On additivity and implementation in games with coalition structure

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
Departamento de Estatística e Investigación Operativa

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obtuvo la máxima calificación de SOBRESALIENTE CUM LAUDE, siendo el Dr. D. Gustavo Bergantiños Cid su director.

Siguiendo las indicaciones de la Comisión de Doutoramento, esta Tesis incluye, bajo el epígrafe “Summary (in Spanish)”, un resumen en castellano de 3.700 palabras.

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To Ulyana and Sasha

To my parents

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Summary

"On additivity and implementation in games with coalition structure"

This dissertation has three main parts. The first part is devoted to transferable utility games with coalition structure. A levels bidding mechanism is presented. This mechanism is a generalization of the bidding mechanism by Pérez-Castrillo and Wettstein (2001). The levels bidding mechanism implements the Owen value (Owen, 1977) and the *levels structure value* (Winter, 1989).

The second part is devoted to non-transferable utility games with coalition structure. The NTU consistent coalitional value is defined for this class of games. The NTU consistent coalitional value generalizes in a natural way both the Owen value to non-transferable utility games and the consistent value (Maschler and Owen, 1989, 1992) for games with coalition structure. In particular, two characterizations are proposed. One of them by means of consistency and the other one by means of balanced contributions.
A *random order value* is also studied. This value arises as the generalization of expected average of marginal contributions which also characterizes both the Shapley value and the Owen value. However, this new value is neither consistent nor satisfies average contributions.

Furthermore, the bargaining mechanism by Hart and Mas-Colell (1996) is generalized for games with coalition structure. It is proved that the NTU consistent coalitional value arises in equilibrium for a wide class of games. A slight modification of this mechanism is also studied. However, the value that arises in equilibrium does not generalize the Owen value for transferable utility games.

In the third part, rules that satisfy the different types of additivity are characterized. The studied games are allocation, bankruptcy, surplus and loss problems, both in the discrete and the continuous case. When additivity in the claims is considered, the rules that arise are the proportional and its weighted versions. When we consider additivity in both the claims and the estate, the arising rules are the equal-sharing rule (Moulin, 1987) and equal-loss (Herrero, Maschler and Villar, 1999) as well as their weighted versions.

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0.1 Acknowledgements

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0.2 Introduction

A game describes a conflict situation among a number of agents or players. Even though players are assumed to have independent interests, they can benefit from cooperation. In this sense, games can be cooperative or non-cooperative.

In non-cooperative games, players are utility maximizers; and a non-cooperative game is the set of rules that describe it. We study subgame perfect Nash equilibria. In a Nash equilibrium, players act in such a way that no one can benefic from deviating. In a subgame perfect equilibrium, players play in this way in every subgame.

In cooperative games, binding agreements are allowed. This means that players may form coalitions and act cooperatively in order to maximize their final payoff. A cooperative game can be described, then, by a characteristic correspondence. This correspondence assigns to each coalition the set of feasible payoffs that can be achieved by its members by acting cooperatively.

When this cooperation is carried out, the question is how the benefit shall be distributed among the players. This problem has been studied from different approaches. The aim is to define a *value* which gives a "fair" (or at least "reasonable") allocation for each problem. This allocation must take into account the contribution of each player to the game.

Within cooperative games, transferable utility (TU) games have been deeply studied. In TU games, utility is freely transferable among members of a coalition. A widely studied value for TU games is the Shapley value (presented by Shapley in 1953). A wider class of cooperative games is the class of non-transferable utility (NTU) games. Some generalizations of the Shapley value to NTU games are, for example, the NTU Shapley value (Aumann, 1985), the Harsanyi value (Harsanyi, 1963) and the consistent value (Maschler and Owen, 1989, 1992).

In his original paper, Shapley (1953) characterizes his value by means of efficiency, symmetry, null player and additivity. Efficiency means that cooperation among all the players occurs. Symmetry assures that symmetric players receive the same. Null player means that players that contribute nothing receive nothing. Finally, additivity means that if we divide the cooperative game as the sum of another two games, the value of the game coincides with the sum of the values of the new games.

Other characterizations of the Shapley value are given, for example, by Myerson (1980) and Hart and Mas-Colell (1989). Myerson (1980) proves that the Shapley value is the only value satisfying efficiency and balanced contributions. Balanced contributions means that the gain or loss that a player i obtains when another player j leaves the game is the same as the gain or loss that player j obtains when player i leaves the game. Hart and Mas-Colell (1989) characterize the Shapley value by using a consistency property. Roughly, consistency means that the payoff does not change if we play a reduced game where some players are removed from the game.

As for NTU games, Hart and Mas-Colell (1996) show that the consistent value is characterized by efficiency and a property that generalizes balanced

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contributions.

Once a value has been established, the implementation for this value aims to state a non-cooperative game such that players, by behaving strategically, get as final outcome the one proposed by the value.

In this context, we say that a mechanism implements the Shapley value (or any other) if two properties are satisfied. First, there must be some kind of equilibrium such that their final payoff is the Shapley value. Second, every equilibrium must have as final payoff the Shapley value. The first property is needed since, even if it is proved that the Shapley value arises in each equilibrium, it may occur that the non-cooperative game has no equilibria.

Implementation for the Shapley value in TU games has been studied by several authors. For example, Gul (1989), Hart and Moore (1990), Winter (1994), Hart and Mas-Colell (1996), Evans (1996), Dasgupta and Chiu (1998), Pérez-Castrillo and Wettstein (2001) or Mutuswami, Pérez-Castrillo and Wettstein (2002).

Hart and Mas-Colell (1996) design a mechanism with multilateral meetings in which a randomly chosen player proposes a feasible payoff. If all the other players accept the offer, the mechanism finishes with this payoff. Otherwise, the mechanism is repeated with probability $\rho \in [0, 1)$, and with probability $1-\rho$ the proposer leaves the game. For monotonic games, this mechanism implements the Shapley value in stationary subgame perfect Nash equilibria. Furthermore, for monotonic NTU games, the consistent value arises in stationary subgame perfect Nash equilibria when ρ approaches 1.

Pérez-Castrillo and Wettstein (2001) design a bidding mechanism which coincides with Hart and Mas-Colell's with $\rho = 0$ (*i.e.* the proposer drops out for sure when his offer is rejected), after a previous stage where players bid for the right to be the proposer. This mechanism implements the Shapley value for subgame perfect Nash equilibria.

Frequently, players do not act individually. Instead, they are exogenously divided into *a priori* coalitions. This division, or coalition structure, may be due, for example, to political affinities or geographical reasons. Owen (1977) generalizes the Shapley value to TU games with coalition structure. The Owen value also satisfies additivity. Moreover, Calvo, Lasaga and Winter (1996) characterize the Owen value as the only value satisfying efficiency, balanced contributions between the coalitions and balanced contributions between the players in the same coalition.

Furthermore, Winter (1991) generalizes the Harsanyi value to NTU games with coalition structure.

When there exist additional subdivisions of the players inside the coalition structure, we say that the players are divided into a levels structure. A generalization of the Shapley value for TU games with levels structure is suggested by Owen (1977) and characterized by Winter (1989).

Allocation problems describe situations in which an *estate* is due to be divided among some players who have *claims* on it. Allocation problems can be considered as a special case of cooperative games, in the sense that it is possible to assign a characteristic function to each allocation problem. In allocation problems, the way to divide the estate among the agents is called a *rule*.

Again, we can consider a property of additivity. A rule can be additive in the estate or additive in the estate and the claims. However, these additivities are unrelated to additivity in the corresponding TU game in characteristic function form.

In Part I, we deal with TU games. In Chapter 1, the non-cooperative game by Pérez-Castrillo and Wettstein (2001) is generalized so that the Owen value is implemented. Chapter 1 is based on Vidal-Puga and Bergantiños (2002a).

In Chapter 2, the non-cooperative game is extended to several levels of cooperation. Chapter 2 is based on Vidal-Puga (2002b).

NTU games are studied in Part II. The consistent value is generalized to NTU games with coalition structure in Chapter 3. The new value, called the consistent coalitional value, is generalized by using the properties of consistency and balanced contributions. Chapter 3 is based on Vidal-Puga and Bergantiños (2002b).

In Chapter 4, the non-cooperative game by Hart and Mas-Colell (1996) is generalized so that the consistent coalitional value arises in equilibrium. Chapter 4 is based on Vidal-Puga (2002a).

Finally, in Part III, we study allocation problems. In Chapter 5, the allocation rules that satisfy additivity are characterized. Chapter 5 is based on Bergantiños and Vidal-Puga (2002).

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Part I

On TU games with coalition structure

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Chapter 1

Implementation of the Owen value

1.1 Introduction

The Shapley value (Shapley, 1953) is one of the most important solution concepts in cooperative game theory. Since the paper of Shapley was published, many authors have studied this value. For instance, Myerson (1980) characterizes it using the property of balanced contributions. Hart and Mas-Colell (1989) do so by using the potential and the property of consistency. Moreover, the Shapley value has been used successfully in cost allocation problems and the analysis of political situations.

Another important aspect of a normative solution is the non-cooperative foundation, or implementation. The idea is to prove that agents can reach the normative solution through a non-cooperative behavior. Indeed, given the cooperative game, a non-cooperative game is associated in such a way that the outcome of some kind of equilibrium of the non-cooperative game coincides with the normative solution of the cooperative game. There are several implementations of the Shapley value (for instance, Gul (1989), Hart and Mas-Colell (1996), and Pérez-Castrillo and Wettstein (2001)).

Owen (1977) studies situations in which players are divided into groups. In this context Owen introduces the Owen value, which is a generalization of the Shapley value. Later, several authors extend other results of the Shapley value to the Owen value. Calvo, Lasaga, and Winter (1996) extend the results about balanced contributions, and Winter (1992) studies the results about the potential and consistency. Also, the Owen value is used in cost allocation problems (Vázquez-Brage, van den Nouweland, and Garcia-Jurado, 1997) and political situations (Carreras and Owen, 1988).

Nevertheless, the non-cooperative foundation of the Shapley value has not been extended to the Owen value. In this Chapter we extend the results obtained by Pérez-Castrillo and Wettstein (2001) and we implement the Owen value.

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Given a cooperative game, Pérez-Castrillo and Wettstein (2001) define a non-cooperative game called the "bidding mechanism". They prove that the payoff of all subgame perfect Nash equilibria outcomes coincides with the Shapley value of the cooperative game. In the bidding mechanism there are three stages. In Stage 1, players bid to become the proposer, where bids can be negative or positive. The player with the highest "net bid" (the difference between the sum of the bids he makes to the others minus the sum of the bids the others make to him) becomes the proposer and pays the bid to the other players. In Stage 2, the proposer makes an offer to the other players. In Stage 3, the other players answer the offer. If everybody accepts the offer, the grand coalition is formed, the proposer pays to the other players according to the offer, and obtains all the resources of the grand coalition. If any player rejects the offer, then the proposer takes his own resources and is out of the game. The rest of the players continue to play the bidding mechanism among themselves.

Given a cooperative game, we define in this Chapter a mechanism[1], the "coalitional bidding mechanism", and we prove that the payoff of all subgame perfect Nash equilibria outcomes coincides with the Owen value of the cooperative game.

The coalitional bidding mechanism has two rounds. Assume that the coalition structure is $\{C_1, ..., C_p\}$. In Round 1, the members of any coalition C_q, independently of the other coalitions, play the bidding mechanism among themselves. Then for any coalition C_q we can find a player, called the representative, who obtains the resources of coalition C_q, or a subcoalition of C_q, if some player is removed because his offer was rejected. In Round 2, the representatives play the bidding mechanism among themselves, taking into account the resources obtained in Round 1.

The Chapter is organized as follows. In Section 1.2 we present the notation and definitions. In Section 1.3 we define formally the coalitional bidding mechanism and we prove that it implements the Owen value. Finally, in Section 1.4 we present some concluding remarks.

1.2 The model

Let (N, v) be a *transferable utility game* (*TU game*), where $N = \{1, 2, ..., n\}$ is the set of *players* and v is the *characteristic function*, which assigns a real number $v(S)$ to every coalition $S \subset N$. We assume that $v(\emptyset) = 0$. Following usual practice, we often refer to "the game v" instead of "the game (N, v)". We denote by $TU\,(N)$ the set of all TU games on the set of players N. We denote by TU the set of all TU games.

For $S \subset N$, when there is no ambiguity, we maintain the notation v when refer to the game v restricted to S as set of players. Otherwise, we use the notation $v|_S$. For simplicity, we denote $v\,(i)$ instead of $v\,(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $S \backslash i$ instead of $S \backslash \{i\}$.

[1] To avoid using the term "game" with different meanings, we say *mechanism* instead of *non-cooperative game*.

A *coalition structure* on N is a partition $\mathcal{C} = \{C_1, ..., C_p\}$, *i. e.* $C_q \cap C_r = \emptyset$ if $q \neq r$ and $\bigcup_{C_q \in \mathcal{C}} C_q = N$. We assume that a coalition structure $\mathcal{C} = \{C_1, ..., C_p\}$ is given and fixed. Given $S \subset N$ we denote by $\mathcal{C}_S$ the restriction of $\mathcal{C}$ to the members of coalition S, i. e. $\mathcal{C}_S = \{C_q \cap S \mid C_q \in \mathcal{C} \text{ and } C_q \cap S \neq \emptyset\}$. Moreover, $\mathcal{C}_{-i} = \mathcal{C}_{N \setminus i}$. We denote by $TU(N, \mathcal{C})$ the set of all triples $(N, v, \mathcal{C})$ where N is the set of players, $\mathcal{C}$ is a coalition structure on N, and v is a characteristic function. We denote by CTU the set of all triples $(N, v, \mathcal{C})$.

To each coalition structure $\mathcal{C}$ and each game v we define by $v/\mathcal{C}$ the *quotient game* induced by v by considering the coalitions of $\mathcal{C}$ as players. Whenever $C_q \in \mathcal{C}$ is considered as a player in $v/\mathcal{C}$, it is denoted by $[C_q]$. Then, for any $Q \subset \mathcal{C}$,

$$(v/\mathcal{C})(Q) = v\left(\bigcup_{[C_q] \in Q} C_q\right).$$

Let $\Pi(N)$ be the set of all orders on N. We say that $\pi \in \Pi(N)$ is *admissible* with respect to the coalition structure $\mathcal{C}$ if for any $i, j, k \in N$, $i, k \in C_q \in \mathcal{C}$, and $\pi(i) < \pi(j) < \pi(k)$ imply that $j \in C_q$, where $\pi(i)$, $\pi(j)$, $\pi(k)$ denote the position of i, j, and k in the order π, respectively. We denote by $\Pi(N, \mathcal{C})$ the set of all admissible orders on N with respect to $\mathcal{C}$.

We say that v is: *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ when $S \cap T = \emptyset$, *strictly superadditive* if $v(S) + v(T) < v(S \cup T)$ when $S \cap T = \emptyset$, *zero-monotonic* if $v(S) + v(i) \leq v(S \cup i)$ when $i \notin S$, and *strictly zero-monotonic* if $v(S) + v(i) < v(S \cup i)$ when $i \notin S$. Note that if the game is (strictly) superadditive then it is (strictly) zero-monotonic.

Given $(N, v, \mathcal{C}) \in TU(N, \mathcal{C})$ the *Owen value* (Owen, 1977) is defined as:

$$\phi_i(N, v, \mathcal{C}) = \frac{1}{|\Pi(N, \mathcal{C})|} \sum_{\pi \in \Pi(N, \mathcal{C})} [v(P_i^\pi \cup i) - v(P_i^\pi)] \quad \text{for all } i \in N$$

where $P_i^\pi = \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of *predecessors* of player i in π, and $|\Pi(N, \mathcal{C})|$ denotes the cardinality of the set $\Pi(N, \mathcal{C})$.

If $\mathcal{C} = \{\{1\}, ..., \{n\}\}$ or $\mathcal{C} = \{N\}$ then the Owen value is given by

$$\phi_i(N, v, \mathcal{C}) = \sum_{S \subseteq N \setminus i} \frac{|S|!\,(n - |S| - 1)!}{n!} [v(S \cup i) - v(S)] \quad \text{for all } i \in N$$

which coincides with the Shapley value of the game v. Usually we denote by $\varphi(N, v)$ the Shapley value of v.

It is well known that for any game $(N, v, \mathcal{C})$ and any $C_q \in \mathcal{C}$, $\sum_{i \in C_q} \phi_i(N, v, \mathcal{C}) = \varphi_{[C_q]}(\mathcal{C}, v/\mathcal{C})$ where $\varphi_{[C_q]}(\mathcal{C}, v/\mathcal{C})$ denotes the Shapley value of $[C_q]$ in the game $v/\mathcal{C}$.

A *value* on $G \subset CTU$ is a map $f : G \longrightarrow \mathbb{R}^N$.

We say that a value f on G satisfies:

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Efficiency. For any $(N, v, \mathcal{C}) \in G$,

$$\sum_{i \in N} f_i(N, v, \mathcal{C}) = v(N).$$

Additivity. For any $(N, v_1, \mathcal{C}), (N, v_2, \mathcal{C}) \in G$ and $i \in N$,

$$f_i(N, v_1 + v_2, \mathcal{C}) = f_i(N, v_1, \mathcal{C}) + f_i(N, v_2, \mathcal{C})$$

where $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \subset N$.

Balanced contributions among players. For any $(N, v, \mathcal{C}) \in G$ and $i, j \in C_q \in \mathcal{C}$,

$$f_i(N, v, \mathcal{C}) - f_i(N \setminus j, v, \mathcal{C}_{-j}) = f_j(N, v, \mathcal{C}) - f_j(N \setminus i, v, \mathcal{C}_{-i}).$$

Balanced contributions among coalitions. For any $(N, v, \mathcal{C}) \in G$ and $C_q, C_r \in \mathcal{C}$,

$$\begin{aligned} &\sum_{i \in C_q} f_i(N, v, \mathcal{C}) - \sum_{i \in C_q} f_i\left(N \setminus C_r, v, \mathcal{C}_{N \setminus C_r}\right) \\ = &\sum_{i \in C_r} f_i(N, v, \mathcal{C}) - \sum_{i \in C_r} f_i\left(N \setminus C_q, v, \mathcal{C}_{N \setminus C_q}\right). \end{aligned}$$

Owen (1977) proves that the Owen value satisfies efficiency and additivity. Later, Calvo, Lasaga and Winter (1996) prove that it also satisfies balanced contributions among players and coalitions.

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1.3 The coalitional bidding mechanism

Given a cooperative game (N, v), Pérez-Castrillo and Wettstein (2001) design a *bidding mechanism*. They prove that the payoff vector of any subgame perfect Nash equilibrium (SPNE) of this mechanism always coincides with the Shapley value of the cooperative game (N, v). Thus, this mechanism implements the Shapley value in subgame perfect Nash equilibria.

In this section we extend the bidding mechanism to cooperative games with a coalition structure. The idea is quite simple. There are Round 1 and Round 2, and Round 1 contains stages 1 through 3. In Round 1, players in the same coalition play the bidding mechanism in order to obtain the resources of this coalition (or a subcoalition if some player is removed). The player who obtains the resources is called the "representative". In Round 2, the representatives play the bidding mechanism with the resources obtained in Round 1.

We now present the *coalitional bidding mechanism (CBM)* formally.

If there is only one player i, he obtains $v(i)$. Assume now that we know the rules of the coalitional bidding mechanism when played by at most $n-1$ players. Then, for a set of players $N = \{1, ..., n\}$ and coalition structure $\mathcal{C} = \{C_1, ..., C_p\}$, the CBM proceeds as follows.

1. Round 1. In this round, the players of any coalition $C_q \in \mathcal{C}$ play the bidding mechanism trying to obtain the resources of C_q. Formally, if there is only one player i, then this player has his resources. Assume now that we know the rules when played by at most $|C_q| - 1$ players. For C_q the process is as follows.

   (a) Stage 1. Each player $i \in C_q$ makes bids $b_j^i \in \mathbb{R}$ for every $j \in C_q \setminus i$. For each $i \in C_q$, we take $B^i = \sum_{j \in C_q \setminus i} b_j^i - \sum_{j \in C_q \setminus i} b_i^j$. Assume that $\alpha_q = argmax_i \left\{B^i\right\}$. In the case of a non-unique maximizer, α_q is randomly chosen among the maximizing indices.

   (b) Stage 2. Player α_q, called the *proposer*, makes an offer $y_i^{\alpha_q}$ to every player $i \in C_q \setminus \alpha_q$.

   (c) Stage 3. The players of $C_q \setminus \alpha_q$, sequentially, either accept or reject the offer. If a rejection is encountered, we say the offer is rejected. Otherwise, we say the offer is accepted.

   The coalitions of $\mathcal{C}$ play sequentially in the order $C_1, ..., C_p$ until we find C_{q_0} and α_{q_0} such that the offer of α_{q_0} is rejected or for any $C_q \in \mathcal{C}$ the offer of α_q is accepted.

   In the first case, player α_{q_0} pays $b_i^{\alpha_{q_0}}$ to every player $i \in C_{q_0} \setminus \alpha_{q_0}$ and leaves the mechanism obtaining $v(\alpha_{q_0}) - \sum_{i \in C_{q_0} \setminus \alpha_{q_0}} b_i^{\alpha_{q_0}}$. All players other than α_{q_0} proceed to play the CBM with $(N', v', \mathcal{C}')$ where $N' = N \setminus \alpha_{q_0}$, $v' = v$ and $\mathcal{C}' = \mathcal{C}_{-\alpha_{q_0}}$. Any player $i \in C_{q_0} \setminus \alpha_{q_0}$ obtains as final payoff the sum of the bid received, $b_i^{\alpha_{q_0}}$, and the payoff outcome of the mechanism corresponding to $(N', v', \mathcal{C}')$. Any player $i \in N \setminus C_{q_0}$ obtains as final payoff the payoff outcome of the mechanism corresponding to $(N', v', \mathcal{C}')$.

   In the second case, for any $C_q \in \mathcal{C}$, player α_q pays $b_i^{\alpha_q} + y_i^{\alpha_q}$ to every player $i \in C_q \setminus \alpha_q$ and becomes the *representative* of coalition C_q. This means that player α_q goes to Round 2 with all the resources of C_q. Moreover, the payoff obtained by this player in this round is $p_{\alpha_q}^1 = -\sum_{i \in C_q \setminus \alpha_q} \left(b_i^{\alpha_q} + y_i^{\alpha_q}\right)$. Any other player $i \in C_q \setminus \alpha_q$ leaves the mechanism obtaining a final payoff of $b_i^{\alpha_q} + y_i^{\alpha_q}$.

   After finishing Round 1, for any $C_q \in \mathcal{C}$ we can find the representative (denoted by r_q) of this coalition. When there is only one player in a coalition, he becomes the representative of himself. Moreover, we denote by C_q^* the set of players whose resources are obtained by player r_q. Notice that $C_q \setminus C_q^*$ is the set of removed proposers in C_q. Of course $C_q^* \subset C_q$ and $r_q \in C_q^*$.

2. Round 2. The representatives play the bidding mechanism (Pérez-Castrillo and Wettstein, 2001) associated with the game (N^*, v^*) where $N^* =$

$\{r_1, ..., r_p\}$ and for any $S \subset N^*$, $v^*(S) = v\left(\bigcup_{r_q \in S} C_q^*\right)$. For any representative r_q, we denote by $p_{r_q}^2$ the payoff obtained by r_q in Round 2.

The final payoff obtained by any representative r_q is the sum of the payoffs obtained in both rounds, *i. e.* $p_{r_q}^1 + p_{r_q}^2$.

We note that the CBM terminates in a finite number of moves.

Remark 1 *In Round 1, we assume that coalitions play the bidding mechanism independently. Moreover, when Stage 3 of the bidding mechanism of some coalition C_q ends, players of C_q must announce to the other coalitions if they have an agreement (the offer of α_q is accepted) or not (the offer of α_q is rejected). Assume that the offer of player α_q is accepted for any $q < q_0$, but the offer of α_{q_0} is rejected. Then a new subgame begins, which coincides with the CBM associated with $\left(N \setminus \alpha_{q_0}, v, \mathcal{C}_{-\alpha_{q_0}}\right)$. The agreement achieved by every coalition C_q with $q < q_0$ is, thus, of course, cancelled. We can include in the CBM an intermediate step between the rejection of the offer of α_{l_0} and the CBM applied to $\left(N \setminus \alpha_{q_0}, v, \mathcal{C}_{-\alpha_{q_0}}\right)$. Assume that for any coalition C_q, $q \neq q_0$, the players of C_q vote whether they prefer to continue with the agreement achieved or not. If at least one player wants to cancel the agreement, it is cancelled. It is not difficult to check that all the results obtained in this Chapter are also valid if we include this intermediate step.*

Remark 2 *In Round 1, we assume that coalitions play the bidding mechanism in the order $C_1, ..., C_p$. Our results are independent of the order in which coalitions play the bidding mechanism. Moreover, if the order is chosen according to some probability distribution over the set of all possible orders, our results are still valid and independent of the probability distribution.*

Gul (1989) analyzed a cooperative game where random meetings between two agents occur. At each meeting, a player (randomly chosen) makes an offer to the other. If this offer is accepted, the proposer buys the resources of the other player. In the bidding mechanism played by any coalition the situation is, in some way, similar. There is also a player (α_q) who makes an offer trying to obtain the resources of the rest of players ($C_q \setminus \alpha_q$). The differences are that, in the bidding mechanism, it could be possible that more than two players are involved and, moreover, the proposer is not randomly chosen.

Before the characterization of the SPNE outcomes of the coalitional bidding mechanism we need the following result.

Proposition 3 *Given a triple $(N, v, \mathcal{C})$ such that v is zero-monotonic, $j \in C_q \in \mathcal{C}$, and $\{j\} \neq C_q$ then*

$$\sum_{i \in C_q} \phi_i(N, v, \mathcal{C}) \geq \sum_{i \in C_q \setminus j} \phi_i(N \setminus j, v, \mathcal{C}_{-j}) + v(j).$$

Proof. We take $P = \{1, ..., p\}$ and define the following games on P: $w(R) = v\left(\bigcup_{r \in R} C_r\right)$ for any $R \subset P$; $w_1(R) = v\left(\bigcup_{r \in R} C_r\right)$ if $q \notin T$ and $w_1(R) = v\left(\bigcup_{r \in R \setminus q} C_r \cup (C_q \setminus j)\right)$ if $q \in R$; $w_2(R) = 0$ if $q \notin R$ and $w_2(R) = v(j)$ if $q \in R$; and $w' = w_1 + w_2$.

Since v is zero-monotonic we obtain that $w(R) = w'(R)$ if $q \notin R$ and $w(R) \geq w'(R)$ if $q \in R$. Hence $\phi_q(P, w) \geq \phi_q(P, w')$. Since the Shapley value satisfies additivity we have $\phi_q(P, w') = \phi_q(P, w_1) + \phi_q(P, w_2)$.

We know that

$$\begin{aligned} \phi_q(P, w) &= \sum_{i \in C_q} \phi_i(N, v, \mathcal{C}), \\ \phi_q(P, w_1) &= \sum_{i \in C_q \setminus j} \phi_i(N \setminus j, v, \mathcal{C}_{-j}), \end{aligned}$$

and

$$\phi_q(P, w_2) = v(j).$$

This concludes the proof. ■

Remark 4 *Using similar arguments to those used in the proof of Proposition 3 we can prove that if v is strictly zero-monotonic, $j \in C_q \in \mathcal{C}$ and $\{j\} \neq C_q$, then*

$$\sum_{i \in C_q} \phi_i(N, v, \mathcal{C}) > \sum_{i \in C_q \setminus j} \phi_i(N \setminus j, v, \mathcal{C}_{-j}) + v(j).$$

First, we prove that the Owen value is the payoff of an SPNE outcome.

Proposition 5 *Given a triple $(N, v, \mathcal{C})$ where v is superadditive, the Owen value $\phi(N, v, \mathcal{C})$ is the payoff of an SPNE outcome of the coalitional bidding mechanism.*

Proof. If there is only one player the result is trivial. Assume that the result holds with at most $n - 1$ players.

We consider the following strategies.

- Round 1. First, we define the strategies in the bidding mechanism associated with any $C_q \in \mathcal{C}$.

  Stage 1. For any $i \in C_q$, $b_j^i = \phi_j(N, v, \mathcal{C}) - \phi_j(N \setminus i, v, \mathcal{C}_{-i})$ for any $j \in C_q \setminus i$.

  Stage 2. Player α_q, the proposer, offers $y_j^{\alpha_q} = \phi_j\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$ to every $j \in C_q \setminus \alpha_q$.

  Stage 3. Any player $i \in C_q \setminus \alpha_q$ accepts the offer of α_q if and only if $y_j^{\alpha_q} \geq \phi_j\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$ for every $j \in C_q \setminus \alpha_q$.

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If an offer is rejected, for instance, the offer of α_{q_0}, we go to the subgame where all players other than α_{q_0} play this mechanism in $\left(N \setminus \alpha_{q_0}, v, \mathcal{C}_{-\alpha_{q_0}}\right)$. We assume that players in $N \setminus \alpha_{q_0}$ play according to the strategies profiles of some SPNE with associated payoff $\phi\left(N \setminus \alpha_{q_0}, v, \mathcal{C}_{-\alpha_{q_0}}\right)$ (by induction hypothesis we can find such SPNE).

- Round 2. We assume that players of N^* play according to the strategies described in Pérez-Castrillo and Wettstein (2001). They show that, for any zero-monotonic game, the strategy profile is an SPNE, and the equilibrium payoff vector coincides with the Shapley value of the game.

First, we prove that according to these strategies any player $i \in N$ receives as payoff the Owen value $\phi_i(N, v, \mathcal{C})$. We must note that the offer of any α_q is accepted. Then player α_q goes to Round 2 as the representative of C_q.

Any player $i \in C_q \setminus \alpha_q$ obtains $b_i^{\alpha_q} + y_i^{\alpha_q} =$

$$\phi_i(N, v, \mathcal{C}) - \phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right) + \phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right) = \phi_i(N, v, \mathcal{C}).$$

We now compute the payoff of any representative r_q. As v is superadditive we conclude that v^* is zero-monotonic. By Pérez-Castrillo and Wettstein (2001), we know that the payoff obtained by r_q in Round 2 ($p_{r_q}^2$) coincides with the Shapley value of (N^*, v^*). Then the final payoff obtained by r_q is

$$\begin{aligned}
p_{r_q}^1 + p_{r_q}^2 &= -\sum_{i \in C_q \setminus r_q} b_i^{r_q} - \sum_{i \in C_q \setminus r_q} y_i^{r_q} + \varphi_{r_q}(N^*, v^*) \\
&= -\sum_{i \in C_q \setminus r_q} \left(\phi_i(N, v, \mathcal{C}) - \phi_i\left(N \setminus r_q, v, \mathcal{C}_{-r_q}\right)\right) \\
&\quad - \sum_{i \in C_q \setminus r_q} \phi_i\left(N \setminus r_q, v, \mathcal{C}_{-r_q}\right) + \varphi_{[C_q]}(\mathcal{C}, v/\mathcal{C}) \\
&= -\sum_{i \in C_q \setminus r_q} \phi_i(N, v, \mathcal{C}) + \sum_{i \in C_q} \phi_i(N, v, \mathcal{C}) \\
&= \phi_{r_q}(N, v, \mathcal{C}).
\end{aligned}$$

We now prove that these strategies are an SPNE. As v is superadditive, (N^*, v^*) is always zero-monotonic. By Pérez-Castrillo and Wettstein (2001) we conclude that, in the subgames obtained after Round 2, these strategies induce an SPNE.

By induction hypothesis, we conclude that these strategies induce an SPNE in all the subgames obtained after the rejection of the offer of some proposer α_q.

It remains only to prove that these strategies induce an SPNE in the bidding mechanism associated with any coalition C_q.

Stage 3. Assume that player i rejects the offer of α_q. Then the coalitional bidding mechanism of $\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$ is played and, by induction hypothesis,

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after the rejection player i can obtain at most $\phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$. Hence, if player i rejects the offer of α_q, he obtains, at most,

$$b_i^{\alpha_q} + \phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right) = \phi_i\left(N, v, \mathcal{C}\right).$$

This means that player i does not improve his payoff.

Stage 2. If player α_q offers to some player $i \in C_q$ less than $\phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$, the offer is rejected and, therefore, player α_q obtains a final payoff of

$$v\left(\alpha_q\right) - \sum_{i \in C_q \setminus \alpha_q} \left(\phi_i\left(N, v, \mathcal{C}\right) - \phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)\right).$$

By Proposition 3, this payoff is not larger than $\phi_{\alpha_q}\left(N, v, \mathcal{C}\right)$, which means that player α_q does not improve his payoff.

If player α_q offers to any player $i \in C_q \setminus \alpha_q$ at least $\phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right)$, the offer is accepted. It is straightforward to prove that player α_q obtains at most $\phi_{\alpha_q}\left(N, v, \mathcal{C}\right)$.

Stage 1. First, we prove that for any $i \in C_q \in \mathcal{C}$, $B^i = 0$.

$$\begin{aligned} B^i &= \sum_{j \in C_q \setminus i} b_j^i - \sum_{j \in C_q \setminus i} b_i^j \\ &= \sum_{j \in C_q \setminus i} \left(\phi_j\left(N, v, \mathcal{C}\right) - \phi_j\left(N \setminus i, v, \mathcal{C}_{-i}\right)\right) \\ &\quad - \sum_{j \in C_q \setminus i} \left(\phi_i\left(N, v, \mathcal{C}\right) - \phi_i\left(N \setminus j, v, \mathcal{C}_{-j}\right)\right). \end{aligned}$$

As the Owen value satisfies balanced contributions among players, we know that for any $j \in C_q \setminus i$,

$$\phi_i\left(N, v, \mathcal{C}\right) - \phi_i\left(N \setminus j, v, \mathcal{C}_{-j}\right) = \phi_j\left(N, v, \mathcal{C}\right) - \phi_j\left(N \setminus i, v, \mathcal{C}_{-i}\right)$$

and hence $B^i = 0$.

Assume that player $i \in C_q$ makes a different bid $\widehat{b}^i$. If $\widehat{B}^i < 0$, the proposer will be another player of C_q. Then player i can not increase his payoff.

If $\widehat{B}^i > 0$, he becomes the proposer, but he must pay $\sum_{j \in C_q \setminus i} \widehat{b}_j^i$ to the other players of $C_q \setminus i$. It is straightforward to prove that player i can obtain, at most, a final payoff of

$$\phi_i\left(N, v, \mathcal{C}\right) - \sum_{j \in C_q \setminus i} \widehat{b}_j^i + \sum_{j \in C_q \setminus i} b_j^i$$

which is smaller than $\phi_i\left(N, v, \mathcal{C}\right)$.

If $\widehat{B}^i = 0$ and player i is not the proposer, using similar arguments to those used when $\widehat{B}^i < 0$, we can conclude that player i does not increase his payoff. If $\widehat{B}^i = 0$ and player i is the proposer, using similar arguments to those used when $\widehat{B}^i > 0$, we can conclude that player i does not increase his payoff. ■

There exist superadditive games such that the associated coalitional bidding mechanisms have SPNE outcomes whose payoff is different from the Owen value.

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Example 6 *Consider* $(N, v, \mathcal{C})$, *where* $N = \{1,2,3,4\}$, $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1,2\}$, $C_2 = \{3,4\}$. *Moreover, v is the characteristic function associated with the weighted majority game where the quota is 3 and the weights are 1, 1, 1, and 2 respectively. This means that $v(S) = 1$ if and only if S contains some of the following subsets:* $\{1,2,3\}$, $\{1,4\}$, $\{2,4\}$, *or* $\{3,4\}$.

It is straightforward to prove that

$$
\begin{aligned}
\phi(N, v, \mathcal{C}) &= \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \\
\phi(N \setminus 1, v, \mathcal{C}_{-1}) &= \left(-, 0, \frac{1}{4}, \frac{3}{4}\right) \\
\phi(N \setminus 2, v, \mathcal{C}_{-2}) &= \left(0, -, \frac{1}{4}, \frac{3}{4}\right) \\
\phi(N \setminus 3, v, \mathcal{C}_{-3}) &= \left(\frac{1}{4}, \frac{1}{4}, -, \frac{1}{2}\right) \\
\phi(N \setminus 4, v, \mathcal{C}_{-4}) &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, -\right).
\end{aligned}
$$

We now define an SPNE whose payoff outcome is $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

Round 1. First, we describe the strategies of players 1 and 2. The bids are $b_2^1 = b_1^2 = 0$. *Then, the proposer α_1 is randomly chosen between 1 and 2. Moreover,* $y_j^{\alpha_1} = 0$ *and player j accepts the offer of α_1 if and only if α_1 offers him something positive.*

We now describe the strategies of players 3 and 4. In the subgame obtained after the offer of α_1 is accepted, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the Owen value. By Proposition 5, we know that these strategies exist. In the subgame obtained after the offer of α_1 is rejected, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the Owen value of $(N \setminus \alpha_1, v, \mathcal{C}_{-\alpha_1})$.

Round 2. We assume that players of N^* play according to the strategies described in Pérez-Castrillo and Wettstein (2001), which implement the Shapley value.

It is not difficult to check that these strategies are an SPNE.

According to these strategies, the offer of player α_1 is rejected, which means that player α_1 obtains a final payoff of $v(\alpha_1) = 0$. *Then players of $N \setminus \alpha_1$ obtain as final payoff* $\phi(N \setminus \alpha_1, v, \mathcal{C}_{-\alpha_1})$. *This means that the final payoff induced by these strategies is* $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

If we want to implement the Owen value, we have to make additional assumptions. We make two kinds of assumptions: first, about players' behavior in Round 1, and second, about the class of cooperative games.

About players' behavior Moldovanu and Winter (1994) say, "We assume that each player prefers to be a member of *large* coalitions rather than smaller ones provided that he earns the same payoff in the two agreements". Hart and Mas-Colell (1996) say, "To facilitate exposition we will assume that both proposers and respondents break ties in favor of quick termination of the game".

If we make in our framework the same assumption as Moldovanu and Winter (1994), we implement the Owen value. The same happens with the assumption by Hart and Mas-Colell (1996).

In order to simplify the exposition, in our model we suppose that if some offer y^{α} of proposer α in Round 1 is rejected by player j, then both players (α and j) have a "punishment" or "penalty", $\varepsilon > 0$, where ε is very small. Note that with this penalty players prefer large coalitions rather than smaller ones and "both proposers and respondents break ties in favor of quick termination of the game". We call this modification the ε-CBM.

We now define the ε-CBM formally. The structure of the non-cooperative game, bids and offers is the same in CBM and ε-CBM. This means that the strategies available for players are the same in both mechanism. The only difference between CBM and ε-CBM lies on the following aspect of the payoff function in Round 1. Assume that for any $C_q \in \mathcal{C}, q < q_0$ the offer of player α_q is accepted and the offer of α_{q_0} is rejected by player j. Then α_{q_0} leaves the mechanism obtaining $v(\alpha_{q_0}) - \sum\limits_{i \in C_{q_0} \setminus \alpha_{q_0}} b_i^{\alpha_{q_0}} - \varepsilon$ (in CBM, player α_{q_0} leaves the mechanism obtaining $v(\alpha_{q_0}) - \sum\limits_{i \in C_{q_0} \setminus \alpha_{q_0}} b_i^{\alpha_{q_0}}$). Player $j \in C_{q_0} \setminus \alpha_{q_0}$ obtains as final payoff the sum of $b_j^{\alpha_{q_0}} - \varepsilon$, and the payoff outcome of the mechanism corresponding to $(N', v', \mathcal{C}')$ (in CBM player j obtains as final payoff the sum of $b_j^{\alpha_{q_0}}$ and the payoff outcome of the mechanism corresponding to $(N', v', \mathcal{C}')$).

Theorem 7 *For any $\varepsilon > 0$, the ε-CBM implements the Owen value in SPNE for superadditive games.*

Proof. We proceed by induction on the number of players. If there is only one player the result is trivial. Assume that if there are at most $n-1$ players the ε-CBM implements the Owen value in SPNE and, moreover, all the offers of Round 1 are accepted. We now prove that the same holds when there are n players.

Consider the strategies defined as in Proposition 5 but with $b_j^i = \phi_j(N, v, \mathcal{C}) - \phi_j(N \setminus i, v, \mathcal{C}_{-i}) + \varepsilon$ and $y_j^{\alpha_q} = \phi_j\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right) - \varepsilon$ for any $C_q \in \mathcal{C}$, $i \in C_q$, and $j \in C_q \setminus i$. Using similar arguments to those used in the proof of Proposition 5, we conclude that these strategies are an SPNE with $\phi(N, v, \mathcal{C})$ as payoff outcome.

We now prove that the payoff in all SPNE outcomes coincides with the Owen value. We do so in several steps.

The structure of this proof is similar to that of the main result by Pérez-Castrillo and Wettstein (2001). The proof of steps B, C, and D is similar although the computations are different. The proof of Step A, however, is completely different.

Step A. At every SPNE outcome, and for every $C_q \in \mathcal{C}$, the offer of the proposer α_q to each player $i \in C_q \setminus \alpha_q$ is $y_i^{\alpha_q} = \phi_i\left(N \setminus \alpha_q, v, \mathcal{C}_{-\alpha_q}\right) - \varepsilon$ and every $i \in C_q \setminus \alpha_q$ accepts this offer.

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Assume that in each coalition $C_q \in \{C_1, ..., C_{p-1}\}$, the offer of a proposer $\alpha_q \in C_q$ is accepted, and consider the subgame starting with the last coalition C_p. Let $\alpha_p \in C_p$ be the proposer in C_p. Let y^{α_p} be an offer of α_p. Let the order of reply of the players in $C_p \setminus \alpha_p$ be $i_1, ..., i_k$.

Claim 1: At every SPNE, the strategies of the players in $C_p \setminus \alpha_p$ must be as follows:

(i) If $y_i^{\alpha_p} > \phi_i \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$ for every $i \in C_p \setminus \alpha_p$, then every $i \in C_p \setminus \alpha_p$ accepts y^{α_p}.

(ii) If $y_j^{\alpha_p} < \phi_j \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$ for some $j \in C_p \setminus \alpha_p$, then some player in $C_p \setminus \alpha_p$ rejects y^{α_p}.

(i) Consider the strategy of the last player i_k. Assuming that his decision node is reached, if he accepts the offer y^{α_p}, then he receives $b_{i_k}^{\alpha_p} + y_{i_k}^{\alpha_p}$, whereas if he rejects y^{α_p}, then by the induction hypothesis he obtains $b_{i_k}^{\alpha_p} + \phi_{i_k} \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$. Hence, at any SPNE, if $y_{i_k}^{\alpha_p} > \phi_{i_k} \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$, then i_k accepts the offer. Repeating the same argument backwards, we can show that players $i_{k-1}, ..., i_1$ accept the offer.

(ii) Suppose, to the contrary, that there exists $j \in C_p \setminus \alpha_p$ with $y_j^{\alpha_p} < \phi_j \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$, but all the players in $C_p \setminus \alpha_p$ accept the offer y^{α_p}. Then, player j receives $b_j^{\alpha_p} + y_j^{\alpha_p}$. However, if player j deviates and rejects the offer, then he obtains $b_j^{\alpha_p} + \phi_j \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon$, which is greater than $b_j^{\alpha_p} + y_j^{\alpha_p}$. Hence, the strategies of the players in $C_p \setminus \alpha_p$ cannot constitute an SPNE.

Claim 2: At every SPNE outcome, every $i \in C_p \setminus \alpha_p$ accepts the offer of the proposer α_p.

Suppose, to the contrary, that at some SPNE outcome, there exists $i \in C_p \setminus \alpha_p$ who rejects the offer y^{α_p}. Then, the proposer obtains

$$v\left(\alpha_p\right) - \varepsilon - \sum_{i \in C_p \setminus \alpha_p} b_i^{\alpha_p}.$$

Let $\delta > 0$ be given. For each $i \in C_p \setminus \alpha_p$, define $z_i^{\alpha_p}(\delta)$ by

$$z_i^{\alpha_p}(\delta) = \phi_i \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon + \delta.$$

Suppose that the proposer α_p proposes $z^{\alpha_p}(\delta)$. By Claim 1 (i), for any $\delta > 0$, every $i \in C_p \setminus \alpha_p$ accepts $z^{\alpha_p}(\delta)$. Hence, player α_p is the representative of coalition C_p in Round 2. Now, in Round 2, there are p players $\{\alpha_1, ..., \alpha_p\}$, where, for any coalition $C_q \in \mathcal{C}$, α_q is the representative of coalition C_q. As the representatives are playing an SPNE of the bidding mechanism associated with (N^*, v^*), by Pérez-Castrillo and Wettstein (2001) we know that the payoff obtained by player α_p in Round 2 is $\varphi_{\alpha_p}(N^*, v^*) = \varphi_{[C_p]}(\mathcal{C}, v/\mathcal{C}) = \sum_{i \in C_p} \phi_i(N, v, \mathcal{C})$. Then, the final payoff of player α_p is

$$\sum_{i \in C_p} \phi_i(N, v, \mathcal{C}) - \sum_{i \in C_p \setminus \alpha_p} \left[\phi_i \left(N \setminus \alpha_p, v, \mathcal{C}_{-\alpha_p}\right) - \varepsilon + \delta\right] - \sum_{i \in C_p \setminus \alpha_p} b_i^{\alpha_p}$$

$$= \sum_{i\in C_p}\phi_i(N,v,\mathcal{C}) - \sum_{i\in C_p\setminus\alpha_p}\phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right) + (|C_p|-1)\varepsilon - (|C_p|-1)\delta - \sum_{i\in C_p\setminus\alpha_p} b_i^{\alpha_p}.$$

By Proposition 3, we know that

$$a = \sum_{i\in C_p}\phi_i(N,v,\mathcal{C}) - \sum_{i\in C_p\setminus\alpha_p}\phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right) - v(\alpha_p) \geq 0.$$

Then, if $0<\delta<\frac{a+|C_p|\varepsilon}{(|C_p|-1)}$, the payoff of α_p obtained by offering $z^{\alpha_p}(\delta)$ is greater than that obtained by offering y^{α_p}. Hence, an offer of y^{α_p} cannot be an SPNE strategy of proposer α_p, which is a contradiction.

