CENTRO INTERNACIONAL DE ESTUDOS DE DOUTORAMENTO E AVANZADOS DA USC (CIEDUS)

Tese de Doutoramento

Time depending dynamics of chains of evolution algebras

Sherzod Murodov

ESCOLA DE DOUTORAMENTO INTERNACIONAL

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

2019

DECLARACIÓN DO AUTOR DA TESE

Time depending dynamics of chains of evolution algebras

D. **Sherzod Murodov**

Presento a miña tese, seguindo o procedemento axeitado ao Regulamento, e declaro que:

1) A tese abarca os resultados da elaboración do meu traballo.

2) De selo caso, na tese faise referencia ás colaboracións que tivo este traballo.

3) A tese é a versión definitiva presentada para a súa defensa e coincide coa versión enviada en formato electrónico.

4) Confirmo que a tese non incorre en ningún tipo de plaxio doutros autores nin de traballos presentados por min para a obtención doutros títulos.

En Santiago de Compostela, 9 de abril de 2019

Asdo.: Sherzod Murodov

AUTORIZACIÓN DO DIRECTOR / TITOR DA TESE

Time depending dynamics of chains of evolution algebras

D. **Manuel Ladra Gonzalez**

D. **Utkir Rozikov**

INFORMAN:

Que a presente tese, correspóndese co traballo realizado por D. ***Sherzod Murodov****, baixo a miña dirección, e autorizo a súa presentación, considerando que reúne os requisitos esixidos no Regulamento de Estudos de Doutoramento da USC, e que como director desta non incorre nas causas de abstención establecidas na Lei 40/2015.*

En Santiago de Compostela, 9 de abril de 2019

Asdo.: Manuel Ladra Gonzalez

Asdo.: Utkir Rozikov

DEPARTAMENTO DE MATEMÁTICAS

Time depending dynamics of chains of evolution algebras

by

Sherzod Murodov

DISSERTATION

Submitted for the degree of

DOCTOR EN MATEMÁTICAS

UNIVERSIDAD DE SANTIAGO DE COMPOSTELA

Santiago de Compostela, 2019

The results presented in this thesis were obtained thanks to the project MTM2016-79661-P (European FEDER support included), Agencia Estatal de Investigación, Ministerio de Ciencia, Innovación y Universidades (Spain).

MTM2016-79661-P

Unión Europea
Fondo Europeo de Desarrollo Regional
Una manera de hacer EUROPA

Acknowledgements

I would like to acknowledge my indebtedness and express my sincere gratitude to my advisors Professor Manuel Ladra and Professor Utkir Abdulloyevich Rozikov for the fundamental role and continuous support of my Ph.D. study and research, for their patience, motivations, enthusiasm, and immense knowledge. Their guidance helped me in all the time of research and writing of this thesis.

I owe a lot to my parents, who encouraged and helped me at every stage of my personal and academic life, and longed to see this achievement come true. Very warm thanks go to my family for support and understanding.

I would like to acknowledge my teachers Academician Sh.A. Ayupov, Professors V.I. Chilin, R.N. Ganikhodjaev, F.T. Adilova, B.A. Omirov, D.Q. Durdiyev, Sh.M. Mirzayev, E. Normatov, F. Ibragimov, all members of Mathematics faculty of the National University of Uzbekistan and Bukhara state university and all the members of the Institute of Mathematics.

I would like to thank rector of Bukhara state medical institute Professor A.Sh. Inoyatov, Professor S.X. Umarov, O'.O. Xodjayev, R.R. Hamroyev, F.K. Xalloqov and all the members of the department of "Biophysics and information technologies in medicine".

I thank my teachers at High School R.S. Zabiyeva, Sh.A. Qarshiyeva.

I thank Xabier García-Martínez, Alejandro Fernández Fariña and Maria Pilar Páez-Guillán for making friendly atmosphere and their help and support.

Particular thanks go to all the members of Department of Matemáticas of the University of Santiago de Compostela for providing a great environment to work in. The support from staff of Faculty of Mathematics has been priceless.

I am very thankful to Uygun Jamilov, Rifqat Davronov, G'olib Botirov, Iqboljon Karimjanov and all my friends who gave me enough moral support, encouragement and motivation to accomplish the personal goals.

Contents

Introduction

Historically, mathematical methods have been applied successfully to population genetics for a long time. Investigation of mathematics to population genetics goes to Mendel's laws (Gregor Johann Mendel, 1822–1884), where he exploited symbols that are quite algebraically suggestive to express his genetic laws. Between 1856 and 1863, Mendel studied the hybridization of peas and gave the fundamental concept of the classical genetics, the gene, under the name "constant character" to explain the observed statistics of inheritance. In 1866, Gregor Mendel published the results of years of experimentation in breeding pea plants [32]. He showed that both parents must pass discrete physical factors which transmit information about their traits to their offspring at conception. Mendel first exploited symbols that are quite algebraically suggestive to express his genetic laws. Thus, mathematicians and geneticists once used non-associative algebras to study Mendelian genetics and it was later termed "Mendelian algebras" by several other authors.

In the 1920s and 1930s as other major achievements of theoretical population genetics the Hardy-Weinberg law, the Wright-Fisher selection model, general genetic algebras were introduced. These are basic to many calculations in population genetics. The mathematics used in the classic works of Etherington, Fisher, Haldane and Wright was also not very complicated but was of great help for the theoretical understanding of evolutionary processes. At present, the methods of mathematical genetics are becoming more and more complex: the theory of probability is used, including the theory of random processes, non-linear differential and difference equations, and non-associative algebras.

Serebrowsky [43] was the first to give an algebraic interpretation of the sign $\times$, which indicated sexual reproduction, and to give a mathematical formulation of Mendel's laws in terms of non-associative algebras. Glivenkov [18] introduced the so-called Mendelian algebras for diploid populations with one locus or two unlinked loci. Independently, Kostitzin [25] also introduced a "symbolic multiplication" to express Mendel's laws. The systematic study of algebras occurring in genetics can be attributed to a series of papers of I.M.H. Etherington in 1939–1941. In [11–13] he succeeded in giving a precise mathematical formulation of Mendel's laws in terms of non-associative algebras, also covered many simple cases in which he considered algebras describing the genetics. Besides Etherington, fundamental contributions have been made by Abraham [1], Gonshor [19], Heuch [21], Holgate [22, 23], Reiersöl [36], Schafer [42]. Until 1980s, the most comprehensive reference in this area was Wörz-Busekros's book [46], where the author gave a rather complete presentation of algebras in genetics and applied the theory and the results to various concrete biological situations. More recent results, such as genetic evolution in genetic algebras, can be found in Lyubich's book [31]. A good survey is Reed's article [35].

Thus, Mendelian genetics introduced a new subject to mathematics: general genetic algebras. These algebras are in general commutative but not associative, furthermore they do not belong to any of the well-known classes of non-associative algebras such as Lie algebras, Jordan algebras, or alternative algebras.

Baur [2] and Correns [8] first detected that chloroplast inheritance departed from Mendel's rules, and much later, mitochondrial gene inheritance was also identified in the same way, and non-Mendelian inheritance of organelle genes was recognized with two features – uniparental inheritance and vegetative segregation. Now, non-Mendelian genetics is a basic language of molecular geneticists. Non-Mendelian inheritance plays an important role in several disease processes. Non-Mendelian genetics offers to mathematics new type of genetic algebras, denominated evolution algebras, introduced in [45], these are algebras in which the multiplication tables are motivated by evolution laws of genetics. In [44] J.P. Tian established the foundation of the framework of evolution algebra theory and to discussed some applications of evolution algebras in non-Mendelian genetics, stochastic processes and genetics.

As a new type of algebra – the evolution algebras are non-associative and non-power-associative Banach algebras. Indeed, they are natural examples of non-associative complete normed algebras arising from science. Evolution algebras have connections with other fields of mathematics, including graph theory (particularly, random graphs and networks), group theory, Markov processes, dynamical systems.

Recently in [7] a notion of a chain of evolution algebras (CEA) is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The sequence of matrices of structural constants for this chain of evolution algebras satisfies the Chapman-Kolmogorov equation.

In [34] a chain of n-dimensional evolution algebras corresponding to a permutation of n numbers was defined. They also showed that a chain of evolution algebras (CEA) corresponding to a permutation is trivial (consisting only algebras with zero-multiplication) iff the permutation has not a fixed point.

In [29] the notion of flow (depending on time) of finite-dimensional algebras was introduced. A flow of algebras is a particular case of a continuous-time dynamical system whose states are finite-dimensional algebras with (cubic) matrices of structural constants satisfying an analogue of the Kolmogorov-Chapman equation. Since there are several kinds of multiplications between cubic matrices one has fix a multiplication first and then consider the Kolmogorov-Chapman equation with respect to the fixed multiplication. The existence of a solution for the Kolmogorov-Chapman equation provides the existence of a flow of algebras. The time-dependent behavior of the different examples of flows was constructed and studied in [30].

Outline of the thesis

In Chapter 1, the basic definitions and construction of chains of evolution algebras are given. In Section 1.1 we give a brief review of chains of evolution algebras (CEAs), following [7]. Section 1.2 is devoted to the construction of new two-dimensional real CEAs. In Section 1.3 we study property transitions of the constructed CEAs. In Theorem 1.3.5 baric property transitions of CEAs are given, and also the existence of “uniqueness of absolute nilpotent element” property transition and the time-dynamics of the idempotent elements of CEAs are given in Theorems 1.3.8 and 1.3.9, respectively.

In Chapter 2 we obtain the classification of two-dimensional real evolution algebras and time depending dynamics of such CEAs. In Theorem 2.1.2 we give the classification of two-dimensional real evolution algebras. In Section 2.2 we study time depending dynamics of two-dimensional real CEAs (Theorem 2.2.2). We also construct new classes of two-dimensional CEAs, which will be isomorphic to a given evolution algebra, for some values of time (Theorem 2.2.6). In Section 2.3 we define (linear) Rota-Baxter operators on evolution algebras. Theorem 2.3.2 consists of all matrices of these linear operators.

In Chapter 3 we construct chains of evolution algebras of a “chicken” population, and also study the time depending dynamics of the constructed chains of evolution algebras of a “chicken” population (Proposition 3.3.2 and Theorem 3.3.5).

Chapter 1

Chains of finite-dimensional evolution algebras

In this chapter we give the basic definitions and construction of chains of evolution algebras. First, we provide a brief review of known chains of evolution algebras (CEAs). Second, we construct new two-dimensional real CEAs. Then we study property transitions of constructed CEAs, in particular, baric property transitions of CEAs and the existence of "uniqueness of absolute nilpotent element" property transition and the time-dynamics of the idempotent elements of CEAs are given.

1.1 Basic definitions

Before we discuss the mathematics of genetics, we need to acquaint ourselves with the necessary language from biology. The **genes** consist of the DNA (deoxyribonucleic acid), so called the chemical basis of heredity, which are carry hereditary information passed from parents to children. Genes are carried in the **chromosomes**, the physical basis of heredity, within the cell, which are package of **DNA** that carries genetic information. Each gene on a chromosome has different forms that it can take. These forms are called **alleles**, which produce variations in a genetically inherited trait. E.g. different alleles produce different hair colours – brown, blond, red, black, etc. Since humans are diploid organisms (meaning we have two sets of 23 chromosomes – one set from each parent), hair types are determined by two alleles. Haploid cells (or organisms) carry a single set of chromosomes. When diploid organisms reproduce, a process called meiosis produces gametes (sex cells) which carry a single set of chromosomes. When these gamete cells fuse (e.g. when sperm fertilizes egg), the result is a zygote, which is again a diploid cell, meaning it carries its hereditary information in a double set of chromosomes. When gametes fuse (or reproduce) to form zygotes a natural “multiplication” operation occurs.

Algebras arise in population genetic models in a quite natural way. If populations with one or several genetic traits are considered then each particular population is characterized by the frequencies of these traits, e.g. gene or genotype frequencies. These frequencies are collected into a vector. Convex combinations of such vectors describe mixtures of populations, thus the addition of vectors and the multiplication with constants have a meaning. Sexual reproduction leads to a multiplicative structure, as it is suggested by a formal representation of Mendel’s laws.

Example 1.1.1 ([35]). (**Simple Mendelian Inheritance**). As a natural first example, we consider simple Mendelian inheritance for a single gene with two alleles A and a. In this case, two gametes fusing (or reproducing) to form a zygote gives the multiplication table shown in Table 1.1, which in freshman biology class might be called a Punnett square.

	A	a
A	AA	Aa
a	aA	aa

Table 1.1: Alleles passing from gametes to zygotes

The zygotes AA and aa are called homozygous, since they carry two copies of the same allele. In this case, simple Mendelian inheritance means that there is no chance involved as to what genetic information will be inherited in the next generation; i.e. AA will pass on the allele A and aa will pass on a. However, the zygotes Aa and aA (which are equivalent) each carry two different alleles. These zygotes are called heterozygous.

	A	a
A	A	$\frac{1}{2}(A+a)$
a	$\frac{1}{2}(a+A)$	a

Table 1.2: Multiplication table of the gametic algebra for simple Mendelian inheritance

The rules of simple Mendelian inheritance indicate that the next generation will inherit either A or a with equal frequency. So, when two gametes reproduce, a multiplication is induced which indicates how the hereditary information will be passed down to the next generation. This multiplication is given by the following rules:

$$A \times A = A \tag{1.1.1}$$

$$A \times a = \frac{1}{2}(A+a) \tag{1.1.2}$$

$$a \times A = \frac{1}{2}(a+A) \tag{1.1.3}$$

$$a \times a = a \tag{1.1.4}$$

Rules (1.1.1) and (1.1.4) are expressions of the fact that if both gametes carry the same allele, then the offspring will inherit it. Rules (1.1.2) and (1.1.3) indicate that when gametes carrying A and a reproduce, half of the time the offspring will inherit A and the other half of the time it will inherit a. These rules are an algebraic representation of the rules of simple Mendelian inheritance. This multiplication table is shown in Table 1.2. We should point out that we are only concerning ourselves with genotypes (gene composition) and not phenotypes (gene expression). Hence we have made no mention of the dominant or recessive properties of our alleles.

Now we have defined a multiplication on the symbols A and a we can mathematically define the two-dimensional algebra over $\mathbb{R}$ with basis $\{A, a\}$ and multiplication table as in Table 1.2. This algebra is called the ***gametic algebra*** for simple Mendelian inheritance with two alleles.

It is important to remark that there exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose research has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called "genetic".

The systematic formulation of reproduction in non-Mendelian genetics as multiplication in algebras was introduced in [44] and called "evolution algebras". These are algebras in which the multiplication tables are motivated by evolution laws of genetics.

We define evolution algebras in terms of generators and defining relations. Because the defining relations are unique for an evolution algebra, the generator set can serve as a basis for an evolution algebra [44]. This property gives some advantage in studying evolution algebras.

Definition 1.1.2 ([44]). Let $(E, \cdot)$ be an algebra over a field $\mathbb{K}$. If it admits a basis $\{e_1, e_2, \dots\}$, such that

$$e_i \cdot e_j = \begin{cases} 0, & \text{if } \ i \neq j; \\ \displaystyle\sum_k a_{ik} e_k, & \text{if } \ i = j, \end{cases}$$

then this algebra is called an *evolution algebra*. The basis is called a *natural basis*.

Denote by $\mathcal{M} = (a_{ij})$ the matrix of structural constants of the evolution algebra E.

Evolution algebras have the following properties [44]:

1. Evolution algebras are not associative, in general.
2. Evolution algebras are commutative, flexible.
3. Evolution algebras are not power-associative, in general.
4. The direct sum of evolution algebras is also an evolution algebra.
5. The Kronecker product of evolution algebras is an evolution algebra.

Recently in [7] a notion of chain of evolution algebras is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The chain is defined by the sequence of matrices of structural constants (of the evolution algebras considered in [44]) which satisfies the Chapman-Kolmogorov equation.

Following [7] we consider a family $\{E^{[s,t]} : s,t \in \mathbb{R},\ 0 \le s \le t\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$, with basis $e_1, \dots, e_n$, and the multiplication table

$$e_i e_i = \sum_{j=1}^{n} a_{ij}^{[s,t]} e_j, \quad i = 1, \dots, n; \quad e_i e_j = 0, \quad i \neq j.$$

Here parameters s, t are considered as time.

Denote by $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,\dots,n}$ the matrix of structural constants.

Definition 1.1.3. A family $\{E^{[s,t]} : s,t \in \mathbb{R},\ 0 \le s \le t\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$ is called a *chain of evolution algebras* (CEA) if the matrix $\mathcal{M}^{[s,t]}$ of structural constants satisfies the Chapman-Kolmogorov equation

$$\mathcal{M}^{[s,t]} = \mathcal{M}^{[s,\tau]} \mathcal{M}^{[\tau,t]}, \quad \text{for any} \quad s < \tau < t. \tag{1.1.5}$$

If ρ_i is a projection map of $E^{[s,t]}$, which maps every element of $E^{[s,t]}$ to its e_i component, then equation (1.1.5) can be written as

$$M_i^{[s,t]} = \sum_{j=1}^{n} \rho_j(M_i^{[s,\tau]}) M_j^{[\tau,t]}, \quad \text{for any} \quad s < \tau < t.$$

Definition 1.1.4. A CEA is called a *time-homogenous* CEA if the matrix $\mathcal{M}^{[s,t]}$ depends only on $t - s$. In this case we write $\mathcal{M}^{[t-s]}$.

Definition 1.1.5. A CEA is called *periodic* if its matrix $\mathcal{M}^{[s,t]}$ is periodic with respect to at least one of the variables s, t, i.e. (periodicity with respect to t) $\mathcal{M}^{[s,t+P]} = \mathcal{M}^{[s,t]}$ for all values of t. The constant P is called the period, and is required to be nonzero.

Remark 1.1.6 ([7]). In general, an algebra $\mathcal{A}^{[s,t]}$ can be given by a cubic matrix $\mathcal{M}^{[s,t]} = \left(a_{ijk}^{[s,t]}\right)_{i,j,k=1,\dots,n}$ of structural constants. Our Definition 1.1.3 can be extended to $\mathcal{A}^{[s,t]}$ using analogues of the Chapman-Kolmogorov equations for quadratic operators (see [15, 16, 41]). Since in the general case there are two types of the Chapman-Kolmogorov equations: type A and type B [15], one also can define two types of chain of (general) algebras using the Chapman-Kolmogorov equations of type A and type B, respectively. In this paper we shall only consider CEA, which is more simple than the general case, because it is defined by quadratic matrices.

If $\left\{\mathcal{M}^{[s,t]},\;\; 0\le s\le t\right\}$ is a family of stochastic matrices which satisfies the equation (1.1.5), then it defines a Markov process. Thus we have

Theorem 1.1.7 ([7]). *For each Markov process, there is a CEA whose structural constants are transition probabilities of the process, and whose generator set (basis) is the state space of the Markov process.*

If $\mathcal{M}^{[s,t]}$ does not depend on time (i.e. $\mathcal{M}^{[s,t]}=\mathcal{M}$), then the CEA contains only one evolution algebra E. Note that for a Markov chain defined by $\mathcal{M}$ the corresponding E has been studied in [44].

The following examples are considered in [7].

Example 1.1.8. To show a time dependent CEA we use the following example of time homogenous Markov process (see [24]). For $n=3$, we consider

$$\begin{aligned}
a_{ii}^{[t]} &= \frac{2}{3}e^{-\frac{3}{2}At}\cos(\alpha t)+\frac{1}{3}, \qquad i=1,2,3;\\
a_{12}^{[t]} = a_{23}^{[t]} = a_{31}^{[t]} &= e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}}\sin(\alpha t)-\frac{1}{3}\cos(\alpha t)\right)+\frac{1}{3};\\
a_{21}^{[t]} = a_{32}^{[t]} = a_{13}^{[t]} &= -e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}}\sin(\alpha t)+\frac{1}{3}\cos(\alpha t)\right)+\frac{1}{3},
\end{aligned}$$

where $A>0$, $\alpha=\frac{\sqrt{3}}{2}A$.

Let $E^{[t]}$, $t\ge 0$, be the corresponding CEA. It is easy to see that $E^{[t]}$ has an oscillation behavior depending on time t. Moreover $\lim_{t\to+\infty}E^{[t]}=E$, where E is an evolution algebra with the multiplication table

$$e_1^2=e_2^2=e_3^2=\frac{1}{3}(e_1+e_2+e_3), \quad e_ie_j=0,\; i\ne j.$$

The CEA correspond to a family of matrices which do not define a process.

Example 1.1.9. We shall give a time homogenous CEA which are different from CEAs corresponding to Markov processes. For $n = 2$ take

$$a_{11}^{[t]} = a_{22}^{[t]} = a^{[t]}; \quad a_{12}^{[t]} = a_{21}^{[t]} = b^{[t]}.$$

Then equation (1.1.5) is equivalent to

$$a^{[t]} = a^{[\tau]}a^{[t-\tau]} + b^{[\tau]}b^{[t-\tau]};$$

$$b^{[t]} = a^{[\tau]}b^{[t-\tau]} + b^{[\tau]}a^{[t-\tau]}.$$

Denote $f(t) = a^{[t]}+b^{[t]}$, $\varphi(t) = a^{[t]}-b^{[t]}$, then the last system of functional equations can be written as

$$f(t) = f(\tau)f(t-\tau), \quad \varphi(t) = \varphi(\tau)\varphi(t-\tau).$$

Both these equations are known as exponential Cauchy equation and the system of equations has solution $f(t) = \lambda^t$, $\varphi(t) = \mu^t$, where $\lambda, \mu \geq 0$. Consequently, $a^{[t]} = \frac{1}{2}(\lambda^t + \mu^t)$, $b^{[t]} = \frac{1}{2}(\lambda^t - \mu^t)$. But this solution does not define any Markov process, in general.

Let $E^{[t]}, t \geq 0$, be the corresponding CEA. Depending on parameters λ and μ we get distinct behavior of $E^{[t]}$ for $t \to +\infty$, i.e. we have

$$\lim_{t\to+\infty} E^{[t]} = \begin{cases} E_0 & \text{if } \ 0 < \lambda, \mu < 1, \\ E_1 & \text{if } \ \lambda = \mu = 1, \\ E_{1/2} & \text{if } \ \lambda = 1, 0 \leq \mu < 1, \\ E_{-1/2} & \text{if } \ \mu = 1, 0 \leq \lambda < 1, \\ E_\infty & \text{otherwise}, \end{cases}$$

where E_0 is an evolution algebra with zero multiplication; E_1 is an evolution algebra with the multiplication table

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1e_2 = 0;$$

$E_{1/2}$ is an evolution algebra with the multiplication table

$$e_1^2 = e_2^2 = \frac{1}{2}(e_1 + e_2), \quad e_1e_2 = 0;$$

$E_{-1/2}$ is an evolution algebra with the multiplication table

$$e_1^2 = \frac{1}{2}(e_1 - e_2), \quad e_2^2 = -\frac{1}{2}(e_1 - e_2), \quad e_1e_2 = 0;$$

and E_∞ is a vector space which has "infinity multiplication", or we can say that in E_∞ an algebra structure is not defined. This example shows that a limit of a CEA can be non evolution algebra.

Example 1.1.10. *A two-state evolution.* Now we shall give an example of time non-homogeneous CEA, the matrix of structural constants of which also does not define any (time non homogenous) Markov process in general. Consider $n = 2$ and the matrix $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ with

$$a_{11}^{[s,t]} = \tfrac{1}{2}\left(1 + \alpha(s,t) + \beta(s,t)\right), \quad a_{12}^{[s,t]} = \tfrac{1}{2}\left(1 - \alpha(s,t) - \beta(s,t)\right),$$
$$a_{21}^{[s,t]} = \tfrac{1}{2}\left(1 + \alpha(s,t) - \beta(s,t)\right), \quad a_{22}^{[s,t]} = \tfrac{1}{2}\left(1 - \alpha(s,t) + \beta(s,t)\right).$$

In this case the equation (1.1.5) is equivalent to (see [20])

$$\begin{aligned} \alpha(s,t) &= \alpha(\tau,t) + \alpha(s,\tau)\beta(\tau,t), \\ \beta(s,t) &= \beta(s,\tau)\beta(\tau,t), \qquad s < \tau < t. \end{aligned} \tag{1.1.6}$$

The second equation of the system (1.1.6) is known as Cantor's second equation, it has very rich family of solutions: $\beta(s,t) = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$. Using this function β for the function α we obtain

$$\frac{\alpha(s,t)}{\Phi(t)} = \frac{\alpha(\tau,t)}{\Phi(t)} + \frac{\alpha(s,\tau)}{\Phi(\tau)}.$$

Now denote $\gamma(s,t)=\frac{\alpha(s,t)}{\Phi(t)}$. Then the last equation gets the following form

$$\gamma(s,t)=\gamma(s,\tau)+\gamma(\tau,t).$$

This equation is known as Cantor's first equation which also has very rich family of solutions: $\gamma(s,t)=\Psi(t)-\Psi(s)$, where Ψ is an arbitrary function. Hence a solution $\mathcal{M}^{[s,t]}=\left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ to the equation (1.1.5) is given by

$$a_{11}^{[s,t]}=\frac{1}{2}\left(1+\Phi(t)(\Psi(t)-\Psi(s))+\frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{12}^{[s,t]}=\frac{1}{2}\left(1-\Phi(t)(\Psi(t)-\Psi(s))-\frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{21}^{[s,t]}=\frac{1}{2}\left(1+\Phi(t)(\Psi(t)-\Psi(s))-\frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{22}^{[s,t]}=\frac{1}{2}\left(1-\Phi(t)(\Psi(t)-\Psi(s))+\frac{\Phi(t)}{\Phi(s)}\right).$$

Let $E^{[s,t]}$, $0\leq s\leq t$, be the corresponding to this solution CEA. This CEA varies by two parameters, for example, if $t=s$, we get $E^{[t,t]}=E$ with multiplication table $e_1^2=e_1, e_2^2=e_2, e_1e_2=0$. Moreover, choosing functions Φ and Ψ one can variate the limit behavior of the CEA. For example, if Φ and Ψ are such that $\lim_{t\to+\infty}\Phi(t)\Psi(t)=\lim_{t\to+\infty}\Phi(t)=0$, then for a fixed s we have $\lim_{t\to+\infty}E^{[s,t]}=E_{1/2}$, where $E_{1/2}$ is an evolution algebra with multiplication table

$$e_1^2=e_2^2=\frac{1}{2}(e_1+e_2),\qquad e_1e_2=0.$$

Example 1.1.11. *An n-dimensional time non-homogenous CEA.* Here for arbitrary n we shall give an example of time non-homogenous CEA. Let $\{A^{[t]}, t\geq 0\}$ be a family of invertible $n\times n$ matrices, for all t. Define the following matrix

$$M^{[s,t]}=A^{[s]}(A^{[t]})^{-1},\quad \text{where } \left(A^{[t]}\right)^{-1} \text{ is the inverse of } A^{[t]}.$$

This matrix satisfies the equation (1.1.5). Indeed, using associativity of the multiplication of matrices we get

$$\mathcal{M}^{[s,\tau]}\mathcal{M}^{[\tau,t]} = A^{[s]}\left((A^{[\tau]})^{-1}A^{[\tau]}\right)(A^{[t]})^{-1} = A^{[s]}(A^{[t]})^{-1} = \mathcal{M}^{[s,t]}.$$

Thus each family (with one parameter) of invertible $n \times n$ matrices defines a CEA $E^{[s,t]}$ which is time non-homogenous, in general. But will be a time homogenous CEA, for example, if $A^{[t]}$ is equal to t-th power of an invertible matrix A.

The construction of a family of invertible $n \times n$ matrices $A^{[t]}$ is not difficult, for example, one can take $A^{[t]}$ as a triangular $n \times n$ matrix of the form

$$A^{[t]} = \begin{pmatrix} a_{11}^{[t]} & 0 & 0 & \dots & 0 \\ a_{21}^{[t]} & a_{22}^{[t]} & 0 & \dots & 0 \\ a_{31}^{[t]} & a_{32}^{[t]} & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1}^{[t]} & a_{n2}^{[t]} & \dots & a_{nn-1}^{[t]} & a_{nn}^{[t]} \end{pmatrix}.$$

which is called lower triangular matrix or one can take an upper triangular matrix. Then the matrices are invertible iff $a_{ii}^{[t]} \neq 0$, for all $i = 1, \dots, n$, and t. So this example also gives a very rich class of CEAs.

Example 1.1.12. *Periodic CEA*. To get a periodic CEA, we can consider the $E^{[s,t]}$ constructed in Example 1.1.10, and choose Φ and Ψ as periodic (non-constant) functions. Then the corresponding CEA is periodic. In this case for any fixed s, the limit $\lim_{t\to+\infty} E^{[s,t]}$ does not exist in general, moreover its set of limit points (evolution algebras) can be a continuum set. We shall make this point clear as follows. We construct a time homogenous CEA which is periodic. Consider $n = 2$ and take

$$a_{11}^{[t]} = a_{22}^{[t]} = a^{[t]}; \quad a_{12}^{[t]} = -b^{[t]}, \quad a_{21}^{[t]} = b^{[t]}.$$

Then equation (1.1.5) is equivalent to

$$a^{[t]} = a^{[\tau]}a^{[t-\tau]} - b^{[\tau]}b^{[t-\tau]};$$

$$b^{[t]} = a^{[\tau]}b^{[t-\tau]} + b^{[\tau]}a^{[t-\tau]}.$$

This system reminds the following identities

$$\cos t = \cos\tau\cos(t-\tau) - \sin\tau\sin(t-\tau);$$

$$\sin t = \cos\tau\sin(t-\tau) + \sin\tau\cos(t-\tau).$$

Consequently, one solution $\mathcal{M}^{[t]} = \left(a_{ij}^{[t]}\right)_{i,j=1,2}$ to equation (1.1.5) is

$$\mathcal{M}^{[t]} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Since that matrix is periodic with period $P = 2\pi$, the corresponding CEA $E^{[t]}$ is also periodic. Moreover this CEA is very interesting: for an arbitrary 2-dimensional evolution algebra E_a^+, or E_a^-, $a \in [-1,1]$, with matrix of structural constants

$$\mathcal{M}_a^{\pm} = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \mp\sqrt{1-a^2} & a \end{pmatrix},$$

respectively, there is a sequence $t_n = t_n(a)$ of times such that $\lim_{n\to\infty} E^{[t_n]} = E_a^+$ or E_a^-. We have $E_a^{\pm} \neq E_b^{\pm}$ if $a \neq b$.

Moreover the following is true.

Proposition 1.1.13 ([7]).

(1) For any $a, b \in [-1,1]$, $a \neq \pm b$, the algebras E_a^+ and E_b^+ are not isomorphic. The algebras E_a^+ and E_{-a}^+ are isomorphic.

(2) For any $a, b \in [-1,1]$, $a \neq \pm b$, the algebras E_a^- and E_b^- are not isomorphic. The algebras E_a^- and E_{-a}^- are isomorphic.

1.2 Construction of chains of two-dimensional evolution algebras

In [7] several concrete examples of chains of evolution algebras are given and their time-dynamics are studied. We continue the research of CEAs, in more detail, we study the CEAs generated by two-dimensional evolution algebras.

To construct a chain of two-dimensional evolution algebras one has to solve equation (1.1.5) for the 2×2 matrix $\mathcal{M}^{[s,t]}$. This equation gives the following system of functional equations (with four unknown functions):

$$\begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{12}^{[s,t]} &= a_{11}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{22}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{21}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{22}^{[\tau,t]}. \end{aligned} \tag{1.2.1}$$

But the analysis of the system (1.2.1) is difficult. We shall consider several cases where the system is solvable:

Case 1. $a_{11}^{[s,t]} = a_{22}^{[s,t]} = \alpha(s,t)$, $a_{12}^{[s,t]} = a_{21}^{[s,t]} = \beta(s,t)$. Then equation (1.2.1) is reduced to

$$\alpha(s,t) = \alpha(s,\tau)\alpha(\tau,t) + \beta(s,\tau)\beta(\tau,t);$$

$$\beta(s,t) = \alpha(s,\tau)\beta(\tau,t) + \beta(s,\tau)\alpha(\tau,t).$$

Denote

$$f(s,t) = \alpha(s,t) + \beta(s,t), \quad \varphi(s,t) = \alpha(s,t) - \beta(s,t). \tag{1.2.2}$$

Then the last system of functional equations can be written as

$$f(s,t) = f(s,\tau)f(\tau,t), \quad \varphi(s,t) = \varphi(s,\tau)\varphi(\tau,t), \quad s \le \tau \le t.$$

Both these equations are known as Cantor's second equation which has a very rich family of solutions:

a) $f(s,t) \equiv 0$;

b) $f(s,t) = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$;
c)

$$f(s,t) = \begin{cases} 1, & \text{if } s \le t < a, \\ 0, & \text{if } t \ge a. \end{cases} \quad \text{where } a > 0.$$

Similarly,
a') $\varphi(s,t) \equiv 0$;
b') $\varphi(s,t) = \frac{\Psi(t)}{\Psi(s)}$, where Ψ is an arbitrary function with $\Psi(s) \neq 0$;
c')

$$\varphi(s,t) = \begin{cases} 1, & \text{if } s \le t < b, \\ 0, & \text{if } t \ge b. \end{cases} \quad \text{where } b > 0.$$

Substituting these solutions into (1.2.2) and finding $\alpha(s,t)$ and $\beta(s,t)$ we get the following matrices

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \qquad \mathcal{M}_1^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{\Psi(t)}{\Psi(s)} & -\frac{\Psi(t)}{\Psi(s)} \\ -\frac{\Psi(t)}{\Psi(s)} & \frac{\Psi(t)}{\Psi(s)} \end{pmatrix};$$

$$\mathcal{M}_2^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \text{if } s \le t < b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \ge b. \end{cases}$$

$$\mathcal{M}_3^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \\ \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \end{pmatrix};$$

$$\mathcal{M}_4^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)} & \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} \\ \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} & \frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)} \end{pmatrix};$$

$$\mathcal{M}_5^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} + 1 & \frac{\Phi(t)}{\Phi(s)} - 1 \\ \frac{\Phi(t)}{\Phi(s)} - 1 & \frac{\Phi(t)}{\Phi(s)} + 1 \end{pmatrix}, & \text{if } s \le t < b; \\ \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \\ \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \end{pmatrix}, & \text{if } t \ge b. \end{cases}$$

$$\mathcal{M}_6^{[s,t]} = \frac{1}{2}\begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \ s \le t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \ t \ge a. \end{cases}$$

$$\mathcal{M}_7^{[s,t]} = \frac{1}{2}\begin{cases} \begin{pmatrix} 1+\frac{\Psi(t)}{\Psi(s)} & 1-\frac{\Psi(t)}{\Psi(s)} \\ 1-\frac{\Psi(t)}{\Psi(s)} & 1+\frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if } \ s \le t < a; \\ \begin{pmatrix} \frac{\Psi(t)}{\Psi(s)} & -\frac{\Psi(t)}{\Psi(s)} \\ -\frac{\Psi(t)}{\Psi(s)} & \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if } \ t \ge a. \end{cases}$$

$$\mathcal{M}_8^{[s,t]} = \frac{1}{2}\begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, & \text{if } \ s \le t < \min\{a,b\}; \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \text{if } \ a \le t < b, \ a < b; \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \ b \le t < a, \ a > b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \ t \ge \max\{a,b\}. \end{cases}$$

Thus in this case we have nine CEAs: $E_i^{[s,t]}$, $0 \le s \le t$, which correspond to the $\mathcal{M}_i^{[s,t]}$, $i = 0, 1, \dots, 8$, listed above.

Case 2. $a_{11}^{[s,t]} = a_{22}^{[s,t]} = \alpha(s,t)$, $a_{12}^{[s,t]} = -a_{21}^{[s,t]} = \beta(s,t)$. Then equation (1.2.1) is reduced to

$$\begin{aligned} \alpha(s,t) &= \alpha(s,\tau)\alpha(\tau,t) - \beta(s,\tau)\beta(\tau,t); \\ \beta(s,t) &= \alpha(s,\tau)\beta(\tau,t) + \beta(s,\tau)\alpha(\tau,t). \end{aligned} \tag{1.2.3}$$

It is easy to check that this system of functional equations has a solution

$$\alpha(s,t) = \cos(t-s), \qquad \beta(s,t) = \sin(t-s).$$

This gives the following matrix

$$\mathcal{M}_9^{[s,t]} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}.$$

Note that this matrix defines a periodic CEA. But we do not know another solution (with $\alpha\beta \neq 0$) of the system (1.2.3).

We denote by $E_9^{[s,t]} = E_9^{[t-s]}$ the CEA which corresponds to $\mathcal{M}_9^{[s,t]}$.

