsilon)}(M,L)=K^{(-\frac32-m)}(M,L)\ .\end{aligned}$$

Hence $B_0u_0\in H^{\frac12-m}(M)$ by elliptic regularity. Since B_0 is arbitrary, we get $u_0\in K^{(\frac12-m)}(M,L)$. So $u_0=0$ by the induction hypothesis, yielding $u=\partial_xu_0=0$. □

By Proposition 7.7, the TVS-isomorphism (7.24) restricts to a linear injection

$$\bigoplus_{m=0}^{\infty} C_m^1 \to K(M,L)\ , \tag{7.36}$$

where $C_m^1=C^\infty(L;\Omega^{-1}NL)$ for all $m\in\mathbb{N}_0$, which is easily seen to be continuous.

Proposition 7.26. *The map* (7.36) *is a TVS-isomorphism, which induces TVS-isomorphisms*

$$\bigoplus_{m<-s-\frac12} C_m^1 \xrightarrow{\cong} K^{(s)}(M,L) \quad (s<-1/2)\ . \tag{7.37}$$

Proof. To prove that (7.36) is surjective, take any $u\in K(M,L)$. By Proposition 7.7, we can assume $u\in\partial_x^mC^{-\infty}(L;\Omega NL)$ for some m. For any $A\in\operatorname{Diff}(L;\Omega NL)$, let $B=(1\otimes A^t)^t\in\operatorname{Diff}(T,L)$. Since $u\in K(M,L)$ and B is local, it follows from the definition of $I(M,L)$ and (2.19) that $Bu\in H_L^{-k}(T)\subset C_L'^{-k}(T)$ for some $k\ge m$. On the other hand, $u=\partial_x^m\delta_L^v$ for some $v\in C^{-\infty}(L;\Omega NL)$. Then (4.17) and (7.1) yield

$$Bu=B\partial_x^m\delta_L^v=\partial_x^mB\delta_L^v=\partial_x^m\delta_L^{B'v}=\partial_x^m\delta_L^{Av}\ .$$

Therefore, by Proposition 7.7,

$$Bu\in C_L'^{-k}(M)\cap\partial_x^mC^{-\infty}(L;\Omega NL)=\partial_x^mC'^{\,m-k}(L;\Omega NL)\ .$$

This means that $Av \in C'^{\,m-k}(L;\Omega NL)$. So $v \in C^\infty(L;\Omega NL)$ because A is arbitrary, and therefore $u \in \partial_x^m C^\infty(L;\Omega NL)$.

The surjectivity of (7.37) follows from Lemma 7.25 and the surjectivity of (7.36).

Finally, (7.36) is open like in Proposition 7.7, using that $C^\infty(L;\Omega^{-1}NL)$ is webbed and $K(M,L)$ ultrabornological (Corollary 7.21). So (7.37) is also open. □

Remark 7.27. Proposition 7.26 reconfirms Corollary 7.19. It also follows from Proposition 7.26 that (7.35) is a TVS-isomorphism.

Remark 7.28. In Section 6.13, for any compact manifold with boundary M, the analog of Proposition 7.7 for $\mathcal{K}(M)$ follows from (6.49) and the application of Proposition 7.7 to $K(\breve{M},\partial M)$.

7.12. **The conormal sequence.** The diagram (7.21) has the restriction

$$\begin{array}{ccccccccc} 0 \to \mathcal{K}(\boldsymbol{M}) & \xrightarrow{\iota} & \dot{\mathcal{A}}(\boldsymbol{M}) & \xrightarrow{R} & \mathcal{A}(\boldsymbol{M}) \to 0 \\ \downarrow{\scriptstyle \boldsymbol{\pi}_*} & & \downarrow{\scriptstyle \boldsymbol{\pi}_*} & & \cong\downarrow{\scriptstyle \boldsymbol{\pi}_*} \\ 0 \to K(M,L) & \xrightarrow{\iota} & I(M,L) & \xrightarrow{R} & J(M,L) \to 0\ . \end{array} \tag{7.38}$$

The bottom row of (7.38) will be called the *conormal sequence* of M at L (or of (M,L)). The following analog of Proposition 7.4 holds true with formally the same proof, using (7.38).

