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Holography, dualities and D-branes

D. José Manuel Penín Ascariz

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Holography, dualities and D-branes

D. Alfonso Vázquez Ramallo

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A mis padres

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Abstract

This thesis is based on the gauge/gravity correspondence. It consists of seven chapters and it can be divided into four different parts. After providing a brief Motivation in which the present work is put in context, the first chapter constitutes an introduction to the topics which will be the matter of study in the rest of the thesis. The first part corresponds to the chapters 2, 3 and 4 and is devoted to the construction of solutions in supergravity duals to a system of D3- and D5-branes intersecting along a (2+1)-dimensional spacetime, in which the backreaction of the D5-branes modelling fundamental matter is taken into account. The second chapter focuses on the case in which the D5-branes model massless fundamental matter, in the third chapter a black hole solution is built and its properties are studied and the fourth chapter deals with the case in which the D5-branes add massive fundamental matter. In the second part, which comprises chapter 5, the spectrum of fluctuations of the class of Gaiotto-Maldacena geometries is obtained. The third part corresponds to chapter 6, and it is centered on the addition of supersymmetric defects to the Brandhuber-Oz geometry, analyzing its supersymmetry and its fluctuations. The fourth part covers the analysis of the integrability of the motion of strings propagating in a class of Massive type IIA supergravity backgrounds found by Cremonesi and Tomasiello, and it is the matter of study of chapter 7. A summary with the main findings in the thesis is also provided.

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Resumen

Esta tesis está basada en la correspondencia gravedad/gauge. Consta de siete capítulos y se puede dividir en cuatro partes diferentes. Tras proporcionar una breve Motivación, en la que se sitúa en contexto el presente trabajo, el primer capítulo constituye una introducción a los tópicos que van a ser materia de estudio en el resto de la tesis. La primera parte se corresponde con los capítulos 2, 3 y 4 y está dedicada a la construcción de soluciones de supergravedad duales a un sistema de D3- y D5-branas que intersecan a lo largo de un espacio-tiempo (2+1)-dimensional, en las que se tiene en cuenta el efecto en la geometría (backreaction) de las D5-branas que modelan la materia fundamental. El segundo capítulo se centra en el caso en el que las D5-branas añaden materia fundamental no masiva, en el tercer capítulo se construye una solución de agujero negro y se estudian sus propiedades y el cuarto capítulo trata el caso en el que las D5-branas modelan materia fundamental masiva. La segunda parte comprende el capítulo 5, y en él se obtiene el espectro de fluctuaciones de la clase de geometrías de Gaiotto-Maldacena. La tercera parte se corresponde con el capítulo 6, y está centrada en la adición de defectos supersimétricos a la geometría de Brandhuber-Oz, analizando su supersimetría y sus fluctuaciones. La cuarta parte cubre el análisis de la integrabilidad del movimiento de cuerdas propagándose en una clase de soluciones de supergravedad tipo IIA masiva construidas por Cremonesi y Tomasiello, y es la materia de estudio del capítulo 7. También se proporciona un resumen

con los principales resultados de la tesis.

Resumo

Esta tese está baseada na correspondencia gravidade/gauge. Consta de sete capítulos e pode dividirse en catro partes diferentes. Tras proporcionar unha breve Motivación, na que se sitúa en contexto o presente traballo, o primeiro capítulo constitúe unha introducción aos tópicos que van ser materia de estudo no resto da tese. A primeira parte correspóndese cos capítulos 2, 3 e 4, e está adicada á construcción de solucións de supergravidade duais ao sistema de D3- e D5-branas que intersecan ao longo dun espazo-tempo (2+1)-dimensional, nas que se ten en conta o efecto na xeometría (backreaction) das D5-branas que engaden a materia fundamental. O segundo capítulo céntrase no caso no que as D5-branas modelan materia fundamental non masiva, no terceiro constrúese unha solución de buraco negro e estúdanse as súas propiedades e o cuarto capítulo trata o caso no que as D5-branas modelan materia fundamental masiva. A segunda parte comprende o capítulo 5, e nel obtense o espectro de fluctuacións da clase de xeometrías de Gaiotto-Maldacena. A terceira parte correspóndese co capítulo 6, e está centrada na adición de defectos supersimétricos á xeometría de Brandhuber-Oz, analizando a súa supersimetría e as súas fluctuacións. A cuarta parte cobre a análise da integrabilidade do movemento de cordas propagándose nunha clase de solucións de supergravidade tipo IIA masiva construidas por Cremonesi e Tomasiello. Tamén se proporciona un resumo cos principais resultados da tese.

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Motivation

The context of this thesis is that of the gauge/gravity duality, which made its appearance in String Theory, and gave a completely new way of understanding the theories of gravity, the quantum field theories and the relations between them. In the forthcoming section, we will give a historical introduction to the subject and contextualize the present work.

Back in the twentieth century, it took place the development and consolidation of suitable candidate theories to explain the four fundamental forces. On the one hand, the physics of gravity was succesfully described using the theory of General Relativity, whose keypoint was to describe the gravitational force as having geometrical origin. On the other hand, the description of the forces between fundamental particles was done by using the framework of quantum field theories (QFT). Originally and succesfully applied in the forties to describe the quantum theory of the electromagnetism via Quantum Electrodynamics (QED), these theories were based on the idea of describing each fundamental particle as a field exhibiting quantum behaviour. In the sixties, Glashow, Weinberg and Salam proposed the Electroweak Model, a QFT that unified the electromagnetic and the weak forces, in which QED emerged as a limit. The nature of the strong interaction was the hardest and latest to be understood, and in the seventies, Quantum Chromodynamics (QCD) appeared as the suitable QFT to account for the nuclear interactions.

It was in the context of trying to tackle the strong interactions where String Theory first appeared. As anticipated, getting a consistent theory for them was a difficult task, mainly due to the large number of particles (hadrons) involved in this interaction. At some stage, even the importance of the role played by QFT in its description was questioned and it appeared the idea of constructing the S-matrix of the theory, which gives the scattering properties of the particles, by just knowing its mathematical properties and little experimental data. Such mathematical properties, and in particular the so-called s-t duality led Veneziano in 1968 [1] and Virasoro in 1969 [2] to propose two different amplitudes to describe the experimental data. It turned out that such amplitudes could be obtained by using a formalism involving an infinite number of harmonic oscillators. This could be understood as a QFT defined in an auxiliary 2-dimensional spacetime, the worldsheet. The physical interpretation of this theory was found independently by Nielsen, Nambu and Susskind, and it was that of a theory of strings. The Nambu-Goto action for the bosonic string was formulated [3] and the Veneziano (Virasoro) amplitude was to be understood as the amplitude of dispersion of two open (closed) strings interacting to produce other two open (closed) strings. String Theory had begun.

This picture had many problems to describe the strong force. Consistency forced the hadrons to live a in a spacetime of 26 dimensions. Also the spectrum of the theory had a tachyon and a massless spin-2 particle, absent in the experimental data. Eventually, it was replaced by QCD to explain the strong force. But the interest about these string models did not cease. In 1971, Ramond incorporated fermionic degrees of freedom to the worldsheet, which originated spacetime fermions. The idea was to construct a worldsheet supersymmetric (SUSY) theory with the hope to eliminate the tachyon from the spectrum.

Simultaneously, Neveu and Schwarz provided a very similar model, distinguished by the boundary conditions of the fermionic modes in the string, but which did not eliminate the tachyon and yielded spacetime bosons. Soon, both models were combined [4] and in 1976 Gliozzi, Scherk and Olive imposed the so-called GSO projection [5] which truncated the spectrum of the resulting superstring theory, removing the tachyon and yielding spacetime SUSY, property that became of fundamental importance. This theory was consistent in a 10-dimensional spacetime.

Simultaneously to this development was the one of constructing different supergravity theories, which came as an attempt to construct supersymmetric theories of gravity, with the purpose of constructing a renormalizable theory of gravity. As we will see, they have a crucial importance in String Theory.

The appearance of the spin-2 particle in the string spectrum, which had the property that at low energies interacted in a smilar way as the graviton did, led to Scherk and Schwarz in 1974 [6] to another intepretation of String Theory: that of a quantum theory of gravity. Thus, by changing the scale of the theory from the strong interaction scale to the Planck scale, this theory could be a good candidate for the unification of the four interactions, as the spectrum had enough room to accomodate matter and gauge bosons. Although String Theory lived in a 10-dimensional spacetime, one could go down to four dimensions by compactifying the extra dimensions.

String Theory proved to be a quite complex scenario. It contained strings, which could be open or closed. But consistency of the open strings called for the existence of extended objects in several dimensions, called D-branes. They emerged as non-perturbative objects to which open strings could be attached. These open strings describe the dynamics of the branes and its massless modes describe a gauge theory on the space they span, the so-called worldvolume. The closed string spectrum is constructed roughly as the tensor product of the modes of the open string spectrum, which can be right- or left-moving. On top of that, the ground state of the fermionic sector of the closed string is described by a spinor that can have two chiralities, which upon the GSO projection yield two different spectra, giving the so-called type IIA (different chiralities for left- and right-moving modes) or type IIB (same chiralities) (super-)String Theory, both having $\mathcal{N} = 2$ SUSY in the massless states. A crucial observation about these string theories was the fact that their massless spectra was the same of two known supergravity theories, namely type IIA supergravity and type IIB supergravity. But this was not all and another construction could be done within String Theory, based on the idea that type IIB super-String Theory enjoys the symmetry of exchanging the right- and left-moving sectors of the theory. Due to this fact, the worldsheet has parity symmetry which reverses the orientation of the string, and we can consider the theory obtained by the quotient of type IIB string by this symmetry. Whereas the previously discussed models consisted of oriented strings, this one is a consistent theory of unoriented strings, called type I superstring.

Yet another two constructions (Gross, Harvey, Martinec and Rohm [7] in 1984) could be done, the so-called heterotic $SO(32)$ and heterotic $E_8 \times E_8$ strings, which exploited the fact that since left- and right-movers are independent, one can combine in a consistent way right-moving modes from the superstring in 10 dimensions and left-moving modes from the 26-dimensional bosonic string, where the extra 16 dimensions are compactified on a lattice. This discovery was the “first superstring revolution”, and the interest of these models was the fact that they had enough room for the standard model gauge group to appear. However, it was proved that there were a huge number of ways to compactify the string, so it was unclear how to derive the standard model from there.

So, by 1995, there were five different consistent superstring theories, and nothing was signaling which one was the right model to focus on. But then, it took place the “second superstring revolution” due mainly to Witten [8], who discovered that all of them were related by a web of dualities. Also, it was discovered another relation of these superstrings with 11-dimensional supergravity (constructed by Cremer, Julia and Scherk [9]). After that, Witten’s insight was to conjecture the existence of a theory in 11 dimensions, which unifies the five different theories, the so-called M-theory.

Not so late, in 1997, it took place one of the most important breakthroughs in the last twenty years: Maldacena proposed the AdS/CFT conjecture [10]. As already mentioned, in addition to the strings, String Theory contains non-perturbative extended objects of different dimensionalities. These objects can be studied from the closed string point of view, as emitters of closed strings, or from the open string point of view, as endpoints to which the strings are attached. Indeed, there is a precise duality in the string partition function that exchanges closed strings and open strings. It is this dual picture that allowed Maldacena to formulate AdS/CFT. The conjecture goes as follows. Taking D3-branes in flat space and after applying a particular decoupling limit, two dualistic pictures of the branes emerge. One is the closed string picture described by type IIB closed String Theory in $AdS_5 \times S^5$ space and the other is the open string picture described by the gauge theory on the worldvolume of the D3-brane, which is the (3+1)-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) gauge theory, which is a conformal field theory (CFT). Both descriptions are thought to be equivalent, in a way in which we will explain in full detail. This was the first known example which realized an idea which was suspected to be true before, that was the existence of a relation between String Theory and gauge theories.

AdS/CFT can be thus understood as a tool which allows us to make predictions in a gauge theory by using a gravity theory and viceversa. This duality is useful since it crucially relates two different regimes of both theories. The large coupling regime in the QFT, which is difficult to tackle since it cannot be adressed perturbatively, corresponds to the supergravity limit of the strings, easy to deal with. However, the small coupling regime in the QFT, corresponds to the quantum regime of the strings in $AdS_5 \times S^5$ spacetime, which we do not know how to address. The counterpart of this dualistic picture is that it is difficult to test the conjecture since one of the sides is always in a regime which is difficult to deal with. This conjecture can be trusted in different levels. The strongest assumption is that type IIB String Theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ in 3+1 dimensions, for any value of the string coupling g_s (related to the gauge theory coupling g_{YM}) and for any value of N. A weaker version of the statement follows from taking the low energy limit: type IIB supergravity+quantum corrections is dual to $\mathcal{N}=4$ SYM in the 't Hooft limit, which follows from taking $g_{YM} \to 0$ and $N \to \infty$ but the 't Hooft coupling $\lambda = g_{YM}^2 N$ kept finite. The weakest version of it, which is the most relevant to this thesis, amounts to take the limit in which the length of the string is zero, and states that type IIB supergravity is dual to $\mathcal{N}=4$ SYM in the limit $\lambda \to \infty$.

This conjecture was generalized in several directions, such as changing the dimensionalities of the theories (leading to the more general case $AdS_{d+1} \times CFT_d$, for a generic dimension d), with different amounts of supersymmetry, or to non-conformal field theories. The duality was applied even outside the framework of String Theory to other gravity theories, the so called bottom-up models, (in opposition to the top-down models in which the gravity theory can be embedded in String Theory) relying on the idea that it is the so-called holographic principle, which we will explain in due time, and not necessary the framework of String Theory, what is behind the relation between a gauge theory and a gravity theory. This leads to a more generic notion of duality which is the gauge/gravity duality.

One interesting generalization of AdS/CFT, came to overcome one drawback of the original conjecture, namely the fact that the matter of the field theory dual is in the adjoint representation of the gauge group. Phenomenologically interesting models involve fundamental matter, such as the quarks in QCD or electrons in a model of condensed matter. How to add fundamental matter to the correspondence was not solved until 2002 by Karch and Katz [11], who introduced probe D7-branes in the geometry of $AdS_5 \times S^5$. By probe, we mean that the brane is introduced as an expectator in the geometry, being its effect or backreaction in it negligible. In the field theory dual, this means that the fundamentals are infinitely massive, non-dynamical degrees of freedom. To solve the backreacted problem, one has to solve the equations of motion of supergravity plus sources, which is a daunting task since it involves the use of delta functions on support of the position of the branes. To bypass this difficulty, one can consider having a continuous distribution of branes, which is the so-called "smearing technique" that avoids the delta functions and notably simplifies the equations of motion.

Among the already mentioned dualities relating the different existent string theories, there is one of relevance to this thesis, which is T-duality and its non-Abelian generalization. It appeared as a symmetry of the string spectrum of a compactified theory on a circle, although it is a true symmetry of String Theory. This duality states the equivalence of a theory compactified on a circle of a given radius with a theory compactified on a circle of a radius inversely proportional to the first one. It esentially acts as a reflection of the right-moving modes of the theory, and due to the GSO projection, its effect in the type II superstrings, which is the one we are interested on in this thesis, is the map between type IIB on S^1 with radius R and type IIA on S^1 with radius $\frac{\alpha'}{R}$ (α' is related to the string scale l_s by $\alpha' = l_s^2$). Its complete effect on the geometry is encoded in the Buscher rules [12, 13]. The non-Abelian T-duality came as a generalization of T-duality to non-Abelian isometry groups [14]. Its rules were worked out originally for the so-called Neveu Schwarz-Neveu Schwarz (NSNS) sector of the theory and they were applied to the sigma models. Eventually, its effect on the Ramond-Ramond (RR) sector was found in 2010, in [15]. This duality is yet not completely understood, in particular its global properties. On top of that, it is not known to be a true symmetry of the whole string partition function, although it can be used in the supergravity limit to generate new solutions of the theory.

T-duality allows us to do another interesting construction. One can consider the combined effect of a parity transformation in type IIA superstrings compactified on a circle with a T-duality transformation. As it happens with the type I string, one can take the quotient of the theory by the combined symmetry, reaching the so called type I' superstring theory. However, to make a consistent theory, one has to add additional D8-branes on top of the generated O8 orientifold planes, which are fixed points under the transformation. The corresponding low energy limit of the theory is the type I' supergravity and the region between the D8-branes is described by the so-called Massive type IIA supergravity [16], which will appear many times in this thesis.

One crucial development in the AdS/CFT correspondence was that of finding integrable structures in the duality. The concept of integrability traces back to hamiltonian dynamics. An integrable system has equations of motion which can be solved for any particular set of initial conditions through quadratures. In AdS/CFT, integrability was discovered when trying to solve the problem of computing the spectrum of anomalous dimensions of operators in $\mathcal{N} = 4$ SYM. This problem could be mapped to that of a Hamiltonian description of a spin chain, which turned out to be integrable, being its dual counterpart the dynamics of a large string moving in the supergravity background. Integrability on the field theory side of the correspondence is related to integrability on the gravity side and it can be studied either from an analytical or from a numerical approach. In this thesis, we will test the integrability of a field theory by means of the study of its dual gravity description.

About this thesis

This thesis is based on the papers [17], [18], [19], [20], [21] and [22] and has the following structure:

- The first chapter is an introduction to the gauge/gravity duality. First we review the basics of supersymmetry and supergravity. Second, we briefly introduce the T-duality symmetry and its non-Abelian generalization. Third, we review the AdS/CFT conjecture and the addition of flavor to it. Fourth, we review the basics of integrability in AdS/CFT. Finally we present the supergravity backgrounds that we will work with in this thesis.

- In the second chapter we construct a type IIB supergravity solution which contains backreacted flavor D5-branes in a background sourced by D3-branes, dual to a defect theory to which we add a hypermultiplet of fundamentals. We focus our attention mainly to the case in which the fundamentals are massless.

- In the third chapter we construct the type IIB supergravity background that generalizes the solution obtained in the second chapter to a case of non-zero temperature in the field theory side. We study the thermodynamics and hydrodynamics of this model.

- In the fourth chapter, we construct the type IIB background which generalizes solution obtained in the second chapter to the case in which the fundamentals introduced by the D5-branes are massive. We study some observables in it such as the Wilson loops or the entanglement entropy.

- In the fifth chapter, we sudy the spin-2 excitations of a class of geometries of type IIA supergravity obtained by Gaiotto and Maldacena, and within this class, we particularize our study to the Abelian and non-Abelian T-dual versions of the $AdS_5 \times S^5$ solution in type IIB supergravity.

- In the sixth chapter we probe the so-called Brandhuber-Oz background in Massive type IIA supergravity with D4- and D6-brane defects in different configurations. We study their fluctuations and identify its field theory duals.

- In the seventh chapter we study the integrability of a class of 6-dimensional $\mathcal{N} = (1, 0)$ SCFT's through the study of the integrability of strings moving on its gravity dual, constructed in a series of papers by Cremonesi and Tomasiello, using both analytical and numerical techniques.

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Introduction

1.1 Basics of Supergravity

1.1.1 Supersymmetry

In the sixties, it arised the question of which kind of symmetries were possible in particle physics. On top of the Poincaré group, defined by the set of generators $\{J_{\mu\nu}, P_\mu\}$ which generate the Lorentz group and the translational symmetries, respectively, one can have internal symmetries T_a describing the particle content, such as the local $SU(3)$ of QCD, which form a Lie algebra of the form $[T_a, T_b] = f_{ab}{}^c T_c$. Whether or not they could be combined into a larger group was studied by Coleman and Mandula, stating the negative in a theorem [23], which had among its assumptions the requirement that the generators formed a Lie algebra. A way of evading this no-go theorem was by requiring instead that they closed under a graded Lie algebra, characterized by having some generators satisfying an anticommuting law, thus being of fermionic nature and being represented by spinors. In schematic representation, the structure of this new symmetry, called supersymmetry (SUSY) is:

$$\begin{aligned} [P, P] &= 0, & [P, J] &= P, \\ [J, J] &= J, & [P, Q^I] &= 0, \\ [J, Q^I] &= Q^I, & \{Q^I, \bar{Q}^J\} &= P\delta^{IJ}, \\ \{Q^I, Q^J\} &= Z^{IJ}, & \{\bar{Q}^I, \bar{Q}^J\} &= Z^{IJ}, \end{aligned}$$

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where P denotes translations, J Lorentz generators, Q^I and $\bar{Q}^J$ SUSY generators and Z^{IJ} are the central charges, which commute with all the other generators. Both spacetime and spinor indices have been omitted, and I, J label the different SUSY generators. Since a spinor field times a boson field gives a spinor field, SUSY has the effect of trading a boson by a fermion. Thus an irreducible representation of the SUSY algebra will contain bosons and fermions, forming a supermultiplet. Indeed, each supermultiplet has the same number of bosons and fermions. Since $[P^2, Q] = 0$, all the particles in the same supermultiplet have the same mass.

SUSY theories are interesting from the theoretical point of view since they have a better UV behaviour, due to the fact that fermions and bosons cancel against each other when running in loops, eliminating divergences in perturbation theory. However, from the experimental point of view, no SUSY particles have been observed to the date. A way to evade this problem is to propose that SUSY could be broken at least at the scale of energies probed by the accelerators. Nevertheless, there are other experimental and theoretical reasons to still be interested in SUSY theories. Focusing on the theoretical side, SUSY theories are also simpler than non SUSY ones, due to the constraints imposed by the symmetry. As it was mentioned in the Motivation, in String Theory, SUSY is also needed to eliminate the tachyon. For a review in SUSY, see for example [24].

1.1.2 Supergravity

Supergravity can be understood in two ways: as a supersymmetric theory of gravity, or as a theory of local supersymmetry. To formulate a theory of supergravity, we must introduce the vielbein formalism. Any curved spacetime is locally flat, which means that locally we must expect the Lorentz invariance of relativity. This can be done manifestly with the use of the vielbein $e^a{}_\mu$ which allows us to write the metric as:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{1.1}$$

In this decomposition, μ, ν are *curved* indices, acted upon general coordinate transformations, whereas a, b are *flat* indices, acted upon by a local Lorentz gauge invariance. Thus, the vielbein is a covector (vector) of a general (local) coordinate transformation. This decomposition is crucial to couple gravity with spinor fields, since while there exist spinor representations of the Lorentz group, there are no finite dimensional spinor representations of the general covariance group. To gauge the Lorentz group, one must introduce a connection to define a covariant derivative D_μ acting on spinors. This object will be the spin-connection ω_μ^{ab}.

To make SUSY local, the spinor parameterizing the transformation must be local: $\epsilon^\alpha \to \epsilon^\alpha(x)$. Gauging the global supersymmetry then calls for a gauge field, which must have both a spinor and a gauge index, since SUSY acts on the spinor index α. Such object is the gravitino $\psi_{\mu\alpha}$, a spin-3/2 field. The effect of SUSY is to transform "gravity" into this gravitino. But the index structure immediately forces the gravity object entering the transformation to be the vielbein, instead of the metric, and the SUSY variations involving vielbein and spinor are schematically represented as:

$$\delta e^a{}_\mu \sim \bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta\psi_\mu \sim D_\mu\epsilon, \tag{1.2}$$

where γ^a is a Dirac gamma matrix. An invariant action under such SUSY transformation contains a kinetic term for the gravitino, the so-called Rarita-Schwinger term, plus the Einstein-Hilbert action. This set of fields, namely the multiplet $(e^a_\mu, \psi_{\mu a})$, defines the $\mathcal{N} = 1$ supergravity multiplet with spins (2, 3/2). Here, $\mathcal{N}$ denotes the amount of supersymmetry, being $\mathcal{N} = 1$ the minimal content.

By putting the fields in curved spacetime, we can also couple gravity with other multiplets: the chiral multiplet of spins (1/2, 0), the gauge multiplet (1, 1/2) and the gravitino multiplet (3/2, 1). By adding appropriate numbers of such multiplets one can obtain supergravity theories with $\mathcal{N} = 1, 2, 3, 4, 8$. $\mathcal{N} > 8$ would involve not well defined fields of spin greater than 2. Such constructions are consistent only for a maximum number of spacetime dimensions, 11. In 11 dimensions, there is a unique $\mathcal{N} = 1$ supergravity theory with fields $(e^a_\mu, \psi_{\mu a}, A_{\mu\nu\rho})$, being the last one an antisymmetric tensor field.

In dimensions lower than 11, many consistent supergravity theories can be constructed, and most of them are obtainable by a Kaluza-Klein (KK) reduction of the 11-dimensional supergravity in some compact space. When performing the reduction, many fields appear, as the indices in the spacetime reduced directions turn into internal indices. Important to this thesis is the fact that 11-dimensional supergravity can be reduced to type IIA supergravity in 10 dimensions. Also, if one knows the compactification ansatz that relates the supergravity theories in different dimensions, one can map solutions in the different theories.

The construction of supergravities is motivated by the fact that they could provide renormalizable theories of gravity, due to the already mentioned boson-fermion loop cancellations. Indeed, as an example, $\mathcal{N} = 8$ has been proven to be free of divergences up to a large number of loops in perturbation theory. Supergravity would also be a unification framework, where the fields of the different theories would appear from KK reduction. Yet other catchy features of supergravity theories is the fact that they arise as low energy limits of String Theory, and 11-dimensional supergravity arises as the low energy limit of the conjectured M-theory, and also appealing is the existence of a web of dualities relating them.

In this thesis, we will deal with type IIA (massive and massless) supergravity, type IIB supergravity, as well as 11-dimensional supergravity. In some cases, we will try to find solutions to their equations of motion, often coupled to branes. Such equations are usually hard to solve, since they are second order differential equations coupling all the fields. Therefore, one looks for strategies to simplify them. One such strategy is to look for supersymmetric solutions. The procedure goes as follows. First, to get a set of manageable equations, one has to write an inspired ansatz for the solutions, which restricts the form of them and leaves as unknowns an unspecified set of functions. Then, one passes them through the supersymmetric variations, which are of the form:

$$\begin{aligned}\delta(\text{fermion}) &= B(\text{boson}),\\ \delta(\text{boson}) &= F(\text{fermion}),\end{aligned} \tag{1.3}$$

being B and F some functions, which depend on the supergravity theory we are dealing with. Since we want a classical supergravity background, we want all the fermion fields to be set to zero. The SUSY variations transform the boson fields into fermion fields. Therefore, to still have a classical background, we require the vanishing of such variations, that is, $B(\text{boson}) = 0$, which always leads to first-order differential equations for the functions entering the ansatz, which are easier to solve than the second-order equations. When trying to solve $B(\text{boson}) = 0$, one usually has to impose some projections on the spinor that parameterizes the SUSY transformation, each of which halves the number of supercharges. Several projections can be imposed, and consistency requires that all of them commute among themselves.

Needless to say that not all the solutions are captured by this technique, but only the SUSY ones. Usually, they represent BPS states which saturate some energy bound and are actually true solutions of the second-order equations of motion. For a good notes on supergravity, see for example [25].

Notation

Throughout this thesis, we will use the following notation. We will use differential forms in the notation with and without indices. A differential p-form is defined as:

$$A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{1.4}$$

Here, $\mu_1, \ldots, \mu_p$ are curved indices, referred to the coordinate basis. We will often go to the tangent space basis, and use flat indices $a, b, \ldots$. Curved indices will be raised and lowered with $g_{\mu\nu}$ and flat indices with η_{ab}:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} e^a e^b, \tag{1.5}$$

being $\eta_{ab} = \text{diag}(-1, +1, \ldots, +1)$ and e^b the vielbein. We will use the spin-connection, defined as $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$. The spin-connection can be obtained by solving Cartan's structure equations:

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{1.6}$$

With this object we define the covariant derivative acting on a spinor:

$$D_\mu \epsilon = (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab}) \epsilon, \tag{1.7}$$

where Γ_a are the Dirac matrices and $\Gamma_{\mu_1 \ldots \mu_n}$ will denote the antisymmetrized product of Γ matrices.

Spinors in different dimensions

In this thesis, we will mainly deal with supergravity in 10 dimensions, although we will briefly make contact with 11-dimensional supergravity. These theories will have a certain amount of supersymmetry, characterized by the spinor field parameterizing the transformation. Some comments about spinors are in order.

A spinor representation of the Lorentz group is associated to a Clifford algebra, which in Minkowski signature reads (it can be generalized to arbitrary signatures):

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}, \tag{1.8}$$

where $\mu, \nu = 0, ..., d-1$ are flat tangent spacetime indices. One can convert them to curved spacetime indices with the help of a d-bein. It can be proven that there exists a representation of this algebra in terms of $n \times n$ complex Dirac matrices Γ_μ with $n = 2^{[\frac{d-1}{2}]}$, being $[\frac{d-1}{2}]$ the integer part of $\frac{d-1}{2}$. Therefore, a spinor has $2^{[\frac{d-1}{2}]}$ complex components. However, several conditions can be imposed to the spinor for certain number of dimensions of the spacetime, each of which halves its degrees of freedom, depending on the existence of matrices imposing such conditions:

-Imposing the reality of the spinors produces (pseudo)-Majorana spinors[1]. This condition can be imposed in $d-1 = 0, 1, 2, 3, 4, m$ mod 8.