Claim 3: At every SPNE, and for every $i\in C_p\setminus\alpha_p$,

$$y_i^{\alpha_p} = \phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right) - \varepsilon.$$

Let y^{α_p} be the offer of α_p at an SPNE. By Claim 2, y^{α_p} must be accepted by every $i\in C_p\setminus\alpha_p$. Then, it follows from Claim 1 (ii) that for every $i\in C_p\setminus\alpha_p$, $y_i^{\alpha_p}\geq\phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right)-\varepsilon$. Suppose that for some $j\in C_p\setminus\alpha_p$, $y_j^{\alpha_p}>\phi_j\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right)-\varepsilon$. Let $\tau=y_j^{\alpha_p}-\left[\phi_j\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right)-\varepsilon\right]>0$. For each $i\in C_p\setminus\alpha_p$, define $w_i^{\alpha_p}=\phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right)-\varepsilon+\frac{\tau}{|C_p|}$. Suppose that the proposer α_p deviates and offers w^{α_p}. Then, by Claim 1 (i), every $i\in C_p\setminus\alpha_p$ accepts w^{α_p}. Moreover, since

$$\begin{aligned}\sum_{i\in C_p\setminus\alpha_p} w_i^{\alpha_p} &= \sum_{i\in C_p\setminus\alpha_p}\left[\phi_i\left(N\setminus\alpha_p,v,\mathcal{C}_{-\alpha_p}\right)-\varepsilon\right]+\frac{|C_p|-1}{|C_p|}\tau \\ &< \sum_{i\in C_p\setminus\alpha_p} y_i^{\alpha_p},\end{aligned}$$

the proposer α_p obtains a greater payoff by offering w^{α_p} than by offering y^{α_p}. Hence, to offer y^{α_p} cannot be an SPNE strategy, which is a contradiction.

Repeating the same arguments for coalitions $C_{p-1},...,C_1$, we can prove Step A.

Step B. Assume that we are in Stage 1 of Round 1 of the bidding mechanism associated with $C_q\in\mathcal{C}$. Then in any SPNE, $B^i=0$ for any $i\in C_q$.

It is straightforward to prove that $\sum_{i\in C_q} B^i=0$. We take

$$X=\left\{i\in C_q : B^i=\max_{j\in C_q}B^j\right\}.$$

If $X = C_q$, the result holds because $\sum_{i \in C_q} B^i = 0$.

If $X \neq C_q$, we get a contradiction by proving that player $i \in X$ has a deviation which improves his final payoff. We take $j \in C_q \backslash X$ such that $B^j \geq B^k$ for any $k \in C_q \setminus X$. Assume that player i makes a new bid $\widehat{b}^i$, where $\widehat{b}^i_k = b^i_k + \delta$ if $k \in X \setminus i$, $\widehat{b}^i_j = b^i_j - |X|\,\delta$, and $\widehat{b}^i_k = b^i_k$ if $k \in C_q \setminus (X \cup j)$.

For any $k \in C_q$, we compute $\widehat{B}^k$ assuming that $\widehat{b}^k = b^k$ for any $k \in C_q \setminus i$. Then $\widehat{B}^k = B^k - \delta$ if $k \in X$, $\widehat{B}^j = B^j + |X|\,\delta$, and $\widehat{B}^k = B^k$ if $k \in C_q \setminus (X \cup j)$.

Since $B^j < B^i$, we can find $\delta > 0$ satisfying $B^j + |X|\,\delta < B^i - \delta$. Moreover, $\widehat{X} = \left\{ k \in C_q : \widehat{B}^k = \max_{h \in C_q} \widehat{B}^h \right\} = X$. This means that any player of X is the proposer with the same probability under b^i and $\widehat{b}^i$. When player i is not the proposer, which happens with probability $\frac{|X|-1}{|X|}$, he obtains, by Step A, the same making a bid b^i or $\widehat{b}^i$. But if player i is the proposer, which happens with probability $\frac{1}{|X|}$, he obtains, by Step A, δ units more with $\widehat{b}^i$ than with b^i.

Step C. Assume that we are in Stage 1 of Round 1 of the bidding mechanism associated with $C_q \in \mathcal{C}$. Then, at every SPNE, the payoff of any player $i \in C_q$ is the same regardless of who is chosen as the proposer.

By Step B, we know that $B^i = 0$ for any $i \in C_q$.

Assume that some player i strictly prefers to be (not to be) the proposer. Then player i can improve his payoff by slightly increasing (decreasing) one of his bids b^i_j. But this is impossible in an SPNE.

Step D. In any SPNE outcome of ϵ-CBM any player $i \in N$ obtains as final payoff his Owen value.

Assume that players are playing according to some SPNE. Given $i \in C_q \in \mathcal{C}$, we denote by p_i the final payoff obtained by player i in this SPNE.

By Step B, we know that any player of C_q is the proposer with probability $\frac{1}{|C_q|}$.

If player i is the proposer, we know, by Step A, that his final payoff is

$$\sum_{j \in C_q} \phi_j\left(N, v, \mathcal{C}\right) - \sum_{j \in C_q \setminus i} \phi_j\left(N \setminus i, v, \mathcal{C}_{-i}\right) + \left(|C_q| - 1\right)\epsilon - \sum_{j \in C_q \setminus i} b^i_j.$$

If $j \in C_q \setminus i$ is the proposer then the final payoff of player i is, by Step A,

$$b^j_i + \phi_i\left(N \setminus j, v, \mathcal{C}_{-j}\right) - \epsilon.$$

By Step C, we know that $|C_q|\,p_i =$

$$\sum_{j \in C_q} \phi_j\left(N, v, \mathcal{C}\right) - \sum_{j \in C_q \setminus i} \phi_j\left(N \setminus i, v, \mathcal{C}_{-i}\right) + \left(|C_q| - 1\right)\epsilon - \sum_{j \in C_q \setminus i} b^i_j$$
$$+ \sum_{j \in C_q \setminus i} \left(b^j_i + \phi_i\left(N \setminus j, v, \mathcal{C}_{-j}\right) - \epsilon\right).$$

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By Step B, we know that

$$-\sum_{j\in C_q\setminus i} b_j^i + \sum_{j\in C_q\setminus i} b_i^j = -B^i = 0.$$

Hence, $|C_q|\, p_i =$

$$\sum_{j\in C_q\setminus i} \left(\phi_i\left(N\setminus j, v, \mathcal{C}_{-j}\right) - \phi_j\left(N\setminus i, v, \mathcal{C}_{-i}\right)\right) + \sum_{j\in C_q} \phi_j\left(N, v, \mathcal{C}\right).$$

Since the Owen value satisfies the property of balanced contributions among coalitions, we conclude that

$$\begin{aligned} |C_q|\, p_i &= \sum_{j\in C_q\setminus i} \left(\phi_i\left(N, v, \mathcal{C}\right) - \phi_j\left(N, v, \mathcal{C}\right)\right) + \sum_{j\in C_q} \phi_j\left(N, v, \mathcal{C}\right) \\ &= \left(|C_q| - 1\right)\phi_i\left(N, v, \mathcal{C}\right) - \sum_{j\in C_q\setminus i} \phi_j\left(N, v, \mathcal{C}\right) + \sum_{j\in C_q} \phi_j\left(N, v, \mathcal{C}\right) \\ &= |C_q|\, \phi_i\left(N, v, \mathcal{C}\right). \end{aligned}$$

Then $p_i = \phi_i\left(N, v, \mathcal{C}\right)$. This finishes the proof of Theorem 7. ■

Remark 8 *In the ε-CBM, if an offer is rejected, the proposer and the responder who rejects the offer have a penalty. It is not difficult to check that the result is also true if only proposers (or responders) have a penalty.*

Theorem 7 also holds if the penalty to the proposer is agent-dependent, i. e. *any agent i has a penalty $\varepsilon_i > 0$ for being removed from the game.*

We have just proved that if we make assumptions about player's behavior, which appears in the ε-CBM, we can implement the Owen value in the class of superadditive games.

As we said before, another way to avoid the multiplicity of payoffs associated with SPNE outcomes in the CBM associated with superadditive games is to find a subclass of such games where the Owen value is the unique payoff associated with SPNE outcomes. We have the following result:

Theorem 9 *The CBM implements the Owen value in SPNE for strictly superadditive games.*

Proof. We already know, by Proposition 5, that there is an SPNE outcome of CBM whose payoff is the Owen value.

Using similar arguments to those used in the proof of Theorem 7 we can prove that the payoff associated with every SPNE outcome coincides with the Owen value. ■

Remark 10 *A natural question that arises is: what happens if in Round 1 coalitions announce nothing? (in CBM they announce if there is an agreement or some player is removed). This means that players in a coalition have no* *information about what happens in other coalitions. Later, in Round 2, the representative of any coalition announces to the other representatives the resources that he has. We call this non-cooperative game the "simultaneous coalitional bidding mechanism" (SCBM). We also define the* ϵ*-SCBM in the same way that we have done with the* ϵ*-CBM.*

Note that the only subgames in SCBM and ϵ*-SCBM are the whole game and those obtained after Round 2.*

Using similar arguments to those used in Proposition 5, it is not difficult to prove that the Owen value is the payoff associated with some SPNE outcome of SCBM and ϵ*-SCBM.*

Nevertheless, we have no uniqueness, as we can see in the following example, which works in SCBM and ϵ*-SCBM. We take* $(N, v, \mathcal{C})$ *, where* $N = \{1, 2, 3, 4\}$*,* $\mathcal{C} = \{C_1, C_2\}$*,* $C_1 = \{1, 2\}$*,* $C_2 = \{3, 4\}$*,* $v(S) = 100$ *if* $|S| = 4$*, and* $v(S) = 0$ *if* $|S| < 4$*.*

We consider the following strategies:

Round 1. Players of C_1 *play as follows: in Stage 1* $b_2^1 = b_1^2 = 0$*; in Stage 2 the proposer,* α_1*, offers* $y_i^{\alpha_1} = 0$ *if* $i \neq 1$ *and* $y_i^{\alpha_1} = 50$ *if* $i = 1$*, any player* $j \in C_1 \setminus \alpha_1$ *accepts an offer* y *if and only if* $y_i^{\alpha_1} = 0$ *if* $i \neq 1$ *and* $y_i^{\alpha_1} = 50$ *if* $i = 1$*. Players of* C_2 *play according to the strategies defined in Proposition 5.*

Round 2. The representatives play according to the strategies described in Pérez-Castrillo and Wettstein (2001), which implement the Shapley value.

It is not difficult to prove that these strategies are an SPNE. Moreover, they induce as final payoff $(50, 0, 25, 25)$*. The Owen value is* $(25, 25, 25, 25)$*.*

1.4 Concluding remarks

In this Chapter we define the coalitional bidding mechanism, which generalizes the bidding mechanism of Pérez-Castrillo and Wettstein (2001) for situations where players are divided into fixed groups. We prove that for superadditive games there always exists an SPNE whose payoff outcome coincides with the Owen value. However, unlike the result of Pérez-Castrillo and Wettstein (2001) on implementation of the Shapley value, there exist SPNE whose payoff outcome is different from the Owen value. But if we restrict the behavior of agents (as in Moldovanu and Winter (1994) or Hart and Mas-Colell (1996)) or we restrict the class of games, for example, to strictly superadditive games, we can implement the Owen value.

Chapter 2

Implementation of the levels value

2.1 Introduction

Frequently, we have more available information than those given by the characteristic function of the game. For example, let us consider the members of the European Union Parliament. Even though all of them have the same rights, they do not act independently, since they belong to different political parties. Furthermore, political parties are not completely independent from each other. On a higher level, parties of similar ideology may be formally associated, such like the Social-democratic or the Socialist Parties are, and so on.

We call this cooperation description of the players a *levels structure*. Values which take into account levels structures are the Owen value (presented by Owen in 1977) for a single level, and the levels structure value (suggested by Owen in 1977 and studied by Winter in 1989). The levels structure value is a generalization of the Owen value for more than one level. Furthermore, the Owen value is a generalization of the Shapley value.

In Chapter 1, the bidding mechanism by Pérez-Castrillo and Wettstein is generalized so that a single-level structure is taken into account. The resulting non-cooperative game implements the Owen value.

In this Chapter, we move a step ahead. We modify the bidding mechanism so that a general levels structure is considered. To do so, we generalize the bidding mechanism to a new mechanism, called the *levels bidding mechanism*.

Given a levels structure with h levels, the levels bidding mechanism has h rounds. In Round 1, the members of the same coalition at this level play the bidding mechanism, trying to obtain the resources of the whole coalition. Eventually, we can find a player (called the *representative*) out of each coalition, who obtains the resources of his own coalition, or of a subcoalition of it if one or more players are removed. In Round 2, the representatives who are in the same coalition at the second level repeat the process taking into account the resources obtained in the previous round. The process goes on until reaching the level h.

In Section 2.2 we present the notation and definitions. In Section 2.3 we define formally the coalitional bidding mechanism and prove that it implements the levels structure value.

2.2 The model

We consider a cooperative game in characteristic form (N, v), where $N = \{1, ..., n\}$ is the set of *players* and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. We denote by $TU(N)$ the set of cooperative games.

A *coalition* of (N, v) is a nonempty subset $S \subset N$. For $S \subset N$, we maintain the notation v when refer to the game v restricted to S as set of players. For simplicity, we denote $v(i)$ instead of $v(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $S \backslash i$ instead of $S \backslash \{i\}$.

We say that (N, v) is *zero-monotonic* if $v(S \cup i) \geq v(S) + v(i)$ for every $S \subset N \backslash i$.

We say that v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every $S, T \in N$ such that $S \cap T = \emptyset$.

Notice that superadditivity implies zero-monotonicity.

A *coalition structure* on N is a partition $\mathcal{C} = \{C_1, ..., C_p\}$ of N, *i.e.* $C_q \cap C_r = \emptyset$ when $C_q \neq C_r$, and $\bigcup_{C_q \in \mathcal{C}} C_q = N$.

Given $i \in C_q \in \mathcal{C}$, we denote by $\mathcal{C}_{-i}$ the coalition structure on $N \backslash i$ which equals $\mathcal{C}$ after removing player i, *i.e.* $\mathcal{C}_{-i} = \{C_1, ..., C_{q-1}, C_q \backslash i, C_{q+1} ..., C_p\}$.

Notice that this means that $\mathcal{C}_{-i}$ may have one less coalition than $\mathcal{C}$.

A *levels structure* for N is a sequence $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, ..., \mathcal{C}^h)$, $h \geq 1$ with $\mathcal{C}^l$ $(0 \leq l \leq h)$ coalition structure on N such that:

1. $\mathcal{C}^0 = \{\{1\}, \{2\}, ..., \{n\}\}$.
2. $\mathcal{C}^h = \{N\}$.
3. If $C_q \in \mathcal{C}^l$ with $0 < l \leq h$ then $C_q = \bigcup_{S \in Q} S$ for some $Q \subset \mathcal{C}^{l-1}$.

We call $\mathcal{C}^l$ the *l-th level* of $\mathfrak{C}$. We say that $\mathfrak{C}$ is a levels structure of *degree* h. Thus, the levels structure $\mathfrak{C}$ has $h + 1$ levels.

If $h = 1$, we say that $\mathfrak{C}$ is a *trivial* levels structure.

Given $i \in C_q \in \mathcal{C}^1$ with $n > 1$, we denote by $\mathfrak{C}_{-i}$ the levels structure for $N \backslash i$ which equals $\mathfrak{C}$ after removing player i. Namely, $\mathfrak{C}_{-i} = (\mathcal{C}^0_{-i}, \mathcal{C}^1_{-i}, ..., \mathcal{C}^h_{-i})$.

Given $S \in \mathcal{C}^l$, we denote by $\mathfrak{C}_{-S}$ the levels structure for $N \backslash S$ induced by $\mathfrak{C}$.

Assume $h \geq 2$. We define by $\mathfrak{C}/\mathcal{C}^1$ the levels structure induced by $\mathfrak{C}$ by dropping out the level $\mathcal{C}^0$ and considering the coalitions $C_q \in \mathcal{C}^1$ as players. Whenever $C_q \in \mathcal{C}^1$ is considered as a player in $\mathfrak{C}/\mathcal{C}^1$, it is denoted by $[C_q]$. We also denote by $[\mathcal{C}^l]$ $(1 \leq l \leq h)$ the coalition structure which comes out from $\mathcal{C}^l$ by considering the coalitions of $\mathcal{C}^1$ as players.

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Thus, we have $\mathfrak{C}/\mathcal{C}^1 = \left(\left[\mathcal{C}^1\right], \left[\mathcal{C}^2\right], ..., \left[\mathcal{C}^h\right]\right)$.

In particular, for $l = 1$, if $\mathcal{C}^1 = \{C_1, ..., C_p\}$, we have

$$\left[\mathcal{C}^1\right] = \{\{[C_1]\}, ..., \{[C_p]\}\}.$$

This new levels structure satisfies conditions 1, 2 and 3. Furthermore, $\mathfrak{C}/\mathcal{C}^1$ has degree $h-1$.

Let $LTU(N)$ be the set of all $(N, v, \mathfrak{C})$ with $(N, v) \in TU(N)$ cooperative game and $\mathfrak{C}$ levels structure for N.

The *quotient game* $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$ is the game $LTU\left(\mathcal{C}^1\right)$ defined on the coalition structure $\mathcal{C}^1$ with characteristic function

$$\left(v/\mathcal{C}^1\right)(Q) = v\left(\bigcup_{[C_q]\in Q} C_q\right)$$

for all $Q \subset \mathcal{C}^1$.

A *value* on $LTU(N)$ is a function $f : LTU(N) \to \mathbb{R}^N$ which assigns to each game $(N, v, \mathfrak{C}) \in LTU(N)$ a vector on $\mathbb{R}^N$, so that $f_i(N, v, \mathfrak{C})$ represents the payoff received by player $i \in N$.

In this Chapter, we use two solution concepts for LTU. The *Shapley value* (Shapley, 1953) and the *levels structure value*, suggested by Owen (1977) and characterized by Winter (1989).

The Shapley value is given by next expression. Given $(N, v) \in TU(N)$ with $i \in N$:

$$\varphi_i(N, v) = \sum_{T\subset N\setminus i} \frac{|T|!(n-|T|-1)!}{n!}\left[v(T\cup i) - v(T)\right].$$

The levels structure value is a generalization of the Shapley value to games with levels structure, *i.e.*, when the levels structure is trivial, both solution concepts give the same payoff vector. In order to define it, we need some additional notation.

Given Π the set of all orders on N, and given a levels structure $\mathfrak{C}$, we define by induction $\Pi_1(\mathfrak{C}) \subset \Pi_2(\mathfrak{C}) \subset ... \subset \Pi_h(\mathfrak{C})$ as follows

$$\Pi_h(\mathfrak{C}) = \Pi.$$

Given the sets $\Pi_{l+1}(\mathfrak{C}) \subset \Pi_{l+2}(\mathfrak{C}) \subset ... \subset \Pi_h(\mathfrak{C})$, we define:

$$\Pi_l(\mathfrak{C}) = \left\{\pi \in \Pi_{l+1}(\mathfrak{C}) : \forall j, k \in C_q \in \mathcal{C}^l, \forall i \in N, \pi(j) < \pi(i) < \pi(k) \Rightarrow i \in C_q\right\}.$$

In particular, orders in $\Pi_1(\mathfrak{C})$ are those in which the players in the same coalition on any level appear always together.

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Given $\pi \in \Pi$, $i \in N$, we denote by $P_i^{\pi} = \{j \in N : \pi(j) < \pi(i)\}$ the set of *predecessors* of i under π. We call *levels structure value* (Winter, 1989) to the function $\psi : LTU(N) \to \mathbb{R}^N$ given by

$$\psi_i(N, v, \mathfrak{C}) = \frac{1}{|\Pi_1(\mathfrak{C})|} \sum_{\pi \in \Pi_1(\mathfrak{C})} [v(P_i^{\pi} \cup i) - v(P_i^{\pi})]$$

for all $i \in N$.

This value generalizes the Owen value for $h = 2$ with coalition structure $\mathcal{C}^1$ and the Shapley value for $h = 1$.

A simple and powerful characterization for the levels structure value is as follows (Calvo, Lasaga and Winter, 1996). The levels structure value is the only solution concept on $LTU(N)$ which satisfies efficiency and balanced contributions.

Efficiency. For any game $(N, v, \mathfrak{C}) \in LTU(N)$, we have $\sum_{i \in N} \psi_i(N, v, \mathfrak{C}) = v(N)$.

Balanced contributions. For any $(N, v, \mathfrak{C}) \in LTU(N)$ and any $S, T \in \mathcal{C}^l$ with $0 \leq l < h$ such that $S, T \subset R \in \mathcal{C}^{l+1}$, $S \neq T$, we have

$$\sum_{i \in S} \psi_i(N, v, \mathfrak{C}) - \sum_{i \in S} \psi_i(N \backslash T, v, \mathfrak{C}_{-T}) = \sum_{i \in T} \psi_i(N, v, \mathfrak{C}) - \sum_{i \in T} \psi_i(N \backslash S, v, \mathfrak{C}_{-S}).$$

Furthermore, the levels structure value also satisfies additivity and quotient game property (Winter, 1989).

Additivity. For any $(N, v, \mathfrak{C})$, $(N, w, \mathfrak{C}) \in LTU(N)$, we have

$$\psi(N, v + w, \mathfrak{C}) = \psi(N, v, \mathfrak{C}) + \psi(N, w, \mathfrak{C})$$

with $(N, v + w)$ the TU game defined on N by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$.

Quotient game property. For any $(N, v, \mathfrak{C}) \in LTU(N)$, we have

$$\sum_{i \in C_q} \psi_i(N, v, \mathfrak{C}) = \psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right).$$

for all $C_q \in \mathcal{C}^1$.

2.3 The levels bidding mechanism

Given a cooperative game (N, v), Pérez-Castrillo and Wettstein (2001) design the *bidding mechanism*. In the bidding mechanism, players bid for the right to propose a payoff, which should be accepted by all the other players. Otherwise the prososer leaves the game. Pérez-Castrillo and Wettstein prove that the payoff of any SPNE of this mechanism always coincides with the Shapley value of the cooperative game (N, v). Thus, this mechanism implements the Shapley value in SPNE.

Our mechanism is played in several rounds. In each round, coalitions in each coalition structure play a bidding mechanism in order to obtain the resources of their own coalition. Namely, they bid for the right to propose a payoff. If this offer is not accepted by the other members of the coalition, the proposer leaves the game. If the offer is accepted by all the members of the coalition, and this happens in every coalition, the proposers become representatives of their coalitions and they move to the next round.

We now present the *levels bidding mechanism (LBM)* formally. We proceed by double induction on h (degree of $\mathfrak{C}$) and n (number of players).

For $h = 1$, the players play a single round. This round comprises the bidding mechanism (Pérez-Castrillo and Wettstein, 2001) associated with the game (N, v).

Assume that we know the rules of the LBM when the levels structure has degree $h-1$, and it comprises $h-1$ rounds.

If there is only one player i, he obtains $v(i)$. Assume now that we know the rules of the LBM when played by $n-1$ players. Then, for a set of players $N = \{1, ..., n\}$ and a levels structure $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, ..., \mathcal{C}^h)$ with $\mathcal{C}^1 = \{C_1, ..., C_p\}$, the LBM proceeds as follows,

Round 1. The players of any coalition $C_q \in \mathcal{C}^1$ play the bidding mechanism trying to obtain the resources of C_q. Formally, if there is only one player i, then this player has his resources. Assume now that we know the rules when played by $|C_q| - 1$ players. For $|C_q| > 1$ it proceeds as follows

Stage 1. Each player $i \in C_q$ makes bids $b_j^i \in \mathbb{R}$ for every $j \in C_q \setminus i$. For each $i \in C_q$, we take $B^i = \sum_{j \in C_q \setminus i} b_j^i - \sum_{j \in C_q \setminus i} b_i^j$. Assume that $\alpha_q = argmax_i \left\{B^i\right\}$. In the case of a non-unique maximizer, α_q is randomly chosen among the maximizing indices.

Stage 2. Player α_q, called the *proposer*, makes an offer $y_i^{\alpha_q}$ to every player $i \in C_q \setminus \alpha_q$.

Stage 3. The players of $C_q \setminus \alpha_q$, sequentially, either accept or reject the offer. If a rejection is encountered, we say the offer is rejected. Otherwise, we say the offer is accepted.

The coalitions of $\mathcal{C}^1$ play sequentially in the order $C_1, ..., C_p$ until either we find $C_{q_0} \in \mathcal{C}^1$ and $\alpha_{q_0} \in C_{q_0}$ such that the offer of α_{q_0} is rejected, or for any $C_q \in \mathcal{C}^1$ the offer of α_q is accepted.

In the first case, player α_{q_0} pays $b_i^{\alpha_q}$ to every player $i \in C_q \setminus \alpha_q$ and leaves the non-cooperative game obtaining $v(\alpha_{q_0}) - \sum_{i \in C_{q_0} \setminus \alpha_{q_0}} b_i^{\alpha_{q_0}}$. All players other than α_{q_0} proceed to play the LBM with $(N', v', \mathfrak{C}')$ where $N' = N \setminus \alpha_{q_0}$, $v' = v$, and $\mathfrak{C}' = \mathfrak{C}_{-\alpha_{q_0}}$. Any player $i \in C_{q_0} \setminus \alpha_{q_0}$ obtains as final payoff the sum of the bid received, $b_i^{\alpha_{q_0}}$, and the payoff outcome of the mechanism corresponding to $(N', v', \mathfrak{C}')$. Any player $i \in N \setminus C_{q_0}$ obtains as final payoff the payoff outcome of the mechanism corresponding to $(N', v', \mathfrak{C}')$.

In the second case, for any $C_q \in \mathcal{C}^1$, player α_q pays $b_i^{\alpha_q} + y_i^{\alpha_q}$ to every $i \in C_q \setminus \alpha_q$ and becomes the *representative* of coalition C_q. This means that player α_q goes to Round 2 with all the resources of C_q. Moreover, the payoff obtained by this player in this round is $p^1_{\alpha_q} = -\sum_{i \in C_q \setminus \alpha_q} \left(b_i^{\alpha_q} + y_i^{\alpha_q}\right)$.

Any other player $i \in C_q \setminus \alpha_q$ leaves the non-cooperative game obtaining a final payoff of $b_i^{\alpha_q} + y_i^{\alpha_q}$.

After finishing Round 1, for any $C_q \in \mathcal{C}^1$ we can find the representative (denoted by r_q) of this coalition.

Rounds 2 throught h. The representatives play the LBM associated with the quotient game $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$, where each r_q plays the role of $[C_q]$. These rounds are well defined by induction on h. For any representative r_q, we denote by $p^2_{r_q}$ the payoff obtained by r_q (or $[C_q]$) in these rounds.

The final payoff obtained by any representative r_q is the sum of the payoffs obtained in all the rounds, *i.e.* $p^1_{r_q} + p^2_{r_q}$.

We must note that the LBM terminates in a finite number of moves.

Remark 11 *Assume that in Round 1 the offer of player α_q is accepted for any $q < q_0$, but the offer of α_{q_0} is rejected. Then a new subgame begins, which coincides with the LBM associated with $\left(N \setminus \alpha_{q_0}, v, \mathfrak{C}_{-\alpha_{q_0}}\right)$. Moreover, when all the offers are accepted in Round 1, another subgame begins, which is equivalent to the LBM associated with $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$.*

Before the characterization of the SPNE outcomes of the levels bidding mechanism we need the following result.

Proposition 12 *Given a triple $(N, v, \mathfrak{C}) \in LTU(N)$ such that (N, v) is zero-monotonic, $j \in C_q \in \mathcal{C}^1 \in \mathfrak{C}$ and $\{j\} \subsetneq C_q$ then*

$$\sum_{i \in C_q} \psi_i(N, v, \mathfrak{C}) \geq \sum_{i \in C_q \setminus j} \psi_i(N \setminus j, v, \mathfrak{C}_{-j}) + v(j).$$

Proof. We take $\mathfrak{C}/\mathcal{C}^1 = \left(\left[\mathcal{C}^1\right], ..., \left[\mathcal{C}^h\right]\right)$ levels coalition structure. Assume $\mathcal{C}^1 = \{C_1, ..., C_p\}$ and $P = \{1, 2, ..., p\}$. Let $\mathfrak{Q} = \left(Q^1, ..., Q^h\right)$ be the levels structure for P which equals $\mathfrak{C}/\mathcal{C}^1$ except for the name of the players, *i.e.*

$\{q_1, q_2, ..., q_k\} \in Q^l :\Leftrightarrow \{[C_{q_1}], [C_{q_2}], ..., [C_{q_k}]\} \in [\mathcal{C}^l] \qquad 1 \leq l \leq h.$

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We define the following games on P. For all $R \subset P$,

$$\begin{aligned}
u(R) &= v\left(\bigcup_{r\in R} C_r\right)\\
w_1(R) &= \left\{\begin{array}{ll} v\left(\bigcup_{r\in R} C_r\backslash j\right) & \text{if } q\in R\\ v\left(\bigcup_{r\in R} C_r\right) & \text{if } q\notin R\end{array}\right\}\\
w_2(R) &= \left\{\begin{array}{ll} v(j) & \text{if } q\in R\\ 0 & \text{if } q\notin R\end{array}\right\}\\
w &= w_1+w_2.
\end{aligned}$$

Notice that the game u on P equals the quotient game $v/\mathcal{C}^1$ on $\mathcal{C}^1$. Thus, their levels structure values are the same

$$\psi_q(P,u,\mathfrak{Q}) = \psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right).$$

Given $C_q \in \mathcal{C}^1$, by the quotient game property,

$$\sum_{i\in C_q}\psi_i(N,v,\mathfrak{C}) = \psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right).$$

Thus,

$$\psi_q(P,u,\mathfrak{Q}) = \sum_{i\in C_q}\psi_i(N,v,\mathfrak{C}).$$

Analogously, the game w_1 on P equals the quotient game $v/\mathcal{C}^1_{-j}$ on $\mathcal{C}^1_{-j}$. Thus:

$$\psi_q(P,w_1,\mathfrak{Q}) = \psi_{[C_q\backslash j]}\left(\mathcal{C}^1_{-j}, v/\mathcal{C}^1_{-j}, \mathfrak{C}_{-j}/\mathcal{C}^1_{-j}\right) = \sum_{i\in C_q\backslash j}\psi_i\left(N\backslash j, v, \mathfrak{C}_{-j}\right).$$

Finally, the levels structure value of q for the game w_2 is:

$$\psi_q(P,w_2,\mathfrak{Q}) = v(j).$$

By applying the zero-monotonicity of v, we get $\psi_q(P,u,\mathfrak{Q}) \geq \psi_q(P,w,\mathfrak{Q})$. By additivity of the levels structure value

$$\begin{aligned}
\sum_{i\in C_q}\psi_i(N,v,\mathfrak{C}) &= \psi_q(P,u,\mathfrak{Q}) \geq \psi_q(P,w,\mathfrak{Q}) = \psi_q(P,w_1+w_2,\mathfrak{Q})\\
&= \psi_q(P,w_1,\mathfrak{Q}) + \psi_q(P,w_2,\mathfrak{Q})\\
&= \sum_{i\in C_q\backslash j}\psi_i\left(N\backslash j, v, \mathfrak{C}_{-j}\right) + v(j).
\end{aligned}$$

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In order to cope with technical problems of ties, we need an additional assumption on the SPNE's. These problems appear when players are indifferent between two or more strategies yielding the same payoff. In Example 6 of Chapter 1 we study a game such that the associated LBM's have SPNE outcomes whose payoff is different from the levels structure value.

In Chapter 1, we make a modification to the coalitional bidding mechanism, so that the player who rejects an offer, and the proposer whose offer is rejected, must pay a small penalty $\varepsilon > 0$.

In this Chapter, we do not move in that direction. Moldovanu and Winter (1994) assume that players prefer agreements which involve large coalitions better than smaller ones (provided his final payoff is the same in both agreements). Hart and Mas-Colell (1996) assume that players "break ties in favor of quick termination of the game"[1]. In this Chapter, we make both assumptions.

As a consequence of our assumptions, we can define a tie-breaking rule satisfying:

- If a player is indifferent between accepting or rejecting an offer of a proposer, he always accepts the offer.
- If a proposer $\alpha \in C_q$ is indifferent between offering b^α or $\widetilde{b}^\alpha$ being b^α due to be rejected by some player $i \in C_q \backslash \alpha$, and $\widetilde{b}^\alpha$ being accepted by every player in $C_q \backslash \alpha$, he always offers $\widetilde{b}^\alpha$.

In the rest of the section, by SPNE we mean SPNE satisfying this tie-breaking rule.

A similar approach by means of tie-breaking rule for SPNE can be found in Navarro and Perea (2001). In their model, a player must choose prices, propose offers and accept or reject offers[2]. If a player is indifferent between accepting or rejecting an offer, he is supposed to accept. If, under certain circumstances, a player is indifferent between proposing Δ or $\widetilde{\Delta}$ with $\Delta < \widetilde{\Delta}$, he is supposed to propose $\widetilde{\Delta}$. If a player is indifferent between choosing price p or $\widetilde{p}$ with $p < \widetilde{p}$, he is supposed to choose price p.

Theorem 13 *The LBM implements the levels structure value in SPNE for superadditive games.*

Proof. The structure of this proof is similar to that of Theorem 7. However, the computations are different.

We proceed by double induction on h and n. For $h = 1$, the mechanism coincides with those by Pérez-Castrillo and Wettstein (2001). Thus, we assume the players play according to a strategy profile described in Pérez-Castrillo and Wettstein (2001). It is not difficult to check that this SPNE satisfies the tie-breaking rule. So, the mechanism implements the levels structure value.

Assume the result is true for levels structures of degree at most $h - 1$.

[1]However, tie-breaking rules are not needed in Hart and Mas-Colell's model.

[2]These offers are differences in payoffs to be received at the end of the mechanism.

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We now prove the result when the degree is h. If there is only one player it is trivial. Assume that if there are at most $n-1$ players, the LBM implements the levels structure value in SPNE and, moreover, all the offers of Round 1 are accepted in SPNE. We now prove that the same holds when there are n players.

We first prove that the levels structure value is indeed an SPNE outcome. We explicitly construct an SPNE which yields the levels structure value as an SPNE outcome.

We consider the following strategies.

Round 1. First, we define the strategies in the LBM associated with any $C_q \in \mathcal{C}^1$.

Stage 1. For any $i \in C_q$, $b_j^i = \psi_j(N, v, \mathfrak{C}) - \psi_j(N \setminus i, v, \mathfrak{C}_{-i})$ for any $j \in C_q \setminus i$.

Stage 2. Player α_q, the proposer, offers $y_j^{\alpha_q} = \psi_j\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$ to every $j \in C_q \setminus \alpha_q$.

Stage 3. Any player $i \in C_q \setminus \alpha_q$ accepts the offer of α_q if and only if $y_j^{\alpha_q} \geq \psi_j\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$ for every $j \in C_q \setminus \alpha_q$.

If some offer is rejected, for instance, the offer of α_{q_0}, we go to the subgame where all players other than α_{q_0} play this mechanism in $\left(N \setminus \alpha_{q_0}, v, \mathfrak{C}_{-\alpha_{q_0}}\right)$. We assume that players in $N \setminus \alpha_{q_0}$ play according to the strategies profiles of some SPNE with associated payoff $\psi\left(N \setminus \alpha_{q_0}, v, \mathfrak{C}_{-\alpha_{q_0}}\right)$ (by induction hypothesis on n we can find such SPNE).

Rounds 2 through h. We assume that the representatives play according to the strategies of some SPNE with associated payoff $\psi\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Again, by induction hypothesis on h, we can find such SPNE.

It is straightforward to prove that these strategies satisfy the tie-breaking rule.

First, we prove that according to these strategies any player $i \in N$ receives as payoff the levels structure value $\psi_i(N, v, \mathfrak{C})$. We must note that for any $C_q \in \mathcal{C}^1$ the offer of α_q is accepted. Then, player α_q goes to Round 2 as the representative of C_q.

Given $C_q \in \mathcal{C}^1$ and $i \in C_q \setminus \alpha_q$, the payoff obtained by player i is $b_i^{\alpha_q} + y_i^{\alpha_q} =$

$$\begin{aligned} & \psi_i(N, v, \mathfrak{C}) - \psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right) + \psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right) \\ = \;& \psi_i(N, v, \mathfrak{C}). \end{aligned}$$

We now compute the payoff of any representative r_q. As v is superadditive we conclude that $v/\mathcal{C}^1$ is also superadditive. By induction hypothesis on h, we know that the payoff obtained by r_q in Rounds 2 through h $(p_{r_q}^2)$ coincides with the levels structure value of $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Then, the final payoff obtained by r_q is

$$\begin{aligned} p^1_{r_q} + p^2_{r_q} &= -\sum_{i \in C_q \setminus r_q} b_i^{r_q} - \sum_{i \in C_q \setminus r_q} y_i^{r_q} + \psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right) \\ &= -\sum_{i \in C_q \setminus r_q} \left(\psi_i\left(N, v, \mathfrak{C}\right) - \psi_i\left(N \setminus r_q, v, \mathfrak{C}_{-r_q}\right)\right) \\ &\quad -\sum_{i \in C_q \setminus r_q} \psi_i\left(N \setminus r_q, v, \mathfrak{C}_{-r_q}\right) + \psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right) \end{aligned}$$

by rearranging and applying the quotient game property,

$$= -\sum_{i \in C_q \setminus r_q} \psi_i(N, v, \mathfrak{C}) + \sum_{i \in C_q} \psi_i(N, v, \mathfrak{C}) = \psi_{r_q}(N, v, \mathfrak{C}).$$

We now prove that these strategies are an SPNE. By induction hypothesis on h, we know that in the subgames obtained after Round 2 these strategies induce an SPNE.

By induction hypothesis on n, in all the subgames obtained after the rejection of the offer of some proposer α_q, these strategies induce an SPNE.

We only have to prove that these strategies induce an SPNE in the bidding mechanism associated with any coalition C_q (Round 1).

Stage 3. Assume that player i rejects the offer of α_q. Then, the LBM mechanism of $\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$ is played and, by induction hypothesis on n, after the rejection player i can obtain at most $\psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$. Hence, if player i rejects the offer of α_q, he obtains, at most,

$$b_i^{\alpha_q} + \psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right) = \psi_i(N, v, \mathfrak{C}).$$

This means that player i does not improve his payoff.

Stage 2. If player α_q offers to some player $i \in C_q$ less than $\psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$, the offer is rejected and, therefore, player α_q obtains a final payoff of

$$v\left(\alpha_q\right) - \sum_{i \in C_q \setminus \alpha_q} \left(\psi_i\left(N, v, \mathfrak{C}\right) - \psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)\right).$$

By Proposition 12, this payoff is not larger than $\psi_{\alpha_q}(N, v, \mathfrak{C})$, which means that player α_q does not improve his payoff.

If player α_q offers to any player $i \in C_q \setminus \alpha_q$ at least $\psi_i\left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q}\right)$, the offer is accepted. It is not difficult to prove that player α_q obtains at most $\psi_{\alpha_q}(N, v, \mathfrak{C})$.

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Stage 1. First, we prove that for any $i \in C_q \in \mathcal{C}^1$, $B^i = 0$.

$$\begin{aligned} B^i &= \sum_{j \in C_q \setminus i} b^i_j - \sum_{j \in C_q \setminus i} b^j_i \\ &= \sum_{j \in C_q \setminus i} \big(\psi_j (N, v, \mathfrak{C}) - \psi_j (N \setminus i, v, \mathfrak{C}_{-i}) \big) \\ &\quad - \sum_{j \in C_q \setminus i} \left(\psi_i (N, v, \mathfrak{C}) - \psi_i (N \setminus j, v, \mathfrak{C}_{-j}) \right). \end{aligned}$$

As the levels structure value satisfies balanced contributions, we know that for any $j \in C_q \setminus i$,

$$\psi_i (N, v, \mathfrak{C}) - \psi_i (N \setminus j, v, \mathfrak{C}_{-j}) = \psi_j (N, v, \mathfrak{C}) - \psi_j (N \setminus i, v, \mathfrak{C}_{-i})$$

and hence $B^i = 0$.

Assume that player $i \in C_q$ makes a different bid $\widehat{b}^i$. If $\widehat{B}^i < 0$, then the proposer is another player of C_q. Then player i can not increase his payoff.

If $\widehat{B}^i > 0$, he becomes the proposer but he must pay $\sum_{j \in C_q \setminus i} \widehat{b}^i_j$ to the other players of $C_q \setminus i$. It is straightforward to prove that player i can obtain, at most, a final payoff of

$$\psi_i(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus i} \widehat{b}^i_j + \sum_{j \in C_q \setminus i} b^i_j$$

which is smaller than $\psi_i(N, v, \mathfrak{C})$.

If $\widehat{B}^i = 0$ and player i is not the proposer, using similar arguments to those used when $\widehat{B}^i < 0$, we can conclude that player i does not increase his payoff. If $\widehat{B}^i = 0$ and player i is the proposer, using similar arguments to those used when $\widehat{B}^i > 0$ we can conclude that player i does not increase his payoff.

We now prove that the payoff in all SPNE outcomes coincides with the levels structure value. We do so in several steps.

Step A. At every SPNE outcome, and for every $C_q \in \mathcal{C}^1$, the offer of the proposer α_q to each player $i \in C_q \setminus \alpha_q$ is $y_i^{\alpha_q} = \psi_i \left(N \setminus \alpha_q, v, \mathfrak{C}_{-\alpha_q} \right)$ and every $i \in C_q \setminus \alpha_q$ accepts this offer.

Assume that in each coalition $C_q \in \{C_1, ..., C_{p-1}\}$, the offer of a proposer $\alpha_q \in C_q$ is accepted, and consider the subgame starting with the last coalition C_p. Let $\alpha_p \in C_p$ be the proposer in C_p. Let y^{α_p} be an offer of α_p. Let the order of reply of the players in $C_p \setminus \alpha_p$ be $i_1, ..., i_k$.

Claim 1: At every SPNE, the strategies of the players in $C_p \setminus \alpha_p$ must satisfy the following statements:

(i) If $y_i^{\alpha_p} \geq \psi_i \left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p} \right)$ for every $i \in C_p \setminus \alpha_p$, then every $i \in C_p \setminus \alpha_p$ accepts y^{α_p}.

(ii) If $y_j^{\alpha_p} < \psi_j\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$ for some $j \in C_p \setminus \alpha_p$, then some player in $C_p \setminus \alpha_p$ rejects y^{α_p}.

(i) Consider the strategy of the last player i_k. Assuming that his decision node is reached, if he accepts the offer y^{α_p}, then he receives $b_{i_k}^{\alpha_p} + y_{i_k}^{\alpha_p}$, whereas if he rejects y^{α_p}, then by the induction hypothesis he obtains at most $b_{i_k}^{\alpha_p} + \psi_{i_k}\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$. Hence, at any SPNE,

- if $y_{i_k}^{\alpha_p} > \psi_{i_k}\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$, then i_k accepts the offer because it is optimal;
- if $y_{i_k}^{\alpha_p} = \psi_{i_k}\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$, then i_k accepts the offer because of the tie-breaking rule.

Repeating the same argument backwards, we can show that players $i_{k-1}, ..., i_1$ accept the offer.

(ii) Suppose, to the contrary, that there exists $j \in C_p \setminus \alpha_p$ with $y_j^{\alpha_p} < \psi_j\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$, but all the players in $C_p \setminus \alpha_p$ accept the offer y^{α_p}. Then, player j receives $b_j^{\alpha_p} + y_j^{\alpha_p}$. However, if player j deviates and rejects the offer, then he obtains $b_j^{\alpha_p} + \psi_j\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$, which is larger than $b_j^{\alpha_p} + y_j^{\alpha_p}$. Hence, the strategies of the players in $C_p \setminus \alpha_p$ cannot constitute an SPNE.