Case 3. $a_{11}^{[s,t]} = a_{21}^{[s,t]} = \alpha(s,t)$, $a_{12}^{[s,t]} = a_{22}^{[s,t]} = \beta(s,t)$. In this case the equation (1.2.1) is reduced to

$$\alpha(s,t) = \alpha(\tau,t)(\alpha(s,\tau) + \beta(s,\tau));$$

$$\beta(s,t) = \beta(\tau,t)(\alpha(s,\tau) + \beta(s,\tau)).$$

Denote

$$\gamma(s,t) = \alpha(s,t) + \beta(s,t), \quad \delta(s,t) = \alpha(s,t) - \beta(s,t). \tag{1.2.4}$$

Then the last system of functional equations can be written as

$$\gamma(s,t) = \gamma(s,\tau)\gamma(\tau,t), \quad \delta(s,t) = \gamma(s,\tau)\delta(\tau,t), \quad s \leq \tau \leq t.$$

The first equation is Cantor's second equation which has the solutions:

a) $\gamma(s,t) \equiv 0$;

b) $\gamma(s,t) = \frac{h(t)}{h(s)}$, where h is an arbitrary function with $h(s) \neq 0$;

c)

$$\gamma(s,t) = \begin{cases} 1, & \text{if } \ s \leq t < a, \\ 0, & \text{if } \ t \geq a, \end{cases} \qquad \text{where } \ a > 0.$$

Using these solution from the second equation we find δ:

a') $\delta(s,t) \equiv 0$;

b') $\delta(s,t) = \frac{g(t)}{h(s)}$, where g is an arbitrary function;

c')

$$\delta(s,t) = \begin{cases} \psi(t), & \text{if } \ s \leq t < a, \\ 0, & \text{if } \ t \geq a, \end{cases}$$

where $a > 0$, and $\psi(t)$ is an arbitrary function.

Substituting these solutions into (1.2.4) we get the following (non-zero) matrices

$$\mathcal{M}_{10}^{[s,t]} = \frac{1}{2}\begin{pmatrix} \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \\ \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \end{pmatrix}.$$

$$\mathcal{M}_{11}^{[s,t]} = \frac{1}{2}\begin{cases} \begin{pmatrix} 1+\psi(t) & 1-\psi(t) \\ 1+\psi(t) & 1-\psi(t) \end{pmatrix}, & \text{if } \ s \le t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \ t \ge a. \end{cases}$$

Thus in this case we have two new CEAs: $E_i^{[s,t]}$, $0 \le s \le t$, which correspond to the $\mathcal{M}_i^{[s,t]}$, $i = 10, 11$, listed above.

Case 4. $a_{11}^{[s,t]} = a_{12}^{[s,t]} = \alpha(s,t)$, $a_{21}^{[s,t]} = a_{22}^{[s,t]} = \beta(s,t)$. In this case the equation (1.2.1) is reduced to

$$\alpha(s,t) = \alpha(s,\tau)(\alpha(\tau,t) + \beta(\tau,t));$$

$$\beta(s,t) = \beta(s,\tau)(\alpha(\tau,t) + \beta(\tau,t)).$$

Denote

$$\gamma(s,t) = \alpha(s,t) + \beta(s,t), \qquad \delta(s,t) = \alpha(s,t) - \beta(s,t).$$

Then the last system of functional equations can be written as

$$\gamma(s,t) = \gamma(s,\tau)\gamma(\tau,t), \qquad \delta(s,t) = \delta(s,\tau)\gamma(\tau,t), \quad s \le \tau \le t.$$

The first equation has solutions:

a) $\gamma(s,t) \equiv 0$;

b) $\gamma(s,t) = \frac{h(t)}{h(s)}$, where h is an arbitrary function with $h(s) \neq 0$;

c)

$$\gamma(s,t) = \begin{cases} 1, & \text{if } \ s \le t < a, \\ 0, & \text{if } \ t \ge a, \end{cases} \qquad \text{where } \ a > 0.$$

Using these solution from the second equation we find δ:
a') $\delta(s,t) \equiv 0$;
b') $\delta(s,t) = g(s)h(t)$, where g is an arbitrary function;
c')

$$\delta(s,t) = \begin{cases} \psi(s), & \text{if } \ s \le t < a, \\ 0, & \text{if } \ t \ge a, \end{cases}$$

where $a > 0$, and $\psi(t)$ is an arbitrary function.

Consequently, we get the following (non-zero) new matrices

$$\mathcal{M}_{12}^{[s,t]} = \frac{1}{2} \begin{pmatrix} h(t)\left(\frac{1}{h(s)} + g(s)\right) & h(t)\left(\frac{1}{h(s)} + g(s)\right) \\ h(t)\left(\frac{1}{h(s)} - g(s)\right) & h(t)\left(\frac{1}{h(s)} - g(s)\right) \end{pmatrix};$$

$$\mathcal{M}_{13}^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} 1+\psi(s) & 1+\psi(s) \\ 1-\psi(s) & 1-\psi(s) \end{pmatrix}, & \text{if } \ s \le t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \ t \ge a. \end{cases}$$

Hence in this case we have two new CEAs: $E_i^{[s,t]}$, $0 \le s \le t$, which correspond to the $\mathcal{M}_i^{[s,t]}$, $i = 12, 13$, listed above.

Case 5. $a_{12}^{[s,t]} \equiv 0$, (the case $a_{21}^{[s,t]} \equiv 0$ is similar) then the system (1.2.1) reduced to following

$$\begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{22}^{[s,\tau]} a_{22}^{[\tau,t]}. \end{aligned} \tag{1.2.5}$$

The first and the third equations of the system (1.2.5) are Cantor's second equations. Substituting solutions of these equations into the second equation of the system (1.2.5) we find the function $a_{21}^{[s,t]}$. Note that in many cases the second equation of the system (1.2.5) will be reduced to Cantor's first equation:

$$\gamma(s,t) = \gamma(s,\tau) + \gamma(\tau,t),$$

which also has very rich family of solutions: $\gamma(s,t)=\Psi(t)-\Psi(s)$, where Ψ is an arbitrary function. Thus solving the system (1.2.5) we obtain the following new matrices:

$$\mathcal{M}_{14}^{[s,t]}=\begin{pmatrix}\frac{\Phi(t)}{\Phi(s)} & 0\\ \Phi(t)\psi(s) & 0\end{pmatrix};$$

$$\mathcal{M}_{15}^{[s,t]}=\begin{cases}\begin{pmatrix}1 & 0\\ \psi(s) & 0\end{pmatrix}, & \text{if } \ s\le t<a;\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, & \text{if } \ t\ge a.\end{cases}$$

$$\mathcal{M}_{16}^{[s,t]}=\begin{pmatrix}0 & 0\\ \frac{g(t)}{\psi(s)} & \frac{\psi(t)}{\psi(s)}\end{pmatrix};$$

$$\mathcal{M}_{17}^{[s,t]}=\begin{pmatrix}\frac{\Phi(t)}{\Phi(s)} & 0\\ \frac{\Phi(t)}{\psi(s)}(g(t)-g(s)) & \frac{\psi(t)}{\psi(s)}\end{pmatrix};$$

$$\mathcal{M}_{18}^{[s,t]}=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{h(t)-h(s)}{\psi(s)} & \frac{\psi(t)}{\psi(s)}\end{pmatrix}, & \text{if } \ s\le t<a;\\ \begin{pmatrix}0 & 0\\ \frac{h(t)}{\psi(s)} & \frac{\psi(t)}{\psi(s)}\end{pmatrix}, & \text{if } \ t\ge a.\end{cases}$$

$$\mathcal{M}_{19}^{[s,t]}=\begin{cases}\begin{pmatrix}0 & 0\\ h(t) & 1\end{pmatrix}, & \text{if } \ s\le t<b;\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, & \text{if } \ t\ge b.\end{cases}$$

$$\mathcal{M}_{20}^{[s,t]}=\begin{cases}\begin{pmatrix}\frac{\Phi(t)}{\Phi(s)} & 0\\ \Phi(t)(v(t)-v(s)) & 1\end{pmatrix}, & \text{if } \ s\le t<b;\\ \begin{pmatrix}\frac{\Phi(t)}{\Phi(s)} & 0\\ \Phi(t)w(s) & 0\end{pmatrix}, & \text{if } \ t\ge b.\end{cases}$$

$$\mathcal{M}_{21}^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ v(t)-v(s) & 1 \end{pmatrix}, & \text{if } s \le t < \min\{a,b\}; \\ \begin{pmatrix} 1 & 0 \\ v(s) & 0 \end{pmatrix}, & \text{if } b \le t < a,\ a > b; \\ \begin{pmatrix} 0 & 0 \\ v(t) & 1 \end{pmatrix}, & \text{if } a \le t < b,\ a < b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \ge \max\{a,b\}. \end{cases}$$

Hence, in this case we have eight new CEAs: $E_i^{[s,t]}$, $0 \le s \le t$, which correspond to the $\mathcal{M}_i^{[s,t]}$, $i = 14, \dots, 21$, listed above.

Case 6. $a_{22}^{[s,t]} \equiv 0$, (the case $a_{11}^{[s,t]} \equiv 0$ is similar) then the system (1.2.1) is reduced to the following

$$\begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{12}^{[s,t]} &= a_{11}^{[s,\tau]} a_{12}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]}, \\ 0 &= a_{21}^{[s,\tau]} a_{12}^{[\tau,t]}. \end{aligned}$$

Multiplying the second and the third equations and using the fourth one we get $a_{12}^{[s,t]} a_{21}^{[s,t]} = 0$. Thus this case reduces to the Case 5, hence does not give any new CEA.

Case 7. Assume $a_{11}^{[s,t]} + a_{12}^{[s,t]} = a_{21}^{[s,t]} + a_{22}^{[s,t]} = 1$. Denote $\alpha(s,t) = a_{11}^{[s,t]}$, $\beta(s,t) = a_{21}^{[s,t]}$. Then from (1.2.1) we get

$$\begin{aligned} \alpha(s,t) &= \alpha(s,\tau)\alpha(\tau,t) + (1-\alpha(s,t))\beta(\tau,t); \\ \beta(s,t) &= \beta(s,\tau)\alpha(\tau,t) + (1-\beta(s,\tau))\beta(\tau,t). \end{aligned}$$

Putting

$$\gamma(s,t) = \alpha(s,t) + \beta(s,t), \quad \delta(s,t) = \alpha(s,t) - \beta(s,t), \tag{1.2.6}$$

we obtain

$$\begin{aligned} \gamma(s,t) &= \gamma(s,\tau)\delta(\tau,t) + \gamma(\tau,t) - \delta(\tau,t); \\ \delta(s,t) &= \delta(s,\tau)\delta(\tau,t). \end{aligned} \tag{1.2.7}$$

The second equation of (1.2.7) has the following solutions:

1) $\delta(s,t) \equiv 0$;

2) $\delta(s,t) = \frac{\theta(t)}{\theta(s)}$, with $\theta(t) \neq 0$;

3)

$$\delta(s,t) = \begin{cases} 1, & \text{if } \ s \leq t < a, \\ 0, & \text{if } \ t \geq a. \end{cases} \quad \text{where } \ a > 0.$$

Substituting these solutions into the first equation of (1.2.7) we get the following

1') $\gamma(s,t) = f(t)$, where f is an arbitrary function;

2') For $\delta(s,t) = \frac{\theta(t)}{\theta(s)}$ we get

$$\tilde{\gamma}(s,t) = \tilde{\gamma}(s,\tau) + \tilde{\gamma}(\tau,t) - \frac{1}{\theta(\tau)}, \tag{1.2.8}$$

where $\tilde{\gamma}(s,t) = \frac{\gamma(s,t)}{\theta(t)}$. We shall find a solution of the equation (1.2.8) which has the form

$$\tilde{\gamma}(s,t) = \lambda \cdot u(t) - \mu \cdot u(s), \quad \lambda, \mu \in \mathbb{R}, \, \lambda \neq \mu.$$

We do not know another kind of solutions of the equation (1.2.8).

Substituting this function into equation (1.2.8) we get $u(t) = \frac{1}{(\lambda-\mu)\theta(t)}$. Consequently, we obtain

$$\gamma(s,t) = \frac{\lambda}{\lambda - \mu} - \frac{\mu\theta(t)}{(\lambda-\mu)\theta(s)}.$$

3') In case of 3) we get the following equation

$$\gamma(s,t) = \begin{cases} \gamma(s,\tau) + \gamma(\tau,t) - 1, & \text{if } \ s \leq t < a, \\ \gamma(\tau,t), & \text{if } \ t \geq a. \end{cases}$$

This equation has the following solution

$$\gamma(s,t) = \begin{cases} 1, & \text{if } \ s \leq t < a, \\ g(t), & \text{if } \ t \geq a, \end{cases} \quad \text{where } g(t) \text{ is an arbitrary function.}$$

Using the obtained solutions γ and δ by (1.2.6) we get the following matrices

$$\mathcal{M}_{22}^{[s,t]} = \begin{pmatrix} f(t) & 1-f(t) \\ f(t) & 1-f(t) \end{pmatrix};$$

$$\mathcal{M}_{23}^{[s,t]}(\lambda,\mu) = \begin{pmatrix} 1-\frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) & \frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) \\ \frac{\lambda}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) & 1-\frac{\lambda}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) \end{pmatrix};$$

$$\mathcal{M}_{24}^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } \ s \le t < a; \\ \begin{pmatrix} g(t) & 1-g(t) \\ g(t) & 1-g(t) \end{pmatrix}, & \text{if } \ t \ge a. \end{cases}$$

In this case we have three new CEAs: $E_i^{[s,t]}$, $0 \le s \le t$, which correspond to the $\mathcal{M}_i^{[s,t]}$, $i = 22, 23, 24$, listed above.

1.3 Property transitions of chains of evolution algebras

If a system has parameters (as usually like: temperature, time, interaction, etc.) then a property of the system can variate by a parameter. For example, the behavior of phases (states) of a system in physics, depends on temperature $T > 0$, if for some values of T there is a unique phase and for other values there are several phases, then the physical system has a phase transition [17]. Similar transitions of a property can be seen for systems of biology, chemistry, etc. Here we shall define a notion of property transition for CEA.

In [39] the algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras are studied. Results of [39] also allow a natural introduction of thermodynamics in studying of several systems of biology, physics and mathematics by theory of evolution algebras.

Definition 1.3.1 ([7]). Assume a CEA, $E^{[s,t]}$, has a property, say P, at pair of times (s_0, t_0); we say that the CEA has P *property transition* if there is a pair $(s, t) \neq (s_0, t_0)$ at which the CEA has no the property P.

Denote

$$\begin{aligned}
\mathcal{T} &= \{(s,t) : 0 \leq s \leq t\}; \\
\mathcal{T}_P &= \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has property } P\}; \\
\mathcal{T}_P^0 &= \mathcal{T} \setminus \mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has no property } P\}.
\end{aligned}$$

Definition 1.3.2. We call the set

$\mathcal{T}_P$-the *duration of the property* P;

$\mathcal{T}_P^0$-the *lost duration of the property* P;

The partition $\{\mathcal{T}_P, \mathcal{T}_P^0\}$ of the set $\mathcal{T}$ is called P *property diagram.*

For example, if P = commutativity then since any evolution algebra is commutative, we conclude that any CEA has not commutativity property transition.

1.3.1 Baric property transition

A *character* for an algebra A is a nonzero multiplicative linear form on A, that is, a nonzero algebra homomorphism from A to $\mathbb{R}$ [31]. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character.

Definition 1.3.3 ([31]). A pair (A, σ) consisting of an algebra A and a character σ on A is called a *baric algebra*. The homomorphism σ is called the weight (or baric) function of A and $\sigma(x)$ the weight (baric value) of x.

In [31] for the evolution algebra of a free population it is proven that there is a character $\sigma(x) = \sum_i x_i$, therefore that algebra is baric. But the evolution algebra E introduced in [44] is not baric, in general. The following theorem gives a criterion for an evolution algebra E to be baric.

Theorem 1.3.4 ([7]). *An n-dimensional evolution algebra E, over the field $\mathbb{R}$, is baric if and only if there is a column $(a_{1i_0}, \ldots, a_{ni_0})^T$ of its matrix of structural constants $\mathcal{M} = (a_{ij})_{i,j=1,\ldots,n}$, such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$, for all $i \neq i_0$. Moreover, the corresponding weight function is $\sigma(x) = a_{i_0i_0}x_{i_0}$.*

Since an evolution algebra is not a baric algebra, in general, using Theorem 1.3.4 we can give the baric property diagram. Let us do this for the above given chains $E_i^{[s,t]}$, $i = 0, \ldots, 24$.

Denote by $\mathcal{T}_b^{(i)}$ the baric property duration of the CEA $E_i^{[s,t]}$, $i = 0, \ldots, 24$.

Theorem 1.3.5.

(i) (There is no non-baric property transition)

The algebras $E_i^{[s,t]}$, $i = 0, 1, 2, 3, 6, 10, 11, 14, 22$, are not baric for any time $(s,t) \in \mathcal{T}$;

(ii) (There is no baric property transition)

The algebras $E_i^{[s,t]}$, $i = 16, 17, 18$, and $E_{23}^{[s,t]}(0, \mu)$, $E_{23}^{[s,t]}(2\mu, \mu)$, $\mu \neq 0$, are baric for any time $(s,t) \in \mathcal{T}$;

(iii) (There is baric property transition)

The CEAs $E_i^{[s,t]}$, $i = 4, 5, 7, 8, 9, 12, 13, 15, 19, 20, 21, 24$, and $E_{23}^{[s,t]}(\lambda, \mu)$, with $\lambda \notin \{0, \mu, 2\mu\}$ have baric transition with baric property duration sets as the following

$$\mathcal{T}_b^{(4)} = \left\{(s,t) \in \mathcal{T} : \frac{\Phi(s)}{\Psi(s)} = \frac{\Phi(t)}{\Psi(t)}\right\};$$

$$\mathcal{T}_b^{(5)} = \{(s,t) \in \mathcal{T} : s \leq t < b,\ \Phi(s) = \Phi(t)\};$$

$$\mathcal{T}_b^{(7)} = \{(s,t) \in \mathcal{T} : s \leq t < a,\ \Psi(s) = \Psi(t)\};$$

$$\mathcal{T}_b^{(8)} = \{(s,t) \in \mathcal{T} : s \leq t < \min\{a, b\}\};$$

$$\mathcal{T}_b^{(9)} = \{(s,t) \in \mathcal{T} : t = s + \pi k, k \in \mathbb{Z}\};$$

$$\mathcal{T}_b^{(12)} = \left\{(s,t) \in \mathcal{T} : g(s) = \pm\frac{1}{h(s)}\right\};$$

$$\mathcal{T}_b^{(13)} = \{(s,t) \in \mathcal{T} : s \le t < a,\ \psi(s) = \pm 1\}\,;$$

$$\mathcal{T}_b^{(15)} = \{(s,t) \in \mathcal{T} : s \le t < a,\ \psi(s) = 0\}\,;$$

$$\mathcal{T}_b^{(19)} = \{(s,t) \in \mathcal{T} : s \le t < a\}\,;$$

$$\mathcal{T}_b^{(20)} = \{(s,t) \in \mathcal{T} : s \le t < b\} \cup \{(s,t) \in \mathcal{T} : t \ge b, w(s) = 0\}\,;$$

$$\mathcal{T}_b^{(21)} = \{(s,t) \in \mathcal{T} : s \le t < \max\{a,b\}\}\,;$$

$$\mathcal{T}_b^{(23)}(\lambda,\mu) = \{(s,t) \in \mathcal{T} : \theta(t) = \theta(s)\}\,, \quad \lambda \ne 0,\ \mu,\ 2\mu;$$

$$\mathcal{T}_b^{(24)} = \{(s,t) \in \mathcal{T} : s \le t < a\}\,.$$

Proof. By Theorem 1.3.4 a two-dimensional evolution algebra $E^{[s,t]}$ is baric if and only if $a_{11}^{[s,t]} \ne 0$, $a_{21}^{[s,t]} = 0$ or $a_{22}^{[s,t]} \ne 0$, $a_{12}^{[s,t]} = 0$. The assertions of the theorem are results of the detailed checking of these conditions. □

Note that the sets $\mathcal{T}_b^{(i)}$, $i = 8, 9, 19, 21, 24$, do not depend on any parameter function. But $\mathcal{T}_b^{(i)}$, $i = 4, 5, 7, 12, 13, 15, 20, 23$, depend on some parameter functions and can be controlled by choosing the corresponding parameter functions $\Phi, \Psi, g, h, \psi, w$. These functions are called *baric property controllers* of the CEAs. Because, they really control the baric duration set, for example, if some of them is a strong monotone function then the duration is "minimal", i.e. the line $s = t$, but if a function is a constant function then the baric duration set is "maximal", i.e. it is $\mathcal{T}$. Since these functions are arbitrary functions, we have a rich class of controller functions, therefore we have a "powerful" control on the property to be baric.

Now we shall compute the Lebesgue measure ν of the sets $\mathcal{T}_b^{(i)}$. It is easy to see that

$$\nu\left(\mathcal{T}_b^{(8)}\right) = \frac{1}{2}(\min\{a,b\})^2; \quad \nu\left(\mathcal{T}_b^{(9)}\right) = 0; \quad \nu\left(\mathcal{T}_b^{(19)}\right) = \frac{1}{2}a^2;$$

$$\nu\left(\mathcal{T}_b^{(20)}\right) \ge \frac{1}{2}b^2; \quad \nu\left(\mathcal{T}_b^{(21)}\right) = \frac{1}{2}(\max\{a,b\})^2; \quad \nu\left(\mathcal{T}_b^{(24)}\right) = \frac{1}{2}a^2.$$

The Lebesgue measure of sets $\mathcal{T}_b^{(i)}$, $i = 4, 5, 7, 12, 13, 15, 20, 23$, depend on the corresponding controller functions.

For given functions g, h, ψ one can easily compute $\nu\left(\mathcal{T}_b^{(i)}\right)$, $i = 12, 13, 15$. For example, if g, h, ψ are elementary functions which do not have "constant parts" in their graphs then $\nu\left(\mathcal{T}_b^{(i)}\right) = 0$, $i = 12, 13, 15$.

Definition 1.3.6 ([7])**.** A function θ defined on $\mathbb{R}$ is called a function of *countable variation* if it has the following properties:

1. it is continuous except at most on a countable set, it has only jump-type discontinuities;

2. it has at most a countable set of singular (extremum) points.

The following theorem gives a characteristics of the baric property duration set.

Theorem 1.3.7. *If the controller function of* $\mathcal{T}_b^{(i)}$*,* $i = 4, 5, 7, 23$ *(i.e.* $\frac{\Phi(t)}{\Psi(t)}$ *for* $i = 4$*;* $\Phi(t)$ *for* $i = 5$*;* $\Psi(t)$ *for* $i = 7$*;* $\theta(t)$ *for* $i = 23$*), is a function of countable variation, then the baric duration set* $\mathcal{T}_b^{(i)}$ *has zero Lebesgue measure, that is the corresponding CEA is not baric almost surely.*

Proof. Is similar to the proof of [7, Theorem 4.8]. □

1.3.2 Absolute nilpotent elements transition

Recall that the element x of an algebra A is called an *absolute nilpotent* if $x^2 = 0$.

Let $E = \mathbb{R}^n$ be an evolution algebra over the field $\mathbb{R}$ with coefficients matrix of structural constants $\mathcal{M} = (a_{ij})$. Then for arbitrary $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i \in \mathbb{R}^n$ we have

$$xy = \sum_j \left(\sum_i a_{ij} x_i y_i\right) e_j, \qquad x^2 = \sum_j \left(\sum_i a_{ij} x_i^2\right) e_j.$$

For an n-dimensional evolution algebra $\mathbb{R}^n$ consider the operator $V\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto V(x) = x'$, defined as

$$x'_j = \sum_{i=1}^{n} a_{ij} x_i^2, \quad j = 1, \dots, n. \tag{1.3.1}$$

This operator is called an *evolution operator* [31].

We have $V(x) = x^2$, hence the equation $V(x) = x^2 = 0$ is given by the following system

$$\sum_i a_{ij} x_i^2 = 0, \quad j = 1, \dots, n. \tag{1.3.2}$$

In this section we shall solve the system (1.3.2) for $E_i^{[s,t]}$, $i = 0, \dots, 24$.

For a CEA $E_i^{[s,t]}$ with matrix $\mathcal{M}_i^{[s,t]}$ denote

$$\mathcal{T}_{nil}^{(i)} = \{(s,t) \in \mathcal{T} : E_i^{[s,t]} \text{ has unique absolute nilpotent}\}, \quad \mathcal{T}_{nil}^{0} = \mathcal{T} \setminus \mathcal{T}_{nil}.$$

The following theorem gives an answer on the problem of the existence of "uniqueness of absolute nilpotent element" property transition.

Theorem 1.3.8.

(1) The CEAs $E_i^{[s,t]}$, $i = 3, 4, 5, 9, 10, 17, 22, 23, 24$, have unique absolute nilpotent element $(0,0)$ for any time $(s,t) \in \mathcal{T}$.

(2) The CEAs $E_i^{[s,t]}$, $i = 0, 1, 2, 16, 19$, have infinitely many of absolute nilpotent elements for any time $(s,t) \in \mathcal{T}$.

(3) The CEAs $E_i^{[s,t]}$, $i = 6, 7, 8, 11, 12, 13, 14, 15, 18, 20, 21$, have "uniqueness of absolute nilpotent element" property transition with the property duration sets as the following

$$\mathcal{T}_{nil}^{(i)} = \{(s,t) \in \mathcal{T} : t < a\}, \quad a > 0, \quad i = 6, 7, 8, 11, 18;$$

$$\mathcal{T}_{nil}^{(12)} = \left\{(s,t) \in \mathcal{T} : g^2(t) \le \frac{1}{h^2(s)}\right\};$$

$$\mathcal{T}_{nil}^{(13)} = \left\{(s,t) \in \mathcal{T} : s \le t < a,\, \psi^2(s) \le 1\right\};$$

$$\mathcal{T}_{nil}^{(14)} = \{(s,t) \in \mathcal{T} : \Phi(s)\psi(s) > 0\};$$

$$\mathcal{T}_{nil}^{(15)} = \{(s,t) \in \mathcal{T} : s \le t < a,\ \psi(s) > 0\}\,;$$

$$\mathcal{T}_{nil}^{(20)} = \{(s,t) \in \mathcal{T} : s \le t < b\} \cup \left\{(s,t) \in \mathcal{T} : t \ge b,\ \frac{w(s)}{\Phi(s)} > 0\right\};$$

$$\begin{aligned}\mathcal{T}_{nil}^{(21)} = \{(s,t) \in \mathcal{T} : s \le t < \min\{a,b\}\} \\ \cup \{(s,t) \in \mathcal{T} : b \le t < a,\ b < a,\ v(s) > 0\}\,.\end{aligned}$$

Proof. The proof consists of a simple analysis of the solutions of the system (1.3.2) for each $E_i^{[s,t]}$, $i = 0, \dots, 24$. □

1.3.3 Idempotent elements transition

An element x of an algebra $\mathcal{A}$ is called *idempotent* if $x^2 = x$; such points of an evolution algebra are especially important, because they are the fixed points (i.e. $V(x) = x$) of the evolution operator V, (1.3.1). We denote by $\mathcal{I}d(E)$ the set of idempotent elements of an algebra E. Using (1.3.1) the equation $x^2 = x$ can be written as

$$x_j = \sum_{i=1}^{n} a_{ij} x_i^2, \quad j = 1, \dots, n. \tag{1.3.3}$$

The general analysis of the solutions of the system (1.3.3) is very difficult. We shall solve this problem for the CEAs $E_i^{[s,t]}$, $i = 0, \dots, 24$.

The following theorem gives the time-dynamics of the idempotent elements for the algebras $E_i^{[s,t]}$, $i = 0, \dots, 24$.

Theorem 1.3.9.

(1) The algebras $E_i^{[s,t]}$, $i = 0, 1, 2$, have unique idempotent $(0,0)$ for any time $(s,t) \in \mathcal{T}$.

(2) The algebras $E_i^{[s,t]}$, $i = 3, 10, 12, 14, 16, 22$, have two idempotents $(0,0)$, $(x_i(s,t), y_i(s,t))$ for any time $(s,t) \in \mathcal{T}$. Moreover, an explicit formula of each $x_i(s,t)$ and $y_i(s,t)$ can be given.

(3) We have

$$\mathcal{I}d\left(E_4^{[s,t]}\right)=\begin{cases}\{0,z_1,z_2,z_3\}, & \textit{if}\ \ s\le t<b,\ \ \frac{\Phi(t)}{\Phi(s)}=\frac{\Psi(t)}{\Psi(s)};\\ \{0,z_3\}, & \textit{if}\ \ s\le t<b,\ \ \frac{\Phi(t)}{\Phi(s)}\neq\frac{\Psi(t)}{\Psi(s)},\ D(s,t)<0;\\ \{0,z_3,(x_*,y_*)\}, & \textit{if}\ \ s\le t<b,\ \frac{\Phi(t)}{\Phi(s)}\neq\frac{\Psi(t)}{\Psi(s)},\ D(s,t)=0;\\ \{0,z_3,(x_\pm,y_\pm)\}, & \textit{if}\ \ s\le t<b,\ \frac{\Phi(t)}{\Phi(s)}\neq\frac{\Psi(t)}{\Psi(s)},\ D(s,t)>0,\end{cases}$$

where $0=(0,0)$, $z_1=(0,\frac{\Phi(t)}{\Phi(s)})$, $z_2=(\frac{\Phi(t)}{\Phi(s)},0)$, $z_3=(\frac{\Phi(t)}{\Phi(s)},\frac{\Phi(t)}{\Phi(s)})$, $D(s,t)=\frac{\Phi(t)}{\Phi(s)}\left(2\frac{\Psi(t)}{\Psi(s)}-\frac{\Phi(t)}{\Phi(s)}\right)$*. The explicit formulas of* x_*, y_*, $x_\pm$ *and* $y_\pm$ *are given below. The sets* $\left\{\frac{\Phi(t)}{\Phi(s)}=\frac{\Psi(t)}{\Psi(s)}\right\}$, $\left\{\frac{\Phi(t)}{\Phi(s)}=2\frac{\Psi(t)}{\Psi(s)}\right\}$ *are critical (boundary) sets of the idempotent elements transition.*

(4) We have

$$\mathcal{I}d\left(E_5^{[s,t]}\right)=\begin{cases}\{0,z_1,z_2,z_3\}, \textit{if}\ \ s\le t<b,\ \Phi(t)=\Phi(s);\\ \{0,z_3\}, \quad \textit{if}\ s\le t<b, \Phi(t)\neq\Phi(s),\ D(s,t)<0;\\ \{0,z_3,(x_*,y_*)\},\ \ \textit{if}\ \ s\le t<b,\Phi(t)\neq\Phi(s),\ D(s,t)=0;\\ \{0,z_3,(x_\pm,y_\pm)\},\ \ \textit{if}\ s\le t<b,\ \Phi(t)\neq\Phi(s),\ D(s,t)>0;\\ \{0,z_3\}, \quad \textit{if}\ \ t\ge b,\end{cases}$$

where z_i *are as in (3),* $D(s,t)=\frac{\Phi(t)}{\Phi(s)}\left(2-\frac{\Phi(t)}{\Phi(s)}\right)$.

(5) The algebras $E_i^{[s,t]}$, $i=6,11,13,15,19$*, have two idempotent elements for any time* (s,t) *with* $s\le t<a$ *and a unique idempotent for time* (s,t) *with* $t\ge a$*. The critical line of the transition is* $t=a$.

(6) We have

$$\mathcal{I}d\left(E_7^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, \ \textit{if } s \leq t < a, \ \Psi(t) = \Psi(s); \\ \{0, z_3\}, \ \textit{if } s \leq t < a, \Psi(t) \neq \Psi(s), \ d(s,t) < 0; \\ \{0, z_3, (x_*, y_*)\}, \ \textit{if } \ s \leq t < a, \Psi(t) \neq \Psi(s), \ d(s,t) = 0; \\ \{0, z_3, (x_\pm, y_\pm)\}, \ \textit{if } \ s \leq t < a, \Psi(t) \neq \Psi(s), \ d(s,t) > 0; \\ 0, \quad \textit{if } \ t \geq a, \end{cases}$$

where $d(s,t) = \frac{2\Psi(t)}{\Psi(s)} - 1$*. The critical sets are* $t = a$*,* $\Psi(t) = \Psi(s)$*,* $\Psi(s) = 2\Psi(t)$*.*

(7) For $a \leq b$ *we have*

$$\mathcal{I}d\left(E_8^{[s,t]}\right) = \begin{cases} \{(0,0),(0,1),(1,0),(1,1)\}, & \textit{if } s \leq t < a; \\ (0,0), & \textit{if } \ t \geq a. \end{cases}$$

For $a > b$ *we have*

$$\mathcal{I}d\left(E_8^{[s,t]}\right) = \begin{cases} \{(0,0),(0,1),(1,0),(1,1)\}, & \textit{if } s \leq t < b; \\ \{(0,0),(1,1)\}, & \textit{if } \ b \leq t < a; \\ (0,0), & \textit{if } \ t \geq a. \end{cases}$$

The lines $t = a$ *and* $t = b$ *are critical for the transition.*

(8) The algebra $E_9^{[s,t]}$ *has three idempotent elements* $(0,0),(1,0),(0,1)$ *for any time* (s,t) *with* $t = s + 2\pi n$*; has three idempotent elements* $(0,0)$*,* $(-1,0)$*,* $(0,-1)$ *for any time* (s,t) *with* $t = s + (2n+1)\pi$ *and at least one idempotent for time* (s,t) *with* $t \neq s + \pi n$*,* $n \in \mathbb{Z}$*.*

(9) We have

$$\mathcal{I}d\left(E_{17}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, & \textit{if } \ D(s,t) < 0; \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, \frac{\Psi(s)}{\Psi(t)})\}, & \textit{if } \ D(s,t) = 0; \\ \{(0,0), z_2, (x_\pm, y_\pm)\}, & \textit{if } \ D(s,t) > 0, \end{cases}$$

where $D(s,t) = \frac{4\Phi^2(t)\Psi(s)}{\Phi(s)\Psi^2(t)}(g(t) - g(s)) - 1$.

(10) We have

$$\mathcal{I}d\left(E_{18}^{[s,t]}\right) = \begin{cases} \{(0,0), (1,0)\}, \quad \textit{if } \ s \leq t < a, \, D(s,t) < 0; \\ \{(0,0), (1,0), (\frac{1}{2}, \frac{\Psi(s)}{\Psi(t)})\}, \quad \textit{if } \ s \leq t < a, \, D(s,t) = 0; \\ \{(0,0), (1,0), (x_\pm, \frac{\Psi(s)}{\Psi(t)})\}, \quad \textit{if } \ s \leq t < a, \, D(s,t) > 0; \\ \{(0,0), (\frac{h(t)\Psi(s)}{\Psi^2(t)}, \frac{\Psi(s)}{\Psi(t)})\}, \quad \textit{if } \ t \geq a, \end{cases}$$

where $D(s,t) = 1 - \frac{4\Psi(s)(h(t)-h(s))}{\Psi^2(t)}$.

(11) We have

$$\mathcal{I}d\left(E_{20}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, \quad \textit{if } \ s \leq t < b, \, D(s,t) < 0, \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, 1)\}, \quad \textit{if } \ s \leq t < b, \, D(s,t) = 0, \\ \{(0,0), z_2, (x_\pm, 1)\}, \quad \textit{if } \ s \leq t < a, \, D(s,t) > 0, \\ \{(0,0), z_2\}, \quad \textit{if } \ t \geq b, \end{cases}$$

where $D(s,t) = 1 - \frac{4\Phi^2(t)(v(t)-v(s))}{\Phi(s)}$.

(12) We have

$$\mathcal{I}d\left(E_{21}^{[s,t]}\right)=\begin{cases}\{(0,0),(1,0)\}, & \textit{if } \ s\le t<\min\{a,b\},\\ & \qquad D(s,t)<0;\\ \{(0,0),(1,0),(\frac{1}{2},1)\}, & \textit{if } \ s\le t<\min\{a,b\},\\ & \qquad D(s,t)=0;\\ \{(0,0),(1,0),(x_{\pm},1)\}, & \textit{if } \ s\le t<\min\{a,b\},\\ & \qquad D(s,t)>0;\\ \{(0,0),(1,0)\}, & \textit{if } \ b<a,\ b\le t<a;\\ \{(0,0),(v(t),1)\}, & \textit{if } \ b>a,\ a\le t<b;\\ (0,0), & \textit{if } \ t\ge\max\{a,b\},\end{cases}$$

where $D(s,t)=1-4(v(t)-v(s))$.