-Imposing definite chirality produces Weyl spinors. This can be done when the spacetime dimension is even.

-Combining both conditions, we obtain Majorana-Weyl spinors in $(d-1)$-dimensional spacetimes.

-Furthermore, the degrees of freedom are halved if we require the spinor to satisfy the Dirac equation, which acts as a projection on the spinor.

This counting of degrees of freedom is useful to obtain the number of supercharges in a SUSY theory, given the spacetime dimension, the number of spinors the theory possesses and its type. For a more detailed treatment of spinors and Clifford algebras, see for example chapter 3 in [25].

After this introduction, we will present the different supergravities we will work with in this thesis, their field content and their SUSY variations if they are needed in the chapters to come.

1.2 Supergravity actions

1.2.1 $\mathcal{N} = 1$, 11-dimensional supergravity

We start our discussion with 11-dimensional supergravity, since from it we can construct type IIA supergravity by dimensional reduction and, after performing a T-duality trans-

[1]For simplicity, we will not make a distinction between Majorana and pseudo-Majorana spinors in this brief section.

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formation in the subsequent theory, we can obtain type IIB supergravity.

The existence of 11-dimensional supergravity was predicted by Nahm [26], and its classical action was found by Cremmer, Julia and Scherk (CJS) in 1978 [9]. We will ignore the fermion content of the theory since we will always deal with classical solutions. This theory contains the metric and a 3-form potential A_3 whose corresponding 4-form field strength is: $F_4 = dA_3$. The action for the bosonic fields is:

$$S_{11d} = \frac{1}{2\kappa^2}\int_M d^{11}x\sqrt{-g}(R - \frac{1}{48}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}) + \frac{1}{2\kappa^2}\int_M \frac{1}{6}A_3\wedge F_4\wedge F_4, \tag{1.9}$$

where κ denotes the strength of the gravity interaction and is related to the Newton constant in 11 dimensions. This theory has $\mathcal{N}=1$ SUSY, which in 11 dimensions amounts to have 32 supercharges, corresponding to one Majorana spinor.

1.2.2 Type IIA and Massive type IIA supergravity

In this section, we will present at the same time two theories of supergravity relevant to this thesis. One is type IIA supergravity and the other one is Massive type IIA supergravity. Although they have a different origin, both are quite similar in field content. Indeed, they only differ by a mass term in the action. By setting the so-called Romans' mass (F_0) to zero in the Massive supergravity, we recover the action and equations of motion of type IIA supergravity, so we will present the formulas that correspond to the $F_0 \neq 0$ case, from which the case $F_0 = 0$ directly follows.

Type IIA supergravity can be obtained from the low energy limit of type IIA String Theory, which is connected with 11-dimensional supergravity. Indeed, 11-dimensional supergravity can be dimensionally reduced to type IIA supergravity, yielding a theory with 32 supercharges ($\mathcal{N}=2$) in 10 dimensions. The reduction ansatz for the metric is:

$$ds_{11}^2 = e^{\frac{2}{3}\phi}ds_{10}^2 + e^{\frac{4}{3}\phi}(dz + C_1)^2, \tag{1.10}$$

where ds_{10}^2 (ds_{11}^2) is the metric of the theory in 10 (11) dimensions. One also has to reduce the 11-dimensional 3-form potential, which generates a 10-dimensional 3-form and a 2-form, depending on whether or not the indices of the 3-form are in the reduction direction. As a result, one obtains a theory with a metric, a dilaton ϕ and a RR 1-form C_1 coming from the reduction of the metric, and a NSNS 2-form B_2 and a RR 3-form C_3 coming from the reduction of the 3-form. This reduction ansatz will be relevant to chapter 7.

On the other hand, Massive type IIA supergravity is connected with type I String Theory. One can consider the combined effect of T-dualizing type IIA String Theory and in the resulting type IIB String Theory, do a parity transformation and quotient the theory by this combined symmetry. This produces the so-called type I' String Theory. Under this transformation, there are points in spacetime which remain invariant, which are called O8-orientifold planes, and consistency requires the addition of D8-branes on top of such planes. The low energy limit of this theory is type I' supergravity and the region between the D8-branes is described by Massive type IIA supergravity.

The bosonic part of the action of Massive type IIA supergravity, reads, in the so-called string frame[2]:

$$S_{IIA} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}[e^{-2\phi}(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_3^2) - \frac{1}{2}F_0^2 - \frac{1}{4}F_2^2 - \frac{1}{48}F_4^2] \\ - \frac{1}{4\kappa_{10}^2}\int[dC_3 \wedge dC_3 \wedge B_2 + \frac{F_0}{3}dC_3 \wedge B_2^3 + \frac{F_0^2}{20}B_2^5], \quad (1.11)$$

where we have introduced the corresponding field strengths of the RR and NSNS potentials:

$$\begin{aligned} H_3 &= dB_2, \\ F_2 &= dC_1 + F_0 B_2, \\ F_4 &= dC_3 - H_3 \wedge C_1 + \frac{F_0}{2}B_2 \wedge B_2. \end{aligned} \quad (1.12)$$

The supersymmetry variations for the dilatino λ and gravitino ψ in Einstein frame read [27]:

$$\delta\lambda = [\frac{1}{2}\partial_\mu\Phi\Gamma^\mu + \frac{5}{8}F_0 e^{\frac{5}{4}\phi} - \frac{3}{16}e^{\frac{3}{4}\phi}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma_{11} - \frac{1}{24}e^{-\frac{\phi}{2}}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_{11} + \frac{1}{192}e^{\frac{\phi}{4}}F_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma}]\epsilon,$$

$$\delta\psi_\mu = [\nabla_\mu - \frac{F_0}{32}e^{\frac{5}{4}\phi}\Gamma_\mu - \frac{1}{64}e^{\frac{3\phi}{4}}F_{\nu\rho}({\Gamma_\mu}^{\nu\rho} - 14{\delta_\mu}^{[\nu}\Gamma^{\rho]})\Gamma_{10} + \\ + \frac{1}{96}e^{-\frac{\phi}{2}}F_{\nu\rho\sigma}({\Gamma_\mu}^{\nu\rho\sigma} - 9{\delta_\mu}^{[\nu}\Gamma^{\rho\sigma]})\Gamma_{10} + \frac{e^{\frac{\phi}{4}}}{256}F_{\nu\rho\sigma\gamma}({\Gamma_\mu}^{\nu\rho\sigma\gamma} - \frac{20}{3}{\delta_\mu}^{[\nu}\Gamma^{\rho\sigma\gamma]})]\epsilon, \quad (1.13)$$

where Γ_{11} is the product of all the gamma matrices, and $\mu, \nu, \rho, \sigma, \gamma \in \{0, ..., 9\}$.

1.2.3 Type IIB supergravity

Type IIB supergravity can be obtained from type IIA supergravity after perfoming a T-duality transformation. It cannot be obtained by reduction of 11-dimensional supergravity. It is a chiral theory, since it has two spinors with the same chirality and it has $\mathcal{N} = 2$ SUSY. The bosonic content of the theory is composed by the metric $g_{\mu\nu}$, the dilaton ϕ, a NSNS 2-form B_2 and the RR forms: a 0-form A_0, a 2-form A_2 and a 4-form A_4. The action for these fields (in string frame) is:

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}\chi\sqrt{-g}[e^{-2\phi}(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_3^2) - \frac{1}{2}F_1^2 - \frac{1}{12}F_3^2 - \frac{1}{480}F_5^2] \\ + \frac{1}{4\kappa_{10}^2}\int dA_2 \wedge H_3 \wedge (A_4 + \frac{1}{2}B_2 \wedge A_2). \quad (1.14)$$

[2]Throughout this thesis, we will work in both string g_{str} and Einstein g_E frame indistinctly, which are related by the transformation $g_{E\mu\nu} = e^{-\frac{\phi}{2}}g_{\text{str}\mu\nu}$

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Where we have defined the field strengths:

$$
\begin{aligned}
H_3 &= dB_2,\\
F_1 &= dA_0,\\
F_3 &= dA_2 + C_0 H_3,\\
F_5 &= dA_4 + H_3 \wedge A_2.
\end{aligned}
\tag{1.15}
$$

The equations of motion derived from this action have to be supplemented by a self-duality condition on F_5:

$$
F_5 = \star F_5. \tag{1.16}
$$

The ϵ spinor parameterizing a SUSY transformation is composed by two Majorana-Weyl spinors of well defined 10-dimensional chirality. Such transformations for the dilatino λ and the gravitino ψ_μ, read, in string frame [28]:

$$
\begin{aligned}
\delta_\epsilon \lambda &= [\frac{1}{2}\Gamma^\mu \partial_\mu \phi + \frac{1}{4\cdot 3!} H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_3 - \frac{e^\phi}{2} F_\mu \Gamma^\mu (i\sigma_2) - \frac{e^\phi}{4\cdot 3!} F_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_1]\epsilon,\\
\delta_\epsilon \psi_\mu &= [\nabla_\mu + \frac{1}{4\cdot 2!} H_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3 + \frac{e^\phi}{8}(F_\nu \Gamma^\nu i\sigma_2 + \frac{1}{3!} F_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\sigma_1 + \frac{1}{2\cdot 5!} F_{\nu\rho\sigma\tau\delta}\Gamma^{\nu\rho\sigma\tau\delta} i\sigma_2)\Gamma_\mu]\epsilon,
\end{aligned}
\tag{1.17}
$$

where $\sigma_i, i = 1,2,3$ are the Pauli matrices, and $\mu,\nu,\rho,\sigma,\tau,\delta$ running from $0,...,9$.

1.3 T-duality and non-Abelian T-duality

1.3.1 Abelian T-duality

We will study Abelian T-duality (when dealing with the Abelian case we will refer to it simply as T-duality (ATD) in opposition to the non-Abelian case, which we will refer to as non-Abelian T-duality (NATD)) in the approach which is most useful to our purposes. As already stated, T-duality is a symmetry of String Theory. One can consider a theory with closed strings and do a toroidal compactification. This means that one of the scalars of the worldsheet is periodically identified such that $X^\mu \sim X^\mu + 2\pi R L$, with $L \in \mathbb{Z}$. Thus X^μ parameterizes a 1-dimensional circle of radius R. This identification gives rise to interesting features. First, we see that a new parameter characterizes the compactification, the *winding number* L, which counts how many times the string wraps the compact direction. Requiring single valuedness of the wave function in the mode expansion of the compactified coordinate produces quantization of the internal momentum $p^\mu = \frac{M}{R}$ with $M \in \mathbb{Z}$. The mass spectrum of the resulting closed string remains invariant under the following transformation:

$$
R \to \frac{\alpha'}{R}, \qquad L \leftrightarrow M. \tag{1.18}
$$

This symmetry receives the name of T-duality. Actually, although we have examined its effect in the spectrum, it can be extended to the mode expansion of the string. T-duality is actually a symmetry of the entire String Theory and we can recast it in terms of its effect in the mode expansion, which is to act as an asymmetric $\mathbb{Z}_2$ reflection of the right-moving worldsheet boson X and fermion ψ leaving its left-moving corresponding partner invariant. In other words, labelling the left- (right-) moving sector by the subindex $L(R)$, T-duality acts as:

$$(X_L, X_R) \to (X_L, -X_R), \qquad (\psi_L, \psi_R) \to (\psi_L, -\psi_R). \tag{1.19}$$

This transformation has an immediate effect on the GSO projection, which actually means to exchange type IIB and type IIA superstrings. Summarizing:

$$\text{type IIB on } S^1 \text{ with radius } R \leftrightarrow \text{type IIA on } S^1 \text{ with radius } \frac{\alpha'}{R}. \tag{1.20}$$

This transformation can be generalized to compactifications in higher-dimensional tori, and whereas for an odd number of T-dualities, the type IIA and type IIB theories are exchanged, for an even number of T-dualities they are mapped to themselves.

However, we will be interested in its effect from the point of view of the low energy limit of the superstring, that is, supergravity. There, its effect is defined for a background which has a global $U(1)$ isometry, which leaves invariant the metric and the fields of the solution. T-duality takes a background with such isometry in type IIA (IIB) and maps it to a background with a global isometry in type IIB (IIA). This relation is an involution, that is, by performing two succesive T-dualities we end up with the same background. The conjecture is that T-dual backgrounds are dual, that is, they are two different descriptions of the same physics.

T-duality is given locally by the Buscher rules [12, 13]. They provide the transformation rules for the metric, the dilaton, the NSNS field and the RR fields to go from the solution in one theory to the solution in its dual theory. Therefore, we can understand T-duality as a solution generating technique. We will use such transformation in this thesis, directly in chapter 3 and indirectly in chapter 5.

We can understand the global properties of this transformation as follows. Having the original background a $U(1)$ isometry, it can be equivalently described by a $U(1)$ principal fibration over a 9-manifold. Such kind of fibrations are classified by an integer positive number, the Chern number. If we project the H_3 field over the base of the fibration, we obtain a 2-form field, which upon integration over the base manifold gives an integer number. The effect of T-duality is precisely to exchange these two integer numbers, and therefore the global meaning of the transformation is understood.

1.3.2 Non-Abelian T-duality

As stated before, the input ingredient for T-duality in supergravity is a background with a global $U(1)$ isometry, to which we apply the Buscher rules. But we can consider a further generalization and study the case of having a background with a global G isometry, where G is a Lie group. In particular, the case we will study in this thesis is the one in which $G = SU(2)$. The generalization of this symmetry is called non-Abelian T-duality (NATD).

Locally, there exists a generalization of the Buscher rules to this case [14], which gives the transformation rules for the NSNS fields, metric and dilaton, although the effect on the RR fields was found later in [15]. This transformation is not an involution, since the NATD background does not preserve the isometry after the transformation.

The global properties of NATD are not known. This can be technically rephrased in the statement that the variables of the NATD background are generically non-compact. It is not even known if NATD is in fact a duality, such that it gives two backgrounds which describe the same physics. Neither there exists a statement proving that NATD is a true symmetry in String Theory. However, we can understand it as a solution generating technique in supergravity.

1.4 The AdS/CFT correspondence

The AdS/CFT correspondence was originally conjectured in 1997 by Maldacena in [10] and it states that type IIB String Theory in $AdS_5 \times S^5$ spacetime is dual to a superconformal field theory (SCFT) in 3+1 dimensions, $\mathcal{N} = 4$ super Yang-Mills (SYM).

This duality was obtained after realizing that there exist two different viewpoints to describe a D3-brane. Let us start by considering a stack of N parallel D3-branes in type IIB String Theory placed in (9+1)-dimensional Minkowski spacetime. The degrees of freedom of this system are the open strings attached to the D3-brane and the closed strings propagating in the bulk. The open strings describe the dynamics of the branes, the closed strings the excitations of the empty spacetime, and both degrees of freedom interact. As an example, two open strings on the stack of branes can merge to form a closed string that propagates in the bulk. We can study the system at energies lower than the string scale $1/l_s$. In this regime, only the massless modes of String Theory can be excited. In this low energy limit, the massless states of the open string give the field content of the $\mathcal{N} = 4$ vector supermultiplet in 3+1 dimensions, whose dynamics are described by the $\mathcal{N} = 4$ SYM action with gauge group $SU(N)$ plus higher derivative corrections, corresponding to the low energy limit expansion of the Dirac-Born-Infeld action (DBI), which describes the dynamics of the branes, to be properly discussed in the next section. The $SU(N)$ gauge group arises since the strings attached to the stack have the freedom of having the endpoints in any of the coincident branes in the stack. The closed string states are described by the low energy limit of closed String Theory, which is type IIB supergravity. There are also the interactions between the closed strings and the open strings. Therefore, we can describe the system by the following action:

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{1.21}$$

where S_{bulk} denotes the action describing the bulk system, namely, the action of type IIB supergravity, S_{brane} the action describing the brane excitations and S_{int} describes the interactions between the bulk and the brane. We can now consider the so-called decoupling limit or Maldacena limit which consists in keeping the energy fixed and $\alpha' \equiv l_s^2 \to 0$, keeping fixed N and the string coupling constant g_s. In this limit the interacion term S_{int} vanishes. The higher derivative corrections in the brane action also vanish, thus leaving only the $\mathcal{N} = 4$ SYM action. Therefore, in this decoupling limit we are left with two systems, free supergravity in the bulk and the gauge theory close to the brane.

But there is a second viewpoint to describe the D3-brane. In a sense, one can integrate the stringy effects and simply consider the deformation that the stack of D3-branes induces in the 10-dimensional flat background, in the low energy limit, which is supergravity. For this, one has to compute the solution of the equations of motion of type IIB supergravity with the stack of D3-branes, which is:

$$\begin{aligned} ds^2 &= h(r)^{-\frac{1}{2}}(-dt^2 + dx^2 + dy^2 + dz^2) + h(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_{S^5}^2), \\ F_5 &= (1+\star)d(h(r)^{-1}) \wedge dt \wedge dx \wedge dy \wedge dz, \\ e^{\phi} &= 1, \end{aligned} \tag{1.22}$$

where:

$$h(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s \alpha'^2 N, \tag{1.23}$$

and where all the RR fields but F_5 are zero, since the C_4 naturally couples to the world-volume of the D3-branes. In this metric, the so-called holographic r coordinate is the distance of a point in the 5-dimensional transverse spacetime to the location of the stack.

Now we can study this geometry in its two asymptotic regions. Far away from the brane, in the region $r \to \infty$, the metric becomes flat 10-dimensional spacetime. Close to the brane, in the so-called near horizon limit, in which $r \to 0$, we can approximate the warp factor $h(r)$ by $h(r) = \frac{R^4}{r^4}$, which gives the metric:

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2}(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_{S^5}^2, \\ F_5 &= (1+\star)\frac{4r^3}{R^4}dr \wedge dt \wedge dx \wedge dy \wedge dz, \\ e^{\phi} &= 1, \end{aligned} \tag{1.24}$$

which is the geometry of $AdS_5 \times S^5$, *i.e.*, the direct product of the 5-dimensional anti-de Sitter space and the 5-dimesional sphere, and which is also a solution to the equations of motion. Now we can study the low energy limit of this solution. There are two kinds of excitations of low energy from the point of view of an observer at infinity. Notice that since the g_{tt} component of the metric depends on r, the energy E_r of an object as measured from an observer at r is related to the energy E_∞ measured by an observer at infinity by the warp factor:

$$E_r h(r)^{-\frac{1}{4}} = E_\infty. \tag{1.25}$$

Therefore, an object moving towards $r \to 0$, will have decreasingly energy for an observer at infinity. Thus, we can identify two kinds of excitations. On the one hand, there are excitations propagating in the bulk with large wavelength. These are described by type IIB supergravity in flat space. On the other hand, any excitation coming from full String Theory, close to $r \to 0$, will again be seen with low energy. In the low energy limit, these two kind of excitations decouple. One can see this by checking that the absorption cross-section of waves of large r on the D-branes vanishes. Conversely, any excitation coming from the D3-branes simply cannot climb out the gravitational potential and go off to infinity.

Therefore, we have two descriptions of D3-branes and both lead to two kind of modes that decouple:

- 1st point of view: $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in 3+1 dimensions and 10-dimensional type IIB supergravity in flat space.
- 2nd point of view: type IIB String Theory in $AdS_5 \times S^5$ and 10-dimensional type IIB supergravity in flat space.

It is then natural to identify the different systems. Therefore we arrive to the AdS/CFT conjecture:

-type IIB String Theory in $AdS_5 \times S^5$ is dual to (3+1)-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM.

Before explaining how the duality goes, let us first define the limits of validity of this identification. The Yang-Mills coupling in the gauge theory is related to the string coupling by:

$$g_s = g_{YM}^2. \tag{1.26}$$

The low energy limit of String Theory, giving supergravity, is obtained after setting $\alpha' = l_s^2 \to 0$. But the supergravity approximation is valid if the curvature of the background is large compared to the string length, that is, $R = \sqrt{\alpha'}(g_s N)^{\frac{1}{4}} >> \sqrt{\alpha'}$. Therefore, $g_s N >> 1$, or $g_{YM}^2 N >> 1$. Also, we must require the quantum string corrections, governed by g_s, to be small, thus $g_s \to 0$. To summarize, supergravity is valid if $g_s \to 0$, $N \to \infty$, but $\lambda = g_s N = g_{YM}^2 N$ must be fixed and large. But in an $SU(N)$ gauge theory, at large N, the effective coupling is the 't Hooft coupling, $\lambda \equiv g_{YM}^2 N$. Thus, the supergravity approximation is valid precisely when the 't Hooft coupling is large, which means that we cannot use perturbation theory in the field theory dual. And conversely, when the field theory is weakly coupled, the supergravity approximation is not valid. This is the reason why AdS/CFT is a duality, since both descriptions, gravity and field theory, are valid in opposite regimes. This means that the duality will be hard to check, since we cannot have control of both descriptions in the same regime. But also, this is precisely why it is a powerful tool, since it makes predictions when the usual techniques, such as perturbation theory, break down.

We can consider three different versions of AdS/CFT:

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- The strongest version of the duality states that both the gauge theory and the gravity theory are dual for any value of the parameters g_s and N.

- A weaker version states that (3+1)-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory in the large N limit is dual to type IIB classical String Theory in the limit $g_s \to 0$ on $AdS_5 \times S^5$, but $g_s N$ can be any finite number. This limit is the 't Hooft limit in the gauge theory and it corresponds to a topological expansion of the Feynman diagrams of the field theory in which the coupling constant goes as $\frac{1}{N}$, and in which at leading order, only the planar diagrams contribute. These diagrams map directly to string perturbation diagrams. Indeed, corrections of $\frac{1}{N}$ correspond in the string side to corrections of g_s.

- The weakest version of the duality states that AdS/CFT is valid only at large $g_s N$, where String Theory is just classical supergravity. More precisely: $\mathcal{N} = 4$ $SU(N)$ SYM theory in large N limit and large λ is dual to classical type IIB supergravity on $AdS_5 \times S^5$. This is actually the version of the duality that we will use in this thesis (although we will use its generalization in the most generic context of the gauge/gravity duality, dealing with other than type IIB supergravity theories and other than AdS spacetimes).

Now we have to understand how the duality goes. Given the content of both theories, we expect a relation between the excitations of the supergravity fields and the operators in the field theory dual. This is indeed the case, and we will elaborate the map between both excitations in the following.

Let us first study the field theory dual. The lagrangian for $\mathcal{N} = 4$ SYM can be obtained from a simpler theory in 10 dimensions. Indeed, one can start with $\mathcal{N} = 1$ SYM theory in 10 dimensions, which contains a gauge field and a 32-component spinor. One can then perform a Kaluza-Klein reduction in this theory by compactification on a T^6 torus. This compactification preserves all the supercharges and gives $\mathcal{N} = 4$ SYM theory in 4 dimensions. When performing this reduction, the components of the gauge field in the reduced directions become scalar fields which preserve $SO(6)$ R-symmetry and this has the interpretation of the rotation symmetry in the extra compact dimensions. The resulting theory, as a consequence of nonrenormalization theorems, is a (super)conformal field theory. In a conformal field theory, we will have a set of conformal operators, which are characterized by having a conformal dimension Δ which completely determines their behaviour under conformal transformations. Therefore, one particular operator $\mathcal{O}$, in $\mathcal{N} = 4$ SYM, will be characterized by a conformal dimension Δ and a representation index I_n for the $SO(6) = SU(4)$ symmetry. The conformal group of this theory is $SO(2, 4)$. Such operator can be mapped to a field in the gravity dual.

In the gravity side we will restrict ourselves to the supergravity limit. Then we have supergravity on $AdS_5 \times S^5$. We can perform a Kaluza-Klein reduction on it. This means that we expand the supergravity fields in a complete set of functions on the sphere (the spherical harmonics). Therefore the corresponding field will have the form, in the simplest case of a scalar field:

$$\phi(x,y) = \sum_n \sum_{I_n} \phi_{(n)}^{I_n}(x) Y_{(n)}^{I_n}(y), \tag{1.27}$$

where n is the level, x a coordinate in AdS_5, y a coordinate in S^5 and I_n an index in a representation of the symmetry group, which is $SO(2,4) \times SO(6)$, since the symmetry group of the AdS_5 space is $SO(2,4)$ and the symmetry group of the sphere is $SO(6)$. We immediately see here that the symmetry groups of the supergravity fields match the symmetry groups of the operators in the field theory dual. These bosonic symmetries actually match when they are extended to the full supergroups on both sides of the duality. This supergroup is $SU(2,2|4)$, whose maximal bosonic subgroup is $SO(2,4) \times SO(6)$.

The identification between excitations then follows: the field $\phi_{(n)}^{I_n}$ living in AdS_5, of mass m corresponds to a gauge invariant operator $\mathcal{O}_{(n)}^{I_n}$ in the field theory dual of conformal dimension Δ. The relation between m and Δ is:

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}. \tag{1.28}$$

The level n is related to fields with increasing mass and, by the above equation, to operators with increasing conformal dimension. One can then check if this identification makes sense by trying to find supergravity excitations having a mass that satisfies the above relation. This is an evidence of the AdS/CFT conjecture and indeed one can construct gauge invariant operators in the field theory dual with a given Δ that depends on the number of fields involved and its dimension and find that there indeed exist supergravity excitations which have a mass that satisfies the above relation with Δ. At this point, we can be worried about the renormalization of Δ, but the large amount of symmetry protects against quantum corrections. This expression can be also generalized to fields with a different spin.

We can even provide a formula which summarizes AdS/CFT, which we will particularize here to the case of having a scalar field. Consider a scalar field in AdS_5, $\hat{\phi}(x,r)$, to which we want to associate an operator $\mathcal{O}$ in the field theory. The asymptotic dependence, close to the boundary $r \to \infty$, where the field theory lives, of $\hat{\phi}$ on the radial coordinate is given by:

$$\hat{\phi} \sim r^{\Delta-4}, \qquad \hat{\phi} \sim r^{-\Delta}, \tag{1.29}$$

being $\Delta = 2 + \sqrt{4 + m^2 R^2}$ and m the mass in 5-dimensions. Then, the first solution is dominant and we can write:

$$\hat{\phi}(x,r) = r^{\Delta-4} \hat{\phi}_\infty(x), \tag{1.30}$$

where $\hat{\phi}_\infty(x)$ is a function living on the boundary and is defined only up to conformal transformations, under which it has conformal dimension $\Delta - 4$. The relation between the field $\hat{\phi}$ and the operator $\mathcal{O}$ follows by identifying the generating functional of correlators of $\mathcal{O}$ with the 5-dimensional action $S_{AdS_5}(\hat{\phi})$ restricted to the solutions of $\hat{\phi}$ with boundary value $\hat{\phi}_\infty(x)$:

$$\langle e^{\int \phi \mathcal{O}} \rangle = e^{-S_{AdS_5}(\hat{\phi})}, \tag{1.31}$$

This relation was first introduced in [29] and [30] and receives the name of the GKPW prescription. Although we did it here for a scalar field, this relation can be extended to fields with a different spin.

The AdS/CFT duality realizes the idea of the holographic principle, originally proposed by 't Hooft and interpreted in String Theory by Susskind [31]. As the entropy of a black hole is proportional to the area of its horizon, we are lead to conjecture that the information of the bulk must be entirely encoded in the boundary. AdS/CFT indeed provides a relation between a gravity theory in a given dimension with a gauge theory in a dimension lower by one unit, since the gauge theory in this case lives on the boundary of the AdS space, being the map between the degrees of freedom the one explained before. Yet other arguments concerning non-triviality of the scattering matrix of a given field theory required that, in case that there exists an equivalence between a gauge theory and a gravity theory, this must occur in different dimensionalities of both theories. There are more arguments that pointed through the relation of a gauge theory with a gravity theory, which we will not discuss here.

In this conjectured correspondence, the role of the radial or holographic coordinate of AdS is that of a renormalization scale in the gauge theory. Moving in the holographic coordinate has the interpretation of a flow of the renormalization group. In this interpretation, the couplings of the field theory in the UV are to be identified with the values of the bulk supergravity fields in the boundary, given by $r \to \infty$.

This duality was further developed in several ways. The first obvious direction was to consider the most general case of $AdS_{d+1} \times CFT_d$ for a generic dimension d, which is realized by having a geometry sourced by branes of different dimensionality. Other directions included changing the compact space S^5 to spaces different than the sphere, which modifies the R-symmetry of the field theory dual. The duality was also applied to non-conformal field theories or to theories with a different amount of supersymmetry. And yet the more striking generalization was that of applying it outside the framework of String Theory, to the so-called bottom-up models, where the gravity theory is not embedded in String Theory, in opposition to the top-down models in which the gravity theory can be embedded in String Theory. The advantage of the top-down models is that the field theory dual can be guessed by having knowledge of the brane setup realizing the near horizon geometry. For good reviews on the subject, see [32].