Claim 2: At every SPNE outcome, every $i \in C_p \setminus \alpha_p$ accepts the offer of the proposer α_p.

Suppose, to the contrary, that at some SPNE outcome, there exists $i \in C_p \setminus \alpha_p$ who rejects the offer y^{α_p}. Then, the proposer obtains

$$e = v\left(\alpha_p\right) - \sum_{i \in C_p \setminus \alpha_p} b_i^{\alpha_p}.$$

Suppose that the proposer α_p proposes $z_i^{\alpha_p} = \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$ to every $i \in C_p \setminus \alpha_p$. By Claim 1 (i), every $i \in C_p \setminus \alpha_p$ accepts z^{α_p}. Hence, player α_p is the representative of coalition C_p in Round 2. Now, in Rounds 2 through h, there are p players $\{\alpha_1, ..., \alpha_p\}$, where, for any coalition $C_q \in \mathcal{C}^1$, α_q is the representative of coalition C_q. As the representatives are playing an SPNE of the LBM associated with $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$, by induction hypothesis on h we know that the payoff obtained by player α_p in Rounds 2 through h is $\psi_{[C_p]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$, which, by the quotient game property, equals $\sum_{i \in C_p} \psi_i\left(N, v, \mathfrak{C}\right)$. Then, the final payoff of player α_p is

$$\widetilde{e} = \sum_{i \in C_p} \psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_p \setminus \alpha_p} \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right) - \sum_{i \in C_p \setminus \alpha_p} b_i^{\alpha_p}.$$

By Proposition 12, we know that

$$\sum_{i \in C_p} \psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_p \setminus \alpha_p} \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right) \geq v\left(\alpha_p\right).$$

Thus, $e \leq \widetilde{e}$.

• If $e < \widetilde{e}$, an offer of y^{α_p} cannot be an SPNE strategy of the proposer α_p, which is a contradiction.

• If $e = \widetilde{e}$, then α_p is indifferent between offering y^{α_p} or z^{α_p}. By Claim 1 (i), offer z^{α_p} is accepted by every $i \in C_p \backslash \alpha_p$. By the tie-breaking rule α_p must propose z^{α_p} better than y^{α_p}, which is a contradiction.

Claim 3: At every SPNE, and for every $i \in C_p \backslash \alpha_p$, $y_i^{\alpha_p} = \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$.

Let y^{α_p} be the offer of α_p at an SPNE. By Claim 2, y^{α_p} must be accepted by every $i \in C_p \setminus \alpha_p$. Then, it follows from Claim 1 (ii) that for every $i \in C_p \setminus \alpha_p$, $y_i^{\alpha_p} \geq \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$. Suppose that for some $j \in C_p \setminus \alpha_p$, $y_j^{\alpha_p} > \psi_j\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$. For each $i \in C_p \setminus \alpha_p$, define $w_i^{\alpha_p} = \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right)$. Suppose that the proposer α_p deviates and offers w^{α_p}. Then, by Claim 1 (i), every $i \in C_p \setminus \alpha_p$ accepts w^{α_p}. Moreover, since

$$\sum_{i \in C_p \backslash \alpha_p} w_i^{\alpha_p} = \sum_{i \in C_p \backslash \alpha_p} \psi_i\left(N \setminus \alpha_p, v, \mathfrak{C}_{-\alpha_p}\right) < \sum_{i \in C_p \backslash \alpha_p} y_i^{\alpha_p},$$

the proposer α_p obtains a greater payoff by offering w^{α_p} than by offering y^{α_p}. Hence, an offer of y^{α_p} cannot be an SPNE strategy, which is a contradiction.

Repeating the same arguments for coalitions $C_{p-1}, ..., C_1$, we prove Step A.

Step B. Assume that we are in Stage 1 of Round 1 of the LBM associated with $C_q \in \mathcal{C}^1$. Then in any SPNE, $B^i = 0$ for any $i \in C_q$.

It is not difficult to prove that $\sum_{i \in C_q} B^i = 0$. We take

$$X = \left\{ i \in C_q : B^i = \max_{j \in C_q} B^j \right\}.$$

If $X = C_q$, the result holds because $\sum_{i \in C_q} B^i = 0$.

If $X \neq C_q$, we get a contradiction by proving that player $i \in X$ has a deviation which improves his final payoff. We take $j \in C_q \backslash X$ such that $B^j \geq B^k$ for any $k \in C_q \backslash X$. Assume that player i makes a new bid $\widehat{b}^i$, where $\widehat{b}_k^i = b_k^i + \delta$ if $k \in X \setminus i$, $\widehat{b}_j^i = b_j^i - |X| \, \delta$, and $\widehat{b}_k^i = b_k^i$ if $k \in C_q \setminus (X \cup j)$.

For any $k \in C_q$, we compute $\widehat{B}^k$ assuming that $\widehat{b}^k = b^k$ for any $k \in C_q \setminus i$. Then $\widehat{B}^k = B^k - \delta$ if $k \in X$, $\widehat{B}^j = B^j + |X| \, \delta$, and $\widehat{B}^k = B^k$ if $k \in C_q \setminus (X \cup j)$.

Since $B^j < B^i$, we can find $\delta > 0$ satisfying $B^j + |X| \, \delta < B^i - \delta$. Moreover, $\widehat{X} = \left\{ k \in C_q : \widehat{B}^k = \max_{m \in C_q} \widehat{B}^m \right\} = X$. This means that any player of X is the proposer with the same probability under b^i and $\widehat{b}^i$. When player i is not the proposer, which happens with probability $\frac{|X|-1}{|X|}$, he obtains, by Step A, the same making a bid b^i or $\widehat{b}^i$. But if player i is the proposer, which happens with probability $\frac{1}{|X|}$, he obtains, by Step A, δ units more with $\widehat{b}^i$ than with b^i.

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Step C. Assume that we are in Stage 1 of Round 1 of the LBM associated with $C_q \in \mathcal{C}^1$. Then, at every SPNE, the payoff of any player $i \in C_q$ is the same regardless of who is chosen as the proposer.

By Step B, we know that $B^i = 0$ for any $i \in C_q$.

Assume that some player i strictly prefers to be (not to be) the proposer. Then player i can improve his payoff by slightly increasing (decreasing) one of his bids b_j^i. But this is impossible in an SPNE.

Step D. In any SPNE outcome of LBM any player $i \in N$ obtains as final payoff his levels structure value.

Assume that players are playing according to some SPNE. Given $i \in C_q \in \mathcal{C}^1$, we denote by p_i the final payoff obtained by player i in this SPNE.

By Step B, we know that any player of C_q is the proposer with probability $\frac{1}{|C_q|}$.

If player i is the proposer, we know, by Step A, that his final payoff is

$$\sum_{j \in C_q} \psi_j (N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus i} \psi_j (N \setminus i, v, \mathfrak{C}_{-i}) - \sum_{j \in C_q \setminus i} b_j^i.$$

If $j \in C_q \setminus i$ is the proposer, then the final payoff of player i is, by Step A,

$$b_i^j + \psi_i (N \setminus j, v, \mathfrak{C}_{-j}).$$

By Step C, we know that

$$\begin{aligned} |C_q| \, p_i &= \sum_{j \in C_q} \psi_j (N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus i} \psi_j (N \setminus i, v, \mathfrak{C}_{-i}) - \sum_{j \in C_q \setminus i} b_j^i \\ &\quad + \sum_{j \in C_q \setminus i} \left(b_i^j + \psi_i (N \setminus j, v, \mathfrak{C}_{-j}) \right). \end{aligned}$$

By Step B, we know that $-\sum_{j \in C_q \setminus i} b_j^i + \sum_{j \in C_q \setminus i} b_i^j = -B^i = 0$. Hence,

$$|C_q| \, p_i = \sum_{j \in C_q \setminus i} \left(\psi_i (N \setminus j, v, \mathfrak{C}_{-j}) - \psi_j (N \setminus i, v, \mathfrak{C}_{-i}) \right) + \sum_{j \in C_q} \psi_j (N, v, \mathfrak{C}).$$

Since the levels structure value satisfies the property of balanced contributions, we conclude that

$$\begin{aligned} |C_q| \, p_i &= \sum_{j \in C_q \setminus i} \left[\psi_i (N, v, \mathfrak{C}) - \psi_j (N, v, \mathfrak{C}) \right] + \sum_{j \in C_q} \psi_j (N, v, \mathfrak{C}) \\ &= (|C_q| - 1) \, \psi_i (N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus i} \psi_j (N, v, \mathfrak{C}) + \sum_{j \in C_q} \psi_j (N, v, \mathfrak{C}) \\ &= |C_q| \, \psi_i (N, v, \mathfrak{C}). \end{aligned}$$

Then $p_i = \psi_i (N, v, \mathfrak{C})$. ■

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2.4 Conclusion

In this Chapter we develop a bidding mechanism which implements the levels structure value of every superadditive game with a levels structure of cooperation. The mechanism is a generalization of the bidding model presented by Pérez-Castrillo and Wettstein (2001) and the coalitional bidding mechanism presented in Chapter 1.

In SPNE, we impose that players prefer large coalitions to small ones. This condition is needed, as we can see by using the game in Example 6.

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Part II

On NTU games with coalition structure

Chapter 3

The consistent coalitional value

3.1 Introduction

One of the most important issues of cooperative game theory is to define "good" values, studying which interesting properties are satisfied by these values and obtaining axiomatic characterizations using some of these properties.

In transferable utility games (TU games), Shapley (1953) introduces the Shapley value. He defines this value as the average of marginal contributions of players when all orders are equally likely. Moreover, he characterizes it as the only value satisfying efficiency, null player, symmetry, and additivity. Later, several authors obtain new characterizations of the Shapley value using other properties. For instance, Myerson (1980) characterizes the Shapley value by using balanced contributions; Hart and Mas-Colell (1989) characterize it by consistency.

There are several extensions of TU games. The most natural is to games with non-transferable utility (NTU games). Other extension is to TU games with a coalition structure. Of course, a third extension is to NTU games with coalition structure. Since the Shapley value has many interesting properties in TU games, several authors propose, in these extended models, values which are generalizations of the Shapley value.

In NTU games the Harsanyi value (Harsanyi, 1963), and the Shapley NTU value (Aumann, 1985), are generalizations of the Shapley value. Later, Maschler and Owen (1989, 1992) define the consistent value for hyperplane games and NTU games respectively. The main idea behind this generalization is to maintain (as far as possible) the consistency property from the Shapley value. Maschler and Owen (1989) prove that, for hyperplane games, the consistent value can be obtained in a similar way that the Shapley value, *i.e.*, as the average of marginal contributions of players when all orders are equally likely. Later, Hart and Mas-Colell (1996) develop a bargaining mechanism which implements the consistent value and characterize it by means of balanced contributions.

Owen (1977) characterizes his value using similar axioms to those used by Shapley (1953). Later, Winter (1992) characterizes the Owen value using the consistency property and Calvo, Lasaga, and Winter (1996) using properties of balanced contributions.

NTU games with coalition structure are studied by Winter (1991), where he characterizes the Game Coalition Structure Value. This value is a generalization of the Harsanyi value for NTU games and the Owen value for TU games with coalition structure.

It is interesting to know whether the consistent value and the Owen value can be generalized the same way to games with coalition structure. We know that the Shapley value, the consistent value, and the Owen value are obtained as an average of marginal contributions depending on equal-likely orders. Thus, it seems reasonable to generalize these values in the same way. We call random order coalitional value (Maschler and Owen (1992) also suggest the name random order value for the consistent value) to the value obtained in this way. Remarkably, this value misses most of the nice properties of the previous values (Shapley, Owen, and consistent); namely, it is not consistent, nor satisfies the balanced contributions properties.

Then, we introduce a new value, called the consistent coalitional value. This new value can be characterized in two ways: the first one using the consistency property and the second one using the balanced contributions properties. We must note that our characterizations generalize the results about consistency obtained by Maschler and Owen (1989) for the consistent value and Winter (1992) for the Owen value, and the results about balanced contributions obtained by Hart and Mas-Colell (1996) for the consistent value and Calvo, Lasaga and Winter (1996) for the Owen value. We believe these characterizations make the consistent coalitional value a proper generalization of the consistent and the Owen value for NTU games with coalition structure.

This Chapter is organized as follows. In Section 3.2 we introduce the notation and some previous results. In Section 3.3 we define the consistent coalitional value and the random order coalitional value. In Section 3.4 we give a list of properties and study which are satisfied by both values. In Section 3.5 we present two axiomatic characterizations of the consistent coalitional value. Finally, in the Appendix, we present the proofs of the results obtained in the Chapter.

3.2 Definitions and previous results

Given a set A, $|A|$ denotes the cardinal of A. If $x, y \in \mathbb{R}^N$ we say $y \leq x$ when $y_i \leq x_i$ for each $i \in N$ and $x \cdot y$ is the scalar product $\sum_{i \in N} x_i y_i$. We denote $\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N : x_i \geq 0, \forall i\right\}$ and $\mathbb{R}^N_{++} = \left\{x \in \mathbb{R}^N : x_i > 0, \forall i\right\}$. We say that $x \in \mathbb{R}^N$ is *normalized* if $\sum_{i \in N} |x_i| = 1$ (in this case, $|x_i| = \max\{x_i, -x_i\}$). Let $\lambda \in \mathbb{R}^N$ be a vector orthogonal to some surface on $\mathbb{R}^N$, we say that λ is *orthonormal* if it is normalized.

A *non-transferable utility game*, or simply an *NTU game*, is a pair (N, V) where $N = \{1, 2, ..., n\}$ is the set of *players* and V is a correspondence (*characteristic function*) which assigns to each coalition $S \subset N$ a subset $V(S) \subset \mathbb{R}^S$ which represents all the possible payoffs that members of S can obtain for themselves when play cooperatively. For $S \subset N$, when there is no ambiguity, we maintain the notation V when refer to the application V restricted to S as player set. For simplicity, we denote $V(i)$ instead of $V(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $S \backslash i$ instead of $S \backslash \{i\}$. We also denote $\overline{S} = N \backslash S$.

We impose the next conditions on the function V:

(A1) For each $S \subset N$, the set $V(S)$ is *comprehensive* (*i.e.*, if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and *bounded above* (*i.e.*, for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is compact).

(A2) For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is *smooth* (on each point of the boundary there exists an unique outward orthonormal vector) and *nonlevel* (the outward vector on each point of $\partial V(S)$ has its coordinates positive). We denote these orthonormal vectors as $\lambda^S = \left(\lambda_i^S\right)_{i \in S}$.

(A3) These λ_i^S are continuous functions on $\partial V(S)$.

(A4) There exists a positive number δ, such that for each $S \subset N$ and $i \in S$, $\lambda_i^S > \delta$.

(A5) For each $S \subset N$, the origin $0_S = (0, ..., 0) \in \mathbb{R}^S$ belongs to $V(S)$.

Property (A5) is a normalization and does not affect our results.

We denote by $NTU(N)$ the set of NTU games over N and by NTU the set of all NTU games.

We now introduce two particular subclasses of NTU games studied in this Chapter.

We say that (N, V) is a *transferable utility game* (or *TU game*) if there exists a function $v : 2^N \to \mathbb{R}$, called the *characteristic function*, satisfying that $v(\emptyset) = 0$ and for each $S \subset N$, $V(S) = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$. Usually we represent a TU game as the pair (N, v). We denote by $TU(N)$ the set of TU games over N and by TU the set of all TU games.

We say that (N, V) is a *hyperplane game* if for all $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda^S \cdot x \leq v(S)\right\} \tag{3.1}$$

for some $v : 2^N \to \mathbb{R}$.

Notice that each TU game is a hyperplane game (just take $\lambda_i^S = 1$ for each $S \subset N$ and $i \in S$).

A *coalition structure* $\mathcal{C}$ on N is a partition of the player set, *i.e.*, $\mathcal{C} = \{C_1, C_2, ..., C_p\} \subset 2^N$ where $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We denote by $(N, V, \mathcal{C})$ an NTU game (N, V) with coalition structure $\mathcal{C}$ on N. We denote by $CNTU(N)$ the set of NTU games with coalition structure on N ($CTU\,(N)$ for TU games) and by $CNTU$ the set of all NTU games with a coalition structure (CTU for TU games).

Given $S \subset N$ we denote by $\mathcal{C}_S$ the coalition structure $\mathcal{C}$ restricted to the players in S, *i.e.*, $\mathcal{C}_S = \{C_q \cap S\}_{C_q \in \mathcal{C}}$. Notice that this implies that $\mathcal{C}_S$ may have less or the same number of coalitions as $\mathcal{C}$. For simplicity we, use $\mathcal{C}_{-i}$ instead of $\mathcal{C}_{N \backslash i}$.

A *payoff configuration* for (N, V) is a set of payoffs $x = \left(x^S\right)_{S \subset N}$ with $x^S \in V(S)$ for all $S \subset N$.

Given G a subset of $CNTU$ (or NTU), a *value* Γ on G is a correspondence which assigns to each $(N, V, \mathcal{C}) \in G$ a subset $\Gamma\,(N, V, \mathcal{C}) \subset V\,(N)$. We say that $\left(\Gamma^S\right)_{S \subset N}$ is a payoff configuration associated with Γ if $\Gamma^S \in \Gamma\,(S, V, \mathcal{C}_S)$ for all $S \subset N$. When several NTU games or coalition structures are involved we write $\Gamma^S\,(V)\,,\ \Gamma^S\,(\mathcal{C})\,,$ or $\Gamma^S\,(V, \mathcal{C})$ instead of Γ^S.

If $\Gamma\,(N, V, \mathcal{C})$ is a single point of $V\,(N)$ for all $(N, V, \mathcal{C}) \in G$ we say that Γ is a *single value*. Of course, each single value has an unique payoff configuration associated. Usually we write Γ^N instead of $\Gamma\,(N, V, \mathcal{C})\,.$

We denote by φ^N (or $\varphi^N\,(v)$) the *Shapley value* (Shapley, 1953) of the TU game (N, v).

For TU games with coalition structure ϕ^N, or $\phi^N\,(v, \mathcal{C})\,,$ denotes the *Owen value* (Owen, 1977), which is a generalization of the Shapley value (when $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{1\}\,, ..., \{n\}\}\,,$ the Owen value coincides with the Shapley value).

Let us mention two characterizations of the Owen value. Winter (1992) shows that the Owen value is the only value satisfying efficiency, individual symmetry, covariance, consistency, and GBCP (Game Between Coalitions Property). Later, Calvo, Lasaga and Winter (1996) show that the Owen value is the only value satisfying efficiency, balanced contributions among coalitions, and balanced contributions among players in the same coalition[1].

We say that a single value φ satisfies *balanced contributions among coalitions (BCAC)* if for each $C_q, C_r \in \mathcal{C}$ with $q \neq r$,

$$\sum_{j \in C_q} \varphi_j^N - \sum_{j \in C_q} \varphi_j^{N \backslash C_r} = \sum_{j \in C_r} \varphi_j^N - \sum_{j \in C_r} \varphi_j^{N \backslash C_q}.$$

We say that a single value φ satisfies *balanced contributions among players in the same coalition (BCAP)* if for each $i, j \in C_q \in \mathcal{C}$ with $i \neq j$,

$$\varphi_i^N - \varphi_i^{N \backslash j} = \varphi_j^N - \varphi_j^{N \backslash i}.$$

[1]Even though Calvo, Lasaga and Winter (1996) present these two balanced properties as only one, we think that for our paper is more intuitive the formulation as two properties.

We now present the consistent value for NTU games following Maschler and Owen (1989, 1992).

Let Π be the set of all orders over N. Given $\pi \in \Pi$ we define the set of *predecessors* of i under π as

$$P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}.$$

The *marginal contribution* of player $i \in N$ to the game V in the order π is

$$d_i(\pi) = \max\left\{x_i : \left((d_j(\pi))_{j \in P_i^\pi}, x_i\right) \in V\left(P_i^\pi \cup i\right)\right\}.$$

So, $d_i(\pi)$ is the maximum that player i can obtain in $V(S)$ after his predecessors obtain their respective $d_j(\pi)$'s. We denote $d(\pi) = (d_i(\pi))_{i \in N}$.

It is straightforward to prove that if (N, V) is a hyperplane game,

$$d_i(\pi) = \frac{v(P_i^\pi \cup i) - \sum\limits_{j \in P_i^\pi} \lambda_j^{P_i^\pi \cup i} d_j(\pi)}{\lambda_i^{P_i^\pi \cup i}}.$$

Given a hyperplane game (N, V), Maschler and Owen (1989) define the *consistent value* Ψ^N (or $\Psi^N(V)$) as the vector of expected marginal contributions, where each $\pi \in \Pi$ is equally likely, *i.e.*

$$\Psi^N = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} d(\pi).$$

It is remarkable that Maschler and Owen (1992) even suggest the name random order value instead of consistent value.

Notice that each $d(\pi)$ is an efficient vector (it belongs to the boundary of $V(N)$). Since we are dealing with hyperplane games, this boundary is flat and the consistent value is also an efficient value.

Maschler and Owen (1989) prove that, given $i \in N$,

$$\Psi_i^N = \frac{1}{|N| \lambda_i^N} \left(\sum_{j \in N \setminus i} \lambda_i^N \Psi_i^{N \setminus j} + v(N) - \sum_{j \in N \setminus i} \lambda_j^N \Psi_j^{N \setminus i} \right). \tag{3.2}$$

One way to extend a hyperplane solution to the general class of NTU games with convex $V(S)$'s is to pass arbitrary hyperplanes to the various sets $V(S)$. These hyperplanes determine a hyperplane game for which we know the solution. If this solution belongs to $V(N)$, we say that this is a solution of the NTU game (N, V). This is the way adopted by Maschler and Owen (1992) for extending the consistent value to the class of NTU games.

Formally, given an NTU game (N, V), we say that (N, V') is a *supporting hyperplane game* for (N, V) if for each $S \subset N$,

$$V'(S) = \left\{x \in \mathbb{R}^S : \lambda^S \cdot x \leq v(S)\right\}$$

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where λ^S is orthonormal to the boundary of $V(S)$ and

$$v(S) = \max \left\{ \lambda^S \cdot x : x \in V(S) \right\}.$$

Notice that $V(S) \subset V'(S)$.

Given an NTU game (N,V), a payoff configuration x is a *consistent value* for (N,V) (Maschler and Owen, 1992) if $x^S \in V(S)$ for all $S \subset N$ and there exists a supporting hyperplane (N,V') game for (N,V) such that $x^S = \Psi^S(V')$ for all $S \subset N$.

3.3 The consistent coalitional value

In this section we define two NTU values for NTU games with coalition structure, which generalize the consistent NTU value and the Owen value. The random order coalitional value generalizes the definition of Ψ as the average of marginal contributions. The consistent coalitional value generalizes the expression (3.2) of Ψ.

We first introduce the random order coalitional value for hyperplane games. Let $(N,V,\mathcal{C})$ be an NTU game with coalition structure. We say that an order $\pi \in \Pi$ is *admissible* with respect to $\mathcal{C}$ if given $i,j \in C_q \in \mathcal{C}$ and $k \in N$ such that $\pi(i) < \pi(k) < \pi(j)$ then $k \in C_q$. We denote by $\Pi^{\mathcal{C}}$ the set of all orders over N admissible with respect to $\mathcal{C}$.

Given a hyperplane game $(N,V,\mathcal{C})$, the *random order coalitional value* Υ^N (or $\Upsilon^N(V,\mathcal{C})$) is defined as the vector of expected marginal contributions when all the admissible orders with respect to $\mathcal{C}$ are equally likely, *i.e.*

$$\Upsilon^N = \frac{1}{|\Pi^{\mathcal{C}}|} \sum_{\pi \in \Pi^{\mathcal{C}}} d(\pi).$$

If (N,V) is a TU game then Υ coincides with the Owen value. Moreover if $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{1\},...,\{n\}\}$ then Υ coincides with the consistent value.

Notice that Υ is a single value. Then, there is only one payoff configuration $\Upsilon = \left(\Upsilon^S\right)_{S \subset N}$ associated with Υ, which satisfies that $\Upsilon^S = \Upsilon^S(V,\mathcal{C}_S) \in \partial V(S)$ for all $S \subset N$.

We now define the consistent coalitional value for hyperplane games.

Given a hyperplane game $(N,V,\mathcal{C})$, the *consistent coalitional value* Φ^N (or $\Phi^N(V,\mathcal{C})$) is the only vector satisfying the following two conditions:

For all $C_q \in \mathcal{C}$, $\sum_{j \in C_q} \lambda_j^N \Phi_j^N =$

$$\frac{1}{|\mathcal{C}|}\left[\sum_{C_r \in \mathcal{C}\setminus C_q}\left(\sum_{j \in C_q} \lambda_j^N \Phi_j^{N\setminus C_r}\right) + v(N) - \sum_{C_r \in \mathcal{C}\setminus C_q}\left(\sum_{j \in C_r} \lambda_j^N \Phi_j^{N\setminus C_q}\right)\right]. \quad (3.3)$$

For all $i \in C_q \in \mathcal{C}$,

$$\Phi_i^N = \frac{1}{|C_q| \lambda_i^N} \left(\sum_{j \in C_q \setminus i} \lambda_i^N \Phi_i^{N \setminus j} + \sum_{j \in C_q} \lambda_j^N \Phi_j^N - \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^{N \setminus i} \right). \tag{3.4}$$

Remark 14 *It is straightforward to prove that Φ is well defined and*

$$\sum_{j \in N} \lambda_j^N \Phi_j^N = v(N).$$

Since Φ is a single value, there is only one consistent coalitional payoff configuration $\Phi = \left(\Phi^S\right)_{S \subset N}$ which satisfies that $\Phi^S = \Phi^S(V, \mathcal{C}_S) \in \partial V(S)$ for all $S \subset N$.

The definition of the consistent coalitional value is not so intuitive as the definition of Υ, which is the natural extension to hyperplane games of the expression of the Owen value in terms of expected marginal contributions. Nevertheless, Φ is a more suitable value for hyperplane games (and NTU games) than Υ. The reason is that, as we prove in Section 3.4, Φ satisfies more interesting properties. Moreover, Φ can be characterized generalizing axiomatic characterizations of the Owen value and the consistent value.

The generalization of Φ to NTU games is done analogously to the consistent value. For an NTU game with coalition structure $(N, V, \mathcal{C})$, we take for each coalition $S \subset N$ an orthonormal vector λ^S to the boundary of $V(S)$. Let $(N, V', \mathcal{C})$ be the resulting hyperplane game and $\Phi = \left(\Phi^S\right)_{S \subset N}$ the consistent coalitional payoff configuration associated with $(N, V', \mathcal{C})$. If Φ is feasible in $(N, V, \mathcal{C})$ then we say that Φ is a *consistent coalitional payoff configuration.*

We can extend the random order coalitional value Υ to NTU games in a similar way.

It is straightforward to prove that if $\mathcal{C} = \{N\}$ or $\mathcal{C} = \{\{1\}, \{2\}, ..., \{n\}\}$ then $\Phi^N = \Psi^N$. Thus, the consistent coalitional value is a generalization of the consistent value for NTU games with coalition structure. Moreover, for TU games with coalition structure the consistent coalitional value coincides with the Owen value (see Corollary 27).

The random order coalitional value also generalizes both the consistent NTU value and the Owen value.

We now compute Υ and Φ in the following example:

Example 15 *(Owen, 1972). Let $(N, V, \mathcal{C})$ be the hyperplane game such that $N = \{1, 2, 3\}$ and*

$$\begin{aligned}
V(i) &= \left\{ x_i \in \mathbb{R}^{\{i\}} : x_i \leq 0 \right\}, \ \forall i \in N, \\
V(\{1,2\}) &= \left\{ (x_1, x_2) \in \mathbb{R}^{\{1,2\}} : x_1 + 4x_2 \leq 1, x_1 \leq 1, x_2 \leq \frac{1}{4} \right\}, \\
V(\{1,3\}) &= \left\{ (x_1, x_3) \in \mathbb{R}^{\{1,3\}} : x_1 \leq 0, x_3 \leq 0 \right\}, \\
V(\{2,3\}) &= \left\{ (x_2, x_3) \in \mathbb{R}^{\{2,3\}} : x_2 \leq 0, x_3 \leq 0 \right\},
\end{aligned}$$

and

$$V(N)=\left\{x\in\mathbb{R}^N:\sum_{i\in N}x_i\leq 1;x_i\leq 1\ \forall i\in N;x_i+x_j\leq 1\ \forall i,j\in N\right\}.$$

If $\mathcal{C}=\{\{1,2\},\{3\}\}$, *making some computations we obtain that*

$$\Upsilon^N=\left(\frac{8}{16},\frac{5}{16},\frac{3}{16}\right)\ \text{and}\ \Phi^N=\left(\frac{13}{32},\frac{13}{32},\frac{6}{32}\right).$$

However, for $\mathcal{C}=\{\{1\},\{2,3\}\}$ *both values coincide because*

$$\Phi^N=\Upsilon^N=\left(\frac{8}{16},\frac{5}{16},\frac{3}{16}\right).$$

In the following lemma we prove that the random order coalitional value also satisfies (3.3).

Lemma 16 *Given a hyperplane game* $(N,V,\mathcal{C})$, *for all* $C_q\in\mathcal{C}$, $\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N=$

$$\frac{1}{|\mathcal{C}|}\left[\sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^{N\backslash C_r}\right)+v(N)-\sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\backslash C_q}\right)\right].$$

Proof. See the Appendix. ■

Since Υ and Φ are different (Example 15) we conclude that Υ does not satisfy (3.4).

In next theorem we prove the existence of consistent coalitional payoff configurations.

Theorem 17 *Every NTU game has a consistent coalitional payoff configuration.*

Proof. See the Appendix. ■

Using arguments similar to those used in the proof of Theorem 17 we can conclude that every NTU game has a random order coalitional payoff configuration.

3.4 Properties

In this section we present several desirable properties and study which of them are satisfied by the consistent coalitional value and the random order coalitional value.

We now define some properties of NTU values. Some of them are well known in the literature of NTU games. Others are introduced in this Chapter generalizing properties of TU games. We present the definitions for single values. The definition for payoff configurations associated with general values is straightforward.

We say that a value Γ satisfies *efficiency* (*EF*) if for each $(N,V,\mathcal{C})\in CNTU$, $\Gamma^N\in\partial V(N)$.

Remark 18 *Since V satisfies (A2) we conclude that if Γ satisfies efficiency then for each $(N,V,\mathcal{C}) \in CNTU$ and $S \subset N$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying $\lambda^S \cdot \Gamma^S = v(S)$ where $v(S) = \max\left\{\lambda^S \cdot x : x \in V(S)\right\}$. Of course the reciprocal is also true.*

Given an $CNTU$ game $(N,V,\mathcal{C})$ we say that two players $i,j \in N$ are *symmetric* if: For each $S \subset N\setminus\{i,j\}$ if $x \in V(S\cup i)$ and $y \in \mathbb{R}^{S\cup j}$ such that $y_j = x_i$, and $y_k = x_k$ for each $k \in S$, then $y \in V(S\cup j)$. For each $S \supset \{i,j\}$ if $x \in V(S)$, $y_i = x_j$, $y_j = x_i$, and $x_k = y_k$ for each $k \in S\setminus\{i,j\}$, then $y \in V(S)$.

We say that a value Γ satisfies *individual symmetry (IS)* if for each pair of symmetric players $i,j \in C_q \in \mathcal{C}$,

$$\Gamma_i^N = \Gamma_j^N.$$

We now generalize the property of covariance to hyperplane games following Maschler and Owen (1989). Let $(N,V,\mathcal{C})$ and $(N,\widetilde{V},\mathcal{C})$ be two hyperplane games such that for each $S \subset N$,

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda^S \cdot x \leq v(S)\right\} \text{ and } \widetilde{V}(S) = \left\{x \in \mathbb{R}^S : \widetilde{\lambda}^S \cdot x \leq \widetilde{v}(S)\right\}.$$

We say that $(N,V,\mathcal{C})$ and $(N,\widetilde{V},\mathcal{C})$ are *equivalent under a linear transformation of player i's utility* if there exist two constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that for all $S \subset N : \widetilde{\lambda}_i^S = \dfrac{\lambda_i^S}{a}$, $\widetilde{\lambda}_j^S = \lambda_j^S$ if $j \neq i$, $\widetilde{v}(S) = v(S) + \dfrac{b\lambda_i^S}{a}$ if $i \in S$, and $\widetilde{v}(S) = v(S)$ if $i \notin S$. Notice that if $(N,V,\mathcal{C})$ and $\left(N,\widetilde{V},\mathcal{C}\right)$ are equivalent under a linear transformation of player i's utility, then $\widetilde{x} \in V(S)$ if and only if there exists $x \in V(S)$ satisfying: $\widetilde{x}_i = ax_i + b$ and $\widetilde{x}_j = x_j$ if $j \in S \setminus i$.

We say that a value Γ satisfies *covariance (COV)* if, given two hyperplane games $(N,V,\mathcal{C})$ and $(N,\widetilde{V},\mathcal{C})$, equivalent under a linear transformation of some player i's utility,

$$\begin{aligned}\Gamma_i\left(N,\widetilde{V},\mathcal{C}\right) &= a\Gamma_i(N,V,\mathcal{C}) + b \text{ and}\\ \Gamma_j\left(N,\widetilde{V},\mathcal{C}\right) &= \Gamma_j(N,V,\mathcal{C}) \text{ if } j \in N\setminus i.\end{aligned}$$

Thus, covariance just states that, if we linearly change player i's utility function, his final payoff changes the same way, while other players' payoffs remain constant.

Hart and Mas-Colell (1989) characterize the Shapley value as the only value on TU games satisfying consistency and other properties. Later, Winter (1992) extends the definition of consistency to TU games with coalition structure.

Maschler and Owen (1989) show that if we define the property of consistency of Hart and Mas-Colell (1989) in NTU games as in the TU case, there is no value satisfying consistency and other "basic" properties (for instance, efficiency).

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Then, they provide a weaker definition of consistency for hyperplane games called bilateral consistency.

We now present a generalization of the property of bilateral consistency to hyperplane games with coalition structure. Our bilateral consistency generalizes the bilateral consistency of Maschler and Owen (1989) in the same way that the consistency of Winter (1992) generalizes the consistency of Hart and Mas-Colell (1989).

Given a value Γ, a hyperplane game $(N, V, \mathcal{C})$, and $S \subset C_q \in \mathcal{C}$, the *reduced game* $(S, V_S, \{S\})$ is defined, for each $T \subset S$, as follows:

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \left(x, \left(\Gamma_i^{T \cup \overline{S}} \right)_{i \in \overline{S}} \right) \in V\left(T \cup \overline{S}\right) \right\}.$$

It is straightforward to prove that V_S is the hyperplane game given, for each $T \subset S$, by

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \sum_{i \in T} \lambda_i^{T \cup \overline{S}} x_i \leq v\left(T \cup \overline{S}\right) - \sum_{i \in \overline{S}} \lambda_i^{T \cup \overline{S}} \Gamma_i^{T \cup \overline{S}} \right\}.$$

We say that a value Γ satisfies *l-consistency* if, for each hyperplane game $(N, V, \mathcal{C})$, $C_q \in \mathcal{C}$ with $l \leq |C_q|$, and $i \in C_q$,

$$\sum_{S \subset C_q : i \in S, |S| = l} \Gamma_i^S(V_S) = \binom{|C_q| - 1}{l - 1} \Gamma_i^N(V).$$

We say that a value Γ satisfies *consistency (CONS)* if it is l-consistent for each l with $1 \leq l \leq n$. CONS is a weaker property than the consistency in the axiomatization of the Shapley value (Hart and Mas-Colell, 1989) and the Owen value (Winter, 1992).

For simplicity, we denote $\Gamma_i^S(V_S) = \Gamma_i^S(V_S, \{S\})$ and $\Gamma_i^N(V) = \Gamma_i^N(V, \mathcal{C})$.

We call *bilateral consistency (BCONS)* to 2-consistency.

Myerson (1980) characterizes the Shapley value using efficiency and balanced contributions (BC). Hart and Mas-Colell (1996) introduce the following generalization of BC for NTU games.

We say that a value Γ satisfies *average balanced contributions (ABC)* if, for each $(N, V, \mathcal{C}) \in CNTU$, $S \subset N$, and $i \in S$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ such that

$$\sum_{j \in S \setminus i} \lambda_i^S \left(\Gamma_i^S - \Gamma_i^{S \setminus j} \right) = \sum_{j \in S \setminus i} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \setminus i} \right).$$

Later, Calvo, Lasaga and Winter (1996) generalize the property of balanced contributions for TU games with a coalition structure obtaining two properties: $BCAC$ and $BCAP$.

We now introduce the properties of average balanced contributions among coalitions and average balanced contributions among players in the same coalition for NTU games with coalition structure. Our average balanced properties generalize the balanced properties of Calvo, Lasaga and Winter (1996) in the same way that the average balanced property of Hart and Mas-Colell (1996) generalizes the balanced property of Myerson (1980).

We say that a value Γ satisfies *average balanced contributions among coalitions (ABCAC)* if, for each NTU game $(N, V, \mathcal{C})$, $S \subset N$, and $C'_q = C_q \cap S \in \mathcal{C}_S$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ such that

$$\sum_{C'_r \in \mathcal{C}_S \backslash C'_q} \left[\sum_{j \in C'_q} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash C'_r} \right) \right] = \sum_{C'_r \in \mathcal{C}_S \backslash C'_q} \left[\sum_{j \in C'_r} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash C'_q} \right) \right].$$

We say that a value Γ satisfies *average balanced contributions among players in the same coalition (ABCAP)* if, for each NTU game $(N, V, \mathcal{C})$, $S \subset N$, $C'_q = C_q \cap S \in \mathcal{C}_S$, and $i \in C'_q$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ such that

$$\sum_{j \in C'_q \backslash i} \lambda_i^S \left(\Gamma_i^S - \Gamma_i^{S \backslash j} \right) = \sum_{j \in C'_q \backslash i} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash i} \right).$$

Before studying the properties satisfied by the consistent coalitional value we need a previous result.

Lemma 19 *Given a hyperplane game $(N, V, \mathcal{C})$ and $i \in S \subset C_q \in \mathcal{C}$,*

$$\left(S \backslash i, V_S, \{S \backslash i\} \right) = \left(S \backslash i, V_{S \backslash i}, \{S \backslash i\} \right).$$

Proof. This result is due to Maschler and Owen (1989). ■

Notice that Lemma 19 says that if we pass to the reduced game V_S and then remove a player (i) we obtain the same game as if we remove the player first $(S \backslash i)$ and then pass to the reduced game $V_{S \backslash i}$.

Proposition 20 *The consistent coalitional value satisfies l-consistency for each l with $1 \leq l \leq n$.*

Proof. See the Appendix. ■

In next theorem we study which of these properties are satisfied by the consistent coalitional value.

Theorem 21 *The consistent coalitional value satisfies EF, IS, $ABCAC$, and $ABCAP$. Moreover, in hyperplane games it also satisfies COV and $BCONS$.*

Proof. See the Appendix. ■

Remark 22 *The random order coalitional value satisfies EF, IS, COV (in hyperplane games), and $ABCAC$.*

It is trivial to see that Υ satisfies EF and IS.

Maschler and Owen (1989) show that, for any order π, the vector $d(\pi)$ satisfies COV. Since Υ is the average of some of these $d(\pi)$'s, we conclude that Υ also satisfies COV.

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By Lemma 16, Υ satisfies (3.3). Now using arguments similar to those used in the proof of Theorem 21 for Φ we can conclude that Υ also satisfies $ABCAC$.

Later, we obtain, as a consequence of Theorems 23 and Theorem 25, that Υ neither satisfy $BCONS$ nor $ABCAP$.

By Theorem 21 we know that Φ satisfies, in NTU games or hyperplane games, all the interesting properties that the Owen value satisfies in TU games. However, Υ does not.

3.5 Axiomatic characterizations

In this section we present two axiomatic characterizations of the consistent coalitional value. The first one on the set of hyperplane games using consistency. The second one on the set of NTU games using balanced contributions.

Hart and Mas-Colell (1989) characterize the Shapley value on the class of TU games as the only single value satisfying EF, SYM (if i and j are symmetric players, they must receive the same), COV, and $CONS$. Later, Maschler and Owen (1989) and Winter (1992) extend this result in two different ways.

Maschler and Owen (1989) extend this result to the class of hyperplane games. They prove that the consistent value is the only single value satisfying EF, SYM, COV, and $BCONS$.

Winter (1992) extends it to the class of TU games with coalition structure. He proves that the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and $GBCP$ (*game between coalitions property*).

We say that a single value f (in the class of TU games) satisfies $GBCP$ if for each TU game $(N, v, \mathcal{C})$ and $C_q \in \mathcal{C}$,

$$\sum_{i \in C_q} f_i(N, v, \mathcal{C}) = f_{C_q}(\mathcal{C}, v/\mathcal{C}, \{\mathcal{C}\})$$

where $(v/\mathcal{C})(R) = v\left(\bigcup_{C_r \in R} C_r\right)$ for each $R \subset \mathcal{C}$. This property says that the amount received by a coalition in the game played by the coalitions (every coalition acts as a single player) coincides with the sum of the amounts received by the members of this coalition in the original game.

This property cannot be exported to hyperplane games.

It is not difficult to check that the proof of Winter's result about the characterization of the Owen value is also valid if we replace $GBCP$ by $BCAC$. Then, the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and $BCAC$.

In Theorem 23 below we generalize the results of Hart an Mas-Colell (1989), Maschler and Owen (1989), and Winter (1992) to hyperplane games with coalition structure.

Theorem 23 *The consistent coalitional value is the only single value on the class of hyperplane games satisfying EF, IS, COV, $BCONS$, and $ABCAC$.*

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Proof. See the Appendix. ■

Remark 24 *The properties used in this theorem are independent (see the Appendix).*

Myerson (1980) characterizes the Shapley value on the class of TU games as the only single value satisfying EF and BC. Later, Calvo, Lasaga and Winter (1996) and Hart and Mas-Colell (1996) extend this result in two different ways.

Calvo, Lasaga and Winter (1996) extend it to the class of TU games with coalition structure. They prove that the Owen value is the only single value satisfying EF, $BCAP$, and $BCAC$. Hart and Mas-Colell (1996) extend Myerson's result to the class of NTU games proving that the consistent value is the only value satisfying EF and ABC.

In Theorem 25 below we generalize the results of Myerson (1980), Calvo, Lasaga and Winter (1996), and Hart and Mas-Colell (1996) to NTU games with coalition structure.

Theorem 25 *The consistent coalitional value is the only value on the class of NTU games with coalition structure satisfying* EF, $ABCAC$, *and* $ABCAP$.

Proof. See the Appendix. ■

Remark 26 *The properties used in this theorem are independent (see the Appendix).*

We now prove that the consistent coalitional value generalizes the Owen value.

Corollary 27 *For each TU game with coalition structure, the Owen value is the only consistent coalitional value.*

Proof. See the Appendix. ■

The results obtained in this section about the consistent coalitional value and the relation with other values can be summarized in the following table.

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About consistency			
Whitout coalition structure		With coalition structure	
TU	Hyperplane	TU	Hyperplane
Shapley	**Consistent**	**Owen**	**Consistent Coalitional**
EF *SYM* *COV* *CONS*	*EF* *SYM* *COV* *BCONS*	*EF* *IS* *COV* *CONS* *BCAC*	*EF* *IS* *COV* *BCONS* *ABCAC*
About balanced contributions			
Whitout coalition structure		With coalition structure	
TU	NTU	TU	NTU
Shapley	**Consistent**	**Owen**	**Consistent Coalitional**
EF *BC*	*EF* *ABC*	*EF* *BCAC* *BCAP*	*EF* *ABCAC* *ABCAP*

Then, the consistent coalitional value is the right generalization of the Owen value and the consistent value to NTU games with coalition structure if we focus in the properties of consistency and balanced contributions of both values.

3.6 Appendix

Proof of Lemma 16. Let $\Upsilon = \left(\Upsilon^S\right)_{S\subset N}$ be the random order coalitional payoff configuration for $(N, V, \mathcal{C})$. By definition, Υ_j^N is the expected marginal contribution of player j over all the $\left|\Pi^{\mathcal{C}}\right|$ admissible orders of players with respect to $\mathcal{C}$. We classify these orders in $|\mathcal{C}|$ groups according the last coalition C_r in such orders.

Let $\Pi^{\mathcal{C}}(C_r)$ be the set of admissible orders with respect to $\mathcal{C}$ in which players of coalition C_r are in the last position. Notice that $\left|\Pi^{\mathcal{C}}\right| = |\mathcal{C}| \left|\Pi^{\mathcal{C}}(C_r)\right|$ for each $C_r \in \mathcal{C}$.