Proposition 7.29. *The maps* (7.30) *and* (7.31) *are surjective topological homomorphisms.*

Corollary 7.30. *The conormal sequence of M at L is exact in the category of continuous linear maps between LCSs.*

The surjectivity of (7.31) can be realized with partial extension maps, as stated in the following analog of Proposition 6.29.

Corollary 7.31. *For all $m \in \mathbb{R}$, there is a continuous linear partial extension map $E_m : J^m(M,L) \to I^{(s)}(M,L)$, where $s = 0$ if $m \geq 0$, and $m > s \in \mathbb{Z}^-$ if $m < 0$.*

Proof. By the commutativity of (7.38), we can take E_m equal to the composition

$$J^m(M,L) \xrightarrow{\boldsymbol{\pi}_*^{-1}} \mathcal{A}^m(\boldsymbol{M}) \xrightarrow{E_m} \dot{\mathcal{A}}^{(s)}(\boldsymbol{M}) \xrightarrow{\boldsymbol{\pi}_*} I^{(s)}(M,L)\ ,$$

where this map E_m is given by Proposition 6.29. □

Corollary 7.32. *$C^\infty(M)$ is dense in $J(M,L)$.*

Proof. Apply (7.28), Corollary 4.6 and Proposition 7.29. □

7.13. **Action of** Diff(M) **on the conormal sequence.** According to Section 4.5, every $A \in \mathrm{Diff}(M)$ defines a continuous linear map A on $I(M,L)$, which preserves $K(M,L)$ because A is local. Therefore it induces a linear map A on $J(M,L)$, which is continuous by Proposition 7.29. This map satisfies the analog of (4.16).

The map A on $J(M,L)$ can be also described as a restriction of A on $C^{-\infty}(M,L)$ (Section 7.6). On the other hand, according to Section 6.14, the lift $\widetilde{A} \in \mathrm{Diff}(\boldsymbol{M})$ defines continuous linear maps on the top spaces of (7.38) which correspond to the operators defined by A on the bottom spaces via the maps $\boldsymbol{\pi}_*$. If $A \in \mathrm{Diff}(M,L)$, then it defines continuous endomorphisms A of $J^{(s)}(M,L)$ and $J^m(M,L)$.

7.14. **Pull-back of elements of the conormal sequence.** Consider the notation and conditions of Section 4.6. By locality, the map (4.20) has a restriction $\phi^* : K(M,L) \to K(M',L')$. So it also induces a linear map $\phi^* : J(M,L) \to J(M',L')$, which is continuous by Proposition 7.29.

7.15. **Push-forward of elements of the conormal sequence.** Consider the notation and conditions of Section 4.7. As above, the map (4.22) has a restriction $\phi_* : K_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}}) \to K_{\mathrm{c}}(M,L)$. Thus it induces a linear map $\phi_* : J_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}}) \to J_{\mathrm{c}}(M,L)$, which is continuous by Proposition 7.29.

7.16. **Case where L is not transversely orientable.** If L is not transversely orientable, we still have a tubular neighborhood T of L in M, but there is no defining function x of L in T trivializing the projection $\varpi : T \to L$. We can cut M along L as well to produce a bounded compact manifold, $\boldsymbol{M}$, with a projection $\boldsymbol{\pi} : \boldsymbol{M} \to M$ and a boundary collar $\boldsymbol{T}$ over T.

By using a boundary defining function $\boldsymbol{x}$ of $\boldsymbol{M}$, we get the same definitions, properties and descriptions of $C^{\pm\infty}(M,L)$ and $J(M,L)$ (Sections 7.3, 7.7 and 7.8).

$C^{-\infty}_L(M)$ and $K(M,L)$ also have the same definitions (Sections 7.4 and 7.10). However (7.19) and (7.32) are not true because the covering map $\boldsymbol{\pi} : \partial\boldsymbol{M} \to L$ is not trivial, and the descriptions given in Propositions 7.7 and 7.26 need a slight modification. This problem can be solved as follows.