One relevant generalization which concerns to this thesis is that of adding fundamental matter to the correspondence, which will be the matter of study of the next section.

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1.5 Addition of flavor to the correspondence

The original formulation of the AdS/CFT correspondence contains only fields transforming in the adjoint representation of the gauge group $SU(N)$. However, phenomenologically interesting models should contain matter fields in the fundamental representation (flavor), like the quarks in QCD or the electrons in condensed matter.

Flavor brane construction

To add flavor to the correspondence, one has to add flavor branes to it. The recipe was first explained in [11], and will be briefly reviewed here. Consider a stack of N_c D3-branes, which we will refer to as color branes, which generate the geometry that corresponds to $\mathcal{N} = 4$ SYM $SU(N)$ gauge theory. We can then add a stack of N_f flavor D7-branes to this geometry. These branes will extend along the same directions as the D3-branes and also along 4 transverse spatial directions. This modifies the number of degrees of freedom that we have. On the one hand, we have the closed strings that propagate in the bulk. On the other hand, we have three different types of open strings: those with both ends on D3-branes (3-3 strings), with both ends on D7-branes (7-7 strings), or with one end on a D3-brane and the other on a D7-brane (3-7 strings). We can then consider the low energy limit of the system. The lightest modes of these branes after quantization will lead to field theories in the D3- and in the D7-branes, in the case in which the strings have both endpoints in the same type of brane, which realizes matter transforming in the adjoint representation of the gauge group. However, (3-7) and (7-3) strings give rise to bifundamental matter. More precisely, after quantization we will have:

- $SU(N_c)$ gauge theory on the D3-branes with adjoint matter transforming in the $\mathbf{N_c} \otimes \bar{\mathbf{N}}_\mathbf{c} = \mathbf{1} \oplus (\mathbf{N_c^2} - \mathbf{1})$[3]
- $SU(N_f)$ gauge theory on the D7-branes with adjoint matter transforming in the $\mathbf{N_f} \otimes \bar{\mathbf{N}}_\mathbf{f} = \mathbf{1} \oplus (\mathbf{N_f^2} - \mathbf{1})$
- bifundamental matter $\mathbf{N_c} \otimes \bar{\mathbf{N}}_\mathbf{f}$ and $\mathbf{N_f} \otimes \bar{\mathbf{N}}_\mathbf{c}$ with (3-7) and (7-3) strings

Now we can consider the near horizon limit for the D3-branes. The lightest modes of the 3-3 strings will give $\mathcal{N} = 4$ SYM with gauge group $SU(N_c)$ with adjoint matter, as in the original AdS/CFT correspondence. The lightest modes of the 3-7 and the 7-3 strings give rise to a $\mathcal{N} = 2$ hypermultiplet of matter transforming in the bifundamental representation of $SU(N_c) \times SU(N_f)$. The gauge group of the D7-brane becomes a global flavor group, in the gauge side, and the 7-7 strings become matter in the adjoint representation of $SU(N_f)$.

[3]For $SU(N)$ with $N \geq 3$ there are two fundamental representations, $\mathbf{N}$ and $\bar{\mathbf{N}}$, for $SU(2)$ there is only one and $U(1)$ is not semisimple, so there is no definition of fundamental representation. We will assume $N \geq 3$.

Finding the geometry that corresponds to this setup will be in principle difficult, since after the addition of the D7-branes, the near horizon geometry is no longer $AdS_5 \times S^5$. We can consider the 't Hooft limit in which $N_f << N_c$. In this approximation, the geometry is not modified by the presence of the D7-branes, which can then be treated as probe branes, and we can ignore their backreaction on the geometry. Therefore, the dynamics of the D7-branes will be simply described by the DBI action, to be discussed soon. This has a precise meaning in the field theory dual, and it translates in modelling infinitely massive, non-dynamical quarks. We refer to this regime as the "quenched approximation". For a review on the holographic quenched flavor, see [33].

This was the first construction proposed to add fundamental matter. However, throughout this thesis, we will be interested in other models, such as the D3-D5 system, the D4-NS5-D6 system, the D6-NS5-D8 system or the Brandhuber-Oz geometry to which we will add flavor defects. The details of each particular setup will be discussed in due time, although the general features they have are similar to the already described D3-D7 model.

Unquenched approximation

In the regime in which N_f is comparable to the number N_c, the probe approximation is no longer a good description, since the effect of the flavor branes in the geometry will no longer be ignorable.

In the case in which the number of flavor branes, N_f, is of the same order of N_c, we will be interested in the so-called Veneziano limit [34], in which $N_c, N_f \to \infty$ but $\frac{N_f}{N_c}$ remains finite. This case captures more dynamics in the field theory dual. The fundamentals are no longer infinitely massive, have dynamics and can run in loops. In the gravity theory, this means that we have to take into account the dynamics of the flavor branes, dictated by the DBI+Wess-Zumino action. The DBI action governs the dynamics of the "shape" and the gauge field of the brane, whereas the Wess-Zumino (WZ) part appears due to the fact that the brane is naturally charged under the RR potentials of the supergravity theory. Therefore, to describe the full backreacted system of flavor branes, we must consider the following action:

$$S = S_{IIB} + S_{DBI} + S_{WZ}. \tag{1.32}$$

Solving the equations of motion for this action is usually a hard task. In practice, we will simplify the problem by restricting ourselves to supersymmetric solutions. In that case, under an appropriate ansatz for the supergravity fields, instead of dealing with the full set of second-order equations coming from the above action, it is enough with solving the first-order supersymmetric variations of supergravity. The effect of the backreacted flavor will be precisely encoded in the ansatz we will use.

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The smearing technique

Solving the problem of the backreacted flavor branes proves to be a daunting task. Each brane will enter the equations of motion as a localized source, which will be described by a delta function whose support is the position of the brane, which will make the problem hard to solve. The smearing technique, first used in [35], comes to overcome this difficulty and amounts to consider a delocalized set of branes sources. Mimicking electrodynamics, when we substitute a point charge by a continuous distribution of charge, in this case, we consider a continuous distribution of branes. This eliminates the delta functions in the equations of motion, which undoubtely simplifies the problem. However, this procedure is not innocuous and since the flavor branes are no longer coincident, the flavor group will be broken down to a less symmetric group. Nevertheless, this procedure usually preserves supersymmetry. Also, since by doing this replacement in the source distribution we are eventually solving different problems, the smearing will have an effect in the solution, which can be regarded as an advantadge or a disadvantage, as we will see in this thesis. The smearing technique was succesfully applied to obtain geometries with flavor backreaction in several systems [36–42]. For a review of the different backgrounds constructed through this technique, see [43].

1.6 D-brane action

As already mentioned in the Motivation, consistency of the open string calls for the existence of extended objects to which the open strings are attached. We can reason this by several ways, for instance, noticing that Dirichlet boundary conditions produce a flow of momentum off the endpoints of the string, which must go somewhere. This somewhere is actually a dynamical hyperplane called a D-brane. One can get information about the nature of this object and its excitations by studying its connection with the open string spectrum. Recall that the bosonic open string spectrum in 10 dimensions is that of a vector multiplet and in the low energy limit, the effective field theory for this open string spectrum quantized with Neumann boundary conditions in flat space is $\mathcal{N} = 1$ SYM. This theory contains in the bosonic sector a gauge field. When requiring Dirichlet boundary conditions in some directions, the zero modes of the open string along these directions are non-dynamical, so the low energy fields corresponding to the massless modes do not depend on these directions. Unsuprinsingly, the low energy effective theory describing these strings is then the Kaluza-Klein reduction of the $\mathcal{N} = 1$ SYM. Recall that the components of the gauge field in the reduced directions become scalar fields, which are actually the transverse directions to the brane. The remaining components of the gauge field give us a gauge theory in the worldvolume of the brane. The endpoints of the string act as sources for the gauge field.

Therefore, we must find an action which gives this theory in the low energy limit. In String Theory, the requirement of conformal invariance of the non-linear sigma model describing the propagation of an open string with Dirichlet boundary conditions in a supergravity background gives a set of equations which can be derived from the following action for a Dp-brane (in string frame):

$$S_{DBI} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{-\phi} \sqrt{-\det(g + \mathcal{F})}. \tag{1.33}$$

This is the Dirac-Born-Infeld (DBI) action. T_p is the tension of the Dp-brane, which is given by:

$$T_p = \frac{1}{(2\pi)^p g_s \alpha'^{\frac{p+1}{2}}}. \tag{1.34}$$

This indeed shows that such an object has non-perturbative nature in String Theory, since its action is proportional to g_s^{-1}. Indeed, from a worldsheet point of view, it can be constructed as a coherent state of infinite oscillator operators. In (1.33), Σ_{p+1} is the worldvolume of the brane, described by the set of coordinates ξ^a, $\{a = 0, ..., p\}$ and g is the induced metric on the Dp-brane, which is obtained through the pullback of the ambient 10-dimensional metric G_{MN} to the worldvolume:

$$g_{ab} = G_{MN} \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b}. \tag{1.35}$$

One can make use of the worldvolume and spacetime diffeomorphism invariance to go to a gauge in which the worldvolume of the Dp-brane is aligned with the first $p+1$ spacetime coordinates. In this gauge, the embedding of the Dp-brane is given by the functions $X^m(\xi^a)$, where X^m, $m = \{1, ..., 9-p\}$, are the spacetime coordinates transverse to the Dp-brane. The induced metric for a Dp-brane, in flat space, then reads:

$$g_{ab} = \eta_{ab} + \delta_{mn} \frac{\partial X^m}{\partial \xi^a} \frac{\partial X^n}{\partial \xi^b}. \tag{1.36}$$

The 2-form $\mathcal{F}$ is given by:

$$\mathcal{F}_{ab} = P[B]_{ab} + 2\pi\alpha' F_{ab}, \tag{1.37}$$

where $F = dA$, being A the gauge field living on the worldvolume of the Dp-brane, B is the NSNS 2-form potential, and $P[...]$ denotes the pullback to the worldvolume. This action describes the massless open string modes, namely, the gauge field A_a and the scalars X^m, but also the closed string modes, since the brane is coupled to G_{MN}, B_{MN} and ϕ. This action realizes an abelian $U(1)$ gauge theory, for the case of a single brane. By having a stack of coincident branes, the symmetry group is enhanced to $SU(N)$, since the endpoints of the string can be in any of the coincident branes. After making an expansion in α' and keeping the leading order, we obtain the bosonic part of the SYM, $SU(N)$ action, which reads:

$$S_{DBI} \approx -\int_{\Sigma_{p+1}} d^{p+1}\xi e^{-\phi} [T_p + \frac{1}{g_{YM}^2} (\frac{1}{2} F_{ab} F^{ab} + \partial_a \phi^m \partial^a \phi^m)], \tag{1.38}$$

where we have defined $\phi^m = \frac{1}{2\pi\alpha'}X^m$, and used the fact that $g_{YM}^2 = 2(2\pi)^{p-2}g_s\alpha'^{\frac{p-3}{2}}$. By taking the fermionic superpartners coming from the fermionic completion of the action for the super D-brane, we indeed recover the full supersymmetric theory. Indeed, for the case in which $p = 3$, this reproduces $\mathcal{N} = 4$ SYM $SU(N)$ theory, the one appearing in the original Maldacena conjecture.

As already stated, the brane also couples to the RR potentials through its worldvolume. The action to describe this coupling mimics the case of the coupling of a charge to an electric field in electromagnetism. Therefore, one can expect a term of the form $\sim \int_{\Sigma_{p+1}} C_{p+1}$. The full action describing this coupling is the WZ action. By using T-duality arguments, one can show that it has the form:

$$S_{WZ} = T_p \int_{\Sigma_{p+1}} \sum_{r=0}^{p+1} P[C_{(r)}] \wedge e^{\mathcal{F}}. \tag{1.39}$$

In the integrand we have to make the series expansion of $e^{\mathcal{F}}$, which produces a polyform, but the integrand only picks the terms proportional to the volume form of the worldvolume.

Therefore, the full DBI+WZ action, giving the total brane action, is given by (in string frame):

$$S_{D_p} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{-\det(g+\mathcal{F})} \pm T_p \int_{\Sigma_{p+1}} \sum_{r=0}^{p+1} P[C_{(r)}] \wedge e^{\mathcal{F}}, \tag{1.40}$$

where the + (-) in front of the second term corresponds to a Dp-brane (anti-Dp-brane). For a more detailed treatment of the topic see, for example, [44].

1.6.1 Kappa symmetry

The worldvolume theory exhibits a fermionic gauge symmetry called kappa symmetry. This symmetry plays an important role in the covariant formulation of superstrings and supermembranes [45]. In the present thesis, it will be used as a technique to find supersymmetric embeddings of probe D-branes.

We can understand the bosonic Dp-brane as a map $X^m(\Sigma_{p+1})$, being Σ_{p+1} the worldvolume of the brane and X^m the coordinates of the 10-dimensional target spacetime. Being in a theory of superstrings, we want to make the brane a supersymmetric object. To proceed, let us replace this map by a supermap $\{Z^M\} = (X^m, \theta^\alpha)$, where the θ^α is a fermionic degree of freedom. The bosonic fields are then replaced by the corresponding superfields. In the following we will use the indices $M, N, ...$ for the superspace coordinates and the indices $m, n, ...$ are the bosonic coordinates of the 10-dimensional spacetime. This allows us to construct the coordinate basis dZ^M, in which we can expand the superspace forms, or in the basis of 1-forms: $E^{\underline{M}} = E^{\underline{M}}_N dZ^N$, where the underlined indices are flat and $E^{\underline{M}}_N$ is a supervielbein. $E^{\underline{M}}$ decomposes under the action of the Lorentz group into a vector $E^{\underline{m}}$ and a spinor $E^{\underline{\alpha}}$, which, for the type IIA String Theory is a 32-component Majorana spinor and for the type IIB is a doublet of chiral Majorana spinors.

For flat superspace one can write:

$$E^m = dX^m + \bar{\theta}\Gamma^m d\theta, \qquad\qquad E^\alpha = d\theta^\alpha, \tag{1.41}$$

where the indices are flat. With these definitions, we can generalize the DBI+WZ action for a super Dp-brane as:

$$S = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-\det(g+\mathcal{F})} + T_p \int_{\Sigma_{p+1}} \sum_{r=0}^{p+1} P[C_{(r)}] e^{\mathcal{F}}, \tag{1.42}$$

with:

$$\begin{aligned} g_{ab} &= E_a^{\underline{m}} E_b^{\underline{n}} \eta_{\underline{mn}}, \\ \mathcal{F}_{ab} &= F_{ab} - E_a^{\underline{M}} E_b^{\underline{N}} B_{\underline{MN}}, \\ C_{(r)} &= \frac{1}{r!} dZ^{M_1} \wedge ... \wedge dZ^{M_r} C_{M_1....M_r}, \end{aligned} \tag{1.43}$$

where a, b run over the worldvolume coordinates, F is the field strength of the worldvolume gauge field $F = dA$[4], B is the NSNS 2-form potential superfield, $C_{(r)}$ is the RR r-form gauge potential superfield and g_{ab} is the induced metric on the worldvolume, obtained through the pullback of the supervielbein to the worldvolume:

$$E_a^{\underline{m}} = \partial_a Z^M E_M^{\underline{m}}. \tag{1.44}$$

The action (1.42) is invariant under the local fermionic transformations:

$$\begin{aligned} \delta_\kappa E^{\underline{m}} &= 0, \\ \delta_\kappa E^{\underline{\alpha}} &= (1+\Gamma_\kappa)\kappa, \end{aligned} \tag{1.45}$$

with $\delta_\kappa E^{\underline{M}} = (\delta_\kappa Z^N) E_N^{\underline{M}}$. This expression generalizes the kappa symmetry variations of the flat superspace coordinates to curved spaces:

$$\delta_\kappa \theta = (1+\Gamma_\kappa)\kappa, \qquad \delta_\kappa X^m = \bar{\theta}\Gamma^m \delta_\kappa \theta. \tag{1.46}$$

We can check that these transformations fulfill (1.45). In terms of the induced gamma matrices $\gamma_a = E_a^{\underline{m}} \Gamma_{\underline{m}}$, the matrix Γ_κ takes the form:

$$\Gamma_\kappa = \frac{1}{\sqrt{-\det(g+\mathcal{F})}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \gamma^{a_1 b_1 \ldots a_n b_n} \mathcal{F}_{a_1 b_1} \ldots \mathcal{F}_{a_n b_n} J_{(p)}^{(n)}, \tag{1.47}$$

[4]We have made F dimensionless with respect to (1.40) by the redefinition $2\pi\alpha' F \to F$

where $J^{(n)}_{(p)}$ is the following matrix:

$$J^{(n)}_{(p)} = \begin{cases} (\Gamma_{11})^{n+\frac{p-2}{2}}\Gamma_{(0)} & \text{(IIA)} \\ (-1)^n(\sigma_3)^{n+\frac{p-3}{2}} i\sigma_2 \otimes \Gamma_{(0)} & \text{(IIB)}, \end{cases} \tag{1.48}$$

with $\Gamma_{(0)}$ being:

$$\Gamma_{(0)} = \frac{1}{(p+1)!}\epsilon^{a_1 \ldots a_{p+1}}\gamma_{a_1 \ldots a_{p+1}}, \tag{1.49}$$

and $\gamma_{a_1 \ldots a_{p+1}}$ denotes the antisymmetrized product of the induced gamma matrices. In equation (1.48) σ_2 and σ_3 are Pauli matrices that act on the two Majorana-Weyl components (represented as a 2-dimensional vector) of the type IIB spinors.

Upon gauge fixing, this symmetry eliminates the extra fermionic degrees of freedom of the worldvolume theory, which guarantees the equality of the fermionic and bosonic degrees of freedom on the worldvolume. The counting of the degrees of freedom goes as follows. For the bosons, we have $9-p$ degrees of freedom coming from the transverse scalars, and $p-1$ physical degrees of freedom coming from the worldvolume gauge field. For the fermions, we have 32 spinors θ that are halved by the equation of motion. But by gauge fixing the local kappa symmetry we also halve the spinorial degrees of freedom, which results in 8 physical spinors.

We will use kappa symmetry throughout this thesis to find supersymmetric embeddings of Dp-branes. We will look for bosonic configurations, without fermions, thus $\theta = 0$, thus we need the variations of θ up to linear order terms in θ. The supersymmetry plus kappa symmmetry transformations read:

$$\delta\theta = \epsilon + (1+\Gamma_\kappa)\kappa, \tag{1.50}$$

with ϵ being the SUSY variation parameter. Generically, these transformations do not leave θ invariant, but one can choose κ in such a way that $\delta\theta = 0$. This is equivalent to gauge fixing the kappa symmetry. Let us impose the following gauge fixing condition:

$$\mathcal{P}\theta = 0, \tag{1.51}$$

with $\mathcal{P}$ a field independent projector, which satisfies $\mathcal{P}^2 = 1$, such that the non-vanishing components of θ are given by $(1-\mathcal{P})\theta$. The condition for preserving the gauge fixing condition $\mathcal{P}\delta\theta = 0$ yields:

$$\mathcal{P}\delta\theta = \mathcal{P}\epsilon + \mathcal{P}(1+\Gamma_\kappa)\kappa = 0, \tag{1.52}$$

which determines $\kappa(\epsilon)$. After gauge fixing the kappa symmetry, the transformation becomes a global SUSY transformation. The condition of unbroken SUSY for the non-vanishing components of θ, given by $(1-\mathcal{P})\theta$ reads:

$$(1-\mathcal{P})\delta\theta = (1-\mathcal{P})\epsilon + (1-\mathcal{P})(1+\Gamma_\kappa)\kappa(\epsilon) = \epsilon + (1+\Gamma_\kappa)\kappa(\epsilon) = 0, \tag{1.53}$$

where in the last equality we used (1.52). If we multiply the last equality by $(1-\Gamma_\kappa)$, we get:

$$(1-\Gamma_\kappa)\epsilon = 0. \tag{1.54}$$

The number of supersymmetries preserved by the brane is given by the number of solutions to this equation. It follows from the definition of Γ_κ that it depends on the first derivatives of the embedding through the induced metric and the pullback of the B field. Of course, for backgrounds of reduced supersymmetry, the target space SUSY parameter ϵ, must be substituted by the so-called Killing spinor of the 10-dimensional geometry where the Dp-brane is embedded, which has a reduced number of supersymmetries. The Killing spinor is the spinor which solves the supersymmetric variations and which may have less number of supersymmetries than the generic SUSY parameter, since the presence of branes usually translate in the existence of projections that this spinor has to satisfy, each of which halves the number of supersymmetries.

This symmetry will be used as a tool to compute supersymmetric embeddings in chapters 4 and 6 of the present work.

1.7 Domains of applicability of the gauge/gravity duality

The gauge/gravity duality had become a very powerful tool to study quantum field theories at large coupling. Outside the weak coupling regime, treated with perturbative techniques, very few tools could be used to deal with quantum field theories. These tools were lattice techniques, which ultimately relied on the availability of powerful computational resources, integrability, which we will briefly introduce in the next section, and localization, which is a method to exactly compute correlation functions in certain supersymmetric quantum field theories providing full non-perturbative information. Therefore, the gauge/gravity duality provides a new framework to address the non-perturbative regime in a broad class of quantum field theories.

As explained before, the original appearance of the correspondence is in the context of String Theory. There, the logic to construct the pair gravity and field theory is to first construct the brane setup that engineers the particular field theory and then to replace some of the branes by their backreaction and to look at the near horizon geometry. These are the so-called top-down models. But as already explained, this duality was extended to the point of considering gravity theories not embedded in String Theory, the so-called bottom-up models, with a less clear field theory dual.

Despite this promising landscape, to the moment, the gravity duals of the main phenomenological theories in nature have not been constructed yet. We only have at our disposal gravity duals of highly symmetric theories. The reason why the duality is still useful is because one expects that it will allow us to extract universal features. One can

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consider several field theories with a known gravity dual that share some features with a given phenomenological model, and extract some universal behaviour from them. One can then conjecture that the phenomenological model indeed displays such behaviour. In this sense, the expectation for the gauge/gravity duality is to provide qualitatively, rather than quantitatively good results. In this spirit, the duality has been used in several contexts, some of which will be listed below:

- QCD: the gravity dual of QCD has not been found yet, and it seems to be a challenging problem. A close model is the Sakai-Sugimoto-Witten model [46]. One should talk in general about the AdS/QCD correspondence to refer to the program to describe QCD in terms of a dual gravity theory. Using models that share some properties with QCD, there were several attempts to understand the different aspects of theory, such as its phase diagram [47], the physics of collisions in the accelerators [48] or the particle spectrum [49], among other properties.

- Condensed matter: the AdS/CFT correspondence found many applications in condensed matter. One should talk in general about the AdS/CMT (condense matter theory) to describe the program to apply String Theory to condense matter systems. As an example, within this framework it was found a way to describe the transition of a superfluid to an insulator. Other models were proposed to describe Weyl semimetals, Kondo models, fractional topological insulators... For a review in the subject, see [50].

- Cosmology: since cosmological models are in fact gravitational theories, one can apply holography to them and derive results of the cosmological observables by studying the corresponding field theoretical quantities [51].

- Quantum information theory: Ryu and Takayanagi gave a famous prescription to compute the entanglement entropy in a quantum field theory [52], which amounts to consider the problem of finding a minimal surface in the gravity dual and it became exploited in many subsequent papers.

- Hydrodynamics: we should also mention the fluid/gravity duality as the framework to study hydrodynamics in a field theory by means of the gravity dual. It was in this context were one of the famous successes of the correspondence was found, namely, the computation of the shear viscosity and the derivation of the Kotvun, Son, Starinets (KSS) lower bound for it [53], which turned out to agree with the experimental measurement of the shear viscosity of the quark-gluon plasma of QCD, one of the phases of the phase diagram of QCD, measured at RHIC. For a good review on the subject, see [54].

- Astrophysics: in the spirit of the previous item, the correspondence found an application in the study of the phases of the highly coupled matter in the cores of neutron stars. As an example [55].

- Non-equilibrium systems: dealing with real time evolution of a gravity system usually involves advanced numerical techniques, which gave rise to a new branch of the duality, known as numerical holography. See for example [56, 57].

In this thesis, we will touch several of the above applications, such as the study of the hydrodynamics of a given background in chapter 3 or the computation of the entanglement entropies in chapter 4.

1.8 Integrability in the gauge/gravity duality

The notion of integrability was first introduced by Liouville in the nineteenth century in the context of classical mechanics as a framework to characterize the cases where the equations of motion could be solved by quadratures. A $2n$-dimensional dynamical Hamiltonian system is Liouville integrable if it possesses n independent conserved quantities which have the mathematical property of being in involution. Then, Liouville's theorem demonstrates that its equations of motion can be solved by quadratures. Very few systems turned out to pertain to this class, and the subject stayed dormant until the second half of the twentieth century when the concept of integrability was rephrased with the invention of the Classical Inverse Scattering Method and the appearance of the Lax formulation. This last technique is based on the idea of finding a pair of matrices L, M (the Lax pair), through which the equations of motion can be written as: $\partial_t L = [M, L]$. This pair allows for an easy construction of conserved quantities, which are characterized by the eigenvalues of the matrix L, which allow one to construct the "spectral curve". The main ingredients of integrability are then the group theory involved in the commutator $[M, L]$ and the study of the spectral curve. The Lax pairs are not unique and one can even endow them with an extra degree of freedom, the spectral parameter, that usually provides a larger set of conserved quantities. This Lax formulation is an essential tool in the study of integrable field theories. In (1+1)-dimensional field theories, the Lax equation can be rephrased as a zero curvature condition, and to find the conserved charges, one has to construct another object, the "monodromy matrix". The requirement that the conserved charges are in involution is eventually connected with the famous Yang-Baxter equation. The fact that in two dimensions the conformal group is infinite dimensional pointed out a connection between conformal symmetry and integrability, and in 1989, Zamolodchikov found that many integrable theories in 1+1 (or 2+0) dimensions could be understood as perturbations of conformal theories. Integrability can also be generalized to higher dimensional field theories.

Integrability appeared in the AdS/CFT correspondence through the computation of the spectrum of the planar one-loop anomalous dimensions of local operators in the gauge theory. It can be constructed a one-to-one map between local operators and states of a certain quantum spin chain. The operator which measures the anomalous dimensions corresponds to the Hamiltonian of the spin chain in this picture. It turned out that this Hamiltonian was indeed integrable, and the model can be understood as a generalization of the Heisenberg spin chain. The spectrum can be solved with the so-called Bethe ansatz, and agrees with the expectation from perturbative gauge theory. The dual picture that emerges is that of a long chain of spins that is mapped to the motion of a long string on the String Theory side. Away from planarity, there still exists integrability in some models describing the equivalent longe-range interactions between spin sites. In the gravity side, integrability rephrases as the integrability of the equations of motion of the string obtained from the non-linear 2-dimensional sigma model in the AdS space, a property that also appears on other coset spaces. Integrability then finds a simple formulation in a family of flat connections on the worldsheet and its monodromy around the closed worldsheet.

Integrability stands out to be a very powerful technique with many applications. It can be used for instance to solve the AdS/CFT spectrum in the planar limit. In both gauge and gravity models, there is a 1-dimensional space (spin chain, string) on which some particles (magnons, excitations) propagate. By symmetry and integrability one can derive how they scatter and can obtain a complete and exact description of the spectrum, even for a finite window of the coupling λ. But there were several attempts to extend integrability to other observables beyond the planar spectrum and to other than $AdS_5 \times S^5$ spaces, with and without success. We will mention some developments. For instance, many deformations of $\mathcal{N} = 4$ SYM turned out to preserve integrable structures. Also, other models like $\mathcal{N} = 6$ Chern-Simons theory, with dual given by type IIA superstrings on $AdS_4 \times CP^3$ were found to display integrability. This symmetry also helped and simplified the study of scattering amplitudes in $\mathcal{N} = 4$ SYM. For a review in the topic, explaining in detail the previous results, see for example [58].

Finding integrability is a rather difficult task. However, in some situations, it turns out to be easier to prove that a system is non-integrable. In this thesis, we will prove the non-integrability of a general class of quantum field theories in chapter 7. We will study the integrability on the gravity side. With the holographic duality in mind, this amounts to find a string soliton, whose motion is dictated by the 2-dimensional sigma model in the gravity background in which it propagates, that captures the motion of a long operator in the dual field theory and show that its dynamics is non-integrable. There are several analytical and numerical techniques to prove non-integrability, given the hamiltonian system in question. The ones used in this thesis will be explained in full detail in chapter 7.