(13) We have

$$\mathcal{I}d\left(E_{23}^{[s,t]}(0,\mu)\right)=\begin{cases}\{(0,0),(0,1)\}, & \textit{if } \ D(s,t)<0;\\ \{(0,0),(0,1),(\frac{\theta(s)}{\theta(t)},\frac{1}{2})\}, & \textit{if } \ D(s,t)=0;\\ \{(0,0),(0,1),(\frac{\theta(s)}{\theta(t)},y_{\pm})\}, & \textit{if } \ D(s,t)>0.\end{cases}$$

$$\mathcal{I}d\left(E_{23}^{[s,t]}(2\mu,\mu)\right)=\begin{cases}\{(0,0),(1,0)\}, & \textit{if } \ D(s,t)<0;\\ \{(0,0),(1,0),(\frac{1}{2},\frac{\theta(s)}{\theta(t)})\}, & \textit{if } \ D(s,t)=0;\\ \{(0,0),(1,0),(x_{\pm},\frac{\theta(s)}{\theta(t)})\}, & \textit{if } \ D(s,t)>0,\end{cases}$$

where $D(s,t)=1-\frac{4\theta(s)}{\theta(t)}\left(\frac{\theta(s)}{\theta(t)}-1\right)$.

(14) We have

$$\mathcal{I}d\left(E_{24}^{[s,t]}\right)=\begin{cases}\{(0,0),(0,1),(1,0),(1,1)\}, \ \textit{if } (s,t)\in\mathcal{T}: s\le t<a;\\ \left\{(0,0),\left(\frac{g(t)}{(1-g(t))^2+g^2(t)},\frac{1-g(t)}{(1-g(t))^2+g^2(t)}\right)\right\}, \ \textit{if } t\ge a.\end{cases}$$

Proof. The proof contains detailed analysis of solutions of the system (1.3.3) for each $E_i^{[s,t]}$. We shall give here proof of the assertion (3) which is more substantial. In case of $E_4^{[s,t]}$ the system (1.3.3) has the following form

$$\begin{cases} 2x = \left(\frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)}\right) x^2 + \left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right) y^2; \\ 2y = \left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right) x^2 + \left(\frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)}\right) y^2. \end{cases} \tag{1.3.4}$$

Case 1.1. $\Phi(t) \equiv \Psi(t)$. It is easy to see that in this case the system (1.3.4) has only four solutions $0 = (0,0)$ and

$$z_1 = z_1(s,t) = \left(0, \frac{\Phi(s)}{\Phi(t)}\right), \; z_2 = z_2(s,t) = \left(\frac{\Phi(s)}{\Phi(t)}, 0\right),$$

$$z_3 = z_3(s,t) = \left(\frac{\Phi(s)}{\Phi(t)}, \frac{\Phi(s)}{\Phi(t)}\right).$$

Case 1.2. $\Phi(t) \neq \Psi(t)$. In this case the solutions 0 and z_3 still exist.

Case 1.2.1. $x = 0$. For $x = 0$ we have only solution $(0,0)$ if $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$ and there are two solutions $(0,0)$ and z_1 if $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$.

Case 1.2.2. $y = 0$. For $y = 0$ we have only one solution $(0,0)$ if $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$ and there are two solutions $(0,0)$ and z_2 if $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$.

Case 1.2.3. $xy \neq 0$. Set $u = \frac{x}{y}$. From system (1.3.4) we get

$$(u-1)\left(\left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right)u^2 - 2\frac{\Psi(t)}{\Psi(s)}u + \left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right)\right) = 0. \tag{1.3.5}$$

This equation has unique solution $u = 1$ if $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$ or $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$ and $D(s,t) = \frac{\Phi(t)}{\Phi(s)}\left(2\frac{\Psi(t)}{\Psi(s)} - \frac{\Phi(t)}{\Phi(s)}\right) < 0$. For $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$ it has two solutions $u = 1$ and $u = u_*(s,t)$ if $D(s,t) = 0$ and three solutions $u = 1$, $u = u_\pm(s,t)$ if $D > 0$.

Now one can describe x, y corresponding to the solutions of (1.3.5). The case $u = 1$, i.e. $x = y$ does not give any new solution. For $u = u_*, u_\pm$ we have $x = u_* y$ and $x = u_\pm y$, substituting these in the second equation of (1.3.4) after simple calculations we get the following non-zero solutions to (1.3.4):

$$x_* = \frac{\Phi(s)}{\Phi(t)}, \qquad y_* = \frac{\Psi(s)}{\Psi(t)} - \frac{\Phi(s)}{\Phi(t)};$$

$$x_\pm = \frac{1 \pm \frac{\Psi(s)}{\Psi(t)}\sqrt{D(s,t)}}{\frac{\Phi(t)}{\Phi(s)} \pm \sqrt{D(s,t)}}, \qquad y_\pm = \frac{\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}}{\frac{\Psi(t)}{\Psi(s)}\left(\frac{\Phi(t)}{\Phi(s)} \pm \sqrt{D(s,t)}\right)}.$$

Note that $x_\pm$, $y_\pm$ are well defined for any (s,t) with $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$. Thus the critical (boundary) times of the transition of idempotent elements are points (s,t) which satisfy $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$ or $\frac{\Phi(t)}{\Phi(s)} = \frac{2\Psi(t)}{\Psi(s)}$.

Proofs of the assertions (i), $i = 1, 2, 4 - 14$, are a similar analysis of the system (1.3.3) for each $E_i^{[s,t]}$. $\square$

Chapter 2

Classification dynamics of two-dimensional chains of evolution algebras

In this chapter we will give a classification of evolution algebras and study the time depending dynamics of two-dimensional chains of evolution algebras. Moreover, we construct the matrices of the (linear) Rota-Baxter operators on the 2-dimensional complex evolution algebras.

2.1 Classifications of finite-dimensional evolution algebras

Let E and E' be evolution algebras and $\{e_i\}$ a natural basis of E. A linear map $\phi\colon E \longrightarrow E'$ is called a homomorphism of evolution algebras if $\phi(xy) = \phi(x)\phi(y)$ and the set $\{\phi(e_i)\}$ is a subset of a natural basis of E'. Moreover, if ϕ is bijective, then it is called an isomorphism.

Let E be a two-dimensional complex evolution algebra and $\{e_1, e_2\}$ be a basis of the algebra E.

It is evident that if $\dim E^2 = 0$, then E is an abelian algebra, i.e. an algebra with all products equal to zero.

The next theorem gives the classification of two-dimensional complex evolution algebras, which is proved in [6].

Theorem 2.1.1 ([6])**.** *Any 2-dimensional complex evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*

1. $\dim E^2 = 1$
   - $E_1 : \quad e_1e_1 = e_1,\ e_2e_2 = 0,$
   - $E_2 : \quad e_1e_1 = e_1, \quad e_2e_2 = e_1,$
   - $E_3 : \quad e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2,$
   - $E_4 : \quad e_1e_1 = e_2,\ e_2e_2 = 0.$
2. $\dim E^2 = 2$
   - $E_5 : \quad e_1e_1 = e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + e_2, \quad 1 - a_2a_3 \neq 0,$ *where* $E_5(a_2, a_3) \cong E_5'(a_3, a_2)$,
   - $E_6 : \quad e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2,$ *where for* $a_4 \neq 0$, $E_6(a_4) \cong E_6(a_4') \ \Leftrightarrow\ \frac{a_4'}{a_4} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ *for some* $k = 0, 1, 2.$

The next theorem gives the classification of the two-dimensional real evolution algebras.

Theorem 2.1.2. *Any two-dimensional real evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*

(i) $\dim E^2 = 1$*:*

$E_1 : e_1e_1 = e_1, \quad e_2e_2 = 0;$

$E_2 : e_1e_1 = e_1, \quad e_2e_2 = e_1;$

$E_3 : e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2;$

$E_4 : e_1e_1 = e_2, \quad e_2e_2 = 0;$

$E_5 : e_1e_1 = e_2, \quad e_2e_2 = -e_2;$

(ii) $\dim E^2 = 2$*:*

$E_6 : e_1e_1 = e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + e_2, \quad 1 - a_2a_3 \neq 0, a_2, a_3 \in \mathbb{R}$, *where* $E_6(a_2, a_3) \cong E_6'(a_3, a_2)$*;*

$E_7 : e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2,$ *where* $a_4 \in \mathbb{R}$*.*

Proof. For a general two-dimensional evolution algebra we have $e_1e_1 = a_1e_1 + a_2e_2,\ \ e_2e_2 = a_3e_1 + a_4e_2$ and $e_1e_2 = e_2e_1 = 0, a_i \in \mathbb{R}$.

(i) Since $\dim E^2 = 1$, we have $e_1e_1 = c_1(a_1e_1 + a_2e_2),\ \ e_2e_2 = c_2(a_1e_1 + a_2e_2)$ and $e_1e_2 = e_2e_1 = 0$. Evidently, $(c_1, c_2) \neq (0, 0)$, because otherwise our algebra will be abelian. Since e_1 and e_2 are symmetric, we can suppose $c_1 \neq 0$, and by a simple change of basis we can suppose $c_1 = 1$.

Case 1. $a_1 \neq 0$. We take an appropriate change of basis $e'_1 = a_1e_1 + a_2e_2,\ e'_2 = Ae_1 + Be_2$, where $a_1B - a_2A \neq 0$. Consider the product

$$\begin{aligned} 0 = e'_1e'_2 &= (a_1e_1 + a_2e_2)(Ae_1 + Be_2) \\ &= a_1A(a_1e_1 + a_2e_2) + a_2Bc_2(a_1e_1 + a_2e_2) \\ &= (a_1A + a_2Bc_2)(a_1e_1 + a_2e_2). \end{aligned}$$

Therefore, $a_1A + a_2Bc_2 = 0$, i.e. $A = -\frac{a_2Bc_2}{a_1}$ and $a_1B - a_2A = a_1B + \frac{a_2^2Bc_2}{a_1} \neq 0$.

For $B \neq 0$, it means that in the case when $a_1^2 + a_2^2c_2 \neq 0$ we can take the above change.

Consider the products

$$\begin{aligned} e'_1e'_1 &= (a_1e_1 + a_2e_2)(a_1e_1 + a_2e_2) = a_1^2(a_1e_1 + a_2e_2) + a_2^2c_2(a_1e_1 + a_2e_2) \\ &= (a_1^2 + a_2^2c_2)(a_1e_1 + a_2e_2) = (a_1^2 + a_2^2c_2)e'_1, \end{aligned}$$

$$\begin{aligned} e'_2e'_2 &= (Ae_1 + Be_2)(a_1e_1 + a_2e_2) = A^2(a_1e_1 + a_2e_2) + B^2c2(a_1e_1 + a_2e_2) \\ &= (A^2 + B^2c_2)(a_1e_1 + a_2e_2) = \Big(\frac{a_2^2B^2c_2^2}{a_1^2} + B^2c_2\Big)e'_1 \\ &= \frac{B^2c_2(a_1^2 + a_2^2c_2)}{a_1^2}e'_1. \end{aligned}$$

Case 1.1. $c_2 = 0$. Then $e_1e_1 = a_1^2e_1$ and $e_2e_2 = e_1e_2 = e_2e_1 = 0$. Taking $e'_1 = \frac{e_1}{a_1^2}$, we obtain the algebra E_1.

Case 1.2. $c_2 \neq 0$. Then taking $B = \sqrt{\frac{a_1^2}{|c_2|}}$, we obtain $e_1e_1 = (a_1^2 + a_2^2c_2)e_1,\ e_2e_2 = (a_1^2{+}a_2^2c_2)e_1$, when $c_2 > 0$ and also $e_1e_1 = (a_1^2{+}a_2^2c_2)e_1,\ e_2e_2 = -(a_1^2 + a_2^2c_2)e_1$, when $c_2 < 0$.

If $c_2 > 0$ then $a_1^2 + a_2^2 c_2 \neq 0$ and we can take the change of basis $e_1' = \frac{e_1}{a_1^2+a_2^2c_2}$, $e_2' = \frac{e_2}{a_1^2+a_2^2c_2}$ which gives the algebra E_2 with the multiplication table $e_1e_1 = e_1,\ e_2e_2 = e_1$.

If $a_1^2 + a_2^2 c_2 \neq 0$ when $c_2 < 0$ and we can take the change of basis $e_1' = \frac{e_1}{a_1^2+a_2^2c_2}$, $e_2' = -\frac{e_2}{a_1^2+a_2^2c_2}$ which gives the algebra with the multiplication table $e_1e_1 = e_1,\ e_2e_2 = -e_1$. It is easy to check that this algebra is isomorphic to the algebra E_5 with the change of basis $e_1' = -e_2,\ e_2' = e_1$.

If $a_1^2 + a_2^2 c_2 = 0$ $(c_2 < 0)$, since $a_1 \neq 0$ we have $a_2 \neq 0$ then $c_2 = -\frac{a_1^2}{a_2^2}$ and we have $e_1e_1 = a_1e_1 + a_2e_2$ and $e_2e_2 = -\frac{a_1^3}{a_2^2}e_1 - \frac{a_1^2}{a_2}e_2$. Then the change of basis $e_1' = \frac{e_1}{a_1}, e_2 = \frac{a_2}{a_1^2}e_2$ gives the algebra E_3.

Case 2. $a_1 = 0$. Then we have $e_1e_1 = a_2e_2$ and $e_2e_2 = c_2a_2e_2$, where $a_2 \neq 0$.

If $c_2 = 0$ then by the change $e_1' = \frac{e_1}{\sqrt{|a_2|}}$ we get the algebra E_4 when $a_2 > 0$. When $a_2 < 0$ by this change of basis we get the algebra $e_1e_1 = -e_2, e_2e_2 = 0$ which is isomorphic to the algebra E_4, by the change of basis $e_1' = e_1, e_2' = e_2$.

If $c_2 \neq 0$, then by $e_1' = \frac{e_1}{\sqrt{|c_2|a_2^2}}$ and $e_2' = \frac{e_2}{c_2a_2}$, we get the algebra $e_1e_1 = e_2,\ \ e_2e_2 = e_2$ $(c_2 > 0)$ which is isomorphic to the algebra E_2.

When $c_2 < 0$ by the change of basis $e_1' = \frac{e_1}{\sqrt{|c_2|a_2^2}}$ and $e_2' = \frac{e_2}{c_2a_2}$, we will take the algebra with the multiplication table $e_1e_1 = -e_2,\ e_2e_2 = e_2$ which is isomorphic to E_5.

(ii) Now consider algebras with $\dim E^2 = 2$. Let us write $e_1e_1 = a_1e_1 + a_2e_2,\ \ e_2e_2 = a_3e_1 + a_4e_2$, where $a_1a_4 - a_2a_3 \neq 0$.

Case 1. $a_1 \neq 0$ and $a_4 \neq 0$. Then the change of basis $e_1 = a_1^{-1}e_1, e_2 = a_4^{-1}e_2$ makes possible to suppose $a_1 = a_4 = 1$. Therefore, we have two-parametric family $E_7(a_2, a_3) : e_1e_1 = e_1 + a_2e_2,\ \ e_2e_2 = a_3e_1 + e_2,\ \ 1 - a_2a_3 \neq 0$. Let us take the general change of basis $e_1' = A_1e_1 + A_2e_2,\ \ e_2' = B_1e_1 + B_2e_2$, where

$A_1B_2 - A_2B_1 \neq 0$. Consider the product

$$\begin{aligned} 0 = e_1'e_2' &= (A_1e_1 + A_2e_2)(B_1e_1 + B_2e_2) \\ &= A_1B_1(e_1 + a_2e_2) + A_2B_2(a_3e_1 + e_2) \\ &= (A_1B_1 + A_2B_2a_3)e_1 + (A_1B_1a_2 + A_2B_2)e_2 \end{aligned}$$

Since in this new basis the algebra should be also an evolution algebra, we have $A_1B_1 + A_2B_2a_3 = 0$ and $A_1B_1a_2 + A_2B_2 = 0$. From this we have $A_2B_2(1 - a_2a_3) = 0$ and $A_1B_1(1 - a_2a_3) = 0$. Since $1 - a_2a_3 \neq 0$, we have $A_1B_1 = A_2B_2 = 0$.

Case 1.1. Let $A_2 = 0$. Then $B_1 = 0$. Consider the products
$e_1'e_1' = A_1^2(e_1 + a_2e_2) = e_1' + a_2'e_2' = A_1e_1 + a_2'B_2e_2$
$\Rightarrow A_1^2 = A_1, A_1^2a_2 = a_2'B_2 \Rightarrow A_1 = 1,$
$e_2'e_2' = B_2^2(a_3e_1 + e_2) = a_3'e_1' + e_2' = a_3'A_1e_1 + B_2e_2$
$\Rightarrow B_2^2a_3 = a_3'A_1, B_2^2 = B_2 \Rightarrow B_2 = 1.$

Case 1.2. Let $A_1 = 0$. Then $B_2 = 0$, and from the family of algebras $E_6(a_2, a_3)$ we get the family $E_6(a_3, a_2)$.

Case 2. Let $a_1 = 0$ or $a_4 = 0$. Since e_1 and e_2 are symmetric, without loss of generality we can suppose $a_1 = 0$, i.e. $e_1e_1 = a_2e_2$ and $e_2e_2 = a_3e_1 + a_4e_2$, where $a_2a_3 \neq 0$. Taking the change of basis $e_1' = \sqrt[3]{\frac{1}{a_2^2a_3}}e_1, \quad e_2' = \sqrt[3]{\frac{1}{a_2a_3^2}}e_2$, we obtain one of the parametric family of algebras $E_7(a_4) : \ e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2$. $\square$

Remark 2.1.3. We note that the classification of the two-dimensional complex evolution algebras consists of a complex variant of the algebras $E_i, \ i = 1, 2, 3, 4, 6, 7$ (see [6]). But E_4 is present only in the real case.

The classification of the three-dimensional complex evolution algebras was studied in [4].

The classification of the finite-dimensional complex evolution algebras with maximal nilpotent index was given in [5].

2.2 Dynamics of two-dimensional real chains of evolution algebras

The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of a dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future. The relation is either a differential equation, a difference equation or another time scale. To determine the state for all future times requires iterating the relation many times each advancing time by a small step.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories.

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc.

Dynamical systems generated by the quadratic evolution operators of corresponding constrained algebras was studied in [10].

Discrete-time dynamical systems generated by gonosomal evolution operators of sex linked inheritance and an algebraic model of the biological system corresponding to the hemophilia was studied in [40].

Recently obtained results for discrete-time dynamical systems and evolution algebras of sex linked inheritance reviewed and several open problems related to such inheritance was discussed in [38].

To study time depending dynamics of two-dimensional real chains of evolution algebras we shall prove the next lemma, which is important to prove the main theorem of this section.

Lemma 2.2.1. *An evolution algebra corresponding to the matrix*

$$\text{(i)} \quad \begin{pmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{pmatrix} \ \textit{is isomorphic to} \begin{cases} E_0, & \textit{if } \ \lambda = 0; \\ \\ E_3, & \textit{if } \ \lambda \neq 0; \end{cases}$$

$$\text{(ii)} \quad \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \ \lambda = \mu = 0; \\ E_2, & \textit{if} \ \ \lambda = \mu \neq 0; \\ E_6(\frac{\mu}{\lambda}, \frac{\mu}{\lambda}), & \textit{if} \ \lambda \neq \mu, \lambda \neq 0, \mu \in \mathbb{R}; \\ E_7(0), & \textit{if} \ \lambda = 0, \mu \neq 0; \end{cases}$$

$$\text{(iii)} \quad \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \ \lambda = \mu = 0; \\ E_6(\frac{\mu}{\lambda}, -\frac{\mu}{\lambda}), & \textit{if} \ \ \lambda \neq 0, \mu \in \mathbb{R}; \\ E_7(0), & \textit{if} \ \ \lambda = 0, \mu \neq 0; \end{cases}$$

$$\text{(iv)} \quad \begin{pmatrix} \lambda & \mu \\ \lambda & \mu \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \ \lambda = \mu = 0; \\ E_2, & \textit{otherwise}; \end{cases}$$

$$\text{(v)} \quad \begin{pmatrix} \lambda & \lambda \\ \mu & \mu \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \ \lambda = \mu = 0; \\ E_1, & \textit{if} \ \ \lambda\mu = 0, \lambda^2 + \mu^2 \neq 0; \\ E_2, & \textit{if} \ \ \lambda \neq 0, \ \mu \neq 0; \end{cases}$$

$$\text{(vi)} \quad \begin{pmatrix} \lambda & 0 \\ \mu & 0 \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \ \lambda = \mu = 0; \\ E_1, & \textit{if} \ \ \lambda \neq 0, \mu = 0; \\ E_2, & \textit{if} \ \ \lambda\mu > 0; \\ E_4, & \textit{if} \ \ \lambda = 0, \mu \neq 0; \\ E_5, & \textit{if} \ \ \lambda\mu < 0; \end{cases}$$

$$\textit{(vii)} \quad \begin{pmatrix} \lambda & 0 \\ \mu & \eta \end{pmatrix} \quad \textit{is isomorphic to} \begin{cases} E_0, & \textit{if} \ \lambda = \mu = \eta = 0; \\ E_1, & \textit{if} \ \{\lambda = 0, \eta \neq 0\} \cup \\ & \qquad \{\lambda \neq 0, \mu = \eta = 0\}; \\ E_2, & \textit{if} \ \lambda \neq 0, \mu \neq 0, \eta = 0; \\ E_4, & \textit{if} \ \lambda = \eta = 0, \mu \neq 0; \\ E_6(\frac{\lambda\mu}{\eta^2}, 0), & \textit{if} \ \lambda \neq 0, \eta \neq 0, \mu \in \mathbb{R}; \end{cases}$$

$$\textit{(viii)} \quad \begin{pmatrix} \lambda & 1-\lambda \\ \mu & 1-\mu \end{pmatrix} \quad \textit{is isomorphic to}$$

$$\begin{cases} E_2, & \textit{if} \ \lambda = \mu; \\ E_6(\frac{\lambda\mu}{(1-\mu)^2}, \frac{(1-\lambda)(1-\mu)}{\lambda^2}), \textit{if} & \\ \qquad \lambda \neq \mu \ \textit{and} \ \lambda \neq 0, \mu \neq 1; & \\ E_7(\frac{1-\mu}{\sqrt[3]{\mu^2}}), & \textit{if} \ \lambda = 0, \mu \neq 0, \mu \neq 1; \\ E_7(\frac{\lambda}{\sqrt[3]{(1-\lambda)^2}}), & \textit{if} \ \lambda \neq 0, \lambda \neq 1, \mu = 1; \\ E_7(0), & \textit{if} \ \lambda = 0, \mu = 1, \end{cases}$$

where E_0 is the trivial evolution algebra (i.e. with zero multiplication) and the evolution algebras E_i, $i = 1, \ldots, 7$, are given in Theorem 2.1.2.

Proof. Let

$$\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrices of structural constants of the evolution algebras $E_{\mathcal{M}}$ and $E_{\mathcal{A}}$. The multiplication table in $E_{\mathcal{M}}$ is

$$e_1e_1 = \alpha e_1 + \beta e_2, \quad e_2e_2 = \gamma e_1 + \delta e_2$$

and in $E_{\mathcal{A}}$ is

$$e_1'e_1' = ae_1' + be_2', \quad e_2'e_2' = ce_1' + de_2'.$$

Let

$$e_1' = xe_1 + ye_2, \quad e_2' = ze_1 + ve_2$$

be the change of basis, where $xv - yz \neq 0$.

We have the following equations:

$$\begin{aligned}
0 = e_1'e_2' &= (xe_1 + ye_2)(ze_1 + ve_2) = (\alpha xz + \gamma yv)e_1 + (\beta xz + \delta yv)e_2,\\
e_1'e_1' &= (xe_1 + ye_2)(xe_1 + ye_2) = (\alpha x^2 + \gamma y^2)e_1 + (\beta x^2 + \delta y^2)e_2,\\
e_1'e_1' &= a(xe_1 + ye_2) + b(ze_1 + ve_2) = (ax + bz)e_1 + (ay + bv)e_2.\\
e_2'e_2' &= (ze_1 + ve_2)(ze_1 + ve_2) = (\alpha z^2 + \gamma v^2)e_1 + (\beta z^2 + \delta v^2)e_2,\\
e_2'e_2' &= c(xe_1 + ye_2) + d(ze_1 + ve_2) = (cx + dz)e_1 + (cy + dv)e_2.
\end{aligned}$$

Consequently,

$$\begin{cases}
xv - yz \neq 0\\
\alpha xz + \gamma yv = 0\\
\beta xz + \delta yv = 0\\
\alpha x^2 + \gamma y^2 = ax + bz\\
\beta x^2 + \delta y^2 = ay + bv\\
\alpha z^2 + \gamma v^2 = cx + dz\\
\beta z^2 + \delta v^2 = cy + dv
\end{cases} \tag{2.2.1}$$

Therefore, we should solve system of equations (2.2.1) for each evolution algebra. Since the CEAs listed above have matrices as in Lemma 2.2.1, it will be enough to check only these forms of matrices.

Case (i). Let $\alpha = \delta = \lambda$, $\beta = \gamma = -\lambda$. It is easy to see that when $\lambda = 0$ this algebra will be the trivial EA. Therefore, we will see the case when $\lambda \neq 0$ and check that the algebra corresponding to the matrix (i) is isomorphic to E_3.

Indeed, in this case (2.2.1) will be

$$\begin{cases} xv - yz \neq 0 \\ \lambda xz - \lambda yv = 0 \\ -\lambda xz + \lambda yv = 0 \\ \lambda x^2 - \lambda y^2 = x + z \\ -\lambda x^2 + \lambda y^2 = y + v \\ \lambda z^2 - \lambda v^2 = -x - z \\ -\lambda z^2 + \lambda v^2 = -y - v \end{cases}$$

Since $xv - yz \neq 0$, when $x = v = 0$, we have the solution $y = -z = \frac{1}{\lambda}$. And by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = -\frac{1}{\lambda}e_2$ we can see that these algebras are isomorphic.

Case (ii). Let $\alpha = \delta = \lambda, \beta = \gamma = \mu$. Then (2.2.1) will be

$$\begin{cases} xv - yz \neq 0 \\ \lambda xz + \mu yv = 0 \\ \mu xz + \lambda yv = 0 \\ \lambda x^2 + \mu y^2 = ax + bz \\ \mu x^2 + \lambda y^2 = ay + bv \\ \lambda z^2 + \mu v^2 = cx + dz \\ \mu z^2 + \lambda v^2 = cy + dv \end{cases}$$

Case (ii).1. For $\lambda = \mu = 0$, this algebra will be the trivial EA.

Case (ii).2. Let $\lambda = 0, \mu \neq 0$. Then this algebra will be isomorphic to $E_7(0)$ by the change of basis $e_1' = \frac{1}{\mu}e_1, e_2' = \frac{1}{\mu}e_2$.

Case (ii).3. Let $\lambda \neq 0$. Then this algebra will be isomorphic to $E_6(\frac{\mu}{\lambda}, \frac{\mu}{\lambda})$, by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\lambda}e_2$.

Case (ii).4. When $\lambda = \mu \neq 0$. Since $xv - yz \neq 0$, for $x = v = y = -z$, we have solution $x = v = y = -z = \frac{1}{2\lambda}$. By the change of basis $e_1' = \frac{1}{2\lambda}e_1 + \frac{1}{2\lambda}e_2, e_2' = -\frac{1}{2\lambda}e_1 + \frac{1}{2\lambda}e_2$ we can see that this algebra is isomorphic to E_2.

Case (iii). Let $\alpha = \delta = \lambda,\ \beta = -\gamma = \mu$. Then (2.2.1) will be

$$\begin{cases} xv - yz \neq 0 \\ \lambda xz - \mu yv = 0 \\ \mu xz + \lambda yv = 0 \\ \lambda x^2 - \mu y^2 = ax + bz \\ \mu x^2 + \lambda y^2 = ay + bv \\ \lambda z^2 - \mu v^2 = cx + dz \\ \mu z^2 + \lambda v^2 = cy + dv \end{cases}$$

Case (iii).1. For $\lambda = 0, \mu \neq 0$, this algebra will be isomorphic to $E_7(0)$ by the change of basis $e_1' = -\frac{1}{\mu}e_1, e_2' = \frac{1}{\mu}e_2$.

Case (iii).2. For $\lambda \neq 0$ and $\mu \in \mathbb{R}$, this algebra will be isomorphic to $E_6(\frac{\mu}{\lambda}, -\frac{\mu}{\lambda})$ by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\lambda}e_2$.

Case (iv). Let $\alpha = \gamma = \lambda, \beta = \delta = \mu$. Then (2.2.1) will be

$$\begin{cases} \lambda xz + \lambda yv = 0 \\ \mu xz + \mu yv = 0 \\ \lambda x^2 + \lambda y^2 = ax + bz \\ \mu x^2 + \mu y^2 = ay + bv \\ \lambda z^2 + \lambda v^2 = cx + dz \\ \mu z^2 + \mu v^2 = cy + dv \end{cases}$$

Case (iv).1. When $\lambda = \mu = 0$ this algebra will be the trivial EA.

Case (iv).2. Let $\lambda = 0, \mu \neq 0$ $(\lambda \neq 0, \mu = 0)$. Then this algebra will be isomorphic to E_2 by the change of basis $e_1' = \frac{1}{\mu}e_1, e_2' = \frac{1}{\mu}e_2$ $(e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\lambda}e_2)$.

Case (iv).3. Let $\lambda \neq \mu \neq 0$ then this algebra will be isomorphic to E_2 and the change of basis $e_1' = \frac{\lambda}{\lambda^2+\mu^2}e_1 + \frac{\mu}{\lambda^2+\mu^2}e_2, e_2' = \frac{\mu}{\lambda^2+\mu^2}e_1 - \frac{\lambda}{\lambda^2+\mu^2}e_2$.

Case (iv).4. When $\lambda = \mu \neq 0$ we obtain the Case (ii).4.

Case (v). Let $\alpha = \beta = \lambda, \gamma = \delta = \mu$. Then (2.2.1) will be

$$\begin{cases} \lambda xz + \mu yv = 0 \\ \lambda xz + \mu yv = 0 \\ \lambda x^2 + \mu y^2 = ax + bz \\ \lambda x^2 + \mu y^2 = ay + bv \\ \lambda z^2 + \mu v^2 = cx + dz \\ \lambda z^2 + \mu v^2 = cy + dv \end{cases}$$

Case (v).1. When $\lambda = \mu = 0$ this algebra will be the trivial EA.

Case (v).2. Let $\lambda = 0, \mu \neq 0$ ($\lambda \neq 0, \mu = 0$). Then this algebra will be isomorphic to E_1 by the change of basis $e_1' = \frac{1}{\mu}e_1 + \frac{1}{\mu}e_2, e_2' = e_1$ ($e_1' = \frac{1}{\lambda}e_1 + \frac{1}{\lambda}e_2, e_2' = e_2$).

Case (v).3. When $\lambda\mu > 0$ then this algebra will be isomorphic to E_2, by the change of basis $e_1' = \frac{1}{\lambda+\mu}e_1 + \frac{1}{\lambda+\mu}e_2, e_2' = \frac{|\mu|}{\sqrt{\lambda\mu}(\lambda+\mu)}e_1 - \frac{|\lambda|}{\sqrt{\lambda\mu}(\lambda+\mu)}e_2$, and when $\lambda\mu < 0$ then this algebra will be isomorphic to E_5, by the change of basis $e_1' = \frac{1}{\lambda+\mu}e_1 + \frac{1}{\lambda+\mu}e_2, e_2' = \frac{|\mu|}{\sqrt{|\lambda\mu|}(\lambda+\mu)}e_1 - \frac{|\lambda|}{\sqrt{|\lambda\mu|}(\lambda+\mu)}e_2$.

Case (v).4. When $\lambda = \mu \neq 0$ we get the Case (ii).4.

Case (vi). Let $\alpha = \lambda, \gamma = \mu, \beta = \delta = 0$. Then (2.2.1) will be

$$\begin{cases} \lambda xz + \mu yv = 0 \\ \lambda x^2 + \mu y^2 = ax + bz \\ \lambda z^2 + \mu v^2 = cx + dz \end{cases}$$

Case (vi).1. When $\lambda = \mu = 0$ this algebra will be the trivial EA.

Case (vi).2. Let $\lambda = 0, \mu \neq 0$. Then this algebra will be isomorphic to E_4 by the change of basis $e_1' = e_1 + \frac{1}{\mu}e_2, e_2' = \frac{1}{\mu}e_1$.

Case (vi).3. Let $\lambda \neq 0, \mu = 0$. Then this algebra will be isomorphic to E_1 by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = e_2$.

Case (vi).4. When $\lambda\mu > 0$ then this algebra will be isomorphic to E_2, by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\sqrt{\lambda\mu}}e_2$, and when $\lambda\mu < 0$ then this algebra will be isomorphic to E_5, by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\sqrt{|\lambda\mu|}}e_2$.

Case (vii). Let $\alpha = \lambda, \beta = 0, \gamma = \mu, \delta = \eta$. Then (2.2.1) will be

$$\begin{cases} \lambda xz + \mu yv = 0 \\ \eta yv = 0 \\ \lambda x^2 + \mu y^2 = ax + bz \\ \eta y^2 = ay + bv \\ \lambda z^2 + \mu v^2 = cx + dz \\ \eta v^2 = cy + dv \end{cases}$$

Case (vii).1. When $\lambda = \mu = \eta = 0$ this algebra will be the trivial EA.

Case (vii).2. Let $\lambda = 0, \mu \neq 0, \eta \neq 0$. Then this algebra will be isomorphic to E_1 by the change of basis $e_1' = \frac{\mu}{\eta^2}e_1 + \frac{1}{\eta}e_2, e_2' = e_1$.

Case (vii).3. Let $\mu = 0, \lambda \neq 0, \eta \neq 0$. Then this algebra will be isomorphic to $E_6(0, 0)$ by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\eta}e_2$.

Case (vii).4. When $\eta = 0, \lambda\mu > 0$ then this algebra will be isomorphic to E_2, by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\sqrt{\lambda\mu}}e_2$, and when $\eta = 0, \lambda\mu < 0$ then this algebra will be isomorphic to E_5, by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = \frac{1}{\sqrt{|\lambda\mu|}}e_2$.

Case (vii).5. Let $\lambda = \mu = 0, \eta \neq 0$. Then this algebra will be isomorphic to E_1 by the change of basis $e_1' = \frac{1}{\eta}e_2, e_2' = e_1$.

Case (vii).6. Let $\lambda = \eta = 0, \mu \neq 0$. Then this algebra will be isomorphic to E_4 by the change of basis $e_1' = e_1 + \frac{1}{\mu}e_2, e_2' = \frac{1}{\mu}e_1$.

Case (vii).7. Let $\mu = \eta = 0, \lambda \neq 0$. Then this algebra will be isomorphic to E_1 by the change of basis $e_1' = \frac{1}{\lambda}e_1, e_2' = e_1$.

Case (vii).8. Let $\lambda \neq 0, \mu \neq 0, \eta \neq 0$. Then this algebra will be isomorphic to $E_6(\frac{\lambda\mu}{\eta^2}, 0)$ by the change of basis $e_1' = \frac{1}{\eta}e_2, e_2' = \frac{1}{\lambda}e_1$.

Case (viii). Let $\alpha = \lambda,\ \beta = (1-\lambda),\ \gamma = \mu, \delta = (1-\mu)$. Then (2.2.1) will be

$$\begin{cases} \lambda xz + \mu yv = 0 \\ (1-\lambda)xz + (1-\mu)yv = 0 \\ \lambda x^2 + \mu y^2 = ax + bz \\ (1-\lambda)x^2 + (1-\mu)y^2 = ay + bv \\ \lambda z^2 + \mu v^2 = cx + dz \\ (1-\lambda)z^2 + (1-\mu)v^2 = cy + dv \end{cases}$$

Case (viii).1. Case $\lambda = \mu$. In this case for any λ this algebra will be isomorphic to E_2 by the change of basis $e_1' = \frac{\lambda}{2\lambda^2-2\lambda+1}e_1 + \frac{1-\lambda}{2\lambda^2-2\lambda+1}e_2, e_2' = \frac{\lambda-1}{2\lambda^2-2\lambda+1}e_1 + \frac{\lambda}{2\lambda^2-2\lambda+1}e_2$.

Case (viii).2. For $\lambda = 0, \mu = 0$ $(\lambda = 1, \mu = 1)$, this algebra will be isomorphic to E_2 by the change of basis $e_1' = e_2, e_2' = e_1$ $(e_1' = e_1, e_2' = e_2)$.

Case (viii).3. For $\lambda \neq \mu, \lambda \neq 0, \mu \neq 1$ this algebra will be isomorphic to $E_6(\frac{\lambda\mu}{(1-\mu)^2}, \frac{(1-\lambda)(1-\mu)}{\lambda^2})$ by the change of basis $e_1' = \frac{1}{1-\mu}e_2, e_2' = \frac{1}{\lambda}e_1$.

Case (viii).4. For $\lambda = 0, \mu \neq 0, \mu \neq 1$ this algebra will be isomorphic to $E_7(\frac{1-\mu}{\sqrt[3]{\mu^2}})$ by the change of basis $e_1' = \frac{1}{\sqrt[3]{\mu}}e_1, e_2' = \frac{1}{\sqrt[3]{\mu^2}}e_2$.