If $C_r \neq C_q$, then the expected marginal contribution for each player $j \in C_q$ in the orders of $\Pi^{\mathcal{C}}(C_r)$ coincides with the expected marginal contribution of player j in the game $(N\backslash C_r, V, \mathcal{C}\backslash C_r)$, which is $\Upsilon_j^{N\backslash C_r}$, *i.e.*

$$\frac{1}{\left|\Pi^{\mathcal{C}}(C_r)\right|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_r)} d_j(\pi) = \frac{1}{\left|\Pi^{\mathcal{C}\backslash C_r}\right|}\sum_{\pi\in\Pi^{\mathcal{C}\backslash C_r}} d_j(\pi) = \Upsilon_j^{N\backslash C_r}. \qquad (3.5)$$

Moreover, for each $\pi \in \Pi^{\mathcal{C}}(C_q)$,

$$\sum_{j\in C_q} \lambda_j^N d_j(\pi) = v(N) - \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right).$$

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Then, for each $C_q \in \mathcal{C}$,

$$\frac{1}{|\Pi^{\mathcal{C}}(C_q)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_q)}\left(\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right) \tag{3.6}$$

$$= \frac{1}{|\Pi^{\mathcal{C}}(C_q)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_q)}\left(v(N)-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right)\right)$$

$$= \frac{1}{|\Pi^{\mathcal{C}}(C_q)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_q)} v(N)$$

$$-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\frac{1}{|\Pi^{\mathcal{C}}(C_q)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_q)} d_j(\pi)\right)\right)$$

$$= v(N)-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}\right).$$

We have then:

$$\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N = \sum_{j\in C_q}\lambda_j^N\frac{1}{|\Pi^{\mathcal{C}}|}\sum_{\pi\in\Pi^{\mathcal{C}}} d_j(\pi)$$

$$= \sum_{j\in C_q}\lambda_j^N\left(\sum_{C_r\in\mathcal{C}}\frac{1}{|\Pi^{\mathcal{C}}|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_r)} d_j(\pi)\right)$$

since $\left|\Pi^{\mathcal{C}}\right| = |\mathcal{C}|\left|\Pi^{\mathcal{C}}(C_r)\right|$, the last expression can be rewritten as

$$\sum_{j\in C_q}\lambda_j^N\frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}}\frac{1}{|\Pi^{\mathcal{C}}(C_r)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_r)} d_j(\pi) =$$

$$\frac{1}{|\mathcal{C}|}\left[\sum_{j\in C_q}\lambda_j^N\sum_{C_r\in\mathcal{C}\setminus C_q}\underbrace{\frac{1}{|\Pi^{\mathcal{C}}(C_r)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_r)} d_j(\pi)}+\underbrace{\frac{1}{|\Pi^{\mathcal{C}}(C_q)|}\sum_{\pi\in\Pi^{\mathcal{C}}(C_q)}\left(\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right)}\right]$$

the terms above brackets are those given in (3.5) and (3.6), so:

$$= \frac{1}{|\mathcal{C}|}\left[\sum_{j\in C_q}\lambda_j^N\sum_{C_r\in\mathcal{C}\setminus C_q}\Upsilon_j^{N\setminus C_r}+v(N)-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}\right)\right]$$

$$= \frac{1}{|\mathcal{C}|}\left[\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^{N\setminus C_r}\right)+v(N)-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}\right)\right]$$

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which is precisely the statement of this lemma. ■

Proof of Theorem 17. The structure of the proof is analogous to the proof of Theorem 3.3 in Maschler and Owen (1992), where they prove the existence of the consistent value for general NTU games.

We make use of induction to prove the following claim:

Given $\left(x^T\right)_{T\subsetneq N}$ *with* $x^T \in \mathbb{R}^T$ *such that, for any* $S \subsetneq N$, *the collection* $\left(x^T\right)_{T\subset S}$ *is a consistent coalitional payoff configuration of the game* $(S, V, \mathcal{C}_S)$, *there exists* $x^N \in \partial V(N)$ *such that* $\left(x^T\right)_{T\subset N}$ *is a consistent coalitional payoff configuration of* $(N, V, \mathcal{C})$.

For $n = 1$ the claim is trivially true, being the collection the empty set $\emptyset$.

Assume now the claim holds for less than n players. Thus, there exists a collection $\left(x^T\right)_{T\subsetneq N}$ such that, for any $S \subsetneq N$, $\left(x^T\right)_{T\subset S}$ is a consistent coalitional payoff configuration of the game $(S, V, \mathcal{C}_S)$.

Assume that $z \in \partial V(N)$. For each $T \subsetneq N$, let $\lambda^T = \left(\lambda_i^T\right)_{i\in T}$ be the orthonormal vector outwards x^T. Moreover, $\left(\lambda_i^N\right)_{i\in N}$ is the orthonormal vector outwards z.

Consider the hyperplane game $(N, V^z, \mathcal{C})$ such that, for each $S \subset N$,

$$V^z(S) = \left\{ y \in \mathbb{R}^S : \lambda^S \cdot y \leq v(S) \right\}$$

where $v(S) = \lambda^S \cdot x^S$ when $S \subsetneq N$ and $v(N) = \lambda^N \cdot z$.

Let $\left(\Phi^S(z)\right)_{S\subset N}$ be the (unique) consistent coalitional payoff configuration for the hyperplane game $(N, V^z, \mathcal{C})$. By definition of V^z, $\Phi^S(z) = x^S$ for all $S \subsetneq N$, independently of the chosen z.

We want to show that there exists a point $x^N \in \partial V(N)$ such that the collection $\left(x^T\right)_{T\subset N}$ is a consistent coalitional payoff configuration for $(N, V, \mathcal{C})$. Notice that it is enough to prove that $\Phi^N\left(x^N\right) = x^N$. We make use of a fixed point theorem. Since Φ satisfies (3.3) and (3.4) and the λ_i^S's are strictly positive and continuous functions, $\Phi^N(z)$ is also a continuous function of z.

We define $M = \max\left\{ \frac{\left|x_i^T\right|}{\delta} : i \in T \subsetneq N \right\}$, where δ is given by (A4).

Given $C_q \in \mathcal{C}$, by (3.3), $|\mathcal{C}| \sum_{j\in C_q} \lambda_j^N \Phi_j^N(z) =$

$$\sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda_j^N x_j^{N\setminus C_r} \right) + v(N) - \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda_j^N x_j^{N\setminus C_q} \right).$$

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By (A5), $v(N) \geq 0$, and since the λ_j^N's are positive,

$$\begin{aligned}
&\geq \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N (-M\delta) \right) - \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N (M\delta) \right) \\
&= -(|\mathcal{C}| - 1) M\delta \sum_{j \in C_q} \lambda_j^N - M\delta \sum_{j \in N \setminus C_r} \lambda_j^N \\
&\geq -(|\mathcal{C}| - 1) M\delta - M\delta = -|\mathcal{C}| M\delta
\end{aligned}$$

where the last inequality comes because λ^N is normalized.

So, $\sum_{j \in C_q} \lambda_j^N \Phi_j^N(z) \geq -M\delta$ for each $C_q \in \mathcal{C}$.

Given $i \in C_q \in \mathcal{C}$, by (3.4),

$$\begin{aligned}
|C_q| \Phi_i^N(z) &= \sum_{j \in C_q \setminus i} x_i^{N \setminus j} + \frac{\sum_{j \in C_q} \lambda_j^N \Phi_j^N(z) - \sum_{j \in C_q \setminus i} \lambda_j^N x_j^{N \setminus i}}{\lambda_i^N} \\
&\geq \sum_{j \in C_q \setminus i} (-M\delta) + \frac{-M\delta - \sum_{j \in C_q \setminus i} \lambda_j^N M\delta}{\lambda_i^N} \\
&= -(|C_q| - 1) M\delta - \frac{M\delta}{\lambda_i^N} - \frac{\sum_{j \in C_q \setminus i} \lambda_j^N M\delta}{\lambda_i^N}
\end{aligned}$$

since $\lambda_i^N > \delta$, λ^N is normalized, and $\delta < 1$,

$$\begin{aligned}
&> -(|C_q| - 1) M\delta - M - M \sum_{j \in C_q \setminus i} \lambda_j^N \\
&> -(|C_q| - 1) M\delta - M - M \\
&> -(|C_q| - 1) M - 2M \\
&\geq -2 |C_q| M.
\end{aligned}$$

So, $\Phi_i^N(z) > -2M$.

The rest of the proof is analogous to Maschler and Owen's (1992) and we just give a geometric description for the case $n = 2$.

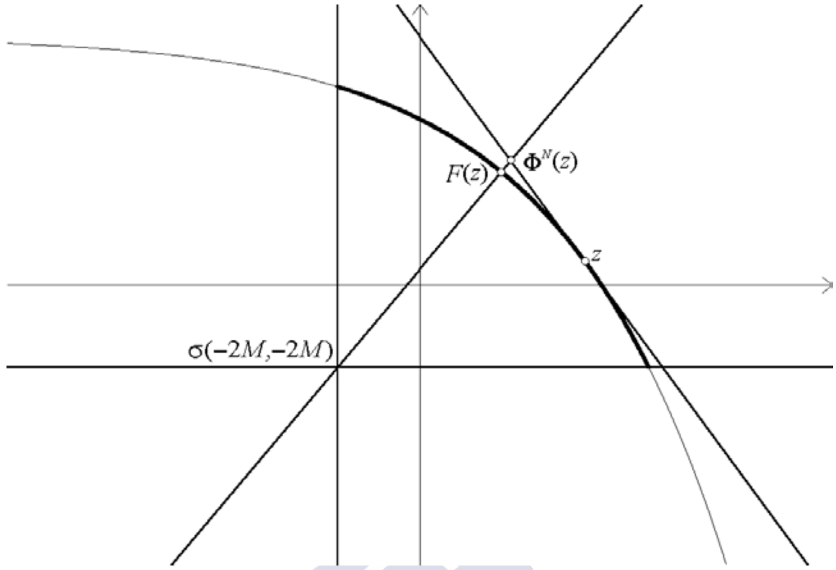

Figure 1

We define $D = \left\{x \in \mathbb{R}^N : x_i \geq -2M \text{ for all } i \in N\right\}$. Given a vector z on $\partial V(N) \cap D$ (which is the thick line in Figure 1), we have proved that $\Phi^N(z) \in D$. Thus, the point $F(z)$ obtained by applying a projection centered at $\sigma = (-2M, ..., -2M) \in \mathbb{R}^N$, also belongs to $\partial V(N) \cap D$ (see Figure 1). By applying a standard fixed point theorem over the (continuous) function F, we find the desired x^N. ■

Proof of Proposition 20. We proceed by induction on l. The theorem is trivially true for $l = 1$. Assume it is true for at most $l - 1$.

If we apply the induction hypothesis to the game $(N\backslash j, V, \mathcal{C}_{-j})$ with $j \in C_q\backslash i$ (if $C_q = \{i\}$, the result is trivially true for C_q) then,

$$\sum_{T \subset C_q\backslash j : i \in T, |T| = l-1} \Phi_i^T(V_T) = \binom{|C_q| - 2}{l - 2} \Phi_i^{N\backslash j}(V). \tag{3.7}$$

We wish to prove that for each $C_q \in \mathcal{C}$ and $i \in C_q$,

$$l\lambda_i^N \sum_{S \subset C_q : i \in S, |S| = l} \Gamma_i^S(V_S) = l\lambda_i^N \binom{|C_q| - 1}{l - 1} \Gamma_i^N(V). \tag{3.8}$$

To do so, we analyze the left side of this expression. Assume that $i \in S \subset C_q$ and $|S| = l$. Applying (3.4) to the game $(S, V_S, \{S\})$, which is also a hyperplane game, we obtain

$$l\lambda_i^N \Phi_i^S(V_S) = \sum_{j \in S\backslash i} \lambda_i^N \Phi_i^{S\backslash j}(V_S) + \sum_{j \in S} \lambda_j^N \Phi_j^S(V_S) - \sum_{j \in S\backslash i} \lambda_j^N \Phi_j^{S\backslash i}(V_S).$$

If we compute Φ in the game V_S we obtain that

$$\sum_{j\in S}\lambda_j^N \Phi_j^S(V_S) = v(N) - \sum_{j\in \overline{S}}\lambda_j^N \Phi_j^N(V).$$

Hence,

$$l\lambda_i^N \Phi_i^S(V_S) = \sum_{j\in S\setminus i}\lambda_i^N \Phi_i^{S\setminus j}(V_S) + v(N) - \sum_{j\in \overline{S}}\lambda_j^N \Phi_j^N(V) - \sum_{j\in S\setminus i}\lambda_j^N \Phi_j^{S\setminus i}(V_S).$$

Since there are $\binom{|C_q|-1}{l-1}$ possible sets $S\subset C_q$ with $i\in S$ and $|S|=l$, $l\lambda_i^N \sum_{S\subset C_q: i\in S,|S|=l}\Phi_i^S(V_S) =$

$$\sum_{S\subset C_q: i\in S,|S|=l}\left(\sum_{j\in S\setminus i}\lambda_i^N \Phi_i^{S\setminus j}(V_S)\right) + \binom{|C_q|-1}{l-1} v(N)$$

$$- \sum_{S\subset C_q: i\in S,|S|=l}\left(\sum_{j\in \overline{S}}\lambda_j^N \Phi_j^N(V)\right) - \sum_{S\subset C_q: i\in S,|S|=l}\left(\sum_{j\in S\setminus i}\lambda_j^N \Phi_j^{S\setminus i}(V_S)\right).$$

rearranging the order of summation, we have:

$$= \sum_{j\in C_q\setminus i}\left(\sum_{S\subset C_q: i,j\in S,|S|=l}\lambda_i^N \Phi_i^{S\setminus j}(V_S)\right) \tag{3.9}$$

$$+\binom{|C_q|-1}{l-1} v(N) \tag{3.10}$$

$$-\sum_{j\in N\setminus i}\left(\sum_{S\subset C_q: i\in S, j\notin S,|S|=l}\lambda_j^N \Phi_j^N(V)\right) \tag{3.11}$$

$$-\sum_{j\in C_q\setminus i}\left(\sum_{S\subset C_q: i,j\in S,|S|=l}\lambda_j^N \Phi_j^{S\setminus i}(V_S)\right). \tag{3.12}$$

We now analyze the four terms separately:

1. Term (3.9) is equal, by Lemma 19, to

$$\sum_{j\in C_q\setminus i}\lambda_i^N\left(\sum_{T\subset C_q\setminus j: i\in T,|T|=l-1}\Phi_i^T(V_T)\right)$$

   which coincides, by (3.7), with

$$\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus i}\lambda_i^N \Phi_i^{N\setminus j}(V).$$

2. Since $v(N) = \lambda_i^N \Phi_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^N(V) + \sum_{j \in N \setminus C_q} \lambda_j^N \Phi_j^N(V)$, term (3.10) is equal to

$$\begin{aligned} &\binom{|C_q|-1}{l-1} \lambda_i^N \Phi_i^N(V) \\ &+ \binom{|C_q|-1}{l-1} \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^N(V) \\ &+ \binom{|C_q|-1}{l-1} \sum_{j \in N \setminus C_q} \lambda_j^N \Phi_j^N(V). \end{aligned}$$

3. Term (3.11) is equal to

$$\begin{aligned} &- \sum_{j \in C_q \setminus i} \left(\sum_{S \subset C_q : i \in S, j \notin S, |S| = l} \lambda_j^N \Phi_j^N(V) \right) \\ &- \sum_{j \in N \setminus C_q} \left(\sum_{S \subset C_q : i \in S, j \notin S, |S| = l} \lambda_j^N \Phi_j^N(V) \right) \end{aligned}$$

since: for each $j \in C_q \setminus i$, there are $\binom{|C_q|-2}{l-1}$ possible sets S such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$, and for each $j \in N \setminus C_q$, there are $\binom{|C_q|-1}{l-1}$ possible sets S such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$, last expression coincides with

$$-\binom{|C_q|-2}{l-1} \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^N(V) - \binom{|C_q|-1}{l-1} \sum_{j \in N \setminus C_q} \lambda_j^N \Phi_j^N(V).$$

4. Term (3.12) is equal, by Lemma 19, to

$$- \sum_{j \in C_q \setminus i} \lambda_j^N \left(\sum_{T \subset C_q \setminus i : j \in T, |T| = l-1} \Phi_j^T(V_T) \right)$$

which coincides, by (3.7), with

$$-\binom{|C_q|-2}{l-2} \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^{N \setminus i}(V).$$

Since $\binom{|C_q|-1}{l-1} = \binom{|C_q|-2}{l-1} + \binom{|C_q|-2}{l-2}$, adding terms (3.10) and (3.11) we obtain

$$\binom{|C_q|-1}{l-1} \lambda_i^N \Phi_i^N(V) + \binom{|C_q|-2}{l-2} \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^N(V).$$

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Then, $l\lambda_i^N \sum_{S\subset C_q: i\in S, |S|=l} \Phi_i^S(V_S) =$

$$\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus i}\lambda_i^N \Phi_i^{N\setminus j}(V) + \binom{|C_q|-1}{l-1}\lambda_i^N\Phi_i^N(V)$$
$$+\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus i}\lambda_j^N\left(\Phi_j^N(V)-\Phi_j^{N\setminus i}(V)\right)$$

In Theorem 21, we prove, without using this proposition, that Φ satisfies $ABCAP$ and hence,

$$= \binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus i}\lambda_i^N \Phi_i^{N\setminus j}(V) + \binom{|C_q|-1}{l-1}\lambda_i^N\Phi_i^N(V)$$
$$+\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus i}\lambda_i^N\left(\Phi_i^N(V)-\Phi_i^{N\setminus j}(V)\right)$$

$$= \binom{|C_q|-1}{l-1}\lambda_i^N\Phi_i^N(V) + \binom{|C_q|-2}{l-2}\left(\sum_{j\in C_q\setminus i}\lambda_i^N\Phi_i^N(V)\right)$$

Since $\binom{|C_q|-1}{l-1}+\binom{|C_q|-2}{l-2}(|C_q|-1) = l\binom{|C_q|-1}{l-1}$ the last expression coincides with

$$l\binom{|C_q|-1}{l-1}\lambda_i^N\Phi_i^N(V)$$

which is precisely the right side of (3.8). ■

Proof of Theorem 21. It is straightforward to prove that Φ satisfies EF and IS. By Proposition 20 we know that Φ satisfies $BCONS$.

We now prove that Φ satisfies $ABCAC$. In order to simplify the notation we assume that $S = N$. By EF, $v(N) = \sum_{C_r\in\mathcal{C}}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^N\right)$. Applying this to (3.3) we obtain that for all $C_q\in\mathcal{C}$, $|\mathcal{C}|\sum_{j\in C_q}\lambda_j\Phi_j^N =$

$$\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Phi_j^{N\setminus C_r}\right)+\sum_{C_r\in\mathcal{C}}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^N\right)$$
$$-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^{N\setminus C_q}\right)$$
$$= \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Phi_j^{N\setminus C_r}\right)+\sum_{j\in C_q}\lambda_j^N\Phi_j^N$$
$$+\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\Phi_j^N-\Phi_j^{N\setminus C_q}\right)\right).$$

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If we subtract $\sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Phi_j^N \right) = (|\mathcal{C}| - 1) \sum_{j \in C_q} \lambda_j^N \Phi_j^N$ in both sides,

$$\begin{aligned} \sum_{j \in C_q} \lambda_j^N \Phi_j^N &= \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \left(\Phi_j^{N \setminus C_r} - \Phi_j^N \right) \right) \\ &\quad + \sum_{j \in C_q} \lambda_j^N \Phi_j^N \\ &\quad + \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\Phi_j^N - \Phi_j^{N \setminus C_q} \right) \right). \end{aligned}$$

Then,

$$0 = \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \left(\Phi_j^{N \setminus C_r} - \Phi_j^N \right) \right) + \sum_{C_r \in \mathcal{C} \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\Phi_j^N - \Phi_j^{N \setminus C_q} \right) \right)$$

which means that Φ satisfies $ABCAC$.

We now prove that Φ satisfies $ABCAP$. In order to simplify the notation we assume that $S = N$. Given $i \in C_q \in \mathcal{C}$, by (3.4), $|C_q| \lambda_i^N \Phi_i^N =$

$$\begin{aligned} & \sum_{j \in C_q \setminus i} \lambda_i^N \Phi_i^{N \setminus j} + \sum_{j \in C_q} \lambda_j^N \Phi_j^N - \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^{N \setminus i} \\ = & \sum_{j \in C_q \setminus i} \lambda_i^N \Phi_i^{N \setminus j} + \lambda_i^N \Phi_i^N + \sum_{j \in C_q \setminus i} \lambda_j^N \left(\Phi_j^N - \Phi_j^{N \setminus i} \right) \\ = & \sum_{j \in C_q \setminus i} \lambda_i^N \left(\Phi_i^{N \setminus j} - \Phi_i^N \right) + |C_q| \lambda_i^N \Phi_i^N + \sum_{j \in C_q \setminus i} \lambda_j^N \left(\Phi_j^N - \Phi_j^{N \setminus i} \right). \end{aligned}$$

Then,

$$0 = \sum_{j \in C_q \setminus i} \lambda_i^N \left(\Phi_i^{N \setminus j} - \Phi_i^N \right) + \sum_{j \in C_q \setminus i} \lambda_j^N \left(\Phi_j^N - \Phi_j^{N \setminus i} \right)$$

which means that Φ satisfies $ABCAP$.

We now prove that Φ satisfies COV. Given $i \in C_q \in \mathcal{C}$, let $\left(N, \widetilde{V}, \mathcal{C} \right)$ be obtained from $(N, V, \mathcal{C})$ by a change in player i's utility. Let a and b be the corresponding constants. We proceed by induction over the number of coalitions of $\mathcal{C}$.

If $\mathcal{C}$ has only one coalition ($\mathcal{C} = \{N\}$) then, $\Phi_i^N \left(\widetilde{V} \right) = \Psi_i^N \left(\widetilde{V} \right) = a \Psi_i^N (V) + b = a \Phi_i^N (V) + b$ and $\Phi_j^N \left(\widetilde{V} \right) = \Psi_j^N \left(\widetilde{V} \right) = \Psi_j^N (V) = \Phi_j^N (V)$ for each $j \in N \setminus i$ because Ψ satisfies COV.

Assume the result holds when $|\mathcal{C}|$ has at most $p - 1$ coalitions. We prove it when $|\mathcal{C}| = p$.

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By (3.3), $|\mathcal{C}| \sum_{j\in C_q} \tilde{\lambda}_j^N \Phi_j^N\left(\widetilde{V}\right) =$

$$\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\tilde{\lambda}_j^N \Phi_j^{N\setminus C_r}\left(\widetilde{V}\right)\right)+\tilde{v}\left(N\right)-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\tilde{\lambda}_j^N \Phi_j^{N\setminus C_q}\left(\widetilde{V}\right)\right).$$

By induction hypothesis $\Phi_i^{N\setminus C_r}\left(\widetilde{V}\right) = a\Phi_i^{N\setminus C_r}\left(V\right)+b$ when $C_r \neq C_q$ and $\Phi_j^{N\setminus C_r}\left(\widetilde{V}\right) = \Phi_j^{N\setminus C_r}\left(V\right)$ when $j \neq i$. Then, $|\mathcal{C}| \sum_{j\in C_q} \tilde{\lambda}_j^N \Phi_j^N\left(\widetilde{V}\right) =$

$$\begin{aligned}
& \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q\setminus i}\lambda_j^N \Phi_j^{N\setminus C_r}\left(V\right)+\lambda_i^N \Phi_i^{N\setminus C_r}\left(V\right)+\frac{b\lambda_i^N}{a}\right) \\
& +v\left(N\right)+\frac{b\lambda_i^N}{a}-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N \Phi_j^{N\setminus C_q}\left(V\right)\right) \\
= & \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N \Phi_j^{N\setminus C_r}\left(V\right)\right)+v\left(N\right) \\
& -\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N \Phi_j^{N\setminus C_q}\left(V\right)\right)+|\mathcal{C}|\frac{b\lambda_i^N}{a} \\
= & |\mathcal{C}|\sum_{j\in C_q}\lambda_j^N \Phi_j^N\left(V\right)+|\mathcal{C}|\frac{b\lambda_i^N}{a}
\end{aligned}$$

where the last equality comes because Φ satisfies (3.3).

Given $k \in C_q$, by (3.4),

$$|C_q|\,\tilde{\lambda}_k^N \Phi_k^N\left(\widetilde{V}\right)=\sum_{j\in C_q\setminus k}\tilde{\lambda}_k^N \Phi_k^{N\setminus j}\left(\widetilde{V}\right)+\sum_{j\in C_q}\tilde{\lambda}_j^N \Phi_j^N\left(\widetilde{V}\right)-\sum_{j\in C_q\setminus k}\tilde{\lambda}_j^N \Phi_j^{N\setminus k}\left(\widetilde{V}\right).$$

If $k=i$, by the induction hypothesis and the previous result, $|C_q|\,\tilde{\lambda}_i^N \Phi_i^N\left(\widetilde{V}\right)=$

$$\begin{aligned}
& \sum_{j\in C_q\setminus i}\left(\lambda_i^N \Phi_i^{N\setminus j}(V)+\frac{b\lambda_i^N}{a}\right) \\
& +\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V)+\frac{b\lambda_i^N}{a}-\sum_{j\in C_q\setminus i}\lambda_j^N\Phi_j^{N\setminus i}(V) \\
= & \sum_{j\in C_q\setminus i}\lambda_i^N\Phi_i^{N\setminus j}(V)+\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V) \\
& -\sum_{j\in C_q\setminus i}\lambda_j^N\Phi_j^{N\setminus i}(V)+\frac{b\lambda_i^N}{a}|C_q| \\
= & \;|C_q|\,\lambda_i^N\Phi_i^N(V)+\frac{b\lambda_i^N}{a}|C_q|
\end{aligned}$$

where the last equality comes because Φ satisfies (3.4).

Then,

$$\Phi_i^N\left(\widetilde{V}\right)=a\Phi_i^N(V)+b.$$

If $k\neq i$, by the induction hypothesis and the previous result, $|C_q|\,\lambda_k^N\Phi_k^N\left(\widetilde{V}\right)=$

$$\begin{aligned}
& \sum_{j\in C_q\setminus k}\lambda_k^N\Phi_k^{N\setminus j}(V)+\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V)+\frac{b\lambda_i^N}{a} \\
& -\sum_{j\in C_q\setminus\{k,i\}}\lambda_j^N\Phi_j^{N\setminus k}(V)-\lambda_i^N\Phi_i^{N\setminus k}(V)-\frac{b\lambda_i^N}{a} \\
= & \sum_{j\in C_q\setminus k}\lambda_k^N\Phi_k^{N\setminus j}(V)+\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V)-\sum_{j\in C_q\setminus k}\lambda_j^N\Phi_j^{N\setminus k}(V) \\
= & \;|C_q|\,\lambda_k^N\Phi_k^N(V).
\end{aligned}$$

Then, $\Phi_k^N\left(\widetilde{V}\right)=\Phi_k^N(V)$.

Given $C_r\in\mathcal{C}\setminus C_q$ using arguments similar to those used for C_q we can conclude that

$$\sum_{j\in C_r}\tilde{\lambda}_j^N\Phi_j^N\left(\widetilde{V}\right)=\sum_{j\in C_r}\lambda_j^N\Phi_j^N(V).$$

Now using (3.4) it is not difficult to conclude that for each $j\in C_r$, $\Phi_j^N\left(\widetilde{V}\right)=\Phi_j^N(V)$.

Then, Φ satisfies COV. ■

Proof of Theorem 23. In Theorem 21 we proved that the consistent coalitional value satisfies these five properties in the class of hyperplane games.

We now prove the reciprocal. Let $\widetilde{\Phi}$ be a single value satisfying these five properties. We will show that $\widetilde{\Phi} = \Phi$. We proceed by induction on the number of players. If there is only one player then, by EF, $\widetilde{\Phi} = \max\{x : x \in V(i)\} = \Phi$.

Assume $n = 2$. We can assume without loss of generality that $\lambda_i^{\{i\}} = \lambda_j^{\{j\}} = 1$. There are two possible coalition structures, $\mathcal{C}^1 = \{i, j\}$ and $\mathcal{C}^2 = \{\{i\}, \{j\}\}$.

Given $a \in \mathbb{R}$, let (N, v^a) be the TU game given by $v^a(i) = v^a(j) = a$ and $v^a(N) = 1$.

Since $\widetilde{\Phi}$ satisfies EF and IS we conclude that

$$\widetilde{\Phi}_i^N\left(v^a, \mathcal{C}^1\right) = \widetilde{\Phi}_j^N\left(v^a, \mathcal{C}^1\right) = \frac{1}{2}.$$

Since $\widetilde{\Phi}$ satisfies EF and $ABCAC$ we conclude that

$$\widetilde{\Phi}_i^N\left(v^a, \mathcal{C}^2\right) = \widetilde{\Phi}_j^N\left(v^a, \mathcal{C}^2\right) = \frac{1}{2}.$$

A similar result can be obtained for Φ.

As any hyperplane game with two players $(N, V, \mathcal{C})$ can be obtained from v^a (for some a) by linear transformation of utilities of players, and Φ and $\widetilde{\Phi}$ satisfy COV it is straightforward to prove that for each $i \in N$,

$$\widetilde{\Phi}_i^N = \frac{v(N) + \lambda_i^N v(i) - \lambda_j^N v(j)}{2\lambda_i^N} = \Phi_i^N.$$

Moreover,

$$\lambda_i^N \Phi_i^N - \lambda_j^N \Phi_j^N = \lambda_i^N \widetilde{\Phi}_i^N - \lambda_j^N \widetilde{\Phi}_j^N = \lambda_i^N v(i) - \lambda_j^N v(j). \tag{3.13}$$

Assume that $\widetilde{\Phi} = \Phi$ for hyperplane games with at most $n-1$ players with $n \geq 3$. We will prove it when $(N, V, \mathcal{C})$ is a hyperplane game with n players.

We first prove that for each $C_q \in \mathcal{C}$,

$$\sum_{j \in C_q} \lambda_j^N \Phi_j^N(V) = \sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^N(V).$$

By induction hypothesis, we know that $\widetilde{\Phi}^S(V) = \Phi^S(V)$ for each $S \subsetneq N$. Given $C_q \in \mathcal{C}$, by (3.3), $\sum\limits_{j \in C_q} \lambda_j^N \Phi_j^N(V) =$

$$\begin{aligned}
& \frac{1}{|\mathcal{C}|}\left[\sum_{C_r \in \mathcal{C} \setminus C_q}\left(\sum_{j \in C_q} \lambda_j^N \Phi_j^{N \setminus C_r}(V)\right) + v(N) - \sum_{C_r \in \mathcal{C} \setminus C_q}\left(\sum_{j \in C_r} \lambda_j^N \Phi_j^{N \setminus C_q}(V)\right)\right] \\
= \; & \frac{1}{|\mathcal{C}|}\left[\sum_{C_r \in \mathcal{C} \setminus C_q}\left(\sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^{N \setminus C_r}(V)\right) + v(N) - \sum_{C_r \in \mathcal{C} \setminus C_q}\left(\sum_{j \in C_r} \lambda_j^N \widetilde{\Phi}_j^{N \setminus C_q}(V)\right)\right].
\end{aligned}$$

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Since $\widetilde{\Phi}$ satisfies EF, $v(N)=\sum_{C_r\in\mathcal{C}}\left(\sum_{j\in C_r}\lambda_j^N\widetilde{\Phi}_j^N(V)\right)$. Then,

$$\begin{aligned}|\mathcal{C}|\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V) &= \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^{N\setminus C_r}(V)\right)+\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V)\\ &+\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Phi}_j^N(V)-\widetilde{\Phi}_j^{N\setminus C_q}(V)\right)\right)\end{aligned}$$

we add and subtract $\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V)\right)=(|\mathcal{C}|-1)\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V)$,

$$\begin{aligned}&= \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\widetilde{\Phi}_j^{N\setminus C_r}(V)-\widetilde{\Phi}_j^N(V)\right)\right)+|\mathcal{C}|\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V)\\ &+\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Phi}_j^N(V)-\widetilde{\Phi}_j^{N\setminus C_q}(V)\right)\right).\end{aligned}$$

So, $|\mathcal{C}|\left(\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V)-\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V)\right)=$

$$\begin{aligned}&= \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Phi}_j^N(V)-\widetilde{\Phi}_j^{N\setminus C_q}(V)\right)\right)\\ &-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\widetilde{\Phi}_j^N(V)-\widetilde{\Phi}_j^{N\setminus C_r}(V)\right)\right).\end{aligned}$$

Since $\widetilde{\Phi}$ satisfies $ABCAC$ we conclude that the last expression is equal to 0. Then,

$$\sum_{j\in C_q}\lambda_j^N\Phi_j^N(V)=\sum_{j\in C_q}\lambda_j^N\widetilde{\Phi}_j^N(V). \tag{3.14}$$

We now prove that $\widetilde{\Phi}_i^N=\Phi_i^N$ for each $i\in C_q\in\mathcal{C}$. We denote by V_S and $\widetilde{V}_S$ the reduced games associated with Φ and $\widetilde{\Phi}$, respectively.

If $C_q=\{i\}$, by (3.14) we conclude that $\widetilde{\Phi}_i^N=\Phi_i^N$.

Assume that $C_q\neq\{i\}$. For each $j\in C_q\setminus i$ we consider $S=\{i,j\}$. We know that V_S and $\widetilde{V}_S$ are hyperplane games. Then, we denote by v_S and $\tilde{v}_S$ the associated functions with V_S and $\widetilde{V}_S$. By the definition of reduced game and the induction hypothesis,

$$\widetilde{V}_S(i)=V_S(i) \text{ and } \widetilde{V}_S(j)=V_S(j).$$

Hence, $v_S(i) = \tilde{v}_S(i)$ and $v_S(j) = \tilde{v}_S(j)$.
Since $\widetilde{\Phi}$ satisfies EF we conclude that $v(N) = \sum_{k \in N} \lambda_k^N \widetilde{\Phi}_k^N(V)$. Then,

$$\widetilde{V}_S(S) = \left\{ (x_i, x_j) \in \mathbb{R}^{\{i,j\}} : \lambda_i^N x_i + \lambda_j^N x_j \leq \lambda_i^N \widetilde{\Phi}_i^N(V) + \lambda_j^N \widetilde{\Phi}_j^N(V) \right\}.$$

By the efficiency of $\widetilde{\Phi}$ and (3.13),

$$\begin{aligned}
\lambda_i^N \widetilde{\Phi}_i^S\left(\tilde{V}_S\right) + \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \lambda_i^N \widetilde{\Phi}_i^N(V) + \lambda_j^N \widetilde{\Phi}_j^N(V) \\
\lambda_i^N \widetilde{\Phi}_i^S\left(\tilde{V}_S\right) - \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \lambda_i^N \tilde{v}_S(i) - \lambda_j^N \tilde{v}_S(j).
\end{aligned}$$

If we sum both expressions on $C_q \setminus i$,

$$\begin{aligned}
\lambda_i^N \sum_{j \in C_q \setminus i} \widetilde{\Phi}_i^S\left(\tilde{V}_S\right) + \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \lambda_i^N (|C_q| - 1) \widetilde{\Phi}_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^N(V) \\
\lambda_i^N \sum_{j \in C_q \setminus i} \widetilde{\Phi}_i^S\left(\tilde{V}_S\right) - \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \lambda_i^N (|C_q| - 1) \tilde{v}_S(i) + \sum_{j \in C_q \setminus i} \lambda_j^N \tilde{v}_S(j).
\end{aligned}$$

Since $\widetilde{\Phi}$ satisfies $BCONS$, $\sum_{j \in C_q \setminus i} \widetilde{\Phi}_i^S(V_S) = (|C_q| - 1) \widetilde{\Phi}_i^N(V)$, and hence

$$\begin{aligned}
\lambda_i^N \widetilde{\Phi}_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^N(V) \\
(|C_q| - 1) \lambda_i^N \widetilde{\Phi}_i^N(V) - \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) &= \lambda_i^N (|C_q| - 1) \tilde{v}_S(i) + \sum_{j \in C_q \setminus i} \lambda_j^N \tilde{v}_S(j).
\end{aligned}$$

A similar analysis for Φ yields,

$$\begin{aligned}
\lambda_i^N \Phi_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^S(V_S) &= \sum_{j \in C_q} \lambda_j^N \Phi_j^N(V) \\
(|C_q| - 1) \lambda_i^N \Phi_i^N(V) - \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^S(V_S) &= \lambda_i^N (|C_q| - 1) v_S(i) + \sum_{j \in C_q \setminus i} \lambda_j^N v_S(j).
\end{aligned}$$

By (3.14),

$$\lambda_i^N \widetilde{\Phi}_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) = \lambda_i^N \Phi_i^N(V) + \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^S(V_S). \quad (3.15)$$

Since $\tilde{v}_S(i) = v_S(i)$ and $\tilde{v}_S(j) = v_S(j)$,

$$\left.\begin{aligned}
&(|C_q| - 1) \lambda_i^N \widetilde{\Phi}_i^N(V) - \sum_{j \in C_q \setminus i} \lambda_j^N \widetilde{\Phi}_j^S\left(\tilde{V}_S\right) \\
&= (|C_q| - 1) \lambda_i^N \Phi_i^N(V) - \sum_{j \in C_q \setminus i} \lambda_j^N \Phi_j^S(V_S).
\end{aligned}\right\} \quad (3.16)$$

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Adding (3.15) and (3.16),

$$|C_q| \lambda_i^N \Phi_i^N (V) = |C_q| \lambda_i^N \widetilde{\Phi}_i^N (V)$$

which means that $\widetilde{\Phi}_i^N (V) = \Phi_i^N (V)$. ■

Proof of Remark 24. $ABCAC$ is independent of the rest of properties because the consistent value satisfies EF, IS, COV, and $BCONS$ but not $ABCAC$.

Using arguments similar to those used by Winter (1992) we can conclude that the rest of properties are independent. ■

Proof of Theorem 25. By Theorem 21 we know that Φ satisfies these properties.

We now prove the reciprocal. We proceed by induction on the number of players. The result is trivially true for $n = 1$. Assume the result holds for each $S \subsetneq N$.

Assume now $\left(\widetilde{\Phi}^S\right)_{S \subset N}$ is a payoff configuration associated with a value $\widetilde{\Phi}$ satisfying these properties. Since $\widetilde{\Phi}$ satisfies EF, by Remark 18, for each $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying $\lambda^S \cdot \widetilde{\Phi}^S = v(S)$ where $v(S) = \max\left\{\lambda^S \cdot x : x \in V(S)\right\}$. Let $(N, V', \mathcal{C})$ be the corresponding hyperplane game, *i.e.* for each $S \subset N$,

$$V'(S) = \left\{y \in \mathbb{R}^S : \lambda^S \cdot y \leq v(S)\right\}.$$

By induction hypothesis, for each $S \subsetneq N$, $\widetilde{\Phi}^S = \Phi^S (V')$. We will show that $\widetilde{\Phi}^N = \Phi^N (V')$. For simplicity, we take $\Phi^N = \Phi^N (V')$. Assume that $i \in C_q \in \mathcal{C}$.

Since $\widetilde{\Phi}$ satisfies EF and $ABCAC$, using arguments similar to those used in the proof of Theorem 23 we can conclude that for each $C_q \in \mathcal{C}$,

$$\sum_{j \in C_q} \lambda_j^N \Phi_j^N = \sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^N.$$

By (3.4),

$$|C_q| \lambda_i^N \Phi_i^N = \sum_{j \in C_q \backslash i} \lambda_i^N \Phi_i^{N \backslash j} + \sum_{j \in C_q} \lambda_j^N \Phi_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N \Phi_j^{N \backslash i}.$$

Since $\sum_{j \in C_q} \lambda_j^N \Phi_j^N = \sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^N$ and the induction hypothesis,

$$\begin{aligned} |C_q| \lambda_i^N \Phi_i^N &= \sum_{j \in C_q \backslash i} \lambda_i^N \widetilde{\Phi}_i^{N \backslash j} + \sum_{j \in C_q} \lambda_j^N \widetilde{\Phi}_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N \widetilde{\Phi}_j^{N \backslash i} \\ &= \sum_{j \in C_q \backslash i} \lambda_i^N \widetilde{\Phi}_i^{N \backslash j} + \lambda_i^N \widetilde{\Phi}_i^N + \sum_{j \in C_q \backslash i} \lambda_j^N \left(\widetilde{\Phi}_j^N - \widetilde{\Phi}_j^{N \backslash i}\right) \end{aligned}$$

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if we add and subtract $\sum_{j\in C_q\setminus i} \lambda_i^N \widetilde{\Phi}_i^N = (|C_q|-1)\lambda_i^N \widetilde{\Phi}_i^N$ we obtain:

$$= \sum_{j\in C_q\setminus i} \lambda_i^N \left(\widetilde{\Phi}_i^{N\setminus j} - \widetilde{\Phi}_i^N\right) + |C_q|\,\lambda_i^N \widetilde{\Phi}_i^N + \sum_{j\in C_q\setminus i} \lambda_j^N \left(\widetilde{\Phi}_j^N - \widetilde{\Phi}_j^{N\setminus i}\right).$$

Then,

$$|C_q|\,\lambda_i^N\left(\Phi_i^N - \widetilde{\Phi}_i^N\right) = \sum_{j\in C_q\setminus i} \lambda_j^N \left(\widetilde{\Phi}_j^N - \widetilde{\Phi}_j^{N\setminus i}\right) - \sum_{j\in C_q\setminus i} \lambda_i^N \left(\widetilde{\Phi}_i^N - \widetilde{\Phi}_i^{N\setminus j}\right).$$

Since $\widetilde{\Phi}$ satisfies $ABCAP$ we conclude that the last expression is equal to 0. Then, $\widetilde{\Phi}_i^N = \Phi_i^N$. ■

Proof of Remark 26. EF is independent of the rest of properties. The value $\Gamma_i^N = 0$ for each NTU game $(N, V, \mathcal{C})$ and $i \in N$ satisfies $ABCAC$ and $ABCAP$ but not EF.

$ABCAP$ is independent of the rest of properties. The random order coalitional value satisfies EF and $ABCAC$ but not $ABCAP$.

$ABCAC$ is independent of the rest of properties.

Given a hyperplane game $(N, V, \mathcal{C})$ we define, for each $i \in N$,

$$\Omega_i^N = \frac{v(N)}{|N|\,\lambda_i^N}.$$

Let $\pi \in \Pi_q$ be a order of players in C_q. We consider $f(\pi) \in \mathbb{R}^{C_q}$ such that for each $i \in C_q$,

$$f_i(\pi) = \max\left\{x_i : \left(\left(\Omega_j^S\right)_{j\in\overline{C_q}}, \left(f_j(\pi)\right)_{j\in P_i^\pi}, x_i\right) \in V(S)\right\}$$

where $S = \overline{C_q} \cup P_i^\pi \cup i$.

It is straightforward to prove that

$$f_i(\pi) = \frac{v(S) - \sum_{j\in\overline{C_q}} \lambda_j^S \Omega_j^S - \sum_{j\in P_i^\pi} \lambda_j^S f_j(\pi)}{\lambda_i^S}.$$

Then, given $i \in C_q \in \mathcal{C}$, we define Γ as follows:

$$\Gamma_i^N = \frac{1}{|\Pi_q|}\sum_{\pi\in\Pi_q} f_i(\pi).$$

Since Ω satisfies EF, for each $C_q \in \mathcal{C}$ and $\pi \in \Pi_q$, $\sum_{j\in C_q} \lambda_j^N \Omega_j^N = \sum_{j\in C_q} \lambda_j^N f_j(\pi)$ and hence, $\sum_{j\in C_q} \lambda_j^N \Omega_j^N = \sum_{j\in C_q} \lambda_j^N \Gamma_j^N$. Then, it is trivial to see that Γ satisfies EF in the class of hyperplane games.

We now prove that Γ satisfies $ABCAP$.

For each $j \in C_q$ we denote by $\Pi_q(j)$ the set of orders of Π_q where j is the last player. If $j \neq i$, then player i's expected marginal contribution conditioned to j being last, is the same as in the game $(N \backslash j, V, \mathcal{C}_{-j})$, which is $\Gamma_i^{N \backslash j}$, *i.e.*

$$\frac{1}{|\Pi_q(j)|} \sum_{\pi \in \Pi_q(j)} f_i(\pi) = \frac{1}{\left|\Pi_q^{\mathcal{C}_{-j}}\right|} \sum_{\pi \in \Pi_q^{\mathcal{C}_{-j}}} f_i(\pi) = \Gamma_i^{N \backslash j}.$$

Given $\pi \in \Pi_q(i)$,

$$\begin{aligned} f_i(\pi) &= \frac{v(N) - \sum_{j \in \overline{C_q}} \lambda_j^N \Omega_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Omega_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N f_j(\pi)}{\lambda_i^N}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} f_i(\pi) &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} \frac{\sum_{j \in C_q \backslash i} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N} \sum_{j \in C_q \backslash i} \lambda_j^N \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} f_j(\pi) \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N} \sum_{j \in C_q \backslash i} \lambda_j^N \Gamma_j^{N \backslash i} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum_{j \in C_q \backslash i} \lambda_j^N \Gamma_j^{N \backslash i}}{\lambda_i^N}. \end{aligned}$$

Thus, for each $i \in C_q \in \mathcal{C}$,

$$\begin{aligned} \Gamma_i^N &= \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q} f_i(\pi) \\ &= \frac{1}{|\Pi_q|} \sum_{j \in C_q \backslash i} \left(\sum_{\pi \in \Pi_q(j)} f_i(\pi) \right) + \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q(i)} f_i(\pi) \end{aligned}$$

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since $|\Pi_q| = |C_q| \, |\Pi_q(j)|$ for each $j \in C_q$, the last expression can be rewritten as

$$= \frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\frac{1}{|\Pi_q(j)|}\sum_{\pi\in\Pi_q(j)} f_i(\pi) + \frac{1}{|\Pi_q(i)|}\sum_{\pi\in\Pi_q(i)} f_i(\pi)\right]$$

$$= \frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\Gamma_i^{N\setminus j} + \frac{\sum\limits_{j\in C_q}\lambda_j^N\Gamma_j^N - \sum\limits_{j\in C_q\setminus i}\lambda_j^N\Gamma_j^{N\setminus i}}{\lambda_i^N}\right].$$

Then,

$$|C_q|\,\lambda_i^N\Gamma_i^N = \sum_{j\in C_q\setminus i}\lambda_i^N\Gamma_i^{N\setminus j} + \sum_{j\in C_q}\lambda_j^N\Gamma_j^N - \sum_{j\in C_q\setminus i}\lambda_j^N\Gamma_j^{N\setminus i}.$$

Since $|C_q|\,\lambda_i^N\Gamma_i^N = \sum\limits_{j\in C_q}\lambda_i^N\Gamma_i^N$ we conclude that Γ satisfies $ABCAP$.