1.9 The D3-D5 system

As explained before, one way to introduce fundamental matter in the AdS/CFT correspondence is through the addition of flavor D7-branes. However, this is not the unique configuration realizing fundamental matter. Indeed, another example are the geometries obtained by adding flavor D5-branes to the D3-brane geometry. The D3-D5 system will be the subject of study of chapters 2, 3 and 4 of this thesis. In the following, we will present these geometries.

The setup of infinite D3- and D5-branes partially overlapping, which has 4-dimensional as well as 3-dimensional dynamics, was studied in [59]. There it was considered a system of N D3-branes filling the directions $\{x^0, x^1, x^2, x^6\}$, intersecting a D5-brane spanning $\{x^0, x^1, x^2, x^3, x^4, x^5\}$, with all branes sitting at the origin of the transverse coordinates. The addition of the D5-brane to the D3-brane background, breaks the supersymmetries and the symmetries between the coordinates in the original background, preserving $SO(2,1)$ symmetry in $\{x^0, x^1, x^2\}$ and $SO(3) \times SO(3) \sim SU(2)_H \times SU(2)_V$ acting on $\{x^3, x^4, x^5\}$ and $\{x^7, x^8, x^9\}$, and preserving 8 supercharges.

There are four kind of string excitations in this system. Closed strings propagating in the bulk produce the fields of type IIB supergravity. 3-3 and 5-5 open strings generate 16-supercharge vector multiplets on the D3- and D5-brane respectively, which split into a vector multiplet and a hypermultiplet under the preserved 8 supercharges. Lastly, 3-5 strings localized on the (2+1)-dimensional intersection of the branes produce a 3-dimensional hypermultiplet, transforming in the fundamental representation of the gauge group of each brane.

Let us now apply the original procedure of Maldacena to this setup. Again, we take $g_s << 1$, $N >> 1$ with $\lambda \equiv g_s N$ fixed. When $g_s N << 1$, the appropriate description of the branes is as flat hypersurfaces. Taking the limit $l_s \to 0$, we decouple the modes of the heavy D5-branes and this produces (3+1)-dimensional $\mathcal{N} = 4$ SYM throughout most of the space, but with a (2+1)-dimensional defect containing a localized interacting fundamental hypermultiplet. In the other regime, $g_s N >> 1$, the appropriate description of the D3-branes is as a supergravity background. However, we still have $g_s << 1$, which means that a single D5-brane must still be described as a hypersurface with propagating open strings. Taking again the limit $l_s \to 0$, we end up with the usual $AdS_5 \times S^5$ near-horizon geometry of the D3-brane with an embedded probe D5-brane. Therefore we have splitted the string modes into two sets: on the gravity side we have closed strings and 5-5 strings whereas the low energy limit of the 3-3 and 3-5 open strings produces the field theory. One then conjectures that both systems are duals of one another.

One can now see that the D5-brane lives on an $AdS_4 \times S^4$ submanifold of $AdS_5 \times S^5$. The isometries of the metric can be mapped to the symmetries of the field theory dual. The $SU(2)_V \times SU(2)_H$ is the R-symmetry of the field theory and $SO(3,2)$ is the 3-dimensional conformal group, suggesting that the field theory should be conformal and contain 8 supercharges. Including the superconformal enhancement to 16 supercharges, it happens that the expectation for the field theory group is actually $OSp(4|4)$.

The field theory action of this system was also obtained in [59] (see also [60,61]), and it can be completely determined using gauge invariance, knowing the preserved supersymmetry and the R-symmetry. This system is actually one of the best studied in holography. The mass spectra of the meson operators in the quenched approximation was first found in [62], by looking at the fluctuations of a probe D5-brane in the $AdS_5 \times S^5$ background. Considered at non-zero temperature, charge density and magnetic field, it was shown to display a rich phase diagram [63–65]. It has also been used to model the quantum Hall effect [66,67] as a holographic model of graphene [68] and it has appeared in the context of bubbling geometries [69].

In this thesis, we construct the supergravity duals to this system that take into account the backreaction of the flavor D5-branes. In chapter 2 we construct the backreacted solutions that include massless fundamentals, and make a first attempt to include massive fundamentals. In chapter 3 we consider the case of having the field theory dual at non-zero temperature. In chapter 4 we consider a backreacted geometry with massive quarks. In all the above generalizations, the price we have to pay to take into account the backreaction of the D5-branes is to modify the compact space, which also modifies the R-symmetry of the field theory dual. We will be concerned about these and other technicalities in the corresponding chapters.

1.10 Gaiotto-Maldacena geometries

Another interesting class of quantum field theories in 4 dimensions are the $\mathcal{N}=2$ theories that arise by the intersection of NS5, D6 and D4 stacks of branes in type IIA String Theory. The gravity duals of these setups will be relevant to our thesis. In chapter 5, we will study the fluctuations of the corresponding supergravity solutions. In what follows, we will describe the brane configurations leading to these theories and the corresponding gravity dual.

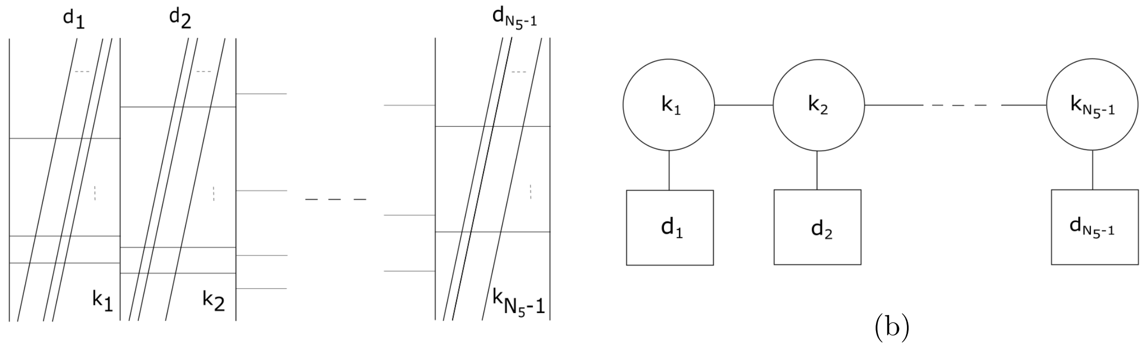
Let us first describe the brane setup [70]. Consider configurations of N_5 NS5-branes filling $\{x^0, x^1, x^2, x^3, x^4, x^5\}$, k_n D4-branes, with $n = 1, ..., N_5 - 1$, spanning the directions $\{x^0, x^1, x^2, x^3, x^6\}$, stretched between the n-th and the $(n+1)$-th NS5-branes and intersecting a stack of d_n D6-branes extended along $\{x^0, x^1, x^2, x^3, x^7, x^8, x^9\}$ in 10-dimensional Minkowski spacetime. We have four kinds of open string excitations in these configurations, to which we can analyze its low energy limit:

- k_n coincident D4-branes allow for 4-4 open strings which turn out to give $\mathcal{N}=2$ SYM theory with gauge group $U(k_n)$. The $U(1)$ factor decouples at low energies due to the coupling to the NS5-branes. The total gauge coupling of the configuration is $\Pi_{n=1}^{N_5-1} SU(k_n)$.

- 4-4 strings between the k_n and the k_{n-1} stack of D4-branes ending on the n-th stack of NS5-branes give a massless hypermultiplet in the bifundamental representation $(\mathbf{k_{n-1}}, \bar{\mathbf{k}}_\mathbf{n})$.

- 4-6 strings stretched between the k_n D4-branes and the d_n D6-branes contribute d_n hypermultiplets in the fundamental representation $\mathbf{k_n}$.

The field theory that corresponds to this configuration is 4-dimensional $\mathcal{N}=2$ gauge theory and can be represented schematically by a quiver diagram as depicted in figure 1.1b. In this representation, circular nodes denote special unitary gauge groups $SU(k_n)$, horizontal lines linking consecutive circular nodes signal bifundamental hypermultiplets, square nodes denote a unitary global symmetry group $U(d_n)$ and a line which connects square and circular nodes represents d_n hypermultiplets in the fundamental representation of $SU(k_n)$. Depending on the particular brane content of the arrangement, one can



(a) (b)

Figure 1.1: Description of a generic brane configuration and the corresponding quiver. In the figure on the left, horizontal lines denote D4-branes, vertical lines NS5-branes and diagonal lines D6-branes. In the figure on the right, the corresponding quiver is depicted.

generate each particular field theory dual. If we require conformality, we must impose a relation between the coefficients d_n and k_n. Further details of the setup can be found in [70].

In the conformal case, these theories arise as a compactification of M5-branes on a punctured Riemann surface [71]. One can now expect that the String Theory backgrounds dual to these SCFT's are given by the near horizon limit of the depicted brane configurations. This limit replaces the D4-branes by their backreacted background, leaving the NS5- and D6-branes as branes in the background. One should expect a background in type IIA String Theory. The bosonic symmetry of the field theory duals $SO(2,4)\times SU(2)_R\times U(1)_R$ requires the dual metric to be a product $AdS_5\times S^2\times S^1\times\Sigma'$, with the other spaces warped over a Riemann surface Σ'. The NS5-branes would wrap an $AdS_5\times S^1$ subspace and the D6-branes an $AdS_5\times S^2$ subspace of this. Depending on the strength of the string coupling, one may be forced to lift this backgrounds to M-theory, instead of working in 10-dimensional supergravity. But for some regime of the parameters, one can indeed provide good holographic duals in type IIA supergravity with these symmetries.

The general solutions in 11-dimensional supergravity preseving the $\mathcal{N}=2$ supersymmetry, with $SO(2,4)\times SO(3)\times U(1)$ isometry were found in [72] and are given by the so-called Lin-Lunin-Maldacena geometries. A solution of these geometries with no sources is given in terms of a single function which satisfies the Toda equation, that completely specifies the metric and all the supergravity fields in 11-dimensional supergravity. This equation is a second-order partial differential equation with three independent variables, one of which is the M-theory circle, and is in principle hard to solve. If we also allow for M5-brane sources, one must modify the Toda equation by adding a source term. We will restrict ourselves to the sourceless case. The solution can be formally reduced to type IIA supergravity if it is independent of some compact internal direction. One can look for

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solutions with such an isometry. Since we have sources, this means that we have to smear our sources along the isometry directions, so the solutions will not correspond directly to physical solutions with localized sources. However, in some range of parameters it turns out that the smeared solutions still provide sensible results. After this assumptions and changing variables in the resulting Toda equation, one ends up with a class of geometries in type IIA supergravity which are specified by a function that solves the axially symmetric Laplace equation in 3 dimensions, the so-called Gaiotto-Maldacena solutions [73]. Generic solutions for this class of geometries and some particular interesting examples were studied in [70, 74, 75] and [76]. We will focus our attention in these geometries, and in chapter 5, we will perform a study of their fluctuations, focusing on finding the spin-2 spectrum.

1.11 The Brandhuber-Oz background

As seen in the previous sections, the gauge/gravity duality allows to have more exotic constructions realizing field theories with matter localized on defects, giving rise to defect quantum field theories coupled with the ambient quantum field theory. As explained, this can be realized with the addition of extra branes intersecting the original stack of branes giving rise to the geometry of the background. In this thesis, we will be particularly interested in realizing such constructions in other geometry, namely, the Brandhuber-Oz solution in type I' String Theory. In the following, we will briefly introduce this background as well as its field theory dual. In chapter 6, we will add defects to this geometry.

The context in which this geometry appears is in the construction of quantum field theories in (4+1)-dimensional spacetimes. QFT's in dimensions larger than 4 have become very important in the modern approach to SUSY QFT's, since they play the role of building blocks which, upon compactification on appropriate surfaces, give rise to virtually all other lower dimensional QFT's. This provides a powerful and deep approach to SUSY QFT's, which, in particular, allows to understand relations among theories. The mere existence of QFT's in dimension larger than 4 is actually remarkable. For instance, gauge theories in 5 dimensions seem a priori uninteresting. Their gauge coupling is irrelevant, which makes them to be not power counting renormalizable and therefore do not seem to be consistent QFT's by themselves. Nevertheless, as first argued in [77, 78], under some favorable circumstances, 5-dimensional gauge theories may be understood as the endpoint of renormalization group flows triggered from UV fixed points where a 5-dimensional SCFT realizing the unique 5-dimensional $F(4)$ superalgebra lives. Also, combinations of String Theory techniques (in particular 5-brane webs in type IIB String Theory [79, 80], as well as exact computations relying on localization, such as the index in [81]), have shown the existence of a wide class of 5-dimensional SCFT's which are intrinsically strongly coupled, some of which admit a mass-deformation triggering a flow into an IR 5-dimensional gauge theory. In parallel, the AdS/CFT correspondence provides complementary means to test the properties of such SCFT's. A large class of AdS_6 backgrounds which can be regarded as the backreacted geometry of certain families of 5-brane webs have been constructed in type IIB String Theory in [82–84] (see also [85]). These geometries contain the (T-dualized) version [86] of the Brandhuber-Oz solution (and its orbifold generalizations) in Type I' String Theory [87, 88] as well as the NATD solutions of [86, 89, 90]. Using these backgrounds it has been possible to perform a very detailed analysis of the dual 5-dimensional theories, for example [91–107].

Let us discuss the construction of the Brandhuber-Oz background. The starting point is type I String Theory on $\mathbb{R}^9 \times S^1$ with N coinciding D5-branes wrapping the circle. The gauge theory on the 6-dimensional brane worldvolume theory has gauge group $Sp(N)$, contains one hypermultiplet in the antisymmetric representation of $Sp(N)$ coming from open strings quantized with Dirichlet-Dirichlet (DD) boundary conditions and 16 hypermultiplets in the fundamental representation coming from strings quantized with Dirichlet-Neumann (DN) boundary conditions and possesses $\mathcal{N} = 1$ SUSY. After performing a T-duality transformation on the circle, we obtain type I' String Theory compactified on $S^1/\mathbb{Z}_2$ with two O8 orientifold planes, located at the fixed points. The D5-branes become D4-branes after the T-duality and we have 16 D8-branes that cancel the -16 units of D8-brane charge that are carried by the two orientifolds. The locations of the D8-branes have a physical meaning, as they correspond to masses for the hypermultiplets in the fundamental representation of the gauge group coming from the 4-8 strings. The hypermultiplet in the antisymmetric representation remains massless. This 5-dimensional theory has again a worldvolume gauge group $Sp(N)$. The field content of the theory, given by the vector multiplet and the hypermultiplets give a theory with $SU(2)_R$ R-symmetry. In addition, the theory has a global $SU(2) \times SO(2N_f) \times U(1)_I$ symmetry.

The low energy description of the system is given in type I' supergravity. The region between two D8-branes is described by Massive type IIA supergravity. Our dual has N_f D8-branes located at one of the orientifolds and $16 - N_f$ at the other. Thus we are always between the D8-branes and the type IIA supergravity description is sufficient. The dual solution is the Brandhuber-Oz geometry and has the form of a fibration of $AdS_6 \times S^4$ space. The AdS_6 part matches the superconformal $SO(2, 5)$ symmetry. Due to the S^4 structure, the $SO(4)$ symmetry group is broken down to a $SU(2) \times SU(2)$ isometry group. This group matches the $SU(2)_R$ R-symmetry and the $SU(2)$ global symmetry. The geometry is singular at the equator of the sphere, which is really a half-S^4, reflecting the presence of the O8-D8's. The space has a boundary which corresponds to the location of the O8 plane, where the dilaton blows up. At this point, the gauge symmetry of the string description is enhanced to E_{N_f+1}. Seiberg argued that the theory at the region corresponding to the field theory limit is a non-trivial fixed point. The dual picture that emerges is that type I' String Theory compactified on the Brandhuber-Oz background is dual to an $\mathcal{N} = 2$ rank N 5-dimensional SCFT fixed point with global symmetry E_{N_f+1}. This solution is the 10-dimensional uplift [108] of the AdS_6 solution of Massive type IIA $F(4)$ supergravity [109]. This theory admits a mass-deformation by turning a Yang-Mills coupling into a conventional 5-dimensional gauge theory with gauge group $USp(2N)$, a hypermultiplet in the 2-index antisymmetric representation and N_f hypermultiplets in the fundamental representation.

In chapter 6, we will put defects on this geometry. Codimension-1 (-2) defects correspond to D6- (D4-)branes, which will be treated as probes in the geometry. We will study the open string degrees of freedom, which will allow us to study the operator content and the meson spectrum of the resulting defect theories.

1.12 Cremonesi-Tomasiello geometries

In this section we will discuss a class of $\mathcal{N} = (1,0)$ 6-dimensional SCFT's which admit a brane description in terms of a system of NS5-, D6- and D8-branes. Such setups resemble the previously discussed Gaiotto-Maldacena geometries, and will be the matter of study of chapter 7. In it, we will test the integrability of the quivers they describe by means of the study of the integrability of the motion of strings in their gravity duals. In the following, we will discuss the brane construction, the field theories they realize and the gravity duals.

QFT's in 6 dimensions were objects of curiosity before the 1990's. Reasoning based on perturbation theory around a Gaussian fixed point (which implied the existence of a continuum Lagrangian) suggested that no such theories existed by themselves, needing some UV completion. But observations based on the possibility of encountering a strongly coupled fixed point and having at the same time anomaly cancellations, gave credibility to the existence of these theories, making them objects of interest [110]. These ideas were supported by the construction of Hanany-Witten brane setups [111] for 6-dimensional field theories. Indeed, the papers [112, 119], found the first of those realisations. These ideas were subsequently developed and in the past few years, the different versions of the $\mathcal{N} = (2,0)$ 6-dimensional SCFT were used as an effective way to organise and understand various features of lower dimensional CFT's (see for example [71]). The Maldacena conjecture [10] gave another important piece of evidence to ascertain the existence of these theories.

For 6-dimensional CFT's with minimal $\mathcal{N} = (1,0)$ SUSY, the same holographic ideas were used by the authors of [113, 114], who constructed the gravitational duals to these field theories in Massive type IIA supergravity, realized in the low energy limit of the 6-dimensional intersection among a setup of D6-NS5-D8-branes. The extension of these ideas to the case in which orientifold planes are present has been carefully discussed in [115].

Let us now describe the brane setup, taking as a reference [116]. Consider a system with NS5-branes which span the directions $\{x^0, x^1, x^2, x^3, x^4, x^5\}$, D6-branes along $\{x^0, x^1, x^2, x^3, x^4, x^5, x^6\}$ and D8-branes along $\{x^0, x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9\}$ placed in 10-dimensional Minkowski spacetime. The presence of the branes give rise to a $SO(1,5) \times SO(3)$ symmetry. One can study the string excitations in an analogous way as we did in the previous section for the Gaiotto-Maldacena geometries:

- 6-6 strings realize the $\mathcal{N} = (1, 0)$ 6-dimensional gauge theory.

- 6-6 strings between consecutive D6-branes provide hypermultiplets in the bifundamental representation.

- 6-8 strings provide the hypermultiplets on the worldvolume of the D6-branes.

A condition of anomaly cancellation must be imposed, and it implies that for every gauge group, the number of flavour fields doubles the number of gauge fields $N_f = 2N_c$. In the brane setup N_c is the number of D6-branes in a given interval $[x_{6,i}, x_{6,i+1}]$ between two NS5-branes. On the other hand, the number $N_f = N_{D6,i+1} + N_{D6,i-1} + N_{D8}$ counts the D6-branes in the two adjacent intervals and the number of D8-branes in the interval. The tensor multiplets living on the NS5-branes play an important role in the cancellation of anomalies and provide self-dual 2-forms, which give place (in the case in which the NS5-branes become coincident) to tensionless strings. In fact, the positions of the NS5-branes are not fixed, but they are represented by a real scalar field Φ_i, which gets a VEV. This scalar field couples to the gauge field strength on the D6-branes, leading to a term in the Lagrangian $L \sim (\Phi_{i+1} - \Phi_i)F^2_{mn} + ...$ When the NS5-branes become coincident, the effective gauge coupling $\frac{1}{g_6^2} = \langle \Phi_{i+1} - \Phi_i \rangle$ diverges. In this limit, the field theory flows to the conjectured CFT [110]. Some pieces of evidence support this proposal: the number of supercharges preserved by the brane setup, the isometries (associated with the unique $\mathcal{N} = (1, 0)$ superconformal algebra), the massless string solitons corresponding with D2-branes that extend in $[x_0, x_1, x_6]$ and end on the 5-branes. A detailed explanation of the story summarised above can be found in [117]. The field theory can be represented in terms of a quiver. One can make an analogous drawing as it was done for the Gaiotto-Maldacena geometries. This schematic representation will be used in chapter 7.

The line of research to find the gravitational duals was started in [113], where the authors searched for supersymmetric AdS_7 solutions in type II supergravities. The proposed metric preserved the $SO(2, 6) \times SO(3)$ isometries and 8 supercharges. The $SO(3)$ corresponds to the symmetry in the 789 directions and is identified with the $Sp(1)$ R-symmetry of the (1,0) SUSY theory. The $SO(2, 6)$ matches the superconformal symmetry. In the particular case of Massive type IIA supergravity a system of BPS equations was written and a family of solutions found. The paper [118], pointed out the type of field theories these Massive type IIA solutions were holographically dual to. Various efforts to solve the BPS equations and interpret the solutions followed [120]. These lead to the formulation by Cremonesi and Tomasiello [114], where a precision test was put forward, calculating the a-central charge, both in the CFT and in holography. Other interesting developments deal with flows away from, and compactifications of the 6-dimensional SCFT's, see for example [121–127]. In chapter 7 we will deal with the Massive type IIA backgrounds as described by Cremonesi and Tomasiello in [114]. These duals are specified by a function which satisfies a ordinary third-order differential equation. Each solution generates a particular quiver, and the precise recipe to construct the gravity dual to any particular quiver was written in [114]. We will revise the construction of such quivers and study its integrability in chapter 7, by means of both analytical and numerical techniques.

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2

Unquenched massless flavored D3-D5 system

2.1 Introduction

The aim of this chapter is to construct a backreacted geometry of the D3-D5 system presented in the first chapter, in which the D5-branes intersect the color D3-branes along a (2+1)-dimensional subspace, where the fundamental matter lives. We will work in the Veneziano limit, a regime in which both the number of flavors N_f and the number of colors N_c is large and their ratio $\frac{N_c}{N_f}$ is fixed. This regime is described by the DBI + WZ action. We will smear the D5-brane sources, which modifies the R-symmetry of the theory and since the smeared flavor branes are not coincident, the flavor symmetry for N_f flavors is also broken down from $U(N_f)$ to $U(1)^{N_f}$.

We will consider the case in which the internal space is a general 5-dimensional Sasaki-Einstein manifold and the corresponding dual field theory is a quiver theory. To preserve some amount of supersymmetry at the probe level the D5-brane must wrap a 3-dimensional special Lagrangian submanifold of the Calabi-Yau cone constructed over the Sasaki-Einstein space [128]. This was checked explicitly in [129–131] for different internal manifolds. To construct a backreacted solution we must include brane sources in the supergravity equations of motion, which induce a violation of the Bianchi identities of some of the RR field strengths. In our case the presence of D5-branes implies that the RR 3-form F_3 is not closed anymore. Actually, the exterior derivative of F_3 is a 4-form which encodes the RR charge distribution of the flavor branes. We will use this fact, together with the intuition obtained from the probe brane analysis, to write an ansatz for F_3 that could preserve some amount of the original supersymmetry. Moreover, the deformation of the metric induced by the flavor in our ansatz is similar to the one corresponding to the D3-D7 system studied in [36], which consists in squashing the internal Sasaki-Einstein space in such a way that its representation as a $U(1)$ bundle over a 4-dimensional Kähler-Einstein manifold is preserved.

Exact localized supergravity solutions for the D3-D5 system have been obtained in references [132, 133]. Their geometry is of the form $AdS_4 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \Sigma$, with Σ being a Riemann surface. These solutions are dual to interface theories or interface operators, preserve 16 supersymmetries and have $SO(2,3) \times SO(3) \times SO(3)$ isometry. Some of these geometries have topologically non-contractible 3-cycles with non-vanishing 3-form RR charge, which can be interpreted as due to the presence of D5-branes. However, the additional degrees of freedom corresponding to the open strings stretched between the D3- and D5-branes do not show up (they are substituted by fluxes). As argued in [134], the supergravity solution by itself should be completed by adding explicit 5-branes in the geometry. This is the point of view adopted in this chapter. Indeed, our D5-branes are dynamical sources contributing to the energy-momentum tensor and violate explicitly the Bianchi identity of the RR 3-form. By smearing these sources we will find simpler solutions which allow to study the effects of dynamical flavors in this class of duals.

The organization of this chapter is the following. In section 2.2 we formulate our setup and write our ansatz for the metric and RR forms of the type IIB theory in the case of massless flavors. We consider the case in which the D3-branes are on the tip of a cone over a general 5-dimensional Sasaki-Einstein space. We also write in this section the system of first-order BPS equations for the different functions of our ansatz and we show that they can be formulated in terms of calibration forms. We end section 2.2 with the demonstration that the equations of motion of supergravity plus brane sources are satisfied by any solution of the BPS system. In section 2.3 we consider the integration of the BPS system. We reduce this first-order system to a unique second-order master equation for a new function. This master equation can be integrated in general for the unflavored system, as shown in section 2.3.1. In section 2.3.2 we find a particular solution for the unflavored system, which leads to a metric displaying an anisotropic scale invariance. In section 2.4 we extend the ansatz to the case of massive flavors. We find the master equation for this massive case and show how to construct solutions that interpolate between the unflavored geometry in the IR and the background corresponding to massless flavors in the UV. Finally, in section 2.5 we discuss our results. The chapter is completed with an appendix, in which we give a detailed derivation of the BPS equations and we write the coordinate representation of two particular Sasaki-Einstein spaces.

2.2 Brane setup and ansatz

Suppose that we have N_c color D3-branes on the tip of a cone over a 5-dimensional Sasaki-Einstein (SE) space $\mathcal{M}_5$ with metric $ds^2_{SE} = ds^2_{KE} + (d\tau + A)^2$, where ds^2_{KE} is the metric of the 4-dimensional Kähler-Einstein (KE) base $\mathcal{M}_4$ and A is a 1-form in $\mathcal{M}_4$. Moreover, we will add N_f flavor D5-branes according to the array:

$$\begin{array}{lccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D3\ (N_c): & \times & \times & \times & \_ & \_ & \_ & \_ & \_ & \_ \\ D5\ (N_f): & \times & \times & \_ & \times & \times & \times & \_ & \_ & \_ \end{array} \tag{2.1}$$

In (2.1) the directions 4-9 are those corresponding to the SE cone. Let us consider the theory at zero temperature. We will adopt the following ansatz for the 10-dimensional metric in Einstein frame:

$$ds^2 = h^{-\frac{1}{2}}\left[-(dx^0)^2+(dx^1)^2+(dx^2)^2+e^{2m}\,(dx^3)^2\right]+h^{\frac{1}{2}}\left[dr^2+e^{2g}\,ds^2_{KE}+e^{2f}\,(d\tau+A)^2\right]\,, \tag{2.2}$$

where m, g and f are squashing functions that depend on the radial variable r and $h = h(r)$ is a warp factor. The type IIB supergravity background corresponding to the array written above should contain a self-dual RR 5-form F_5, induced by the stack of the color D3-branes. We will adopt the following ansatz for F_5:

$$F_5 \,=\, K(r)\,\big(1+*\big)\,d^4x\wedge dr\,, \tag{2.3}$$

where $K = K(r)$ is a function to be determined. Actually, from the Bianchi identity of F_5 ($dF_5 = 0$) we can relate $K(r)$ to the functions appearing in the metric (2.2). We get:

$$K\,h^2\,e^{-m}\,e^{4g+f} = Q_c\,, \tag{2.4}$$

where Q_c is a constant that can be related to the number of colors N_c by employing the flux quantization condition of the 5-form, namely:

$$Q_c \,=\, \frac{(2\pi)^4\,g_s\,\alpha'^{\,2}\,N_c}{\mathrm{Vol}(\mathcal{M}_5)}\,. \tag{2.5}$$

The total action of the system is the sum of the one corresponding to 10-dimensional type IIB supergravity and the action of the branes:

$$S \,=\, S_{IIB}\,+\,S_{branes}\,, \tag{2.6}$$

where S_{branes} denotes the sum of the DBI and WZ actions for the flavor branes. The N_f flavor branes of our setup act as sources of the RR 3-form F_3. Indeed, the D5-branes couple naturally to the RR 6-form potential C_6 through the WZ term of their worldvolume action, which is given by:

$$S_{WZ} \,=\, T_5\sum^{N_f}\int_{\mathcal{M}_6}\hat{C}_6\,, \tag{2.7}$$

where the hat over C_6 denotes its pullback to the worldvolume and T_5 is the tension of the D5-brane, given by:

$$\frac{1}{T_5} \,=\, (2\pi)^5\,g_s\,(\,\alpha'\,)^3. \tag{2.8}$$

In the smearing approach, valid when N_f is large, we substitute the discrete distribution of flavor branes by a continuous distribution with the appropriate normalization, in such a way that the smearing amounts to performing the substitution:

$$\sum^{N_f} \int_{\mathcal{M}_6} \hat{C}_6 \implies \int_{\mathcal{M}_{10}} \Xi \wedge C_6 \, , \tag{2.9}$$

where Ξ is a 4-form (the smearing form) with components along the the directions orthogonal to the worldvolume of the flavor branes. The coupling of the flavor branes to C_6 modifies the Bianchi identity for F_3, which gets a source term proportional to Ξ. In order to determine this modification, let us write the supergravity plus branes action (2.6) in terms of the RR 7-form F_7 and its 6-form potential C_6. This action contains a contribution of the form:

$$-\frac{1}{2\kappa_{10}^2}\,\frac{1}{2}\int_{\mathcal{M}_{10}} e^{-\phi}\; F_7 \wedge *F_7 \,+\, T_5 \int_{\mathcal{M}_{10}} C_6 \wedge \Xi \, , \tag{2.10}$$

where $2\,\kappa_{10}^2 \;=\; (2\pi)^7\, g_s^2\, (\,\alpha'\,)^4$. The equation of motion of C_6 derived from (2.10) gives rise to the Maxwell equation for F_7 with Ξ playing the role of a source, which is just:

$$d\Big(e^{-\phi}\, * F_7\Big) \;=\; -2\kappa_{10}^2\, T_5\; \Xi \, . \tag{2.11}$$

Taking into account that $F_3 = -e^{-\phi} * F_7$, we get that (2.11) is equivalent to the following violation of Bianchi identity of F_3:

$$dF_3 \;=\; 2\,\kappa_{10}^2\, T_5\, \Xi \, . \tag{2.12}$$

The 4-form Ξ is just the RR charge distribution due to the presence of the D5-branes. Clearly, Ξ is non-zero on the location of the sources. In a localized setup, in which the N_f branes are on top of each other, Ξ will contain Dirac δ-functions and finding the corresponding backreacted geometry is technically a very complicated task although, as discussed above, supergravity solutions for the D3-D5 intersection have indeed been found [132–134]. Here we avoid this difficulty by separating the N_f branes and distributing them homogeneously along the internal manifold in such a way that, in the limit in which N_f is large, they can be described by a continuous charge distribution Ξ.