Case (viii).5. For $\lambda \neq 0, \lambda \neq 1, \mu = 1$ this algebra will be isomorphic to $E_7(\frac{\lambda}{\sqrt[3]{(1-\lambda)^2}})$ by the change of basis $e_1' = \frac{1}{\sqrt[3]{1-\lambda}}e_1, e_2' = \frac{1}{\sqrt[3]{(1-\lambda)^2}}e_2$.

Case (viii).6. For $\lambda = 0, \mu = 1$ this algebra will be isomorphic to $E_7(0)$ by the change of basis $e_1' = e_1, e_2' = e_2$. □

By this lemma we will prove the next theorem which gives the time depending dynamics of the chains of evolution algebras $E_i^{[s,t]}, i = 0, \dots, 24$, constructed in Section 1.2.

Theorem 2.2.2.

$E_1^{[s,t]}$ *is isomorphic to* E_3 *for any* $0 \leq s \leq t$.

$$E_2^{[s,t]} \simeq \begin{cases} E_3 \ \text{ for all } \ (s,t) \in \{(s,t) : s \leq t < b\} \ ; \\ E_0 \ \text{ for all } \ (s,t) \in \{(s,t) : t \geq b\} \ . \end{cases}$$

$E_3^{[s,t]} \simeq E_2$ *for any* $s, t \in \mathcal{T}$.

$$E_4^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}, \frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}\right) \\ \quad \textit{for all} \ \ (s,t) \in \left\{(s,t) : \frac{\Phi(t)}{\Phi(s)} \neq -\frac{\Psi(t)}{\Psi(s)}\right\}; \\ E_7(0) \ \ \textit{for all} \ \ (s,t) \in \left\{(s,t) : \frac{\Phi(t)}{\Phi(s)} = -\frac{\Psi(t)}{\Psi(s)}\right\} \ . \end{cases}$$

$$E_5^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}, \frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}\right) \\ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < b, \ \ \Phi(t) \neq -\Phi(s)\}; \\ E_7(0) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < b, \ \ \Phi(t) = -\Phi(s)\} \ ; \\ E_2 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq b\} \ . \end{cases}$$

$$E_6^{[s,t]} \simeq \begin{cases} E_2 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a\} \ ; \\ E_0 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$$E_7^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Psi(s)-\Psi(t)}{\Psi(s)+\Psi(t)}, \frac{\Psi(s)-\Psi(t)}{\Psi(s)+\Psi(t)}\right) \\ \quad \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a, \ \ \Psi(t) \neq -\Psi(s)\}; \\ E_7(0) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a, \ \ \Psi(t) = -\Psi(s)\} \ ; \\ E_3 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$$E_8^{[s,t]} \simeq \begin{cases} E_6(0,0) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < \min\{a,b\}\} \ ; \\ E_3 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : a \leq t < b, \ \ a < b\} \ ; \\ E_2 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : b \leq t < a, \ \ b < a\} \ ; \\ E_0 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq \max\{a,b\}\} \ . \end{cases}$$

$$E_9^{[s,t]} \simeq \begin{cases} E_6(\tan(t-s), -\tan(t-s)) \\ \quad \textit{for all} \;\; (s,t) \in \left\{(s,t) : t \neq s + \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\right\}; \\ \\ E_7(0) \;\; \textit{for all} \;\; (s,t) \in \left\{(s,t) : t = s + \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\right\} \;. \end{cases}$$

$E_{10}^{[s,t]} \simeq E_2$ *for any* $s, t \in \mathcal{T}$.

$$E_{11}^{[s,t]} \simeq \begin{cases} E_2 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : s \leq t < a\} \;; \\ \\ E_0 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq a\} \;. \end{cases}$$

$$E_{12}^{[s,t]} \simeq \begin{cases} E_1 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq a, \;\; h(s)g(s) = \pm 1\} \;; \\ \\ E_2 \;\; \textit{for all} \;\; (s,t) \in \left\{(s,t) : s \leq t < a, \;\; h^2(s)g^2(s) < 1\right\}; \\ \\ E_3 \;\; \textit{for all} \;\; (s,t) \in \left\{(s,t) : s \leq t < a, \;\; h^2(s)g^2(s) > 1\right\} \;. \end{cases}$$

$$E_{13}^{[s,t]} \simeq \begin{cases} E_1 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : s \leq t < a, \;\; \psi(s) = \pm 1\} \;; \\ \\ E_2 \;\; \textit{for all} \;\; (s,t) \in \left\{(s,t) : s \leq t < a, \;\; \psi^2(s) < 1\right\} \;; \\ \\ E_3 \;\; \textit{for all} \;\; (s,t) \in \left\{(s,t) : s \leq t < a, \;\; \psi^2(s) > 1\right\} \;; \\ \\ E_0 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq a\} \;. \end{cases}$$

$$E_{14}^{[s,t]} \simeq \begin{cases} E_1 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : \;\; \psi(s) = 0\} \;; \\ \\ E_2 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : \;\; \Phi(s)\psi(s) > 0\}; \\ \\ E_5 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : \;\; \Phi(s)\psi(s) < 0\} \;. \end{cases}$$

$$E_{15}^{[s,t]} \simeq \begin{cases} E_1 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a, \ \ \psi(s) = 0\} \ ; \\ \\ E_2 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a, \ \ \psi(s) > 0\} \ ; \\ \\ E_5 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a, \ \ \psi(s) < 0\} \ ; \\ \\ E_0 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$E_{16}^{[s,t]} \simeq E_1$ *for any* $s, t \in \mathcal{T}$.

$E_{17}^{[s,t]} \simeq E_6\left(\frac{\Phi^2(t)\psi(s)(g(t)-g(s))}{\Phi(s)\psi^2(t)}, 0\right)$ *for any* $s, t \in \mathcal{T}$.

$$E_{18}^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\psi(s)(h(t)-h(s))}{\psi^2(t)}, 0\right) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < a\} \ ; \\ \\ E_1 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$$E_{19}^{[s,t]} \simeq \begin{cases} E_1 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < b\} \ ; \\ \\ E_0 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq b\} \ . \end{cases}$$

$$E_{20}^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Phi^2(t)(v(t)-v(s))}{\Phi(s)}, 0\right) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \leq t < b\} \ ; \\ \\ E_1 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \geq b, \ \ w(s) = 0\} \ ; \\ \\ E_2 \ \ \textit{for all} \ \ (s,t) \in \left\{(s,t) : t \geq b, \frac{\Phi^2(t)w(s)}{\Phi(s)} > 0\right\} \ ; \\ \\ E_5 \ \ \textit{for all} \ \ (s,t) \in \left\{(s,t) : t \geq b, \frac{\Phi^2(t)w(s)}{\Phi(s)} < 0\right\} \ . \end{cases}$$

$$E_{21}^{[s,t]} \simeq \begin{cases} E_6(v(t)-v(s),0) \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : s \leq t < \min\{a,b\}\} \;; \\ E_1 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : a \leq t < b, \;\; a < b\} \cup \\ \qquad\qquad \{(s,t) : b \leq t < a, \;\; a > b, \;\; v(s) = 0\} \,; \\ E_2 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : b \leq t < a, \;\; a > b, \;\; v(s) > 0\} \;; \\ E_5 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : b \leq t < a, \;\; a > b, \;\; v(s) < 0\} \;; \\ E_0 \;\; \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq \max\{a,b\}\} \;. \end{cases}$$

$E_{22}^{[s,t]} \simeq E_2$ *for any* $s,t \in \mathcal{T}$.

$$E_{23}^{[s,t]}(\lambda,\mu) \simeq$$

$$\begin{cases} E_6(\frac{2\theta(s)(\theta(s)-\theta(t))}{(\theta(s)+\theta(t))^2},0) \;\; \textit{for all} \;\; \lambda = 2\mu \;\; (s,t) \in \{(s,t) : s \leq t < a\} \,; \\ E_6(\frac{\xi\zeta}{(1-\zeta)^2}, \frac{(1-\xi)(1-\zeta)}{\xi^2}) \;\; \textit{for all} \;\; \lambda \neq 2\mu \\ (s,t) \in \left\{(s,t) : \theta(t) \neq \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\right\} ; \\ E_7(\frac{1-\zeta}{\sqrt[3]{\zeta^2}}) \;\; \textit{for all} \;\; \lambda \neq 2\mu, \; \lambda \neq 0 \\ (s,t) \in \left\{(s,t) : \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \theta(s), \;\; \theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\right\} ; \\ E_7(\frac{\xi}{\sqrt[3]{(1-\xi)^2}}) \;\; \textit{for all} \;\; \lambda \neq 2\mu \\ \;\; (s,t) \in \left\{(s,t) : \theta(t) \neq \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \theta(s), \;\; \theta(t) = \frac{2\mu-\lambda}{\lambda}\theta(s)\right\} ; \\ E_7(0) \;\; \textit{for all} \;\; \lambda \neq 2\mu, \\ (s,t) \in \left\{(s,t) : \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) = \frac{2\mu-\lambda}{\lambda}\theta(s)\right\} , \end{cases}$$

where $\xi = 1 - \frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1 - \frac{\theta(t)}{\theta(s)}\right), \; \zeta = \frac{\lambda}{2(\lambda-\mu)}\left(1 - \frac{\theta(t)}{\theta(s)}\right)$.

$$E_{24}^{[s,t]} \simeq \begin{cases} E_6(0,0) \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : s \le t < a\} \ ; \\ \\ E_2 \ \ \textit{for all} \ \ (s,t) \in \{(s,t) : t \ge a\} \ . \end{cases}$$

Remark 2.2.3. We note that the CEAs which can be isomorphic to E_6 and E_7 are interesting having an uncountable family of pairwise non-isomorphic EAs, while other CEAs are generated by finitely many pairwise non-isomorphic EAs.

Proof.

1. From the proved lemma it is easy to see that $E_1^{[s,t]}$ is isomorphic to E_3, $\forall s,t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{2}\frac{\Psi(s)}{\Psi(t)}e_1, \ \ e_2' = -\frac{1}{2}\frac{\Psi(s)}{\Psi(t)}e_2$.

2. $E_2^{[s,t]}$ is isomorphic to E_3, $\forall s,t \in \mathcal{T}, s \le t < b$, by the change of basis $e_1' = \frac{1}{2}e_1, \ \ e_2' = -\frac{1}{2}e_2$, in time when $t \ge b$ will be E_0.

3. $E_3^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{4}\frac{\Phi(s)}{\Phi(t)}e_1 + \frac{\Phi(s)}{\Phi(t)}e_2, \ \ e_2' = -\frac{1}{4}\frac{\Phi(s)}{\Phi(t)}e_1 + \frac{\Phi(s)}{\Phi(t)}e_2$.

4. $E_4^{[s,t]}$ is isomorphic to $E_6\left(\frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}, \frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}\right)$, when $\frac{\Phi(t)}{\Phi(s)} \neq -\frac{\Psi(t)}{\Psi(s)}$, $\forall s,t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{2}\frac{\Phi(s)\Psi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}e_1$, $e_2' = \frac{1}{2}\frac{\Phi(s)\Psi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}e_2$, and it is isomorphic to $E_7(0)$, when $\frac{\Phi(t)}{\Phi(s)} = -\frac{\Psi(t)}{\Psi(s)}$, $\forall s,t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{2}\frac{\Phi(s)\Psi(s)}{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}e_1, \ \ e_2' = \frac{1}{2}\frac{\Phi(s)\Psi(s)}{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}e_2$.

5. $E_5^{[s,t]}$ is isomorphic to $E_6\left(\frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}, \frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}\right)$, when $\Phi(t) \neq -\Phi(s)$, $\forall s,t \in \mathcal{T}, s \le t < b$, by the change of basis $e_1' = \frac{1}{2}\frac{\Phi(s)}{\Phi(t)+\Phi(s)}e_1, \ \ e_2' = \frac{1}{2}\frac{\Phi(s)}{\Phi(t)+\Phi(s)}e_2$, and it is isomorphic to $E_7(0)$, when $\Phi(t) = -\Phi(s)$, $\forall s,t \in \mathcal{T}, s \le t < b$, by the change of basis $e_1' = -e_1, \ \ e_2' = -e_2$. It is isomorphic to E_2, $\forall s,t \in \mathcal{T}, t \ge b$, by the change of basis $e_1' = \frac{1}{2}\frac{\Phi(s)}{\Phi(t)}e_1 + \frac{\Phi(s)}{\Phi(t)}e_2, \ \ e_2' = -\frac{1}{2}\frac{\Phi(s)}{\Phi(t)}e_1 + \frac{\Phi(s)}{\Phi(t)}e_2$.

6. $E_6^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}, s \leq t < a$, by the change of basis $e'_1 = \frac{1}{4}e_1 + \frac{1}{4}e_2,\quad e'_2 = -\frac{1}{4}e_1 + \frac{1}{4}e_2$, and in time when $t \geq a$ this algebra will be E_0.

7. $E_7^{[s,t]}$ is isomorphic to $E_6\left(\frac{\Psi(s)-\Psi(t)}{\Psi(t)+\Psi(s)}, \frac{\Psi(t)-\Psi(s)}{\Psi(t)+\Psi(s)}\right)$, when $\Psi(t) \neq -\Psi(s)$, $\forall s,t \in \mathcal{T}, s \leq t < b$, by the change of basis $e'_1 = \frac{1}{2}\frac{\Psi(s)}{\Psi(t)+\Psi(s)}e_1,\quad e'_2 = \frac{1}{2}\frac{\Psi(s)}{\Psi(t)+\Psi(s)}e_2$, and it is isomorphic to $E_7(0)$, when $\Psi(t) = -\Psi(s),\ \forall s,t \in \mathcal{T}, s \leq t < b$, by the change of basis $e'_1 = -e_1,\quad e'_2 = -e_2$. Isomorphic to E_3, for all $s,t \in \mathcal{T}, t \geq b$, by change of basis $e'_1 = \frac{1}{2}\frac{\Psi(s)}{\Psi(t)}e_1,\ \ e'_2 = -\frac{\Psi(s)}{\Psi(t)}e_2$.

8. $E_8^{[s,t]}$ is isomorphic to $E_6(0,0)$, $\forall s,t \in \mathcal{T}, s \leq t < \min\{a,b\}$, by the change of basis $e'_1 = e_1,\quad e'_2 = e_2$, and it is isomorphic to E_3 $\forall s,t \in \mathcal{T}, a \leq t < b,\quad a < b$, by the change of basis $e'_1 = \frac{1}{2}e_1,\quad e'_2 = -\frac{1}{2}e_2$. By the change of basis $e'_1 = \frac{1}{4}e_1 + \frac{1}{4}e_2,\quad e'_2 = -\frac{1}{4}e_1 + \frac{1}{2}e_2$, this algebra will be isomorphic to E_2, $\forall s,t \in \mathcal{T}, b \leq t < a,\quad b < a$. And this algebra will be E_0, $\forall s,t \in \mathcal{T}, t \geq \max\{a,b\}$.

9. $E_9^{[s,t]}$ is isomorphic to $E_6(\tan(t-s), -\tan(t-s))$, $\forall s,t \in \mathcal{T}$, when $t \neq s + \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$, by the change of basis $e'_1 = \frac{1}{\cos(t-s)}e_1,\ \ e'_2 = \frac{1}{\cos(t-s)}e_2$, and it is isomorphic to $E_7(0)$ $\forall s,t \in \mathcal{T}$, when $t = s + \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$, by the change of basis $e'_1 = -e_1,\quad e'_2 = e_2$.

10. $E_{10}^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}$, by the change of basis:

$$e'_1 = \frac{1}{4}\left(\frac{h(s)(h(t)+g(t))}{h^2(t)+g^2(t)}e_1 + \frac{h(s)(h(t)-g(t))}{h^2(t)+g^2(t)}e_2\right),$$
$$e'_2 = \frac{1}{4}\left(\frac{h(s)(h(t)+g(t))}{h^2(t)+g^2(t)}e_1 - \frac{h(s)(h(t)-g(t))}{h^2(t)+g^2(t)}e_2\right).$$

11. $E_{11}^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}, s \leq t < a$, by the change of basis $e'_1 = \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 + \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right),\ \ e'_2 = \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 - \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right)$, in time when $t \geq 0$ will be E_0.

12. $E_{12}^{[s,t]}$ is isomorphic to E_1, $\forall s,t \in \mathcal{T}$, when $g(s)h(s) = \pm 1$. In the case when $g(s)h(s) = 1$, the change of basis will be $e_1' = \frac{1}{h(t)}e_1 + \frac{1}{h(t)}e_2,\ e_2' = e_2$, and when $g(s)h(s) = -1$, the change of basis will be $e_1' = -\frac{1}{h(t)}e_1 - \frac{1}{h(t)}e_2,\ e_2' = e_1$.

    When $g^2(s)h^2(s) < 1$, $E_{12}^{[s,t]}$ is isomorphic to E_2, for all $s,t \in \mathcal{T}$, by the change of basis

$$
\begin{aligned}
e_1' &= \frac{1}{2}\left(\frac{h(s)}{2h(t)}e_1 + \frac{h(s)}{2h(t)}e_2\right),\\
e_2' &= \frac{1}{4}\left(\frac{h(s)\sqrt{1-g(s)h(s)}}{2h(t)\sqrt{1+g(s)h(s)}}e_1 - \frac{h(s)\sqrt{1+g(s)h(s)}}{2h(t)\sqrt{1-g(s)h(s)}}e_2\right).
\end{aligned}
$$

    When $g^2(s)h^2(s) > 1$, $E_{12}^{[s,t]}$ is isomorphic to E_3, $\forall s,t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 + \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right),\ \ e_2' = \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 - \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right)$, in time when $t \geq 0$ will be E_0.

13. $E_{13}^{[s,t]}$ is isomorphic to E_1, $\forall s,t \in \mathcal{T}, s \leq t < a$, when $\psi(s) = \pm 1$. In the case when $\psi(s) = 1$, the change of basis will be $e_1' = e_1 + e_2,\ e_2' = e_2$, and when $\psi(s) = -1$, the change of basis will be $e_1' = e_1 + e_2,\ e_2' = e_1$.

    When $\psi^2(s) < 1$, $E_{13}^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}$, by the change of basis

$$
\begin{aligned}
e_1' &= \frac{1}{2}\left(\frac{h(s)}{2h(t)}e_1 + \frac{h(s)}{2h(t)}e_2\right),\\
e_2' &= \frac{1}{2}\left(\frac{\sqrt{1-\psi(s)}}{2h(t)\sqrt{1+\psi(s)}}e_1 - \frac{h(s)\sqrt{1+\psi(s)}}{2h(t)\sqrt{1-\psi(s)}}e_2\right),
\end{aligned}
$$

    When $\psi^2(s) > 1$, $E_{13}^{[s,t]}$ is isomorphic to E_3, $\forall s,t \in \mathcal{T}$, by the change of basis

$$
\begin{aligned}
e_1' &= \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 + \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right),\\
e_2' &= \frac{1}{4}\left(\frac{1+\psi(t)}{1+\psi^2(t)}e_1 - \frac{1-\psi(t)}{1+\psi^2(t)}e_2\right), \quad \text{in time when } t \geq 0 \text{ will be } E_0.
\end{aligned}
$$

14. When $\psi(s)=0$, $E_{14}^{[s,t]}$ is isomorphic to E_1, $\forall s,t\in\mathcal{T}$, by the change of basis $e_1'=\frac{\Phi(s)}{\Phi(t)}e_1,\ \ e_2'=e_2$. When $\Phi(s)\psi(s)>0$, $E_{14}^{[s,t]}$ is isomorphic to E_2, $\forall s,t\in\mathcal{T}$, by the change of basis $e_1'=\frac{\Phi(s)}{\Phi(t)}e_1,\ \ e_2'=\frac{\sqrt{|\Phi(s)|}}{\Phi(t)\sqrt{|\psi(s)|}}e_2$, and when $\Phi(s)\psi(s)<0$ it is isomorphic to E_5, $\forall s,t\in\mathcal{T}$, by the change of basis $e_1'=\frac{\Phi(s)}{\Phi(t)}e_1,\ \ e_2'=\frac{\sqrt{|\Phi(s)|}}{\Phi(t)\sqrt{|\psi(s)|}}e_2$.

15. When $\psi(s)=0$, $E_{15}^{[s,t]}$ is isomorphic to E_1, $\forall s,t\in\mathcal{T}, s\le t<a$, by the change of basis $e_1'=e_1,\ \ e_2'=e_2$. When $\psi(s)>0$, $E_{15}^{[s,t]}$ is isomorphic to E_2, $\forall s,t\in\mathcal{T}, s\le t<a$, by the change of basis $e_1'=e_1,\ \ e_2'=\frac{1}{\sqrt{|\psi(s)|}}e_2$, and in time $\forall s,t\in\mathcal{T}, s\le t<a$, when $\psi(s)<0$ it is isomorphic to E_5, by the change of basis $e_1'=e_1,\ \ e_2'=\frac{1}{\sqrt{|\psi(s)|}}e_2$, and in time when $t\ge a$ this algebra will be E_0.

16. $E_{16}^{[s,t]}$ is isomorphic to E_1, $\forall s,t\in\mathcal{T}$, by the change of basis $e_1'=\frac{g(t)\psi(s)}{\psi^2(t)}e_1+\frac{\psi(s)}{\psi(t)}e_2,\ \ e_2'=e_1$.

17. $E_{17}^{[s,t]}$ is isomorphic to $E_6(\frac{\Phi^2(t)\psi(s)}{\Phi(s)\psi^2(t)}(g(t)-g(s)),0)$, $\forall s,t\in\mathcal{T}$, by the change of basis $e_1'=\frac{\psi(s)}{\psi(t)}e_2,\ \ e_2'=\frac{\Phi(s)}{\Phi(t)}e_1$.

18. $E_{18}^{[s,t]}$ is isomorphic to $E_6(\frac{\psi(s)}{\psi^2(t)}(h(t)-h(s)),0)$, $\forall s,t\in\mathcal{T}, s\le t<a$, by the change of basis $e_1'=\frac{\psi(s)}{\psi(t)}e_2,\ \ e_2'=e_1$ and it is isomorphic to E_1 $\forall s,t\in\mathcal{T}, t\ge a$, by the change of basis $e_1'=\frac{h(t)\psi(s)}{\psi^2(t)}e_1+\frac{\psi(s)}{\psi(t)}e_2,\ \ e_2'=e_1$.

19. $E_{19}^{[s,t]}$ is isomorphic to E_1, $\forall s,t\in\mathcal{T}, s\le t<b$, by the change of basis $e_1'=h(t)e_1+e_2,\ \ e_2'=e_1$, in time when $t\ge b$ will be E_0.

20. $E_{20}^{[s,t]}$ is isomorphic to $E_6(\frac{\Phi^2(t)}{\Phi(s)}(v(t)-v(s)),0)$, $\forall s,t\in\mathcal{T}, s\le t<b$, by the change of basis $e_1'=e_2,\ \ e_2'=\frac{\Phi(s)}{\Phi(t)}e_1$.

    When $w(s)=0$, $E_{20}^{[s,t]}$ is isomorphic to E_1, $\forall s,t\in\mathcal{T}, t\ge b$, by the change of basis $e_1'=\frac{\Phi(s)}{\Phi(t)}e_1,\ \ e_2'=e_2$. And when $\Phi(s)w(s)>0$, $E_{20}^{[s,t]}$ is isomorphic to E_2, $\forall s,t\in\mathcal{T}, t\ge b$, by the change of basis $e_1'=$

$\frac{\Phi(s)}{\Phi(t)}e_1,\ \ e'_2 = \frac{\sqrt{|\Phi(s)|}}{\Phi(t)\sqrt{|w(s)|}}e_2$, and when $\Phi(s)\psi(s) < 0$ it is isomorphic to E_5, $\forall s,t \in \mathcal{T}, t \geq b$, by the change of basis $e'_1 = \frac{\Phi(s)}{\Phi(t)}e_1,\ \ e'_2 = \frac{\sqrt{|\Phi(s)|}}{\Phi(t)\sqrt{|w(s)|}}e_2$.

21. $E_{21}^{[s,t]}$ is isomorphic to $E_6(v(t) - v(s), 0)$, $\forall s,t \in \mathcal{T}, s \leq t < \min\{a,b\}$, by the change of basis $e'_1 = e_2,\ \ e'_2 = e_1$.

    When $v(s) = 0$, $E_{21}^{[s,t]}$ is isomorphic to E_1, $\forall s,t \in \mathcal{T}, b \leq t < a,\ \ a > b$, by the change of basis $e'_1 = e_1,\ \ e'_2 = e_2$. And when $v(s) > 0$, $E_{21}^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}, b \leq t < a,\ \ a > b$, by the change of basis $e'_1 = e_1,\ \ e'_2 = \frac{1}{\sqrt{|v(s)|}}e_2$, and when $v(s) < 0$, it is isomorphic to E_5, $\forall s,t \in \mathcal{T}, b \leq t < a,\ \ a > b$, by the change of basis $e'_1 = e_1,\ \ e'_2 = \frac{1}{\sqrt{|v(s)|}}e_2$.

    $E_{21}^{[s,t]}$ is isomorphic to E_1, $\forall s,t \in \mathcal{T}, b \leq t < a,\ \ a < b$, by the change of basis $e'_1 = v(t)e_1 + e_2,\ \ e'_2 = e_1$. And this algebra will be E_0, $\forall s,t \in \mathcal{T}, t \geq \max\{a,b\}$.

22. $E_{22}^{[s,t]}$ is isomorphic to E_2, $\forall s,t \in \mathcal{T}$, by the change of basis

$$\begin{aligned} e'_1 &= \frac{f(t)}{2f^2(t) - 2f(t) + 1}e_1 + \frac{1 - f(t)}{2f^2(t) - 2f(t) + 1}e_2 \\ e'_2 &= -\frac{1 - f(t)}{2f^2(t) - 2f(t) + 1}e_1 + \frac{f(t)}{2f^2(t) - 2f(t) + 1}e_2. \end{aligned}$$

23. $E_{23}^{[s,t]}$ is isomorphic to $E_6(\frac{2\theta(s)(\theta(s)-\theta(t))}{(\theta(s)+\theta(t))^2}, 0)$ for any $s,t \in \mathcal{T}$, when $\lambda = 2\mu$, by the change of basis $e'_1 = \frac{2\theta(s)}{\theta(s)+\theta(t)}e_2,\ \ e'_2 = e_1$.

    $E_{23}^{[s,t]}$ is isomorphic to $E_6(\frac{\xi\zeta}{(1-\zeta)^2}, \frac{(1-\xi)(1-\zeta)}{\xi^2})$ for all $\lambda \neq 2\mu$, $(s,t) \in \left\{(s,t) : \theta(t) \neq \frac{2\lambda}{2\mu-\lambda}\theta(s), \theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\right\}$, by the change of basis $e'_1 = \frac{1}{1-\zeta}e_2, e'_2 = \frac{1}{\xi}e_1$.

$E_{23}^{[s,t]}$ is isomorphic to $E_7\Big(\frac{1-\zeta}{\sqrt[3]{\zeta^2}}\Big)$ for all $\lambda \neq 2\mu$,

$$(s,t) \in \Big\{(s,t):\ \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s),\ \lambda \neq 0,\ \theta(t) \neq \theta(s),$$
$$\theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\Big\},$$

by the change of basis $e_1' = \frac{1}{\sqrt[3]{\zeta}}e_1, e_2' = \frac{1}{\sqrt[3]{\zeta^2}}e_2$.

$E_{23}^{[s,t]}$ is isomorphic to $E_7\Big(\frac{1-\zeta}{\sqrt[3]{\zeta^2}}\Big)$ for all $\lambda \neq 2\mu$,

$$(s,t) \in \Big\{(s,t):\ \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s),\ \lambda \neq 0,\ \theta(t) \neq \theta(s),$$
$$\theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\Big\},$$

by the change of basis $e_1' = \frac{1}{\sqrt[3]{1-\xi}}e_1, e_2' = \frac{1}{\sqrt[3]{(1-\xi)^2}}e_2$.

$E_{23}^{[s,t]}$ is isomorphic to $E_7(0)$ for all $\lambda \neq 2\mu$,
$(s,t) \in \Big\{(s,t) : \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s), \theta(t) = \frac{2\mu-\lambda}{\lambda}\theta(s)\Big\}$, by the change of basis $e_1' = e_1, e_2' = e_2$.

24. $E_{24}^{[s,t]}$ is isomorphic to $E_6(0,0)$, $\forall s,t \in \mathcal{T}, s \leq t < a$, by the change of basis $e_1' = e_1,\ \ e_2' = e_2$, and it is isomorphic to E_2, $\forall s,t \in \mathcal{T}, t \geq a$, by the change of basis

$$e_1' = \frac{g(t)}{2g^2(t)-2g(t)+1}e_1 + \frac{1-g(t)}{2g^2(t)-2g(t)+1}e_2,$$
$$e_2' = -\frac{1-g(t)}{2g^2(t)-2g(t)+1}e_1 + \frac{g(t)}{2g^2(t)-2g(t)+1}e_2. \qquad \square$$

From the previous theorem we see that the constructed chains of evolution algebras never contain an evolution algebra isomorphic to E_4 in any time s,t.

Before constructing the chain of evolution algebras, which will be isomorphic to E_4 for some period of time, we need the next theorem which gives all classes of evolution algebras isomorphic to E_4.

Theorem 2.2.4. *An evolution algebra $E_{\mathcal{M}}$ is isomorphic to E_4 iff $E_{\mathcal{M}}$ has the matrix of structural constants in the following form:*

$$\mathcal{M}_1 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad or \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \tag{2.2.2}$$

where $\beta, \gamma \in \mathbb{R}$.

Proof. Let

$$\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the matrix of structural constants of an evolution algebra $E_{\mathcal{M}}$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the matrix of structural constants of the evolution algebra E_4. The multiplication table in $E_{\mathcal{M}}$ is

$$e_1e_1 = \alpha e_1 + \beta e_2, \quad e_2e_2 = \gamma e_1 + \delta e_2$$

and in E_4 is

$$e_1'e_1' = e_2', \quad e_2'e_2' = 0.$$

Let

$$e_1' = xe_1 + ye_2, \quad e_2' = ze_1 + ve_2$$

be the change of basis, where $xv - yz \neq 0$.

Thus we have the following equations:

$$\begin{aligned}
0 = e_1'e_2' &= (xe_1 + ye_2)(ze_1 + ve_2) = (\alpha xz + \gamma yv)e_1 + (\beta xz + \delta yv)e_2,\\
e_1'e_1' &= (xe_1 + ye_2)(xe_1 + ye_2) = (\alpha x^2 + \gamma y^2)e_1 + (\beta x^2 + \delta y^2)e_2,\\
e_1'e_1' &= e_2' = ze_1 + ve_2,\\
e_2'e_2' &= (ze_1 + ve_2)(ze_1 + ve_2) = (\alpha z^2 + \gamma v^2)e_1 + (\beta z^2 + \delta v^2)e_2,\\
e_2'e_2' &= 0.
\end{aligned}$$

Consequently,

$$\begin{cases} xv - yz \neq 0 \\ \alpha xz + \gamma yv = 0 \\ \beta xz + \delta yv = 0 \\ \alpha x^2 + \gamma y^2 = z \\ \beta x^2 + \delta y^2 = v \\ \alpha z^2 + \gamma v^2 = 0 \\ \beta z^2 + \delta v^2 = 0. \end{cases} \tag{2.2.3}$$

Therefore, we should solve the system of equations (2.2.3).

Case 1. Let $\det(\mathcal{M}) \neq 0$. Then we have the solution $x = y = z = v = 0$, which does not satisfy $xv - yz \neq 0$.

Case 2. Let $\det(\mathcal{M}) = 0$, from this we have $\alpha\delta = \beta\gamma$.

Without loss of generality suppose $\gamma \neq 0, \delta \neq 0$, then $\frac{\alpha}{\gamma} = \frac{\beta}{\delta} = k$ and $\alpha = k\gamma, \beta = k\delta$.

Our system of equations will be

$$\begin{cases} xv - yz \neq 0 \\ k\gamma xz + \gamma yv = 0 \\ k\delta xz + \delta yv = 0 \\ k\gamma x^2 + \gamma y^2 = z \\ k\delta x^2 + \delta y^2 = v \\ k\gamma z^2 + \gamma v^2 = 0 \\ k\delta z^2 + \delta v^2 = 0 \end{cases} \Rightarrow \begin{cases} xv - yz \neq 0 \\ kxz + yv = 0 \\ kxz + yv = 0 \\ kx^2 + y^2 = \frac{z}{\gamma} \\ kx^2 + y^2 = \frac{v}{\delta} \\ kz^2 + v^2 = 0 \\ kz^2 + v^2 = 0. \end{cases}$$

Let $z \neq 0$. From $kz^2 + v^2 = 0 \Rightarrow v = \pm\sqrt{|k|}|z|$, $xv - yz \neq 0 \Rightarrow \pm x\sqrt{|k|}|z| - yz \neq 0 \Rightarrow \pm x\sqrt{|k|} \neq y\frac{z}{|z|}$, $kxz + yv = 0 \Rightarrow kxz = \mp y\sqrt{|k|}|z| \Rightarrow \mp y = -\sqrt{|k|}x\frac{z}{|z|}$.

Case 2.1. Let $z > 0$. Then we have

$$\begin{cases} \pm x\sqrt{|k|} \neq y \\ \mp y = -\sqrt{|k|}x \end{cases} \Rightarrow \begin{cases} y \neq \pm\sqrt{|k|}x \\ y = \pm\sqrt{|k|}x \end{cases},$$

which has no solution.

Case 2.2. Let $z < 0$. Then we get

$$\begin{cases} \pm x\sqrt{|k|} \neq -y \\ \mp y = \sqrt{|k|}x \end{cases} \Rightarrow \begin{cases} y \neq \pm\sqrt{|k|}x \\ y = \pm\sqrt{|k|}x \end{cases},$$

which has no solution.

Case 3. Let $\det \mathcal{M} = 0, \gamma \neq 0, \delta \neq 0, z = 0$. Then from (2.2.3) we have

$$\begin{cases} xv \neq 0 \\ yv = 0 \\ kx^2 + y^2 = 0 \\ kx^2 + y^2 = \frac{v}{\delta} \\ v^2 = 0. \end{cases}$$

From $v^2 = 0 \Rightarrow v = 0$ which does not satisfy $xv \neq 0$. Thus we have no solution in this case too.

Case 4. Let $\det \mathcal{M} = 0, \alpha = \beta = 0$ (the case $\gamma = 0, \delta = 0$ is similar). Then from (2.2.3) we have

$$\begin{cases} xv - yz \neq 0 \\ \gamma yv = 0 \\ \delta yv = 0 \\ \gamma y^2 = z \\ \delta y^2 = v \\ \gamma v^2 = 0 \\ \delta v^2 = 0. \end{cases}$$

Case 4.1. Let $\delta = 0$ (the case $v = 0$ is similar). We have

$$\begin{cases} xv - yz \neq 0 \\ \gamma yv = 0 \\ \gamma y^2 = z \\ v = 0 \\ \gamma v^2 = 0, \end{cases}$$

which has solution $x, y \neq 0, z \neq 0, \gamma = \frac{z}{y^2}$.

Case 4.2. When $\gamma = 0$ our system of equations has no solution.

Case 5. Let $\det \mathcal{M} = 0, \alpha = \gamma = 0$ (the case $\beta = 0, \delta = 0$ is similar). Then from (2.2.3) we have

$$\begin{cases} xv - yz \neq 0 \\ \beta xz + \delta yv = 0 \\ z = 0 \\ \beta x^2 + \delta y^2 = v \\ \beta z^2 + \delta v^2 = 0 \end{cases} \Rightarrow \begin{cases} xv \neq 0 \\ \delta yv = 0 \\ z = 0 \\ \beta x^2 + \delta y^2 = v \\ \delta v^2 = 0, \end{cases}$$

which has solution $\forall y, x \neq 0, v \neq 0, \beta = \frac{v}{x^2}$.

Thus we have proved that an evolution algebra $E_{\mathcal{M}}$ is isomorphic to E_4 iff $E_{\mathcal{M}}$ has the matrix of structural constants in the following form:

$$\mathcal{M}_1 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix},$$

where $\beta, \gamma \in \mathbb{R}$.

Consequently we have only two evolution algebras which are isomorphic to E_4. They are

$$E_{\mathcal{M}_1}: \ e_1e_1 = 0, \quad e_2e_2 = \gamma e_1; \qquad E_{\mathcal{M}_2}: \ e_1e_1 = \beta e_2, \quad e_2e_2 = 0.$$

$\square$

Thus, we should construct CEAs with the matrix of structural constant showed in (2.2.2).