If we proceed with Γ in the same way that we did with Φ we can extend Γ to the set of NTU games and prove that Γ also satisfies EF and $ABCAP$ in the class of NTU games. ■

Proof of Corollary 27. Since each TU game is a hyperplane game we conclude that the consistent coalitional value is a single value. Repeating the same arguments that in the proof of Theorem 25 for TU games we can obtain that there is at most a value (on the set of TU games) satisfying EF, $ABCAC$, and $ABCAP$. Then, we only need to prove that the Owen value ϕ satisfies these properties.

We know that ϕ satisfies EF. We now prove that ϕ satisfies $ABCAC$ and $ABCAP$. For simplicity, we assume that $S = N$.

Since ϕ satisfies $BCAC$, for each $C_q, C_r \in \mathcal{C}$

$$\sum_{j\in C_q}\left(\phi_j^N - \phi_j^{N\setminus C_r}\right) = \sum_{j\in C_r}\left(\phi_j^N - \phi_j^{N\setminus C_q}\right).$$

Then,

$$\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\left(\phi_j^N - \phi_j^{N\setminus C_r}\right)\right) = \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\left(\phi_j^N - \phi_j^{N\setminus C_q}\right)\right)$$

which means that ϕ satisfies $ABCAC$ in TU games.

Since ϕ satisfies $BCAP$, for each $C_q \in \mathcal{C}$ and $i, j \in C_q$

$$\phi_i^N - \phi_i^{N\setminus j} = \phi_j^N - \phi_j^{N\setminus i}.$$

Then,

$$\sum_{j\in C_q\setminus i}\left(\phi_i^N - \phi_i^{N\setminus j}\right) = \sum_{j\in C_q\setminus i}\left(\phi_j^N - \phi_j^{N\setminus i}\right)$$

which means that ϕ satisfies $ABCAP$ in TU games. ■

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Chapter 4

A bargaining approach

4.1 Introduction

Hart and Mas-Colell (1996) develop a bargaining mechanism which yields the consistent value (Maschler and Owen 1989, 1992) for NTU games. First, a player is randomly chosen in order to propose a payoff. In case this proposal not be accepted by all other players, the mechanism is played again under the same conditions with probability $\delta \in [0, 1)$. With probability $1 - \delta$, the proposer leaves the game and the mechanism is repeated with the rest of the players. Hart and Mas-Colell consider that the consistent value is a very appropriate generalization for the Shapley (1953) value (used in TU games) to NTU games.

Other non-cooperative mechanisms which implement the Shapley value are, for example, Gul (1989), Hart and Moore (1990), Winter (1994), Evans (1996), Dasgupta and Chiu (1998), Pérez-Castrillo and Wettstein (2001) and Mutuswa mi, Pérez-Castrillo and Wettstein (2002). Navarro and Perea (2001) design a mechanism which implements the Myerson (1977) value, which is an extension of the Shapley value to graph-restricted games.

Sometimes, however, players are associated in *a priori* coalitions. Owen (1977) studies them in TU games. He proposes a value, called the Owen value, which generalizes the Shapley value for games with a coalition structure. Later, Winter (1991) proposes a value, called the Game Coalition Structure value, which is a generalization of the Harsanyi (1963) value NTU games and the Owen value for TU games with a coalition structure.

A non-cooperative mechanism which implements the Owen value in the TU case is given in Chapter 1.

In this Chapter, we develop a mechanism that takes into account the coalition structure and implements both the consistent value for NTU games, and the Owen value for TU games.

The mechanism is as follows: First, a player is randomly chosen out of each coalition and proposes a payoff. Then, each proposal is voted by the rest of the members of its own coalition. If one of them rejects the proposed payoff, the mechanism is either played again under the same conditions (probability ρ), or the proposer leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$). If there is no rejection, the proposal of one of the coalitions is randomly chosen. If this proposal is not accepted by all other coalitions, the mechanism is played again under the same conditions (probability ρ), or the entire proposing coalition leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$).

When the coalition structure is trivial (*i.e.*, either there is a single grand coalition or all the coalitions are singletons), this mechanism coincides with Hart and Mas-Colell's. Thus, the consistent value arises in equilibrium. Furthermore, when the mechanism is applied to a transferable utility (TU) game with coalition structure, the Owen value is implemented.

As for general NTU games with coalition structure, the arising equilibrium payoff is the consistent coalitional value, studied in Chapter 3.

Assume we change the mechanism so that, before any proposal is set, all the players know who is bound to be the proposer. This new mechanism also coincides with Hart and Mas-Colell's when the coalition structure is trivial. However, for general NTU games with coalition structure, a new coalitional value arises. We study this value in Section 4.4.

The structure of this Chapter is as follows: In Section 4.2 we give the definitions and results used in the Chapter. In Section 4.3 we describe the coalitional mechanism and give the main results: Theorem 38 deals with the existence of equilibria. Theorem 39 proves the result for hyperplane games. Theorem 41 gives the general convergence result. In Section 4.4, we present a slight modification in the coalitional mechanism. Finally, the proofs are located in the Appendix.

4.2 Definitions and previous results

In this Chapter, we follow the notation in Hart and Mas-Colell (1996). In particular, given a vector $x \in \mathbb{R}^N$, we denote by x^i the ith coordinate of x. Let $N = \{1, 2, ..., n\}$ and $2^N = \{S : S \subset N\}$. Given $x, y \in \mathbb{R}^N$, we say $y \leq x$ when $y^i \leq x^i$ for every $i \in N$. We denote by $x \cdot y$ the scalar product $\sum_{i \in N} x^i y^i$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x^i \geq 0, \forall i\}$, and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x^i > 0, \forall i\}$.

A *non-transferable utility game*, or *NTU game*, is a pair (N, V) where N is the set of *players* and V is a correspondence which assigns to each *coalition* $S \subset N$, $S \neq \emptyset$ a subset $V(S) \subset \mathbb{R}^S$ representing all the possible payoffs that the members of S can obtain for themselves when play cooperatively. For $S \subset N$, we maintain the notation V when refer to the application V restricted to S as player set. For simplicity, we denote $V(i)$ instead of $V(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $N \backslash i$ instead of $N \backslash \{i\}$.

We impose next conditions on V:

(A.1) For each $S \subset N$, the set $V(S)$ is closed, convex, *comprehensive* (*i.e.*, if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and *upper bounded* (*i.e.*, for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is bounded).

(A.2) For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is *smooth* (this means that on each point of the boundary there exists an unique outward ortonormal vector) and *nonlevel* (this means that the outward vector on each point of $\partial V(S)$ has its coordinates positive).

(A.3) *Monotonicity*: For each $T \subset S$, $V(T) \times \left\{0^{S \backslash T}\right\} \subset V(S)$.

(A.4) *Normalization*: For each $S \subset N$, 0^S belongs to $V(S)$.

For each $i \in N$, let $r^i := \max\{x : x \in V(i)\}$ (notice that, by (A.4), $r^i \geq 0$). When

$$V(S) = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x^i \leq v(S)\right\}$$

for some $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, we say that (N, V) is a *transferable utility game* (or *TU game*) and it is represented by (N, v). We denote by $TU(N)$ the set of TU games over N.

When

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S)\right\} \tag{4.1}$$

for some $\lambda_S \in \mathbb{R}^S_{++}$ and $v : 2^N \to \mathbb{R}$, we say that (N, V) is a *hyperplane game*.

Notice that every TU game is a hyperplane game with $\lambda^i_S = 1$ for every $S \subset N$ and $i \in S$.

If $r^S \in \partial V(S)$ for all $S \subsetneq N$ and $r^N \in V(N)$, we say that (N, V) is a *pure bargaining game*.

We say that an NTU game is *totally essential* if $r^S \in V(S)$ for all $S \subset N$. We say that an NTU game is *zero-monotonic* if $V(i) \times V(S \backslash i) \subset V(S)$ for all $i \in S \subset N$.

Given N, we call *coalition structure* on N a partition of the player set, *i.e.*, $\mathcal{C} = \{C_1, C_2, ..., C_p\} \subset 2^N$ is a coalition structure on N if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$.

We denote by $(N, V, \mathcal{C})$ an NTU game (N, V) with coalition structure $\mathcal{C}$ on N. We denote by $CNTU(N)$ the set of NTU games with coalition structure on N. For coalitions $S \subset N$, we denote by $\mathcal{C}_S$ the restriction of $\mathcal{C}$ to the players in S (notice that this implies that $\mathcal{C}_S$ may have less or the same number of coalitions as $\mathcal{C}$). We also denote $\mathcal{C}_{-i} := \mathcal{C}_{N \backslash i}$.

Given G a subset of $NTU(N)$ or $CNTU(N)$, a *value* on G is a correspondence which assigns to each element in G a subset of $\mathbb{R}^N$. When these subsets are singletons we call the value a *single value*. A well known single value for TU games is the *Shapley value* (Shapley, 1953). We denote by $\varphi_N \in \mathbb{R}^N$ the Shapley value of the TU game (N, v). For TU games with coalition structure, Owen (1977) proposes a single value based on Shapley's which takes into account the coalition structure $\mathcal{C}$. We call this value the *Owen coalitional value*, or simply the *Owen value*. We denote by $\phi_N \in \mathbb{R}^N$ the Owen value of the TU game with coalition structure $(N, v, \mathcal{C})$.

The *consistent value* for NTU games is introduced by Maschler and Owen (1989, 1992). Let (N, V) be a hyperplane game defined as in (4.1). Given $i \in N$, the consistent value Ψ is defined recursively as follows

$$\Psi^i_{\{i\}} = r^i.$$

Assume we know Ψ^j_S for all $S \subsetneq N$ and $j \in S$. Then,

$$\Psi^i_N = \frac{1}{|N|\,\lambda^i_N}\left(\sum_{j\in N\setminus i}\lambda^i_N \Psi^i_{N\setminus j} - \sum_{j\in N\setminus i}\lambda^j_N \Psi^j_{N\setminus i} + v\left(N\right)\right).$$

For a general NTU game (N, V), Maschler and Owen (1992) take for each coalition $S \subset N$ a vector λ_S normal to the boundary of $V(S)$. Let (N, V') be the resulting hyperplane game, *i.e.* $V'(S) = \left\{x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S, \lambda_S)\right\} \supset V(S)$, with

$$v(S, \lambda_S) := \max\left\{\lambda_S \cdot x : x \in V(S)\right\}.$$

Let $\Psi = (\Psi_S)_{S\subset N}$ with Ψ_S the (only) consistent value for (S, V'). If Ψ is a feasible payoff in (N, V) (*i.e.*, $\Psi_S \in V(S), \forall S \subset N$) then Ψ_N is a *consistent value* for V.

The consistent value coincides with the Shapley value for TU games. Maschler and Owen (1992) also show that the consistent value exists (it is not always unique though) for any NTU game.

Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure. In Chapter 3, we define recursively the *consistent coalitional value* as follows. Given $i \in C_q \in \mathcal{C}$:

$$\Phi^i_{\{i\}} = r^i.$$

Assume we know Φ^j_S for all $S \subsetneq N$ and $j \in S$. Then,

$$\left.\begin{aligned}\Phi^i_N = \;& \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N}\left(\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j_N\Phi^j_{N\setminus C_r} - \sum_{j\in C_r}\lambda^j_N\Phi^j_{N\setminus C_q}\right)\right)\\ &+\frac{1}{|C_q|\,\lambda^i_N}\left(\sum_{j\in C_q\setminus i}\lambda^i_N\Phi^i_{N\setminus j} - \sum_{j\in C_q\setminus i}\lambda^j_N\Phi^j_{N\setminus i}\right)\\ &+\frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N}v\left(N\right).\end{aligned}\right\}\quad (4.2)$$

Following the usual practice, we consider a *payoff configuration* as a set of payoffs $x = (x_S)_{S\subset N}$ with $x_S \in V(S)$ for all $S \subset N$.

The generalization of Φ to NTU games (not necessarily hyperplane games) is done analogously to the consistent value. For an NTU game with coalition structure $(N, V, \mathcal{C})$, we take for each coalition $S \subset N$ a normal vector λ_S to the boundary of $V(S)$. Let $(N, V', \mathcal{C})$ be the resulting hyperplane game. Let $\Phi := (\Phi_S)_{S\subset N}$ for all $S \subset N$ be the (unique) consistent coalitional payoff configuration for V'. If Φ is a feasible payoff configuration for $(N, V, \mathcal{C})$, then Φ is a consistent coalitional payoff configuration for V.

In Chapter 3, we prove that the consistent coalitional value exists for any NTU game (although it is not necessarily unique) and give the following characterization. Given $S \subset N$ player set, we denote by $C'_q := C_q \cap S$ (when different from $\emptyset$) the restriction of C_q in $\mathcal{C}_S$.

The set $\Phi = (\Phi_S)_{S\subset N}$ is a consistent coalitional payoff configuration for $(N, V, \mathcal{C})$ if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$, orthogonal to $V(S)$, such that:

(B.1) $\Phi_S \in \partial V(S)$;

(B.2) $\sum_{C'_r\in\mathcal{C}_S\setminus C'_q}\left[\sum_{i\in C'_q}\lambda^i_S(\Phi^i_S - \Phi^i_{S\setminus C'_r})\right] = \sum_{C'_r\in\mathcal{C}_S\setminus C'_q}\left[\sum_{i\in C'_r}\lambda^i_S(\Phi^i_S - \Phi^i_{S\setminus C'_q})\right]$ for every $C'_q \in \mathcal{C}_S$;

(B.3) $\sum_{j\in C'_q\setminus i}\lambda^i_S(\Phi^i_S - \Phi^i_{S\setminus j}) = \sum_{j\in C'_q\setminus i}\lambda^j_S(\Phi^j_S - \Phi^j_{S\setminus i})$ for every $i \in C'_q \in \mathcal{C}_S$.

Thus, in this Chapter, *EF*, *ABCAC*, and *ABCAP* are denoted by (B.1), (B.2), and (B.3), respectively.

4.3 The coalitional mechanism

In this section we describe the coalitional mechanism. This mechanism is a modification of the bargaining mechanism presented by Hart and Mas-Colell (1996).

In order to characterize the equilibria, we need to restrict the class of games. This restriction is given by property (A.5) below. We claim that this property is not too restrictive by showing that a significative class of games (including TU zero-monotonic games and pure bargaining games) satisfies it. Then, we characterize the equilibria and show that there exits at least an equilibrium. Finally, we prove that the equilibria yield the consistent coalitional value.

Given an NTU game (N, V) and $\rho \in [0, 1)$, Hart and Mas-Colell (1996) define the following bargaining mechanism (associated with (N, V) and ρ):

> "In each round there is a set of *active* players, and a *proposer* $i \in S$. In the first round $S = N$. The proposer is chosen at random out of S, with all players in S being equally likely to be selected. The proposer makes a proposal which is feasible, *i.e.* a payoff vector in $V(S)$. If all the members of S accept it – they are asked in some prespecified order – then the game ends with these payoffs. If it is rejected by even one member of S, then we move to the next round where, with probability ρ, the set of active players is again S and,

> with probability $1-\rho$, the proposer i *drops out* and the set of active players becomes $S\backslash i$. In the latter case the dropped i gets a final payoff of 0."

Hart and Mas-Colell (1996) prove that, for each hyperplane game, and for each $\rho \in [0,1)$, the bargaining mechanism implements the consistent value for SPNE.

Furthermore, for a general NTU game (N,V), if for each $S \subset N$, $a_S(\rho)$ is the payoff of a SPNE for $\rho \in [0,1)$ and a_S is a limit point of $a_S(\rho)$ as $\rho \to 1$, then $(a_S)_{S\subset N}$ is a consistent payoff configuration of the NTU game (N,V).

Now we describe the *coalitional bargaining mechanism* formally. For each $S \subset N$, we denote by Γ_S the set of applications $\gamma : \mathcal{C}_S \to S$ satisfying $\gamma(C'_q) \in C'_q$ for each $C'_q \in \mathcal{C}_S$. For simplicity, we denote $\Gamma = \Gamma_N$.

The coalitional bargaining mechanism associated with $(N,V,\mathcal{C})$ and $\rho \in [0,1)$ is defined as follows:

> In each round there is a set $S \subset N$ of active players. At first round, $S = N$. Each round has two stages. On the first stage, a *proposer* is randomly chosen out of each coalition. Namely, a function $\gamma \in \Gamma_S$ is randomly chosen, being each γ equally likely to be chosen. Players in $C'_q \in \mathcal{C}_S$ are aware of the identity of proposer $\gamma\left(C'_q\right)$, but not of the proposers in other coalitions. The coalitions play sequentially (say, for example, in the order $\left(C'_1, C'_2, ..., C'_p\right)$) on the following way: Proposer $\gamma\left(C'_1\right)$ proposes a feasible payoff, *i.e.* a vector in $V(S)$. The members of $C'_1\backslash\gamma\left(C'_1\right)$ are then asked in some prespecified order. If one of them rejects the proposal, then we move to the next round where the set of active players is S with probability ρ and $S\backslash\gamma\left(C'_1\right)$ with probability $1-\rho$. In the latter case, player $\gamma\left(C'_1\right)$ gets a final payoff of 0. If all of them accept the proposal, the game moves to the next coalition C'_2. Then, players of C'_2, unaware of $\gamma\left(C'_1\right)$'s identity and his proposal, proceed to repeat the process under the same conditions, and so on. If all the proposals are accepted in each coalition, the proposers are called *representatives*. We denote by $a\left(S,\gamma\left(C'_q\right)\right) \in V(S)$ the proposal of $\gamma\left(C'_q\right)$.
>
> On the second stage, a proposal $a\left(S,\gamma\left(C'_q\right)\right)$ is randomly chosen, being each proposal equally likely to be chosen. We call $\gamma\left(C'_q\right)$ the *representative-proposer*, or simply *r.p.* If all the members of $S\backslash C'_q$ accept $a\left(S,\gamma\left(C'_q\right)\right)$ – they are asked in some prespecified order – then the game ends with these payoffs. If it is rejected by at least one member of $S\backslash C'_q$, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1-\rho$, the entire coalition C'_q *drops out* and the set of active players becomes $S\backslash C'_q$. In the latter case each member of the dropped coalition C'_q gets a final payoff of 0.

Clearly, given any set of strategies, this mechanism finishes in a finite number of rounds with probability 1.

Also note that the proposed payoff of $\gamma(C'_q)$ is independent of who are the proposers in other coalitions.

Remark 28 *The normalization given by property (A.4) does not affect our results, although the bargaining mechanism must be changed as follows: The player* $i \in N$ *who drops out, receives an amount* $x^i \in \mathbb{R}$ *such that* $x^i \in V(i)$*. This* x^i *can be considered as a "penalty payoff". Also, the monotonic property must be changed to* $V(T) \times \left(x^{S \setminus T}\right) \subset V(S)$ *for each* $T \subset S$*.*

The coalitional bargaining mechanism may be interpreted as the mechanism by Hart and Mas-Colell played on two stages, one of them by the coalitions and another by the players inside the same coalition. On the second stage, the coalitions play Hart and Mas-Colell's mechanism. This means that a coalition is randomly chosen to propose a payoff. The disagreement on this payoff by at least one of the other coalitions puts the whole proposing coalition in jeopardy. In order to decide the proposals, the members of each coalition play Hart and Mas-Colell's mechanism on a first stage. Thus, a player is randomly chosen inside each coalition and proposes a feasible payoff. Only if all the rest of the members of his coalition agree to this payoff, the proposal goes on to the second stage. Otherwise the proposer is in jeopardy. However, once the proposal is presented on the second stage, it is backed by the whole proposing coalition, so that its rejection may imply the whole coalition leaves the game.

In our study, as in Hart and Mas-Colell's, we consider stationary SPNE's. In this context, an SPNE is stationary if the players strategies depend only on the set S of active players. It does not depend, however, on the previous history nor the number of played rounds.

We also assume, as Hart and Mas-Colell, that players break ties in favor of quick termination of the game. We must note that this assumption is not needed in Hart and Mas-Colell's model. However, Example 42 shows that we cannot avoid it in our coalitional mechanism.

¿From now on, when we say SPNE, we mean stationary SPNE satisfying this tie-breaking rule.

Given a set of stationary strategies, let S denote the set of active players.

We denote by $a(S,i) \in V(S)$ the payoff proposed by $i \in C'_q \in \mathcal{C}_S$ when the set of proposers is determined by some $\gamma \in \Gamma_S$ with $\gamma(C'_q) = i$. We also define, for a given $\gamma \in \Gamma_S$:

$$a(S)_\gamma = \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} a\left(S, \gamma(C'_q)\right).$$

Since $V(S)$ is a convex set and each $a(S,\gamma(C'_q))$ belongs to $V(S)$, their average also belongs to $V(S)$. When all the proposals are accepted, $a(S)_\gamma$ is the expected final payoff when γ determines the set of proposers (or representatives).

Given $i \in C'_q \in \mathcal{C}_S$, let $\Gamma_{S,i}$ $(\Gamma_i = \Gamma_{N,i})$ be the subset of functions $\gamma \in \Gamma_S$ such that $\gamma(C'_q) = i$. Notice that $|\Gamma_S| = |\Gamma_{S,i}| \left|C'_q\right|$ for all $i \in C'_q \in \mathcal{C}_S$. Then,

$$a\left(S|_i\right) = \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a\left(S, \gamma\left(C'_r\right)\right) \in V\left(S\right)$$

is the expected final payoff when all the proposals are accepted and player i is the proposer (and representative) of his coalition.

We denote:

$$a(S) = \frac{1}{|\Gamma_S|} \sum_{\gamma \in \Gamma_S} a\left(S\right)_\gamma \in V(S)$$

as the expected final payoff when all the proposals are accepted. Given $C'_q \in \mathcal{C}_S$, it is straightforward to prove that $a(S)$ may also be expressed as:

$$a(S) = \frac{1}{\left|C'_q\right|} \sum_{i \in C'_q} a\left(S|_i\right).$$

It is also straightforward to prove:

$$a\left(S\right) = \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} \frac{1}{\left|C'_q\right|} \sum_{i \in C'_q} a\left(S, i\right). \tag{4.3}$$

Proposition 1 in Hart and Mas-Colell (1996) characterizes the proposals corresponding to an equilibrium by (1) $a\left(S,i\right) \in \partial V\left(S\right)$ and (2) $a\left(S,i\right)^j = \delta a\left(S\right)^j + \left(1-\rho\right) a\left(S \backslash i\right)^j$.

We now introduce some properties which generalize (1) and (2) in Hart and Mas-Colell (1996) to games with coalition structure.

We consider the following properties:

(C.1) $a(S,i) \in \partial V(S)$ for every $i \in N$;

(C.2) $a(S|_i)^j = \rho a(S)^j + (1-\rho) a(S \backslash i)^j$ for every $i, j \in C'_q \in \mathcal{C}_S$ with $j \neq i$;

(C.2') $a(S,i)^j = \rho a(S)^j + (1-\rho)\left[|\mathcal{C}_S|\, a(S\backslash i)^j - \sum_{C'_r \in \mathcal{C}_S \backslash C'_q} a(S \backslash C'_r)^j\right]$ for every $i, j \in C'_q \in \mathcal{C}_S$;

(C.3) $a(S,i)^j = \rho a(S)^j + (1-\rho) a(S \backslash C'_q)^j$ for every $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

Of course (C.1) coincides with Property (1) of Proposition 1 in Hart and Mas-Colell (1996). Property (2) is split into two properties: (C.2) or (C.2'), and (C.3) following usual practice in the literature on games with coalition structure.

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Proposition 29 *If (C.3) holds, then (C.2) is equivalent to (C.2').*

Proof. See the Appendix. ■

Proposition 30 *If (C.3) holds, then* $a(S)^j = a(S|_i)^j$ *for every* $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

Proof. See the Appendix. ■

Proposition 31 *Let* $(N, V, \mathcal{C})$ *be a hyperplane game with coalition structure. Assume a set of strategies* $\left(a(S,i)_{i \in S}\right)_{S \subset N}$ *satisfies (C.1), (C.2) and (C.3). Then,* $(a(S))_{S \subset N}$ *is the consistent coalitional value for the game* $(N, V, \mathcal{C})$.

Proof. See the Appendix. ■

By Proposition 29, Proposition 31 also holds if we replace (C.2) by (C.2').

However, in order to characterize the equilibria, properties (C.1), (C.2) and (C.3) are not enough in general. Thus, we impose an additional condition to the NTU games considered.

Given $\left(a(S,i)_{i \in S}\right)_{S \subset N}$ set of proposals, we define the vector $c(S,i) \in \mathbb{R}^S$ with $S \subset N$ and $i \in S$ as follows:

$$\left.\begin{aligned} c(S,i)^i &= -\sum_{C'_r \in \mathcal{C}_S \setminus C'_q} a(S \setminus C'_r)^i & & \\ c(S,i)^j &= |\mathcal{C}_S|\, a(S \setminus i)^j - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} a(S \setminus C'_r)^j & & \text{for all } j \in C'_q \setminus i \\ c(S,i)^j &= a\left(S \setminus C'_q\right)^j & & \text{for all } j \in S \setminus C'_q. \end{aligned}\right\} \tag{4.4}$$

We consider the following property:

(A.5) For any $\left(a(S,i)_{i \in S}\right)_{S \subset N}$ set of proposals satisfying (C.1), (C.2) and (C.3), and for every $S \subset N$, $i \in C'_q \in \mathcal{C}_S$, the vector $c(S,i)$ belongs to $V(S)$.

This property is not satisfied by general NTU games, as next example shows:

Example 32 *Let* $(N, V, \mathcal{C})$ *be such that* $N = \{1,2,3\}$, $\mathcal{C} = \{\{1,2\},\{3\}\}$ *and* V *be defined as follows,*

$V(i) = 0 - \mathbb{R}_+$, $i = 1,2,3$;
$V(\{1,2\}) = V(\{1,3\}) = (0,0) - \mathbb{R}^2_+$;
$V(\{2,3\}) = \left\{(x_2, x_3) : \frac{4}{5}x_2 + 2x_3 \leq 1, x_2 + x_3 \leq \frac{7}{8}\right\}$ *and*
$V(N) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1\}$.

It can be easily checked that this game is superadditive[1]. *Furthermore, if we take* $S = N$, $C_q = \{1,2\}$ *and* $i = 1$, *it can be checked that, given* $\rho \in \left(\frac{25-\sqrt{145}}{24}, 1\right)$ *and a set of proposals satisfying (C.1), (C.2) and (C.3), we have*

$$\left(-a(\{1,2\})^1, 2a(\{2,3\})^2 - a(\{1,2\})^2, a(\{3\})^3\right) = \left(0, \frac{70-55\rho}{12\rho^2 - 80\rho + 80}, 0\right) \notin V(N).$$

[1]It is not smooth, but we can make it smooth by a small modification which does not change our result.

Nevertheless, next proposition shows that several interesting subclasses of NTU games satisfy (A.5).

Proposition 33 *Property (A.5) is satisfied by the next class of games,*

- *zero-monotonic TU games;*
- *totally essential three-player hyperplane*[2] *games; and*
- *pure bargaining games.*

Proof. See the Appendix. ■

Proposition 34 *The proposals in any SPNE of an NTU game satisfying (A.5) are characterized by (C.1), (C.2) and (C.3). Moreover, all the proposals are accepted and* $a(S) \geq 0^S$ *for all* $S \subset N$*.*

Proof. See the Appendix. ■

Remark 35 *There is a subtle difference between the result given by Proposition 34 and Proposition 1 in Hart and Mas-Colell (1996). In Hart and Mas-Colell's model, the proposals* $a(S,i)$ *are nonnegative. In our model, the proposals do not need to be nonnegative, as it can be checked in Example 43. However, their (weighted) average* $a(S)$ *is always nonnegative in SPNE.*

Now, we see two important corollaries of Proposition 34.

Corollary 36 *Let* $(N, V, \mathcal{C})$ *be an NTU game with coalition structure satisfying (A.5). Then, a player's expected payoff in SPNE is independent of who is the proposer in other coalitions. Namely:*

$$a(S)^j = a(S|_i)^j \qquad \forall i \in C'_q \in \mathcal{C}_S; j \notin C'_q.$$

The proof of Corollary 36 is immediate from Proposition 30 and Proposition 34.

Hart and Mas-Colell say: "if ρ is close to 1– *i.e.*, the 'cost of delay' is low – then there is little dispersion among individual proposals: all the $a(N,i)$ constitute[3] small deviations of $a(N)$. This implies, first, that $a(N)$ *is almost Pareto optimal* (since the $a(N,i)$ are Pareto optimal). And second, that *there is no substantial advantage or disadvantage to being the proposer*; the 'first-mover' effect vanishes."

Next corollary states that the coalitional bargaining mechanism behaves in the same way.

Corollary 37 *There exists* $M \in \mathbb{R}$ *such that* $\left|a(N,i)^j - a(N)^j\right| < M(1-\rho)$ *for all* $i, j \in N$*.*

[2]It is enough that the game coincides with a hyperplane game in $V(S) \cap \mathbb{R}^S_+$ for all $S \subset N$.

[3]Hart and Mas-Colell denote $a(N,i)$ and $a(N)$ as $a_{N,i}$ and a_N, respectively.

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Proof. See the Appendix. ■

In next theorem we prove the existence of equilibria.

Theorem 38 *Let $(N, V, \mathcal{C})$ an NTU game with coalition structure satisfying (A.5). Then, for each $\rho \in [0, 1)$, there exists an SPNE.*

Proof. See the Appendix. ■

Next results characterize the SPNE payoffs.

Theorem 39 *Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure satisfying (A.5). Then, for each $\rho \in [0, 1)$, there exists a unique SPNE. Furthermore, the SPNE payoff configuration equals the unique consistent coalitional payoff configuration of $(N, V, \mathcal{C})$.*

Theorem 39 is an immediate consequence of Proposition 31, Proposition 34 and Theorem 38.

Corollary 40 *The coalitional mechanism, when applied to zero-monotonic TU games, implements the Owen value.*

Since the consistent coalitional value coincides with the Owen value in TU games with coalition structure, Corollary 40 is an immediate consequence of Proposition 33 and Theorem 39.

Notice that the coalitional bargaining mechanism implements the Shapley value for zero-monotonic games because the Shapley value coincides with the Owen value when the coalition structure is trivial.

Theorem 41 *Let $(N, V, \mathcal{C})$ be an NTU game with coalition structure satisfying (A.5). If $a_\rho := (a_\rho(S))_{S \subset N}$ is an SPNE payoff configuration for each ρ and a is the limit of a_ρ when $\rho \to 1$, then a is a consistent coalitional payoff configuration of $(N, V, \mathcal{C})$.*

Proof. See the Appendix. ■

If we do not assume the tie-breaking rule, the consistent coalitional value is still an SPNE payoff. However, there can be other SPNE's which do not yield the consistent coalitional value, as next example shows.

Example 42 *Consider $(N, v, \mathcal{C})$, where $N = \{1, 2, 3, 4\}$, $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$. Moreover, v is the characteristic function associated with the weighted majority game where the quota is 3 and the weights are 1, 1, 1, and 2 respectively. This means that $v(S) = 1$ if and only if S contains some of the following subsets: $\{1, 2, 3\}$, $\{1, 4\}$, $\{2, 4\}$, or $\{3, 4\}$.*

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It is straightforward to prove that

$$\begin{aligned}
\Phi_N &= \left(0,0,\frac{1}{2},\frac{1}{2}\right)\\
\Phi_{N\backslash 1} &= \left(-,0,\frac{1}{4},\frac{3}{4}\right)\\
\Phi_{N\backslash 2} &= \left(0,-,\frac{1}{4},\frac{3}{4}\right)\\
\Phi_{N\backslash 3} &= \left(\frac{1}{4},\frac{1}{4},-,\frac{1}{2}\right)\\
\Phi_{N\backslash 4} &= \left(\frac{1}{4},\frac{1}{4},\frac{1}{2},-\right).
\end{aligned}$$

We now define an SPNE whose payoff outcome is $\left(0,0,\frac{1}{4},\frac{3}{4}\right)$.

First, we describe the strategies of players 1 and 2. When one of them is chosen as proposer, his proposal is $a(N,\gamma(C_1)) = \left(0,0,\frac{1}{2},\frac{1}{2}\right)$. *Moreover, players* 1 *and* 2 *accept an offer if and only if it offers them something positive. In the subgame obtained after* $\gamma(C_1)$ *drops out of the game the strategy of player* j *coincides with the strategy with* $\Phi_{N\backslash\gamma(C_1)}$ *as payoff outcome. In the subgame obtained after* C_2 *drops out of the game the strategies of players* 1 *and* 2 *coincide with the strategies with* $\Phi_{N\backslash C_2}$ *as payoff outcome.*

We now describe the strategies of players 3 *and* 4. *In the subgame obtained after the offer of* $\gamma(C_1)$ *is accepted, the strategies of players* 3 *and* 4 *coincide with the strategies with* Φ_N *as payoff outcome. In the subgame obtained after* $\gamma(C_1)$ *drops out the game, the strategies of players* 3 *and* 4 *coincide with the strategies with* $\Phi_{N\backslash\gamma(C_1)}$ *as payoff outcome. In the subgame obtained after* C_1 *drops out the game, the strategies of players 3 and 4 coincide with the strategies with* $\Phi_{N\backslash C_1}$ *as payoff outcome.*

It is not difficult to check that these strategies are an SPNE.

According to these strategies, the offer of player $\gamma(C_1)$ *is rejected, which means that player* $\gamma(C_1)$ *obtains a final payoff of* 0. *Then players of* $N\setminus\gamma(C_1)$ *obtain* $\Phi_{N\backslash\gamma(C_1)}$ *as final payoff. This means that the final payoff induced by these strategies is* $\left(0,0,\frac{1}{4},\frac{3}{4}\right)$.

4.4 A modification in the coalitional mechanism

In this section we present a slight modification of the coalitional bargaining mechanism defined previously. The new mechanism is simpler. Unfortunately, when we restrict it to TU games with coalition structure, the payoffs of the SPNE's can be different from the Owen value.

We assume that a single proposer is chosen, and his proposal is voted first by the members of his own coalition and then by the members of the other coalitions.

Formally,

In each round there is a set $S \subset N$ of active players. At first round, $S = N$. First, a coalition C'_q out of $\mathcal{C}_S$ is randomly chosen, being each coalition equally likely to be chosen. Then, a *proposer* is randomly chosen out of C'_q, being each player equally likely to be chosen. We denote by q^* this proposer. Player q^* proposes a feasible payoff, *i.e.* a vector in $V(S)$. The members of $S \backslash q^*$ are then asked in some prespecified order, but beginning with the members of $C'_q \backslash q^*$. If one of the members of $C'_q \backslash q^*$ rejects the proposal, then we move to the next round where the set of active players is S with probability ρ and $S \backslash q^*$ with probability $1 - \rho$. In the latter case, player q^* gets a final payoff of 0. If the offer is accepted by all the members of $C'_q \backslash q^*$ and rejected by at least one member of $S \backslash C'_q$, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1 - \rho$, the entire coalition C'_q *drops out* and the set of active players becomes $S \backslash C'_q$. In the latter case each member of the dropped coalition C'_q gets a final payoff of 0. If all the members of $S \backslash q^*$ accept the proposal, then the game ends with these payoffs.

This mechanism also generalizes Hart and Mas-Colell's bargaining mechanism.

The main difference between the bargaining coalitional mechanism and this new mechanism is that, in the latter, when the players of a coalition accept the proposal of one of their members, they know that this proposal is due to be voted by the other coalitions. In the first mechanism, however, players only know this proposal would have a chance to be voted by the other coalitions.

This slight difference is not innocuous and affects in an important way to the behavior of agents, as we can see in the following example:

Example 43 *Consider* $(N, v, \mathcal{C})$, *where* $N = \{1, 2, 3\}$, $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1, 2\}$, $C_2 = \{3\}$. *Moreover, v is the characteristic function associated with the weighted majority game where the quota is 3 and the weights are 2, 1, and 1 respectively. This means that* $v(S) = 1$ *if and only if S contains* $\{1, 2\}$ *or* $\{1, 3\}$. *Otherwise,* $v(S) = 0$.

The Owen value for this game is $\left(\frac{3}{4}, \frac{1}{4}, 0\right)$.

Assume they play the bargaining coalitional mechanism with $\rho = 0$. *Player* 3 *would propose* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, *since this is the payoff players in* C_1 *would get in absence of him. Player* 2 *would propose* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *for a similar reason. Player* 1, *however, would propose* $\left(\frac{3}{2}, -\frac{1}{2}, 0\right)$, *and player* 2 *accepts! Notice that, by rejecting, player* 2 *gets* 0, *and by accepting, his final payoff is* $\frac{1}{2}$ *if the r.p. is player* 3, *and* $-\frac{1}{2}$ *if the r.p. is player* 1. *In expected terms, player* 2 *gets* 0.

The expected final payoff is the Owen value:

$$\frac{1}{4}\left(\frac{3}{2}, -\frac{1}{2}, 0\right) + \frac{1}{4}\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(\frac{3}{4}, \frac{1}{4}, 0\right).$$

Assume now they play the new mechanism. Again, both player 3 *and* 2 *are due to propose* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *should any of them be the proposer. In this case, however, player* 1 *cannot expect player* 2 *to accept a negative payoff. Player* 1 *proposes* $(1, 0, 0)$*, which seems more standard. However, the final expected payoff is* $\left(\frac{5}{8}, \frac{3}{8}, 0\right)$*, i.e. different from the Owen value.*

We proceed now to characterize the SPNE's in this new mechanism. To do so, for each $S \subset N$, we keep the notation $a(S, i)$ for the proposal made by player i if he is chosen as proposer.

Given a hyperplane game $(N, V, \mathcal{C})$, we inductively define the following solution concept. For all $i \in C_q \in \mathcal{C}$:

$$\chi^i_{\{i\}} = r^i.$$

Assume we know χ^j_S for all $S \subsetneq N$ and $j \in S$. Then,

$$\begin{aligned}
\chi^i_N = & \frac{1}{|\mathcal{C}| \, |C_q| \, \lambda^i_N} \sum_{C_r \in \mathcal{C} \setminus C_q} \left(|C_q| \, \lambda^i_N \chi^i_{N \setminus C_r} - \sum_{j \in C_r} \lambda^j_N \chi^j_{N \setminus C_q} \right) \\
& + \frac{1}{|\mathcal{C}| \, |C_q| \, \lambda^i_N} \left(\sum_{j \in C_q \setminus i} \lambda^i_N \chi^i_{N \setminus j} - \sum_{j \in C_q \setminus i} \lambda^j_N \chi^j_{N \setminus i} \right) \\
& + \frac{1}{|\mathcal{C}| \, |C_q| \, \lambda^i_N} v(N).
\end{aligned}$$

It is straightforward to prove that $\chi_N \in \partial V(N)$.

We can also generalize χ to any NTU game analogously to Ψ and Φ – *i.e.* by means of supporting hyperplanes.

Let $a(S)$ be defined as in (4.3). We consider the next property:

(C.4) $a(S, i)^j = \rho a(S)^j + (1 - \rho) a(S \setminus i)^j$ for every $i, j \in C'_q \in \mathcal{C}_S$ with $j \neq i$.

Proposition 44 *Let* $(N, V, \mathcal{C})$ *be a hyperplane game with coalition structure. Assume a set of strategies* $\left(a(S, i)_{i \in S}\right)_{S \subset N}$ *for the new mechanism satisfies (C.1), (C.3), and (C.4). Then,* $(a(S))_{S \subset N} = (\chi_S)_{S \subset N}$.

Proof. See the Appendix. ■

Notice the differences between the characterizations in both models. In both mechanisms, the proposals are Pareto efficient (property (C.1)) and satisfy (C.3). However, in the new mechanism, property (C.2) is replaced by (C.4). Now, the members of the proposer's coalition know that the proposal would also be proposed to the other coalitions should they accept it.

Given $\left(a(S, i)_{i \in S}\right)_{S \subset N}$ set of proposals, we define the vector $d(S, i) \in R^S$ with $S \subset N$ and $i \in S$ as follows:

$$\begin{aligned}
& d(S, i)^i = 0 \\
& d(S, i)^j = a(S \setminus i)^j \qquad \text{for all } j \in C'_q \setminus i \\
& d(S, i)^j = a\left(S \setminus C'_q\right)^j \quad \text{for all } j \in S \setminus C'_q
\end{aligned}$$

Again, we consider a new property:

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(A.6) For any $(a(S,i))_{S\subset N, i\in S}$ set of proposals satisfying (C.1), (C.3), and (C.4), and for every $S\subset N$, $i\in C_q^{'}\in\mathcal{C}_S$, the vector $d(S,i)$ belongs to $V(S)$.

The proofs of Proposition 45, Theorem 46, Theorem 47 and Theorem 48 are analogous to those of Proposition 34, Theorem 38, Theorem 39 and Theorem 41, respectively, and we omit them.

Proposition 45 *The proposals in any SPNE of the new mechanism of an NTU game satisfying (A.6) are characterized by (C.1), (C.3), and (C.4). Moreover, all the proposals are accepted and* $a(S)\geq 0^S$ *for all* $S\subset N$.

Theorem 46 *Let* $(N,V,\mathcal{C})$ *be an NTU game with coalition structure satisfying (A.6). Then, for each* $\rho\in[0,1)$, *there exists an SPNE.*

Theorem 47 *Let* $(N,V,\mathcal{C})$ *be a hyperplane game with coalition structure satisfying (A.6). Then, for each* $\rho\in[0,1)$, *there exists a unique SPNE. Furthermore, the SPNE payoff configuration equals* $(\chi_S)_{S\subset N}$.

Theorem 48 *Let* $(N,V,\mathcal{C})$ *be an NTU game with coalition structure satisfying (A.6). If* $a_\rho := (a_\rho(S))_{S\subset N}$ *is an SPNE payoff configuration for the new mechanism for each* ρ *and* a *is the limit of* a_ρ *when* $\rho\to 1$, *then* $a=(\chi_S)_{S\subset N}$.

We must note that, however χ does not generalize the Owen value for TU games, it does generalize the consistent value for NTU games with trivial coalition structure.

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4.5 Appendix

***Proof* of Proposition 29.** Fix $i,j\in C_q'\in\mathcal{C}_S$ with $j\neq i$. By definition of $a(S|_i)$:

$$\begin{aligned}
a(S|_i)^j &= \frac{1}{|\mathcal{C}_S|}\sum_{C_r'\in\mathcal{C}_S}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_r'))^j\\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C_r'\in\mathcal{C}_S\setminus C_q'}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_r'))^j+\frac{1}{|\mathcal{C}_S|}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_q'))^j\\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C_r'\in\mathcal{C}_S\setminus C_q'}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_r'))^j+\frac{1}{|\mathcal{C}_S|}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,i)^j\\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C_r'\in\mathcal{C}_S\setminus C_q'}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_r'))^j+\frac{1}{|\mathcal{C}_S|}a(S,i)^j.
\end{aligned}$$

So,

$$a(S,i)^j=|\mathcal{C}_S|\,a(S|_i)^j-\sum_{C_r'\in\mathcal{C}_S\setminus C_q'}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C_r'))^j. \tag{4.5}$$

We first prove that, under (C.3), (C.2) implies (C.2').