Instead of trying to specify explicitly the family of flavor branes of our setup, let us formulate directly an ansatz for the F_3 leading to a smearing form compatible with the preservation of some amount of supersymmetry. We will assume that our flavors are massless, which implies that the flavor branes reach the origin $r = 0$ and that the smearing form Ξ is homogeneous in r, *i.e.*, independent of the radial coordinate. From our array we notice that x^3 is a direction orthogonal to the D5-branes. Therefore, one of the legs of the 4-form Ξ (and of F_3) should be along the x^3 direction, whereas the others should lie along the internal space. In order to specify this internal structure, let $\{e^i\}$ $(i = 1, \cdots, 4)$

be a canonical basis of vielbein 1-forms for the KE space ($ds^2_{KE} = \sum_i (e^i)^2$). In this basis the Kähler 2-form J_{KE} of $\mathcal{M}_4$ can be written simply as:

$$J_{KE} \,=\, e^1 \wedge e^2 + e^3 \wedge e^4 \,, \tag{2.13}$$

and it is related to the 1-form A in (2.2) as:

$$J_{KE} \,=\, \frac{dA}{2} \,. \tag{2.14}$$

The explicit coordinate form of the e^i's and A for the cases in which $\mathcal{M}_5 = T^{1,1}, \mathbb{S}^5$ is given in the appendix. Let us next introduce the complex 2-form Ω_2 as:

$$\Omega_2 \,=\, (e^1 + ie^2) \wedge (e^3 + ie^4) \,. \tag{2.15}$$

This form satisfies:

$$d\Omega_2 \,=\, 3\, i\, \Omega_2 \wedge A \,. \tag{2.16}$$

Therefore, if we define $\hat{\Omega}_2$ as:

$$\hat{\Omega}_2 \,=\, e^{3i\tau}\, \Omega_2 \,, \tag{2.17}$$

it follows from (2.16) that the exterior derivative of $\hat{\Omega}_2$ is given by:[1]

$$d\hat{\Omega}_2 \,=\, 3\, i\, \hat{\Omega}_2 \wedge (d\tau + A) \,. \tag{2.19}$$

Let us now separate real and imaginary parts of $\hat{\Omega}_2$. From (2.19) we obtain:

$$d\,\mathrm{Im}\, \hat{\Omega}_2 \,=\, 3\, \mathrm{Re}\, \hat{\Omega}_2 \wedge (d\tau + A) \,. \tag{2.20}$$

Let us now write our ansatz for F_3 as:

$$F_3 \,=\, Q_f\, dx^3 \wedge \mathrm{Im}\, \hat{\Omega}_2 \,, \tag{2.21}$$

where Q_f is a constant proportional to the number of flavors N_f. More explicitly, F_3 can be written as:

$$F_3 \,=\, Q_f\, dx^3 \wedge \Big[e^1 \wedge (\cos(3\tau)\, e^4 + \sin(3\tau)\, e^3 \,) \,+\, e^2 \wedge (\cos(3\tau)\, e^3 - \sin(3\tau)\, e^4 \,) \Big] \,. \tag{2.22}$$

It follows from (2.20) that the modified Bianchi identity for F_3 is:

$$dF_3 \,=\, -3\, Q_f\, dx^3 \wedge \mathrm{Re}\, \hat{\Omega}_2 \wedge (d\tau + A) \,. \tag{2.23}$$

[1] The 2-form $\hat{\Omega}_2$ is related to the holomorphic $(3,0)$-form of the Calabi-Yau cone with metric $ds^2_{CY} = dr^2 + r^2\, ds^2_{SE}$ as:

$$\Omega_{CY} \,=\, r^2\, \hat{\Omega}_2 \wedge (dr + ir\, (d\tau + A)) \,. \tag{2.18}$$

The closure of Ω_{CY} implies (2.16).

The smearing form Ξ can be read from the right-hand side of (2.23). Notice that Ξ does not depend on x^3 (it only depends on dx^3), which means that we are homogeneously distributing our flavor branes in the x^3 direction.

To find a supersymmetric solution for our ansatz for the metric, dilaton and RR forms, we have to consider the supersymmetric variations of type IIB supergravity and find the corresponding first-order BPS equations ensuring the existence of Killing spinors. The detailed analysis of these SUSY variations is performed in the appendix. The resulting BPS system for the different functions of the ansatz is:

$$
\begin{aligned}
h' &= -Q_c\, e^{-4g-f} \,-\, Q_f\, e^{\frac{\phi}{2}-m-2g}\, h \ , \\
\phi' &= Q_f\, e^{\frac{\phi}{2}-m-2g} \ , \\
m' &= -\,Q_f\, e^{\frac{\phi}{2}-m-2g} \ , \\
g' &= e^{f-2g} \ , \\
f' &= 3\,e^{-f} \,-\, 2e^{f-2g} + \frac{Q_f}{2}\, e^{\frac{\phi}{2}-m-2g} \ .
\end{aligned}
\tag{2.24}
$$

In the vielbein basis (2.94), the Killing spinor takes the form:

$$
\epsilon = h^{-\frac{1}{8}}\, e^{\frac{3}{2}\, i\sigma_2\, \tau}\, \eta \ , \tag{2.25}
$$

where σ_2 is a Pauli matrix and η is a doublet of constant Majorana-Weyl spinors, characterized by the four projections in (2.128) and (2.129). Therefore, the solutions of (2.24) give rise to a 10-dimensional supersymmetric background which preserves two supercharges.

Interestingly, one can write the BPS equations in terms of generalized calibration forms. As we are dealing with a system with two types of branes, we expect to have two of such calibration forms. To begin with we will have a 4-form $\mathcal{K}_4$ that will calibrate the geometry of the D3-branes within our background. This form can be defined as a fermion bilinear constructed from the Killing spinor η as:

$$
\mathcal{K}_4 = \frac{1}{4!}\, \mathcal{K}_{a_1\cdots a_4}\, E^{a_1\cdots a_4} \ , \qquad\qquad \mathcal{K}_{a_1\cdots a_4} = \eta^{\dagger}\, i\sigma_2\, \Gamma_{a_1\cdots a_4}\, \eta \ , \tag{2.26}
$$

where $E^{a_1 a_2 \cdots}$ denotes $E^{a_1} \wedge E^{a_2} \wedge \cdots$, with the E^a's being the 1-forms of our 10-dimensional vielbein basis. From the projections imposed on our Killing spinors, we get that $\mathcal{K}_4$ is given by:

$$
\mathcal{K}_4 = E^{x^0 x^1 x^2 x^3} \ , \tag{2.27}
$$

which is quite natural given the fact that our D3-branes wrap the Minkowski part of the 10-dimensional spacetime. Moreover, we should also have a 6-form $\mathcal{K}_6$ calibrating the worldvolume of the D5-branes. In terms of fermion bilinears, $\mathcal{K}_6$ is defined as:

$$\mathcal{K}_6 \,=\, \frac{1}{6!}\,\mathcal{K}_{a_1\cdots a_6}\,E^{a_1\cdots a_6}\;, \qquad\qquad \mathcal{K}_{a_1\cdots a_6} \,=\, \eta^\dagger\,\sigma_1\,\Gamma_{a_1\cdots a_6}\,\eta\;. \tag{2.28}$$

The explicit form of $\mathcal{K}_6$ can also be determined from the projections satisfied by the spinor η. Our D5-branes are extended along 3 Minkowski and 3 internal directions. We expect the G-structure of our geometry to play a key role in determining the components of $\mathcal{K}_6$ along the the internal manifold. We are dealing here with an $SU(3)$ structure generated by D5-branes wrapping a 3-cycle. This $SU(3)$ structure is endowed with a Kähler 2-form J and a holomorphic 3-form Ω, given by:

$$\begin{aligned}
J \,&=\, E^1\wedge E^2 \,+\, E^3\wedge E^4 \,+\, E^r\wedge E^5\;,\\
\Omega \,&=\, e^{3i\tau}\left(E^1+iE^2\right)\wedge\left(E^3+iE^4\right)\wedge\left(E^r+iE^5\right)\;.
\end{aligned} \tag{2.29}$$

It can be checked from the projections satisfied by the spinor η that $\mathcal{K}_6$ can be written in terms of the real part of Ω as:

$$\mathcal{K}_6 \,=\, E^{x^0\,x^1\,x^2}\;\wedge\;\mathrm{Re}\,\Omega\;. \tag{2.30}$$

Moreover we can write the full set of SUSY-preserving conditions (2.24) as:

$$\begin{aligned}
d\,\mathcal{K}_4 + *\,d\mathcal{K}_4 \,&=\, F_5\,,\\
e^{-\phi}\,d\left(e^{\frac{\phi}{2}}\,\mathcal{K}_6\right) \,&=\, *\,F_3\;,\\
d\big(e^{\frac{\phi}{2}}\,h^{-\frac{1}{2}}\,*\mathcal{K}_6\big) \,&=\, 0\;,\\
d\left(h^{-\frac{1}{2}}\,J\right) \,&=\, 0\;,\\
d\left(e^{\phi+m}\right) \,&=\, 0.
\end{aligned} \tag{2.31}$$

Notice that (2.31) agrees with the classification obtained in [135]. As a non-trivial check of our BPS system of first-order differential equations, let us verify that the second-order equations of motion of the supergravity plus branes system are satisfied if (2.25) holds. These equations of motion follow from the action (2.6), which we now write explicitly. The action of type IIB supergravity in Einstein frame for the fields turned on reads:

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\left[\int d^{10}x\,\sqrt{-g}\left(R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right)-\int\left(\frac{1}{2}\,e^{\phi}F_3\wedge *F_3\,+\,\frac{1}{4}\,F_5\wedge *F_5\right)\right]\;, \tag{2.32}$$

whereas the action of our calibrated distribution of flavor branes reads

$$S_{\text{branes}} = -T_5 \int \left(e^{\phi/2} \mathcal{K}_6 - C_6 \right) \wedge \Xi \ , \tag{2.33}$$

where $\mathcal{K}_6$ is the calibration 6-form written in (2.30) and the smearing form Ξ can be read from the right-hand side of (2.23). In terms of our vielbein basis, Ξ can be written as:

$$\kappa_{10}^2 \, T_5 \, \Xi \ = \ -\frac{3}{2} \, Q_f \, h^{-\frac{1}{2}} \, e^{-f-2g-m} \, E^{x^3} \wedge \mathrm{Re}\Big[e^{3i\tau} \, (E^1 + iE^2) \wedge (E^3 + iE^4) \wedge E^5 \Big] \ . \tag{2.34}$$

The equation of motion of the RR flux F_5 is $dF_5 = 0$ and it is satisfied by our ansatz if (2.4) is fulfilled. Moreover, the equation of motion for the 3-form flux F_3 reads:

$$d \left(e^{\phi} * F_3 \right) = 0 \ . \tag{2.35}$$

and it is satisfied if the BPS system holds since, according to the second equation in (2.31), $e^{\phi} * F_3$ is an exact 7-form. In order to write the dilaton and the Einstein equations, we define first the following notation:

$$\omega_{(p)} \lrcorner \lambda_{(p)} = \frac{1}{p!} \omega^{\mu_1 \ldots \mu_p} \lambda_{\mu_1 \ldots \mu_p} \ , \tag{2.36}$$

for any two p-forms $\omega_{(p)}$ and $\lambda_{(p)}$. One can easily prove that, in a 10-dimensional manifold, one has:

$$\int \omega_{(p)} \wedge \lambda_{(10-p)} = - \int d^{10}x \, \sqrt{-g} \lambda \lrcorner (*\omega) \ . \tag{2.37}$$

Using these results, we can write the equation of motion of the dilaton as:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = \frac{1}{12} e^{\phi} F_3^2 - \kappa_{10}^2 \, T_5 \, e^{\frac{\phi}{2}} \ \Xi \lrcorner (*\mathcal{K}_6) \ , \tag{2.38}$$

and one can check that it is satisfied when the functions of our ansatz are solutions of the system (2.24). Let us now write the Einstein equations that follow from the action (2.6) as:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, R \ = \ & \frac{1}{2} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \, g_{\mu\nu} \, \partial_\lambda \phi \, \partial^\lambda \phi \right) + \frac{1}{2 \cdot 3!} \left(3 F_{\mu\rho\sigma} \, F_\nu{}^{\rho\sigma} - \frac{1}{2} \, g_{\mu\nu} \, F_{\lambda\rho\sigma} \, F^{\lambda\rho\sigma} \right) \ + \\ & + \frac{1}{4 \cdot 5!} \left(5 F_{\mu\rho\sigma\alpha\beta} \, F_\nu{}^{\rho\sigma\alpha\beta} - \frac{1}{2} \, g_{\mu\nu} \, F_{\lambda\rho\sigma\alpha\beta} \, F^{\lambda\rho\sigma\alpha\beta} \right) \ + \ T_{\mu\nu}^{flav} \ , \end{aligned} \tag{2.39}$$

where $T_{\mu\nu}^{\text{flav}}$ is the energy-momentum tensor coming from the action of the flavor branes. It is given by:

$$T_{\mu\nu}^{\text{flav}} = \kappa_{10}^2 \, T_5 \, e^{\frac{\phi}{2}} \Big[g_{\mu\nu} \, \Xi \lrcorner (*\mathcal{K}_6) - \frac{1}{3!} \, \Xi_{\mu\rho_1\rho_2\rho_3} \, (*\mathcal{K}_6)_\nu{}^{\rho_1\rho_2\rho_3} \Big] \ . \tag{2.40}$$

It follows from (2.40) that the non-zero components of $T^{\text{flav}}_{\mu\nu}$, in flat coordinates, are given by:

$$\begin{aligned}
T^{\text{flav}}_{x^\mu x^\nu} &= -3\,Q_f\,h^{-\frac{1}{2}}\,e^{\frac{\phi}{2}-f-2g-m}\,\eta_{\mu\nu}\;, \qquad\qquad (\mu,\nu=0,1,2)\;, \\
T^{\text{flav}}_{rr} &= -3\,Q_f\,h^{-\frac{1}{2}}\,e^{\frac{\phi}{2}-f-2g-m}\;, \\
T^{\text{flav}}_{ij} &= -\frac{3\,Q_f}{2}\,h^{-\frac{1}{2}}\,e^{\frac{\phi}{2}-f-2g-m}\,\delta_{ij}\;, \qquad\qquad (i,j=1,\cdots,4)\;,
\end{aligned} \tag{2.41}$$

and one can readily demonstrate that (2.39) holds along the different directions.

2.3 Integration of the BPS system

Let us now consider the integration of the first-order system (2.24). Clearly, the warp factor h only appears in the first equation in (2.24), which can be integrated once the other functions are known. Moreover, since $\phi' = -m'$ in (2.24), we can take without loss of generality:

$$\phi = -m\;, \tag{2.42}$$

and, thus, we are left with three equations for ϕ', g' and f'. We can also rewrite the function $K(r)$ appearing in the ansatz (2.3) for F_5 in a very convenient way. First, we notice that using (2.42) the first equation in (2.24) can be written as:

$$e^{-4g-f}\,Q_c = -h' - \phi'\,h\;. \tag{2.43}$$

Plugging this result in (2.4) we arrive at the following expression of $K(r)$:

$$K(r) = -h^{-2}\,e^{-\phi}\left(h' + \phi'\,h\right)\;, \tag{2.44}$$

which can be simply recast as:

$$K(r) = \partial_r\left(e^{-\phi}\,h^{-1}\right)\;. \tag{2.45}$$

Thus, the RR 5-form F_5 for our solutions can be written as:

$$F_5 = \partial_r\left(e^{-\phi}\,h^{-1}\right)\left(1+*\right)d^4x\wedge dr\;. \tag{2.46}$$

In order to continue with the integration of the BPS equations, let us next introduce a new radial variable ρ related to the old one as:

$$\frac{d\rho}{dr} = e^{-2g}\,. \tag{2.47}$$

Denoting with a dot the differentiation with respect to ρ, the BPS system reduces to:

$$\begin{aligned}
\dot{\phi} &= Q_f \, e^{\frac{3\phi}{2}} \, , \\
\dot{g} &= e^f \, , \\
\dot{f} &= 3\, e^{-f+2g} - 2\, e^f + \frac{Q_f}{2}\, e^{\frac{3\phi}{2}} \, .
\end{aligned} \tag{2.48}$$

Let us integrate first the equation of the dilaton in (2.48). Let ϕ_0 be the value of the dilaton at $\rho = 0$ and let us define the flavor deformation parameter ε as:

$$\varepsilon = \frac{3}{2} Q_f \, e^{\frac{3}{2}\phi_0} \, . \tag{2.49}$$

Then, as a function of ρ, the dilaton can be written as:

$$e^{\frac{3}{2}\phi} = \frac{e^{\frac{3}{2}\phi_0}}{1 - \varepsilon\, \rho} \, . \tag{2.50}$$

Since the left-hand side of (2.50) cannot be negative, it follows that the allowed range of the ρ variable is $-\infty < \rho < \varepsilon^{-1}$ (the IR corresponds to $\rho \to -\infty$, while the UV is the region $\rho \to \epsilon^{-1}$). Notice that ϕ grows when ρ is increased and that ϕ diverges at the UV endpoint $\rho = \varepsilon^{-1}$. Using (2.50) the remaining equations for g and f are:

$$\begin{aligned}
\dot{g} &= e^f \, , \\
\dot{f} &= 3\, e^{-f+2g} - 2\, e^f + \frac{1}{3} \frac{\varepsilon}{1 - \varepsilon\, \rho} \, .
\end{aligned} \tag{2.51}$$

One can readily show that this system of two first-order equations is equivalent to the following second-order equation for g:

$$\ddot{g} + 2(\dot{g})^2 - 3e^{2g} - \frac{1}{3} \frac{\varepsilon}{1 - \varepsilon\, \rho}\, \dot{g} = 0 \, . \tag{2.52}$$

Let us next rewrite (2.52) in terms of the new function G, defined as:

$$G \equiv e^{2g} \, . \tag{2.53}$$

We arrive at the following non-linear master equation:

$$\ddot{G} - 6\, G^2 - \frac{1}{3} \frac{\varepsilon}{1 - \varepsilon\rho}\, \dot{G} = 0 \, . \tag{2.54}$$

Notice that, if we know G, we can easily get the two functions f and g as:

$$e^g = \sqrt{G} \, , \qquad e^f = \frac{\dot{G}}{2G} \, . \tag{2.55}$$

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2.3.1 The unflavored solution

Let us consider for a while the unflavored case $\varepsilon = 0$. In this case the dilaton is constant and the master equation (2.54) becomes:

$$\ddot{G}_0 - 6\,G_0^2 = 0\,, \tag{2.56}$$

where $G_0(\rho) \equiv G(\rho)_{\varepsilon=0}$. By multiplying (2.56) by $\dot{G}_0$ one immediately realizes that this equation can be integrated once as:

$$\dot{G}_0^2 = 4\,G_0^3 - g_3\,, \tag{2.57}$$

where g_3 is an integration constant. The first-order differential equation (2.57) is well-known. Indeed, the Weierstrass function $\wp(\rho; g_2, g_3)$ satisfies the differential equation:

$$\dot{\wp}^2 = 4\,\wp^3 - g_2\,\wp - g_3\,, \tag{2.58}$$

where g_2 and g_3 are the so-called lattice invariants. Clearly, our first integral (2.57) is just (2.58) for $g_2 = 0$. Therefore, up to a constant shift in the ρ coordinate, we can write the general solution of (2.56) as:

$$G_0(\rho) = \wp(\rho; 0, g_3)\,. \tag{2.59}$$

In order to write neatly the metric corresponding to the solution (2.59), we change to a new radial variable ζ, defined as:

$$\zeta = \sqrt{G_0} = e^g\,. \tag{2.60}$$

Then, one can show that:

$$e^f = \zeta\,\sqrt{k(\zeta)}\,, \qquad\qquad dr = \frac{d\zeta}{\sqrt{k(\zeta)}}\,, \tag{2.61}$$

where $k(\zeta)$ is defined as:

$$k(\zeta) \equiv 1 - \frac{b^6}{\zeta^6}\,, \tag{2.62}$$

with $b^6 = g_3/4$. Then, the unflavored 10-dimensional metric takes the form:

$$ds^2_{unflav} = h^{-\frac{1}{2}}\,dx^2_{1,3} + h^{\frac{1}{2}}\left[\frac{d\zeta^2}{k(\zeta)} + \zeta^2\,ds^2_{KE} + \zeta^2\,k(\zeta)\,(d\tau + A)^2\right]\,, \tag{2.63}$$

where the warp factor h can be determined by integrating the first equation in (2.24) (see, for example appendix B in [36] for its explicit expression). The geometry (2.63) for b real and positive describes N_c smeared D3-branes on the blown-up 4-cycle of the Calabi-Yau [136, 137]. Indeed, in the 6-dimensional part of the metric (2.63) the Kähler-Einstein cycle is blown up at $\zeta = b$. The gauge theory dual to this local Kähler deformation of the Calabi-Yau cone is a deformation of the superconformal theory due to the insertion of a

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VEV of a dimension 6 operator [137]. By choosing appropriately the integration constant in the warp factor h one can show that the solution becomes asymptotically $AdS_5 \times \mathcal{M}_5$ for large values of the radial variable ζ. In the particular case with $b = g_3 = 0$ we recover the conformal $AdS_5 \times \mathcal{M}_5$ solution. Interestingly, in this case the master function $G_0(\rho)$ is simply:

$$G_0(\rho)_{g_3=0} \;=\; \wp(\rho;0,0) \;=\; \frac{1}{\rho^2}\;. \tag{2.64}$$

2.3.2 A massless flavored solution

Let us now come back to the $\varepsilon \neq 0$ model. The master equation (2.54) can be regarded a deformation of the unflavored one. We have not been able to solve analytically (2.54) in general. However, it is rather easy to find a simple analytic solution similar to the scaling one in (2.64). This solution is simply:

$$G \;=\; \frac{8}{9}\,\frac{1}{(\varepsilon^{-1}\,-\,\rho)^2}\;. \tag{2.65}$$

By using (2.55) we can straightforwardly obtain the functions f and g as:

$$e^g \;=\; \frac{2\sqrt{2}}{3}\,\frac{1}{\varepsilon^{-1}-\rho}\;, \qquad\qquad e^f \;=\; \frac{1}{\varepsilon^{-1}-\rho}\;, \tag{2.66}$$

where we have taken into account that $-\infty < \rho < \varepsilon^{-1}$. Let us write this solution in terms of the original variable r, which is related to ρ as:

$$r \;=\; \int G(\rho)\,d\rho \;=\; \frac{8}{9}\,\frac{1}{\varepsilon^{-1}-\rho}\;, \tag{2.67}$$

where we have adjusted the integration constant in such a way that the radial variable r takes values in the range $0 \leq r < \infty$. In terms of r, the dilaton is:

$$e^{\frac{3\phi}{2}} \;=\; \frac{3}{4\,Q_f}\,r\;, \tag{2.68}$$

where we have used (2.49). The functions f and g are:

$$e^g \;=\; \frac{3}{2\sqrt{2}}\,r\;, \qquad\qquad e^f \;=\; \frac{9}{8}\,r\;. \tag{2.69}$$

The warp factor h is the solution of the following first-order differential equation:

$$h' + \frac{2}{3r}\,h \;=\; -\frac{512}{792}\,\frac{Q_c}{r^5}\;, \tag{2.70}$$

whose general solution is:

$$h \;=\; \frac{256}{1215}\,\frac{Q_c}{r^4} + \frac{C}{r^{\frac{2}{3}}}\;, \tag{2.71}$$

where C is an integration constant. In what follows we will take $C = 0$.

Let us write the resulting total 10-dimensional metric as:

$$ds^2_{10} = ds^2_5 + d\hat{s}^2_5 \,, \tag{2.72}$$

where ds^2_5 ($d\hat{s}^2_5$) is the metric of the non-compact (compact) 5-dimensional part of the 10-dimensional space. If we define the radius R as:

$$R^4 = \frac{256}{1215}\, Q_c \,, \tag{2.73}$$

then ds^2_5 can be simply written as:

$$ds^2_5 = \frac{r^2}{R^2}\left[dx^2_{1,2} + \left(\frac{4Q_f}{3}\right)^{\frac{4}{3}} \frac{(dx^3)^2}{r^{\frac{4}{3}}}\right] + R^2\,\frac{dr^2}{r^2} \,. \tag{2.74}$$

This metric is an anisotropic version of AdS_5. Let us next write the compact part of the metric. First we define a new radius $\bar{R}$ as:

$$\bar{R}^2 = \frac{9}{8}\, R^2 \,. \tag{2.75}$$

More explicitly, $\bar{R}^4$ is:

$$\bar{R}^4 = \frac{4}{15}\, Q_c \,. \tag{2.76}$$

Then, we can write the compact metric $d\hat{s}^2_5$ as a squashed version of the original SE space:

$$d\hat{s}^2_5 = \bar{R}^2\left[\,ds^2_{KE} + \frac{9}{8}\,(d\tau + A)^2\right] \,. \tag{2.77}$$

Notice that the squashing factor in (2.77) is constant and does not depend on the number of flavors. Moreover, the non-compact part of the metric (written in (2.74)) is invariant under the following anisotropic scale transformations:

$$r \to r/\lambda \,, \qquad x^{0,1,2} \to \lambda\, x^{0,1,2} \,, \qquad x^3 \to \lambda^{\frac{1}{3}}\, x^3 \,, \tag{2.78}$$

where λ is an arbitrary positive constant. This means that, effectively, the x^3 direction has an anomalous scale dimension. According to the standard notation, in a general Lifshitz-like anisotropic scaling the coordinates transform as in (2.78), with the anisotropic coordinate changing as $x^3 \to \lambda^{\frac{1}{z}}\, x^3$, where z is an exponent which measures the degree of anisotropy of this coordinate. It is clear from (2.78) that $z = 3$ for our system. Notice also that the dilaton is not invariant under the scale transformation (2.78). Indeed, one can check easily that it transforms as $e^{\phi} \to \lambda^{-\frac{2}{3}}\, e^{\phi}$. For other examples of scaling anisotropic backgrounds constructed from brane intersections see [138].

The existence of scale invariant solutions of our BPS equations is quite remarkable and it is consistent with the field theory analysis performed in [60] of a bulk $\mathcal{N} = 4$, $d = 4$ theory coupled to a $\mathcal{N} = 4$, $d = 3$ hypermultiplet living on a defect. Indeed, in this last reference it was shown that the model remains superconformal after the addition of the hypermultiplet, giving rise to a defect conformal field theory. It would be very interesting to understand the value of the Lifshitz-like exponent z in this context.