Consider (1.2.1) with $a_{11}^{[s,t]} = \alpha(s,t),\ a_{12}^{[s,t]} = \beta(s,t),\ a_{21}^{[s,t]} = \gamma(s,t),\ a_{22}^{[s,t]} = \delta(s,t)$. Therefore, to find a CEA, we should solve the next equation:

$$\begin{pmatrix} \alpha(s,\tau) & \beta(s,\tau) \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} \alpha(\tau,t) & \beta(\tau,t) \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} \alpha(s,t) & \beta(s,t) \\ \gamma(s,t) & \delta(s,t) \end{pmatrix}. \quad (2.2.4)$$

Case 1.1. Consider in (2.2.4), $\alpha(s,t) = \gamma(s,t) \equiv 0,\ \beta(s,t) \neq 0,\ \delta(s,t) \neq 0.$, Then we have the following:

$$\begin{pmatrix} 0 & \beta(s,\tau) \\ 0 & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} 0 & \beta(\tau,t) \\ 0 & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(s,t) \\ 0 & \delta(s,t) \end{pmatrix}. \quad (2.2.5)$$

From (2.2.5), we get the next system of functional equations:

$$\begin{cases} \beta(s,\tau)\delta(\tau,t) = \beta(s,t), \\ \delta(s,\tau)\delta(\tau,t) = \delta(s,t). \end{cases} \quad (2.2.6)$$

The second equation of the system (2.2.6) is known as Cantor's second equation, which has the next solutions:
(1) $\delta(s,t) \equiv 0$;
(2) $\delta(s,t) = \frac{\phi(t)}{\phi(s)}$, where ϕ is an arbitrary function with $\phi(s) \neq 0$;
(3) $\delta(s,t) = \begin{cases} 1, & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$

Substituting theses solutions in the first equation of (2.2.6), we find $\beta(s,t)$:
(1) $\beta(s,t) \equiv 0$;
(2) $\beta(s,t) = \rho(s)\phi(t)$, where ρ is an arbitrary function;
(3) $\beta(s,t) = \begin{cases} \sigma(s), & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a, \end{cases}$

where σ is an arbitrary function;

From these solutions we have the next matrices of structural constants of CEAs:

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$\mathcal{M}_{25}^{[s,t]} = \begin{pmatrix} 0 & \rho(s)\phi(t) \\ 0 & \frac{\phi(t)}{\phi(s)} \end{pmatrix},$$

where ρ, ϕ are arbitrary functions, with $\phi(s) \neq 0$;

$$\mathcal{M}_{26}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & \sigma(s) \\ 0 & 1 \end{pmatrix}, & \text{if } 0 < s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a, \end{cases}$$

where $a > 0$ and σ is an arbitrary function.

Case 1.2. Consider the case $\alpha(s,t) = \beta(s,t) \equiv 0$, $\gamma(s,t) \neq 0$, $\delta(s,t) \neq 0$. Then from (2.2.4) we have the following:

$$\begin{pmatrix} 0 & 0 \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma(s,t) & \delta(s,t) \end{pmatrix}.$$

From the last equality, we have following system of equations:

$$\begin{cases} \delta(s,\tau)\gamma(\tau,t) = \gamma(s,t), \\ \delta(s,\tau)\delta(\tau,t) = \delta(s,t). \end{cases} \tag{2.2.7}$$

The second equation of the system (2.2.7), known as Cantor's second equation, which has the next solutions:

(1) $\delta(s,t) \equiv 0$;

(2) $\delta(s,t) = \frac{\varphi(t)}{\varphi(s)}$, where φ is an arbitrary function with $\varphi(s) \neq 0$;

(3) $\delta(s,t) = \begin{cases} 1, & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$

Substituting these solutions in the first equation of (2.2.7), we find $b(s,t)$:

(1) $\gamma(s,t) \equiv 0$;

(2) $\gamma(s,t) = \frac{f(t)}{\varphi(s)}$, where f is an arbitrary function;

(3) $\gamma(s,t) = \begin{cases} g(t), & \text{if } 0 < s \leq t < a; \\ 0, & \text{if } t \geq a. \end{cases}$

where g is an arbitrary function.

From these solutions we have the next matrices of structural constants of CEAs:

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$
$$\mathcal{M}_{27}^{[s,t]} = \begin{pmatrix} 0 & 0 \\ \frac{f(t)}{\varphi(s)} & \frac{\varphi(t)}{\varphi(s)} \end{pmatrix},$$

where f, φ are arbitrary functions, $\varphi(s) \neq 0$;

$$\mathcal{M}_{28}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ g(t) & 1 \end{pmatrix}, & \text{if } 0 < s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a, \end{cases}$$

where $a > 0$ and g is an arbitrary function.

Case 1.3. Let us try to find the solution satisfying the following:

$$\begin{pmatrix} \alpha(s,\tau) & \beta(s,\tau) \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} \alpha(\tau,t) & \beta(\tau,t) \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(s,t) \\ 0 & 0 \end{pmatrix}. \tag{2.2.8}$$

From (2.2.8) we have the next system of functional equations:

$$\begin{cases} \alpha(s,\tau)\alpha(\tau,t) + \beta(s,\tau)\gamma(\tau,t) = 0, \\ \alpha(s,\tau)\beta(\tau,t) + \beta(s,\tau)\delta(\tau,t) = \beta(s,t), \\ \gamma(s,\tau)\alpha(\tau,t) + \delta(s,\tau)\gamma(\tau,t) = 0, \\ \gamma(s,\tau)\beta(\tau,t) + \delta(s,\tau)\delta(\tau,t) = 0. \end{cases} \tag{2.2.9}$$

Let $\alpha(s,t) = \gamma(s,t) = 0$. Then we get:

$$\begin{cases} \beta(s,\tau)\delta(\tau,t) = \beta(s,t), \\ \delta(s,\tau)\delta(\tau,t) = 0. \end{cases} \tag{2.2.10}$$

To find a non-zero solution of the system of equations (2.2.10) we should prove that the equation

$$\delta(s,\tau)\delta(\tau,t) = 0, \quad \text{for all } \ s<\tau<t, \tag{2.2.11}$$

has a non-zero solution. Indeed, take $C > 0$ and

$$\delta(s,t) = \begin{cases} 0, & \text{if } \ 0 < C \le s < t \ \text{ or } \ 0 < s < t \le C; \\ f(s,t), & \text{if } \ 0 < s < C < t, \end{cases} \tag{2.2.12}$$

where $f(s,t)$ is an arbitrary non-zero function.

Now, we show that independently on $f(s,t)$ the function (2.2.12) satisfies (2.2.11): take an arbitrary τ such that $s<\tau<t$, then for given $C>0$, we have only two possibilities:

Case 1.3.1. Let $\tau \le C$. By the defined function (2.2.12), we have that $\delta(s,\tau) = 0$ and for $\delta(\tau,t)$:

$$\delta(\tau,t) = \begin{cases} 0, & \text{if } t \le C; \\ f(\tau,t), & \text{if } \ t > C, \end{cases} \tag{2.2.13}$$

where $f(\tau,t)$ is the function fixed in (2.2.12).

Therefore, $\delta(s,\tau)\delta(\tau,t) = 0$.

Case 1.3.2. $\tau > C$. Also from (2.2.12), we have that $\delta(\tau,t) = 0$ and for $\delta(s,\tau)$:

$$\delta(s,\tau) = \begin{cases} f(s,\tau), & \text{if } \ s < C; \\ 0, & \text{if } s \ge C, \end{cases}$$

where $f(s,\tau)$ is the function fixed in (2.2.12).

Therefore, $\delta(s,\tau)\delta(\tau,t) = 0$.

Thus we have proved that the function (2.2.12) satisfies equation (2.2.11).

Now we should find solutions of the first equation of the system (2.2.10):

$$\beta(s,\tau)\delta(\tau,t) = \beta(s,t), \quad s<\tau<t, \tag{2.2.14}$$

where $\delta(\tau,t)$ given by (2.2.12).

To find solution we have the next possibilities:

Case 1.3.3. If $\tau \le C$, then by the defined function (2.2.12) we have that $\delta(s,\tau) = 0$ and from (2.2.13) in period of time $t \le C$, $\delta(\tau,t) = 0$, then from (2.2.14) we have that $\beta(s,t) = 0$. When $t > C$, $\delta(\tau,t) = f(\tau,t)$ and by (2.2.14) we have to solve the next equation:

$$\beta(s,\tau)f(\tau,t) = \beta(s,t), \quad s < \tau < t. \tag{2.2.15}$$

We solve (2.2.15) for some particular cases:

Case 1.3.3.1 Consider $\beta(s,t) = f(s,t)$. Then from (2.2.15), we have $f(s,\tau)f(\tau,t) = f(s,t)$, which is Cantor's second equation. As $f(s,t)$ is a non-zero function, then we have the next solution:

$$f(s,t) = \frac{\Phi(t)}{\Phi(s)},$$

where Φ is an arbitrary function, with $\Phi(s) \neq 0$.

Thus we have the following solution of the system (2.2.9):

$$\alpha(s,t) \equiv 0,$$

$$\beta(s,t) = \begin{cases} 0, & \text{if } \ s < t \le C \\ \frac{\Phi(t)}{\Phi(s)}, & \text{if } \ t > C, \end{cases}$$

$$\gamma(s,t) \equiv 0,$$

$$\delta(s,t) = \begin{cases} 0, & \text{if } \ 0 < C \le s < t \ \text{ or } \ 0 < s < t \le C; \\ \frac{\Phi(t)}{\Phi(s)}, & \text{if } \ s < C < t, \end{cases}$$

where $C > 0$ and Φ is an arbitrary function, with $\Phi(s) \neq 0$.

Then we have the next matrix of structural constants:

$$\mathcal{M}_{29}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \quad s < t \le C; \\ \begin{pmatrix} 0 & \frac{\Phi(t)}{\Phi(s)} \\ 0 & 0 \end{pmatrix}, & \text{if } \quad t > C, \end{cases}$$

where $C > 0$ and Φ is an arbitrary function, with $\Phi(t) \neq 0$.

Case 1.3.3.2. Let $\beta(s,t) \neq f(s,t)$. As $f(\tau,t)$ is an arbitrary non-zero function, consider $f(\tau,t) = \frac{\phi(\tau)}{\phi(t)}$, with $\phi(t) \neq 0$. Then from (2.2.15) we have the next:

$$\beta(s,\tau) \cdot \frac{\phi(\tau)}{\phi(t)} = \beta(s,t),$$

$$\beta(s,t)\phi(t) = \beta(s,\tau)\phi(\tau).$$

From the last equality, we can see that $\beta(s,t)\phi(t)$ does not depend on t, i.e. there exists a function $\rho(s)$ such that, $\beta(s,t)\phi(t) = \rho(s)$. Therefore, $\beta(s,t) = \frac{\rho(s)}{\phi(t)}$.

Then we get the following solution of the system (2.2.9):

$$\begin{aligned}
&\alpha(s,t) \equiv 0,\\
&\beta(s,t) = \begin{cases} 0, & \text{if } \ s < t \leq C;\\ \frac{\rho(s)}{\phi(t)}, & \text{if } \ t > C, \end{cases}\\
&\gamma(s,t) \equiv 0,\\
&\delta(s,t) = \begin{cases} 0, & \text{if } \ 0 < C \leq s < t \ \text{ or } \ 0 < s < t \leq C;\\ \frac{\phi(s)}{\phi(t)}, & \text{if } \ s < C < t, \end{cases}
\end{aligned}$$

where $C > 0$ and ϕ, ρ are arbitrary functions with $\phi(t) \neq 0$.

Then we have, respectively, the next matrix of structural constants to the solution:

$$\mathcal{M}_{30}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \quad s < t \leq C;\\ \begin{pmatrix} 0 & \frac{\rho(s)}{\phi(t)} \\ 0 & 0 \end{pmatrix}, & \text{if } \quad t > C, \end{cases}$$

where $C > 0$ and ϕ, ρ are arbitrary functions with $\phi(t) \neq 0$.

Case 1.3.4. When $\tau > C$, then by the defined function (2.2.12) we have that $\delta(\tau,t) = 0$. Then from (2.2.14) we have that $\beta(s,t) = 0$. Thus we get the trivial CEA.

Case 1.4. Let us try to find the solution satisfying:

$$\begin{pmatrix} \alpha(s,\tau) & \beta(s,\tau) \\ \gamma(s,\tau) & \delta(s,\tau) \end{pmatrix} \cdot \begin{pmatrix} \alpha(\tau,t) & \beta(\tau,t) \\ \gamma(\tau,t) & \delta(\tau,t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma(s,t) & 0 \end{pmatrix}. \tag{2.2.16}$$

From the equality (2.2.16) we have the next system of functional equations:

$$\begin{cases} \alpha(s,\tau)\alpha(\tau,t) + \beta(s,\tau)\gamma(\tau,t) = 0, \\ \alpha(s,\tau)\beta(\tau,t) + \beta(s,\tau)\delta(\tau,t) = 0, \\ \gamma(s,\tau)\alpha(\tau,t) + \delta(s,\tau)\gamma(\tau,t) = \gamma(s,t), \\ \gamma(s,\tau)\beta(\tau,t) + \delta(s,\tau)\delta(\tau,t) = 0. \end{cases}$$

Let $\alpha(s,t) = \beta(s,t) = 0$. Then we have the next system:

$$\begin{cases} \delta(s,\tau)\gamma(\tau,t) = \gamma(s,t), \\ \delta(s,\tau)\delta(\tau,t) = 0. \end{cases}$$

The analysis of this system is similar to the (2.2.10) and we get the following CEAs:

$$\mathcal{M}_{31}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ \frac{\Psi(t)}{\Psi(s)} & 0 \end{pmatrix}, & \text{if} \quad s < C; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if} \quad s \geq C, \end{cases}$$

where $C > 0$ and Ψ is an arbitrary function, with $\Psi(t) \neq 0$;

$$\mathcal{M}_{32}^{[s,t]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ \frac{\sigma(t)}{\varphi(s)} & 0 \end{pmatrix}, & \text{if} \quad s < C; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if} \quad s \geq C, \end{cases}$$

where $C > 0$ and φ, σ are arbitrary functions with $\varphi(s) \neq 0$.

Denote by $E_i^{[s,t]}$ the CEA with matrix $\mathcal{M}_i^{[s,t]}$.

Remark 2.2.5. We should note that, from the CEAs $E_i^{[s,t]}, i = 25, \dots, 32$, only $E_{27}^{[s,t]}$ coincides with the CEA $E_{16}^{[s,t]}$ constructed in Section 1.2 and it has the same dynamic. All other CEAs are different from CEAs constructed in Section 1.2 and have different dynamics.

The next theorem gives the time-depending dynamics of these CEAs:

Theorem 2.2.6. *For the next CEAs hold:*

$$E_{25}^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t, \quad \rho(s) = 0\}, \\ E_2 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t, \quad \rho(s) \neq 0\}. \end{cases}$$

$$E_{26}^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t < a, \quad \sigma(s) = 0\}, \\ E_2 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t < a, \quad \sigma(s) \neq 0\}, \\ E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq a\} \;. \end{cases}$$

$E_{27}^{[s,t]}$ *isomorphic to* E_1 *for any* $(s,t) \in \mathcal{T}$.

$$E_{28}^{[s,t]} \simeq \begin{cases} E_1 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t < a\}, \\ E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : t \geq a\}. \end{cases}$$

$$E_{29}^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t \leq C\}, \\ E_4 & \textit{for all} \;\; (s,t) \in \{(s,t) : t > C\}. \end{cases}$$

$$E_{30}^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < t \leq C\}, \\ E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : t > C, \quad \rho(s) = 0\}, \\ E_4 & \textit{for all} \;\; (s,t) \in \{(s,t) : t > C, \quad \rho(s) \neq 0\} \;. \end{cases}$$

$$E_{31}^{[s,t]} \simeq \begin{cases} E_4 & \textit{for all} \;\; (s,t) \in \{(s,t) : s < C\}, \\ E_0 & \textit{for all} \;\; (s,t) \in \{(s,t) : s \geq C\}. \end{cases}$$

$$E_{32}^{[s,t]} \simeq \begin{cases} E_0 & \textit{for all} \quad (s,t) \in \{(s,t) : s < C, \quad \sigma(t) = 0\}, \\ E_4 & \textit{for all} \quad (s,t) \in \{(s,t) : s < C, \quad \sigma(t) \neq 0\}, \\ E_0 & \textit{for all} \quad (s,t) \in \{(s,t) : s \geq C\} \ . \end{cases}$$

Proof. When $\rho(s) = 0$, then $E_{25}^{[s,t]}$ is isomorphic to E_1, $\forall s, t \in \mathcal{T}$, by the change of basis $e_1' = e_1$, $e_2' = \frac{\phi(s)}{\phi(t)} e_2$, and when $\rho(s) \neq 0$, it is isomorphic to E_2, $\forall s, t \in \mathcal{T}$, by the change of basis $e_1' = \frac{1}{\rho(s)\phi(t)} e_1$, $e_2' = \frac{\phi(s)}{\phi(t)} e_2$.

When $\sigma(s) = 0$, then $E_{26}^{[s,t]}$ is isomorphic to E_1, $\forall s, t \in \mathcal{T}$, $s < t < a$, by the change of basis $e_1' = e_1$, $e_2' = e_2$, and when $\sigma(s) \neq 0$, it is isomorphic to E_2, $\forall s, t \in \mathcal{T}$, $s < t < a$, by the change of basis $e_1' = \frac{1}{\sigma(s)} e_1$, $e_2' = e_2$. In period of time $t \geq a$, it will be isomorphic to the trivial evolution algebra E_0.

$E_{27}^{[s,t]}$ is isomorphic to E_1, $\forall s, t \in \mathcal{T}$, by the change of basis $e_2' = \frac{f(t)\varphi(s)}{\varphi^2(t)} e_1 + \frac{\varphi(s)}{\varphi(t)} e_2$, $e_2' = e_1$.

$E_{28}^{[s,t]}$ is isomorphic to E_1, $\forall s, t \in \mathcal{T}$, $s < t < a$, by the change of basis $e_1' = \sigma(t) e_1 + e_2$, $e_2' = e_1$, in period of time $t \geq a$, it will be isomorphic to the trivial evolution algebra E_0.

$E_{29}^{[s,t]}$ is isomorphic to E_4, $\forall s, t \in \mathcal{T}$, $t > C$, by the change of basis $e_1' = \frac{\Phi(s)}{\Phi(t)} e_1$, $e_2' = e_2$, in period of time $s < t \leq C$, it will be isomorphic to the trivial evolution algebra E_0.

When $\rho(s) \neq 0$, then $E_{30}^{[s,t]}$ is isomorphic to E_4, $\forall s, t \in \mathcal{T}$, $t > C$, by the change of basis $e_1' = \frac{\phi(t)}{\rho(s)} e_1$, $e_2' = e_2$, in period of time $s < t \leq C$ and when $\rho(s) = 0$, then it will be isomorphic to the trivial evolution algebra E_0.

$E_{31}^{[s,t]}$ isomorphic to E_4, $\forall s, t \in \mathcal{T}$, $s < C$, by the change of basis $e_1' = \frac{\Psi(s)}{\Psi(t)} e_1$, $e_2' = e_2$, in period of time $s \geq C$, it will be isomorphic to the trivial evolution algebra E_0.

When $\sigma(t) \neq 0$, then $E_{32}^{[s,t]}$ is isomorphic to E_4, $\forall s, t \in \mathcal{T}$, $s < C$, by the change of basis $e_1' = \frac{\varphi(s)}{\sigma(t)} e_1$, $e_2' = e_2$, in period of time $s \geq C$, and when $\sigma(t) = 0$, then it will be isomorphic to the trivial evolution algebra E_0. $\square$

Thus we proved that there exists CEAs which for some values of time will be isomorphic to E_4.

2.3 Rota-Baxter operators on Evolution Algebras

In this section we will study Rota-Baxter operators on evolution algebras.

Originally the Rota-Baxter operators were defined on associative algebras by G. Baxter to solve a problem in probability [3] and then developed by the Rota school [37]. These operators have connections with many areas of mathematics and mathematical physics, such as number theory, combinatorics, quantum field theory.

Definition 2.3.1. Let $\mathcal{F}$ be a field. A *Rota-Baxter operator* of weight $\lambda \in \mathcal{F}$ on an evolution algebra $(E, \cdot)$ over $\mathcal{F}$ is a linear map $P : E \longrightarrow E$ satisfying

$$P(x) \cdot P(y) = P(x \cdot P(y) + P(x) \cdot y + \lambda \cdot x \cdot y) \ \text{ for all } x, y \in E.$$

Note that, if P is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}P$ is a Rota-Baxter operator P of weight 1. Therefore, one only needs to consider Rota-Baxter operators of weight 0 and 1. We also assume that $\mathcal{F} = \mathbb{C}$.

To study Rota-Baxter operators on an evolution algebra $(E, \cdot)$ over the field $\mathbb{C}$, we need the next theorem, which gives the classification of two-dimensional complex evolution algebras.

For an n-dimensional evolution algebra we have that $e_ie_i = \sum_{j=1}^{n} a_{ij}e_j$, for all i, $e_ie_j = 0, i \neq j$, and for the Rota-Baxter operator $P(e_i) = \sum_{j=1}^{n} r_{ij}e_j$.

To find a Rota-Baxter operator of weight λ on evolution algebra we should solve the system of equations obtained from the next equality:

$$P(e_i)P(e_j) = P(e_iP(e_j) + P(e_i)e_j + \lambda e_ie_j), \quad i, j = 1, \dots, n. \tag{2.3.1}$$

For $i = j$, the LHS of (2.3.1) equals to:

$$P(e_i)P(e_i) = \sum_{j=1}^{n}\sum_{k=1}^{n} r_{ij}^2 a_{jk} e_k \tag{2.3.2}$$

And the RHS (2.3.1), for $i = j$, is equal to:

$$P(e_i P(e_i) + P(e_i)e_i + \lambda e_i e_i) = (2r_{ii} + \lambda)\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij} r_{jk} e_k$$

For $i \neq j$, we have:

$$P(e_i)P(e_j) = \sum_{k=1}^{n}\sum_{s=1}^{n} r_{ik} r_{jk} a_{ks} e_s$$

$$P(e_i P(e_j) + P(e_i)e_j + \lambda e_i e_j) = r_{ji}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} r_{kl} e_l + r_{ij}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{js} r_{sl} e_l \tag{2.3.3}$$

Thus, to find an operator of Rota-Baxter on an evolution algebra we have to solve system of equations (2.3.1) with equalities (2.3.2)–(2.3.3).

Without loss of generality for the Rota-Baxter operator with $\lambda \neq 0$ we can assume $\lambda = 1$, therefore we will find the Rota-Baxter operators corresponding to a given evolution algebra with weight $\lambda = 0$ and $\lambda = 1$.

Now we will find the Rota-Baxter operators corresponding to the two-dimensional complex evolution algebras E_i, $i = 1, \ldots, 6$, with weight $\lambda = 0$ and $\lambda = 1$.

Consider the Rota-Baxter operator

$$\begin{pmatrix} P(e_1) \\ P(e_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{C}$.

- For the algebra $E_1 : e_1 e_1 = e_1$ to find the matrix form of the Rota-Baxter operator of weight 0 we should solve the next system of equations:

$$\begin{cases} a^2 = 0, \\ 2ab = 0, \\ c^2 = 0, \\ ac = ac, \\ bc = 0. \end{cases}$$

This system of equations has the next solution:

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}.$$

For the algebra $E_1 : e_1e_1 = e_1$ to find the matrix form of the Rota-Baxter operator of weight $\lambda = 1$ we should solve the next system of equations:

$$\begin{cases} a^2 + a = 0, \\ (2a+1)b = 0, \\ c^2 = 0, \\ ac = ac, \\ bc = 0. \end{cases}$$

This system of equations has the next solutions:

$$\begin{pmatrix} -1 & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

- For the algebra $E_2 : e_1e_1 = e_1, \quad e_2e_2 = e_1$ to find the matrix form of the Rota-Baxter operator of weight 0 we should solve next system of equations:

$$\begin{cases} b^2 = a^2, \\ 2ab = 0, \\ c^2 + d^2 = 2ad, \\ 2bd = 0, \\ bd = ab, \\ (b+c)b = 0. \end{cases}$$

This system of equations has the next solution:

$$\begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}.$$

For the algebra $E_2 : e_1e_1 = e_1, \quad e_2e_2 = e_1$ to find the matrix form of the Rota-Baxter operator of weight 1 we should solve the next system of equations:

$$\begin{cases} b^2 = a^2 + a, \\ (2a+1)b = 0, \\ c^2 + d^2 = (2d+1)a, \\ (2d+1)b = 0, \\ bd = ab, \\ (b+c)b = 0. \end{cases}$$

This system of equations has the next solutions:

$$\begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ c & -1+ic \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ c & -1-ic \end{pmatrix}.$$

- For the algebra $E_3 : e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2$ to find the matrix form of the Rota-Baxter operator of weight 0 we should solve the next system of equations:

$$\begin{cases} a^2 - b^2 = 2a(a+c), \\ a^2 - b^2 = 2a(b+d), \\ c^2 - d^2 = -2d(a+c), \\ c^2 - d^2 = -2d(b+d), \\ ac - bd = (c-b)(a+c), \\ ac - bd = (c-b)(b+d). \end{cases}$$

This system of equations has the next solutions:

$$\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}, \quad \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}.$$

For the algebra $E_3 : e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2$, to find the matrix form of the Rota-Baxter operator of weight 1 we should solve the next system of equations:

$$\begin{cases} a^2 - b^2 = (2a+1)(a+c), \\ a^2 - b^2 = (2a+1)(b+d), \\ c^2 - d^2 = -(2d+1)(a+c), \\ c^2 - d^2 = -(2d+1)(b+d), \\ ac - bd = (c-b)(a+c), \\ ac - bd = (c-b)(b+d). \end{cases}$$

This system of equations has the following solutions:

$$\begin{pmatrix} -1+b & b \\ -b & -1-b \end{pmatrix}, \begin{pmatrix} -1-b & b \\ b & -1-b \end{pmatrix},$$

$$\begin{pmatrix} b & b \\ -b & -b \end{pmatrix}, \quad \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}.$$

- For the algebra $E_4 : e_1e_1 = e_2$ to find the matrix form of the Rota-Baxter operator of weight 0 we should solve the next system of equations:

$$\begin{cases} a^2 = 2ad, \\ 2ac = 0, \\ ac = cd, \\ c^2 = 0. \end{cases}$$

This system of equations has the next solutions:

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & \frac{a}{2} \end{pmatrix}.$$

For the algebra $E_4 : e_1e_1 = e_2$ to find the matrix form of the Rota-Baxter operator of weight 1 we should solve the next system of equations:

$$\begin{cases} a^2 = (2a+1)d, \\ (2a+1)c = 0, \\ ac = cd, \\ c^2 = 0. \end{cases}$$

This system of equations has the next solution:

$$\begin{pmatrix} a & b \\ 0 & \frac{a^2}{1+2a} \end{pmatrix},$$

where $a \neq -\frac{1}{2}$. The system has not solution when $a = -\frac{1}{2}$.

- For the algebra $E_5(x, y) : e_1e_1 = e_1 + xe_2, \quad e_2e_2 = ye_1 + e_2, \quad 1 - xy \neq 0, x, y \in \mathbb{C}$, to find the matrix form of the Rota-Baxter operator of weight 1 we should solve the next system of equations:

$$\begin{cases} a^2 + b^2y = (2a+1)(a+xc), \\ a^2x + b^2 = (2a+1)(b+xd), \\ c^2 + d^2y = (2d+1)(ay+c), \\ c^2x + d^2 = (2d+1)(by+d), \\ ac + bdy = c(a+cx) + b(ay+c), \\ acx + bd = c(b+dx) + b(by+d). \end{cases} \tag{2.3.4}$$

To find solution of the system (2.3.4) consider the following:

Case 1. Let $a = 0$. Then from (2.3.4)

$$\begin{cases} b^2y = xc, \\ b^2 = b + xd, \\ c^2 + d^2y = (2d+1)c, \\ c^2x + d^2 = (2d+1)(by+d), \\ bdy = c^2x + bc, \\ bd = c(b+dx) + b(by+d). \end{cases} \tag{2.3.5}$$

From this system consider:

Case 1.1. Let $b = 0$. Then

$$\begin{cases} 0 = xc, \\ 0 = xd, \\ c^2 + d^2y = (2d+1)c, \\ c^2x + d^2 = (2d+1)d, \\ 0 = c^2x, \\ 0 = cdx. \end{cases}$$

In case when $x = 0$ we have:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_2 & -1 \end{pmatrix},$$

where $c_{1,2} = \frac{-1\pm\sqrt{1-4y}}{2}$.
When $x \neq 0$ we have the trivial solution of the system $a = b = c = d = 0$.

Case 1.2. $b \neq 0$. Then in (2.3.5) consider:

Case 1.2.1. $c = 0$. Then we get:

$$\begin{cases} b^2 y = 0, \\ b^2 = b + xd, \\ d^2 y = 0, \\ d^2 = (2d+1)(by+d), \\ bdy = 0, \\ b^2 y = 0. \end{cases}$$

Since $b \neq 0$, we have $y = 0$. Then we have:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b_1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & b_2 \\ 0 & -1 \end{pmatrix},$$

where $b_{1,2} = \frac{1 \pm \sqrt{1-4x}}{2}$.

Case 1.2.2. Let $c \neq 0$. Then from (2.3.5) consider:

Case 1.2.2.1. When $x = 0$, then $y = 0$. And we get the system

$$\begin{cases} b^2 = b, \\ c^2 = (2d+1)c, \\ d^2 + d = 0, \\ bc = 0, \end{cases}$$

which has not solution.

Case 1.2.2.2. Let $x \neq 0$. Since $b \neq 0,\ c \neq 0$, then $y \neq 0$. Then from (2.3.5) consider:

Case 1.2.2.2.1. $d = 0$. And we have the system

$$\begin{cases} b^2y = cx, \\ b^2 = b, \\ c^2 = c, \\ c^2x = by, \\ c^2x + bc = 0, \\ b^2y + bc = 0, \end{cases}$$

which has solution $b = c = 1, x = y = -1$. But this solution does not satisfy the condition $xy \neq 1$.

Case 1.2.2.2.2. $d \neq 0$. Then from (2.3.5) consider:

Case 1.2.2.2.2.1. When $d = -\frac{1}{2}$ we get the solution, which also contradicts the condition $xy \neq 1$.

Case 1.2.2.2.2.2. When $d \neq -\frac{1}{2}$. One have to solve the system:

$$\begin{cases} b^2y = xc, \\ b^2 = b + xd, \\ c^2 + d^2y = 2cd + c, \\ c^2x = d^2 + 2bdy + by + d, \\ bdy = c^2x + bc, \\ bc + cdx + b^2y = 0. \end{cases}$$

The system has the next solution: $b = -1,\ c = 1,\ d = -2,\ x = -1,\ y = -1$. But this solution also contradicts the condition $xy \neq 1$.

Case 2. Let $a \neq 0$. Then from (2.3.4):

Case 2.1. For $a = -\frac{1}{2}$ we get:

$$\begin{cases} \frac{1}{4} + b^2 y = 0, \\ \frac{x}{4} + b^2 = 0, \\ c^2 + d^2 y = (2d+1)(-\frac{y}{2} + c), \\ c^x + d^2 = (2d+1)(by+d), \\ -\frac{c}{2} + bdy = c(-\frac{1}{2} + cx) + b(-\frac{y}{2} + c), \\ -\frac{cx}{2} + bd = c(b + xd) + b(by + d). \end{cases}$$

From the first two equations of the system, we can see that $xy = 1$, which contradicts the condition $xy \neq 1$.

Case 2.2. Let $a \neq -\frac{1}{2}$. Then from (2.3.4):

Case 2.2.1. When $b = 0$, we get

$$\begin{cases} a^2 + 2acx + a + cx = 0, \\ a^2 x - 2axd - xd = 0, \\ c^2 + d^2 y = 2ady + 2cd + ay + c, \\ c^2 x = d^2 + d, \\ c^2 x = 0, \\ acx - cdx = 0. \end{cases}$$

For this system, when $c = 0$, we have the next solutions:

$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, which corresponds to the evolution algebra $E(0,0)$.

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which corresponds to the evolution algebra $E(x,y)$.

When $c \neq 0$, it implies that $x = 0$. Thus we have the next solutions which correspond to $E(0,y)$:

$$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ c_2 & 0 \end{pmatrix},$$

where $c_{1,2} = \frac{1 \pm \sqrt{1-4y}}{2}$ and $y \neq 0$.

Case 2.2.2. Let $b \neq 0$. Then from (2.3.4) consider:

Case 2.2.2.1. $c = 0$. Then we have

$$\begin{cases} b^2y = a^2 + a, \\ a^2x + b^2 = 2ab + 2adx + b + xd, \\ d^2y = 2ady + ay, \\ d^2 + 2bdy + by + d = 0, \\ bdy = aby, \\ b^2y = 0. \end{cases}$$

From the last equation of the system, it implies that $y = 0$. Thus we have the next solutions corresponding to the evolution algebra $E(x, 0)$:

$$\begin{pmatrix} -1 & b_{1,2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$

where $b_{1,2} = \frac{-1\pm\sqrt{1-4x}}{2}$.

Case 2.2.2.2. Let $c \neq 0$. Then from (2.3.4) consider:

Case 2.2.2.2.1. $d = 0$. Then we have

$$\begin{cases} b^2y = a^2 + 2acx + a + cx, \\ a^2x + b^2 = 2ab + b, \\ c^2 = ay + c, \\ c^2x = by, \\ c^2x + aby + bc = 0, \\ acx = bc + b^2y. \end{cases}$$

When $x = 0$, it is easy to see that the system of equations has not solution.

From $x \neq 0$ implies that $y \neq 0$. Then

Case 2.2.2.2.2. Let $d \neq 0$. When $d = -\frac{1}{2}$ we get the solution, which also contradicts the condition $xy \neq 1$. Then from (2.3.4) consider:

Case 2.2.2.2.2.1. $x = 0$. Then we have

$$\begin{cases} b^2y = a^2 + a, \\ b^2 = 2ab + b, \\ c^2 + d^2y = 2ady + 2cd + ay + c, \\ d^2 + 2bdy + by + d = 0, \\ bdy = aby + bc, \\ bc + b^2y = 0. \end{cases}$$

When $y = 0$ and $y = -\frac{1}{4}$ one can check that system has not solutions. In the case, when $y \neq 0$ and $y \neq \frac{1}{4}$, we have the next solutions of the system:

$$\begin{pmatrix} \frac{1-4y+\sqrt{1-4y}}{8y-2} & -\frac{1}{\sqrt{1-4y}} \\ \frac{y}{\sqrt{1-4y}} & \frac{1-4y-\sqrt{1-4y}}{8y-2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1-4y-\sqrt{1-4y}}{8y-2} & \frac{1}{\sqrt{1-4y}} \\ -\frac{y}{\sqrt{1-4y}} & \frac{1-4y+\sqrt{1-4y}}{8y-2} \end{pmatrix}.$$

Case 2.2.2.2.2.2. When $x \neq 0$. Then from (2.3.4) consider:

Case 2.2.2.2.2.2.1. $y = 0$, then we get

$$\begin{cases} a^2 + 2acx + a + cx = 0, \\ a^2x + b^2 = 2ab + 2adx + b + dx, \\ c^2 = 2cd + c, \\ c^2x = d^2 + d, \\ c^2x + bc = 0, \\ acx = bc + cdx. \end{cases}$$

The system of equations has the following solutions:

$$\begin{pmatrix} \frac{1-4x+\sqrt{1-4x}}{8x-2} & -\frac{x}{\sqrt{1-4x}} \\ \frac{1}{\sqrt{1-4x}} & \frac{1-4x-\sqrt{1-4x}}{8x-2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1-4x-\sqrt{1-4x}}{8x-2} & \frac{x}{\sqrt{1-4x}} \\ -\frac{1}{\sqrt{1-4x}} & \frac{1-4x+\sqrt{1-4x}}{8x-2} \end{pmatrix},$$

where $x \neq \frac{1}{4}$.

Case 2.2.2.2.2.2.2. When $y \neq 0$, then we should solve (2.3.4) for all non-zero unknowns.

$$\begin{cases} a^2 + b^2 y = (2a+1)(a+xc), \\ a^2 x + b^2 = (2a+1)(b+xd), \\ c^2 + d^2 y = (2d+1)(ay+c), \\ c^2 x + d^2 = (2d+1)(by+d), \\ ac + bdy = c(a+cx) + b(ay+c), \\ acx + bd = c(b+dx) + b(by+d). \end{cases} \tag{2.3.6}$$

From the second and third equations of the system $(a^2 - d - 2ad)x = b + 2ab - b^2, (d^2 - a - 2ad)y = c - c^2 + 2cd$, consider the next cases:

Case A. Let $a^2 - d - 2ad = 0, b + 2ab - b^2 = 0, d^2 - a - 2ad = 0, c - c^2 + 2cd = 0$. From which we will get the next solutions:

a) $a = b = c = d = -1$;

b) $a = \frac{-3+i\sqrt{3}}{6},\ b = \frac{i}{\sqrt{3}},\ c = -\frac{i}{\sqrt{3}},\ d = \frac{-3-i\sqrt{3}}{6}$;

c) $a = \frac{-3-i\sqrt{3}}{6},\ b = -\frac{i}{\sqrt{3}},\ c = \frac{i}{\sqrt{3}}, d = \frac{-3+i\sqrt{3}}{6}$.