Let $\check\pi : \check L \to L$ denote the two-fold covering of transverse orientations of L, and let $\check\sigma$ denote its deck transformation different from the identity. Since the lift of NL to $\check L$ is trivial, $\check\pi$ on $\check L \equiv \{0\} \times \check L$ can be extended to a two-fold covering $\check\pi : \check T := (-\epsilon,\epsilon)_x \times \check L \to T$, for some $\epsilon > 0$. Its deck transformation different from the identity is an extension of $\check\sigma$ on $\check L \equiv \{0\} \times \check L$, also denoted by $\check\sigma$. Then $\check L$ is transversely oriented in $\check T$; i.e., its normal bundle $N\check L$ is trivial. Thus $C^{-\infty}_{\check L}(\check T)$ and $K(\check T,\check L)$ satisfy (7.19), (7.32) and Propositions 7.7 and 7.26. Since $N\check L \equiv \check\pi^*NL$, the map $\check\sigma$ lifts to a homomorphism of $N\check L$, which induces a homomorphism of $\Omega^{-1}NL$ also denoted by $\check\sigma$. Let L_{-1} be the union of non-transversely oriented connected components of L, and L_1 the union of its transversely oriented components. Correspondingly, let $\check L_{\pm1} = \check\pi^{-1}(L_{\pm1})$ and $\check T_{\pm1} = (-\epsilon,\epsilon) \times \check L_{\pm1}$. Since $\check\sigma^*x = \pm x$ on $T_{\pm1}$, Propositions 7.7 and 7.26 become true in this case by replacing $C^r(L;\Omega^{-1}NL)$ ($r \in \mathbb{Z} \cup \{\pm\infty\}$) with the direct sum of the spaces

$$\{\, u \in C^r(\check L_{\pm1};\Omega^{-1}N\check L_{\pm1}) \mid \check\sigma^*u = \pm u \,\} \,.$$

Now the other results about $C^{-\infty}_L(M)$ and $K(M,L)$, indicated in Sections 7.4, 7.5, 7.10 and 7.11, can be obtained by using these extensions of Propositions 7.7 and 7.26 instead of (7.19) and (7.32). Sections 7.12 to 7.15 also have straightforward extensions.

8. Dual-conormal sequence

8.1. **The spaces $K'(M,L)$ and $J'(M,L)$.** Consider the notation of Section 7 assuming that L is transversely oriented; the extension to the non-transversely orientable case can be made like in Section 7.16. Like in Sections 5.1 and 6.22, let

$$K'(M,L) = K(M,L;\Omega)' \,, \quad J'(M,L) = J(M,L;\Omega)' \,.$$

By (7.26) and (7.32),

$$\mathcal{K}'(\boldsymbol{M}) \equiv K'(M,L) \oplus K'(M,L) \,, \quad \dot{\mathcal{A}}'(\boldsymbol{M}) \equiv J'(M,L) \,. \tag{8.1}$$

Let also

$$\begin{cases} K'^{(s)}(M,L) = K^{(-s)}(M,L;\Omega)' , & K'^{m}(M,L) = K^{-m}(M,L;\Omega)' , \\ J'^{(s)}(M,L) = J^{(-s)}(M,L;\Omega)' , & J'^{m}(M,L) = J^{-m}(M,L;\Omega)' , \end{cases} \tag{8.2}$$

which satisfy the analogs of (8.1). Like in Section 6.22, for $s < s'$ and $m < m'$, we get continuous linear restriction maps

$$K'^{(s')}(M,L) \to K'^{(s)}(M,L) , \quad K'^{m}(M,L) \to K'^{m'}(M,L) ,$$

and continuous injections

$$J'^{(s')}(M,L) \subset J'^{(s)}(M,L) , \quad J'^{m'}(M,L) \subset J'^{m}(M,L) ,$$

forming projective spectra. By (8.1), its analogs for the spaces (8.2) and according to Section 6.22, the spaces $K'^{(s)}(M,L)$ and $K'^{m}(M,L)$ satisfy the analogs of (5.4) and (5.5), and the spaces $J'^{(s)}(M,L)$ and $J'^{m}(M,L)$ satisfy the analogs of (6.55) and (6.56). Using (8.1), we get the following consequences of Proposition 6.42 and Corollaries 6.43 and 6.44.