We have checked that the 10-dimensional metric (2.72) is free of curvature singularities. The Ricci scalar and the squares of the Ricci and Riemann tensors are constant and given by:

$$R^{\mu}{}_{\mu} \,=\, \frac{7\sqrt{15}}{4}\, Q_c^{-\frac{1}{2}} \;, \qquad R_{\mu\nu}\, R^{\mu\nu} \,=\, \frac{3975}{8}\, Q_c^{-1} \;, \qquad R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} \,=\, \frac{10095}{16}\, Q_c^{-1} \;. \tag{2.79}$$

Finally, let us point out that the Ricci scalar for a generic solution of the flavored BPS equations can be written in terms of G and its radial derivative $G' = dG/dr$ as:

$$R^{\mu}{}_{\mu} \,=\, \frac{Q_f\, e^{\frac{3\phi}{2}}}{h^{\frac{1}{2}}\, G} \left[\, \frac{9}{G'} \,+\, \frac{Q_f\, e^{\frac{3\phi}{2}}}{G} \,\right] . \tag{2.80}$$

Equation (2.80) shows that the solution is well-defined in the region in which G is positive and monotonic ($R^{\mu}{}_{\mu}$ diverges if $G \neq 0$ and $G' = 0$).

2.4 Massive flavors

In this section we discuss the generalization of the previous results to the situation in which the flavors added by the D5-branes are massive. In this case the flavor branes do not reach the origin at $r = 0$ and one expects to have a smearing 4-form Ξ depending on the radial coordinate r. As noticed in [36] for other brane intersections, there is an easy way to incorporate this radial dependence of Ξ by performing the following substitution in our ansatz:

$$Q_f \;\to\; Q_f\, p(r) \;, \tag{2.81}$$

where $p(r)$ is a radial profile which depends on the particular distribution of the flavor branes. We will assume that $p(r)$ is a monotonic function of r, which vanishes for r smaller than certain fixed value $r = r_q$ and becomes equal to one in the deep UV region $r \to \infty$, where the quarks are effectively massless:

$$p(r < r_q) \,=\, 0 \;, \qquad\qquad p(r \to \infty) \,=\, 1 \;. \tag{2.82}$$

Notice that now our problem has an explicit scale r_q, which corresponds to the mass of the quarks. In what follows we will assume that the function $p(r)$ is known. To obtain $p(r)$ for a given mass distribution of the flavors we would have to perform a microscopic calculation of the charge density of the D5-branes (see [139] for a similar calculation for the D3-D7 system), something that we will not attempt to do here. It follows from this discussion that the new ansatz for F_3 in this massive case is:

$$F_3 \,=\, Q_f\, p(r)\, dx^3 \wedge \mathrm{Im}\,\hat{\Omega}_2 \;\;. \tag{2.83}$$

The expression of the smearing 4-form Ξ can be immediately obtained by computing the exterior derivative of F_3 in (2.83). We get:

$$2\,\kappa_{10}^2\,T_5\,\Xi \;=\; -Q_f\,\Big[\,3\,p(r)\,dx^3\wedge \mathrm{Re}\,\hat{\Omega}_2\wedge(d\tau+A)\;+\;p'(r)\,dx^3\wedge dr\wedge \mathrm{Im}\,\hat{\Omega}_2\,\Big]\;. \tag{2.84}$$

Moreover, it is straightforward to check that the BPS equations are obtained from the ones in (2.24) by performing the substitution (2.81). It is also easy to verify that any solution of the BPS system also solves the equations of motion of the gravity-plus-branes system for any profile function $p(r)$. Indeed, $dF_5 = 0$ by construction and equations (2.35), (2.38) and (2.39) are fulfilled if $\mathcal{K}_6$ is taken as in (2.30) and the energy-momentun tensor of the flavor brane is given by (2.40), where the smearing form Ξ is the one written in (2.84). In this massive case, the non-zero components of $T^{\text{flav}}_{\mu\nu}$, in flat coordinates, are given by:

$$\begin{aligned}
T^{\text{flav}}_{x^\mu\,x^\nu} \;&=\; -\,Q_f\,h^{-\frac12}\,e^{\frac{\phi}{2}-f-2g-m}\,\big(3\,p\;+\;e^f\,p'\,\big)\eta_{\mu\nu}\;, \qquad\qquad (\mu,\nu=0,1,2)\;,\\
T^{\text{flav}}_{rr} \;&=\; -\,3\,Q_f\,p\,h^{-\frac12}\,e^{\frac{\phi}{2}-f-2g-m}\;,\\
T^{\text{flav}}_{ij} \;&=\; -\,\frac{Q_f}{2}\,h^{-\frac12}\,e^{\frac{\phi}{2}-f-2g-m}\,\big(3\,p\;+\;e^f\,p'\,\big)\delta_{ij}\;, \qquad\qquad (i,j=1,\cdots,4)\;,\\
T_{55} \;&=\; -Q_f\,h^{-\frac12}\,e^{\frac{\phi}{2}-2g-m}\,p'\;.
\end{aligned} \tag{2.85}$$

Let us now sketch the integration of the BPS system for any profile function $p(r)$. First of all, it is clear (2.42) continues to hold, *i.e.*, we can take $m=-\phi$ also in this massive case. Moreover, the expression of F_5 can also be written in terms of ϕ and h as in (2.46). Changing the radial coordinate as in (2.47), we get the following equation for the dilaton:

$$\dot{\phi}\;=\;Q_f\,p\,e^{\frac{3\phi}{2}}\;, \tag{2.86}$$

where the dot denotes the derivative with respect to the new radial coordinate ρ. This equation can be readily integrated, with the result:

$$e^{\frac{3\phi}{2}}\;=\;\frac{e^{\frac{3\phi_q}{2}}}{1\;-\;\frac32\,Q_f\,e^{\frac{3\phi_q}{2}}\,\int_{\rho_q}^{\rho}\,p(\bar\rho)\,d\bar\rho}\;, \tag{2.87}$$

with $\phi_q\;=\;\phi(\rho=\rho_q)$ and ρ_q is the value of ρ corresponding to the threshold value $r=r_q$ of the original r variable, *i.e.*, $p(\rho<\rho_q)=0$. It is also straightforward to find a master equation generalizing (2.54). The relation between G and the functions f and g is the same as in (2.55). Then, G satisfies the following non-linear second-order differential equation:

$$\ddot{G}\;-\;6\,G^2\;=\;\frac{\dot{\phi}}{2}\,\dot{G}\;. \tag{2.88}$$

Notice that the right-hand side of (2.88) vanishes for $\rho < \rho_q$ and (2.88) becomes, in this region, identical to the unflavored equation (2.56). This is quite natural since there are no flavor sources when $\rho < \rho_q$. On the contrary, when $\rho > \rho_q$ the dilaton runs according to (2.87) and the master equation gets deformed with respect to its unflavored version. Actually, it is quite convenient to write $\dot{\phi}$ in the following way. Let ε_q be the flavor deformation parameter at the threshold, namely:

$$\varepsilon_q \equiv \frac{3}{2} Q_f \, e^{\frac{3\phi_q}{2}} \; . \tag{2.89}$$

Then, by combining (2.86) and (2.87), we get:

$$\frac{\dot{\phi}}{2} = \frac{1}{3} \frac{\varepsilon_q \, p(\rho)}{1 - \varepsilon_q \int_{\rho_q}^{\rho} p(\bar{\rho}) \, d\bar{\rho}} \; . \tag{2.90}$$

Equation (2.88) must be solved numerically. Actually, we know analytically its solution in

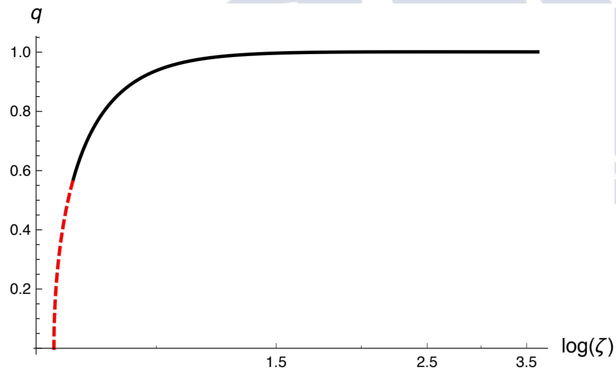

Figure 2.1: In this plot we depict the squashing function q versus the holographic coordinate $\zeta = \sqrt{G}$ for massive flavors with a Heaviside profile function p. The unflavored curve is the dashed line drawn in red. It corresponds to the Weierstrass function (2.59) with $g_3 = 0.5$. The flavor sources are placed at $\zeta \geq \zeta_q$, with $\zeta_q = 0.755$. In this region the profile function p is equal to one. The flavor deformation parameter for this plot is $\varepsilon_q = 0.05$. The $q = q(\zeta)$ curve in this plot interpolates smoothly between $q = 0$ and $q = \frac{3}{2\sqrt{2}} \approx 1.06$.

the region $\rho < \rho_q$, where $G(\rho)$ is the Weierstrass function written on the right-hand side of (2.59). We are interested in solutions of (2.88) approaching the scaling background described in section 2.3.2 in the deep UV region $\rho \approx \varepsilon^{-1}$. To find this interpolating background we must solve numerically (2.88) for $\rho > \rho_q$ by imposing the initial conditions at $\rho = \rho_q$ corresponding to the unflavored function (2.59) for some value of the lattice invariant g_3. Next, we must find the value of g_3 which leads to the desired UV behaviour. This can be done by means of the standard shooting technique. A good strategy to perform this calculation is by considering the relative squashing of the internal manifold, defined as:

$$q \equiv e^{f-g} \ . \tag{2.91}$$

From (2.55) we obtain that q is related to our master function G as:

$$q = \frac{\dot{G}}{2G^{\frac{3}{2}}} \ . \tag{2.92}$$

The flavored solution of section 2.3.2 has a constant squashing given by $q = \frac{3}{2\sqrt{2}}$, while for the unflavored general solution of section 2.3.1 $q = \sqrt{k}$, *i.e.*, $q \to 0(1)$ in the IR (UV) if the parameter $b \neq 0$ and it takes the constant value $q = 1$ if $b = 0$. For a given profile function p we want to integrate (2.88) in such a way that q interpolates between the value $q = 0$ in the IR and $q = \frac{3}{2\sqrt{2}}$ in the UV. We have verified numerically that this is possible for the simplified case in which $p(\rho)$ is a Heaviside step function ($p(\rho) = \Theta(\rho - \rho_q)$). This fact is illustrated in figure 2.1 for some particular values of the parameters.

2.5 Discussion

In this chapter we found supersymmetric solutions generated by the intersection of color D3-branes and flavor D5-branes, in which the latter create a codimension-1 defect on the worldvolume of the color branes. Our backgrounds solve the equations of motion of supergravity with sources corresponding to smeared flavor branes. We considered a generic case in which the color D3-branes are placed on the tip of a Calabi-Yau cone constructed with a general Sasaki-Einstein space. We found a system of first-order BPS equations, which we were able to integrate in general in the unflavored case. Moreover, we obtained a particular solution of the flavored equations for massless quarks which gives rise to a metric displaying anisotropic scale invariance. Finally, we extended our ansatz to the case of massive flavors.

The backgrounds constructed here could be generalized in several directions. The most obvious one would be the construction of a black hole solution for the D3-D5 system, which would be the dual of the unquenched defect theory at non-zero temperature (see [37] for a similar problem in the D3-D7 system). This will indeed be the main subject of the next chapter.

In the case of massive flavors we only sketched the form of the solutions. What remains to be done is the calculation of the profile $p(r)$ for a given quark mass and the integration of the master equation. Once this is done we could study the dependence of the observables on the quark mass. When the mass of the quarks is increased the flavor effects should decrease and, in fact, they should disappear completely when the quarks are infinitely massive. Therefore, by varying the quark mass we introduce a renormalization group flow and we could analyze, for example, how the entanglement entropy changes with the energy scale or how the anisotropy of the system evolves with the quark mass. This will be the matter of chapter 4.

2.A Supersymmetry and Killing spinors

In this appendix we analyze the Killing spinor equations for our ansatz. We will begin by writing the supersymmetry variations of the dilatino λ and gravitino ψ in a background of type IIB supergravity with RR 3- and 5-forms. In the Einstein frame, these variations are:

$$\begin{aligned}\delta\lambda &= \frac{1}{2}\Gamma^a\partial_a\phi - \frac{e^{\phi/2}}{24}F_{abc}\Gamma^{abc}\sigma_1\epsilon\,,\\ \delta\psi_a &= D_a\epsilon + \frac{1}{4}\frac{1}{480}F_{bcdef}\Gamma^{bcdef}\Gamma_a i\sigma_2\epsilon - \frac{1}{96}e^{\phi/2}F_{bcd}\left(\Gamma_a{}^{bcd} - 9\delta_a{}^b\Gamma^{cd}\right)\sigma_1\epsilon\,, \end{aligned} \tag{2.93}$$

where σ_1 and σ_2 are Pauli matrices and ϵ is a doublet of Majorana-Weyl spinors. In (2.93) $D_a\epsilon = (\partial_a + \frac{1}{4}\omega_a{}^{bc}\Gamma_{bc})\,\epsilon$ is the covariant derivative acting on the Killing spinor ϵ. We will work in flat components with respect to the following vielbein basis:

$$\begin{aligned} E^{x^\mu} &= h^{-\frac{1}{4}}\,dx^\mu\,, \qquad (\mu = 0,1,2), \qquad E^{x^3} = h^{-\frac{1}{4}}\,e^m\,dx^3\,,\\ E^r &= h^{\frac{1}{4}}\,dr\,,\\ E^i &= h^{\frac{1}{4}}\,e^g\,e^i\,, \qquad (i = 1,\cdots,4)\,, \qquad E^5 = h^{\frac{1}{4}}\,e^f\,(d\tau + A)\,. \end{aligned} \tag{2.94}$$

Let us write the RR forms in flat components. The 5-form (2.3) can be written as:

$$F_5 = h^{\frac{3}{4}}\,e^{-m}\,K\,(1+*)\,E^{x^0}\wedge E^{x^1}\wedge E^{x^2}\wedge E^{x^3}\wedge E^r \tag{2.95}$$

Using (2.4), this last expression becomes:

$$F_5 = Q_c\,h^{-\frac{5}{4}}\,e^{-4g-f}\,(1+*)\,E^{x^0}\wedge E^{x^1}\wedge E^{x^2}\wedge E^{x^3}\wedge E^r\,. \tag{2.96}$$

Moreover, the 3-form F_3 written in (2.22) can be recast as:

$$F_3 = Q_f\,h^{-\frac{1}{4}}\,e^{-m-2g}\,E^{x^3}\wedge\Big[E^1\wedge(\cos(3\tau)\,E^4 + \sin(3\tau)\,E^3\,) + E^2\wedge(\cos(3\tau)\,E^3 - \sin(3\tau)\,E^4\,)\Big]\,. \tag{2.97}$$

Let us start our analysis by considering the condition $\delta\lambda = 0$. First we impose to the spinor ϵ the projection required by the Kähler condition:

$$\Gamma^{12}\,\epsilon = \Gamma^{34}\,\epsilon\,. \tag{2.98}$$

Then, it is straightforward to show that $\delta\lambda = 0$ implies:

$$\phi'\,\epsilon - Q_f\,e^{\frac{\phi}{2}-m-2g}\Big[\cos(3\tau)\,\Gamma^{14} - \sin(3\tau)\,\Gamma^{24}\Big]\Gamma^{r\,x^3}\,\sigma_1\,\epsilon = 0\,. \tag{2.99}$$

Therefore, for consistency, we must impose the following additional projection to ϵ:

$$\Big[\cos(3\tau)\,\Gamma^{14}\;-\;\sin(3\tau)\,\Gamma^{24}\Big]\,\Gamma^{r\,x^3}\,\sigma_1\,\epsilon\;=\;\epsilon\;, \tag{2.100}$$

and we get the following differential equation for the dilaton:

$$\phi'\;=\;Q_f\,e^{\frac{\phi}{2}\,-\,m\,-2g}\;. \tag{2.101}$$

We go on by requiring that $\delta\psi_a\;=\;0$ for the different directions of the 10-dimensional spacetime. Due to the presence of the covariant derivative of the spinor, we need the different components of the spin-connection 1-form. The non-vanishing components are easily computed for our vielbein basis by using Cartan's structure equation, with the result:

$$\begin{aligned}
\omega^{x^\mu}{}_{r}\;&=\;-\frac{1}{4}\,h'\,h^{-\frac{5}{4}}\,E^{x^\mu}\;, && (\mu=0,1,2)\;,\\
\omega^{x^3}{}_{r}\;&=\;\Big(-\,\frac{1}{4}\,h'\,h^{-\frac{5}{4}}+m'\,h^{-\frac{1}{4}}\Big)\,E^{x^3}\,,\\
\omega^{i}{}_{r}\;&=\;\Big(\frac{1}{4}\frac{h'}{h}\,+\,g'\Big)\,h^{-\frac{1}{4}}\,E^{i}\;, && (i=1,\cdots,4)\;,\\
\omega^{5}{}_{r}\;&=\;\Big(\frac{1}{4}\frac{h'}{h}\,+\,f'\Big)\,h^{-\frac{1}{4}}\,E^{5}\;,\\
\omega^{5}{}_{i}\;&=\;e^{f-2g}\,h^{-\frac{1}{4}}\,J_{ij}\,E^{j}\;, && (i=1,\cdots,4)\;,\\
\omega^{i}{}_{j}\;&=\;\hat\omega^{i}{}_{j}\;-\;e^{f-2g}\,h^{-\frac{1}{4}}\,J^{i}{}_{j}\,E^{5}\;,
\end{aligned} \tag{2.102}$$

where $\hat\omega^{i}{}_{j}$ is the spin-connection 1-form of the KE base, which satisfies $de^i+\hat\omega^{i}{}_{j}\wedge e^j\;=\;0$.

Let us next require that $\delta\psi_{x^\mu}=0$, for $\mu=0,1,2$. We get:

$$\begin{aligned}
&h^{-\frac{5}{4}}\,h'\,\Gamma_{x^\mu\,r}\,\epsilon\;-\;\frac{Q_c}{2}\,e^{-4g-f}\,h^{-\frac{5}{4}}\,\Big(\Gamma^{x^0\,x^1\,x^2\,x^3\,r}\;+\;\Gamma^{12345}\Big)\,\Gamma_{x^\mu}\,i\sigma_2\,\epsilon\;+\\
&\qquad -Q_f\,e^{\frac{\phi}{2}\,-\,m\,-2g}\,h^{-\frac{1}{4}}\,\Big[\cos(3\tau)\,\Gamma^{14}\;-\;\sin(3\tau)\,\Gamma^{24}\Big]\,\Gamma^{x^\mu\,x^3}\,\sigma_1\,\epsilon\;=\;0\;,
\end{aligned} \tag{2.103}$$

We now take into account the 10-dimensional chirality condition satisfied by the spinor ϵ:

$$\Gamma^{x^0\,x^1\,x^2\,x^3\,r\,1\,2\,3\,4\,5}\epsilon\;=\;\epsilon\;, \tag{2.104}$$

which implies that:

$$\Gamma^{x^0\,x^1\,x^2\,x^3\,r}\epsilon\;=\;-\Gamma^{1\,2\,3\,4\,5}\epsilon\;, \tag{2.105}$$

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and we impose the following projection:

$$\Gamma_{x^0\,x^1\,x^2\,x^3}\,(i\sigma_2)\,\epsilon \,=\, \epsilon\;, \tag{2.106}$$

which corresponds to having a D3-brane extended in the $x^0\,x^1\,x^2\,x^3$ directions. Using (2.105), (2.106) and (2.100) we conclude that (2.103) implies the following differential equation for the warp factor h:

$$h' \,=\, -Q_c\,e^{-4g-f} \,-\, Q_f\,e^{\frac{\phi}{2}-m-2g}\,h\;. \tag{2.107}$$

We now consider the supersymmetric variation of the gravitino along the x^3 direction. We get:

$$\Big(h^{-\frac{5}{4}}\,h' \,-\, 4\,h^{-\frac{1}{4}}\,m'\Big)\,\Gamma_{x^3 r}\,\epsilon \,-\, \frac{Q_c}{2}\,e^{-4g-f}\,h^{-\frac{5}{4}}\,\Big(\Gamma^{x^0\,x^1\,x^2\,x^3\,r} \,+\, \Gamma^{12345}\Big)\,\Gamma_{x^3}\,(i\sigma_2)\,\epsilon \,-$$
$$-3\,Q_f\,e^{\frac{\phi}{2}\,-\,m\,-2g}\,h^{-\frac{1}{4}}\,\Big[\cos(3\tau)\,\Gamma^{14} \,-\, \sin(3\tau)\,\Gamma^{24}\Big]\,\sigma_1\,\epsilon \,=\, 0\;. \tag{2.108}$$

Using (2.105), (2.106) and (2.100), we get the following differential equation for m:

$$m' \,=\, \frac{h'}{4h} \,+\, \frac{Q_c}{4}\,e^{-4g-f}\,h^{-1} \,-\, \frac{3}{4}\,Q_f\,e^{\frac{\phi}{2}\,-m\,-2g}\;. \tag{2.109}$$

Plugging (2.107) on the right-hand side of this last equation, we get

$$m' \,=\, -\,Q_f\,e^{\frac{\phi}{2}\,-\,m\,-2g}\;. \tag{2.110}$$

By comparing with (2.101) we conclude that $m' = -\phi'$.

Let us next consider the variation of the gravitino along the radial direction, which leads to the following equation:

$$h^{-\frac{1}{4}}\,\partial_r\,\epsilon + \frac{Q_c}{16}\,e^{-4g-f}\,h^{-\frac{5}{4}}\,\Gamma^r\,\Big(\Gamma^{x^0\,x^1\,x^2\,x^3\,r} \,-\, \Gamma^{12345}\Big)\,i\sigma_2\,\epsilon \,-$$
$$-\frac{Q_f}{8}e^{\frac{\phi}{2}\,-\,m\,-2g}\,h^{-\frac{1}{4}}\,\Gamma^{r\,x^3}\,\Big[\cos(3\tau)\,\Gamma^{14} \,-\, \sin(3\tau)\,\Gamma^{24}\Big]\,\sigma_1\,\epsilon \,=\, 0\;. \tag{2.111}$$

After imposing again (2.100), (2.105) and (2.106), this equation becomes:

$$\partial_r\,\epsilon \,=\, \Big(\frac{Q_c}{8}\,e^{-4g-f}\,h^{-1} \,+\, \frac{Q_f}{8}\,e^{\frac{\phi}{2}\,-\,m\,-2g}\Big)\epsilon\;. \tag{2.112}$$

Comparing with (2.107), the right-hand side of (2.112) can be written in terms of the derivative of the warp factor h:

$$\partial_r\,\epsilon \,=\, -\frac{h'}{8h}\,\epsilon\;. \tag{2.113}$$

This equation can be integrated as:

$$\epsilon = h^{-\frac{1}{8}}\,\tilde{\epsilon}\,, \tag{2.114}$$

with $\tilde{\epsilon}$ being a spinor independent of the radial coordinate.

Let us next analyze the variations of the components of the gravitino along the internal SE space. We first consider one of the KE directions (say the e^1 direction). We get:

$$e^{-g}\,\Gamma_{1r}\left(\hat{D}_1\,\epsilon - A_1\partial_\tau\epsilon\right) + \frac{1}{2}\,e^{f-2g}\,\Gamma_{5r12}\,\epsilon - \frac{1}{2}\left(\frac{h'}{4h}+g'\right)\epsilon -$$
$$-\frac{Q_c}{8}\,e^{-4g-f}\,h^{-1}\,\epsilon - \frac{Q_f}{8}\,e^{\frac{\phi}{2}-m-2g}\,\epsilon = 0\,, \tag{2.115}$$

where $\hat{D}$ is the spinor covariant derivative on the KE base (*i.e.*, with the spin-connection $\hat{\omega}^{ij}$) and A is the 1-form potential of J_{KE} (see (2.14)). From (2.115) it is clear that we must impose a further projection on ϵ, namely:

$$\Gamma_{5r12}\,\epsilon = \epsilon\,. \tag{2.116}$$

One can check that this new projection combined with the ones previously imposed to ϵ implies:

$$\Gamma_{12}\,\epsilon = \Gamma_{34}\,\epsilon = \Gamma_{r5}\,\epsilon = i\sigma_2\,\epsilon\,. \tag{2.117}$$

We now use the fact that in any KE space there is a covariantly constant spinor which satisfies the condition:

$$\hat{D}_i\,\epsilon = \frac{3}{2}\,\Gamma_{12}\,A_i\,\epsilon = \frac{3}{2}\,(i\sigma_2)\,A_i\,\epsilon\,. \tag{2.118}$$

Actually, in the vielbein basis of the KE space we are using it turns out that ϵ can be taken to be independent of the KE coordinates and (2.118) follows from our projections. Moreover, the difference appearing on the first term in (2.115) (for any KE direction) becomes:

$$\hat{D}_i\,\epsilon - A_i\,\partial_\tau\,\epsilon = A_i\left(\frac{3}{2}\,(i\sigma_2)\,\epsilon - \partial_\tau\,\epsilon\right)\,, \tag{2.119}$$

and clearly vanishes if we require that ϵ depends on τ in such a way that:

$$\partial_\tau\,\epsilon = \frac{3}{2}\,(i\sigma_2)\,\epsilon = \frac{3}{2}\,\Gamma_{12}\,\epsilon\,. \tag{2.120}$$

Using (2.116) and (2.120), we arrive at the following differential equation for the function g:

$$g' = -\frac{h'}{4h} + e^{f-2g} - \frac{Q_c}{4}\,e^{-4g-f}\,h^{-1} - \frac{Q_f}{4}\,e^{\frac{\phi}{2}-m-2g}\,. \tag{2.121}$$

Let us now substitute the value (2.107) of h' on the right-hand side of (2.121). We get the following equation for g:

$$g' = e^{f-2g}\,. \tag{2.122}$$

Let us finally consider the variation of the gravitino along the SE fiber τ. We get:

$$\Gamma_{5r}\Big(e^{-f}\,\partial_\tau\epsilon - e^{f-2g}\,\Gamma_{12}\,\epsilon\Big) - \frac{1}{2}\left(\frac{h'}{4h}+f'\right)\epsilon - \frac{Q_c}{8}\,e^{-4g-f}\,h^{-1}\,\epsilon + \frac{Q_f}{8}\,e^{\frac{\phi}{2}-m-2g}\,\epsilon = 0\ , \quad (2.123)$$

After using (2.116) and (2.120), we arrive at the following first-order equation for f:

$$f' = -\frac{h'}{4h} + 3e^{-f} - 2e^{f-2g} - \frac{Q_c}{4}\,e^{-4g-f}\,h^{-1} + \frac{Q_f}{4}\,e^{\frac{\phi}{2}-m-2g}\ . \quad (2.124)$$

which, after using (2.107), becomes:

$$f' = 3\,e^{-f} - 2e^{f-2g} + \frac{Q_f}{2}\,e^{\frac{\phi}{2}-m-2g}\ . \quad (2.125)$$

Collecting (2.101), (2.107), (2.110), (2.122) and (2.125), we obtain the BPS first-order system (2.24).

Let us finally find the expression of the spinor ϵ satisfying all the conditions we have imposed. We first notice that (2.100) can be written as:

$$\Gamma^{rx^3 14}\,e^{-3\tau\,\Gamma_{12}}\,\sigma_1\,\epsilon = \epsilon\ , \quad (2.126)$$

and is solved by a spinor of the form:

$$\epsilon = h^{-\frac{1}{8}}\,e^{\frac{3}{2}\,\Gamma_{12}\,\tau}\,\eta\ , \quad (2.127)$$

where η is a constant spinor which satisfies the following projection equation:

$$\Gamma_{rx^3 14}\,\sigma_1\,\eta = \eta\ . \quad (2.128)$$

In (2.127) we have already taken into account the dependence of ϵ on the radial coordinate written in (2.114). It is now immediate to check that all the conditions required are fulfilled if η is constant and, besides (2.128), satisfies the equation:

$$\Gamma_{12}\,\eta = \Gamma_{34}\,\eta = \Gamma_{r5}\,\eta = i\sigma_2\,\eta\ . \quad (2.129)$$

Notice that, after using (2.129), the expression of ϵ written in (2.127) is the same as in (2.25).