$a = b = c = d = -1$ will be solution of the system of equations (2.3.6) iff, when $x = y = 1$, which contradicts the condition $xy \neq 1$.

$a = \frac{-3+i\sqrt{3}}{6},\ b = \frac{i}{\sqrt{3}},\ c = -\frac{i}{\sqrt{3}}, d = \frac{-3-i\sqrt{3}}{6}$ and $a = \frac{-3-i\sqrt{3}}{6},\ b = -\frac{i}{\sqrt{3}},\ c = \frac{i}{\sqrt{3}},\ d = \frac{-3+i\sqrt{3}}{6}$, will be solutions of the system of equations (2.3.6) iff, when $x = 1 - y$. From the condition $xy \neq 1$ we get $x \neq \frac{1 \pm i\sqrt{3}}{2}$.

Case B. Let $a^2 - d - 2ad = 0,\ b + 2ab - b^2 = 0,\ y = \frac{c - c^2 + 2cd}{d^2 - a - 2ad}$. Then substituting $d = \frac{a^2}{2a+1},\ b = 2a+1,\ y = \frac{c-c^2+2cd}{d^2-a-2ad}$, we get the next:

$$\begin{cases} a+a^2+\frac{(1+2a)^3(1+2a^2-2a(-1+c)-c)c}{a(1+4a+6a^2+3a^3)}+cx+2acx=0 \ ; \\ \\ \frac{(1+2a)^2\left((1+2a)(1+\frac{2a^2}{1+2a})c(-1-\frac{2a^2}{1+2a}+c)+\frac{a(1+4a+6a^2+3a^3)(\frac{a^4}{(1+2a)^2})+\frac{a^2}{1+2a}-c^2x}{(1+2a)^2}\right)}{a(1+4a+6a^2+3a^3)}=0; \\ \frac{c(2a^3+c(1+x)+a(1+c(4+3x))+a^2(3+c(4+3x)))}{1+3a+3a^2}=0 \ ; \\ \\ c(-1-2a+\frac{(1+2a)^3(1+2a^2-2a(-1+c)-c)}{a(1+4a+6a^2+3a^3)}+\frac{a(1+a)x}{1+2a})=0 \ , \end{cases}$$

where $a\neq -\frac{1}{2},\ a\neq -1,\ a\neq \frac{-3+i\sqrt{3}}{6}, a\neq \frac{-3-i\sqrt{3}}{6}$.

From which we will get the next solution:

$x_1=-\frac{(1+2a)^2}{a^2},\ c_1=\frac{a^3}{(1+2a)^2},$

$x_2=-\frac{(1+2a)^2}{(1+a)^2},\ c_2=\frac{1+3a+3a^2+a^3}{(1+2a)^2}.$

Thus we will get the solution of the system in the next form:

$b=1+2a,\ c_1=\frac{a^3}{(1+2a)^2},\ d=\frac{a^2}{1+2a},\ x_1=-\frac{(1+2a)^2}{a^2},\ y_1=-\frac{a^2}{(1+2a)^2},$

$b=1+2a,\ c_2=\frac{1+3a+3a^2+a^3}{(1+2a)^2},\ d=\frac{a^2}{1+2a},\ x_2=-\frac{(1+2a)^2}{(1+a)^2},\ y_2=-\frac{(1+a)^2}{(1+2a)^2}.$

But these solutions contradict the condition $xy\neq 1$.

Case C. Let $x=\frac{b+2ab-b^2}{a^2-d-2ad},\ c-c^2+2cd=0,\ d^2-a-2ad=0$. Similar to the Case B we get the next solutions of the system:

$a=\frac{d^2}{1+2d},\ b_1=\frac{d^3}{(1+2d)^2},\ c=1+2d,\ x_1=-\frac{d^2}{(1+2d)^2},\ y_1=-\frac{(1+2d)^2}{d^2},$

$a=\frac{d^2}{1+2d},\ b_2=\frac{1+3d+3d^2+d^3}{(1+2d)^2},\ c=1+2d,\ x_2=-\frac{(1+d)^2}{(1+2d)^2},\ y_2=-\frac{(1+2d)^2}{(1+d)^2}.$

Also these solutions contradict the condition $xy\neq 1$.

Thus we can not take these solutions as a solution of the system (2.3.6).

Case D. Let $x=\frac{b+2ab-b^2}{a^2-d-2ad},\ y=\frac{c-c^2+2cd}{d^2-a-2ad}$.

In this case we have the next solution of the system (2.3.6):

$a=-1-d,$

$b=-\frac{d(1+d)}{c},$

$x = \frac{d(1+d)(c+2cd-d(1+d))}{c^2(1+3d+3d^2)}$,

$y = \frac{c(1-c+2d)}{c^2(1+3d+3d^2)}$,

where $d \neq -1,\ d \neq -\frac{3\pm i\sqrt{3}}{6},\ c \neq \frac{d(1+d)}{1+2d},\ c \neq 1+2d$.

For the algebra $E_5(x,y): e_1e_1 = e_1 + xe_2, \quad e_2e_2 = ye_1 + e_2, \quad 1 - xy \neq 0, x, y \in \mathbb{C}$, to find the matrix form of the Rota-Baxter operator of weight 0 we should solve the next system of equations:

$$\begin{cases} b^2y = a^2 + 2acx, \\ a^2x + b^2 = 2ab + 2adx, \\ c^2 + d^2y = 2ady + 2cd, \\ c^2x = 2bdy + d^2, \\ bdy = c^2x + aby + bc, \\ acx = bc + cdx + b^2y. \end{cases} \tag{2.3.7}$$

One can check that this system of equations has only the trivial solution, in the case when one of unknowns of a, b, c, d will equal to 0.

Also, when $x = y = 0$ we have only the trivial solution of the system.

For the algebra $E_5(0, \frac{1}{4})$, system (2.3.7) has the next solution:

$$\begin{pmatrix} a & 2a \\ -\frac{a}{2} & -a \end{pmatrix}.$$

And for the algebra $E_5(\frac{1}{4}, 0)$, system (2.3.7) has the next solution:

$$\begin{pmatrix} a & \frac{a}{2} \\ -2a & -a \end{pmatrix}.$$

Consider for the system (2.3.7), that $a \neq 0, b \neq 0, c \neq 0, d \neq 0$, and for the algebra $E(x,y)$, let $x \neq 0, y \neq 0$.

From the first and the fifth equations of the system (2.3.7), let $b^2y - a^2 \neq 0, \quad c^2x + aby + bc \neq 0$ (otherwise we get contradiction). Then,

$$c = \frac{b^2y - a^2}{2ax}, \; d = \frac{c^2x + aby + bc}{by}.$$

After replacing these values to the second equation of the system (2.3.7), we have

$$x = \frac{-a^4 + 2a^3b - 2a^2b^2y - b^4y^2}{2a^3by}.$$

Then substituting this values to the $3, 4, 6$-th equations of the system (2.3.7) we get

$$y = \frac{-a^2 + 2ab}{3b^2}.$$

Then substituting the value of y to the value of x, we get

$$x = \frac{(2a - b)b}{3a^2}.$$

For the parameters x, y we should check that $xy \neq 1$. Substituting the values of x and y, we have

$$xy = -\frac{(a - 2b)(2a - b)}{9ab} \neq 1.$$

Then, we get $a \neq -b$.

Substituting the values of x, y to values of c, d, we get $d = -a, \; c = -\frac{a^2}{b}$.

Thus, we have the next solution of the system (2.3.7):

$$\begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix},$$

where $a \neq -b$.

- For the algebra $E_6(x) : e_1e_1 = e_2, \quad e_2e_2 = e_1 + xe_2, \ x \in \mathbb{C}$, to find the matrix form of the Rota-Baxter operator of weight 1 we should solve the next system of equations:

$$\begin{cases} b^2 = (2a+1)c, \\ a^2 + b^2x = (2a+1)d, \\ d^2 = (2d+1)(a+cx), \\ c^2 + d^2x = (2d+1)(b+dx), \\ bd = ab + c^2 + bcx, \\ ac = cd + b^2. \end{cases} \tag{2.3.8}$$

Case 1. Let $a = 0$. Then

$$\begin{cases} b^2 = c, \\ b^2x = d, \\ d^2 = 2cdx + cx, \\ c^2 + d^2x = 2bd + 2d^2x + b + dx, \\ bd = c^2 + bcx, \\ cd + b^2 = 0. \end{cases}$$

Case 1.1. Consider $b = 0$. Then it is easy to see that the system has only the trivial solution.

Case 1.2. In the case when $b \neq 0$, one can check that the system has not solutions.

Case 2. Let $a \neq 0$ and $a \neq -\frac{1}{2}$ (in the case when $a = -\frac{1}{2}$, it is easy to see that (2.3.8) has not solution).

Case 2.1. Let $b = 0$. Then from (2.3.8),

$$\begin{cases} (2a+1)c = 0, \\ a^2 = (2a+1)d, \\ d^2 = (2d+1)(a+cx), \\ c^2 + d^2x = 2d^2x + dx, \\ c^2 = 0, \\ ac = cd. \end{cases}$$

And this system has the next solutions:

For the algebra $E_6(x)$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and for the algebra $E_6(0)$

$$\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \quad \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}.$$

Case 2.2. Let $b \neq 0$.

Case 2.2.1. Let $c = 0$. Then we get $b = 0$, which contradicts the **Case 2.2.**. Thus we do not have solution.

Case 2.2.2. Let $c \neq 0$. Then for the case, when $d = 0$ the system has not solution. Consider $d \neq 0$.

Case 2.2.2.1. Consider $x = 0$, then from (2.3.8) we get the following system of equations:

$$\begin{cases} b^2 = (2a+1)c, \\ a^2 = (2a+1)d, \\ d^2 = 2ad + a, \\ c^2 = 2bd + b, \\ bd = ab + c^2, \\ ac = cd + b^2. \end{cases} \tag{2.3.9}$$

This system has the following solutions:

$$\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \quad \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}, \quad \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{\sqrt[6]{-1}}{\sqrt{3}} \\ \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix},$$

$$\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{\sqrt[6]{-1}}{\sqrt{3}} \\ -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \quad \begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \quad \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}.$$

Case 2.2.2.2. Consider $x \neq 0$, then we get the following solution of the system (2.3.9) for the algebra $E_6(\frac{-b^3-c^3}{bc^2})$:

$$\begin{pmatrix} \frac{b^2-c}{2c} & b \\ c & \frac{-b^2-c}{2c} \end{pmatrix},$$

where the parameters b, c are solutions of the next system of equations

$$\begin{cases} \frac{b^6+5b^3c^3+4c^6}{c} & = \frac{c(b^3+c^3)}{b}, \\ \frac{b^4}{c} + 4bc^2 & = c. \end{cases}$$

For the algebra $E_6(x) : e_1e_1 = e_2, \ \ e_2e_2 = e_1 + xe_2, \ x \in \mathbb{C}$, to find the matrix form of the Rota-Baxter operator of weight 0 we should solve the following system of equations:

$$\begin{cases} b^2 = 2ac, \\ a^2 + b^2x = 2ad, \\ d^2 = 2ad + 2cdx, \\ c^2 = 2bd + d^2x, \\ bd = ab + c^2 + bcx, \\ ac = cd + b^2. \end{cases}$$

Which will have the next solution for the algebra $E_6(-\frac{3b^2}{4c^2})$:

$$\begin{pmatrix} \frac{b^2}{2c} & b \\ c & \frac{-b^2}{2c} \end{pmatrix},$$

where the parameters b, c are solutions of the next system of equations

$$\begin{cases} \frac{3b^6}{c} + 16b^3c^2 + 16c^5 & = 0, \\ \frac{b^3}{c} + 4c^2 & = 0. \end{cases}$$

Thus we have proved the next theorem, which gives all matrices form of the Rota-Baxter operators on 2-dimensional complex evolution algebras.

Theorem 2.3.2. *The matrices of the Rota-Baxter operators on the two-dimensional complex evolution algebras are given in the next table, with parameters* $a, b, c, d, x, y \in \mathbb{C}$.

Evolution Algebra	*Matrices of RBOs of weight 0 on the evolution algebra*
E_1	$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$
E_2	$\begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}$
E_3	$\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}, \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$
E_4	$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & \frac{a}{2} \end{pmatrix}$
$E_5(\frac{1}{4}, 0)$	$\begin{pmatrix} a & \frac{a}{2} \\ -2a & -a \end{pmatrix}$
$E_5(0, \frac{1}{4})$	$\begin{pmatrix} a & 2a \\ -\frac{a}{2} & -a \end{pmatrix}$
$E_5\left(\frac{(2a-b)b}{3a^2}, \frac{-a^2+2ab}{3b^2}\right)$ $a \neq 2b, b \neq 2a,\ a \neq -b$ $a \neq 0, b \neq 0$	$\begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}$
$E_6(-\frac{3b^2}{4c^2})$ $b \neq 0,\ c \neq 0$	$\begin{pmatrix} \frac{b^2}{2c} & b \\ c & \frac{-b^2}{2c} \end{pmatrix}$ *where the parameters* b, c *are solutions of the following system of equations* $\begin{cases} \frac{3b^6}{c} + 16b^3c^2 + 16c^5 &= 0, \\ \frac{b^3}{c} + 4c^2 &= 0. \end{cases}$

Evolution Algebra	*Matrices of RBOs of weight 1 on the evolution algebra*
E_1	$\begin{pmatrix} -1 & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$
E_2	$\begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix},$ $\begin{pmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix},$ $\begin{pmatrix} -1 & 0 \\ c & -1+ic \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ c & -1-ic \end{pmatrix}$
E_3	$\begin{pmatrix} -1+b & b \\ -b & -1-b \end{pmatrix}, \begin{pmatrix} -1-b & b \\ b & -1-b \end{pmatrix},$ $\begin{pmatrix} b & b \\ -b & -b \end{pmatrix}, \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}$
E_4	$\begin{pmatrix} a & b \\ 0 & \frac{a^2}{1+2a} \end{pmatrix}, \ a \neq -\frac{1}{2}$
$E_5(0,y)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_{1,2} & -1 \end{pmatrix},$ *where* $c_{1,2} = \frac{-1\pm\sqrt{1-4y}}{2}$.
$E_5(0,y)$ $y \neq 0$	$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ c_{1,2} & 0 \end{pmatrix},$ *where* $c_{1,2} = \frac{1\pm\sqrt{1-4y}}{2}, \ \ c_{1,2} \neq 0$

$E_5(0,y)$ $y \neq 0,\ y \neq \frac{1}{4}$	$\begin{pmatrix} \frac{1-4y+\sqrt{1-4y}}{8y-2} & -\frac{1}{\sqrt{1-4y}} \\ \frac{y}{\sqrt{1-4y}} & \frac{1-4y-\sqrt{1-4y}}{8y-2} \end{pmatrix},$ $\begin{pmatrix} \frac{1-4y-\sqrt{1-4y}}{8y-2} & \frac{1}{\sqrt{1-4y}} \\ -\frac{y}{\sqrt{1-4y}} & \frac{1-4y+\sqrt{1-4y}}{8y-2} \end{pmatrix}$
$E_5(x,0)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b_{1,2} \\ 0 & -1 \end{pmatrix},$ $\begin{pmatrix} -1 & -b_{1,2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$ *where* $b_{1,2} = \frac{1\pm\sqrt{1-4x}}{2},\ \ b_{1,2} \neq 0.$
$E_5(x,0)$ $x \neq 0,\ x \neq \frac{1}{4}$	$\begin{pmatrix} \frac{1-4x+\sqrt{1-4x}}{8x-2} & -\frac{x}{\sqrt{1-4x}} \\ \frac{1}{\sqrt{1-4x}} & \frac{1-4x-\sqrt{1-4x}}{8x-2} \end{pmatrix},$ $\begin{pmatrix} \frac{1-4x-\sqrt{1-4x}}{8x-2} & \frac{x}{\sqrt{1-4x}} \\ -\frac{1}{\sqrt{1-4x}} & \frac{1-4x+\sqrt{1-4x}}{8x-2} \end{pmatrix}$
$E_5(0,0)$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$
$E_5(x,y)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$E_5(x,1-x)$ $x \neq \frac{1\pm i\sqrt{3}}{2}$	$\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix},$ $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$

$E_5(x,y)$ $x=\frac{d(1+d)(c+2cd-d(1+d))}{c^2(1+3d+3d^2)}$, $y=\frac{c(1-c+2d)}{c^2(1+3d+3d^2)}$	$\begin{pmatrix} -1-d & -\frac{d(1+d)}{c} \\ c & d \end{pmatrix}$, $d\neq 0,\ d\neq -1,\ d\neq -\frac{3\pm i\sqrt{3}}{6}$, $c\neq 0,\ c\neq \frac{d(1+d)}{1+2d},\ c\neq 1+2d.$
$E_6(0)$	$\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{\sqrt[6]{-1}}{\sqrt{3}} \\ \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{\sqrt[6]{-1}}{\sqrt{3}} \\ -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}$.
$E_6(x)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$E_6(\frac{-b^3-c^3}{bc^2})$ $-b^3-c^3\neq 0$ $b\neq 0,\ c\neq 0$	$\begin{pmatrix} \frac{b^2-c}{2c} & b \\ c & \frac{-b^2-c}{2c} \end{pmatrix}$ *where the parameters b, c are solutions* *of the following system of equations* $\begin{cases} \frac{b^6+5b^3c^3+4c^6}{c} & = \frac{c(b^3+c^3)}{b}, \\ \frac{b^4}{c}+4bc^2 & = c. \end{cases}$

Chapter 3

Chain of evolution algebras of a "chicken" population

In this chapter we study an evolution algebra corresponding to a bisexual population. We construct chains of evolution algebras of a "chicken" population and study their time depending dynamics.

3.1 Basic definitions of chains of evolution algebras of a "chicken" population

In [28] an (evolution) algebra identifying the coefficients of inheritance of a bisexual population as the structure constants of the algebra was introduced. They proved that this algebra is commutative (and hence flexible), not associative and not necessarily power associative. They also showed that the evolution algebra of a bisexual population is not a baric algebra, but a dibaric algebra and hence its square is baric. Moreover, they showed that the algebra is a Banach algebra. The set of all derivations of the evolution algebra was described. We find necessary conditions for a state of the population to be a fixed point or a zero point of the evolution operator which corresponds to the evolution algebra. We also establish upper estimate of the limit points set for trajectories of the evolution operator. Using the necessary conditions we give a detailed analysis of a special case of the evolution algebra (bisexual population of which has a preference on type "1" of females and males). For such a special case we describe the full set of idempotent elements and the full set of absolute nilpotent elements.

The theory of dibaric algebras by applying some results to an evolution algebra of a bisexual population defined using inheritance coefficients of the population was developed in [27].

In [26] a notion of evolution algebra of a "chicken" population (EACP) was introduced. The algebra is given by a rectangular matrix of structural constants. In this paper we introduce a notion of chain of evolution algebras of a "chicken" population (CEACP). The sequence of matrices of structural constants for this CEACPs satisfies an analogue of Chapman-Kolmogorov equation (with a specific multiplication defined for rectangular matrices).

The simple description of the complex EACP by using the Jordan form of the matrix of structural constants was obtained in [9]. The right and plenary periods for basis elements of the EACP were calculated in [33].

Following [26] we consider a set $\{h_i, i = 1, \dots, n\}$ (the set of "hen"s) and r (a "rooster").

Definition 3.1.1 ([26]). Let $(\mathcal{E}, \cdot)$ be an algebra over a field K of characteristic $\neq 2$. If it admits a basis $\{h_1, \dots, h_n, r\}$, such that

$$\begin{aligned} h_i r = r h_i &= \tfrac{1}{2}\left(\textstyle\sum_{j=1}^n a_{ij} h_j + b_i r\right), \\ h_i h_j &= 0, \quad i, j = 1, \dots, n; \quad rr = 0, \end{aligned}$$

then this algebra is called an *evolution algebra of a "chicken" population* (EACP). We call the basis $\{h_1, \dots, h_n, r\}$ a *natural basis.*

Thus an algebra EACP, $\mathcal{E}$, is defined by a rectangular $n \times (n+1)$-matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix},$$

which is called the matrix of structural constants of the algebra $\mathcal{E}$.

We write the matrix M in the form $M = A \oplus \mathbf{b}$ where $A = (a_{ij})_{i,j=1,\dots,n}$ and $\mathbf{b}^T = (b_1, \dots, b_n)$.

Assume we have two rectangular $n \times (n+1)$-matrices $M = A \oplus \mathbf{b}$ and $H = B \oplus \mathbf{c}$. Then we define a multiplication of such matrices by

$$MH = AB \oplus A\mathbf{c}, \qquad HM = BA \oplus B\mathbf{b}. \tag{3.1.1}$$

We note that this multiplication agrees with the usual multiplication of $(n+1) \times (n+1)$-matrices with zero $(n+1)$-th row.

Consider a family $\left\{\mathcal{E}^{[s,t]} : \ s,t \in \mathbb{R}, \ 0 \le s \le t\right\}$ of $(n+1)$-dimensional evolution algebras over the field $\mathbb{R}$, with basis $\{h_1, \dots, h_n, r\}$ and multiplication table

$$h_i r = r h_i = \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij}^{[s,t]} h_j + b_i^{[s,t]} r\right); \quad h_i h_j = 0, \quad 1 \le i,j \le n; \quad rr = 0.$$

Here the parameters s, t are considered as time.

Denote by $M^{[s,t]} = A^{[s,t]} \oplus \mathbf{b}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,\dots,n} \oplus (b_i^{[s,t]})_{i=1,\dots,n}$ the matrix of structural constants.

Definition 3.1.2. A family $\left\{\mathcal{E}^{[s,t]} : \ s,t \in \mathbb{R}, \ 0 \le s \le t\right\}$ of $(n+1)$-dimensional EACP over the field $\mathbb{R}$ is called a *chain of evolution algebras of a "chicken" population* (CEACP) if the matrix $M^{[s,t]}$ of structural constants satisfies the Chapman-Kolmogorov equation

$$M^{[s,t]} = M^{[s,\tau]} M^{[\tau,t]}, \quad \text{for any} \quad s < \tau < t. \tag{3.1.2}$$

By the rule (3.1.1) of multiplication we get from (3.1.2) the following

$$A^{[s,t]} = A^{[s,\tau]} A^{[\tau,t]}, \quad \mathbf{b}^{[s,t]} = A^{[s,\tau]} \mathbf{b}^{[\tau,t]} \quad \text{for any} \quad s < \tau < t. \tag{3.1.3}$$

Definition 3.1.3. A CEACP is called a *time-homogenous* if the matrix $M^{[s,t]}$ depends only on $t - s$. In this case we write $M^{[t-s]}$.

Definition 3.1.4. A CEACP is called *periodic* if its matrix $M^{[s,t]}$ is periodic with respect to at least one of the variables s, t, i.e. (periodicity with respect to t) $M^{[s,t+P]} = M^{[s,t]}$ for all values of t. The constant P is called the period, and is required to be nonzero.

Remark 3.1.5. To define a CEACP one has to solve the system (3.1.3). Since the first equation of the system does not depend on $\mathbf{b}^{[s,t]}$, one can solve it firstly and then, for each solution, from the second equation one must find the corresponding $\mathbf{b}^{[s,t]}$. We note that the first equation is the usual Chapman-Kolmogorov equation (for quadratic matrices) and a wide class of their solutions are known in [7] and in Sections 1.2 and 2.2. In the next section we shall construct several examples of chains of evolution algebras of a “chicken” population (CEACPs).

3.2 Constructions of chains of evolution algebras of a “chicken” population

In this section we construct several examples CEACPs corresponding to a Markov process and to a family of matrices which does not define a process.

The CEACP corresponding to a Markov process. We recall that a left (right) stochastic matrix is a square matrix of non-negative real numbers, with each column (row) summing to 1. A stochastic vector is a vector whose elements are non-negative real numbers which sum to 1. Thus, each row of a right stochastic matrix (or column of a left stochastic matrix) is a stochastic vector.

If $\left\{A^{[s,t]},\ \ 0 \leq s \leq t\right\}$ is a family of stochastic matrices which satisfies the first equation of the system (3.1.3), then it defines a Markov process. We are going to solve the second equation of (3.1.3).

Lemma 3.2.1. *If* $\left\{A^{[s,t]},\ \ 0 \leq s \leq t\right\}$ *is a family of left stochastic matrices then* $\sum_{i=1}^{n} b_i^{[s,t]}$ *does not depend on* s. *Moreover if* $\mathbf{b}^{[s,t]}$ *is a stochastic vector for some time* (s_0, t_0) *then it is a stochastic vector for any time* (s, t_0) *with* $s < t_0$.

Proof. The second equation of (3.1.3) in the coordinates has the following form

$$b_i^{[s,t]} = \sum_{j=1}^{n} a_{ij}^{[s,\tau]} b_j^{[\tau,t]}, \qquad \text{for any} \quad s < \tau < t. \tag{3.2.1}$$

From this we get

$$\sum_{i=1}^{n} b_i^{[s,t]} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}^{[s,\tau]} \right) b_j^{[\tau,t]} = \sum_{j=1}^{n} b_j^{[\tau,t]}. \tag{3.2.2}$$

Now if $\mathbf{b}^{[s,t]}$ is a stochastic vector for some time (s_0, t_0) then $b_i^{[s_0,t_0]} \geq 0$, $\sum_{i=1}^{n} b_i^{[s_0,t_0]} = 1$, this by (3.2.2) gives $\sum_{i=1}^{n} b_i^{[s,t_0]} = 1$ for any $s < t_0$. If write the formula (3.2.1) for $\tau = s_0$ and $t = t_0$, then we get $b_i^{[s,t_0]} \geq 0$ for any $s < t_0$. Thus the vector $\mathbf{b}^{[s,t_0]}$ is a stochastic vector for any $s < t_0$. □

Consider the matrix M_0 which is the $(n+1) \times (n+1)$ matrix constructed from M by adding $(n+1)$-th zero-row. The following theorem follows from Lemma 3.2.1.

Theorem 3.2.2. *For each Markov process given by M_0, there is a chain of evolution algebras of a "chicken" population (CEACP) whose structural constants are transition probabilities of the process, and whose generator set (basis) is the state space of the Markov process.*

The CEACP corresponding to a family of matrices which does not define a process. Now we consider several concrete examples.

Example 3.2.3. Here we describe all 2-dimensional CEACP with basis $\{h, r\}$. Such chains are given by vector $(a^{[s,t]}, b^{[s,t]})$ for which the equation (3.1.2) has the following form

$$a^{[s,\tau]} a^{[\tau,t]} = a^{[s,t]}, \quad a^{[s,\tau]} b^{[\tau,t]} = b^{[s,t]}. \tag{3.2.3}$$

The first equation of the system (3.2.3) is known as Cantor's second equation which has a very rich family of solutions:

a) $a^{[s,t]} = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$;

b)

$$a^{[s,t]} = \begin{cases} 1, & \text{if} \quad s \leq t < \theta, \\ 0, & \text{if} \quad t \geq \theta, \end{cases} \qquad \text{where} \quad \theta > 0.$$

In the case a) from the second equation of (3.2.3) we get $b^{[s,t]} = \alpha(t)/\Phi(s)$, where α is an arbitrary function.

In the case b) from the second equation of (3.2.3) we get

$$b^{[s,t]} = \begin{cases} b^{[\tau,t]}, & \text{if } \; s < \tau < \min\{\theta, t\}, \\ 0, & \text{if } \; \theta \le \tau < t. \end{cases}$$

This has the following solution

$$b^{[s,t]} = \begin{cases} \beta(t), & \text{if } \; s < \min\{\theta, t\}, \\ 0, & \text{if } \; \theta \le s, \end{cases}$$

where β is an arbitrary function. Thus we obtain two CEACP:

$$\mathcal{E}_1^{[s,t]}, \quad \text{with } \; hr = rh = \frac{1}{2\Phi(s)}(\Phi(t)h + \alpha(t)r), \; h^2 = r^2 = 0; \tag{3.2.4}$$

$$\mathcal{E}_2^{[s,t]}, \quad \text{with } \; hr = rh = \frac{1}{2}\begin{cases} h + \beta(t)r, & \text{if } \; s < \min\{\theta, t\}, \\ 0 & \text{if } \; \theta \le s, \end{cases} \qquad h^2 = r^2 = 0. \tag{3.2.5}$$

Example 3.2.4. Consider

$$A^{[t]} = \frac{1}{2}\begin{pmatrix} \lambda^t + \mu^t & \lambda^t - \mu^t \\ \lambda^t - \mu^t & \lambda^t + \mu^t \end{pmatrix}, \qquad \text{where } \; \lambda, \mu > 0.$$

This matrix satisfies the first equation of (3.1.3) with $A^{[s,t]} = A^{[t-s]}$. Now we shall find solutions of the second equation corresponding to $A^{[t-s]}$. The equation has the following form

$$\begin{aligned} (\lambda^{\tau-s} + \mu^{\tau-s})b_1^{[\tau,t]} + (\lambda^{\tau-s} - \mu^{\tau-s})b_2^{[\tau,t]} &= 2b_1^{[s,t]} \\ (\lambda^{\tau-s} - \mu^{\tau-s})b_1^{[\tau,t]} + (\lambda^{\tau-s} + \mu^{\tau-s})b_2^{[\tau,t]} &= 2b_2^{[s,t]} \end{aligned} \; .$$

Denoting $\psi(s,t) = b_1^{[s,t]} + b_2^{[s,t]}$ from the last system we obtain

$$\lambda^\tau \psi(\tau,t) = \lambda^s \psi(s,t).$$

Hence $\psi(s,t)$ has the form $\psi(s,t) = \lambda^{-s}\alpha(t)$, where α is an arbitrary function. Consequently $b_2^{[s,t]} = \lambda^{-s}\alpha(t) - b_1^{[s,t]}$. Then for the function $b_1^{[s,t]}$ we have

$$(\lambda^{\tau-s} + \mu^{\tau-s})b_1^{[\tau,t]} + (\lambda^{\tau-s} - \mu^{\tau-s})(\lambda^{-\tau}\alpha(t) - b_1^{[\tau,t]}) = 2b_1^{[s,t]},$$

i.e.

$$2\mu^\tau b_1^{[\tau,t]} + (\lambda^{\tau-s} - \mu^{\tau-s})\lambda^{-\tau}\mu^s\alpha(t) = 2\mu^s b_1^{[s,t]}.$$

Denote $f(s,t) = 2\mu^s b_1^{[s,t]}$. Then from the last equation we get

$$f(\tau,t) - f(s,t) = ((\mu/\lambda)^\tau - (\mu/\lambda)^s)\alpha(t).$$

This equation has the following solution

$$f(s,t) = (\mu/\lambda)^s\alpha(t) + \beta(t),$$

where β is an arbitrary function.

Hence

$$b_1^{[s,t]} = \frac{1}{2}(\lambda^{-s}\alpha(t) + \mu^{-s}\beta(t)), \qquad b_2^{[s,t]} = \frac{1}{2}(\lambda^{-s}\alpha(t) - \mu^{-s}\beta(t)).$$

Thus we get the following solution of (3.1.2), i.e. the system (3.1.3)

$$M^{[s,t]} = \frac{1}{2}\begin{pmatrix} \lambda^{t-s} + \mu^{t-s} & \lambda^{t-s} - \mu^{t-s} & \lambda^{-s}\alpha(t) + \mu^{-s}\beta(t) \\ \lambda^{t-s} - \mu^{t-s} & \lambda^{t-s} + \mu^{t-s} & \lambda^{-s}\alpha(t) - \mu^{-s}\beta(t) \end{pmatrix}.$$

Let $\mathcal{E}^{[s,t]}$, $0 \leq s \leq t$, be the CEACP corresponding to this matrix. In spite of $A^{[t-s]}$ is a time-homogenous, we note that this CEACP is not a time-homogenous, in general. But if $\alpha(t) = \lambda^t$, $\beta(t) = \mu^t$, then the CEACP is a time-homogenous.

Depending on parameters λ, μ and parameter-functions α, β we get distinct behavior of $\mathcal{E}^{[s,t]}$ for $t - s \to +\infty$. In the case if one of the following limits does not exist

$$\lim_{t-s\to+\infty}(\lambda^{-s}\alpha(t) + \mu^{-s}\beta(t)) = b_1, \quad \lim_{t-s\to+\infty}(\lambda^{-s}\alpha(t) + \mu^{-s}\beta(t)) = b_2$$

then a limiting EACP does not exist. If the limits do exist, then we have

$$\lim_{t-s\to+\infty} \mathcal{E}^{[s,t]} = \begin{cases} \mathcal{E}_0 & \text{if } \ 0 < \lambda, \mu < 1, \\ \mathcal{E}_1 & \text{if } \ \lambda = \mu = 1, \\ \mathcal{E}_{1/2} & \text{if } \ \lambda = 1, 0 \leq \mu < 1, \\ \mathcal{E}_{-1/2} & \text{if } \ \mu = 1, 0 \leq \lambda < 1, \\ \mathcal{E}_{\infty} & \text{otherwise}, \end{cases}$$

where $\mathcal{E}_0$ is an EACP with multiplication (omitted multiplications are zero):

$$h_1 r = b_1 r, \quad h_2 r = b_2 r;$$

$\mathcal{E}_1$ is an EACP with the multiplication table

$$h_1 r = h_1 + b_1 r, \quad h_2 r = h_2 + b_2 r;$$

$\mathcal{E}_{1/2}$ is an EACP with

$$h_1 r = \frac{1}{2}(h_1 + h_2) + b_1 r, \quad h_2 r = \frac{1}{2}(h_1 + h_2) + b_2 r;$$

$\mathcal{E}_{-1/2}$ is an EACP with

$$h_1 r = \frac{1}{2}(h_1 - h_2) + b_1 r, \quad h_2 r = -\frac{1}{2}(h_1 - h_2) + b_2 r;$$

and $\mathcal{E}_\infty$ is a vector space which has an "infinity multiplication", or we can say that in $\mathcal{E}_\infty$ an algebra structure is not defined. This example shows that a limit of a CEACP can be non evolution algebra.

Example 3.2.5. Consider the solution $A^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ to the first equation of (3.1.3) given in Example 1.1.10 (see Section 1.1).

$$\begin{aligned}
a_{11}^{[s,t]} &= \tfrac{1}{2}\left(1+\Phi(t)(\Psi(t)-\Psi(s))+\tfrac{\Phi(t)}{\Phi(s)}\right),\\
a_{12}^{[s,t]} &= \tfrac{1}{2}\left(1-\Phi(t)(\Psi(t)-\Psi(s))-\tfrac{\Phi(t)}{\Phi(s)}\right),\\
a_{21}^{[s,t]} &= \tfrac{1}{2}\left(1+\Phi(t)(\Psi(t)-\Psi(s))-\tfrac{\Phi(t)}{\Phi(s)}\right),\\
a_{22}^{[s,t]} &= \tfrac{1}{2}\left(1-\Phi(t)(\Psi(t)-\Psi(s))+\tfrac{\Phi(t)}{\Phi(s)}\right),
\end{aligned} \tag{3.2.6}$$

where $\Phi \neq 0$ and Ψ are arbitrary functions.