Corollary 8.1. *$K'(M,L)$ and $J'(M,L)$ are complete Montel spaces.*

Corollary 8.2. *$K'^{(s)}(M,L)$ and $J'^{(s)}(M,L)$ are bornological and barreled.*

Corollary 8.3. *$K'(M,L) \equiv \varprojlim K'^{(s)}(M,L)$ and $J'(M,L) = \bigcap_s J'^{(s)}(M,L)$.*

Like in Section 6.22, the versions of (7.15), (7.28) and Corollary 7.14 with ΩM induce continuous inclusions

$$C^{-\infty}(M) \supset J'(M,L) \supset C^{\infty}(M,L) . \tag{8.3}$$

8.2. **A description of $J'(M,L)$.** With the notation and conditions of Section 7.8, the identity (8.1) and Corollaries 6.48 to 6.50 have the following consequences.

Corollary 8.4. *$J'^{m}(M,L) \cong \boldsymbol{x}^m H_{\mathrm{b}}^{-\infty}(\boldsymbol{M}) = \boldsymbol{x}^{m-\frac{1}{2}} H^{-\infty}(\mathring{\boldsymbol{M}})$ ($m \in \mathbb{R}$).*

Corollary 8.5. *$J'(M,L) \cong \bigcap_m \boldsymbol{x}^m H_{\mathrm{b}}^{-\infty}(\boldsymbol{M}) = \bigcap_m \boldsymbol{x}^m H^{-\infty}(\mathring{\boldsymbol{M}})$.*

Corollary 8.6. *$C^{\infty}_{\mathrm{c}}(M \setminus L)$ is dense in every $J'^{m}(M,L)$ and in $J'(M,L)$. Therefore the right-hand side inclusion of (8.3) is also dense.*

The analog of Remark 6.51 makes sense for $J'(M,L)$.

8.3. **Description of $K'(M,L)$.** The version of Proposition 7.26 with ΩM has the following direct consequence, where we set

$$C^2_m = C^{\infty}(L; \Omega^{-1} NL \otimes \Omega M)' = C^{\infty}(L;\Omega)' = C^{-\infty}(L)$$

for every $m \in \mathbb{N}_0$.

Corollary 8.7. *The transposes of the versions of (7.36) and (7.37) with ΩM are TVS-isomorphisms,*

$$K'(M,L) \xrightarrow{\cong} \prod_{m=0}^{\infty} C^2_m , \quad K'^{(s)}(M,L) \xrightarrow{\cong} \prod_{m<s-1/2} C^2_m \quad (s > 1/2) .$$

8.4. **Dual-conormal sequence.** Transposing the density-bundles version of (7.38), we get the commutative diagram

$$\begin{array}{ccccc} 0 \leftarrow \mathcal{K}'(\boldsymbol{M}) & \xleftarrow{\iota^t} & \mathcal{A}'(\boldsymbol{M}) & \xleftarrow{R^t} & \dot{\mathcal{A}}'(\boldsymbol{M}) \leftarrow 0 \\ \boldsymbol{\pi}^* \big\uparrow & & \boldsymbol{\pi}^* \big\uparrow & & \boldsymbol{\pi}^* \big\uparrow \cong \\ 0 \leftarrow K'(M,L) & \xleftarrow{\iota^t} & I'(M,L) & \xleftarrow{R^t} & J'(M,L) \leftarrow 0\,. \end{array} \tag{8.4}$$

Its maps are compatible with the Sobolev and symbol order filtrations. Its bottom row will be called the *dual-conormal sequence* of M at L (or of (M,L)). The following analog of Proposition 6.45 holds true with formally the same proof, using Proposition 7.29 and Corollaries 7.13, 7.15 and 7.31.

Proposition 8.8. *The dual-conormal sequence of M at L is exact in the category of continuous linear maps between LCSs.*

8.5. **Action of** $\mathrm{Diff}(M)$ **on the dual-conormal sequence.** With the notation of Section 7.13, consider the actions of A^t and $\widetilde{A}^t$ on the bottom and top spaces of the version of (7.38) with ΩM and $\Omega\boldsymbol{M}$. Taking transposes again, we get induced actions of A and $\widetilde{A}$ on the bottom and top spaces of (8.4), which correspond one another via the linear maps $\boldsymbol{\pi}^*$. These maps satisfy the analogs of (5.7).