By (C.3), we know $a(S,\gamma(C'_r))^j = \rho a(S)^j + (1-\rho)a(S\backslash C'_r)^j$ for any $C'_r \neq C'_q$, so:

$$\begin{aligned} a(S,i)^j &= |\mathcal{C}_S|\, a(S|_i)^j - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q} \frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} \left[\rho a(S)^j + (1-\rho)a(S\backslash C'_r)^j\right] \\ &= |\mathcal{C}_S|\, a(S|_i)^j - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q} \left[\rho a(S)^j + (1-\rho)a(S\backslash C'_r)^j\right] \end{aligned}$$

by (C.2), we know $a(S|_i)^j = \rho a(S)^j + (1-\rho)a(S\backslash i)^j$, so:

$$\begin{aligned} &= |\mathcal{C}_S|\left[\rho a(S)^j + (1-\rho)a(S\backslash i)^j\right] - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left[\rho a(S)^j + (1-\rho)a(S\backslash C'_r)^j\right] \\ &= \rho a\,(S)^j + (1-\rho)\left[|\mathcal{C}_S|\, a(S\backslash i)^j - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S\backslash C'_r)^j\right]. \end{aligned}$$

Which is condition (C.2').

We now prove the reciprocal. By (4.5):

$$a(S|_i)^j = \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S,\gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|}a(S,i)^j$$

by (C.3) and (C.2'),

$$\begin{aligned} &= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}\left[\rho a(S)^j + (1-\rho)a(S\backslash C'_r)^j\right] \\ &\quad + \frac{1}{|\mathcal{C}_S|}\left\{\rho a(S)^j + (1-\rho)\left[|\mathcal{C}_S|\, a(S\backslash i)^j - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S\backslash C'_r)^j\right]\right\} \end{aligned}$$

$$\begin{aligned} &= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S)^j + (1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S\backslash C'_r)^j \\ &\quad + \rho\frac{1}{|\mathcal{C}_S|}a(S)^j + (1-\rho)\frac{1}{|\mathcal{C}_S|}\,|\mathcal{C}_S|\, a(S\backslash i)^j - (1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S\backslash C'_r)^j \end{aligned}$$

$$\begin{aligned} &= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S)^j + (1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S\backslash C'_r)^j \\ &\quad + \rho\frac{1}{|\mathcal{C}_S|}a(S)^j + (1-\rho)a(S\backslash i)^j - (1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S\backslash C'_r)^j \end{aligned}$$

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$$
\begin{aligned}
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q} a(S)^j + \rho\frac{1}{|\mathcal{C}_S|}a(S)^j + (1-\rho)a(S\backslash i)^j \\
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S} a(S)^j + (1-\rho)a(S\backslash i)^j \\
&= \rho a(S)^j + (1-\rho)a(S\backslash i)^j.
\end{aligned}
$$

Which is condition (C.2). ■

Proof of Proposition 30. We first prove that $a(S|_i)^j = a(S|_k)^j, \forall i,k \in C'_q \in \mathcal{C}_S; j \notin C'_q$:

$$
\begin{aligned}
a(S|_i)^j &= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S,\gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S,\gamma(C'_q))^j \\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}} a(S,\gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|}a(S,i)^j \\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,k}|}\sum_{\gamma\in\Gamma_{S,k}} a(S,\gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|}a(S,i)^j
\end{aligned}
$$

by (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho)a(S\backslash C'_q)^j = a(S,k)^j$, so:

$$
= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,k}|}\sum_{\gamma\in\Gamma_{S,k}} a(S,\gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|}a(S,k)^j = a(S|_k)^j.
$$

Now, we have:

$$
a(S)^j = \frac{1}{|C'_q|}\sum_{k\in C'_q} a(S|_k)^j = \frac{1}{|C'_q|}\sum_{k\in C'_q} a(S|_i)^j = a(S|_i)^j.
$$

■

Proof of Proposition 31. We proceed by induction. The case of one player is trivial. Assume the result is true for hyperplane games with less than n players. Assume $V(N) = \left\{x \in \mathbb{R}^N : \lambda \cdot x \leq v(N)\right\}$ for some $\lambda \in \mathbb{R}^N_{++}$.

By Theorem 25, it is enough to prove that $a(N)$ satisfies (B.1), (B.2) and (B.3).

We know that $a(N) = \frac{1}{|\mathcal{C}|}\sum_{C_q\in\mathcal{C}}\frac{1}{|C_q|}\sum_{i\in C_q} a(N,i)$. Moreover, $\lambda \cdot a(N,i) = v(N)$ for each $i \in N$ because $a(N,i) \in \partial V(N)$ by (C.1). Then, $\lambda \cdot a(N) = v(N)$ and hence $a(N)$ satisfies (B.1).

We now prove that $a(N)$ satisfies (B.2). For each $\gamma \in \Gamma$ with $\gamma(C_q) = i \in$

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$C_q \in \mathcal{C}$, $|\mathcal{C}| \sum_{j\in C_q} \lambda^j a\,(N)^j_\gamma =$

$$\begin{aligned}
&\sum_{j\in C_q}\left(\sum_{C_r\in\mathcal{C}}\lambda^j a(N,\gamma(C_r))^j\right)\\
&= \sum_{j\in C_q}\lambda^j a(N,i)^j + \sum_{j\in C_q}\left(\sum_{C_r\in\mathcal{C}\backslash C_q}\lambda^j a(N,\gamma(C_r))^j\right)\\
&= \lambda^i a(N,i)^i + \sum_{j\in C_q\backslash i}\lambda^j a(N,i)^j + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j a(N,\gamma(C_r))^j\right)
\end{aligned}$$

by (C.1), $\lambda\cdot a(N,i) = v\,(N)$ and then:

$$\begin{aligned}
&= v\,(N) - \sum_{j\in N\backslash i}\lambda^j a(N,i)^j + \sum_{j\in C_q\backslash i}\lambda^j a(N,i)^j + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j a(N,\gamma(C_r))^j\right)\\
&= v\,(N) - \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda^j a(N,i)^j\right) + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j a(N,\gamma(C_r))^j\right)
\end{aligned}$$

by (C.3),

$$\begin{aligned}
&= v\,(N) - \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda^j\left[\rho a(N)^j + (1-\rho)a(N\backslash C_q)^j\right]\right)\\
&\quad + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j\left[\rho a(N)^j + (1-\rho)a(N\backslash C_r)^j\right]\right).
\end{aligned}$$

This amount is independent of γ. Then:

$$\begin{aligned}
|\mathcal{C}|\sum_{j\in C_q}\lambda^j a(N)^j &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}|\mathcal{C}|\sum_{j\in C_q}\lambda^j a\,(N)^j_\gamma\\
&= v\,(N) - \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda^j\left[\rho a(N)^j + (1-\rho)a(N\backslash C_q)^j\right]\right)\\
&\quad + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j\left[\rho a(N)^j + (1-\rho)a(N\backslash C_r)^j\right]\right).
\end{aligned}$$

Since $a\,(N)$ satisfies (B.1):

$$v\,(N) = \sum_{j\in N}\lambda^j a(N)^j = \sum_{j\in C_q}\lambda^j a(N)^j + \sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda^j a(N)^j\right).$$

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Hence, $|\mathcal{C}| \sum_{j\in C_q} \lambda^j a\left(N\right)^j =$

$$\sum_{j\in C_q}\lambda^j a(N)^j + \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j a(N)^j\right)$$
$$-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j\left[\rho a(N)^j+(1-\rho)a(N\setminus C_q)^j\right]\right)$$
$$+\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j\left[\rho a(N)^j+(1-\rho)a(N\setminus C_r)^j\right]\right)$$

$$= \sum_{j\in C_q}\lambda^j a(N)^j + \sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j\left[(1-\rho)a(N)^j-(1-\rho)a(N\setminus C_q)^j\right]\right)$$
$$-\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j\left[(1-\rho-1)a(N)^j-(1-\rho)a(N\setminus C_r)^j\right]\right)$$

$$= \sum_{j\in C_q}\lambda^j a(N)^j + (1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j\left[a(N)^j-a(N\setminus C_q)^j\right]\right)$$
$$-(1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j\left[a(N)^j-a(N\setminus C_r)^j\right]\right)+\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j a(N)^j\right)$$

$$= (1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j\left[a(N)^j-a(N\setminus C_q)^j\right]\right)$$
$$-(1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j\left[a(N)^j-a(N\setminus C_r)^j\right]\right)+(|\mathcal{C}|-1)\sum_{j\in C_q}\lambda^j a\left(N\right)^j.$$

¿From where we get (dividing by $(1-\rho)$):

$$\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_r}\lambda^j\left[a(N)^j-a(N\setminus C_q)^j\right]\right)=\sum_{C_r\in\mathcal{C}\setminus C_q}\left(\sum_{j\in C_q}\lambda^j\left[a(N)^j-a(N\setminus C_r)^j\right]\right)$$

which is precisely property (B.2) when $S=N$ and $\lambda_N=\lambda$.

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We now prove that $a(N)$ satisfies (B.3). Given $i \in C_q$, we know that:

$$|C_q|\,\lambda^i a(N)^i = \sum_{j\in C_q} \lambda^i a(N|_j)^i = \lambda^i a(N|_i)^i + \sum_{j\in C_q\backslash i} \lambda^i a(N|_j)^i.$$

Since $a(N|_i) = \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}} \frac{1}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i} a(S,\gamma(C_r))$ and $a(N,\gamma(C_r)) \in \partial V(N)$ for each $\gamma \in \Gamma_i$ and $C_r \in \mathcal{C}$, we conclude that $\sum_{j\in N} \lambda^j a(N|_i)^j = v(N)$. Then,

$$|C_q|\,\lambda^i a(N)^i = v(N) - \sum_{j\in N\backslash C_q} \lambda^j a(N|_i)^j - \sum_{j\in C_q\backslash i} \lambda^j a(N|_i)^j + \sum_{j\in C_q\backslash i} \lambda^i a(N|_j)^i$$

we know that $a(N) \in \partial V(N)$ and hence,

$$\begin{aligned} &= \lambda^i a(N)^i + \sum_{j\in C_q\backslash i} \lambda^j a(N)^j + \underbrace{\sum_{j\in N\backslash C_q} \lambda^j a(N)^j} \\ &\quad - \underbrace{\sum_{j\in N\backslash C_q} \lambda^j a(N|_i)^j} - \sum_{j\in C_q\backslash i} \lambda^j a(N|_i)^j + \sum_{j\in C_q\backslash i} \lambda^i a(N|_j)^i \end{aligned}$$

the terms over the brackets are equal because $a(N)^j = a(N|_i)^j$ for all $j \in N\backslash C_q$ (Proposition 30)

$$= \lambda^i a(N)^i + \sum_{j\in C_q\backslash i} \lambda^j a(N)^j - \sum_{j\in C_q\backslash i} \lambda^j a(N|_i)^j + \sum_{j\in C_q\backslash i} \lambda^i a(N|_j)^i$$

by (C.2):

$$\begin{aligned} &= \lambda^i a(N)^i + \sum_{j\in C_q\backslash i} \lambda^j a(N)^j \\ &\quad - \sum_{j\in C_q\backslash i} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N\backslash i)^j\right] + \sum_{j\in C_q\backslash i} \lambda^i \left[\rho a(N)^i + (1-\rho) a(N\backslash j)^i\right] \end{aligned}$$

we add and subtract $\sum_{j\in C_q\backslash i} \lambda^i a(N)^i$ and gather terms to obtain:

$$\begin{aligned} &= \sum_{j\in C_q} \lambda^i a(N)^i + (1-\rho) \sum_{j\in C_q\backslash i} \lambda^j a(N)^j \\ &\quad -(1-\rho) \sum_{j\in C_q\backslash i} \lambda^j a(N\backslash i)^j - (1-\rho) \sum_{j\in C_q\backslash i} \lambda^i a(N)^i + (1-\rho) \sum_{j\in C_q\backslash i} \lambda^i a(N\backslash j)^i. \end{aligned}$$

This first term is $|C_q|\,\lambda^i a(N)^i$. So, the rest of terms must equal zero. Dividing by $(1-\rho)$:

$$\sum_{j\in C_q\backslash i} \lambda^j a(N)^j - \sum_{j\in C_q\backslash i} \lambda^j a(N\backslash i)^j - \sum_{j\in C_q\backslash i} \lambda^i a(N)^i + \sum_{j\in C_q\backslash i} \lambda^i a(N\backslash j)^i = 0.$$

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Or:

$$\sum_{j\in C_q\backslash i}\lambda^j\left[a(N)^j-a(N\backslash i)^j\right]=\sum_{j\in C_q\backslash i}\lambda^i\left[a(N)^i-a(N\backslash j)^i\right].$$

which is property (B.3) when $S=N$ and $\lambda_N=\lambda$. ■

Proof of Proposition 33. *Zero-monotonic TU games*: By Proposition 31, we know that $a(S)=\phi_S$ for all $S\subset N$.

By Proposition 3, given a triple $(N,v,\mathcal{C})$ such that (N,v) is a zero-monotonic TU game, $S\subset N$ and $i\in C_q'\in\mathcal{C}_S$, then

$$\sum_{j\in C_q'}\phi_S^j\geq\sum_{j\in C_q'\backslash i}\phi_{S\backslash i}^j+v(i).$$

By normalization, $v(i)\geq 0$ and thus

$$\sum_{j\in C_q'}\phi_S^j\geq\sum_{j\in C_q'\backslash i}\phi_{S\backslash i}^j. \tag{4.6}$$

Now, we have $\sum_{j\in S}c(S,i)^j=$

$$-\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\phi_{S\backslash C_r'}^i+\sum_{j\in C_q'\backslash i}\left(|\mathcal{C}_S|\,\phi_{S\backslash i}^j-\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\phi_{S\backslash C_r'}^j\right)+\sum_{j\in S\backslash C_q'}\phi_{S\backslash C_q'}^j$$

$$=\sum_{j\in C_q'\backslash i}|\mathcal{C}_S|\,\phi_{S\backslash i}^j+\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\left(\sum_{j\in C_r'}\phi_{S\backslash C_q'}^j-\sum_{j\in C_q'}\phi_{S\backslash C_r'}^j\right)$$

by (B.2), $\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\left(\sum_{j\in C_r'}\phi_{S\backslash C_q'}^j-\sum_{j\in C_q'}\phi_{S\backslash C_r'}^j\right)=\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\left(\sum_{j\in C_r'}\phi_S^j-\sum_{j\in C_q'}\phi_S^j\right)$ and thus,

$$\begin{aligned}
&=\sum_{j\in C_q'\backslash i}|\mathcal{C}_S|\,\phi_{S\backslash i}^j+\sum_{C_r'\in\mathcal{C}_S\backslash C_q'}\left(\sum_{j\in C_r'}\phi_S^j-\sum_{j\in C_q'}\phi_S^j\right)\\
&=\sum_{j\in C_q'\backslash i}|\mathcal{C}_S|\,\phi_{S\backslash i}^j+\sum_{j\in S\backslash C_q'}\phi_S^j-(|\mathcal{C}_S|-1)\sum_{j\in C_q'}\phi_S^j\\
&=|\mathcal{C}_S|\left(\sum_{j\in C_q'\backslash i}\phi_{S\backslash i}^j-\sum_{j\in C_q'}\phi_S^j\right)+\sum_{j\in S}\phi_S^j
\end{aligned}$$

by (4.6):

$$\leq\sum_{j\in S}\phi_S^j=v(S)$$

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which means that (A.5) holds for $(N, v, \mathcal{C})$.

Essential three-player hyperplane games: By Proposition 31, we know that $a(S) = \Phi_S$ for all $S \subset N$.

We consider $i = 1$ and the coalition structure $\mathcal{C} = \{\{1,2\}, \{3\}\}$. The other possibilities are equivalent or trivial.

Let $(\lambda_S)_{S \subset N}$ be the coefficients of a hyperplane game $(N, V, \mathcal{C})$. We want to prove that $-\lambda_N^1 \Phi_{12}^1 + 2\lambda_N^2 \Phi_{23}^2 - \lambda_N^2 \Phi_{12}^2 + \lambda_N^3 \Phi_3^3 \leq v(N)$.

By applying the inductive formula given by (4.2), we have,

$$\begin{aligned} -\lambda_N^1 \Phi_{12}^1 &= -\frac{\lambda_N^1 v(12)}{2\lambda_{12}^1} - \frac{\lambda_N^1 r^1}{2} + \frac{\lambda_{12}^2 \lambda_N^1 r^2}{2\lambda_{12}^1} \\ 2\lambda_N^2 \Phi_{23}^2 &= \frac{\lambda_N^2 v(23)}{\lambda_{23}^2} + \lambda_N^2 r^2 - \frac{\lambda_{23}^3 \lambda_N^2 r^3}{\lambda_{23}^2} \\ -\lambda_N^2 \Phi_{12}^2 &= -\frac{\lambda_N^2 v(12)}{2\lambda_{12}^2} - \frac{\lambda_N^2 r^2}{2} + \frac{\lambda_{12}^1 \lambda_N^2 r^1}{2\lambda_{12}^2} \\ \lambda_N^3 \Phi_3^3 &= \lambda_N^3 r^3. \end{aligned}$$

So, their sum is

$$\left.\begin{array}{l} -\dfrac{\lambda_N^1 v(12)}{2\lambda_{12}^1} - \dfrac{\lambda_N^1 r^1}{2} + \dfrac{\lambda_{12}^2 \lambda_N^1 r^2}{2\lambda_{12}^1} + \dfrac{\lambda_N^2 v(23)}{\lambda_{23}^2} + \lambda_N^2 r^2 \\ -\dfrac{\lambda_{23}^3 \lambda_N^2 r^3}{\lambda_{23}^2} - \dfrac{\lambda_N^2 v(12)}{2\lambda_{12}^2} - \dfrac{\lambda_N^2 r^2}{2} + \dfrac{\lambda_{12}^1 \lambda_N^2 r^1}{2\lambda_{12}^2} + \lambda_N^3 r^3 \end{array}\right\} \quad (4.7)$$

Let $\left(\frac{v(23) - \lambda_{23}^3 r^3}{\lambda_{23}^2}, r^3\right) \in V(23)$. By monotonicity,

$$\left(0, \frac{v(23) - \lambda_{23}^3 r^3}{\lambda_{23}^2}, r^3\right) \in V(N).$$

Thus,

$$\frac{\lambda_N^2 v(23)}{\lambda_{23}^2} - \frac{\lambda_N^2 \lambda_{23}^3 r^3}{\lambda_{23}^2} + \lambda_N^3 r^3 \leq v(N).$$

So, the amount given in (4.7) is not larger than

$$\left.\begin{array}{l} -\dfrac{\lambda_N^1 v(12)}{2\lambda_{12}^1} - \dfrac{\lambda_N^1 r^1}{2} + \dfrac{\lambda_{12}^2 \lambda_N^1 r^2}{2\lambda_{12}^1} + \lambda_N^2 r^2 \\ -\dfrac{\lambda_N^2 v(12)}{2\lambda_{12}^2} - \dfrac{\lambda_N^2 r^2}{2} + \dfrac{\lambda_{12}^1 \lambda_N^2 r^1}{2\lambda_{12}^2} + v(N). \end{array}\right\} \quad (4.8)$$

By essentiality, $(r^1, r^2) \in V(12)$. So, $\lambda_{12}^1 r^1 + \lambda_{12}^2 r^2 \leq v(12)$. Thus, the expression given by (4.8) is not more than

$$-\lambda_N^1 r^1 + v(N) \leq v(N).$$

Pure bargaining games: We first prove by induction that, for $S \subsetneq N$, $a(S) = r^S$. By (C.1), the result is trivial for $n = 1$. Assume that $a(T) = r^T$ for all $T \subsetneq S$. Then, given $i \in C'_q \in \mathcal{C}_S$:

By (C.2'), $a(S,i)^j = \rho a(S)^j + (1-\rho)\left(|\mathcal{C}_S| r^j - \sum_{C_r \in \mathcal{C}_S \backslash C_q} r^j\right) = \rho a(S)^j + (1-\rho) r^j$ for all $j \in C'_q \backslash i$.

By (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho) r^j$ for all $j \in S \backslash C'_q$.

Thus, $a(S,i)$ coincide with $\rho a(S) + (1-\rho) r$ in all coordinates but (at most) the ith. Moreover, both $a(S)$ and r belong to $V(S)$, and so $\rho a(S) + (1-\rho) r$ does. Thus, by (C.1), $a(S,i)^i \geq \rho a(S)^i + (1-\rho) r^i$. By averaging over i, we have $a(S)^i \geq \rho a(S)^i + (1-\rho) r^i$ and thus $a(S)^i \geq r^i$. We have then $a(S) \geq r^S$. Since $r^S \in \partial V(S)$ and $a(S) \in V(S)$, we conclude that $a(S) = r^S$.

Now, we have:

$$\begin{aligned}
c(S,i) &= \left(-\sum_{C_r \in \mathcal{C}_S \backslash C_q} r^i, \left(|\mathcal{C}_S| r^j - \sum_{C_r \in \mathcal{C}_S \backslash C_q} r^j\right)_{j \in C'_q \backslash i}, \left(r^j\right)_{j \in S \backslash C'_q}\right) \\
&= \left(-\left(|\mathcal{C}|-1\right) r^i, \left(r^j\right)_{j \in C'_q \backslash i}, \left(r^j\right)_{j \in S \backslash C'_q}\right) \\
&= \left(-\left(|\mathcal{C}|-1\right) r^i, r^{S \backslash i}\right).
\end{aligned}$$

By (A.4), $r^i \geq 0$ and thus $c(S,i) \leq \left(0, r^{S \backslash i}\right)$. By (A.3), $\left(0, r^{S \backslash i}\right) \in V(S)$. By comprehensiveness, $c(S,i) \in V(S)$. ■

Proof of Proposition 34. We proceed by induction. The result holds trivially when $n = 1$. Assume that it is true when there are at most $n-1$ players.

Assume we are in an SPNE. By induction hypothesis, the expected payoff for the players in $S \subsetneq N$ in any SPNE with S as set of active players is $a(S)$. Let $b_N \in V(N)$ be the expected payoff when N is the set of active players. We must prove that (C.1), (C.2), and (C.3) hold for $S = N$.

We proceed by a series of Claims:

Claim (A): Given $C_q \in \mathcal{C}$ on the second stage, assume the proposers are determined by $\gamma \in \Gamma$ and the r.p. is $\gamma(C_q)$. Then, all players in $N \backslash C_q$ accept $\gamma(C_q)$'s proposal if $a(N, \gamma(C_q))^i \geq \rho b_N^i + (1-\rho) a(N \backslash C_q)^i$ for every $i \in N \backslash C_q$. Otherwise, the proposal is rejected.

Notice that, in the case of rejection on the second stage, the expected payoff of a player $i \in N \backslash C_q$ is, by induction hypothesis, $\rho b_N^i + (1-\rho) a(N \backslash C_q)^i$.

Suppose we reach the second stage. We assume without loss of generality that $C_q = \{1, 2, ..., c_q\}$ and $(c_q + 1, ..., n)$ is the order in which the players in $N \backslash C_q$ are asked.

If the game reaches player n, *i.e.*, there has been no previous rejection, his optimal strategy involves accepting the proposal if $a(N, \gamma(C_q))^n$ is equal (by the tie-breaking rule) or higher than $\rho b_N^n + (1-\rho) a(N \backslash C_q)^n$ and rejecting it if it is lower than $\rho b_N^n + (1-\rho)a(N\backslash C_q)^n$. Player $n-1 \in N\backslash C_q$, anticipates reaction of player n. Hence, if $a(N,\gamma(C_q))^n \geq \rho b_N^n + (1-\rho)a(N\backslash C_q)^n$ and $a(N,\gamma(C_q))^{n-1} \geq \rho b_N^{n-1} + (1-\rho)a(N\backslash C_q)^{n-1}$, and the game reaches player $n-1$, he accepts the proposal. If $a(N,1)^n < \rho b_N^n + (1-\rho)a(N\backslash C_q)^n$, player $n-1$ is indifferent between accepting or rejecting the proposal, since he knows player n is bound to reject the proposal should the game reach him. In any case, the proposal is rejected. By going backwards, we prove the result for all players in $N\backslash C_q$ on the second stage.

Claim (B): Let $\gamma \in \Gamma$ be the correspondence which determines the set of proposers on the first stage. Given any $C_q \in \mathcal{C}$, assume all the coalitions which choose representative after C_q are bound to choose their proposer as representative should the game reach them. Given $i \in C_q$, let $b_{N,i}$ be the expected final payoff in SPNE restricted to i be a representative. Then, all players in $C_q\backslash\gamma(C_q)$ accept $\gamma(C_q)$'s proposal if $b_{N,\gamma(C_q)}^j \geq \rho b_N^j + (1-\rho)\,a\,(N\backslash\gamma\,(C_q))^j$ for every $j \in C_q\backslash\gamma(C_q)$. Otherwise, the proposal is rejected.

Notice that, under our hypothesis, in the case of rejection of $\gamma(C_q)$'s proposal on the first stage, the expected payoff to a player $j \in C_q\backslash\gamma(C_q)$ is $\rho b_N^j + (1-\rho)a(N\backslash\gamma(C_q))^j$.

We assume without loss of generality that $C_q = \{1,...,c_q\}$, $\gamma(C_q) = 1$ and players in $C_q\backslash 1$ are asked in the order $(2,...,c_q)$.

If the game reaches player c_q, *i.e.*, there has been no previous rejection, his optimal strategy involves accepting any proposal of player 1 satisfying $b_{N,1}^{c_q} \geq \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$ and rejecting any proposal such that $b_{N,1}^{c_q} < \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$. Player $c_q - 1 \in C_q$, anticipates reaction of player c_q. Hence, if $b_{N,1}^{c_q} \geq \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$ and $b_{N,1}^{c_q-1} \geq \rho b_N^{c_q-1} + (1-\rho)a(N\backslash 1)^{c_q-1}$, and the game reaches player $c_q - 1$, he accepts the proposal. If $b_{N,1}^{c_q} < \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$, player $c_q - 1$ is indifferent between accepting or rejecting the proposal, since he knows player c_q is bound to reject the proposal should the game reach him. In any case, the proposal is rejected. By going backwards, we prove the result for all players in $C_q\backslash 1$ on the first stage.

Claim (C): All the offers on the first stage are accepted.

Assume coalitions play the first stage in the order $(C_1, C_2, ..., C_p)$ and that the mechanism reaches coalition C_p; *i.e.* there has been no previous rejection. Assume the proposal of $\gamma(C_p)$ is rejected. This means the final payoff for player $\gamma(C_p)$ is $\rho b_N^{\gamma(C_p)}$.

We can assume without loss of generality that $C_p = \{1,2,...,c_p\}$, $\gamma(C_p) = 1$ and players are asked in the order $(2,...,c_p)$.

We define a new proposal $\widehat{a}(N,1)$ for player 1 as follows. Let $c(N,1)$ be defined as in (4.4). By (A.5) and induction hypothesis, $c(N,1) \in V(N)$. By convexity, $\rho b_N + (1-\rho)\,c(N,1) \in V(N)$. Let $\widehat{a}(N,1) = \rho b_N + (1-\rho)\,c(N,1)$.

Assume the mechanism reaches c_p; *i.e.* has not been previous rejection. Then, by rejecting $\widehat{a}(N,1)$, the expected final payoff for player c_p is $\rho b_N^{c_p} + (1-\rho)\,\widehat{a}\,(N\backslash 1)^{c_p}$.

If c_p accepts $\widehat{a}(N,1)$ and the chosen proposal in the second stage is from $C_r \neq C_q$, then c_p can obtain $\rho b_N^{c_p} + (1-\rho)\, a(N \backslash C_r)^{c_p}$ by rejecting it. If the proposal chosen in the second stage is from C_q, then it is accepted (by *Claim (A)*).

Thus, if c_p accepts $\widehat{a}(N,1)$, his expected final payoff is at least

$$\begin{aligned} & \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}\backslash C_p}\left[\rho b_N^{c_p} + (1-\rho)\, a(N\backslash C_r)^{c_p}\right] + \frac{1}{|\mathcal{C}|}\widehat{a}(N,1)^{c_p} \\ = & \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}\backslash C_p}\rho b_N^{c_p} + \frac{1}{|\mathcal{C}|}(1-\rho)\sum_{C_r \in \mathcal{C}\backslash C_p} a(N\backslash C_r)^{c_p} \\ & + \frac{1}{|\mathcal{C}|}\rho b_N^{c_p} + (1-\rho)\left(a(N\backslash 1)^{c_p} - \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}\backslash C_p} a(N\backslash C_r)^{c_p}\right) \\ = & \rho b_N^{c_p} + (1-\rho)\, a(N\backslash 1)^{c_p}. \end{aligned}$$

Thus, by the tie-breaking rule, it is optimal for c_p to accept $\widehat{a}(N,1)$. By going backwards, we can prove that it is optimal for $c_p - 1, c_p - 2, ..., 2$ to accept $\widehat{a}(N,1)$. Furthermore, the expected final payoff for player 1 is not less than

$$\begin{aligned} & \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}\backslash C_p}\left[\rho b_N^{1} + (1-\rho)\, a(N\backslash C_r)^{1}\right] + \frac{1}{|\mathcal{C}|}\widehat{a}(N,1)^{1} \\ = & \frac{1}{|\mathcal{C}|}\sum_{C_r \in \mathcal{C}\backslash C_p}\left[\rho b_N^{1} + (1-\rho)\, a(N\backslash C_r)^{1}\right] \\ & + \frac{1}{|\mathcal{C}|}\left[\rho b_N^{1} - (1-\rho)\sum_{C_r \in \mathcal{C}\backslash C_p} a(N\backslash C_r)^{1}\right] \\ = & \rho b_N^{1}. \end{aligned}$$

So, by the tie-breaking rule, it is optimal for player 1 to change his proposal to $\widehat{a}(N,1)$. This contradiction proves that there are not proposals rejected on the first stage in C_p. By going backwards, we prove that no proposal is rejected on the first stage in $C_{p-1}, ..., C_1$.

Claim (D): All the offers on the second stage are accepted.

Suppose the proposal of $\gamma(C_q)$ is rejected on the second stage. Then, the final payoff for the members of C_q (including $\gamma(C_q)$) is 0 with probability $\frac{1}{|\mathcal{C}|} > 0$. By *Claim (B)*, we know that $b^j_{N,\gamma(C_q)} \geq \rho b_N^j + (1-\rho)\, a(N\backslash\gamma(C_q))^j$ for all $j \in C_q \backslash \gamma(C_q)$. Assume that $\gamma(C_q)$ changes his strategy and proposes

$$\widehat{a}(N,\gamma(C_q)) = \left(0^{C_q}, \rho b_N^{N\backslash C_q} + (1-\rho)\, a(N\backslash C_q)^{N\backslash C_q}\right).$$

By convexity and monotonicity, $\widehat{a}(N,\gamma(C_q)) \in V(N)$. By *Claim (A)*, this proposal is bound to be accepted should $\gamma(C_q)$ be the r.p. on the second

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stage. However, $b_{N,\gamma(C_q)}$ remains unaltered. So, by *Claim (B)*, $\widehat{a}(N,\gamma(C_q))$ is also accepted on the first stage. Moreover, the expected final payoff for $\gamma(C_q)$ also remains the same. By the tie-breaking rule, we are not in an SPNE. This contradiction proves that the proposals on the second stage are always accepted.

Since all the proposals are accepted, we can assure that $b_N = a(N)$ and $b_{N,i} = a(N|_i)$ for all $i \in N$.

We show now (C.1), (C.2), and (C.3) hold.

Suppose (C.1) does not hold, *i.e.*, there exists a player $i \in C_q$ such that $a(N,i)$ is not Pareto optimal. Thus, $a(N,i)$ belongs to the interior of $V(N)$; so, there exists $\varepsilon > 0$ such that $\widehat{a}(N,i) := a(N,i) + \left(\varepsilon, 0^{N\setminus i}\right) \in V(N)$.

Notice that, since the proposal $a(N,i)$ of player i is accepted, by *Claim (B)* , together with *Claim (C)* and *Claim (D),* we know that $a(N|_i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus i)^j$ for every $j \in C_q\setminus i$ and, by *Claim (A)*, $a(N,i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus C_q)^j$ for every $j \in N\setminus C_q$. So $\widehat{a}(N,i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus i)^j$ for every $j \in C_q\setminus i$ and $\widehat{a}(N,i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus C_q)^j$ for every $j \in N\setminus C_q$. By *Claim (A)* and *Claim (B)*, if player i changes his proposal to $\widehat{a}(N,i)$, it is bound to be accepted and his expected final payoff improves by $\dfrac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.1).

Suppose (C.2) does not hold. Let $j_0 \in C_q\setminus i$ be a player such that $a(N|_i)^{j_0} = \rho a(N)^{j_0} + (1-\rho)a(N\setminus i)^{j_0} + \alpha$ with $\alpha \neq 0$. By *Claim* (B), $\alpha > 0$.

By comprehensiveness and nonlevelness, we have $a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\setminus j_0}\right)$ belongs to the interior of $V(N)$. Thus, there exists an $\varepsilon > 0$ such that $\widehat{a}(N,i) := a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\setminus j_0}\right) + \left(\varepsilon, 0^{N\setminus i}\right)$ belongs to $V(N)$. Suppose player i changes his proposal to $\widehat{a}(N,i)$. The new value $\widehat{a}(N|_i)$ satisfies:

$$\begin{aligned}
\widehat{a}(N|_i)^i &= a(N|_i)^i + \frac{\varepsilon}{|\mathcal{C}|};\\
\widehat{a}(N|_i)^{j_0} &= a(N|_i)^{j_0} - \alpha = \rho b_N^{j_0} + (1-\rho)a(N\setminus i)^{j_0};\\
\widehat{a}(N|_i)^j &= a(N|_i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus i)^j \text{ for all } j \in C_q\setminus\{i, j_0\};\\
\widehat{a}(N,i)^j &= a(N,i)^j \geq \rho b_N^j + (1-\rho)a(N\setminus C_q)^j \text{ for all } j \in N\setminus C_q.
\end{aligned}$$

So, by *Claim (A)* and *Claim (B)*, the new proposal of player i is due to be accepted. Also, player i improves his expected payoff by $\dfrac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.2).

The reasoning for (C.3) is similar to (C.2) and we omit it.

It remains to show that $a(N) \geq 0$. Notice that player $i \in N$ can guarantee a payoff of at least 0 by proposing always 0^N and accepting only proposals which give him a nonnegative expected payoff. Thus, $a(N) \geq 0$.

Conversely, we show that proposals $(a(S,i)_{i\in S})_{S\subset N}$ satisfying (C.1), (C.2) and (C.3) can be supported as an SPNE.

First, we prove that $a(S) \geq 0$ for all $S \subset N$. By induction hypothesis, this is true for any $S \subsetneq N$. Given $i \in C_q \in \mathcal{C}$, by (A.5), we have $c(N,i) \in V(N)$. By convexity, $\widetilde{c}(N,i) := \rho a(N) + (1-\rho)c(N,i) \in V(N)$.

Since $a(N,i)$ satisfies (C.2) and (C.3), by Proposition 29, $a(N,i)$ also satisfies (C.2'). Then, $a(N,i)^{N\backslash i} = \widetilde{c}(N,i)^{N\backslash i}$. We now conclude that $a(N,i) \geq \widetilde{c}(N,i)$ because $a(N,i) \in \partial V(N)$ and $\widetilde{c}(N,i) \in V(N)$. Hence,

$$a(N,i)^i \geq \widetilde{c}(N,i)^i = \rho a(N)^i - (1-\rho) \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^i .$$

So,

$$\begin{aligned} a(N|_i)^i &= \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}} \frac{1}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i} a(N,\gamma(C_r))^i \\ &= \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_q} \frac{1}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i} a(N,\gamma(C_r))^i + \frac{1}{|\mathcal{C}|}a(N,i)^i \end{aligned}$$

by (C.3),

$$\begin{aligned} &= \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_q}\left[\rho a(N)^i + (1-\rho)a(N\backslash C_r)^i\right] + \frac{1}{|\mathcal{C}|}a(N,i)^i \\ &\geq \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_q}\left[\rho a(N)^i + (1-\rho)a(N\backslash C_r)^i\right] \\ &\quad +\frac{1}{|\mathcal{C}|}\left[\rho a(N)^i - (1-\rho)\sum_{C_r\in\mathcal{C}\backslash C_q} a(N\backslash C_r)^i\right] \\ &= \rho a(N)^i . \end{aligned}$$

Furthermore, by (C.2) and $a(N\backslash j) \geq 0$, we have $a(N|_j)^i \geq \rho a(N)^i$ for all $j \in C_q\backslash i$. Thus,

$$a(N)^i = \frac{1}{|C_q|}\sum_{j\in C_q} a(N|_j)^i \geq \frac{1}{|C_q|}\sum_{j\in C_q}\rho a(N)^i = \rho a(N)^i$$

and so $a(N)^i \geq 0$. Moreover,

$$a(N|_i)^i \geq \rho a(N)^i \geq 0.$$

We now follow the same reasoning of Hart and Mas-Colell to verify that the strategies corresponding to these proposals form an SPNE. By the induction hypothesis, this is so in any subgame after a player (or coalition) has dropped out. Fix a player $i \in C_q$. If he rejects the proposal from a proposer $j \in C_q\backslash i$, his expected final payoff is $\rho a(N)^i + (1-\rho)a(N\backslash j)^i$. If he rejects the proposal from a r.p. $j \in C_r \neq C_q$, his expected final payoff is $\rho a(N)^i + (1-\rho)a(N\backslash C_r)^i$.

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In any case, his expected final payoff is the same as that the other player is offering. Since the rest of the players accept the proposal, he does not improve his expected final payoff by rejecting it. If the proposer is player i himself, the strategies of the other players do not allow him to decrease his proposal to any of them (since it would be rejected by *Claim (A)* and *Claim* (B)). Moreover, increasing one or more of his offers to the other players keeping the rest unaltered implies his own payment decreases (by (C.1) and nonlevelness). Finally, by offering an unacceptable proposal, he may be removed and his expected final payoff becomes 0, which does not improve his final payoff (because $a\left(N|_i\right)^i \geq 0$). Thus, the proposals do form an SPNE. ■

Proof of Corollary 37. Fix $i \in C_q \in \mathcal{C}$. Given $j \in N \backslash C_q$, by (C.3):

$$\begin{aligned} \left|a(N,i)^j - a(N)^j\right| &= \left|\rho a(N)^j + (1-\rho) a(N\backslash C_q)^j - a(N)^j\right| \\ &= (1-\rho)\left|a(N\backslash C_q)^j - a(N)^j\right|. \end{aligned}$$

So, we take $M_1 \in \mathbb{R}$ as the maximum of the set:

$$\left\{\left|a(N\backslash C_q)^j - a(N)^j\right| : C_q \in \mathcal{C}, j \in N\backslash C_q\right\}.$$

This maximum exists because $a\left(S\right) \geq 0$ for all $S \subset N$.[4]
We have then $\left|a(N,i)^j - a(N)^j\right| \leq M_1(1-\rho)$ for all $j \in N\backslash C_q$.
Given $j \in C_q \backslash i$, by (C.2'), $\left|a(N,i)^j - a(N)^j\right| =$

$$\begin{aligned} &\left|\rho a(N)^j + (1-\rho)\left[|\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j\right] - a(N)^j\right| \\ =\ & (1-\rho)\left|\,|\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j - a(N)^j\right| \end{aligned}$$

Thus, we take $M_2 \in \mathbb{R}$ as the maximum of the set:

$$\left\{\left|\,|\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j - a(N)^j\right| : i, j \in C_q, j \neq i\right\}$$

This maximum exists because $a\left(S\right) \geq 0$ for all $S \subset N$.
We have then $\left|a(N,i)^j - a(N)^j\right| \leq M_2(1-\rho)$.
We now study $\left|a(N,i)^i - a(N)^i\right|$. We know:

$$a(N)^i = \sum_{C_r \in \mathcal{C}} \left[\sum_{j \in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} a(N,j)^i\right].$$

[4] This set may be no finite, because we are considering the proposals for any ρ.

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Then,

$$a(N,i)^i = |\mathcal{C}|\,|C_q| \left[a(N)^i - \sum_{C_r \in \mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} a(N,j)^i \right) - \sum_{j \in C_q \setminus i} \frac{1}{|\mathcal{C}|\,|C_q|} a(N,j)^i \right].$$

So, $\left|a(N,i)^i - a(N)^i\right| =$

$$\begin{aligned}
& |\mathcal{C}|\,|C_q| \left| a(N)^i - \sum_{C_r \in \mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} a(N,j)^i \right) - \sum_{j \in C_q \setminus i} \frac{1}{|\mathcal{C}|\,|C_q|} a\,(N,j)^i - \frac{1}{|\mathcal{C}|\,|C_q|} a(N)^i \right| \\
= \; & |\mathcal{C}|\,|C_q| \left| \sum_{C_r \in \mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} \left[a(N)^i - a(N,j)^i\right] \right) - \sum_{j \in C_q \setminus i} \frac{1}{|\mathcal{C}|\,|C_q|} \left[a(N)^i - a(N,j)^i\right] \right| \\
\le \; & \sum_{C_r \in \mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \frac{|C_q|}{|C_r|} \left|a(N)^i - a(N,j)^i\right| \right) + \sum_{j \in C_q \setminus i} \left|a(N)^i - a(N,j)^i\right| \\
\le \; & \sum_{C_r \in \mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \frac{|C_q|}{|C_r|} M_1 \right) + \sum_{j \in C_q \setminus i} M_2 = |C_q|\,(|\mathcal{C}|-1)\,M_1 + (|C_q|-1)\,M_2.
\end{aligned}$$

So, we take $M = \max\{|C_q|\,(|\mathcal{C}|-1)M_1 + (|C_q|-1)M_2 : C_q \in \mathcal{C}\}$. ■

Proof of Theorem 38. By Proposition 34, we only need to prove that there exist proposals satisfying (C.1), (C.2), and (C.3). We proceed by induction on the number of players. Clearly, the result is true for $n = 1$. Assume now that we have $a\,(S,i)$ for each $S \subset N$ and each $i \in S$ satisfying (C.1), (C.2), and (C.3) when $S \subsetneq N$. By Proposition 34, $a\,(S) \geq 0$ for all $S \subsetneq N$.

For each $i \in C_q \in \mathcal{C}$, by property (A.5) the vector $c\,(N,i)$ belongs to $V\,(N)$. By (A.2), there exists an unique $z_i \in \partial V(N)$ such that $z_i^j = c\,(N,i)^j$ for all $j \in N\setminus i$. We define:

$$\begin{aligned}
\beta_1 &= \min\left\{a^i\,(S) : i \in S \subsetneq N\right\} \in \mathbb{R}, \\
\beta_2 &= \min\left\{z_j^i : i,j \in N\right\} \in \mathbb{R}, \\
\beta &= \min\,(\beta_1,\beta_2) \in \mathbb{R}, \\
K &= \{x \in V(N) : x \geq (\beta,...,\beta)\}\,.
\end{aligned}$$

This set K is nonempty ($z_i \in K$ for all $i \in N$), closed (because $V(N)$ is closed) and bounded (by (A.1)). Thus, K is a compact set. Furthermore, K is convex (because $V(N)$ is convex).

We define n functions $\alpha_i : K \to K$ as follows. Given $i \in C_q \in \mathcal{C}$, $\alpha_i^j(x) := \rho x^j + (1-\rho)c\,(N,i)^j$ for each $j \in N\setminus i$ and $\alpha_i^i\,(x)$ is defined in such a way that $\alpha_i\,(x) \in \partial V\,(N)$.

These functions are well defined, because $y_i := \rho x + (1-\rho)z_i$ belongs to K (by convexity) and $\alpha_i(x)$ equals y_i in all coordinates but i's, which we increase until reaching the boundary of $V(N)$.

Also, because of the smoothness of property (A.2) the functions α_i are continuous. By the convexity of the domain, $\frac{1}{|\mathcal{C}|}\sum_{C_q\in\mathcal{C}}\frac{1}{|C_q|}\sum_{i\in C_q}\alpha_i(x)\in K$ for each $x\in K$. By a standard fix point theorem, there exists a vector $a(N)\in K$ satisfying $a(N)=\frac{1}{|\mathcal{C}|}\sum_{C_q\in\mathcal{C}}\frac{1}{|C_q|}\sum_{i\in C_q}\alpha_i(a(N))$.

We define $a(N,i)=\alpha_i(a(N))$ for each $i\in N$. It is trivial to see that $(a(N,i))_{i\in N}$ satisfies (C.1), (C.2'), and (C.3). By Proposition 29, $(a(N,i))_{i\in N}$ also satisfies (C.2). ■

Proof of Theorem 41. By Theorem 25, it is enough to prove that $a=(a(S))_{S\subset N}$ satisfies (B.1), (B.2), and (B.3). By Corollary 37, $a_\rho(S,i)\to a(S)$ for any $i\in S\subset N$. Since $a_\rho(S,i)\in\partial V(S)$ for every $S\subset N$ and every $\rho\in[0,1)$, and $\partial V(S)$ is closed, we conclude that $a(S)\in\partial V(S)$ for every $S\subset N$. Thus, a satisfies property (B.1) of the characterization of the consistent coalitional payoff configuration.

Let λ_S be the unit length normal to $\partial V(S)$ at $a(S)$ for each $S\subset N$. We associate to each ρ a hyperplane game with coalition structure $(N,V_\rho,\mathcal{C})$ as follows:

Given $\rho\in[0,1)$ and $S\subset N$ with $|S|$ elements, there exists at least one hyperplane on $\mathbb{R}^S$ containing the $|S|$ points $\{a_\rho(S,i):i\in S\}$. If there are more than one hyperplane, we take the one whose unit length outward orthogonal vector $\lambda_S(\rho)$ is closest to λ_S.