Now we shall find solutions of the second equation of (3.1.3) corresponding to $A^{[s,t]}$. Denoting $f(s,t) = b_1^{[s,t]} + b_2^{[s,t]}$, $g(s,t) = b_1^{[s,t]} - b_2^{[s,t]}$ the second equation can be written as

$$f(\tau,t) + \Phi(\tau)(\Psi(\tau)-\Psi(s))g(\tau,t) = f(s,t),$$

$$(\Phi(\tau)/\Phi(s))g(\tau,t) = g(s,t).$$

From the second equation of this system we get $\Phi(s)g(s,t) = \alpha(t)$, where α is an arbitrary function. Substituting $g(\tau,t) = \alpha(t)/\Phi(\tau)$ in the first equation of the last system we get $f(s,t) = \beta(t) - \Psi(s)\alpha(t)$, where β is an arbitrary function. Finally we obtain the following

$$\begin{aligned}
b_1^{[s,t]} &= \tfrac{1}{2}\left(\beta(t)+(\tfrac{1}{\Phi(s)}-\Psi(s))\alpha(t)\right),\\
b_2^{[s,t]} &= \tfrac{1}{2}\left(\beta(t)-(\tfrac{1}{\Phi(s)}+\Psi(s))\alpha(t)\right).
\end{aligned} \tag{3.2.7}$$

Thus we showed that the matrix $A^{[s,t]}$ given by (3.2.6) and the vector $\mathbf{b}^{[s,t]}$ given by (3.2.7) generate a CEACP, $\mathcal{E}^{[s,t]}$, $0 \le s \le t$. This CEACP varies by four parameter-functions, for example, if Φ, Ψ, α and β such that

$$\lim_{t\to+\infty}\Phi(t)\Psi(t)=\lim_{t\to+\infty}\Phi(t)=\lim_{t\to+\infty}\alpha(t)=\lim_{t\to+\infty}\beta(t)=0,$$

then for a fixed s we have $\lim\limits_{t\to\infty}\mathcal{E}^{[s,t]}=\mathcal{E}_{1/2}$, where $\mathcal{E}_{1/2}$ is an EACP with the multiplication table

$$h_1r=h_2r=\frac{1}{2}(h_1+h_2).$$

Example 3.2.6. For any n we shall give an example of a time non-homogenous $(n+1)$-dimensional CEACP. Let $\{T^{[t]},\, t\geq 0\}$ be a family of invertible $n\times n$ matrices, for all t. Define the following matrix

$$A^{[s,t]}=T^{[s]}(T^{[t]})^{-1}, \tag{3.2.8}$$

where $(T^{[t]})^{-1}$ is the inverse of $T^{[t]}$.

We note that a construction of a family of invertible $n\times n$ matrices $T^{[t]}$ is not difficult, for example, one can take $T^{[t]}$ as a lower or upper triangular $n\times n$ matrix. Then the matrices are invertible iff $\det(T^{[t]})\neq 0$ for all t, i.e. the diagonal elements of the triangular matrix are non-zero.

Let $\mathbf{c}^{[t]}$, $t>0$, be an arbitrary family of vectors. Define

$$\mathbf{b}^{[s,t]}=T^{[s]}\mathbf{c}^{[t]}. \tag{3.2.9}$$

Proposition 3.2.7. *The matrix $M^{[s,t]}=A^{[s,t]}\oplus\mathbf{b}^{[s,t]}$ given by (3.2.8) and (3.2.9) generates a $(n+1)$-dimensional CEACP.*

Proof. We shall show that the matrix $M^{[s,t]}$ given in the statement of the proposition satisfies the equation (3.1.2):

$$\begin{aligned}M^{[s,\tau]}M^{[\tau,t]}&=(A^{[s,\tau]}\oplus\mathbf{b}^{[s,\tau]})(A^{[\tau,t]}\oplus\mathbf{b}^{[\tau,t]})=A^{[s,\tau]}A^{[\tau,t]}\oplus A^{[s,\tau]}\mathbf{b}^{[\tau,t]}\\&=T^{[s]}\left((T^{[\tau]})^{-1}T^{[\tau]}\right)(T^{[t]})^{-1}\oplus T^{[s]}\left((T^{[\tau]})^{-1}T^{[\tau]}\right)\mathbf{c}^{[t]}\\&=T^{[s]}(T^{[t]})^{-1}\oplus T^{[s]}\mathbf{c}^{[t]}=M^{[s,t]}.\end{aligned}$$

□

Thus each family (with one parameter) of invertible $n \times n$ matrices together with a family of vectors (with one parameter) define a CEACP $\mathcal{E}^{[s,t]}$ which is time non-homogenous, in general.

Example 3.2.8. It is easy to see that the matrix

$$A^{[s,t]} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$$

satisfies the equation(3.1.2). We are going to find corresponding solutions of the second equation of (3.1.3).

This equation has the following form

$$\begin{aligned} \cos(\tau - s) b_1^{[\tau,t]} + \sin(\tau - s) b_2^{[\tau,t]} &= b_1^{[s,t]} \\ -\sin(\tau - s) b_1^{[\tau,t]} + \cos(t - s) b_2^{[\tau,t]} &= b_2^{[s,t]}. \end{aligned}$$

It is easy to check that this system has the following solution

$$b_1^{[s,t]} = c_1 \sin(t-s) - c_2 \cos(t-s), \quad b_2^{[s,t]} = c_1 \cos(t-s) + c_2 \sin(t-s),$$

where c_1 and c_2 are arbitrary numbers.

Thus the following matrix satisfies (3.1.2):

$$M^{[s,t]} = \begin{pmatrix} \cos(t-s) & \sin(t-s) & c_1 \sin(t-s) - c_2 \cos(t-s) \\ -\sin(t-s) & \cos(t-s) & c_1 \cos(t-s) + c_2 \sin(t-s) \end{pmatrix}. \quad (3.2.10)$$

Since this matrix is periodic with period $P = 2\pi$, the corresponding CEACP $\mathcal{E}^{[t]}$ is also periodic. Moreover, for arbitrary 3-dimensional EACP $\mathcal{E}_a^+$, or $\mathcal{E}_a^-$, $a \in [-1, 1]$, with matrix of structural constants

$$M_a^{\pm} = \begin{pmatrix} a & \pm\sqrt{1-a^2} & \pm c_1\sqrt{1-a^2} - c_2 a \\ \mp\sqrt{1-a^2} & a & c_1 a \pm c_2\sqrt{1-a^2} \end{pmatrix}$$

respectively, there is a sequence $t_n = t_n(a)$ of times such that $\lim_{n\to\infty} \mathcal{E}^{[t_n]} = \mathcal{E}_a^+$ or $\mathcal{E}_a^-$. We have $\mathcal{E}_a^{\pm} \neq \mathcal{E}_b^{\pm}$ if $a \neq b$.

Moreover the following is true.

Proposition 3.2.9.

(1) For any $a, b \in [-1,1]$, $a \neq b$, the algebras $\mathcal{E}_a^+$ and $\mathcal{E}_b^+$ are not isomorphic.

(2) For any $a, b \in [-1,1]$, $a \neq b$, the algebras $\mathcal{E}_a^-$ and $\mathcal{E}_b^-$ are not isomorphic.

Proof. (1) Let $\varphi = (\alpha_{ij})_{i,j=1,2,3}$ be an isomorphism of the evolution algebra $\mathcal{E}_a^+$ to the evolution algebra $\mathcal{E}_b^+$. Here $\det(\varphi) \neq 0$. Since $\det(A^{[s,t]}) = 1$, from $\varphi(h_1h_2) = \varphi(h_1)\varphi(h_2) = 0$, $\varphi(h_1^2) = 0$, $\varphi(h_2^2) = 0$ and $\varphi(rr) = 0$ we get

$$\alpha_{11}\alpha_{23} + \alpha_{21}\alpha_{13} = 0, \quad \alpha_{12}\alpha_{23} + \alpha_{22}\alpha_{13} = 0;$$

$$\alpha_{11}\alpha_{13} = 0, \quad \alpha_{12}\alpha_{13} = 0; \quad \alpha_{21}\alpha_{23} = 0, \quad \alpha_{22}\alpha_{23} = 0;$$

$$\alpha_{31}\alpha_{33} = 0, \quad \alpha_{32}\alpha_{33} = 0.$$

Since $\det(\varphi) \neq 0$ the last equations give the following possibilities to φ:

$$F_1 = \left\{ \varphi = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} : (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\alpha_{33} \neq 0 \right\},$$

$$F_2 = \left\{ \varphi = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix} : (\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\alpha_{23} \neq 0 \right\},$$

$$F_3 = \left\{ \varphi = \begin{pmatrix} 0 & 0 & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix} : (\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31})\alpha_{13} \neq 0 \right\}.$$

We note that the classes F_i are the same up to renumbering of indexes. Therefore, we consider only class F_1. If we take $\varphi \in F_1$ then we get the following relation between the matrices M_a^+ and

$$M_b^+ = \varphi(M_a^+) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}, \tag{3.2.11}$$

where m_{ij} is function of a given by the following formulas

$$\begin{aligned}
m_{11} &= \frac{(\alpha_{11}a - \alpha_{12}\sqrt{1-a^2})\alpha_{22} - (\alpha_{11}\sqrt{1-a^2} + \alpha_{12}a)\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \\
&= a - \frac{(\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22})\sqrt{1-a^2}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \\
m_{12} &= \frac{-(\alpha_{11}a - \alpha_{12}\sqrt{1-a^2})\alpha_{12} + (\alpha_{11}\sqrt{1-a^2} + \alpha_{12}a)\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \\
&= \frac{(\alpha_{11}^2 + \alpha_{12}^2)\sqrt{1-a^2}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \\
m_{13} &= \frac{\alpha_{11}(c_1\sqrt{1-a^2} - c_2 a) + \alpha_{12}(c_1 a + c_2\sqrt{1-a^2})}{\alpha_{33}}, \\
m_{21} &= \frac{(\alpha_{21}a - \alpha_{22}\sqrt{1-a^2})\alpha_{22} - (\alpha_{21}\sqrt{1-a^2} + \alpha_{22}a)\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \\
&= -\frac{(\alpha_{21}^2 + \alpha_{22}^2)\sqrt{1-a^2}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \\
m_{22} &= \frac{-(\alpha_{21}a - \alpha_{22}\sqrt{1-a^2})\alpha_{12} + (\alpha_{21}\sqrt{1-a^2} + \alpha_{22}a)\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \\
&= a + \frac{(\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22})\sqrt{1-a^2}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}, \\
m_{23} &= \frac{\alpha_{21}(c_1\sqrt{1-a^2} - c_2 a) + \alpha_{22}(c_1 a + c_2\sqrt{1-a^2})}{\alpha_{33}}.
\end{aligned}$$

By (3.2.11) we should have $m_{11} = m_{22} = b$ and $m_{12} = -m_{21} = \sqrt{1-b^2}$. These equalities for $a \neq \pm 1$ give

$$\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0, \quad \alpha_{11}^2 + \alpha_{12}^2 = \alpha_{21}^2 + \alpha_{22}^2.$$

Consequently, if $a \neq \pm 1$ then $b = a$. Moreover, if $a = \pm 1$ then again we get $m_{11} = m_{22} = b = a = \pm 1$. Hence isomorphisms of the class F_1 can not give an isomorphism between $\mathcal{E}_a^+$ and $\mathcal{E}_b^+$ if $a \neq b$. Similar argument works for classes F_2 and F_3.

(2) The proof is similar to that of (1). □

Consider now discrete time n, $n \in \mathbb{N}$ and the CEACP $\{\mathcal{E}^{[n]}, n \in \mathbb{N}\}$ given by the matrix (3.2.10) with $t - s = n$.

Proposition 3.2.10. *The discrete time CEACP $\mathcal{E}^{[n]}$, $n \in \mathbb{N}$, is dense in the set $\{\mathcal{E}_a^{\pm}, a \in [-1, 1]\}$ of EACP, i.e. for an arbitrary EACP $\mathcal{E}_a^{\pm}$ there exists a sequence $\{n_k\}_{k=1,2,\ldots}$ of natural numbers such that $\lim_{k\to\infty} \mathcal{E}^{[n_k]} = \mathcal{E}_a^+$ or $\mathcal{E}_a^-$.*

Proof. It is known that the sequences $\{\sin n\}$ and $\{\cos n\}, n \in \mathbb{N}$, are dense in $[-1, 1]$ (see e.g. [14]). Hence for any $a \in [-1, 1]$ there is a sequence $\{n_k\}_{k=1,2,\ldots}$ of natural numbers such that $\lim_{k\to\infty} \cos(n_k) = a$. The same sequence can be used to get $\lim_{k\to\infty} \mathcal{E}^{[n_k]} = \mathcal{E}_a^+$ or $\mathcal{E}_a^-$. □

3.3 Time depending dynamics of a chain of evolution algebras of a "chicken" population

In this section for 2 and 3-dimensional algebras of CEACP we shall study the time depending dynamics of isomorphic EACPs in the chain.

Let $\mathcal{E}$ be a 2-dimensional EACP and $\{h, r\}$ be a basis of this algebra.

It is evident that if $\dim \mathcal{E}^2 = 0$ then $\mathcal{E}$ is an abelian algebra, i.e. an algebra with all products equal to zero.

Proposition 3.3.1 ([26]). *Any 2-dimensional, non-trivial EACP $\mathcal{E}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

E_1: $\quad rh = hr = h, \quad h^2 = r^2 = 0,$

E_2: $\quad rh = hr = \frac{1}{2}(h + r), \quad h^2 = r^2 = 0.$

The following proposition gives the time dynamics of (3.2.4) and (3.2.5).

Proposition 3.3.2. *We have*

$$\mathcal{E}_1^{[s,t]} \cong \begin{cases} E_1, & \text{if } t \in \{t : \alpha(t) = 0\}, \\ E_2, & \text{if } t \in \{t : \alpha(t) \neq 0\}. \end{cases}$$

$$\mathcal{E}_2^{[s,t]} \cong \begin{cases} E_1, & \text{if } (s,t) \in \{(s,t) : \beta(t) = 0, s < \min\{\theta, t\}\}, \\ E_2, & \text{if } (s,t) \in \{(s,t) : \beta(t) \neq 0, s < \min\{\theta, t\}\}, \\ E_0, & \text{if } (s,t) \in \{(s,t) : s \geq 0\}, \end{cases}$$

where E_0 is the algebra with zero multiplication.

Proof. Since $\Phi(t) \neq 0$ if $\alpha(t) = 0$, then by the change of basis $h' = h$ and $r' = \frac{2\Phi(s)}{\Phi(t)} r$, we get the algebra E_1. In the case $\alpha(t) \neq 0$, the change $h' = \frac{\Phi(s)}{\Phi(t)} r$ and $r' = \frac{\Phi(s)}{\alpha(t)} h$ implies the algebra E_2.

For $\mathcal{E}_2^{[s,t]}$ the proof is similar. $\square$

To give some illustration of Proposition 3.3.2 we consider the following example.

Example 3.3.3. Take

$$\alpha(t) = \begin{cases} (5-t)(20-t)(2019-t), & \text{if } 0 \leq t \leq 2019, \\ 0, & \text{if } t > 2019. \end{cases}$$

By the proposition the corresponding CEACP will be isomorphic to E_1 if time $t \in \{5, 20\} \cup [2019, +\infty)$. So there are three critical times 5, 20, 2019, at which the chain changes the algebras and remain with the same algebra (up to isomorphism) between critical times. This is like a phase transition property of physical systems [17]. The parameter is the temperature and the system changes its phase (state) at critical temperatures.

Let now $\mathcal{E}$ be a 3-dimensional EACP and $\{h_1, h_2, r\}$ be a basis of this algebra.

Theorem 3.3.4 ([26]). *Any 3-dimensional EACP $\mathcal{E}$ with* $\dim(\mathcal{E}^2) = 1$ *is isomorphic to one of the following pairwise non isomorphic algebras:*

$\mathcal{E}_1$: $h_1 r = r$;

$\mathcal{E}_2$: $h_1 r = h_2$;

$\mathcal{E}_3$: $h_1 r = h_1 + r$.

In each algebra $rh_i = h_i r$, $i = 1, 2$, *and all omitted products are zero.*

In order to use this theorem first we give a class of 3-dimensional CEACP $\mathcal{E}^{[s,t]}$ with $\dim\big(\big(\mathcal{E}^{[s,t]}\big)^2\big) = 1$ for any (s,t). Such a CEACP has the matrix $M^{[s,t]}$ in the following form

$$M^{[s,t]} = \begin{pmatrix} a_1^{[s,t]} & a_2^{[s,t]} & b^{[s,t]} \\ ca_1^{[s,t]} & ca_2^{[s,t]} & cb^{[s,t]} \end{pmatrix},$$

where $c = c(s,t)$ is a given function.

For this operator the equation (3.1.3) can be written as

$$\begin{aligned}
&a_1^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = a_1^{[s,t]},\\
&a_2^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = a_2^{[s,t]},\\
&b^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = b^{[s,t]},\\
&c(s,\tau)a_1^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = c(s,t)a_1^{[s,t]},\\
&c(s,\tau)a_2^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = c(s,t)a_2^{[s,t]},\\
&c(s,\tau)b^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau,t)a_2^{[s,\tau]}\right) = c(s,t)b^{[s,t]}.
\end{aligned} \tag{3.3.1}$$

Comparing the first and the fourth equations we get $c(s,\tau) = c(s,t)$, i.e. c should not depend on t. So denote this function as $c(s)$. Then from system (3.3.1) it remains the following system

$$
\begin{aligned}
a_1^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau)a_2^{[s,\tau]}\right) &= a_1^{[s,t]},\\
a_2^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau)a_2^{[s,\tau]}\right) &= a_2^{[s,t]},\\
b^{[\tau,t]}\left(a_1^{[s,\tau]} + c(\tau)a_2^{[s,\tau]}\right) &= b^{[s,t]}.
\end{aligned}
$$

Denote $\gamma(s,t) = a_1^{[s,t]} + c(t)a_2^{[s,t]}$ and $\delta(s,t) = a_1^{[s,t]} - c(t)a_2^{[s,t]}$. Then from first and second equations of (3.3.1) we get

$$\gamma(s,t) = \gamma(s,\tau)\gamma(\tau,t), \quad \delta(s,t) = \gamma(s,\tau)\delta(\tau,t), \;\; s \le \tau \le t.$$

The first equation is Cantor's second equation which has a solution $\gamma(s,t) = \frac{\phi(t)}{\phi(s)}$, where ϕ is an arbitrary function with $\phi(s) \neq 0$.

Using the solution from the second equation we find $\delta(s,t) = \frac{\psi(t)}{\phi(s)}$, where ψ is an arbitrary function. Then we obtain

$$a_1^{[s,t]} = \begin{cases} \frac{\phi(t)+\psi(t)}{2\phi(s)}, & \text{if } \; c(t) \neq 0, \\ \frac{\phi(t)}{\phi(s)}, & \text{if } \; c(t) = 0, \end{cases}$$

$$a_2^{[s,t]} = \begin{cases} \frac{\phi(t)-\psi(t)}{2c(t)\phi(s)}, & \text{if } \; c(t) \neq 0 \\ \frac{\psi(t)}{\phi(s)}, & \text{if } \; c(t) = 0, \end{cases}, \qquad b^{[s,t]} = \frac{\alpha(t)}{\phi(s)}.$$

Consequently, for t such that $c(t) \neq 0$, we have

$$M^{[s,t]} = \frac{1}{2\phi(s)} \begin{pmatrix} \phi(t)+\psi(t) & \frac{\phi(t)-\psi(t)}{c(t)} & 2\alpha(t) \\ c(t)(\phi(t)+\psi(t)) & \phi(t)-\psi(t) & 2c(t)\alpha(t) \end{pmatrix} \tag{3.3.2}$$

and for t such that $c(t) = 0$ we have

$$M^{[s,t]} = \frac{1}{\phi(s)} \begin{pmatrix} \phi(t) & \psi(t) & \alpha(t) \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3.3}$$

Thus we proved the following

Theorem 3.3.5. *The matrices* (3.3.2) *and* (3.3.3) *generate a CEACP* $\mathcal{E}^{[s,t]}$ *with* $\dim\left(\left(\mathcal{E}^{[s,t]}\right)^2\right) = 1$ *for any* (s,t).

Consider a 3-dimensional EACP $\mathcal{E}$ with $\dim(\mathcal{E}^2) = 1$. Then it has the following form

$$\begin{aligned} h_1 r = r h_1 &= \frac{1}{2}(ah_1 + bh_2 + Ar), \\ h_2 r = r h_2 &= \frac{c}{2}(ah_1 + bh_2 + Ar), \\ h_1^2 = h_2^2 &= h_1 h_2 = r^2 = 0. \end{aligned}$$

We note that the non-zero coefficients of $h_1 r$ can be taken equal to 1. Indeed, if $abA \neq 0$, then the change of basis $h_1' = \frac{2}{A}h_1$, $h_2' = \frac{2b}{aA}h_2$, $r' = \frac{2}{a}r$, makes all coefficients of $h_1 r$ equal 1, i.e. we obtain

$$\begin{aligned} h_1 r = r h_1 &= h_1 + h_2 + r, \\ h_2 r = r h_2 &= \frac{bc}{a}(h_1 + h_2 + r), \\ h_1^2 = h_2^2 &= h_1 h_2 = r^2 = 0. \end{aligned} \tag{3.3.4}$$

In case some a, b, A are equal 0 then one can choose a suitable change of basis to make non-zero coefficients of $h_1 r$ equal to 1. But we do not consider these particular cases here, because they are simpler than the case $abA \neq 0$.

Proposition 3.3.6. *Assume there is no zero in the first row of matrices (3.3.2) and (3.3.3). Then for the corresponding CEACP* $\mathcal{E}^{[s,t]}$ *we have* $\mathcal{E}^{[s,t]} \cong \mathcal{E}_3$ *for any* (s,t).

Proof. By the above-mentioned change of the basis, the algebra $\mathcal{E}^{[s,t]}$ can be written as (3.3.4). In this case we have

$$\frac{bc}{a} = \begin{cases} \frac{\phi(t)-\psi(t)}{\phi(t)+\psi(t)}, & \text{if } c(t) \neq 0, \\ 0, & \text{if } c(t) = 0. \end{cases}$$

By the change

$$h_1' = \frac{\phi(t)+\psi(t)}{2\phi(t)}(h_1+h_2), \qquad h_2' = \frac{\psi(t)-\phi(t)}{2\phi(t)}h_1 + \frac{\psi(t)+\phi(t)}{2\phi(t)}h_2,$$
$$r' = \frac{\phi(t)+\psi(t)}{2\phi(t)}r,$$

we get the algebra $\mathcal{E}_3$. □

Remark 3.3.7.

1. In [28] it was shown that the evolution algebra of a bisexual population is not a baric algebra. Since the EACP is an algebra of a bisexual population it is not baric. So CEACP is not baric any time, i.e. it has not baric property transition.

2. The matrices of structural constants (depending on (s,t)) of a CEACP satisfy the Chapman-Kolmogorov equation. To say it another way, a CEACP is a continuous-time dynamical system which in a fixed time is an EACP. It is well known that if a matrix satisfying the Chapman-Kolmogorov equation is stochastic, then it generates a Markov process. In [20], the authors investigated the time evolution of stochastic *non*-Markov processes as they occur in the coarse-grained description of open and closed systems. Some aspects of the theory are illustrated for the two-state process and the Gauss process. Since the matrix of structural constants of each EACP of a CEACP has a general form, the non-Markov processes of [20] can be, in particular, obtained by matrices of structural constants of a CEACP. Thus CEACP can be used in biology and physics.

Resumen de la Tesis Doctoral (in Spanish)

Time depending dynamics of chains of evolution algebras

Resumen abreviado: Dinámica dependiente del tiempo de cadenas de álgebras de evolución

Esta tesis está dedicada al estudio de la dinámica dependiente del tiempo de cadenas de álgebras de evolución. En [7] se introduce una noción de una cadena de álgebras de evolución (CEA). Esta cadena es un sistema dinámico cuyo estado en cada momento dado es un álgebra de evolución. Las sucesiones de matrices de las constantes estructurales para estas cadenas de álgebras de evolución satisfacen la ecuación de Chapman-Kolmogorov.

Al principio se dan las definiciones básicas y la construcción de cadenas de álgebras de evolución. Damos un breve repaso de las cadenas de álgebras de evolución (CEAs), siguiendo [7]. Construimos nuevas CEAs reales de dimensión 2 y estudiamos las transiciones de varias propiedades de estas CEAs construidas, en concreto, transiciones de la propiedad bárica, la existencia de la "unicidad del elemento nilpotente absoluto" y la dinámica temporal de los elementos idempotentes.

Obtenemos la clasificación de álgebras de evolución reales de dimensión 2 y dinámicas dependientes del tiempo de dichas CEAs y construimos nuevas clases de CEAs de dimensión 2, las cuales serán isomorfas a un álgebra de evolución dada, para algunos valores del tiempo. También definimos operadores (lineales) Rota-Baxter sobre álgebras de evolución.

Finalmente, construimos cadenas de álgebras de evolución de la población de "pollos", y también estudiamos la dinámica dependiente del tiempo de cadenas de álgebras de evolución de la población de "pollos".

La genética mendeliana introdujo una nuevo tema en las matemáticas: álgebras genéticas generales. Estas álgebras son en general conmutativas pero no asociativas, además no pertenecen a ninguna de las conocidas clases de álgebras no asociativas como las álgebras de Lie, las álgebras de Jordan o las álgebras alternativas.

Baur [2] y Correns [8] detectaron por primera vez que la herencia de cloroplastos se apartaba de las leyes de Mendel. Hoy en día, la genética no mendeliana es un lenguaje básico de los genetistas moleculares. La genética no mendeliana ofrece a las matemáticas un nuevo tipo de álgebras genéticas, denominadas álgebras de evolución, introducidas en [45]; son álgebras en las que las tablas de multiplicación están motivadas por las leyes evolutivas de la genética.

Recientemente en [7] se introduce una noción de una cadena de álgebras de evolución (CEA). Esta cadena es un sistema dinámico cuyo estado en cada momento es un álgebra de evolución. La sucesión de matrices de las constantes estructurales para esta cadena de álgebras de evolución satisface le ecuación de Chapman-Kolmogorov. En [29] se introduce la noción de flujo (dependiente del tiempo) de álgebras de dimensión finita. Un flujo de álgebras es un caso particular de un sistema dinámico de tiempo continuo cuyos estados son álgebras de dimensión finita.

En el Capítulo 1 damos las definiciones básicas y la construcción de cadenas de álgebras de evolución. En primer lugar damos un breve repaso de las cadenas conocidas de álgebras de evolución (CEAs). En segundo lugar construimos nuevas CEAs reales de dimensión 2. Luego estudiamos las transiciones de las propiedades de las CEAs construidas, en particular, las transiciones de la propiedad bárica de las CEAs y la existencia de la "unicidad del elemento nilpotente absoluto" y se da la dinámica dependiente del tiempo de los elementos idempotentes de las CEAs.

Definición 1.1.2 ([44]). Sea $(E, \cdot)$ un álgebra sobre un cuerpo $\mathbb{K}$. Si ella admite una base $\{e_1, e_2, \dots\}$, tal que

$$e_i \cdot e_j = \begin{cases} 0, & si \;\; i \neq j; \\ \sum_k a_{ik} e_k, & si \;\; i = j, \end{cases}$$

entonces esta álgebra se llama un *álgebra de evolución*. Esta base se llama una *base natural*.

Denotamos por $\mathcal{M} = (a_{ij})$ la matriz de las constantes estructurales del álgebra de evolución E.

Siguiendo [7] consideramos una familia $\left\{E^{[s,t]} : \; s,t \in \mathbb{R}, \; 0 \leq s \leq t\right\}$ de álgebras de evolución de dimensión n sobre el cuerpo $\mathbb{R}$, con base $e_1, \dots, e_n$, y tabla de multiplicación

$$e_i e_i = \sum_{j=1}^{n} a_{ij}^{[s,t]} e_j, \quad i = 1, \dots, n; \quad e_i e_j = 0, \quad i \neq j.$$

Aquí los parámetros s, t son considerados como el tiempo.

Denotamos por $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,\dots,n}$ la matriz de las constantes estructurales.

Definición 1.1.3. Una familia $\left\{E^{[s,t]} : \; s,t \in \mathbb{R}, \; 0 \leq s \leq t\right\}$ de álgebras de evolución de dimensión n sobre el cuerpo $\mathbb{R}$ se llama una *cadena de álgebras de evolución (CEA)* si la matriz $\mathcal{M}^{[s,t]}$ de constantes estructurales satisface la ecuación de Chapman-Kolmogorov

$$\mathcal{M}^{[s,t]} = \mathcal{M}^{[s,\tau]} \mathcal{M}^{[\tau,t]}, \;\; \text{para cualquier} \;\; s < \tau < t.$$

El siguiente teorema da el diagrama de la propiedad bárica para cadenas $E_i^{[s,t]}$, $i = 0, \dots, 24$, construidas en la Sección 1.2.

Denotamos por $\mathcal{T}_b^{(i)}$ la duración de la propiedad bárica de la CEA $E_i^{[s,t]}$, $i = 0, \dots, 24$.

Teorema 1.3.5.

(i) (No hay transición de la propiedad no bárica)

Las álgebras $E_i^{[s,t]}$, $i = 0, 1, 2, 3, 6, 10, 11, 14, 22$, *no son báricas para cualquier tiempo* $(s,t) \in \mathcal{T}$;

(ii) (No hay transición de la propiedad bárica)

Las álgebras $E_i^{[s,t]}$, $i = 16, 17, 18$, *y* $E_{23}^{[s,t]}(0,\mu)$, $E_{23}^{[s,t]}(2\mu,\mu)$, $\mu \neq 0$, *son báricas para cualquier tiempo* $(s,t) \in \mathcal{T}$;

(iii) (Hay una transición de la propiedad bárica)

Las CEAs $E_i^{[s,t]}$, $i = 4, 5, 7, 8, 9, 12, 13, 15, 19, 20, 21, 24$, *y* $E_{23}^{[s,t]}(\lambda,\mu)$, *con* $\lambda \notin \{0, \mu, 2\mu\}$ *tienen una transición de propiedad bárica con una duración de propiedad bárica establecida de la siguiente manera*

$$\mathcal{T}_b^{(4)} = \left\{(s,t) \in \mathcal{T} : \frac{\Phi(s)}{\Psi(s)} = \frac{\Phi(t)}{\Psi(t)}\right\};$$

$$\mathcal{T}_b^{(5)} = \{(s,t) \in \mathcal{T} : s \leq t < b,\ \Phi(s) = \Phi(t)\};$$

$$\mathcal{T}_b^{(7)} = \{(s,t) \in \mathcal{T} : s \leq t < a,\ \Psi(s) = \Psi(t)\};$$

$$\mathcal{T}_b^{(8)} = \{(s,t) \in \mathcal{T} : s \leq t < \min\{a,b\}\};$$

$$\mathcal{T}_b^{(9)} = \{(s,t) \in \mathcal{T} : t = s + \pi k, k \in \mathbb{Z}\};$$

$$\mathcal{T}_b^{(12)} = \left\{(s,t) \in \mathcal{T} : g(s) = \pm\frac{1}{h(s)}\right\};$$

$$\mathcal{T}_b^{(13)} = \{(s,t) \in \mathcal{T} : s \leq t < a,\ \psi(s) = \pm 1\};$$

$$\mathcal{T}_b^{(15)} = \{(s,t) \in \mathcal{T} : s \leq t < a,\ \psi(s) = 0\};$$

$$\mathcal{T}_b^{(19)} = \{(s,t) \in \mathcal{T} : s \leq t < a\};$$

$$\mathcal{T}_b^{(20)} = \{(s,t) \in \mathcal{T} : s \leq t < b\} \cup \{(s,t) \in \mathcal{T} : t \geq b, w(s) = 0\};$$

$$\mathcal{T}_b^{(21)} = \{(s,t) \in \mathcal{T} : s \leq t < \max\{a,b\}\}\,;$$

$$\mathcal{T}_b^{(23)}(\lambda,\mu) = \{(s,t) \in \mathcal{T} : \theta(t) = \theta(s)\}\,, \quad \lambda \neq 0,\ \mu,\ 2\mu;$$

$$\mathcal{T}_b^{(24)} = \{(s,t) \in \mathcal{T} : s \leq t < a\}\,.$$

El siguiente teorema da para las CEAs construidas una respuesta sobre la transición de la propiedad del problema de la existencia de la “unicidad del elemento nilpotente absoluto”.

Teorema 1.3.8.

(1) Las CEAs $E_i^{[s,t]}$*,* $i = 3,4,5,9,10,17,22,23,24$*, tienen un único elemento nilpotente absoluto* $(0,0)$ *para cualquier tiempo* $(s,t) \in \mathcal{T}$*.*

(2) Las CEAs $E_i^{[s,t]}$*,* $i = 0,1,2,16,19$*, tienen infinitos elementos nilpotentes absolutos para cualquier tiempo* $(s,t) \in \mathcal{T}$*.*

(3) Las CEAs $E_i^{[s,t]}$*,* $i = 6,7,8,11,12,13,14,15,18,20,21$*, tienen la transición de la propiedad de “unicidad del elemento nilpotente absoluto” con la duración de la propiedad establecida de la siguiente manera*

$$\mathcal{T}_{nil}^{(i)} = \{(s,t) \in \mathcal{T} : t < a\}, \quad a > 0, \quad i = 6,7,8,11,18;$$

$$\mathcal{T}_{nil}^{(12)} = \left\{(s,t) \in \mathcal{T} : g^2(t) \leq \frac{1}{h^2(s)}\right\};$$

$$\mathcal{T}_{nil}^{(13)} = \left\{(s,t) \in \mathcal{T} : s \leq t < a,\ \psi^2(s) \leq 1\right\};$$

$$\mathcal{T}_{nil}^{(14)} = \{(s,t) \in \mathcal{T} : \Phi(s)\psi(s) > 0\}\,;$$

$$\mathcal{T}_{nil}^{(15)} = \{(s,t) \in \mathcal{T} : s \leq t < a, \psi(s) > 0\}\,;$$

$$\mathcal{T}_{nil}^{(20)} = \{(s,t) \in \mathcal{T} : s \leq t < b\} \cup \left\{(s,t) \in \mathcal{T} : t \geq b,\ \frac{w(s)}{\Phi(s)} > 0\right\};$$

$$\mathcal{T}_{nil}^{(21)} = \{(s,t) \in \mathcal{T} : s \leq t < \min\{a,b\}\} \cup \{(s,t) \in \mathcal{T} : b \leq t < a,\ b < a,\ v(s) > 0\}\,.$$

El siguiente teorema da la dinámica temporal de los elementos idempotentes para las álgebras $E_i^{[s,t]}$, $i = 0,\ldots,24$.

Teorema 1.3.9.

(1) Las álgebras $E_i^{[s,t]}$*,* $i = 0, 1, 2$*, tienen un único idempotente* $(0, 0)$ *para cualquier tiempo* $(s, t) \in \mathcal{T}$*.*

(2) Las álgebras $E_i^{[s,t]}$*,* $i = 3, 10, 12, 14, 16, 22$*, tienen dos idempotentes* $(0, 0)$*,* $(x_i(s, t), y_i(s, t))$ *para cualquier tiempo* $(s, t) \in \mathcal{T}$*. Además se puede dar una fórmula explícita para cada* $x_i(s, t)$ *e* $y_i(s, t)$*.*

(3) Tenemos

$$\mathcal{I}d\left(E_4^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & si \;\; s \le t < b, \;\; \frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}; \\ \{0, z_3\}, & si \;\; s \le t < b, \;\; \frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, \; D(s,t) < 0; \\ \{0, z_3, (x_*, y_*)\}, & si \;\; s \le t < b, \frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, \; D(s,t) = 0; \\ \{0, z_3, (x_\pm, y_\pm)\}, & si \;\; s \le t < b, \;\; \frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, \; D(s,t) > 0, \end{cases}$$

donde $0 = (0, 0)$*,* $z_1 = (0, \frac{\Phi(t)}{\Phi(s)})$*,* $z_2 = (\frac{\Phi(t)}{\Phi(s)}, 0)$*,* $z_3 = (\frac{\Phi(t)}{\Phi(s)}, \frac{\Phi(t)}{\Phi(s)})$*, y* $D(s,t) = \frac{\Phi(t)}{\Phi(s)}\left(2\frac{\Psi(t)}{\Psi(s)} - \frac{\Phi(t)}{\Phi(s)}\right)$*.*

Las fórmulas explícitas de x_*, y_**,* $x_\pm$ *e* $y_\pm$ *son dadas más abajo. Los conjuntos* $\left\{\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}\right\}$*,* $\left\{\frac{\Phi(t)}{\Phi(s)} = 2\frac{\Psi(t)}{\Psi(s)}\right\}$ *son conjuntos críticos (frontera) de la transición de los elementos idempotentes.*

(4) Tenemos

$$\mathcal{I}d\left(E_5^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & si \;\; s \le t < b, \; \Phi(t) = \Phi(s); \\ \{0, z_3\}, & si \; s \le t < b, \Phi(t) \neq \Phi(s), \; D(s,t) < 0; \\ \{0, z_3, (x_*, y_*)\}, & si \;\; s \le t < b, \Phi(t) \neq \Phi(s), \; D(s,t) = 0; \\ \{0, z_3, (x_\pm, y_\pm)\}, & si \; s \le t < b, \; \Phi(t) \neq \Phi(s), \; D(s,t) > 0; \\ \{0, z_3\}, & si \;\; t \ge b, \end{cases}$$

donde z_i *son como en (3),* $D(s,t) = \frac{\Phi(t)}{\Phi(s)}\left(2 - \frac{\Phi(t)}{\Phi(s)}\right)$*.*

(5) *Las álgebras* $E_i^{[s,t]}$, $i = 6, 11, 13, 15, 19$, *tienen dos idempotentes para cualquier tiempo* (s,t) *con* $s \leq t < a$ *y un único elemento idempotente para el tiempo* (s,t) *con* $t \geq a$. *La línea crítica de la transición es* $t = a$.