2.A.1 Some Sasaki-Einstein spaces

Let us finish this appendix by writing the coordinate representation of the metric of two particularly relevant Sasaki-Einstein spaces. First of all we consider the case in which $\mathcal{M}_5 = T^{1,1}$, whose KE base is the product $\mathbb{S}^2\times\mathbb{S}^2$. The vielbein 1-forms e^i can be parameterized as:

$$e^1 = \frac{1}{\sqrt{6}}\sin\theta_1\,d\phi_1\ , \qquad e^2 = \frac{1}{\sqrt{6}}\,d\theta_1\ ,$$
$$e^3 = \frac{1}{\sqrt{6}}\sin\theta_2\,d\phi_2\ , \qquad e^4 = \frac{1}{\sqrt{6}}\,d\theta_2\ , \quad (2.130)$$

where θ_i and ϕ_i are angles which take values in the range $0 \le \theta_1, \theta_2 \le \pi$ and $0 \le \phi_1, \phi_2 < 2\pi$. The fiber coordinate τ in the $T^{1,1}$ space is usually represented as $\tau = \psi/3$, where $0 \le \psi < 4\pi$, and the 1-form A takes the form:

$$A \,=\, \frac{1}{3}\big(\cos\theta_1\, d\phi_1 \,+\, \cos\theta_2\, d\phi_2\big)\ . \tag{2.131}$$

From this coordinate representation it is straightforward to compute the volume of the $T^{1,1}$ space, as well as the value of corresponding constant Q_c. We get:

$$\mathrm{Vol}\big(T^{1,1}\big) \,=\, \frac{16}{27}\,\pi^3\ , \qquad\qquad Q_c\big(T^{1,1}\big) \,=\, 27\,\pi\, g_s\, \alpha'^{\,2}\, N_c\ . \tag{2.132}$$

Our second example is the 5-sphere $\mathbb{S}^5$, which is a SE space with $\mathbb{C}\,\mathbb{P}^2$ base. In order to represent the 4-dimensional metric of the $\mathbb{C}\,\mathbb{P}^2$ base, let us consider an angular coordinate χ taking values in the range $0 \le \chi \le \pi$, as well as a set of $SU(2)$ left-invariant 1-forms ω^i $(i = 1, 2, 3)$ satisfying $d\omega^i \,=\, \frac{1}{2}\,\epsilon^{ijk}\,\omega^j \wedge \omega^k$. The ω^i can be represented in terms of three angles (θ, φ, ψ) as:

$$\begin{aligned}
\omega^1 \,&=\, \cos\psi\, d\theta \,+\, \sin\psi\, \sin\theta\, d\varphi \\
\omega^2 \,&=\, \sin\psi\, d\theta \,-\, \cos\psi\, \sin\theta\, d\varphi \\
\omega^3 \,&=\, d\psi \,+\, \cos\theta\, d\varphi\ ,
\end{aligned} \tag{2.133}$$

where $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$, and $0 \le \psi < 4\pi$. The coordinate representation of the canonical vielbein basis of $\mathbb{C}\,\mathbb{P}^2$ is:

$$\begin{aligned}
e^1 \,&=\, \frac{1}{2}\cos\Big(\frac{\chi}{2}\Big)\,\omega^1\ , & e^2 \,&=\, \frac{1}{2}\cos\Big(\frac{\chi}{2}\Big)\,\omega^2\ , \\
e^3 \,&=\, \frac{1}{2}\cos\Big(\frac{\chi}{2}\Big)\,\sin\Big(\frac{\chi}{2}\Big)\,\omega^3\ , & e^4 \,&=\, \frac{1}{2}\,d\chi\ .
\end{aligned} \tag{2.134}$$

In this case the fiber τ takes values in the range $0 \le \tau \le 2\pi$ and the 1-form A is:

$$A \,=\, \frac{1}{2}\,\cos^2\Big(\frac{\chi}{2}\Big)\,\omega^3\ . \tag{2.135}$$

Finally, after a simple calculation, we readily obtain:

$$\mathrm{Vol}\big(\mathbb{S}^5\big) \,=\, \pi^3\ , \qquad\qquad Q_c\big(\mathbb{S}^5\big) \,=\, 16\,\pi\, g_s\, \alpha'^{\,2}\, N_c\ . \tag{2.136}$$

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3

A D3-D5 black hole

3.1 Introduction

In this chapter, we construct a backreacted solution with an event horizon for the D3-D5 intersection studied in the previous chapter. Having this black hole geometry amounts to consider the field theory dual at non-zero temperature. We will work in the Veneziano limit, and we will smear the D5-branes in the same way as we did in the setup of the previous chapter, creating a (2+1)-dimensional, codimension-1 defect on the worldvolume of the D3-branes, whose field theory dual was already described in chapter 1.

Most of the smeared flavored geometries found in the literature preserve some amount of supersymmetry. Indeed, in these models the preservation of supersymmetry is a crucial guide to find the deformation induced by the flavor branes. However, there are other solutions which are not supersymmetric and correspond to systems at finite temperature and/or finite baryon density (see [37, 47, 140–142]). It turns out that adding an event horizon to the geometry of the previous chapter is straightforward and amounts to adding a blackening factor to the metric. This blackening factor has a non-standard power dependence on the radial coordinate due to the spatial anisotropy of the geometry.

In our background the D5-branes are homogeneously distributed along the internal directions, as well as across the cartesian direction transverse to the defect. Therefore, our gravitational solution should be regarded as the holographic dual of a multilayered system. The different layers are created by the stack of flavor D5-branes distributed in parallel 2-dimensional planes inside the 3-dimensional space. The resulting system has one distinguished direction and thus it is clearly anisotropic. We want to explore its properties for observables living in a single layer and also for those connecting two different layers. We will find that, non-trivially, the intra-layer dynamics is the same as that of a stack of effective D2-branes, which means that strongly coupled (2+1)-dimensional SYM can be used to describe our system. We will also be able to study some inter-layer properties.

We will start our analysis by studying the thermodynamics of the D3-D5 black hole and by computing by different methods the VEV of the stress-energy tensor of the dual theory. This analysis will serve us to characterize the anisotropy of the system from the holographic perspective. There is an extensive literature on anisotropic holography. In a by no means exhaustive list, let us mention the articles [138, 143–153], where other backgrounds dual to anisotropic theories have been obtained (some of these geometries are also generated by the backreaction of branes). We will also be able to compute the transport coefficients up to second order for perturbations that propagate along the (2+1)-dimensional intersection of the D3- and D5-branes. We will find that these transport coefficients are the same as those of a D2-brane, a result which is not expected a priori.

The organization of the rest of this chapter is the following. In section 3.2 we present our black hole background, whose thermodynamic properties are analyzed in section 3.3. Besides its temperature and entropy, we obtain the chemical potential associated to the D5-brane charge. This allows us to obtain the Helmhotz and Gibbs free energies and find the speed of sound in the directions parallel and orthogonal to the defect. We will check these results by computing the VEV of the stress-energy tensor from the regularized Brown-York tensor of the gravity theory. In section 3.4 we obtain an effective gravitational action for our problem in 4 dimensions, which we renormalize holographically by means of a suitable boundary counterterm constructed from a superpotential. In section 3.5 we present a 5-dimensional gravitational action for our system, which includes a smeared codimension-1 DBI contribution due to the D5-branes. The regulating boundary term for this action contains a bulk superpotential, as well as a superpotential generated by the flavor branes. We use both the 4- and 5-dimensional regulated actions to calculate the VEV of the stress-energy tensor and to confirm the values obtained in the thermodynamic analysis. In section 3.6 we use the 4-dimensional effective action to compute the transport coefficients in the shear and sound channels. Finally, in section 3.7 we summarize our results. The chapter is completed with three appendices with details of the calculations presented in the main text.

3.2 The black hole geometry

In this section we present the metric of our black hole geometry, as well as its forms. Our background is based on the array (2.3) of D3- and D5-branes, where the N_c D3-branes are color branes and the N_f D5-branes are flavor branes. As it is clear from (2.1), the D5-branes create a (2+1)-dimensional defect in the (3+1)-dimensional bulk gauge theory. In general, the directions 4-9 correspond to a Sasaki-Einstein cone, with the D3-branes located at the tip of the cone. For concreteness we will consider here the case in which the

D3-branes are in flat space and, therefore, the base of the cone will be just the 5-sphere $\mathbb{S}^5$.

The 10-dimensional metric of our geometry in Einstein frame has the factorized form:

$$ds_{10}^2 = ds_5^2 + d\hat{s}_5^2 \, , \tag{3.1}$$

where ds_5^2 is:

$$ds_5^2 = \frac{r^2}{R^2}\Big[-b\,(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + e^{-2\phi}\,(dx^3)^2 \Big] + R^2 \frac{dr^2}{b\,r^2} \, , \tag{3.2}$$

where R is a constant radius and $b = b(r)$ is the blackening factor, given by:

$$b = 1 - \left(\frac{r_h}{r}\right)^{\frac{10}{3}} \, , \tag{3.3}$$

with r_h being the horizon radius. The function ϕ multiplying the metric (3.2) along the x^3 direction is the type IIB supergravity dilaton, which is not constant due to the presence of the D5-branes. The running of ϕ characterizes the anisotropy introduced by the flavor branes in the (3+1)-dimensional gauge theory.

The metric $d\hat{s}_5^2$ in (3.1) corresponds to the internal part of the 10-dimensional geometry. As in the smeared solution of the previous chapter this internal metric is just a deformed $\mathbb{S}^5$. This deformation can be easily described when the $\mathbb{S}^5$ is represented as a $U(1)$ bundle over $\mathbb{CP}^2$: the deformation is just a squashing of the $U(1)$ fiber relative to the $\mathbb{CP}^2$ base. Actually, the internal part of our metric is:

$$d\hat{s}_5^2 = \bar{R}^2 \left[ds^2_{\mathbb{CP}^2} + \frac{9}{8}(d\tau + A)^2 \right] \, , \tag{3.4}$$

where $\bar{R}^2$ is a constant related to the radius R as:

$$\bar{R}^2 = \frac{9}{8} R^2 \, . \tag{3.5}$$

and the $\mathbb{CP}^2$ base is given by:

$$ds^2_{\mathbb{CP}^2} = d\chi^2 + \frac{\cos^2\chi}{4}((\omega^1)^2 + (\omega^2)^2) + \frac{\cos^2\chi \sin^2\chi}{4}(\omega^3)^2 \, , \tag{3.6}$$

where χ is an angular coordinate taking values in the range $0 \leq \chi \leq \pi$ and ω^1, ω^2 and ω^3 are the three $SU(2)$ left-invariant 1-forms introduced in (2.133). The fiber τ in (3.4) takes values in the range $0 \leq \tau \leq 2\pi$ and the 1-form A is given by (2.135).

Our backreacted background is a solution of the equations of motion derived from the total action of the system, which is the sum of the type IIB supergravity action and of the action of the D5-branes:

$$S = S_{IIB} + S_{branes} \, . \tag{3.7}$$

The action of type IIB supergravity in Einstein frame is:

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\left[\int d^{10}x\,\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) - \int\left(\frac{1}{2}\,e^{\phi}F_3\wedge *F_3 \,+\, \frac{1}{4}\,F_5\wedge *F_5\right)\right]\,, \tag{3.8}$$

while the action of the branes is given by the sum of DBI and WZ terms:

$$S_{branes} \,=\, -T_5\sum_{N_f}\left(\,\int_{\mathcal{M}_6} d^6\xi\, e^{\frac{\phi}{2}}\,\sqrt{-\hat{g}_6}\,-\,\int_{\mathcal{M}_6}\,\hat{C}_6\right)\,, \tag{3.9}$$

where T_5 is the tension of the D5-brane, given by (2.8), $\hat{g}_6$ is the determinant of the induced metric on the worldvolume $\mathcal{M}_6$ and $\hat{C}_6$ is the pullback to $\mathcal{M}_6$ of the RR 6-form potential of the type IIB theory. In (3.8) we have only included the RR 3- and 5-forms F_3 and F_5, which are the only non-trivial ones for our D3-D5 geometry.

As seen in the previous chapter, the stack of color D3-branes induces a self-dual RR 5-form F_5, whose form is given by (2.3), where $K = K(r)$ is a function of the radial variable whose explicit expression can be found in (2.45). Moreover, the N_f flavor D5-branes act as a source of the RR 3-form F_3 through the WZ term of the action (3.9).

We will work as in the previous chapter, in the smearing approach, valid when N_f is large, and we substitute the discrete distribution of flavor branes by a continuous distribution with the appropriate normalization, in such a way that the smearing amounts to performing the substitution given in (2.9). The resulting smearing 4-form Ξ has components along the directions orthogonal to the worldvolume of the flavor branes, which characterizes the charge distribution of the flavor branes. As shown in chapter 2, this WZ coupling induces a violation of the Bianchi identity of F_3, given by (2.12).

The vielbein basis of $\mathbb{C}\,\mathbb{P}^2$ is the one shown in (2.134). With them we construct Ω_2 and $\hat{\Omega}_2$ as given by equations (2.15) and (2.17). F_3 is then given by (2.21), and the Bianchi identity takes the form (2.23), from which we can read the smearing form. Therefore we have:

$$F_3 \,=\, Q_f\,dx^3\wedge\mathrm{Im}\,\hat{\Omega}_2\,, \tag{3.10}$$

where Q_f is a constant proportional to the number of flavors N_f. The precise relation between Q_f and N_f can be obtained by analyzing the embeddings of the family of flavor branes that source the background. For the case of flavor branes dual to massless quarks we get:

$$Q_f \,=\, \frac{4\,\pi\,N_f}{9\,\sqrt{3}}\,. \tag{3.11}$$

The smearing form Ξ does not depend on x^3 (it only depends on dx^3), which means that we are homogeneously distributing our flavor branes in the x^3 direction and, therefore, we can regard our setup as dual to a multilayer system. Moreover, Ξ is also independent of the radial coordinate r, as expected for a charge distribution corresponding to massless quarks. The radii R and $\bar{R}$ depend on the number of color branes N_c, through the relations (2.73) and (2.76). where Q_c is proportional to N_c and given by (2.136).

In what follows we will take $g_s = \alpha' = 1$. The dilaton for our solution is given by (2.68). It is important to point out that our solution is non-analytic in N_f, which means that we cannot take the unflavored limit $N_f = 0$ and recover the isotropic $AdS_5 \times \mathbb{S}^5$ background.[1]

When $r_h = 0$ (and $b = 1$) our solution is supersymmetric, as shown in chapter 2, and can be found by solving a set of first-order BPS equations and is invariant under a set of Lifshitz-like anisotropic scale transformations in which the x^3 coordinate transforms with an anomalous exponent $z = 3$.

In order to explore the physical consequences of the anisotropy of our background, we have computed in appendix 3.A the potential energy, at zero temperature, for a quark-antiquark pair, following the holographic prescription of references [154, 155]. We have considered the cases in which the charges are in the same layer (*i.e.*, when they have the same value of x^3) and when they are separated along x^3. Let us summarize here the results. The intra-layer potential takes the form:

$$V_{q\bar{q}} \sim \frac{N_c^{\frac{2}{3}}}{N_f^{\frac{1}{3}}} \frac{1}{d_\parallel^{\frac{4}{3}}} , \tag{3.12}$$

where $d_\parallel$ is the $q\bar{q}$ distance in the $x^1 x^2$ plane. Moreover, for charges with the same values of (x^1, x^2) and separated a distance $d_\perp$ along the coordinate x^3, we obtain:

$$V_{q\bar{q}} \sim \frac{N_c^2}{N_f^3} \frac{1}{d_\perp^4} . \tag{3.13}$$

The different behaviours (3.12) and (3.13) gives us a measure of the effects of the anisotropy on physical observables. Another effect of this anisotropy is encoded in the entanglement entropies for slab regions and their complements at zero temperature. For a slab with a finite width $l_\parallel$ in the plane, the entanglement entropy behaves as (see appendix 3.A for details):

$$S_\parallel \sim \frac{N_f^{\frac{2}{3}} N_c^{\frac{5}{3}}}{l_\parallel^{\frac{4}{3}}} , \tag{3.14}$$

whereas if the slab has a finite width $l_\perp$ along x^3 we get:

$$S_\perp \sim \frac{N_c^4}{N_f^4} \frac{1}{l_\perp^6} . \tag{3.15}$$

Equations (3.14) and (3.15) contain information about the quantum correlations of the model. In particular, the dependence of the entropies on the length determines the critical behaviour of the mutual information. Interestingly, $S_\parallel$ depends on N_c and $l_\parallel$ as in the case

[1] One can take this $N_f = 0$ limit in the equations of motion but not in their particular solution corresponding to our background.

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of a D2-brane. We will find several times in this chapter this equivalence of the intra-layer physics with the one corresponding to an effective D2-brane.

When $r_h \neq 0$ our solution has a horizon and becomes a black hole with a non-zero temperature. In this case one can show that it solves the Einstein equations with sources that follow from the action (3.7). In particular the DBI term of (3.9) contributes to the energy-momentun tensor and, as already mentioned, the WZ term induces a violation of the Bianchi identity of F_3. In the next section we explore the thermodynamic properties of this black hole.

3.3 Thermodynamics of the black hole

Let us now work out the thermodynamics of the black hole presented in the previous section. First of all, we recall that the temperature T is given by the general formula:

$$T = \frac{1}{2\pi} \left[\frac{1}{\sqrt{g_{rr}}} \frac{d}{dr} \left(\sqrt{-g_{x^0x^0}} \right) \right]_{r=r_h} , \tag{3.16}$$

which leads to the following relation between T and the horizon radius r_h:

$$T = \frac{5\, r_h}{6\pi\, R^2} \, . \tag{3.17}$$

Using (2.73) we can recast this relation in terms of Q_c as:

$$r_h = \frac{2^5\, \pi}{3^{\frac{3}{2}}\, 5^{\frac{3}{2}}}\, Q_c^{\frac{1}{2}}\, T \, . \tag{3.18}$$

The entropy density s is given by the Bekenstein-Hawking formula:

$$s = \frac{2\pi}{\kappa_{10}^2} \frac{A_8}{V_3} , \tag{3.19}$$

where A_8 is the volume at the horizon of the 8-dimensional space orthogonal to t and r and V_3 is the infinite constant volume of the 3-dimensional Minkowski directions. For our black hole geometry we get:

$$\frac{A_8}{V_3} = 2^{-\frac{11}{3}}\, 3^{\frac{17}{6}}\, 5^{-\frac{1}{2}}\, \pi^3\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{1}{2}}\, r_h^{\frac{7}{3}} \, . \tag{3.20}$$

After using (3.18) to relate r_h and T, we arrive at:

$$s = \frac{2^3}{5^4\, 3^{\frac{2}{3}}\, \pi^{\frac{2}{3}}}\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}\, T^{\frac{7}{3}} \, . \tag{3.21}$$

Notice the fractional powers of Q_c and Q_f in (3.21), which mean that s has a non-standard dependence on N_c and N_f. To explore further this dependence, let us rewrite (3.21) in

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terms of N_c and N_f. With this purpose we use the relations (2.5) and (3.11), from which we get that the combination appearing in (3.21) is given by:

$$Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}} \;=\; \frac{256}{3\, 3^{\frac{2}{3}}}\, \pi^{\frac{7}{3}}\, N_f^{\frac{2}{3}}\, N_c^{\frac{5}{3}} \;, \tag{3.22}$$

and the entropy density can be written as:

$$s \;=\; \alpha_s\, N_f^{\frac{2}{3}}\, N_c^{\frac{5}{3}}\, T^{\frac{7}{3}} \;, \tag{3.23}$$

where α_s is the following numerical coefficient:

$$\alpha_s \;=\; \frac{2048}{5625}\, \frac{\pi^{\frac{5}{3}}}{3^{\frac{1}{3}}} \;\approx\; 1.701 \;. \tag{3.24}$$

The ADM energy of the background is given by the standard equation:

$$E_{ADM} \;=\; -\frac{1}{\kappa_{10}^2}\, \sqrt{|g_{tt}|}\, \int_{\mathcal{M}_{t,r_\infty}} \sqrt{\det g_8}\, \left(K_T \,-\, K_0 \right) \,, \tag{3.25}$$

where the symbols K_T and K_0 denote the extrinsic curvatures of the 8-dimensional subspace within the 9-dimensional (constant time) space, at finite and zero temperature, respectively. For an arbitrary hypersurface the extrinsic curvature K is given by:

$$K \;=\; \frac{1}{\sqrt{\det g_9}}\, \partial_\mu \left(\sqrt{\det g_9}\; n^\mu \right) \,, \tag{3.26}$$

with n^μ being a normalized vector perpendicular to the surface. For a constant r hypersurface, we have:

$$n^\mu \;=\; \frac{1}{\sqrt{g_{rr}}}\, \delta_r^\mu \;. \tag{3.27}$$

For our geometry it is straightforward to prove that:

$$K \;=\; \frac{7}{3R}\, \sqrt{b} \,, \tag{3.28}$$

where b is the blackening factor (3.3). From this result it follows that:

$$K_T \;=\; \frac{7}{3R}\, \sqrt{1 - \left(\frac{r_h}{r}\right)^{\frac{10}{3}}} \,, \qquad\qquad K_0 \;=\; \frac{7}{3R} \,, \tag{3.29}$$

and thus the difference of the extrinsic curvatures appearing in (3.25) is:

$$K_T \,-\, K_0 \;\approx\; -\frac{7}{6R}\, \left(\frac{r_h}{r}\right)^{\frac{10}{3}} \,, \qquad\qquad (r \to \infty) \;. \tag{3.30}$$

The energy density ϵ can now be easily computed, with the result:

$$\epsilon \,=\, \frac{E_{ADM}}{V_3} \,=\, \frac{7}{10}\,\alpha_s\,N_f^{\frac{2}{3}}\,N_c^{\frac{5}{3}}\,T^{\frac{10}{3}} \,=\, \beta_s\,Q_f^{\frac{2}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{10}{3}}\;, \tag{3.31}$$

where α_s is the numerical coefficient (3.24) and we have introduced a new numerical factor β_s, given by:

$$\beta_s \,=\, \frac{28}{3125\,(3\pi)^{\frac{2}{3}}}\;. \tag{3.32}$$

Notice that the entropy density (3.21) can be rewritten as:

$$s \,=\, \frac{10}{7}\,\beta_s\,Q_f^{\frac{2}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{7}{3}}\;. \tag{3.33}$$

The free energy density f in the canonical ensemble is defined as:

$$f \,=\, \epsilon \,-\, T\,s\;. \tag{3.34}$$

By using (3.23) and (3.31) we readily obtain:

$$f \,=\, -\frac{3}{10}\,\alpha_s\,N_f^{\frac{2}{3}}\,N_c^{\frac{5}{3}}\,T^{\frac{10}{3}} \,=\, -\frac{3}{7}\,\beta_s\,Q_f^{\frac{2}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{10}{3}}\;. \tag{3.35}$$

To explore the complete thermodynamics of the system it is convenient to consider the situation in which the number of flavor D5-branes can change. In our setup this number of flavor branes is determined by Q_f. Therefore, we allow Q_f to vary and we will introduce the chemical potential Φ, conjugate to Q_f. The first law of thermodynamics for these variables becomes:

$$d\epsilon \,=\, T\,ds \,+\, \Phi\,dQ_f\;. \tag{3.36}$$

Clearly, the chemical potential Φ measures the energy cost of introducing additional flavor branes in the system. After performing the Legendre transform as in (3.34), we can write the variation of the free energy f in the canonical ensemble as:

$$df \,=\, -s\,dT \,+\, \Phi\,dQ_f\;. \tag{3.37}$$

It follows immediately from (3.37) that s and Φ are given by the following partial derivatives of f:

$$s \,=\, -\Big(\frac{\partial f}{\partial T}\Big)_{Q_f}\;, \qquad\qquad \Phi \,=\, \Big(\frac{\partial f}{\partial Q_f}\Big)_{T}\;. \tag{3.38}$$

By using (3.35), it is now straightforward to compute the partial derivative of f with respect to T and check the first equation in (3.38). Moreover, by computing the derivative of (3.35) with respect to Q_f we obtain the expression of the chemical potential Φ:

$$\Phi \,=\, -\frac{2}{7}\,\beta_s\,Q_f^{-\frac{1}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{10}{3}}\;. \tag{3.39}$$

The Gibbs free energy, *i.e.*, the thermodynamic potential in the grand canonical ensemble, is defined as:

$$g = f - \Phi\, Q_f \ . \tag{3.40}$$

Plugging (3.35) and (3.39) on the right-hand side of (3.40) we get the value of g for our system:

$$g = -\frac{1}{7}\,\beta_s\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}\, T^{\frac{10}{3}} \ . \tag{3.41}$$

As argued in [143] (see also [156]), the two thermodynamic potentials f and g are related to the pressure in the x^1x^2 plane (p_{xy}) and in the x^3 direction (p_z) as:

$$f = -p_{xy} \ , \qquad\qquad g = -p_z \ . \tag{3.42}$$

To demonstrate these identifications of the free energies with the pressures one should take into account the extensivity of the energy and the anisotropic character of our system (details can be found in [143]). In our system these pressures are thus given by:

$$p_{xy} = \frac{3}{7}\,\beta_s\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}\, T^{\frac{10}{3}} = \frac{3}{7}\,\epsilon \ , \qquad\qquad p_z = \frac{1}{7}\,\beta_s\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}\, T^{\frac{10}{3}} = \frac{1}{7}\,\epsilon \ . \tag{3.43}$$

The speeds of sound along the x^1x^2 and x^3 are defined as:

$$v_{xy}^2 = \left(\frac{\partial p_{xy}}{\partial \epsilon}\right)_{Q_f} \ , \qquad\qquad v_z^2 = \left(\frac{\partial p_z}{\partial \epsilon}\right)_{Q_f} \ . \tag{3.44}$$

Using (3.43) we can readily evaluate the derivatives on the right-hand side of (3.44), with the result:

$$v_{xy}^2 = \frac{3}{7} \ , \qquad\qquad v_z^2 = \frac{1}{7} \ , \tag{3.45}$$

to be compared with the value $v_s^2 = 1/2$ for a 2-dimensional CFT and $v_s^2 = 1/3$ for a 3-dimensional CFT.[2]

The pressure difference is a manifestation of the anisotropy of the system and is measured by the non-vanishing chemical potential. Actually, it is straightforward to verify that, for our system, one has:

$$p_z - p_{xy} = \Phi\, Q_f \ . \tag{3.46}$$

Moreover, we have the following equation of state:

$$\epsilon = 2\, p_{xy} + p_z \ . \tag{3.47}$$

By combining (3.46) and (3.47) we can obtain the two pressures as functions of ϵ and Q_f:

$$p_{xy} = \frac{1}{3}\,\epsilon - \frac{1}{3}\,\Phi\, Q_f \ , \qquad\qquad p_z = \frac{1}{3}\,\epsilon + \frac{2}{3}\,\Phi\, Q_f \ . \tag{3.48}$$

[2]The speed of sound for a Dp-brane is $v_s^2 = \frac{5-p}{9-p}$. Therefore v_{xy} coincides with the speed of sound of a D2-brane.

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It is also easy to relate the different quantities to the entropy:

$$\epsilon = \frac{7}{10}\,T s\;, \qquad f\,=\,-\frac{3}{10}\,T s\;, \qquad g\,=\,-\frac{1}{10}\,T s\;, \qquad \Phi\,Q_f\,=-\frac{1}{5}\,T s\;. \tag{3.49}$$

From these equations one can show that the following relation holds:

$$\epsilon\,=\,\frac{3}{4}\,T s\,+\,\frac{1}{4}\,\Phi\,Q_f\;, \tag{3.50}$$

as well as the so-called Gibbs-Duhem relations:

$$\epsilon\,+\,p_{xy}\,=\,T s\;, \qquad\qquad \epsilon\,+\,p_z\,=\,T s\,+\,\Phi\,Q_f\;. \tag{3.51}$$

Finally, the heat capacity is:

$$c_v\,=\,\partial_T\,\epsilon\,=\,\frac{7}{3}\,\alpha_s\,N_f^{\frac{2}{3}}\,N_c^{\frac{5}{3}}\,T^{\frac{7}{3}}\,=\,\frac{10}{3}\,\beta_s\,Q_f^{\frac{2}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{7}{3}}\;. \tag{3.52}$$

To get some insight on the nature of our solution, let us analyze the dependence of the entropy density s on N_c, N_f and T and let us compare it with some known results for other gravity duals. It follows from (3.33) that s behaves with the temperature as $s \sim T^{\frac{7}{3}}$. For a Dp-brane background $s \sim T^{\frac{9-p}{5-p}}$ [157]. Taking $p = 2$ in this last formula we obtain the same behaviour as in (3.33). This is an indication that our geometry is related to the one generated by D2-branes. Actually, if we define λ as:

$$\lambda\,=\,\frac{N_c}{N_f^2}\;, \tag{3.53}$$

then the entropy density (3.33) can be written as:

$$s \sim N_c^2\,\lambda^{-\frac{1}{3}}\,T^{\frac{7}{3}}\;, \tag{3.54}$$

which is exactly the form of the entropy of a D2-brane black hole if λ is interpreted as a 't Hooft coupling [157].[3] In the case of a stack of N_c D2-branes, realizing (2+1)-dimensional SYM, the 't Hooft coupling is $\lambda = g_{YM}^2\,N_c$ ($\lambda = N_c$ in our units). Our result suggests that, in our flavored system, the relevant scaling of the coupling with N_c and N_f is the one written in (3.53). Notice that having a ratio of the numbers of color and flavors as parameter is very natural in a limit of the Veneziano type. Notice also [158] that the dimensionless parameter controlling the backreaction of the flavor D5-branes is $\kappa_{10}^2\,N_f\,T_{D5}\,R^{-2} \sim N_f/\sqrt{N_c}$. In this parameter N_c and N_f scale precisely as in (3.53).