We define

$$V_\rho(S):=\left\{x\in\mathbb{R}^S:\lambda_S(\rho)\cdot x\le\lambda_S(\rho)\cdot a_\rho(S,i),i\in S\right\}.$$

The half-space $V_\rho(S)$ is well defined because $\lambda_S(\rho)\cdot a_\rho(S,i)=\lambda_S(\rho)\cdot a_\rho(S,j)$ for all $i,j\in S$.

By Corollary 37, $a_\rho(S,i)\to a(S)$. By the smoothness of $\partial V(S)$, $\lambda_S(\rho)\to\lambda_S$. Therefore,

$$V_\rho(S)\to V'(S):=\left\{x\in\mathbb{R}^S:\lambda_S\cdot x\le\lambda_S\cdot a(S)\right\}.$$

By Proposition 34, the proposals $\{a_\rho(S,i):S\subset N,i\in S\}$ satisfy (C.1), (C.2), and (C.3) for $(N,V,\mathcal{C})$. But these properties are the same for $(N,V_\rho,\mathcal{C})$. Thus, by Proposition 34, a_ρ is an SPNE payoff configuration for $(N,V_\rho,\mathcal{C})$. By Theorem 39, this implies that a_ρ is the only consistent coalitional payoff configuration for $(N,V_\rho,\mathcal{C})$.

Hence, each a_ρ satisfies properties (B.1), (B.2), and (B.3) for these vectors $(\lambda_S(\rho))_{S\subset N}$. Given $S\subset N$, $x:=(x_S)_{S\subset N}\in\mathbb{R}^{2^S}$ (with $x_\emptyset=0$), $\mu:=(\mu_S)_{S\subset N}\in\mathbb{R}^{2^S}$ (with $\mu_\emptyset=0$), and $i\in C'_q\in\mathcal{C}_S$, we define the following functions:

$$
\begin{aligned}
F_1(S,x,\mu) &= \sum_{C_r' \in \mathcal{C}_S \setminus C_q'} \left[\sum_{j \in C_q'} \mu_S^j \left(x_S^j - x_{S \setminus C_r'}^j \right) \right] \\
&\quad - \sum_{C_r' \in \mathcal{C}_S \setminus C_q'} \left[\sum_{j \in C_r'} \mu_S^j \left(x_S^j - x_{S \setminus C_q'}^j \right) \right] \\
F_2(S,i,x,\mu) &= \sum_{j \in C_q' \setminus i} \mu_S^i \left(x_S^i - x_{S \setminus j}^i \right) - \sum_{j \in C_q' \setminus i} \mu_S^j \left(x_S^j - x_{S \setminus i}^j \right) \\
F(x,\mu) &= \sum_{S \subset N} \left[\sum_{C_q' \in \mathcal{C}_S} \left(F_1(S,x,\mu)^2 + \sum_{i \in C_q'} F_2(S,i,x,\mu)^2 \right) \right].
\end{aligned}
$$

These functions are continuous. Thus, $F^{-1}(0) := \{(x,\mu) : F(x,\mu) = 0\}$ is a closed set. Since (a_ρ, λ_ρ) satisfies (B.2) and (B.3) we conclude that $F_1(S, a_\rho, \lambda_\rho) = 0$ and $F_2(S, i, a_\rho, \lambda_\rho) = 0$ for all $S \subset N$ and $i \in C_q' \in \mathcal{C}_S$. Then, $(a_\rho, \lambda_\rho) \in F^{-1}(0)$. We know that $F^{-1}(0)$ is a closed set and $(a_\rho, \lambda_\rho) \to (a, \lambda)$. Then, $(a, \lambda) \in F^{-1}(0)$. So, a satisfies (B.2) and (B.3).

Since a satisfies (B.1), (B.2), and (B.3), we conclude that this vector is the consistent coalitional value of the hyperplane game $(N, V', \mathcal{C})$. Now it is not difficult to conclude (by the definition of Φ) that a is a consistent coalitional payoff configuration of $(N, V, \mathcal{C})$. ■

Proof of Proposition 44. We proceed by induction on n. For $n = 1$, the result is trivial, because $\chi_{\{i\}}^i = r^i$. Assume the result is true for at most $n-1$ players.

By induction hypothesis, $a(S) = \chi_S$ for all $S \subsetneq N$. By (C.1), given $i \in C_q \in \mathcal{C}$,

$$
\lambda_N^i a(N,i)^i = v(N) - \sum_{C_r \in \mathcal{C} \setminus C_q} \left[\sum_{j \in C_r} \lambda_N^j a(N,i)^j \right] - \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j .
$$

Thus, $|\mathcal{C}| \lambda_N^i a(N)^i =$

$$
\sum_{C_r \in \mathcal{C}} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i
$$

$$
\begin{aligned}
= & \sum_{C_r \in \mathcal{C} \setminus C_q} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i + \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^i a(N,j)^i \\
& + \frac{1}{|C_q|} v(N) - \sum_{C_r \in \mathcal{C} \setminus C_q} \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N,i)^j - \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j
\end{aligned}
$$

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$$= \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N,i)^j \right]$$
$$+ \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i a(N,j)^i - \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j \right] + \frac{1}{|C_q|} v(N).$$

by (C.3), (C.4) and induction hypothesis

$$= \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i \left(\rho a(N)^i + (1-\rho)\chi^i_{N\setminus C_r} \right) - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j \left(\rho a(N)^j + (1-\rho)\chi^j_{N\setminus C_q} \right) \right]$$
$$+ \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i \left(\rho a(N)^i + (1-\rho)\chi^i_{N\setminus j} \right) - \sum_{j \in C_q \setminus i} \lambda_N^j \left(\rho a(N)^j + (1-\rho)\chi^j_{N\setminus i} \right) \right] + \frac{1}{|C_q|} v(N$$

$$= \rho \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N)^i - \rho \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N)^j$$
$$+ \rho \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^i a(N)^i - \rho \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^j a(N)^j$$
$$+ (1-\rho) \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i \chi^i_{N\setminus C_r} - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j \chi^j_{N\setminus C_q} \right]$$
$$+ (1-\rho) \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i \chi^i_{N\setminus j} - \sum_{j \in C_q \setminus i} \lambda_N^j \chi^j_{N\setminus i} \right] + \frac{1}{|C_q|} v(N).$$

By (C.1), $v(N) = \sum_{j \in N} \lambda_N^j a(N,i)^j$ for all $i \in N$. By (4.3) and averaging over i we conclude that $v(N) = \sum_{j \in N} \lambda_N^j a(N)^j = \sum_{C_r \in \mathcal{C}\setminus C_q} \sum_{j \in C_r} \lambda_N^j a(N)^j + \sum_{j \in C_q \setminus i} \lambda_N^j a(N)^j + \lambda_N^i a(N)^i$ and so, $|\mathcal{C}| \lambda_N^i a(N)^i =$

$$\rho \left[\sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N)^i + \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^i a(N)^i + \frac{1}{|C_q|} \lambda_N^i a(N)^i - \frac{1}{|C_q|} v(N) \right]$$
$$+ (1-\rho) \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i \chi^i_{N\setminus C_r} - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j \chi^j_{N\setminus C_q} \right]$$
$$+ (1-\rho) \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i \chi^i_{N\setminus j} - \sum_{j \in C_q \setminus i} \lambda_N^j \chi^j_{N\setminus i} \right] + \frac{1}{|C_q|} v(N)$$

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$$\begin{aligned}
&= \rho\left[\sum_{C_r\in\mathcal{C}}\frac{1}{|C_r|}\sum_{j\in C_r}\lambda_N^i a(N)^i - \frac{1}{|C_q|}v(N)\right]\\
&\quad +(1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right]\\
&\quad +(1-\rho)\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{1}{|C_q|}v(N)\\
&= \rho\,|\mathcal{C}|\,\lambda_N^i a(N)^i - \frac{\rho}{|C_q|}v(N)\\
&\quad +(1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right]\\
&\quad +(1-\rho)\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{1}{|C_q|}v(N).
\end{aligned}$$

By gathering terms and dividing by $(1-\rho)$ we have

$$\begin{aligned}
|\mathcal{C}|\,\lambda_N^i a(N)^i &= \sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right]\\
&\quad +\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{v(N)}{|C_q|}.
\end{aligned}$$

And so,

$$\begin{aligned}
a(N)^i &= \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i}\sum_{C_r\in\mathcal{C}\setminus C_q}\left(|C_q|\,\lambda_N^i\chi_{N\setminus C_r}^i - \sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right)\\
&\quad +\frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i}\sum_{j\in C_q\setminus i}\left(\lambda_N^i\chi_{N\setminus j}^i - \lambda_N^j\chi_{N\setminus i}^j\right)\\
&\quad +\frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i}v(N)\\
&= \chi_N^i.
\end{aligned}$$

■

Part III

On allocation problems

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Chapter 5

Additive rules in allocation problems

5.1 Introduction

Many economic situations can be modelled as a problem of how to divide a resource (divisible or indivisible) among agents who have claims on it. In this Chapter we study problems where an estate E must be divided among a finite group of agents N, c_i being the claim of agent i.

We study four kinds of problems. The difference among them is the way in which we must divide the estate. In bankruptcy problems (introduced by O'Neill, 1982, and studied later by Aumann and Maschler, 1985) any agent must receive at least 0 and at most his claim. In allocation problems (Chun, 1988, and Herrero, Maschler, and Villar, 1999) agents can receive anything. In surplus problems (Moulin, 1987) every agent must receive at least 0. In loss problems, every agent must receive at most his claim. Notice that with the four classes of problems we cover all the possibilities.

One of the most important topics of these problems is the axiomatic characterizations of rules. The idea is to propose desirable properties and to find out which of them characterize every rule. Properties often help agents to compare different rules and to decide which rule is preferred for a particular situation[1].

A dual approach is to study what the rules satisfying a set of properties are. For instance, Young (1988) characterizes the rules satisfying continuity, symmetry, and consistency; de Frutos (1999) characterizes the rules satisfying non-manipulability; and Moulin (2000) characterizes the rules satisfying consistency, composition up, composition down, and scale invariance.

In this Chapter we adopt both approaches. We characterize the rules satisfying additivity in each of the four problems mentioned above. Moreover, using

[1] See Thomson (2000) for a complete survey of the axiomatic analyses of bankruptcy and taxation problems.

these additivity properties, we characterize the well-known rules based on the principles of "proportionality", "equal award", and "equal loss".

Additivity is a standard property. It has been used in many situations. Although the justification of additivity is not so clear as other properties (for example, efficiency or symmetry) in most cases it produces very interesting classes of rules. For instance, the Shapley value, the most important value in transferable utility games, is characterized by additivity and other properties. If we compare the Shapley value with other prominent values (for example, the nucleolus) we realize that these values satisfy all the properties characterizing the Shapley value except additivity.

In this Chapter we use two definitions of additivity: additivity on the estate (Moulin, 1987, and Chun, 1988), called $A1$, and additivity on the estate and the claims (Bergantiños and Méndez-Naya, 2001), called $A2$. In the four kind of problems we characterize the rules satisfying $A1$ and $A2$ in the continuous case (the estate is perfectly divisible) and the discrete case (the estate comes in units which cannot be divided).

The rules satisfying $A1$ in the discrete case are as follows. In allocation problems they are characterized by the product of the estate and a claims-depending function. In surplus problems all the estate is given to a player, who is selected depending on the claims. In loss and bankruptcy problems there are no rules.

The rules satisfying $A2$ in the discrete case are as follows. In allocation problems they are characterized by the sum of two parts: one depending on the estate and other depending on the claims. In surplus problems the estate is always given to a fixed player. In loss problems all the loss is always suffered by a fixed player. There is no bankruptcy rule satisfying $A2$.

The rules satisfying $A1$ in the continuous case are as follows. In allocation problems they are similar to the ones of the discrete case. In surplus problems the estate is divided among agents according to a weight system, which depends on the claims. In loss and bankruptcy problems only the proportional rule satisfies $A1$.

The rules satisfying $A2$ in the continuous case are as follows. In allocation problems they are similar to the ones of the discrete case. In surplus (loss) problems the estate (loss) is divided among agents according to a weight system, independent of the claims. There is no bankruptcy rule satisfying $A2$.

In the continuous case we obtain axiomatic characterizations of well-known rules. In allocation problems and surplus problems, the proportional rule is characterized by $A1$ and other properties. In allocation and loss problems, the rights-egalitarian rule (Herrero, Maschler and Villar, 1999) is characterized by $A2$ and other properties. Moreover, $A2$ and other properties also characterize the equal-sharing rule (Moulin, 1987) in surplus problems.

As a consequence of our results we can say that additivity properties also support the use of rules based on three classical principles. Property $A1$ is related to the principle of "proportionality"; property $A2$ is related to the principles of "equal award" and "equal loss".

This Chapter is organized as follows. Section 5.2 introduces the problems studied in this Chapter. Section 5.3 studies the discrete case and Section 5.4 the continuous case. Section 5.5 is devoted to concluding remarks.

5.2 Preliminaries

We introduce some notation. $\mathbb{Z}$ denotes the set of integer numbers and $\mathbb{N}$ denotes the set of non-negative integer numbers. $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{Q}_+$ denotes the set of non-negative rational numbers. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ the set of non-negative real numbers.

$\mathbb{N}$ also denotes the set of potential agents. Let N be any finite subset of $\mathbb{N}$. Given $x, y \in \mathbb{R}^N$, $x \geq y$ means $x_i \geq y_i$ for all $i \in N$; and $x + y = (x_i + y_i)_{i \in N}$. Moreover, $0_N = (0, ..., 0) \in \mathbb{R}^N$. Given $S \subset N$, $1_S = (x_i)_{i \in N}$ such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$.

We study problems where an estate E must be divided among a group of agents N, c_i being the claim of agent $i \in N$, $c = (c_i)_{i \in N}$ the vector of claims, and $C = \sum_{i \in N} c_i$ the sum of the claims. We assume that the estate and the claims are non-negative. The question that arises is: how to divide the estate among agents? The way to answer this question is by defining rules. A rule, f, is a map which assigns to any problem (c, E) a vector $f(c, E)$ where $f_i(c, E)$ denotes the part of the estate received by agent $i \in N$.

Through this Chapter, we study continuous and discrete problems. In continuous problems the estate and the claims are perfectly divisible (for instance, money). Thus, $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$. Continuous problems had been widely studied in the literature.

In discrete problems, both claims and estate to be distributed come in indivisible units and hence $(c, E) \in \mathbb{N}^N \times \mathbb{N}$. Allocation of organs for transplants, college admissions, and in general queuing problems where individual claims consist of a finite number of jobs, could be some problems of the discrete model. In Moulin (2000, 2002) and Moulin and Stong (2002) some of these discrete problems are studied.

We now give a list of problems that fit in our general framework. We give the definition for continuous problems (by changing $\mathbb{R}_+$ for $\mathbb{N}$ and $\mathbb{R}$ for $\mathbb{Z}$ the definition of the discrete problems is obtained). Notice that the difference among these problems is, mainly, in the definition of what a rule is.

A *bankruptcy problem*, shortly BP, is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ where $C \geq E$. We denote by $\mathcal{B}$ the set of all bankruptcy problems. A *bankruptcy rule* is a function $f : \mathcal{B} \to \mathbb{R}^N$ satisfying that for all $(c, E) \in \mathcal{B}$, $\sum_{i \in N} f_i(c, E) = E$ and $0 \leq f_i(c, E) \leq c_i$ for all $i \in N$.

A *surplus problem*, shortly SP, is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$. We denote by $\mathcal{S}$ the set of all surplus problems. A *surplus rule* is a function $f : \mathcal{S} \to \mathbb{R}^N$ satisfying that for all $(c, E) \in \mathcal{S}$, $\sum_{i \in N} f_i(c, E) = E$ and $0 \leq f_i(c, E)$ for all $i \in N$.

An *allocation problem*, shortly AP, is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$. We denote by $\mathcal{A}$ the set of all allocation problems. An *allocation rule* is a function $f : \mathcal{A} \to \mathbb{R}^N$ satisfying that for all $(c, E) \in \mathcal{A}$, $\sum_{i \in N} f_i(c, E) = E$.

These problems have been studied in the literature. For instance: O'Neill (1982) and Aumann and Maschler (1985) study bankruptcy problems, Moulin (1987) study surplus problems, and Chun (1988) and Herrero, Maschler and Villar (1999) study allocation problems[2]. See Thomson (2000) for a survey of the axiomatic analyses of bankruptcy problems. This surveys also analyses surplus and allocation problems.

As far as we know the next class of problems has not been explicitly studied. A *loss problem*, shortly LP, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ where $C \geq E$. We denote by $\mathcal{L}$ the set of all loss problems. A *loss rule* is a function $f : \mathcal{L} \to \mathbb{R}^N$ satisfying that for all $(c, E) \in \mathcal{L}$, $\sum_{i \in N} f_i(c, E) = E$ and $f_i(c, E) \leq c_i$ for all $i \in N$.

Remark 49 *These four kinds of problems cover all possible definitions of a rule. Given $i \in N$, a bankruptcy rule satisfies $0 \leq f_i(c, E) \leq c_i$, a surplus rule satisfies $0 \leq f_i(c, E)$, a loss rule satisfies $f_i(c, E) \leq c_i$, and an allocation rule has no restrictions at all. Notice that any bankruptcy rule is a loss rule and any surplus rule is an allocation rule.*

Herrero, Maschler and Villar (1999) consider the case where $E < 0$. It is not difficult to check that our results are also valid if we allow the estate to be negative in AP and LP.

In BP and LP we need to impose the condition $C \geq E$ because otherwise it is not possible to find f satisfying $\sum_{i \in N} f_i(c, E) = E$ and $f_i(c, E) \leq c_i$ for all $i \in N$.

We now present some rules for the continuous problem studied in this Chapter. The *proportional rule, P*, divides the estate proportionally to the claims when their sum C is different from zero. Formally, for all $i \in N, c \neq 0_N$,

$$P_i(c, E) = \lambda c_i \text{ where } \lambda = \frac{E}{C}.$$

Remark 50 *Assume that $c = 0_N$ and (c, E) is a BP or a LP. Since $C \geq E \geq 0$, we conclude that $E = 0$. Then, any bankruptcy rule or loss rule satisfies $f(0_N, 0) = 0_N$. Thus, $P(0_N, 0) = 0_N$.*

If $c = 0_N$ and (c, E) is a SP or an AP, we define $P_i(0_N, E) = \frac{E}{n}$ for all $i \in N$. It is not difficult to check that the results obtained in this Chapter are also true if we define $P(0_N, E)$ in a different way.

The *equal-sharing rule, ES,* divides the estate equally among agents. Formally, for all $i \in N$,

$$ES_i(c, E) = \frac{E}{n}.$$

[2] Chun (1988) uses the name *rights problems*.

Notice that the equal-sharing rule is both an allocation rule and a surplus rule. Nevertheless, it is not a bankruptcy rule or a loss rule because $ES_i(c,E)$ could be larger than c_i.

We present a family of rules closely related to ES. We say that ω is a *weight system* if $\omega \in \mathbb{R}^N_+$ and $\sum_{i\in N} \omega_i = 1$.

The *weighted-sharing rule* WS^ω *with weight system* ω is defined for all $i \in N$ by

$$WS_i^\omega(c,E) = \omega_i E.$$

Notice that WS^ω is an allocation rule and a surplus rule but it is neither a bankruptcy rule nor a loss rule. Of course, if $\omega_i = \frac{1}{n}$ for all $i \in N$, then $WS^\omega = ES$.

Herrero, Maschler and Villar (1999) define the *rights-egalitarian rule*, RE, for all $i \in N$ as

$$RE_i(c,E) = c_i - \frac{1}{n}(C-E).$$

Notice that the rights-egalitarian rule is an allocation rule and a loss rule (in LP, $C \geq E$). Nevertheless, it is neither a bankruptcy rule nor a surplus rule because $RE_i(c,E)$ could be negative.

We present other rules closely related to RE. The *rights-weighted rule* RW^ω *with weight system* ω, is defined for all $i \in N$ by

$$RW_i^\omega(c,E) = c_i - \omega_i(C-E).$$

Notice that RW^ω is an allocation rule and a loss rule but not a bankruptcy rule or a surplus rule. Of course, if $\omega_i = \frac{1}{n}$ for all $i \in N$ then, $RW^\omega = RE$.

Remark 51 *Given a bankruptcy rule f, Aumann and Maschler (1985) define the* dual rule f^D *as*

$$f^D(c,E) = c - f(c,C-E) \text{ for all } (c,E) \in \mathcal{B}.$$

Notice that f^D assigns "awards" (E) in the same way that f assigns "losses" $(C-E)$. When $C - E \geq 0$, we can extend this definition to AP, SP, and LP.

It is trivial to see that for any weight system ω, the weighted-sharing rule WS^ω and the rights-weighted rule RW^ω are dual rules, i. e., $(WS^\omega)^D = RW^\omega$ *and* $(RW^\omega)^D = WS^\omega$. *Moreover, the proportional rule is self-dual* $\left(P^D = P\right)$.

We now present some properties used in this Chapter. First we introduce the two additivity properties considered in this Chapter.

Additivity on E *(A1).* For all (c,E) and (c,E'),

$$f(c,E+E') = f(c,E) + f(c,E').$$

Moulin (1987) and Chun (1988) use this property in surplus problems and in allocation problems, respectively. Property $A1$ says that dividing the estate among the agents is the same as dividing, first, one part of the estate and, afterwards, the remaining estate.

Additivity on (c, E) $(A2)$. For all (c, E) and (c', E'),

$$f(c + c', E + E') = f(c, E) + f(c', E').$$

Bergantiños and Méndez-Naya (2001) introduce this property in bankruptcy problems and in allocation problems. Suppose that the product sold by a firm depends on several parts (quality and marketing, for instance). The total revenue of the firm, $E+E'$, can be divided into two parts: one motivated by quality (E) and the other by marketing (E'). We can also determine the contribution of every agent of the firm to quality (c) and marketing (c'). Now we can allocate the revenue according to two procedures. First, we allocate the total revenue $(E + E')$ according to the total contribution $(c + c')$. Second, we allocate the revenue motivated by quality (E) according to the contribution to quality (c), and the revenue of marketing (E') according to the contribution to marketing (c'). $A2$ guarantees that both procedures coincide.

Usually it is not very difficult to determine the contribution of the agents to each part (for instance, hours worked) and the total revenue. But sometimes it seems impossible to know exactly the contribution of each part to the total revenue. Under these circumstances it seems that we cannot apply the second procedure. Nevertheless, if the allocation rule satisfies $A2$ we can do this, because both procedures coincide.

There is no relation between $A1$ and $A2$. Later, we see examples of rules satisfying $A1$ but not $A2$ and rules satisfying $A2$ but not $A1$.

Remark 52 *Another possibility is to define additivity on c. For all* (c, E) *and* (c', E), $f(c + c', E) = f(c, E) + f(c', E)$. *Nevertheless, no rule satisfies this property because* $\sum_{i \in N} f_i(c + c', E) = E$, $\sum_{i \in N} f_i(c, E) = E$, *and* $\sum_{i \in N} f_i(c', E) = E$.

We now consider more properties. Symmetry and continuity are standard properties that can be defined in each of the four problems studied in this Chapter.

Symmetry (SYM). For any problem (c, E) if $c_i = c_j$ then $f_i(c, E) = f_j(c, E)$.

Continuity on E $(C1)$. For any sequence of problems (c, E^l) and any problem (c, E), if $E^l \to E$, then $f(c, E^l) \to f(c, E)$.

Continuity on (c, E) $(C2)$. For any sequence of problems (c^l, E^l) and any problem (c, E), if $(c^l, E^l) \to (c, E)$, then $f(c^l, E^l) \to f(c, E)$.

Of course, if a rule satisfies $C2$ it also satisfies $C1$.

Assume that agent $i's$ claim is at least as large as agent $j's$ claim. Order preservation says that agent i must receive at least the same amount as agent j.

Order preservation (OP). For every problem (c, E), if $c_i \geq c_j$ then $f_i(c, E) \geq f_j(c, E)$.

Of course, OP is a generalization of SYM.

Suppose that the estate, E, equals the sum of the claims, C. Compatibility says that each agent gets exactly the amount that he claims.

Compatibility (COM). For every problem (c, C), $f(c, C) = c$.

Next property appears in a preliminary draft of Herrero, Maschler and Villar (1999), but not in the final version. Condition i) says that no agent gets less than he has right to. Condition ii) says that no agent gets more than he claims.

Claims boundedness (CB). For all $(c, E) \in \mathcal{A}$:
i) $f(c, E) \geq c$ if $C \leq E$.
ii) $f(c, E) \leq c$ if $C \geq E$.

Notice that all bankruptcy rules and loss rules satisfy CB.

5.3 The discrete problem

In this section we characterize the set of additive rules in the four discrete problems. In Theorem 53 we characterize the rules satisfying $A1$ and in Theorem 56 the rules satisfying $A2$. Finally, we briefly compare the rules satisfying $A1$ with the rules satisfying $A2$.

Theorem 53 *a) An allocation rule f satisfies $A1$ if and only if for all $(c, E) \in \mathcal{A}$,*

$$f_i(c, E) = E\alpha_i(c) \text{ for all } i \in N$$

where $\alpha : \mathbb{N}^N \to \mathbb{Z}^N$ satisfies $\sum_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{N}^N$.

b) A surplus rule f satisfies $A1$ if and only if for all $(c, E) \in \mathcal{S}$,

$$f_i(c, E) = \begin{cases} E & \text{if } i = s(c) \\ 0 & \text{otherwise} \end{cases}$$

where $s : \mathbb{N}^N \to N$.

c) There is no loss rule satisfying $A1$.

d) There is no bankruptcy rule satisfying $A1$.

Proof. a) It is trivial to prove that if $f(c, E) = E\alpha(c)$ then f satisfies $A1$.

We now prove the reciprocal. Suppose that f is an allocation rule satisfying $A1$. Let $\alpha : \mathbb{N}^N \to \mathbb{Z}^N$ be such that $\alpha(c) = f(c, 1)$. Since f is an allocation rule, we conclude that $\sum_{i \in N} \alpha_i(c) = 1$. Given $(c, E) \in \mathcal{A}$ and $i \in N$, by $A1$, $f_i(c, E) = Ef_i(c, 1) = E\alpha_i(c)$.

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b) Since all surplus rules are allocation rules, the allocation rules of a) satisfying $f(c,E) \geq 0_N$ for all $(c,E) \in \mathcal{S}$ are all the surplus rules satisfying $A1$.

Assume that f is as in a) and $f(c,E) \geq 0_N$ for all $(c,E) \in \mathcal{S}$. Since $\sum\limits_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{N}^N$, it is not difficult to conclude that, given $c \in \mathbb{N}^N$, there exists $i_c \in N$ such that $\alpha_{i_c}(c) = 1$ and $\alpha_i(c) = 0$ if $i \neq i_c$. Considering $s : \mathbb{N}^N \to N$ such that $s(c) = i_c$ the result holds trivially.

c) Using arguments similar to those used in a) we can conclude that if f is a loss rule satisfying $A1$ then for all $(c,E) \in \mathcal{L}$, $f(c,E) = E\alpha(c)$ where $\sum\limits_{i \in N} \alpha_i(c) = 1$. Then, given $c \in \mathbb{N}^N$, there exists $j \in N$ such that $\alpha_j(c) \geq 1$. If $E > c_j$ then $f_j(c,E) = E\alpha_j(c) > c_j$. Hence, f is not a loss rule.

d) Since all bankruptcy rules are loss rules, the result is a consequence of c).

■

Remark 54 *Theorem 53 c) cannot be proved using a) because the set of allocation problems $(\mathcal{A})$ is different from the set of loss problems $(\mathcal{L})$. If $(c,E) \in \mathcal{L}$ then $C \geq E$, but in $\mathcal{A}$, $C < E$ is also possible. This means that an allocation rule could satisfy $A1$ in $\mathcal{L}$ but not in $\mathcal{A}$. This comment can be applied also to the proofs of Theorems 56, 57, and 59.*

The allocation and surplus rules satisfying $A1$ have a special structure. Given a vector of claims c, what really matters is the way in which we divide one unit among agents because when there are E units, any agent receives E times what he receives when there is one unit.

Notice that in SP all the estate is received by one agent, meanwhile the rest receive 0. This agent is selected depending on c.

Consider the family F of surplus rules where an agent with the highest claim receives all the estate and the rest of agents receive nothing. Formally,

$$F = \left\{ \begin{array}{c} f: \text{ for all } c \in \mathbb{N}^N \text{ there exists } s(c) \in N \text{ such that } c_{s(c)} = \max\limits_{j \in N} c_j, \\ f_{s(c)}(c,E) = E \text{ and } f_i(c,E) = 0 \text{ if } i \neq s(c) \end{array} \right\}.$$

Next corollary characterizes F as the set of rules satisfying $A1$ and OP.

Corollary 55 *A surplus rule f satisfies $A1$ and OP if and only if $f \in F$.*

Proof. It is clear that the rules of F satisfy $A1$ and OP.

Assume that f is a surplus rule satisfying $A1$ and OP. By Theorem 53 we know that there exists $s(c) \in N$ such that $f_{s(c)}(c,E) = E$ and $f_i(c,E) = 0$ if $i \neq s(c)$. Since f satisfies OP we conclude that $c_{s(c)} = \max\limits_{j \in N} c_j$. Hence, $f \in F$.

■

It is trivial to see that this corollary is a tight characterization result.

Next theorem characterizes the rules satisfying $A2$.

Theorem 56 *a) An allocation rule f satisfies $A2$ if and only if for all $(c, E) \in \mathcal{A}$,*

$$f_i(c, E) = \beta_i(c) + E x_i \text{ for all } i \in N$$

where $\beta : \mathbb{N}^N \to \mathbb{Z}^N$ satisfies $\sum_{i \in N} \beta_i(c) = 0$ for all $c \in \mathbb{N}^N$ and $\beta(c + c') = \beta(c) + \beta(c')$ for all $c, c' \in \mathbb{N}^N$. Moreover, $x \in \mathbb{Z}^N$ and $\sum_{i \in N} x_i = 1$.

b) A surplus rule f satisfies $A2$ if and only if there exists $i_0 \in N$ such that for all $(c, E) \in \mathcal{S}$,

$$f_i(c, E) = \begin{cases} E & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

c) A loss rule f satisfies $A2$ if and only if there exists $i_0 \in N$ such that for all $(c, E) \in \mathcal{L}$,

$$f_i(c, E) = \begin{cases} c_i - (C - E) & \text{if } i = i_0 \\ c_i & \text{otherwise.} \end{cases}$$

d) There is no bankruptcy rule satisfying $A2$.

Proof. a) It is straightforward to prove that if $f(c, E) = \beta(c) + Ex$ then f satisfies $A2$.

We now prove the reciprocal. Suppose that f is an allocation rule satisfying $A2$. Given an allocation problem (c, E) and a rule f satisfying $A2$,

$$f(c, E) = f(c, 0) + f(0_N, E).$$

Since f satisfies $A2$, $f(0_N, E) = Ef(0_N, 1)$. Consider $x = f(0_N, 1)$. Then, $x \in \mathbb{Z}^N$ and $\sum_{i \in N} x_i = \sum_{i \in N} f_i(0_N, 1) = 1$.

Consider $\beta : \mathbb{N}^N \to \mathbb{Z}^N$ such that $\beta(c) = f(c, 0)$ for all $c \in \mathbb{N}^N$. Then, $\sum_{i \in N} \beta_i(c) = \sum_{i \in N} f_i(c, 0) = 0$. Moreover, for all $c, c' \in \mathbb{N}^N$, $\beta(c+c') = f(c+c', 0) = f(c, 0) + f(c', 0) = \beta(c) + \beta(c')$.

b) Since every surplus rule is an allocation rule, the allocation rules of a) satisfying $f(c, E) \geq 0_N$ for all $(c, E) \in \mathcal{S}$ are all the surplus rules satisfying $A2$.

Assume that f is as in a) and $f(c, E) \geq 0_N$ for all $(c, E) \in \mathcal{S}$. Given $c \in \mathbb{N}^N$, $\beta(c) = f(c, 0) \geq 0_N$ and $\sum_{i \in N} f_i(c, 0) = 0$. Then $\beta(c) = 0_N$. Since $x = f(0_N, 1) \geq 0_N$, and $\sum_{i \in N} x_i = 1$, there exists $i_0 \in N$ such that $x_{i_0} = 1$ and $x_i = 0$ for all $i \in N \setminus i_0$. Now the result holds trivially.

c) It is trivial to prove that if $f(c, E) = c - (C - E) 1_{\{i_0\}}$ then f satisfies $A2$.

We now prove the reciprocal. Assume that f satisfies $A2$. Since $f(c, E) \leq c$ and $\sum_{i \in N} f_i(c, E) = E$ we conclude that for all $i \in N$:

- $f\left(1_{\{i\}},1\right)=1_{\{i\}}$,
- $f\left(1_{\{i\}},0\right)=0_N$ or $f\left(1_{\{i\}},0\right)=1_{\{i\}}-1_{\{i'\}}$ where $i'\in N\setminus i$.

We first prove that there exists $i\in N$ such that $f\left(1_{\{i\}},0\right)\neq 0_N$. We prove it by contradiction. Assume that $f\left(1_{\{i\}},0\right)=0_N$ for all $i\in N$. Since f satisfies $A2$, for all $i\in N$,

$$f\left(1_N,1\right)=f\left(1_{\{i\}},1\right)+\sum_{j\in N\setminus i} f\left(1_{\{j\}},0\right)=1_{\{i\}}$$

which is a contradiction.

Consider $i\in N$ such that $f\left(1_{\{i\}},0\right)=1_{\{i\}}-1_{\{i'\}}$. Assume that there exists $j\in N\setminus i$ such that $f\left(1_{\{j\}},0\right)=1_{\{j\}}-1_{\{j'\}}$. Since f satisfies $A2$,

$$\begin{aligned} f\left(1_{\{i,j\}},1\right) &= f\left(1_{\{i\}},1\right)+f\left(1_{\{j\}},0\right)=1_{\{i\}}+1_{\{j\}}-1_{\{j'\}} \\ f\left(1_{\{i,j\}},1\right) &= f\left(1_{\{j\}},1\right)+f\left(1_{\{i\}},0\right)=1_{\{j\}}+1_{\{i\}}-1_{\{i'\}} \end{aligned}$$

which means that $i'=j'$.

We have proved that there exists $i_0\in N$ such that if $i\in N$ and $f\left(1_{\{i\}},0\right)\neq 0_N$ then $f\left(1_{\{i\}},0\right)=1_{\{i\}}-1_{\{i_0\}}$.

Assume now that there exists $j\in N$, $j\neq i_0$ such that $f\left(1_{\{j\}},0\right)=0_N$. We already know that there exists $i\in N$ such that $f\left(1_{\{i\}},0\right)=1_{\{i\}}-1_{\{i_0\}}$. Since f satisfies $A2$,

$$\begin{aligned} f\left(1_{\{i,j\}},1\right) &= f\left(1_{\{i\}},1\right)+f\left(1_{\{j\}},0\right)=1_{\{i\}} \\ f\left(1_{\{i,j\}},1\right) &= f\left(1_{\{j\}},1\right)+f\left(1_{\{i\}},0\right)=1_{\{j\}}+1_{\{i\}}-1_{\{i_0\}} \end{aligned}$$

which is a contradiction because $j\neq i_0$.

Then, for all $i\in N\setminus i_0$, $f\left(1_{\{i\}},0\right)=1_{\{i\}}-1_{\{i_0\}}$. Now it is not difficult to conclude that $f\left(1_{\{i_0\}},0\right)=0_N=1_{\{i_0\}}-1_{\{i_0\}}$.

Given $(c,E)\in\mathcal{L}$, we can find a partition $\{N_1,\{i\},N_2\}$ of N such that $c_i=c_i^1+c_i^2$, $c_i^1\in\mathbb{N}$, $c_i^2\in\mathbb{N}$, and $E=\sum_{j\in N_1} c_j+c_i^1$. Since f satisfies $A2$,

$$\begin{aligned} f(c,E) &= \sum_{j\in N_1} f\left(c_j 1_{\{j\}},c_j\right)+f\left(c_i^1 1_{\{i\}},c_i^1\right)+f\left(c_i^2 1_{\{i\}},0\right)+\sum_{j\in N_2} f\left(c_j 1_{\{j\}},0\right) \\ &= \sum_{j\in N_1} c_j f\left(1_{\{j\}},1\right)+c_i^1 f\left(1_{\{i\}},1\right)+c_i^2 f\left(1_{\{i\}},0\right)+\sum_{j\in N_2} c_j f\left(1_{\{j\}},0\right) \\ &= \sum_{j\in N_1} c_j 1_{\{j\}}+c_i^1 1_{\{i\}}+c_i^2\left(1_{\{i\}}-1_{\{i_0\}}\right)+\sum_{j\in N_2} c_j\left(1_{\{j\}}-1_{\{i_0\}}\right) \\ &= c-(C-E)\,1_{\{i_0\}}. \end{aligned}$$

d) Suppose that $N=\{1,2\}$, $E=5$, and $c=(7,7)$. We can find $i\in N$ such that $f_i(c,E)\geq 3$. Assume without loss of generality that $i=1$.

Since f satisfies $A2$, $f_1(c,E) = f_1((6,1),1) + f_1((1,6),4) \leq 1+1 = 2$, which is a contradiction. ■

The allocation rules satisfying $A2$ can be divided into two terms: the first term depending on c $(\beta(c))$ and the second term depending on E (Ex). In the first term, given a vector of claims c, the function β reassigns units of the indivisible good among agents in such a way that some agents must provide to other agents some units of this good. In the second term any agent receives E times the amount he would receive when all agents claim 0 and only 1 unit is available. Notice that the class of allocation rules satisfying $A2$ is unrelated to the class of allocation rules satisfying $A1$ (there are allocation rules satisfying $A2$ but not $A1$ and vice versa). As a consequence of Theorem 53 and Theorem 56, an allocation rule f satisfies $A1$ and $A2$ if and only if for all $(c,E) \in \mathcal{A}$, $f(c,E) = Ex$ where $x \in \mathbb{Z}^N$ and $\sum_{i\in N} x_i = 1$.

The surplus rules satisfying $A2$ are those in which an agent receives all the estate and the rest receive nothing. Notice that the class of surplus rules satisfying $A2$ is a subset of the class of surplus rules satisfying $A1$.

The loss rules satisfying $A2$ are those in which an agent loses the total loss $(C-E)$ and the rest lose nothing. Notice that there is no loss rule satisfying $A1$. Moreover, it is not difficult to check that the surplus rules satisfying $A2$ are dual of the loss rules satisfying $A2$, when we consider both as allocation rules.

In BP there is no bankruptcy rule satisfying $A1$ or $A2$.

5.4 The continuous problem

In this section we characterize the set of additive rules in the four continuous problems. In Theorem 57 we characterize the rules satisfying $A1$ and in Theorem 59 the rules satisfying $A2$.

Later, we characterize more rules using these additive properties. In Theorem 61, we characterize the set of allocation rules satisfying $A2$ and CB. In Corollary 62, we characterize the proportional rule in allocation and surplus problems using $A1$ and other properties. Finally, in Corollary 63, we characterize, using $A2$ and other properties, the rights-egalitarian rule in allocation and loss problems and the equal-sharing rule in surplus problems.

Next theorem characterizes the rules satisfying $A1$ in the continuous problem

Theorem 57 *a) An allocation rule f satisfies $A1$ and $C1$ if and only if*

$$f_i(c,E) = E\alpha_i(c) \text{ for all } i \in N,\ (c,E) \in \mathcal{A}$$

where $\alpha : \mathbb{R}^N_+ \to \mathbb{R}^N$ satisfies $\sum_{i\in N} \alpha_i(c) = 1$ for all $c \in \mathbb{R}^N_+$.

b) A surplus rule f satisfies $A1$ if and only if for all $(c,E) \in \mathcal{S}$,

$$f_i(c,E) = E\omega_i(c) \text{ for all } i \in N$$

where $\omega : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is such that $\omega(c)$ is a weight system for every $c \in \mathbb{R}^N_+$.

c) The proportional rule is the only loss rule satisfying $A1$.
d) The proportional rule is the only bankruptcy rule satisfying $A1$.

Proof. a) It is trivial to prove that any rule given by $f(c,E) = E\alpha(c)$ satisfies $A1$ and $C1$.

We now prove the reciprocal. Assume that f is an allocation rule satisfying $A1$ and $C1$. Since f satisfies $A1$ we conclude that:

- $f(c,E) = Ef(c,1)$ for all $E \in \mathbb{N}$.
- If $E = \frac{1}{q}$, where $q \in \mathbb{N}$, then $f(c,E) = Ef(c,1)$.
- Given $E \in \mathbb{Q}_+$, $f(c,E) = Ef(c,1)$.

Since f satisfies $C1$, $f(c,E) = Ef(c,1)$ for all $E \in \mathbb{R}_+$.

If we take $\alpha(c) = f(c,1)$ the result holds.

b) It is trivial to prove that any function given by $f(c,E) = E\omega(c)$ satisfies $A1$.

We now prove the reciprocal. Assume that f is a surplus rule satisfying $A1$. Using arguments similar to those used in a) we can conclude that given $E \in \mathbb{Q}_+$, $f(c,E) = Ef(c,1)$.

Let E be a non-negative irrational number ($E \in \mathbb{R}_+ \setminus \mathbb{Q}_+$). Then, there exists $\left(E^l\right)_{l\in\mathbb{N}}$ such that $E^l \in \mathbb{Q}_+$, $0 < E^l < E$, and $\lim_{l\to\infty} E^l = E$. Thus, for all $l \in \mathbb{N}$,

$$f\left(c, E - E^l\right) = f(c,E) - f\left(c, E^l\right) = f(c,E) - E^l f(c,1).$$

Since $f\left(c, E - E^l\right) \geq 0_N$ and $\sum_{i\in N} f_i\left(c, E - E^l\right) = E - E^l$, for all $i \in N$,

$$f_i\left(c, E - E^l\right) \leq E - E^l.$$

Thus, for all $i \in N$,

$$0 \leq \lim_{l\to\infty} f_i\left(c, E - E^l\right) \leq E - \lim_{l\to\infty} E^l = 0.$$

So, for all $i \in N$,

$$0 = \lim_{l\to\infty} f_i\left(c, E - E^l\right) = f_i(c,E) - Ef_i(c,1).$$

Hence, $f(c,E) = Ef(c,1)$ for all $(c,E) \in \mathcal{S}$.

If we take $\omega(c) = f(c,1)$ the result holds.

c) It is trivial to prove that the proportional rule satisfies $A1$.

We now prove the reciprocal. Assume that f is a loss rule satisfying $A1$. We first prove that $f(c,\varepsilon E) = Ef(c,\varepsilon)$ when $(c,\varepsilon E) \in \mathcal{L}$, $(c,\varepsilon) \in \mathcal{L}$, $E \in \mathbb{R}_+$, and $\varepsilon \in \mathbb{Q}_+$.

Using arguments similar to those used in a) we can conclude that, given $c \in \mathbb{R}^N_+$ such that $C > 0$, $(c, \varepsilon E) \in \mathcal{L}$, $(c, \varepsilon) \in \mathcal{L}$, $E \in \mathbb{Q}_+$,and $\varepsilon \in \mathbb{Q}_+$,

$$f(c, \varepsilon E) = E f(c, \varepsilon).$$

Given $(c, \varepsilon E) \in \mathcal{L}$ with E a non-negative irrational number, there exists $\left(E^l\right)_{l \in \mathbb{N}}$ such that $E^l \in \mathbb{Q}_+$, $0 < E^l < E$, and $\lim_{l \to \infty} E^l = E$. Thus, for all $l \in \mathbb{N}$,

$$f\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) = c - f(c, \varepsilon E) + E^l f(c, \varepsilon).$$

Since $f\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) \leq c$ and $\sum_{i \in N} f_i\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) = C - \left(\varepsilon E - \varepsilon E^l\right)$, for all $i \in N$,

$$f_i\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) \geq c_i - \left(\varepsilon E - \varepsilon E^l\right).$$

Thus, for all $l \in \mathbb{N}$,

$$c - \left(\varepsilon E - \varepsilon E^l\right) 1_N \leq f\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) \leq c.$$

So, $c = \lim_{l \to \infty} f\left(c, C - \left(\varepsilon E - \varepsilon E^l\right)\right) = c - f(c, \varepsilon E) + E f(c, \varepsilon)$. Hence, $f(c, \varepsilon E) = E f(c, \varepsilon)$ for all $E \in \mathbb{R}_+$ such that $(c, \varepsilon E) \in \mathcal{L}$ and $(c, \varepsilon) \in \mathcal{L}$.

We now prove that f is the proportional rule.

If $c = 0_N$ and $(c, E) \in \mathcal{L}$ then $E = 0$ and $f(0_N, 0) = 0_N$.