(6) *Tenemos*

$$\mathcal{I}d\left(E_7^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & si\ s \leq t < a,\ \Psi(t) = \Psi(s); \\ \{0, z_3\}, & si\ s \leq t < a, \Psi(t) \neq \Psi(s),\ d(s,t) < 0; \\ \{0, z_3, (x_*, y_*)\}, & si\ \ s \leq t < a, \Psi(t) \neq \Psi(s),\ d(s,t) = 0; \\ \{0, z_3, (x_\pm, y_\pm)\}, & si\ \ s \leq t < a, \Psi(t) \neq \Psi(s),\ d(s,t) > 0; \\ 0, & si\ \ t \geq a, \end{cases}$$

donde $d(s,t) = \frac{2\Psi(t)}{\Psi(s)} - 1$. *Los conjuntos críticos son* $t = a$, $\Psi(t) = \Psi(s)$, $\Psi(s) = 2\Psi(t)$.

(7) *Para* $a \leq b$ *tenemos*

$$\mathcal{I}d\left(E_8^{[s,t]}\right) = \begin{cases} \{(0,0), (0,1), (1,0), (1,1)\}, & si\ s \leq t < a; \\ (0,0), & si\ \ t \geq a. \end{cases}$$

Para $a > b$ *tenemos*

$$\mathcal{I}d\left(E_8^{[s,t]}\right) = \begin{cases} \{(0,0), (0,1), (1,0), (1,1)\}, & si\ s \leq t < b; \\ \{(0,0), (1,1)\}, & si\ \ b \leq t < a; \\ (0,0), & si\ \ t \geq a. \end{cases}$$

Las líneas $t = a$ *y* $t = b$ *son críticas para la transición.*

(8) El álgebra $E_9^{[s,t]}$ *tiene tres elementos idempotentes* $(0,0),(1,0),(0,1)$ *para cualquier tiempo* (s,t) *con* $t = s+2\pi n$*; tiene tres elementos idempotentes* $(0,0),(-1,0),(0,-1)$ *para cualquier tiempo* (s,t) *con* $t = s+(2n+1)\pi$ *y al menos un idempotente para el tiempo* (s,t) *con* $t \neq s+\pi n$*,* $n \in \mathbb{Z}$*.*

(9) Tenemos

$$\mathcal{I}d\left(E_{17}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, & si \;\; D(s,t) < 0; \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, \frac{\Psi(s)}{\Psi(t)})\}, & si \;\; D(s,t) = 0; \\ \{(0,0), z_2, (x_\pm, y_\pm)\}, & si \;\; D(s,t) > 0, \end{cases}$$

donde $D(s,t) = \frac{4\Phi^2(t)\Psi(s)}{\Phi(s)\Psi^2(t)}(g(t)-g(s)) - 1.$

(10) Tenemos

$$\mathcal{I}d\left(E_{18}^{[s,t]}\right) = \begin{cases} \{(0,0),(1,0)\}, & si \;\; s \leq t < a,\ D(s,t) < 0; \\ \{(0,0),(1,0),(\frac{1}{2}, \frac{\Psi(s)}{\Psi(t)})\}, & si \;\; s \leq t < a,\ D(s,t) = 0; \\ \{(0,0),(1,0),(x_\pm, \frac{\Psi(s)}{\Psi(t)})\}, & si \;\; s \leq t < a,\ D(s,t) > 0; \\ \{(0,0),(\frac{h(t)\Psi(s)}{\Psi^2(t)}, \frac{\Psi(s)}{\Psi(t)})\}, & si \;\; t \geq a, \end{cases}$$

donde $D(s,t) = 1 - \frac{4\Psi(s)(h(t)-h(s))}{\Psi^2(t)}.$

(11) Tenemos

$$\mathcal{I}d\left(E_{20}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, & si \;\; s \leq t < b,\ D(s,t) < 0, \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, 1)\}, & si \;\; s \leq t < b,\ D(s,t) = 0, \\ \{(0,0), z_2, (x_\pm, 1)\}, & si \;\; s \leq t < a,\ D(s,t) > 0, \\ \{(0,0), z_2\}, & si \;\; t \geq b, \end{cases}$$

donde $D(s,t) = 1 - \frac{4\Phi^2(t)(v(t)-v(s))}{\Phi(s)}.$

(12) Tenemos

$$\mathcal{I}d\left(E_{21}^{[s,t]}\right)=\begin{cases}\{(0,0),(1,0)\}, & si\ \ s\leq t<\min\{a,b\},\\ & \qquad D(s,t)<0;\\ \{(0,0),(1,0),(\frac{1}{2},1)\}, & si\ \ s\leq t<\min\{a,b\},\\ & \qquad D(s,t)=0;\\ \{(0,0),(1,0),(x_{\pm},1)\}, & si\ \ s\leq t<\min\{a,b\},\\ & \qquad D(s,t)>0;\\ \{(0,0),(1,0)\}, & si\ \ b<a,\ b\leq t<a;\\ \{(0,0),(v(t),1)\}, & si\ \ b>a,\ a\leq t<b;\\ (0,0), & si\ \ t\geq\max\{a,b\},\end{cases}$$

donde $D(s,t)=1-4(v(t)-v(s))$.

(13) Tenemos

$$\mathcal{I}d\left(E_{23}^{[s,t]}(0,\mu)\right)=\begin{cases}\{(0,0),(0,1)\}, & si\ \ D(s,t)<0;\\ \{(0,0),(0,1),(\frac{\theta(s)}{\theta(t)},\frac{1}{2})\}, & si\ \ D(s,t)=0;\\ \{(0,0),(0,1),(\frac{\theta(s)}{\theta(t)},y_{\pm})\}, & si\ \ D(s,t)>0.\end{cases}$$

$$\mathcal{I}d\left(E_{23}^{[s,t]}(2\mu,\mu)\right)=\begin{cases}\{(0,0),(1,0)\}, & si\ \ D(s,t)<0;\\ \{(0,0),(1,0),(\frac{1}{2},\frac{\theta(s)}{\theta(t)})\}, & si\ \ D(s,t)=0;\\ \{(0,0),(1,0),(x_{\pm},\frac{\theta(s)}{\theta(t)})\}, & si\ \ D(s,t)>0,\end{cases}$$

donde $D(s,t)=1-\frac{4\theta(s)}{\theta(t)}\left(\frac{\theta(s)}{\theta(t)}-1\right)$.

(14) Tenemos

$$\mathcal{I}d\left(E_{24}^{[s,t]}\right)=\begin{cases}\{(0,0),(0,1),(1,0),(1,1)\}, & si\ (s,t)\in\mathcal{T}:s\leq t<a;\\ \{(0,0),\left(\frac{g(t)}{(1-g(t))^2+g^2(t)},\frac{1-g(t)}{(1-g(t))^2+g^2(t)}\right)\}, & si\ t\geq a.\end{cases}$$

En el Capítulo 2 se obtiene la clasificación de las álgebras de evolución reales de dimensión 2 y la dinámica dependiente del tiempo de dichas CEAs.

También serán construidas nuevas clases de CEAs de dimensión 2, que serán isomorfas a un álgebra de evolución dada, para algunos valores del tiempo. Además, serán definidos operadores (lineales) de Rota-Baxter sobre álgebras de evolución.

El siguiente teorema da la clasificación de las álgebras de evolución reales de dimensión 2.

Teorema 2.1.2. *Cualquier álgebra de evolución real E de dimensión 2 es isomorfa a una de las siguientes álgebras no isomorfas entre ellas:*

(i) $\dim E^2 = 1$*:*

$E_1 : e_1e_1 = e_1, \quad e_2e_2 = 0;$

$E_2 : e_1e_1 = e_1, \quad e_2e_2 = e_1;$

$E_3 : e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2;$

$E_4 : e_1e_1 = e_2, \quad e_2e_2 = 0;$

$E_5 : e_1e_1 = e_2, \quad e_2e_2 = -e_2;$

(ii) $\dim E^2 = 2$*:*

$E_6 : e_1e_1 = e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + e_2, \quad 1 - a_2a_3 \neq 0, a_2, a_3 \in \mathbb{R}$, *donde* $E_6(a_2, a_3) \cong E_6'(a_3, a_2)$*;*

$E_7 : e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2,$ *donde* $a_4 \in \mathbb{R}$.

El siguiente teorema da la dinámica dependiente del tiempo de las cadenas de álgebras de evolución $E_i^{s,t}, i = 0, \ldots, 24$, construidas en la Sección 1.2.

Teorema 2.2.2. $E_1^{[s,t]}$ *es isomorfa a* E_3 *para cualquier* $0 \leq s \leq t$.

$$E_2^{[s,t]} \simeq \begin{cases} E_3 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < b\} \ ; \\ \\ E_0 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \geq b\} \ . \end{cases}$$

$E_3^{[s,t]} \simeq E_2$ *para cualquier* $s, t \in \mathcal{T}$.

$$E_4^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}, \frac{\Phi(t)\Psi(s)-\Psi(t)\Phi(s)}{\Phi(t)\Psi(s)+\Psi(t)\Phi(s)}\right) \\ \quad \textit{para todo} \quad (s,t) \in \left\{(s,t) : \frac{\Phi(t)}{\Phi(s)} \neq -\frac{\Psi(t)}{\Psi(s)}\right\}; \\ E_7(0) \quad \textit{para todo} \quad (s,t) \in \left\{(s,t) : \frac{\Phi(t)}{\Phi(s)} = -\frac{\Psi(t)}{\Psi(s)}\right\} \ . \end{cases}$$

$$E_5^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}, \frac{\Phi(t)-\Phi(s)}{\Phi(t)+\Phi(s)}\right) \\ \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < b, \quad \Phi(t) \neq -\Phi(s)\}; \\ E_7(0) \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < b, \quad \Phi(t) = -\Phi(s)\} \ ; \\ E_2 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \geq b\} \ . \end{cases}$$

$$E_6^{[s,t]} \simeq \begin{cases} E_2 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < a\} \ ; \\ E_0 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$$E_7^{[s,t]} \simeq \begin{cases} E_6\left(\frac{\Psi(s)-\Psi(t)}{\Psi(s)+\Psi(t)}, \frac{\Psi(s)-\Psi(t)}{\Psi(s)+\Psi(t)}\right) \\ \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < a, \quad \Psi(t) \neq -\Psi(s)\}; \\ E_7(0) \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < a, \quad \Psi(t) = -\Psi(s)\} \ ; \\ E_3 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \geq a\} \ . \end{cases}$$

$$E_8^{[s,t]} \simeq \begin{cases} E_6(0,0) \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \leq t < \min\{a,b\}\} \ ; \\ E_3 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : a \leq t < b, \quad a < b\} \ ; \\ E_2 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : b \leq t < a, \quad b < a\} \ ; \\ E_0 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \geq \max\{a,b\}\} \ . \end{cases}$$

$$E_9^{[s,t]} \simeq \begin{cases} E_6(\tan(t-s), -\tan(t-s)) \\ \quad \textit{para todo} \quad (s,t) \in \left\{(s,t) : t \neq s + \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\right\}; \\ E_7(0) \quad \textit{para todo} \quad (s,t) \in \left\{(s,t) : t = s + \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\right\} \ . \end{cases}$$

$E_{10}^{[s,t]} \simeq E_2$ *para cualquier* $s, t \in \mathcal{T}$.

$$E_{11}^{[s,t]} \simeq \begin{cases} E_2 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a\} \; ; \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\} \; . \end{cases}$$

$$E_{12}^{[s,t]} \simeq \begin{cases} E_1 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a, \;\; h(s)g(s) = \pm 1\} \; ; \\ E_2 & \textit{para todo} \;\; (s,t) \in \big\{(s,t) : s \leq t < a, \;\; h^2(s)g^2(s) < 1\big\} ; \\ E_3 & \textit{para todo} \;\; (s,t) \in \big\{(s,t) : s \leq t < a, \;\; h^2(s)g^2(s) > 1\big\} \; . \end{cases}$$

$$E_{13}^{[s,t]} \simeq \begin{cases} E_1 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a, \;\; \psi(s) = \pm 1\} \; ; \\ E_2 & \textit{para todo} \;\; (s,t) \in \big\{(s,t) : s \leq t < a, \;\; \psi^2(s) < 1\big\} \; ; \\ E_3 & \textit{para todo} \;\; (s,t) \in \big\{(s,t) : s \leq t < a, \;\; \psi^2(s) > 1\big\} \; ; \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\} \; . \end{cases}$$

$$E_{14}^{[s,t]} \simeq \begin{cases} E_1 & \textit{para todo} \;\; (s,t) \in \{(s,t) : \;\; \psi(s) = 0\} \; ; \\ E_2 & \textit{para todo} \;\; (s,t) \in \{(s,t) : \;\; \Phi(s)\psi(s) > 0\} ; \\ E_5 & \textit{para todo} \;\; (s,t) \in \{(s,t) : \;\; \Phi(s)\psi(s) < 0\} \; . \end{cases}$$

$$E_{15}^{[s,t]} \simeq \begin{cases} E_1 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a, \;\; \psi(s) = 0\} \; ; \\ E_2 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a, \;\; \psi(s) > 0\} \; ; \\ E_5 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a, \;\; \psi(s) < 0\} \; ; \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\} \; . \end{cases}$$

$E_{16}^{[s,t]} \simeq E_1$ *para cualquier* $s, t \in \mathcal{T}$.

$$E_{17}^{[s,t]} \simeq E_6\big(\tfrac{\Phi^2(t)\psi(s)(g(t)-g(s))}{\Phi(s)\psi^2(t)},0\big) \text{ } \textit{para cualquier } s,t \in \mathcal{T}.$$

$$E_{18}^{[s,t]} \simeq \begin{cases} E_6\big(\frac{\psi(s)(h(t)-h(s))}{\psi^2(t)},0\big) \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \le t < a\} \;; \\ \\ E_1 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \ge a\} \;. \end{cases}$$

$$E_{19}^{[s,t]} \simeq \begin{cases} E_1 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \le t < b\} \;; \\ \\ E_0 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \ge b\} \;. \end{cases}$$

$$E_{20}^{[s,t]} \simeq \begin{cases} E_6(\frac{\Phi^2(t)(v(t)-v(s))}{\Phi(s)},0) \quad \textit{para todo} \quad (s,t) \in \{(s,t) : s \le t < b\} \;; \\ \\ E_1 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \ge b, \quad w(s) = 0\} \;; \\ \\ E_2 \quad \textit{para todo} \quad (s,t) \in \Big\{(s,t) : t \ge b, \frac{\Phi^2(t)w(s)}{\Phi(s)} > 0\Big\} \;; \\ \\ E_5 \quad \textit{para todo} \quad (s,t) \in \Big\{(s,t) : t \ge b, \frac{\Phi^2(t)w(s)}{\Phi(s)} < 0\Big\} \;. \end{cases}$$

$$E_{21}^{[s,t]} \simeq \begin{cases} E_6(v(t)-v(s),0) \\ \qquad \textit{para todo} \quad (s,t) \in \{(s,t) : s \le t < \min\{a,b\}\}\,; \\ E_1 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : a \le t < b, \quad a < b\} \cup \\ \qquad\qquad \{(s,t) : b \le t < a, \quad a > b, \quad v(s) = 0\}\,; \\ E_2 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : b \le t < a, \quad a > b, \quad v(s) > 0\}\,; \\ \\ E_5 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : b \le t < a, \quad a > b, \quad v(s) < 0\}\,; \\ \\ E_0 \quad \textit{para todo} \quad (s,t) \in \{(s,t) : t \ge \max\{a,b\}\}\,. \end{cases}$$

$$E_{22}^{[s,t]} \simeq E_2 \text{ } \textit{para cualquier } s,t \in \mathcal{T}.$$

$$E_{23}^{[s,t]}(\lambda,\mu) \simeq$$

$$\begin{cases} E_6(\frac{2\theta(s)(\theta(s)-\theta(t))}{(\theta(s)+\theta(t))^2},0) \;\; \textit{para todo} \;\; \lambda = 2\mu \;\; (s,t) \in \{(s,t) : s \leq t < a\} \;; \\[2ex] E_6(\frac{\xi\zeta}{(1-\zeta)^2}, \frac{(1-\xi)(1-\zeta)}{\xi^2}) \;\; \textit{para todo} \;\; \lambda \neq 2\mu \\ (s,t) \in \left\{(s,t) : \theta(t) \neq \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\right\}; \\[1ex] E_7\left(\frac{1-\zeta}{\sqrt[3]{\zeta^2}}\right) \;\; \textit{para todo} \;\; \lambda \neq 2\mu, \; \lambda \neq 0 \\ (s,t) \in \left\{(s,t) : \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \theta(s), \;\; \theta(t) \neq \frac{2\mu-\lambda}{\lambda}\theta(s)\right\}; \\[1ex] E_7\left(\frac{\xi}{\sqrt[3]{(1-\xi)^2}}\right) \;\; \textit{para todo} \;\; \lambda \neq 2\mu \\ \;(s,t) \in \left\{(s,t) : \theta(t) \neq \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) \neq \theta(s), \;\; \theta(t) = \frac{2\mu-\lambda}{\lambda}\theta(s)\right\}; \\[1ex] E_7(0) \;\; \textit{para todo} \;\; \lambda \neq 2\mu, \\ (s,t) \in \left\{(s,t) : \theta(t) = \frac{2\lambda}{2\mu-\lambda}\theta(s), \;\; \theta(t) = \frac{2\mu-\lambda}{\lambda}\theta(s)\right\}, \end{cases}$$

donde $\xi = 1 - \frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1 - \frac{\theta(t)}{\theta(s)}\right)$, $\zeta = \frac{\lambda}{2(\lambda-\mu)}\left(1 - \frac{\theta(t)}{\theta(s)}\right)$.

$$E_{24}^{[s,t]} \simeq \begin{cases} E_6(0,0) \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : s \leq t < a\} \;; \\[2ex] E_2 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\} \;. \end{cases}$$

El siguiente teorema da la dinámica dependiente del tiempo de las nuevas CEAs construidas en la Sección 2.2.

Teorema 2.2.6. *Para las siguientes CEAs se verifica:*

$$E_{25}^{[s,t]} \simeq \begin{cases} E_1 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t, \;\; \rho(s) = 0\}, \\[2ex] E_2 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t, \;\; \rho(s) \neq 0\}. \end{cases}$$

$$E_{26}^{[s,t]} \simeq \begin{cases} E_1 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t < a, \;\; \sigma(s) = 0\}, \\[2ex] E_2 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t < a, \;\; \sigma(s) \neq 0\}, \\[2ex] E_0 \;\; \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\} \;. \end{cases}$$

$E_{27}^{[s,t]}$ *es isomorfa a* E_1 *para cualquier* $(s,t) \in \mathcal{T}$.

$$E_{28}^{[s,t]} \simeq \begin{cases} E_1 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t < a\}\,, \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t \geq a\}\,. \end{cases}$$

$$E_{29}^{[s,t]} \simeq \begin{cases} E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t \leq C\}\,, \\ E_4 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t > C\}\,. \end{cases}$$

$$E_{30}^{[s,t]} \simeq \begin{cases} E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < t \leq C\}\,, \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t > C, \;\; \rho(s) = 0\}\,, \\ E_4 & \textit{para todo} \;\; (s,t) \in \{(s,t) : t > C, \;\; \rho(s) \neq 0\} \;\; . \end{cases}$$

$$E_{31}^{[s,t]} \simeq \begin{cases} E_4 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < C\}\,, \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \geq C\}\,. \end{cases}$$

$$E_{32}^{[s,t]} \simeq \begin{cases} E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < C, \;\; \sigma(t) = 0\}\,, \\ E_4 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s < C, \;\; \sigma(t) \neq 0\}\,, \\ E_0 & \textit{para todo} \;\; (s,t) \in \{(s,t) : s \geq C\} \;\; . \end{cases}$$

Así, hemos demostrado que existen CEAs que para algunos valores de tiempo son isomorfas a E_4.

Definición 2.3.1. Sea $\mathcal{F}$ un cuerpo. Un *operador de Rota-Baxter* de peso $\lambda \in \mathcal{F}$ sobre un álgebra de evolución $(E, \cdot)$ sobre $\mathcal{F}$ es una aplicación lineal $P \colon E \to E$ satisfaciendo

$$P(x) \cdot P(y) = P(x \cdot P(y) + P(x) \cdot y + \lambda \cdot x \cdot y), \quad \text{para todo } x, y \in E.$$

El siguiente teorema da todas las formas de matrices de los operadores de Rota-Baxter sobre las álgebras de evolución complejas de dimensión 2.

Teorema 2.3.2. *Se dan en la siguiente tabla las matrices de los operadores de Rota-Baxter sobre las álgebras de evolución complejas de dimensión 2, donde los parámetros* $a, b, c, d, x, y \in \mathbb{C}$.

Álgebra de Evolución	*Matrices de RBOs de peso* 0 *del álgebra de evolución*
E_1	$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$
E_2	$\begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}$
E_3	$\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}, \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$
E_4	$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & \frac{a}{2} \end{pmatrix}$
$E_5(\frac{1}{4}, 0)$	$\begin{pmatrix} a & \frac{a}{2} \\ -2a & -a \end{pmatrix}$
$E_5(0, \frac{1}{4})$	$\begin{pmatrix} a & 2a \\ -\frac{a}{2} & -a \end{pmatrix}$
$E_5\big(\frac{(2a-b)b}{3a^2}, \frac{-a^2+2ab}{3b^2}\big)$ $a \neq 2b, b \neq 2a,\ a \neq -b$ $a \neq 0, b \neq 0$	$\begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}$
$E_6(-\frac{3b^2}{4c^2})$ $b \neq 0,\ c \neq 0$	$\begin{pmatrix} \frac{b^2}{2c} & b \\ c & \frac{-b^2}{2c} \end{pmatrix}$ *donde los parámetros* b, c *son soluciones del siguiente sistema de ecuaciones* $\begin{cases} \frac{3b^6}{c} + 16b^3c^2 + 16c^5 & = 0, \\ \frac{b^3}{c} + 4c^2 & = 0. \end{cases}$

Álgebra de Evolución	*Matrices de RBOs de peso 1 del álgebra de evolución*
E_1	$\begin{pmatrix} -1 & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$
E_2	$\begin{pmatrix} 0 & 0 \\ c & ic \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c & -ic \end{pmatrix},$ $\begin{pmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix},$ $\begin{pmatrix} -1 & 0 \\ c & -1+ic \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ c & -1-ic \end{pmatrix}$
E_3	$\begin{pmatrix} -1+b & b \\ -b & -1-b \end{pmatrix}, \begin{pmatrix} -1-b & b \\ b & -1-b \end{pmatrix},$ $\begin{pmatrix} b & b \\ -b & -b \end{pmatrix}, \quad \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}$
E_4	$\begin{pmatrix} a & b \\ 0 & \frac{a^2}{1+2a} \end{pmatrix}, \ a \neq -\frac{1}{2}$
$E_5(0,y)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_{1,2} & -1 \end{pmatrix},$ *donde* $c_{1,2} = \frac{-1\pm\sqrt{1-4y}}{2}$.
$E_5(0,y)$ $y \neq 0$	$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ c_{1,2} & 0 \end{pmatrix},$ *donde* $c_{1,2} = \frac{1\pm\sqrt{1-4y}}{2}, \ \ c_{1,2} \neq 0$

$E_5(0,y)$ $y\neq 0,\ y\neq \frac{1}{4}$	$\begin{pmatrix} \frac{1-4y+\sqrt{1-4y}}{8y-2} & -\frac{1}{\sqrt{1-4y}} \\ \frac{y}{\sqrt{1-4y}} & \frac{1-4y-\sqrt{1-4y}}{8y-2} \end{pmatrix}$, $\begin{pmatrix} \frac{1-4y-\sqrt{1-4y}}{8y-2} & \frac{1}{\sqrt{1-4y}} \\ -\frac{y}{\sqrt{1-4y}} & \frac{1-4y+\sqrt{1-4y}}{8y-2} \end{pmatrix}$
$E_5(x,0)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & b_{1,2} \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -b_{1,2} \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, *donde* $b_{1,2}=\frac{1\pm\sqrt{1-4x}}{2}$, $b_{1,2}\neq 0$.
$E_5(x,0)$ $x\neq 0,\ x\neq \frac{1}{4}$	$\begin{pmatrix} \frac{1-4x+\sqrt{1-4x}}{8x-2} & -\frac{x}{\sqrt{1-4x}} \\ \frac{1}{\sqrt{1-4x}} & \frac{1-4x-\sqrt{1-4x}}{8x-2} \end{pmatrix}$, $\begin{pmatrix} \frac{1-4x-\sqrt{1-4x}}{8x-2} & \frac{x}{\sqrt{1-4x}} \\ -\frac{1}{\sqrt{1-4x}} & \frac{1-4x+\sqrt{1-4x}}{8x-2} \end{pmatrix}$
$E_5(0,0)$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$
$E_5(x,y)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$E_5(x,1-x)$ $x\neq \frac{1\pm i\sqrt{3}}{2}$	$\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}$, $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}$

$E_5(x,y)$ $x=\frac{d(1+d)(c+2cd-d(1+d))}{c^2(1+3d+3d^2)}$, $y=\frac{c(1-c+2d)}{c^2(1+3d+3d^2)}$	$\begin{pmatrix} -1-d & -\frac{d(1+d)}{c} \\ c & d \end{pmatrix}$, $d\neq 0,\ d\neq -1,\ d\neq -\frac{3\pm i\sqrt{3}}{6}$, $c\neq 0,\ c\neq \frac{d(1+d)}{1+2d},\ c\neq 1+2d.$
$E_6(0)$	$\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & 0 \\ 0 & \frac{-3+i\sqrt{3}}{6} \end{pmatrix},$ $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix},$ $\begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{\sqrt[6]{-1}}{\sqrt{3}} \\ \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}, \begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{\sqrt[6]{-1}}{\sqrt{3}} \\ -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix},$ $\begin{pmatrix} \frac{-3+i\sqrt{3}}{6} & \frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3-i\sqrt{3}}{6} \end{pmatrix}, \begin{pmatrix} \frac{-3-i\sqrt{3}}{6} & -\frac{(\sqrt[6]{-1})^5}{\sqrt{3}} \\ \frac{\sqrt[6]{-1}}{\sqrt{3}} & \frac{-3+i\sqrt{3}}{6} \end{pmatrix}.$
$E_6(x)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$E_6(\frac{-b^3-c^3}{bc^2})$ $-b^3-c^3\neq 0$ $b\neq 0,\ c\neq 0$	$\begin{pmatrix} \frac{b^2-c}{2c} & b \\ c & \frac{-b^2-c}{2c} \end{pmatrix}$ *donde los parámetros* b,c *son soluciones del siguiente* *sistema de ecuaciones* $\begin{cases} \frac{b^6+5b^3c^3+4c^6}{c} &= \frac{c(b^3+c^3)}{b}, \\ \frac{b^4}{c}+4bc^2 &= c. \end{cases}$

En el Capítulo 3 estudiamos el álgebra de evolución correspondiente a una población bisexual. Construimos cadenas de álgebras de evolución de la población del “pollo” y estudiamos su dinámica dependiente del tiempo.

La siguiente proposición da la dinámica temporal de las dos siguientes CEACP, (3.2.4) y (3.2.5).

$$\mathcal{E}_1^{[s,t]}, \quad \text{con} \quad hr = rh = \frac{1}{2}\Phi(s)(\Phi(t)h + \alpha(t)r),\, h^2 = r^2 = 0;$$

$$\mathcal{E}_2^{[s,t]}, \quad \text{con} \quad hr = rh = \frac{1}{2}\begin{cases} h + \beta(t)r, & \text{si } s < \min\{\theta, t\}, \\ 0, & \text{si } \theta \leq s, \end{cases} \qquad h^2 = r^2 = 0.$$

Proposición 3.3.2. *Tenemos*

$$\mathcal{E}_1^{[s,t]} \cong \begin{cases} E_1, & si \;\; t \in \{t : \alpha(t) = 0\}, \\ E_2, & si \;\; t \in \{t : \alpha(t) \neq 0\}. \end{cases}$$

$$\mathcal{E}_2^{[s,t]} \cong \begin{cases} E_1, & si \;\; (s,t) \in \{(s,t) : \beta(t) = 0, s < \min\{\theta, t\}\}, \\ E_2, & si \;\; (s,t) \in \{(s,t) : \beta(t) \neq 0, s < \min\{\theta, t\}\}, \\ E_0, & si \;\; (s,t) \in \{(s,t) : s \geq 0\}, \end{cases}$$

donde E_0 es el álgebra con multiplicación cero.

Teorema 3.3.5. *Las matrices*

$$M^{[s,t]} = \frac{1}{2\phi(s)} \begin{pmatrix} \phi(t) + \psi(t) & \frac{\phi(t) - \psi(t)}{c(t)} & 2\alpha(t) \\ c(t)(\phi(t) + \psi(t)) & \phi(t) - \psi(t) & 2c(t)\alpha(t) \end{pmatrix} \tag{3.3.2}$$

y

$$M^{[s,t]} = \frac{1}{\phi(s)} \begin{pmatrix} \phi(t) & \psi(t) & \alpha(t) \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3.3}$$

generan un CEACP $\mathcal{E}^{[s,t]}$ con $\dim\left(\left(\mathcal{E}^{[s,t]}\right)^2\right) = 1$ para cualquier (s,t).

Bibliography

[1] Abraham, V.M. Linearizing quadratic transformations in genetic algebras. *Proc. London Math. Soc. (3)* **40** (1980), no. 2, 346–363.

[2] Baur, E. Das Wesen und die Erblichkeitsverhältnisse der "Varietates albomarginatae hort." von Pelargonium zonale. *Zeitschr f ind Abst u Vererbungsl* **1** (1909), 330–351.

[3] Baxter G. An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math.* **10** (1960), 731–742.

[4] Cabrera Casado, Y., Siles Molina, M., Velasco, M.V. Evolution algebras of arbitrary dimension and their decompositions. *Linear Algebra Appl.* **495** (2016), 122–162.

[5] Casas J.M., Ladra M., Omirov B.A., Rozikov U.A. On nilpotent index and dibaricity of evolution algebras. *Linear Algebra Appl.* **439** (2013), no. 1, 90–105.

[6] Casas J.M., Ladra M., Omirov B.A., Rozikov U.A. On evolution algebras. *Algebra Colloquium* **21** (2014), no. 2, 331–342.

[7] Casas J.M., Ladra M., Rozikov U.A. A chain of evolution algebras. *Linear Algebra Appl.* **435** (2011), no. 4, 852–870.

[8] Correns, C. Vererbungsversuche mit blass(gelb)grünen und buntblättrigen Sippen bei Mirabilis jalapa, Urtica pilulifera und Lunaria annua. *Zeitschr f ind Abst u Vererbungsl* **1** (1909), 291–329.

[9] Dzhumadil'daev A., Omirov B.A., Rozikov U.A. On a class of evolution algebras of "chicken" population. *Internat. J. Math.* **25** (2014), no. 8, 1450073 (19 pages).

[10] Dzhumadil'daev A., Omirov B.A., Rozikov U.A. Constrained evolution algebras and dynamical systems of a bisexual population. *Linear Algebra Appl.* **496** (2016), 351–380.

[11] Etherington I.M.H. Genetic algebras. *Proc. Roy. Soc. Edinburgh* **59** (1939), 242–258.

[12] Etherington I.M.H. Duplication of linear algebras. *Proc. Edinburgh Math. Soc. (2)* **6** (1941), 222–230.

[13] Etherington I.M.H. Non-associative algebra and the symbolism of genetics. *Proc. Roy. Soc. Edinburgh. Sect. B.* **61** (1941), 24–42.

[14] Galperin G., Zemlyakov A. *Mathematical billiards.* Nauka, Moscow, 1990 (in Russian).

[15] Ganikhodjaev N.N. On stochastic processes generated by quadric operators. *J. Theoret. Probab.* **4** (1991), no. 4, 639–653.

[16] Ganikhodjaev N.N., Akin H., Mukhamedov F.M. On the ergodic principle for Markov and quadratic stochastic processes and its relations. *Linear Algebra Appl.* **416** (2006), no. 2-3, 730–741.

[17] Georgii H.-O. *Gibbs measures and phase transitions.* De Gruyter, Studies in Mathematics 9, Walter de Gruyter & Co., Berlin, 1988.

[18] Glivenkov, V. Algèbre Mendelienne. *C. R. (Doklady) Acad. Sci. URSS* **4** (1936), 385–386.

[19] Gonshor, H. Contributions to genetic algebras. II. *Proc. Edinburgh Math. Soc (2)* **18** (1973), 273–279.

[20] Hänggi P., Thomas H. Time evolution, correlations, and linear response of non-Markov processes. *Zeitschrift Phys. B.* **26** (1977), 85–92.

[21] Heuch, I. Sequences in genetic algebras for overlapping generations. *Proc. Edinburgh Math. Soc. (2)* **18** (1972), 19–29.

[22] Holgate, P. Sequences of powers in genetic algebras. *J. London Math. Soc.* **42** (1967), 489–496.

[23] Holgate, P. Selfing in genetic algebras. *J. Math. Biology* **6** (1978), 197–206.

[24] Kolmogorov A.N. On analytic methods in probability theory. *Uspekhi Mat. Nauk.* **5** (1938), 5–41 (in Russian), English transl., Selected works of A.N. Kolmogorov, Vol. II, Kluwer, Dordrecht 1992, article 9.

[25] Kostitzin, V.A. Sur les coefficients mendéliens d'hérédité. *C. R. Acad. Sci., Paris* **206** (1938), 883–885.

[26] Labra A., Ladra M., Rozikov U.A. An evolution algebra in population genetics. *Linear Algebra Appl.* **457** (2014), 348–362.

[27] Ladra M., Omirov B.A., Rozikov U.A. Dibaric and evolution algebras in biology. *Lobachevskii J. Math.* **35** (2014), no. 3, 198–210.

[28] Ladra M., Rozikov U. A. Evolution algebra of a bisexual population. *J. Algebra* **378** (2013), 153–172.

[29] Ladra M., Rozikov U.A. Flow of finite-dimensional algebras. *J. Algebra* **470** (2017), 263–288.

[30] Ladra M., Rozikov U.A. Construction of flows of finite-dimensional algebras. *J. Algebra* **492** (2017), 475–489.

[31] Lyubich Y.I. *Mathematical structures in population genetics.* Springer-Verlag, Berlin, 1992.

[32] Mendel, G. *Experiments in plant-hybridization.* (1865). The Electronic Scholarly Publishing Project `http://www.esp.org/foundations/genetics/classical/gm-65.pdf`

[33] Omirov B.A., Rozikov U.A. On subalgebras of an evolution algebra of a "chicken" population. *J. Algebra Relat. Topics* **5** (2017), no. 2, 13–24.

[34] Omirov B.A., Rozikov U.A., Tulenbayev K.M. On real chains of evolution algebras. *Linear Multilinear Algebra* **63** (2015), no. 3, 586–600.

[35] Reed M.L. Algebraic structure of genetic inheritance. *Bull. Amer. Math. Soc. (N.S.)* **34** (1997), no. 2, 107–130.

[36] Reiersöl, O. Genetic algebras studied recursively and by means of differential operators. *Math. Scand.* **10** (1962), 25–44.

[37] Rota G.-C. *Baxter operators, an introduction.* Gian-Carlo Rota on combinatorics, pp. 504–512, Contemp. Mathematicians, Birkhäuser Boston, Boston, 1995.

[38] Rozikov U.A. Evolution operators and algebras of sex linked inheritance. *Asia Pacific Math. Newsletter* **3** (2013), no. 1, 6–11.

[39] Rozikov U.A., Tian J.P. Evolution algebras generated by Gibbs measures. *Lobachevskii J. Math.* **32** (2011), no. 4, 270–277.

[40] Rozikov U.A., Varro R. Dynamical systems generated by a gonosomal evolution operator. *Discontinuity, Nonlinearity, and Complexity* **5** (2016), no. 2, 173–185.

[41] Sarymsakov T.A., Ganikhodjaev N.N. Analytic methods in the theory of quadric stochastic operators. *J. Theoret. Probab.* **3** (1990), no. 1, 51–70.

[42] Schafer, R.D. *An introduction to nonassociative algebras.* Academic Press, New York, 1966.

[43] Serebrowsky, A. On the properties of the Mendelian equations. *C. R. (Doklady) Acad. Sci. URSS* **2** (1934), 33–39 (in Russian).

[44] Tian J.P. *Evolution algebras and their applications.* Lecture Notes in Mathematics 1921, Springer-Verlag, Berlin, 2008.

[45] Tian J.P., Vojtechovsky P. Mathematical concepts of evolution algebras in non-Mendelian genetics. *Quasigroups Related Systems* **14** (2006), 111–122.

[46] Wörz-Busekros A. *Algebras in genetics.* Lecture Notes in Biomathematics 36, Springer-Verlag, Berlin-New York, 1980.