8.6. **Pull-back of elements of the dual-conormal sequence.** With the notation and conditions of Section 5.3, besides (5.9), we get continuous linear pull-back maps $\phi^* : K'(M,L) \to K'(M',L')$ and $\phi^* : J'(M,L) \to J'(M',L')$.

8.7. **Push-forward of elements of the dual-conormal sequence.** With the notation and conditions of Section 5.4, besides (5.11), we get continuous linear push-forward maps $\phi_* : K'_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}}) \to K'_{\mathrm{c}}(M,L)$ and $\phi_* : J'_{\mathrm{c}}(M',L';\Omega_{\mathrm{fiber}}) \to J'_{\mathrm{c}}(M,L)$.

8.8. $I(M,L)$ **vs** $I'(M,L)$**.**

Lemma 8.9. *For all $m \in \mathbb{N}_0$, $\boldsymbol{\pi}^*\big(H^{-m}(M) \cap I'(M,L)\big) \subset \dot{H}^{-m}(\boldsymbol{M}) \cap \mathcal{A}'(\boldsymbol{M})$.*

Proof. Using a volume form on M and its lift to $\boldsymbol{M}$ to define a scalar product of $L^2(M)$ and $L^2(\boldsymbol{M})$, it follows that $\boldsymbol{\pi}^* : C^\infty(M) \to C^\infty(\boldsymbol{M})$ induces a unitary isomorphism $\boldsymbol{\pi}^* : L^2(M) \to L^2(\boldsymbol{M})$. Hence the statement is true for $m = 0$ because $L^2(\boldsymbol{M}) \equiv \dot{H}^0(\boldsymbol{M})$ (Section 6.4). Then, for arbitrary m, by (2.22) and (6.25), and according to Section 7.1,

$$\begin{aligned} &\boldsymbol{\pi}^*\big(H^{-m}(M) \cap I'(M,L)\big) \\ &\qquad = \boldsymbol{\pi}^*\big(\mathrm{Diff}^m(M)\, L^2(M) \cap I'(M,L)\big) \subset \mathrm{Diff}^m(\boldsymbol{M})\, \boldsymbol{\pi}^* L^2(M) \cap \mathcal{A}'(\boldsymbol{M}) \\ &\qquad = \mathrm{Diff}^m(\boldsymbol{M})\, \dot{H}^0(\boldsymbol{M}) \cap \mathcal{A}'(\boldsymbol{M}) \subset \dot{H}^{-m}(\boldsymbol{M}) \cap \mathcal{A}'(\boldsymbol{M})\,. \quad \square \end{aligned}$$

Lemma 8.10. $\boldsymbol{\pi}^*\big(I(M,L) \cap I'(M,L)\big) \subset C^\infty(\boldsymbol{M})$.

Proof. For every $u \in I(M,L) \cap I'(M,L)$, there is some $m \in \mathbb{N}_0$ such that $u \in I^{(-m)}(M,L)$. Then, by Lemma 8.9, for any $B \in \mathrm{Diff}(M,L)$,

$$\widetilde{B}\boldsymbol{\pi}^* u = \boldsymbol{\pi}^* B u \in \boldsymbol{\pi}^*\big(H^{-m}(M) \cap I'(M,L)\big) \subset \dot{H}^{-m}(\boldsymbol{M}) \cap \mathcal{A}'(\boldsymbol{M})\,.$$

Since the operators $\widetilde{B}$ ($B \in \mathrm{Diff}(M,L)$) generate $\mathrm{Diff}(\widetilde{M},\widetilde{L})$ as $C^\infty(\widetilde{M})$-module, it follows that $u \in \dot{\mathcal{A}}(\boldsymbol{M}) \cap \mathcal{A}'(\boldsymbol{M}) = C^\infty(\boldsymbol{M})$ by (6.59). $\square$