Assume that $(c, E) \in \mathcal{L}$ and $c \neq 0_N$. Let $\varepsilon \in \mathbb{Q}_+$ be such that $C > \varepsilon$. Then,

$$f(c, E) = f\left(c, \frac{E}{\varepsilon}\varepsilon\right) = \frac{E}{\varepsilon} f(c, \varepsilon).$$

We know that,

$$c = f\left(c, \frac{C}{\varepsilon}\varepsilon\right) = \frac{C}{\varepsilon} f(c, \varepsilon)$$

which implies that $f(c, \varepsilon) = \frac{\varepsilon}{C} c$. Then,

$$f(c, E) = \frac{E}{\varepsilon}\frac{\varepsilon}{C} c = \frac{E}{C} c = P(c, E).$$

d) Since every bankruptcy rule is a loss rule, d) is an immediate consequence of c). ■

Remark 58 *For $n > 1$, condition $C1$ in a) is needed in order to avoid an infinite family of meaningless solutions. Let $\left(B^l\right)_{l \in \mathbb{L}}$ be a Hamel basis of $\mathbb{R}$ as vector space over $\mathbb{Q}$ with $B^l > 0$ for all $l \in \mathbb{L}$ (see, for instance, Aczél and Dhombres (1989) for a detailed explanation of Hamel bases). Let $\gamma : \mathbb{R}^N_+ \times \mathbb{L} \to \mathbb{R}^N$ be any function satisfying*

$$\sum_{i \in N} \gamma_i(c, l) = B^l \qquad \text{for all}\,(c, l) \in \mathbb{R}^N_+ \times \mathbb{L}.$$

We define f as follows. Given $E \in \mathbb{R}_+$, there exists a unique $\left\{q^1, ..., q^m\right\} \subset \mathbb{Q}$ and $\left\{B^{l_1}, ..., B^{l_m}\right\}$ such that $E = \sum\limits_{k=1}^{m} q^k B^{l_k}$. Thus,

$$f(c, E) = \sum_{k=1}^{m} q^k \gamma(c, l_k)$$

is well-defined, it satisfies $A1$, and $\sum\limits_{i \in N} f_i(c, E) = E$ for all $E \in \mathbb{R}_+$. However, by choosing an appropriate function γ, it is not of the form $f(c, E) = E\alpha(c)$. For example, given $l_0 \in \mathbb{L}$, we define γ as follows

$$\gamma(c, l) = \begin{cases} 1_{\{1\}} B^{l_0} & \text{if } l = l_0 \\ 1_N \frac{B^l}{n} & \text{otherwise.} \end{cases}$$

The associated function f is an allocation rule satisfying $A1$. However, it cannot be written as $f(c, E) = E\alpha(c)$.

Chun (1988) characterizes, in Theorem 4, the class of allocation rules satisfying $A1$, $C2$, SYM[3], and Pareto optimality. Notice that here Pareto optimality is already included in the definition of an allocation rule.

Assume that we restrict ourselves to allocation problems where $C > 0$, as Chun (1988) does. Using arguments similar to those used in the proof of Theorem 57 a), we can conclude the following. An allocation rule satisfy $A1$, $C2$, and SYM if and only if, for all $(c, E) \in \mathcal{A}$

$$f(c, E) = E\alpha(c)$$

where $\alpha : \mathbb{R}_+^N \to \mathbb{R}^N$ is a continuous function satisfying $\sum\limits_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{R}_+^N$ and $\alpha_i(c) = \alpha_j(c)$ whenever $c_i = c_j$.

Of course, these rules coincide with the class of rules characterized in Theorem 4 of Chun (1988), although the formulation is different.

It is straightforward to prove that an allocation rule f satisfies $A1$, $C1$, and SYM if and only if for all $(c, E) \in \mathcal{A}$, $f(c, E) = E\alpha(c)$ where $\alpha : \mathbb{R}_+^N \to \mathbb{R}^N$ satisfies $\sum\limits_{i \in N} \alpha_i(c) = 1$ for all $c \in \mathbb{R}_+^N$ and $\alpha_i(c) = \alpha_j(c)$ whenever $c_i = c_j$.

Notice that in Theorem 57 d) we obtain a new characterization of the proportional bankruptcy rule.

If we compare the rules satisfying $A1$ in the continuous model (Theorem 57) with the rules satisfying $A1$ in the discrete model (Theorem 53) we obtain the following. In AP and SP these rules are, basically, the same. In LP and BP of the discrete model there are no rules but in the continuous model there is only a rule, the proportional rule.

Next theorem characterizes the rules satisfying $A2$.

[3] Chun (1988) uses Anonymity instead of Symmetry. Nevertheless our comments are also valid if we write Anonymity instead of Symmetry.

Theorem 59 *a) An allocation rule f satisfies $A2$ and $C1$ if and only if*

$$f_i(c,E) = \beta_i(c) + Ex_i \text{ for all } i \in N,\ (c,E) \in \mathcal{A}$$

where $\beta : \mathbb{R}^N_+ \to \mathbb{R}^N$ is a function satisfying $\sum_{i\in N} \beta_i(c) = 0$ for all $c \in \mathbb{R}^N_+$ and $\beta(c+c') = \beta(c) + \beta(c')$ for all $c, c' \in \mathbb{R}^N_+$. Moreover, $x \in \mathbb{R}^N$ and $\sum_{i\in N} x_i = 1$.

b) A surplus rule f satisfies $A2$ if and only if it is a weighted-sharing rule WS^ω for some weight system ω.

c) A loss rule f satisfies $A2$ if and only if it is a rights-weighted rule RW^ω for some weight system ω.

d) There is no bankruptcy rule satisfying $A2$.

Proof. a) It is trivial to prove that if $f(c,E) = \beta(c) + Ex$ then f satisfies $A1$ and $C1$.

We now prove the reciprocal. Since f satisfies $A2$ we conclude that $f(c,E) = f(0_N, E) + f(c,0)$.

Using arguments similar to those used in the proof of Theorem 57 a) we can conclude that $f(0_N, E) = Ef(0_N, 1)$ for all $E \in \mathbb{R}_+$.

If we define $\beta(c) = f(c,0)$ and $x = f(0_N, 1)$, the result holds.

b) It is trivial to prove that every weighted-sharing rule WS^ω satisfies $A2$.

We now prove the reciprocal. Assume that f is a surplus rule satisfying $A2$.

Using arguments similar to those used in the proof of Theorem 57 b) we can conclude that given $E \in \mathbb{R}_+$, $f(0_N, E) = Ef(0_N, 1)$.

If we define $\beta(c) = f(c,0)$ and $x = f(0_N, 1)$, every surplus rule has the form given in a). Since $f(c,E) \geq 0_N$ for all $(c,E) \in \mathcal{S}$, $\beta(c) = f(c,0) \geq 0_N$. As $\sum_{i\in N} f_i(c,0) = 0$ we conclude that $f(c,0) = 0_N$. Now it is not difficult to prove that $f = WS^\omega$ where $\omega = f(0_N, 1)$ is a weight system.

c) It is trivial to prove that every rights-weighted rule RW^ω satisfies $A2$.

We now prove the reciprocal. Assume that f is a loss rule satisfying $A2$.

We first prove that for all $i \in N$ and $x \in \mathbb{R}_+$, $f\left(x1_{\{i\}}, 0\right) = xf\left(1_{\{i\}}, 0\right)$.

Using arguments similar to those used in Theorem 57 a) we can conclude that given $x \in \mathbb{Q}_+$ and $i \in N$,

$$f\left(x1_{\{i\}}, 0\right) = xf\left(1_{\{i\}}, 0\right).$$

Let x be a non-negative irrational number. Then, there exists $\left(x^l\right)_{l\in\mathbb{N}}$ such that $x^l \in \mathbb{Q}_+$, $0 < x^l < x$, and $\lim_{l\to\infty} x^l = x$. Thus, for all $i \in N$ and $l \in \mathbb{N}$,

$$f\left(\left(x - x^l\right)1_{\{i\}}, 0\right) = f\left(x1_{\{i\}}, 0\right) - f\left(x^l 1_{\{i\}}, 0\right) = f\left(x1_{\{i\}}, 0\right) - x^l f\left(1_{\{i\}}, 0\right).$$

Since $f\left(\left(x - x^l\right)1_{\{i\}}, 0\right) \leq \left(x - x^l\right)1_{\{i\}}$ and $\sum_{j\in N} f_j\left(\left(x - x^l\right)1_{\{i\}}, 0\right) = 0$, for all $i \in N$ and $l \in \mathbb{N}$,

$$f\left(\left(x - x^l\right)1_{\{i\}}, 0\right) \geq -\left(x - x^l\right)1_N.$$

Thus, for all $i \in N$,

$$-\left(x - x^l\right) 1_N \leq f\left(\left(x - x^l\right) 1_{\{i\}}, 0\right) \leq \left(x - x^l\right) 1_{\{i\}}.$$

Then,

$$0_N = \lim_{l \to \infty} f\left(\left(x - x^l\right) 1_{\{i\}}, 0\right) = f\left(x 1_{\{i\}}, 0\right) - x f\left(1_{\{i\}}, 0\right).$$

Since $f(c,E) \leq c$ and $\sum_{i \in N} f_i(c,E) = E$ for all $(c,E) \in \mathcal{L}$, we conclude that for all $i \in N$, $f\left(x 1_{\{i\}}, x\right) = x 1_{\{i\}}$; and $f\left(1_{\{i\}}, 0\right) = 1_{\{i\}} - y^i$, where $y^i \in \mathbb{R}^N_+$.

We now prove that $y^i = y^j$ for all $i, j \in N$, $i \neq j$. Since f satisfies $A2$,

$$\begin{aligned} f\left(1_{\{i,j\}}, 1\right) &= f\left(1_{\{i\}}, 1\right) + f\left(1_{\{j\}}, 0\right) = 1_{\{i\}} + 1_{\{j\}} - y^j \\ f\left(1_{\{i,j\}}, 1\right) &= f\left(1_{\{i\}}, 0\right) + f\left(1_{\{j\}}, 1\right) = 1_{\{i\}} - y^i + 1_{\{j\}}. \end{aligned}$$

Then, $y^i = y^j$.

Given $i \in N$ we define $\omega = y^i$. It is trivial to see that ω is a weight system.

We now prove that $f = RW^\omega$. Given $(c,E) \in \mathcal{L}$, we consider a partition $\{N_1, \{i\}, N_2\}$ of N such that $c_i = c_i^1 + c_i^2$, $c_i^1 \geq 0$, $c_i^2 \geq 0$, and $E = \sum_{j \in N_1} c_j + c_i^1$. Since f satisfies $A2$,

$$\begin{aligned} f(c,E) &= \sum_{j \in N_1} f\left(c_j 1_{\{j\}}, c_j\right) + f\left(c_i^1 1_{\{i\}}, c_i^1\right) + f\left(c_i^2 1_{\{i\}}, 0\right) + \sum_{j \in N_2} f\left(c_j 1_{\{j\}}, 0\right) \\ &= \sum_{j \in N_1} c_j 1_{\{j\}} + c_i^1 1_{\{i\}} + c_i^2 f\left(1_{\{i\}}, 0\right) + \sum_{j \in N_2} c_j f\left(1_{\{j\}}, 0\right) \\ &= \sum_{j \in N_1} c_j 1_{\{j\}} + c_i^1 1_{\{i\}} + c_i^2 \left(1_{\{i\}} - \omega\right) + \sum_{j \in N_2} c_j \left(1_{\{j\}} - \omega\right). \end{aligned}$$

Now it is straightforward to prove that $f(c,E) = RW^\omega(c,E)$.

d) Using the same example as in Theorem 56 d) we conclude that there is no bankruptcy rule satisfying $A2$. ■

Remark 60 *$C1$ is again needed in Theorem 59 a). We can use the same example as in Remark 58.*

We now compare the results obtained in Theorem 56 and Theorem 59 (the discrete and the continuous case). For AP the rules satisfying $A2$ in the discrete case coincide, basically, with the rules satisfying $A2$ and $C1$ in the continuous case. For SP and LP the statement of Theorem 59 applied to the discrete problem implies the statement of Theorem 56. The reason is that the rules described in Theorem 56 b) and 56 c) can be obtained as a weighted-sharing rule WS^ω and a rights-weighted rule RW^ω where $\omega = 1_{\{i_0\}}$. Moreover, when the weight system ω has a different expression, WS^ω and RW^ω do not induce a discrete rule.

If we compare the rules satisfying $A1$ (Theorem 57) with the rules satisfying $A2$ (Theorem 59) in the four kinds of problems we obtain the following:

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In AP both class of rules are unrelated. Moreover, an allocation rule f satisfies $A1$, $A2$, and $C1$ if and only if for all $(c,E)\in\mathcal{A}$, $f(c,E)=Ex$ where $x\in\mathbb{R}^N$ and $\sum_{i\in N}x_i=1$.

In SP the class of rules satisfying $A2$ is a subset of the class of rules satisfying $A1$ (just take $\omega(c)=\omega$ for all c).

In LP the proportional rule, the only rule satisfying $A1$, does not satisfy $A2$. Moreover, the rights-weighted rules, which satisfy $A2$, do not satisfy $A1$.

In BP the proportional rule, the only rule satisfying $A1$, does not satisfy $A2$.

Notice that in the discrete problem the relations among rules satisfying $A1$ and rules satisfying $A2$ are similar to those described before in the continuous AP and SP, but different in LP and BP.

In Theorem 59 c) we characterize the class of rights-weighted rules in LP. Notice that the rights-weighted rules also satisfy $A2$ and $C1$ in AP (take $\beta_i(c)=c_i-\omega_iC$ and $x_i=\omega_i$ for all $i\in N$). Nevertheless, there are more allocation rules satisfying both properties. Next theorem characterizes the rights-weighted allocation rules as the only allocation rules satisfying $A2$ and CB.

Theorem 61 *An allocation rule f satisfies $A2$ and CB if and only if f is a rights-weighted rule RW^{ω} for some weight system ω.*

Proof. If ω is a weight system it is straightforward to prove that RW^w satisfies $A2$ and CB.

We now prove the reciprocal. Let f be an allocation rule satisfying both properties. We define ω as $f(0_N,1)$. Since f satisfies CB we conclude that ω is a weight system.

Using arguments similar to those used in the proof of Theorem 59 b) we conclude that for all $E\in\mathbb{R}_+$, $f(0_N,E)=Ef(0_N,1)=E\omega$.

Let (c,E) be an allocation problem where $E\leq C$. As f satisfies CB we know that $f(c,C)=c$. Since f satisfies $A2$,

$$\begin{aligned} f(c,E) &= f(c,C)-f(0_N,C-E)\\ &= c-\omega(C-E)\\ &= RW^{\omega}(c,E). \end{aligned}$$

The case $E>C$ is similar to the case $E\leq C$. ■

We now prove that this theorem is a tight characterization result.

ES satisfies $A2$ but not CB.

Given $(c,E)\in\mathcal{A}$ and $i\in N$, we define the allocation rule ψ as

$$\psi_i(c,E)=\begin{cases}\min\{x,c_i\} & if\ C\geq E\\ RE_i(c,E) & if\ C<E\end{cases}$$

where $\sum_{i\in N}\min\{x,c_i\}=E$. The allocation rule ψ satisfies CB but not $A2$.

Finally, we present some characterizations of well known rules using $A1$ and $A2$.

In Theorem 57 c) and d), we characterize the proportional rule in LP and BP as the only rule satisfying $A1$. Nevertheless, in AP and in SP there are more rules satisfying these two properties. Next corollary shows that if we add COM to $A1$ and $C1$ ($A1$), the proportional rule becomes the only rule satisfying these properties in AP (SP). In this proposition we assume that $C > 0$[4].

Corollary 62 *a) The proportional rule is the only allocation rule satisfying $A1$, $C1$, and COM.*

b) The proportional rule is the only surplus rule satisfying $A1$ and COM.

Proof. a) Of course the proportional rule satisfies $A1$, $C1$, and COM.

Let f be an allocation rule satisfying these properties. It is trivial to see that Theorem 57 a) is also true if we restrict to the case $C > 0$. Then we conclude that for any $(c, E) \in \mathcal{A}$, $f(c, E) = E\alpha(c)$ where $\sum_{i \in N} \alpha_i(c) = 1$. Since f satisfies COM, $f(c, C) = c$. Now it is not difficult to conclude that $\alpha(c) = \frac{c}{C}$ and, hence, $f(c, E) = \frac{E}{C}c = P(c, E)$.

b) It is similar to the proof of a). ■

Corollary 62 a) is a tight characterization result. RE satisfies $C1$ and COM but not $A1$. Because of Remark 58 there exist rules satisfying $A1$ and COM but not $C1$. We take the same example but with the function γ defined as follows:

$$\gamma(c, l) = \begin{cases} c\frac{B^l}{C} & \text{if } C's\ B^l\text{-th coordinate is non-zero} \\ 1_N \frac{B^l}{n} & \text{otherwise.} \end{cases}$$

Finally, ES satisfies $A1$ and $C1$ but not COM.

Corollary 62 b) is also a tight characterization result. RE satisfies COM but not $A1$. ES satisfies $A1$ but not COM.

Moulin (1987) characterizes (Theorem 2) the proportional surplus rule using $A1$ and other properties. These properties are completely different from $C1$ and COM, used in Corollary 62 b).

Corollary 62 a) is closely related to Theorem 5 in Chun (1988) where he proved that the proportional rule is the only allocation rule satisfying $A1$, $C2$, and COM[5]. Since $C1$ is weaker than $C2$, Corollary 62 a) implies Theorem 5 in Chun (1988). It is trivial to prove that Corollary 62 a) is also true if we replace COM by CB.

In Theorem 59 b) and c) and in Theorem 61 we characterize the class of weighted-sharing rules and rights-weighted rules in AP, SP, and LP using $A2$ and other properties. Unfortunately, in BP there are no rules satisfying $A2$. Next corollary shows that if we add SYM to the properties used in Theorem 59 and Theorem 61, we can characterize the equal-sharing rule in SP and the rights-egalitarian rule in AP and LP.

[4]In the general case ($C \geq 0$) we obtain that there exist many rules satisfying these properties. Nevertheless, all of them satisfy $f(c, E) = \frac{E}{C}c$ when $C > 0$.

[5]Compatibility is called Exact Clearence in Chun (1988).

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Corollary 63 *a) The rights-egalitarian rule is the only allocation rule satisfying* $A2$, CB, *and* SYM.

b) The equal-sharing rule is the only surplus rule satisfying $A2$ *and* SYM.

c) The rights-egalitarian rule is the only loss rule satisfying $A2$ *and* SYM.

Proof. a) It is clear that RE satisfies these properties.

Let f be an allocation rule satisfying these properties. By Theorem 61 we conclude that there exists a weight system ω such that $f = RW^{\omega}$.

For any $i, j \in N$, $f_i(0_N, 1) = f_j(0_N, 1) = \frac{1}{n}$ because f is symmetric. Since $\omega = f(0_N, 1)$, we conclude that $f = RE$.

b) It is cleat that ES satisfies these properties.

Let f be a surplus rule satisfying these properties. By Theorem 59, there exists a weight system ω such that $f = WS^{\omega}$. Consider the surplus problem $(1_N, E)$, where $E > 0$. Since f satisfies SYM, $f_i(1_N, E) = f_j(1_N, E)$ for all $i, j \in N$. Now it is not difficult to conclude that $\omega_i = \frac{1}{n}$ for all $i \in N$. Hence, $f = ES$.

c) It is similar to the proof of b). ■

Corollary 63 a) is a tight characterization result. ψ satisfies CB and SYM but not $A2$. ES satisfies $A2$ and SYM but not CB. Any RW^{ω} different from RE satisfies $A2$ and CB but not SYM.

Corollary 63 b) is a tight characterization result. P satisfies SYM but not $A2$. Any WS^{ω} different from ES satisfies $A2$ but not SYM.

Corollary 63 c) is a tight characterization result. P satisfies SYM but not $A2$. Any RW^{ω} different from RE satisfies $A2$ but not SYM.

Since CB implies COM and RE satisfies CB, Corollary 63 a) can also be obtained as a consequence of Proposition 3 in Bergantiños and Méndez-Naya (2001), where it is proved that RE is the only allocation rule satisfying $A2$, COM, and SYM.

Moulin (1987) characterizes (Theorem 2) the equal-sharing surplus rule using several properties, which are different from the properties used Corollary 63 b). Nevertheless, Moulin (1987) also uses an additivity property. Moulin uses $A1$ instead of $A2$.

5.5 Concluding remarks

In bankruptcy problems, both $A1$ and $A2$ are very restrictive properties. Only the proportional rule, in the continuous case, satisfies $A1$.

In loss and surplus problems with $A2$ we characterize interesting classes of rules. In the discrete case, we characterize the rules "everything for one player". In the continuous case, we characterize the weigthted versions of "equal loss" and "equal award". In both cases, the rules satisfying $A2$ in loss problems are dual of the rules satisfying $A2$ in surplus problems.

In loss and surplus problems with $A1$ we characterize the proportional rule in loss problems. Notice that the results obtained with $A1$ are not so "homogeneous" as with $A2$. For instance, the comments about duality are not true with $A1$.

In allocation problems there are many rules satisfying both additivity properties. Among these rules we can identify some interesting subclasses, for instance the loss and surplus rules satisfying these additivity properties.

In the continuous case we can characterize well-known rules using these additivity properties. With $A1$ and other properties we characterize the proportional rule in each of the four problems. With $A2$ and other properties we characterize the "egalitarian" rules, the equal-sharing rule in surplus problems and the rights-egalitarian rule in loss and allocation problems. This allows us to say that additivity properties support the use of rules based on three classical principles: $A1$ supports the principle of "proportionality", and $A2$ supports the principles of "equal award" and "equal loss".

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Summary (in Spanish)

Un juego describe una situación conflictiva entre un conjunto de *agentes* o *jugadores*. Aunque estos jugadores tienen intereses independientes, pueden salir todos beneficiados si colaboran. En este sentido, los juegos pueden ser cooperativos o no cooperativos.

En juegos no cooperativos, los jugadores son maximizadores de utilidad; y un juego no cooperativo es el conjunto de reglas que lo describen. Estudiaremos equilibrios de Nash perfectos en subjuegos. En un equilibrio de Nash, los jugadores actúan de tal forma que ninguno de ellos puede beneficiarse al cambiar su estrategia. En un equilibrio perfecto en subjuegos, los jugadores actúan de esta manera en cada subjuego.

En los juegos cooperativos, sin embargo, están permitidos los acuerdos vinculantes. Esto significa que los jugadores pueden formar coaliciones y actuar cooperativamente con el objetivo de maximizar su pago final. Entonces, un juego cooperativo está totalmente determinado por una función característica. Esta función asigna a cada coalición el conjunto de pagos admisibles que pueden ser alcanzados por sus miembros en caso de actuar cooperativamente.

Cuando esta cooperación se produce, la pregunta es cómo se debe distribuir entre los jugadores el pago resultante. Este problema ha sido estudiado de diversas formas. El objetivo es definir una regla o *valor* que proporcione un reparto "justo" (o, al menos, "razonable") para cada problema. Este reparto debe tener en cuenta la contribución de cada jugador al juego.

Dentro de los juegos cooperativos, los juegos de utilidad transferible (o juegos TU) han sido estudiados detalladamente. En los juegos TU, existe una utilidad (por ejemplo, dinero) que es común a todos los jugadores y puede ser libremente distribuida entre los miembros de una coalición. Un valor ampliamente estudiado para juegos TU es el valor de Shapley (presentado por Shapley en 1953). Una clase más extensa de juegos cooperativos son los juegos de hiperplano. En los juegos de hiperplano, el conjunto de pagos admisibles es un semiespacio (esto es, está delimitado por un hiperplano). Finalmente, los juegos sin utilidad transferible (o juegos NTU) son juegos más generales e incluyen a los juegos de hiperplano (y, por tanto, también a los juegos TU).

Decimos que un valor f definido en la clase de juegos NTU *generaliza* el valor g si ambos valores coinciden para juegos TU. Algunas generalizaciones del valor de Shapley para juegos NTU son, por ejemplo, el valor de Shapley NTU

(Aumann, 1985), el valor de Harsanyi (Harsanyi, 1963) y el valor consistente (Maschler y Owen, 1989, 1992).

Shapley (1953) caracteriza su valor utilizando eficiencia, simetría, jugador nulo y aditividad. Eficiencia significa que todos los jugadores cooperan. Simetría significa que dos jugadores simétricos deben recibir lo mismo. Jugador nulo significa que los jugadores que no contribuyen nada al juego no deben recibir nada. Finalmente, aditividad significa que si dividimos el juego cooperativo como la suma de dos nuevos juegos, el valor del juego coincide con la suma de los valores de los estos nuevos juegos.

Myerson (1980) y Hart y Mas Colell (1989) proporcionan otras caracterizaciones del valor de Shapley. Myerson (1980) demuestra que el valor de Shapley es el único valor que verifica eficiencia y contribuciones equilibradas. La contribución de un jugador i con respecto al jugador j es lo que el jugador j gana o pierde cuando el jugador i abandona el juego. Contribuciones equilibradas significa que, dados dos jugadores i, j, la contribución de i con respecto a j es igual a la contribución de j con respecto a i. Hart y Mas Colell (1989) caracterizan el valor de Shapley mediante una propiedad de consistencia. A grandes rasgos, puede decirse que, bajo consistencia, el pago final no cambia si jugamos en un juego reducido, con algunos jugadores fuera del juego. A los jugadores que no están fuera del juego los denominaremos *jugadores activos*.

Para juegos NTU, Maschler y Owen (1989) definen el valor consistente para juegos de hiperplano intentando generalizar la propiedad de consistencia. Esta propiedad no es directamente generalizable a juegos de hiperplano (por tanto, tampoco a juegos NTU). Sin embargo, si consideramos todos los juegos reducidos de dos jugadores, hallamos su valores consistentes, y calculamos su término medio, el valor resultante coincide con el valor consistente del juego original. Maschler y Owen (1989) llaman a esta propiedad *consistencia bilateral*, y demuestran que el valor consistente está caracterizado por eficiencia, simetría, *covarianza* y consistencia bilateral. La propiedad de covarianza significa que si la utilidad de un jugador cambia linealmente y la de los demás jugadores no varía, entonces el valor de este jugador cambia linealmente, mientras que el valor de los demás jugadores permanece constante.

Además, Hart y Mas Colell (1996) demuestran que el valor consistente está caracterizado por eficiencia y una propiedad que generaliza contribuciones equilibradas. En particular, esta propiedad, que llamaremos *contribuciones equilibradas promediadas*, significa que, dado un jugador i y una coalición S con $i \in S$, el promedio de las contribuciones de los jugadores de $S\setminus\{i\}$ con respecto a i, coincide con el promedio de las contribuciones de i con respecto a los jugadores de $S\setminus\{i\}$.

Una vez que tenemos un valor establecido, la implementación de este valor consiste en definir un juego no cooperativo o mecanismo tal que los jugadores, comportándose estratégicamente, obtengan en equilibrio el valor propuesto.

En este contexto, decimos que un mecanismo[6] *implementa* el valor de Shap-

[6]Diremos *mecanismo* en lugar de *juego no cooperativo* para evitar ambigüedades con el término "juego".

ley (o cualquier otro) si se verifican dos condiciones. Primero, debe existir algún tipo de equilibrio tal que su pago final coincida con el valor de Shapley. Segundo, cada equilibrio debe tener como pago final el valor de Shapley. La primera propiedad es necesaria ya que, aunque se demuestre que el valor de Shapley es el pago de cada equilibrio, puede ocurrir que el mecanismo no tenga equilibrios de ese tipo.

La implementación del valor de Shapley ha sido estudiada por varios autores. Por ejemplo, Gul (1989), Hart y Moore (1990), Winter (1994), Hart y Mas Colell (1996), Evans (1996), Dasgupta y Chiu (1998), Pérez Castrillo y Wettstein (2001) o Mutuswami, Pérez Castrillo y Wettstein (2002).

Nos centraremos en las implementaciones de Hart y Mas Colell (1996) y de Pérez Castrillo y Wettstein (2001).

En la implementación de Hart y Mas-Colell (1996), un jugador, escogido al azar, propone un reparto admisible. Si todos los demás jugadores aceptan la oferta, el mecanismo termina con este pago. En caso contrario, el mecanismo se repite con probabilidad $\rho \in [0, 1)$, y con probabilidad $1 - \rho$ el proponente abandona el juego. Para juegos TU monótonos, este mecanismo implementa el valor de Shapley en equilibrios de Nash perfectos en subjuegos estacionarios. Además, para juegos NTU monótonos, el valor consistente aparece como límite de los pagos cuando ρ tiende a 1.

Pérez Castrillo y Wettstein (2001) diseñan un mecanismo de subastas que coincide con el de Hart y Mas Colell para $\rho = 0$ (esto es, el proponente deja el juego de forma segura cuando su oferta es rechazada) después de una etapa previa en la que los jugadores subastan el derecho a ser el proponente. Este mecanismo implementa el valor de Shapley para juegos cero-monótonos en equilibrios de Nash perfectos en subjuegos.

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Estructuras coalicionales

Frecuentemente, los jugadores no son independientes unos de los otros. Es posible que estén exógenamente divididos en coaliciones *a priori*. Esta división, o *estructura coalicional*, puede ser debida, por ejemplo, a afinidades políticas o a razones geográficas. Cuando todos los jugadores pertenecen a una única coalición *a priori*, o cada coalición *a priori* está formada por un único jugador, decimos que la estructura coalicional es *trivial*, ya que no aporta información adicional. Cuando un valor f definido para juegos con estructura coalicional coincide con otro valor g (por ejemplo, el valor de Shapley) para juegos con estructura coalicional trivial, decimos que f *generaliza* g.

El valor de Owen (Owen, 1977) generaliza el valor de Shapley a juegos TU con estructura coalicional. El valor de Owen, al igual que el valor de Shapley, también verifica aditividad.

Además, Winter (1989) caracteriza el valor de Owen como el único valor que verifica eficiencia, simetría individual, covarianza, una variante de consistencia, y una propiedad que llama *propiedad del juego entre coaliciones*. La *simetría individual* significa que jugadores simétricos dentro de la misma coalición *a priori* reciben lo mismo. La consistencia de Winter coincide con la consistencia en juegos sin estructura coalicional, pero refiriéndose solo a juegos reducidos en los que los jugadores activos pertenecen a la misma coalición *a priori*. Llamaremos a esta propiedad *consistencia coalicional*[7]. Por último, la propiedad del juego entre coaliciones está relacionado con el juego cociente que resulta de considerar cada coalición *a priori* como un único jugador.

Por su parte, Calvo, Lasaga y Winter (1996) caracterizan el valor de Owen como el único valor que verifica eficiencia, contribuciones equilibradas entre las coaliciones y contribuciones equilibradas entre los jugadores de la misma coalición.

Por otro lado, Winter (1991) generaliza el valor de Harsanyi para juegos NTU con estructura coalicional.

Cuando existen subdivisiones adicionales de los jugadores dentro de la estructura coalicional, decimos que los jugadores están divididos en una *estructura de niveles*. Por ejemplo, dentro del Parlamento Europeo, los jugadores (parlamentarios) están divididos en partidos políticos. Sin embargo, estos partidos políticos también pueden a su vez estar agrupados en asociaciones más extensas, tales como el Partido Popular Europeo, el Partido Socialista o los Verdes. Una generalización del valor de Shapley para juegos TU con estructura de niveles es el *valor de estructura de niveles*, sugerida por Owen (1977) y caracterizada por Winter (1989).

Problemas de asignación

Los problemas de asignación describen situaciones en las cuales un *bien* debe ser repartido entre varios jugadores que tienen *demandas* sobre él. Tanto el bien como las demandas se suponen cantidades no negativas.

Los problemas de asignación pueden ser considerados como un caso especial de juegos cooperativos, ya que es posible asignar una función característica a cada problema de asignación. En los problemas de asignación, el modo de dividir el bien entre los agentes se llama una *regla*.

Podemos considerar cuatro clases de problemas de asignación. Estos problemas se diferencian en forma en la que debemos dividir el bien. En los *problemas de bancarrota* (presentados por O'Neill, 1982; y estudiados por Aumann y Maschler, 1985) cada agente debe recibir como mínimo 0 y como máximo su demanda. En los *problemas de asignación* generales (Chun, 1988; y Herrero, Maschler y Villar, 1999) no hay restricciones en la cantidad que puede recibir cada jugador. En los *juegos de excedente* (Moulin, 1987) cada jugador debe recibir al menos 0. En *problemas de pérdida*, cada jugador debe recibir como máximo su demanda. Por tanto, estas cuatro clases de problemas cubren todas las posibilidades.

De nuevo, podemos considerar una propiedad de aditividad. Una regla puede ser aditiva en el bien o aditiva en el bien y las demandas. Sin embargo, estas aditividades no están relacionadas con la aditividad del correspondiente juego TU en forma característica.

[7]Winter (1992) la llama simplemente *consistencia*.

Implementación en juegos TU con estructura coalicional

Los capítulos 1 y 2 están dedicados a juegos TU con estructura coalicional.

En el capítulo 1, generalizamos el mecanismo de Pérez Castrillo y Wettstein (2001) de forma que el valor de Owen es implementado. El nuevo mecanismo se llama *mecanismo de subastas coalicional*.

El mecanismo de subastas coalicional tiene dos etapas. En la primera etapa, cada coalición, secuencialmente, debe escoger un representante. Para ello, utilizamos el mecanismo de Pérez Castrillo y Wettstein (2001). Primero, los jugadores se subastan el derecho a ser proponente. Para ello, cada jugador i ofrece una *subasta* b_j^i a cada jugador j perteneciente a su misma coalición. La *subasta neta* de cada jugador se calcula como la diferencia entre la suma de las subastas hechas a los demás jugadores menos la suma de las subastas que los demás jugadores le hacen. Aquel jugador α con la subasta neta mayor es escogido como proponente (en caso de empate, el proponente se escoge al azar entre aquellos con subasta neta máxima). El proponente α debe entonces hacer una oferta y_j^α a cada jugador j perteneciente a la misma coalición. Los jugadores deben entonces votar a favor o en contra de la propuesta. Si todos aceptan, el proponente α se llama *representante* de su coalición y el turno para a la siguiente coalición, que procede a elegir su representante de la misma forma. Continuamos así hasta que todas las coaliciones han elegido sus respectivos representantes, o bien un jugador j_0 vota en contra de la propuesta de su proponente α_0.

En este último caso, el proponente α_0 paga sus subastas $b_j^{\alpha_0}$ y abandona el juego. Los demás jugadores continúan jugando con un jugador menos.

En el primer caso, es decir, si todas las coaliciones han elegido sus respectivos representantes, cada representante α paga $b_j^\alpha + y_j^\alpha$ a cada miembro de su coalición y pasa a la segunda etapa con todos los recursos de su coalición (menos los de los jugadores que han abandonado el juego). Los demás jugadores dejan el mecanismo con estos pagos.

Finalmente, los representantes juegan entre ellos el mecanismo de Pérez Castrillo y Wettstein (2001).

En la proposición 5, se demuestra que el valor de Owen es un pago en equilibrio de Nash perfecto en subjuegos para juegos superaditivos.

Sin embargo, este mecanismo no implementa el valor de Owen. En el ejemplo 6 se muestra que otros pagos aparecen en equilibrio. Por tanto, debemos hacer suposiciones adicionales, bien sobre el propio mecanismo, bien sobre la clase de juegos.

Primero, modificamos el mecanismo, de forma que tanto el jugador que es expulsado como el jugador que vota en contra, deben pagar una pequeña multa $\varepsilon > 0$, tan pequeña como se desee. En el teorema 7 se demuestra que el mecanismo así modificado implementa el valor de Owen para juegos superaditivos.

Además, este resultado se mantiene si la multa debe pagarla sólo uno de los dos jugadores (el que es expulsado o el que vota en contra). También se mantiene si la multa que debe pagar el jugador expulsado depende de la identidad del jugador.

Esta multa puede interpretarse como un pequeño trámite burocrático que el jugador que vota en contra debe hacer para explicar su decisión. A menos que algún jugador realmente disfrute escribiendo papeles, el mecanismo implementa el valor de Owen.

También podemos restringir la clase de juegos. En el teorema 9, se demuestra que el mecanismo original (sin multas) implementa el valor de Owen para juegos estrictamente superaditivos.

En el capítulo 2, extendemos el mecanismo de subastas coalicional a juegos con una estructura de niveles. Si tenemos h niveles, jugaremos h etapas. En cada etapa, cada coalición escoge un representante para la siguiente etapa mediante el mecanismo de subastas, de la misma forma que en el capítulo 1.

De nuevo, el ejemplo 6 nos muestra que puede haber equilibrios cuyo pago final no coincide con el valor de estructura de niveles. En este capítulo, hacemos suposiciones sobre el comportamiento de los jugadores. En concreto, suponemos que los jugadores, en caso de indiferencia, prefieren estrictamente pertenecer a coaliciones grandes en lugar de pequeñas (esta suposición también la hacen Moldovanu y Winter, 1994) y prefieren acabar pronto el juego (suposición también hecha por Hart y Mas Colell, 1996).

En el teorema 13 se demuestra que, bajo estas suposiciones, el mecanismo implementa el valor de estructura de niveles.

Implementación en juegos NTU con estructura coalicional

En los capítulos 3 y 4 se estudian juegos NTU con estructura coalicional.

En el capítulo 3 se define un valor para juegos NTU con estructura coalicional que generaliza el valor consistente y el valor de Owen. Este nuevo valor, llamado *valor consistente coalicional*, verifica muchas propiedades que cabrían esperarse de una generalización de este tipo.

En particular, el valor consistente coalicional verifica eficiencia, contribuciones equilibradas promediadas entre los jugadores dentro de la misma coalición, y contribuciones equilibradas promediadas entre las coaliciones (teorema 21). Además, el valor consistente coalicional está caracterizado por estas tres propiedades (teorema 25).

Por otro lado, el valor consistente coalicional verifica simetría individual y covarianza (teorema 21). Verifica también consistencia bilateral al considerar sólo como jugadores activos coaliciones pertenecientes a la misma coalición *a priori*. Esto es la consistencia de Winter aplicada a juegos con estructura coalicional. Respecto a la propiedad del juego entre coaliciones, no puede ser exportada a juegos de hiperplano (ni, por tanto, a juegos NTU). Sin embargo, la caracterización de Winter puede modificarse sustituyendo la propiedad del juego entre coaliciones por la propiedad de contribuciones equilibradas entre las coaliciones. Así, el valor de Owen es el único valor en la clase de juegos TU que verifica eficiencia, simetría individual, covarianza, consistencia y contribuciones equilibradas entre las coaliciones.

El teorema 23 demuestra que el valor consistente coalicional es el único valor en la clase de juegos de hiperplano que verifica eficiencia, simetría individual, covarianza, consistencia bilateral y contribuciones equilibradas promediadas entre las coaliciones.

Por tanto, desde el punto de vista de la consistencia y de las contribuciones equilibradas, este valor generaliza el valor consistente (a juegos con estructura coalicional) de la misma forma que el valor de Owen generaliza el valor de Shapley. Además, generaliza el valor de Owen (a juegos NTU) de la misma forma que el valor consistente generaliza el valor de Shapley.

En el capítulo 4 se generaliza el mecanismo de Hart y Mas Colell (1996) para juegos NTU con estructura coalicional. El mecanismo tiene dos fases. En la primera fase, al igual que en el capítulo 1, las coaliciones eligen representante de forma secuencial. Un jugador es escogido proponente en la primera coalición, y propone un reparto admisible. Si uno de los demás miembros de la coalición rechaza la oferta, el juego se repite bajo las mismas condiciones con probabilidad $\rho \in [0, 1)$; y con probabilidad $1 - \rho$, el proponente abandona el juego, y el mecanismo se repite con los demás jugadores. Si todos aceptan la propuesta, el turno pasa a la siguiente coalición, y así sucesivamente hasta que todas las coaliciones hayan aceptado su propuesta respectiva. En la segunda fase, se escoge una de las propuestas (por ejemplo, la de la coalición C_q), y se somete a la votación de los jugadores de $N \backslash C_q$. Si todos aceptan la propuesta, el mecanismo termina con estos pagos. Si al menos uno de los jugadores de $N \backslash C_q$ la rechaza, el juego se repite bajo las mismas condiciones con probabilidad ρ; y con probabilidad $1 - \rho$, todos los jugadores de la coalición C_q abandonan el juego, mientras el mecanismo se repite con los demás jugadores.

Bajo ciertas condiciones no muy restrictivas, los pagos en equilibrio de este mecanismo coinciden con el valor consistente coalicional para juegos de hiperplano. Para juegos NTU generales, los pagos en equilibrio se aproximan al valor consistente coalicional cuando ρ tiende a 1.

En la sección 4.4 se estudia una variante de este mecanismo y se caracteriza su equilibrio. El nuevo mecanismo generaliza el de Hart y Mas Colell (1996), pero sin embargo no implementa el valor de Owen.

Aditividad en problemas de asignación

Una forma de abordar los problemas de asignación es mediante la axiomatización de reglas. La idea es proponer propiedades deseables y estudiar cuáles de ellas caracterizan cada regla. A menudo, las propiedades ayudan a comparar diferentes reglas y decidir cuál de ellas es la más apropiada para cada situación particular.

Otra forma es estudiar qué reglas verifican un conjunto de propiedades. Por ejemplo, Young (1988) caracteriza las reglas que verifican continuidad, simetría y consistencia. De Frutos (1999) caracteriza las reglas que verifican no-manipulabilidad. Moulin (2000) caracteriza la reglas que verifican consistencia, composición arriba, composición abajo, e invarianza de escala.

En el capítulo 5, se caracterizan las reglas que verifican aditividad en cada una de las cuatro clases de problemas mencionados anteriormente. Además, por medio de estas propiedades, se caracterizan las reglas basadas en los principios de "proporcionalidad", "igual ganancia" e "igual pérdida".

La propiedad de aditividad ha sido utilizada con frecuencia. Pese a que no es tan justificable como otras propiedades (por ejemplo, eficiencia o simetría), con frecuencia produce clases de reglas muy interesantes. Por ejemplo, el valor de Shapley y el valor se Owen se pueden caracterizar mediante aditividad y otras propiedades. Si se compara el valor de Shapley con otros valores (por ejemplo, el nucleolo), se concluye que estos valores verifican todas las propiedades que caracterizan el valor de Shapley excepto aditividad.

Se consideran dos definiciones de aditividad: aditividad en el bien (Moulin, 1987; y Chun, 1988), denominada $A1$, y aditividad en el bien y las demandas (Bergantiños y Méndez Naya, 2001), denominada $A2$. En las cuatro clases de problemas, se caracterizan las reglas verificando $A1$ y $A2$ en el caso continuo (el bien es perfectamente divisible) y en el caso discreto (el bien está expresado en unidades que no pueden ser divididas).

El teorema 53 caracteriza las reglas que verifican $A1$ en el caso discreto. En problemas de asignación generales, estas reglas están caracterizadas por el producto del bien y un vector que depende de las demandas. En problemas de excedente, todo el bien es para un solo jugador, que es seleccionado a partir del vector de demandas. En problemas de pérdida y de bancarrota, no existen reglas verificando $A1$.

Las reglas verificando $A2$ en el caso discreto vienen dadas en el teorema 56. En problemas de asignación generales, estas reglas están caracterizadas por la suma de dos partes: una dependiente del bien y otra dependiente de las demandas. En problemas de excedente, el bien es para un jugador fijo. En problemas de pérdida, toda la pérdida recae sobre un jugador fijo. En problemas de bancarrota, no existen reglas verificando $A2$.

Las reglas verificando $A1$ en el caso continuo son caracterizadas en el teorema 57. En problemas de asignación generales, estas reglas son parecidas a las del caso discreto. En problemas de excedente, el bien se divide entre los agentes siguiendo una ponderación que depende del vector de demandas. En problemas de pérdida y de bancarrota, el bien se divide proporcionalmente entre los jugadores.

El teorema 59 caracteriza las reglas que verifican $A2$ en el caso continuo. En problemas de asignación generales, estas reglas son parecidas a las del caso discreto. En problemas de excedente (pérdida), el bien (la pérdida) se divide entre los agentes siguiendo una ponderación fija, independiente del vector de demandas. Además, no existen reglas de bancarrota verificando $A2$.

En el caso continuo se obtienen caracterizaciones axiomáticas de reglas conocidas (corolarios 62 y 63). En problemas de asignación generales y en problemas de excedente, repartir el bien de forma proporcional a las demandas es la única regla que verifica $A1$ y otras propiedades. En problemas de asignación generales y en problemas de pérdida, repartir las pérdidas equitativamente entre todos los jugadores (Herrero, Maschler y Villar, 1999) es la única regla que verifica $A2$ y otras propiedades. Además, en problemas de excedente, $A2$ y otras propiedades también caracterizan la regla que consiste que repartir el bien equitativamente entre todos los jugadores (Moulin, 1987).

Como consecuencia de estos resultados, se puede concluir que las propiedades de aditividad apoyan el uso de reglas basadas en tres principios clásicos. La propiedad $A1$ está relacionada con el principio de "proporcionalidad"; y la propiedad $A2$ está relacionada con los principios de "igual ganancia" e "igual pérdida".

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