Theorem 8.11. *$I(M,L)\cap I'(M,L)=C^\infty(M)$.*

Proof. Suppose there is some non-smooth $u\in I(M,L)\cap I'(M,L)$. However $\boldsymbol{\pi}^*u\in C^\infty(\boldsymbol{M})$ by Lemma 8.10. Then there is a chart (V,y) of L such that, for the induced chart $(U\equiv(-\epsilon,\epsilon)\times V,(x,y))$ of M, the function u is smooth on $((-\epsilon,0)\cup(0,\epsilon))\times V$, and has smooth extensions to $(-\epsilon,0]\times V$ and $[0,\epsilon)\times V$, but $\partial_x^m u(0^-,y_0)\neq\partial_x^m(0^+,y_0)$ for some $m\in\mathbb{N}_0$ and $y_0\in V$. After multiplying u by a smooth function supported in U whose value at y_0 is nonzero, we can assume u is supported in $(-\epsilon/2,\epsilon/2)\times V$. Then there is some $v\in C^\infty(L;\Omega)$ such that $\operatorname{supp}v\subset V$ and

$$\int_{y\in V}(u(0^-,y)-u(0^+,y))\,v(y)\neq 0\,. \tag{8.5}$$

On the other hand, there is a sequence $\phi_k\in C^\infty_{\mathrm{c}}(-\epsilon,\epsilon)$ so that the restrictions of mth derivatives $\phi_k^{(m)}$ to $(-\epsilon/2,\epsilon/2)$ are compactly supported and converge to δ_0 in $C^{-\infty}(-\epsilon/2,\epsilon/2)$ as $k\to\infty$. For instance, we may take

$$\phi_k(t)=h(t)\int_0^t\int_0^{t_{m-1}}\cdots\int_0^{t_1}f_k(t_0)\,dt_0\cdots dt_{m-1}\,,$$

where $h,f_k\in C^\infty_{\mathrm{c}}(-\epsilon,\epsilon)$ with $h=1$ on $(-\epsilon/2,\epsilon/2)$, $\operatorname{supp}f_k\subset(-\epsilon/2,\epsilon/2)$, f_k is even, and $f_k\to\delta_0$ in $C^{-\infty}_{\mathrm{c}}(-\epsilon/2,\epsilon/2)$ and $f_k(0)\to\infty$ as $k\to\infty$. Thus

$$\phi_k^{(m)}(0)=f_k(0)\to\infty\,, \tag{8.6}$$

$$\int_{-\infty}^0 a(t)\phi_k^{(m)}(t)\,dt\to\frac{a(0)}{2}\,,\quad\int_0^\infty b(t)\phi_k^{(m)}(t)\,dt\to\frac{b(0)}{2}\,, \tag{8.7}$$

for all $a\in C^\infty_{\mathrm{c}}(-\infty,0]$ and $b\in C^\infty_{\mathrm{c}}[0,\infty)$.

The sequence $w_k\equiv\phi_k(x)\,v(y)\otimes|dx|\in C^\infty_{\mathrm{c}}(T;\Omega)\subset C^\infty(M;\Omega)$ satisfies

$$\partial_x^m w_k\equiv\phi_k^{(m)}(x)\,v(y)\otimes|dx|\to\delta_0(x)\,v(y)\otimes|dx|\equiv\delta_L^v$$

in $I(M,L;\Omega)$ as $k\to\infty$. Since $u\in I'(M,L)$ and $\partial_x^{m+1}w_k\in I(M,L;\Omega)$, it follows that $\langle u,\partial_x^{m+1}w_k\rangle\to\langle u,\partial_x\delta_L^v\rangle$ as $k\to\infty$. But

$$\begin{aligned}\langle u,\partial_x^{m+1}w_k\rangle&=\int_{y\in V}\int_{-\epsilon/2}^0 u(x,y)\,\phi_k^{(m+1)}(x)\,v(y)\,dx\\&\quad+\int_{y\in V}\int_0^{\epsilon/2}u(x,y)\,\phi_k^{(m+1)}(x)\,v(y)\,dx\\&=\phi_k^{(m)}(0)\int_{y\in V}\big(u(0^-,y)-u(0^+,y)\big)\,v(y)\\&\quad-\int_{y\in V}\int_{-\epsilon/2}^0\partial_xu(x,y)\,\phi_k^{(m)}(x)\,v(y)\,dx\\&\quad-\int_{y\in V}\int_0^{\epsilon/2}\partial_xu(x,y)\,\phi_k^{(m)}(x)\,v(y)\,dx\,,\end{aligned}$$

which is divergent by (8.5)–(8.7). ∎

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