The matching we found of the entropy with the one corresponding to a D2-brane is an indication that the dynamics in the x^1x^2 plane (at least its deviation from conformality)

[3]Equivalently, if we define the temperature-dependent effective dimensionless coupling as $\lambda_{eff}(T) = \lambda/T$, the entropy density (3.54) can be written as $s \sim N_c^2\,[\lambda_{eff}(T)]^{-\frac{1}{3}}\,T^2$. We are grateful to Javier Tarrío for suggesting this interpretation of our entropy formula.

is governed by a (2+1)-dimensional SYM theory in the strongly coupled regime. The value of the speed of sound v_{xy} found above points in the same direction. In section 3.6 we will confirm this fact by computing the hydrodynamic transport coefficients for perturbations propagating in the x^1x^2 plane. Actually, there is a direct way to relate our setup to a system of D2-branes. Indeed, by performing a T-duality transformation along the x^3 direction we can convert our D3-D5 solution into a D2-D6 geometry, in which the D2's are the color branes and the D6's are the flavor branes. In this D2-D6 solution the x^3 direction is now a distinguished coordinate transverse to the color branes. The corresponding 10-dimensional metric of type IIA supergravity in the Einstein frame takes the form:

$$ds^2_{IIA} = \Big(\frac{4\,Q_f}{3}\Big)^{\frac{1}{4}}\,\frac{r^{\frac{9}{4}}}{R^{\frac{5}{2}}}\Bigg[-\,b\,(dx^0)^2\,+\,(dx^1)^2\,+\,(dx^2)^2\,+\,\frac{R^4}{r^4}\frac{dr^2}{b}\,+ \\ +\,\frac{9}{8}\frac{R^4}{r^2}\,\Big(\frac{(d\bar{x}^3)^2}{r^{\frac{4}{3}}}\,+\,ds^2_{\mathbb{CP}^2}+\frac{9}{8}\,(d\tau+A)^2\Big)\Bigg]\,, \tag{3.55}$$

where $b = b(r)$ is the blackening factor (3.3) and the coordinate $\bar{x}^3$ is related to the original cartesian coordinate x^3 by the following rescaling:

$$\bar{x}^3 \,=\, \Big(\frac{4\sqrt{2}}{9\,Q_f}\Big)^{\frac{1}{3}}\,x^3\,. \tag{3.56}$$

Notice that the D2-branes in this D2-D6 solution are smeared in x^3, since none of the functions of the metric depends on this coordinate.

This type IIA background is also endowed with a running dilaton ϕ_{IIA}, as well as RR 2- and 4-forms, given by:

$$\begin{aligned} e^{2\phi_{IIA}} &= \Big(\frac{3}{4\,Q_f}\Big)^{\frac{7}{3}}\,R^2\,r^{\frac{1}{3}}\;, \\ F_2 &= Q_f\,\mathrm{Im}\,(\hat{\Omega}_2)\;, \\ F_4 &= \frac{20}{3}\Big(\frac{2\,Q_f^2}{9}\Big)^{\frac{1}{3}}\,\frac{r^{\frac{7}{3}}}{R^4}\,dr\wedge dx^0\wedge dx^1\wedge dx^2\;. \end{aligned} \tag{3.57}$$

3.3.1 Stress-energy tensor

The energy density and the pressures of our model can also be obtained by calculating the holographic stress-energy tensor. We will compute this tensor by using several methods and we will check that one gets the same results as those we obtained in the previous subsection by using anisotropic thermodynamics. In this subsection we will compute the VEV of the stress-energy tensor from the Brown-York tensor at the boundary, following the prescription of [159]. In sections 3.4 and 3.5 we will dimensionally reduce our 10-dimensional theory and will calculate the stress-energy tensor by holographic renormalization, after adding suitable boundary terms to the reduced actions.

The Brown-York tensor of the 10-dimensional gravity theory is:

$$\tau_{ij} \,=\, \frac{1}{\kappa_{10}^2}\,\big(K_{ij} \,-\, K\,\gamma_{ij}\big)\;, \tag{3.58}$$

where γ_{ij} is the induced metric at a $r =$ constant surface, K_{ij} is the extrinsic curvature of the surface and $K = \gamma^{ij}\,K_{ij}$. The VEV of the stress-energy tensor of the dual theory is related to the Minkowski components of the Brown-York tensor at the boundary [159]:

$$\langle T^{\mu}{}_{\nu}\rangle \,=\, V_{SE}\,\sqrt{-\gamma_{Min}}\;\tau^{\mu}{}_{\nu}\Big|_{reg\,,\,r_\Lambda\to\infty}\;, \tag{3.59}$$

where V_{SE} is the volume for the compact 5-dimensional part of the metric, which for the S^5 is

$$V_{SE} \,=\, \Big(\frac{9\,\pi}{8}\Big)^3\,R^5\;. \tag{3.60}$$

In (3.59) γ_{Min} is the determinant of the Minkowski part of the induced metric. The right-hand side of (3.59) is divergent at the UV boundary. We will give below a precise prescription to eliminate this divergence.

The extrinsic curvature tensor K_{ij} can be obtained from the covariant expression:

$$K_{ij} \,=\, -\frac{1}{2}\,\big(\nabla_i\,n_j \,+\, \nabla_j\,n_i\big)\;, \tag{3.61}$$

where n_i are the components of the normal vector to the $r =$ constant surface ($n^i\,n_i \,=\, 1$). In a diagonal metric as the one we have in (3.2), the vector n_i is given by:

$$n_i \,=\, \sqrt{g_{rr}}\;\delta_i^r\;. \tag{3.62}$$

Let us now introduce the notation:

$$g_{x^0x^0} \,\equiv\, -k_1^2 \,=\, -\frac{r^2}{R^2}\,b\;, \qquad\qquad g_{x^1x^1} \,=\, g_{x^2x^2} \,\equiv\, k_2^2 \,=\, \frac{r^2}{R^2}\;,$$

$$g_{x^3x^3} \,\equiv\, k_3^2 \,=\, \frac{1}{R^2}\,\Big(\frac{4Q_f}{3}\Big)^{\frac{4}{3}}\,r^{\frac{2}{3}}\;, \qquad\qquad g_{rr} \,\equiv\, k_r^2 \,=\, \frac{R^2}{r^2\,b}\;, \tag{3.63}$$

where we are assuming that the metric is given by (3.2) and (3.4). With these notations, we have:

$$\sqrt{-\gamma_{Min}} \,=\, k_1\,k_2^2\,k_3 \,=\, \frac{1}{R^4}\,\Big(\frac{4Q_f}{3}\Big)^{\frac{2}{3}}\,r^{\frac{10}{3}}\,b^{\frac{1}{2}}\;, \tag{3.64}$$

and it is straightforward to compute the components of the extrinsic curvature along the Minkowski directions. The non-vanishing components are:

$$K_{x^0x^0} \,=\, \frac{k_1\,k_1'}{k_r}\;, \qquad\qquad K_{x^1x^1} \,=\, K_{x^2x^2} = -\frac{k_2\,k_2'}{k_r}\;,$$

$$K_{x^3x^3} \,=\, -\frac{k_3\,k_3'}{k_r}\;, \qquad\qquad K \,=\, -\frac{1}{k_r}\,\partial_r\log\big(k_1\,k_2^2\,k_3\big)\;. \tag{3.65}$$

Plugging these results in (3.58) we get explicitly the non-zero components of the Brown-York tensor:

$$
\begin{aligned}
\tau^{x^0}{}_{x^0} &= \frac{1}{\kappa_{10}^2}\,\frac{1}{k_r}\,\partial_r \log\left(k_2^2\, k_3\right) \;=\; \frac{1}{\kappa_{10}^2\, R}\,\frac{7}{3}\, b^{\frac{1}{2}} \;, \\
\tau^{x^1}{}_{x^1} &= \tau^{x^2}{}_{x^2} \;=\; \frac{1}{2\,\kappa_{10}^2\, R}\, r\, b^{\frac{1}{2}}\, \partial_r \log\left(r^{\frac{14}{3}}\, b\right) \;=\; \frac{1}{\kappa_{10}^2\, R}\,\frac{1}{3\, b^{\frac{1}{2}}}\,\left[7 - 2\left(\frac{r_h}{r}\right)^{\frac{10}{3}}\right] \;, \\
\tau^{x^3}{}_{x^3} &= \frac{1}{2\,\kappa_{10}^2\, R}\, r\, b^{\frac{1}{2}}\, \partial_r \log\left(r^6\, b\right) \;=\; \frac{1}{\kappa_{10}^2\, R}\,\frac{1}{b^{\frac{1}{2}}}\left[3 \,-\, \frac{4}{3}\left(\frac{r_h}{r}\right)^{\frac{10}{3}}\right] \;. \qquad (3.66)
\end{aligned}
$$

Let us now specify the regulating procedure we will employ to compute $\langle T^{\mu}{}_{\nu}\rangle$. Since we are interested in matching the thermodynamic values found above, it is enough to subtract the zero temperature supersymmetric value, as it was done in [37] for the D3-D7 system. More concretely, we will take $\langle T^{\mu}{}_{\nu}\rangle$ to be given by:

$$
\langle T^{\mu}{}_{\nu}\rangle \;=\; V_{SE} \lim_{r_\Lambda \to \infty} \left[\sqrt{-\gamma_{Min}}\;\tau^{\mu}{}_{\nu} \;-\; b^{\frac{1}{2}} \lim_{r_h \to 0}\left(\sqrt{-\gamma_{Min}}\;\tau^{\mu}{}_{\nu}\right)\right]_{r=r_\Lambda} \;, \qquad (3.67)
$$

where the $b^{\frac{1}{2}}$ factor is introduced to match the geometries at the cutoff. Using (3.64) and (3.66) we get that the only non-zero components of $\langle T^{\mu}{}_{\nu}\rangle$ are:

$$
\langle T^{x^0}{}_{x^0}\rangle \;=\; -\epsilon \;, \qquad \langle T^{x^1}{}_{x^1}\rangle \;=\; \langle T^{x^2}{}_{x^2}\rangle \;=\; \frac{3\,\epsilon}{7} \;, \qquad \langle T^{x^3}{}_{x^3}\rangle \;=\; \frac{\epsilon}{7} \;, \qquad (3.68)
$$

where ϵ is the ADM energy density (3.31). Equivalently, we can write the VEV of the stress-energy tensor as:

$$
\langle T^{\mu}{}_{\nu}\rangle \;=\; \mathrm{diag}\big(-\epsilon\,,\, p_{xy}\,,\, p_{xy}\,,\, p_z\big) \;, \qquad (3.69)
$$

where p_{xy} and p_z are precisely the values of the pressures found before by introducing the chemical potential.

Notice that the calculation of p_{xy} and p_z using the Brown-York tensor depends on the behaviour of the geometry as we increase the holographic coordinate r and approach the boundary. On the contrary, the calculation of the pressures based on Φ is determined by the behaviour of the geometry as we vary the flavor charge Q_f. The agreement of the results found by these two methods is a non-trivial consistency check of our gravity dual.

3.4 Reduced action in 4 dimensions

In order to apply the full machinery of the holographic duality to our system it is quite convenient to integrate the action over the internal manifold and convert our problem into a system of low dimensional gravity. There are two possible approaches to carry out this reduction. First of all, we could consider the x^3 coordinate as internal and reduce the system to a 4-dimensional system in the coordinates (t, x^1, x^2, r). This is the point of view we will adopt in this section. This approach is very useful to study the dynamics of the system in the (x^1, x^2) plane and, indeed, we will use the results of this section in our analysis of the hydrodynamic modes of section 3.6. Alternatively, we could include x^3 in our set of reduced coordinates and deal with a 5-dimensional anisotropic problem. We will analyze this 5-dimensional reduction in the next section.

The reduction of our problem to a low dimensional gravity system will allow us to implement a holographic renormalization procedure. We will be able to compute in this framework the VEV of the stress-energy tensor and to confirm the thermodynamic results of section 3.3.1. Moreover, in section 3.6 we will study the fluctuations of the 4-dimensional fields and we will obtain some hydrodynamic coefficients.

Let us consider the following reduction ansatz to 4 dimensions of the 10-dimensional metric:

$$ds^2_{10} \,=\, e^{\frac{10}{3}\gamma - \beta}\, g_{mn} dz^m\, dz^n + e^{\frac{10}{3}\gamma + 2\beta}\,(dx^3)^2 + e^{-2(\gamma+\lambda)}\, ds^2_{\mathbb{CP}^2} + e^{2(4\lambda-\gamma)}\,(d\tau + A)^2 \;, \tag{3.70}$$

where $g_{mn} = g_{mn}(z)$ is a 4-dimensional metric and the scalar fields γ, λ and β depend on the 4-dimensional coordinates $z^m = (t, x^1, x^2, r)$. In addition, in the reduced theory we have the dilaton field $\phi = \phi(z)$. The action of this 4-dimensional gravity theory can be obtained from the one of type IIB supergravity. The details of this calculation are given in appendix 3.B. The expression of this effective action is:

$$S_{eff} \,=\, \frac{V_5\, V_{x^3}}{2\,\kappa_{10}^2} \int d^4 z\, \sqrt{-g_4}\, \Big[R_4 - \frac{40}{3}\,(\partial\gamma)^2 - 20\,(\partial\lambda)^2 - \frac{3}{2}\,(\partial\beta)^2 \,-\, \frac{1}{2}\,(\partial\phi)^2 \,-\, V\Big] \;, \tag{3.71}$$

where V is the following potential for the scalar fields ϕ, γ, λ and β:

$$V = 4\, e^{\frac{16}{3}\gamma + 12\lambda - \beta} - 24\, e^{\frac{16}{3}\gamma + 2\lambda - \beta} + Q_f^2\, e^{4\gamma + 4\lambda - 3\beta + \phi} + \frac{Q_c^2}{2}\, e^{\frac{40}{3}\gamma - \beta} + 6\, Q_f\, e^{\frac{14}{3}\gamma - 2\lambda - 2\beta + \frac{\phi}{2}} \;. \tag{3.72}$$

In order to write the equations of motion of the reduced theory in a compact form, let us collect the four scalar fields in a single vector Ψ with components:

$$\Psi = (\phi, \gamma, \lambda, \beta) \;. \tag{3.73}$$

Moreover, we define a coefficient α_Ψ which takes the following values for the different scalar fields:

$$(\alpha_\phi\,,\, \alpha_\gamma\,,\, \alpha_\lambda\,,\, \alpha_\beta) \,=\, (1\,,\, \frac{3}{80}\,,\, \frac{1}{40}\,,\, \frac{1}{3}) \;. \tag{3.74}$$

Then, Einstein equations can be compactly written in terms of Ψ as:

$$R_{mn} \,=\, \sum_{\Psi} \frac{1}{2\alpha_\Psi}\, \partial_m \Psi\, \partial_n \Psi \,+\, \frac{1}{2}\, g_{mn}\, V \;. \tag{3.75}$$

Moreover, if we define the d'Alembertian of any scalar field Ψ as:

$$\Box\,\Psi \equiv \frac{1}{\sqrt{-g_4}}\,\partial_m\left(\sqrt{-g_4}\,g^{mn}\,\partial_n\,\Psi\right)\,, \tag{3.76}$$

then, the equations for the scalar fields are:

$$\Box\,\Psi \;=\; \alpha_\Psi\,\partial_\Psi\,V\;. \tag{3.77}$$

Let us now write our black hole solution in terms of the 4-dimensional variables. The 4-dimensional metric takes the diagonal form:

$$ds_4^2 \;=\; -c_1^2(r)\,dt^2 \;+\; c_2^2(r)[(dx^1)^2 \;+\; (dx^2)^2] \;+\; c_3^2(r)\,dr^2\;. \tag{3.78}$$

The actual values of the c_i coefficients for our background are:

$$\begin{aligned}
c_1^2(r) \;&=\; \left(\frac{9}{8}\right)^3\left(\frac{4\,Q_f}{3}\right)^{\frac{2}{3}} R^2\,b(r)\,r^{\frac{7}{3}}\;,\\
c_2^2(r) \;&=\; \left(\frac{9}{8}\right)^3\left(\frac{4\,Q_f}{3}\right)^{\frac{2}{3}} R^2\,r^{\frac{7}{3}} \;=\; \frac{c_1^2(r)}{b(r)}\;,\\
c_3^2(r) \;&=\; \left(\frac{9}{8}\right)^3\left(\frac{4\,Q_f}{3}\right)^{\frac{2}{3}} \frac{R^6}{r^{\frac{5}{3}}\,b(r)} \;=\; \frac{R^4}{r^4\,b^2(r)}\,c_1^2(r)\;,
\end{aligned} \tag{3.79}$$

where $b(r)$ is the blackening factor defined in (3.3). Moreover, in our geometry the different scalars take the values:

$$\begin{aligned}
e^\phi \;&=\; \left(\frac{3}{4\,Q_f}\right)^{\frac{2}{3}} r^{\frac{2}{3}}\;, \qquad\qquad & e^\gamma \;&=\; \left(\frac{8}{9}\right)^{\frac{3}{5}}\frac{1}{R}\;,\\
e^\lambda \;&=\; \left(\frac{9}{8}\right)^{\frac{1}{10}}\;, & e^\beta \;&=\; \frac{9}{8}\left(\frac{4\,Q_f}{3}\right)^{\frac{2}{3}} R^{\frac{2}{3}}\;r^{\frac{1}{3}}\;.
\end{aligned} \tag{3.80}$$

One can easily verify that these metric and scalar fields solve (3.75) and (3.77).

Let us have a closer look at the 4-dimensional metric we obtained. Plugging the $c_i(r)$ functions (3.79) into (3.78), we get:

$$ds_4^2 \;\sim\; r^{\frac{7}{3}}\left[-\,b(r)\,dt^2 \;+\; (dx)^1 + (dx^2)^2 \;+\; R^4\,\frac{dr^2}{b(r)\,r^4}\right]\,. \tag{3.81}$$

It is easy to check that this metric is equivalent to the one obtained when the 10-dimensional geometry of the D2-brane is reduced to 4 dimensions (change to the new radial coordinate $\rho = r^{\frac{3}{2}}$ and compare with the reduced metric written in [160]). Another way of reaching the same conclusion is by noting that under a scale transformation of the type:

$$t \;\to\; \lambda\,t\,, \qquad x^{1,2} \;\to\; \lambda\,x^{1,2}\,, \qquad r \;\to\; r/\lambda\;, \tag{3.82}$$

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the zero-temperature metric changes homogeneously as:

$$ds_4^2 \;\to\; \lambda^{-\frac{1}{3}}\, ds_4^2\;. \tag{3.83}$$

This behaviour corresponds to a hyperscaling violation of the type $ds_4^2 \;\to\; \lambda^{\theta}\, ds_4^2$, with hyperscaling violation exponent $\theta = -\frac{1}{3}$ which, as shown in [161], is the θ exponent corresponding to a D2-brane. However, our 4-dimensional theory has more scalars than the reduced theory of a D2-brane and, therefore, even if the metrics are equal, both problems are not equivalent in principle.

3.4.1 Stress-energy tensor

We now compute the VEV of the stress-energy tensor in this dimensionally reduced gravity theory. First of all we need to renormalize holographically the on-shell action by adding boundary terms. Besides the standard Gibbons-Hawking term, we will add a counterterm constructed with the superpotential for the potential V written in (3.72) [162]. This superpotential will be denoted by W_{4d} and must satisfy:

$$V \,=\, \frac{1}{2}\Big[\frac{3}{80}\,\big(\partial_\gamma\, W_{4d}\big)^2 \,+\, \frac{1}{40}\,\big(\partial_\lambda\, W_{4d}\big)^2 \,+\, \frac{1}{3}\,\big(\partial_\beta\, W_{4d}\big)^2 \,+\, \big(\partial_\phi\, W_{4d}\big)^2\Big] \,-\, \frac{3}{8}\, W_{4d}^2\;. \tag{3.84}$$

It can be readily checked that the function:

$$W_{4d} \,=\, -6\, e^{\frac{8}{3}\,\gamma\,-\,4\lambda\,-\,\frac{\beta}{2}} \,-\, 4\, e^{\frac{8}{3}\,\gamma\,+\,6\lambda\,-\,\frac{\beta}{2}} \,+\, Q_c\, e^{\frac{20}{3}\gamma\,-\,\frac{\beta}{2}} \,+\, 2\, Q_f\, e^{2\gamma+2\lambda+\frac{\phi}{2}\,-\,\frac{3\beta}{2}}\;, \tag{3.85}$$

solves (3.84). Moreover, one can verify that W_{4d} gives rise to the BPS equations satisfied by the zero temperature supersymmetric solution of the second chapter.

In terms of W_{4d} the boundary action takes the form:

$$S_{boundary} \,=\, \frac{V_5\, V_{x^3}}{2\,\kappa_{10}^2}\,\int_{r\to\infty} d^3x\,\sqrt{\gamma}\,\Big(2\,K \,+\, W_{4d}\Big)\;, \tag{3.86}$$

where γ is the determinant of the induced metric on constant-r slices and $K = K^{\mu}{}_{\mu}$ is the trace of the extrinsic curvature of these slices. One can check that, after dividing by the infinite volume V_3 of the (2+1)-dimensional Minkowski spacetime, the sum of the actions (3.71) and (3.86) evaluated on-shell is finite. We get:

$$\frac{S_{renormalized}}{V_3\, V_{x^3}} \,=\, \frac{S_{eff,on-shell} \,+\, S_{boundary,on-shell}}{V_3\, V_{x^3}} \,=\, \frac{3}{7}\,\beta_s\, Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}\, T^{\frac{10}{3}} \tag{3.87}$$

where β_s is the constant defined in (3.32). To obtain (3.87) we have integrated from $r = r_h$ to $r = \infty$. Notice that $S_{renormalized}$ is equal, as it should, to minus the free energy density f (compare with (3.35)). The minus sign in this relation is due to the fact that we are working in Minkowski signature.

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By taking the functional derivative of the on-shell renormalized action with respect to the boundary metric we obtain the expectation value of the field theory stress-energy tensor:

$$\langle T^{\mu}{}_{\nu} \rangle \,=\, \frac{V_5\, V_{x^3}}{2\,\kappa_{10}^2}\,\sqrt{\gamma}\,\Big[-2\,K^{\mu}{}_{\nu}\,+\,\delta^{\mu}{}_{\nu}\,\big(2\,K\,+\,W_{4d}\big)\Big]_{r\to\infty}\;. \tag{3.88}$$

Evaluating the right-hand side of (3.88) for our solution, we get:

$$\langle T^{\mu}{}_{\nu} \rangle \,=\, \text{diag}\big(-\epsilon\,,\, p_{xy}\,,\, p_{xy}\,\big)\;, \tag{3.89}$$

where ϵ is the ADM energy density (3.31) and p_{xy} is the pressure in the xy plane written in (3.43).

3.5 Reduced action in 5 dimensions

Let us now reduce our system to 5 dimensions, namely those corresponding to the coordinates $z^m \,=\, (t, x^1, x^2, x^3, r)$. In principle this reduction would allow us to study the inter-layer properties and could be used to analyze the consequences of the anisotropy of the model. In this section we will use this 5-dimensional formalism to compute the complete stress-energy tensor and to establish the holographic dictionary for the D5-brane chemical potential.

Let us adopt the following reduction ansatz for the metric:

$$ds^2_{10} \,=\, e^{\frac{10}{3}\gamma}\, g_{pq}\, dz^p\, dz^q \,+\, e^{-2(\gamma+\lambda)}\, ds^2_{\mathbb{CP}^2} \,+\, e^{2(4\lambda-\gamma)}\,(d\tau + A)^2\;, \tag{3.90}$$

where g_{pq} is a 5-dimensional metric and the scalar fields γ and λ depend on the 5-dimensional coordinates z^m. It is important to notice that the RR 3-form F_3 for our solution has a leg in x^3, as well as two legs in the internal space (see (2.21)). Therefore, when it is reduced to 5 dimensions it gives rise to a 1-form $\mathcal{F}_1$, which we will represent in terms of a scalar potential $\mathcal{V}$ as:

$$\mathcal{F}_1 \,=\, d\,\mathcal{V}\;. \tag{3.91}$$

Moreover, our D5-branes are codimension-1 objects (extended along the hypersurface $x^3 =$ constant and smeared over x^3). The corresponding DBI action contains the determinant of the induced metric on this 4-dimensional surface, which we will denote by $\hat{g}_4$, integrated over x_3 to take into account the smearing. In addition to the metric and $\mathcal{V}$, the 5-dimensional theory has three scalar fields (γ, λ and the dilaton ϕ). The total effective action is worked out in appendix 3.B and takes the form:

$$S_{eff} \,=\, \frac{V_5}{2\,\kappa_{10}^2}\int d^5z\,\sqrt{-g_5}\,\Big[R_5 \,-\, \frac{40}{3}\,(\partial\gamma)^2 \,-\, 20\,(\partial\lambda)^2 \,-\, \frac{1}{2}\,(\partial\phi)^2$$
$$-\,\frac{1}{2}\,e^{4\gamma+4\lambda+\phi}\,(\partial\mathcal{V})^2 \,-\, U\Big] \,-\, \frac{V_5}{2\,\kappa_{10}^2}\int d^5z\,\sqrt{-\hat{g}_4}\,\big[\,6\,Q_f\, e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\,\big]\;, \tag{3.92}$$

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where U is the potential:

$$U = 4\, e^{\frac{16}{3}\,\gamma+12\lambda} - 24\, e^{\frac{16}{3}\gamma+2\lambda} + \frac{Q_c^2}{2}\, e^{\frac{40}{3}\gamma}\ . \tag{3.93}$$

For our D3-D5 black hole solution the 5-dimensional metric takes the form:

$$ds_5^2 \,=\, -d_1^2(r)\, dt^2 \,+\, d_2^2(r)\left[(dx^1)^2 \,+\, (dx^2)^2\right] \,+\, d_3^2(r)\,(dx^3)^2 \,+\, d_r^2(r)\,(dr)^2\ , \tag{3.94}$$

where the different d functions are given by:

$$\begin{aligned} d_1^2(r) \,&=\, \left(\frac{9}{8}\right)^2 r^2\, R^{\frac{4}{3}}\, b(r)\ , \\ d_2^2(r) \,&=\, \left(\frac{9}{8}\right)^2 r^2\, R^{\frac{4}{3}}\ , \\ d_3^2(r) \,&=\, \left(\frac{9}{8}\right)^2 \left(\frac{4\,Q_f}{3}\right)^{\frac{4}{3}} R^{\frac{4}{3}}\, r^{\frac{2}{3}}\ , \\ d_r^2(r) \,&=\, \left(\frac{9}{8}\right)^2 \frac{R^{\frac{16}{3}}}{r^2\, b(r)}\ . \end{aligned} \tag{3.95}$$

In (3.95) the function $b(r)$ is the blackening function (3.3). Moreover, the scalar fields corresponding to the D3-D5 black hole are:

$$e^{\phi} \,=\, \left(\frac{3}{4\,Q_f}\right)^{\frac{2}{3}} r^{\frac{2}{3}}\ , \qquad e^{\gamma} \,=\, \left(\frac{8}{9}\right)^{\frac{3}{5}} \frac{1}{R}\ , \qquad e^{\lambda} \,=\, \left(\frac{9}{8}\right)^{\frac{1}{10}}\ . \tag{3.96}$$

Notice that they are the same as in (3.80). The function $\mathcal{V}$ is given by:

$$\mathcal{V} \,=\, \sqrt{2}\, Q_f\, x^3\ . \tag{3.97}$$

It can be easily checked that the metric written in (3.94) and (3.95), together with the scalars written in (3.96) and the function $\mathcal{V}$ written in (3.97), satisfy the equations of motion derived from the action (3.92) (these equations have been explicitly written in appendix 3.B).

It is also interesting to relate these fields to the ones corresponding to the 4-dimensional approach for our solution. The 5-dimensional to 4-dimensional reduction is analyzed in appendix 3.B (section 3.B.3). As mentioned above, the scalars (ϕ, γ, λ) take the same values in 4 and in 5 dimensions. Moreover, the 4-dimensional scalar β is related to d_3 as:

$$e^{\beta} \,=\, d_3\ , \tag{3.98}$$

while the functions c_1, c_2 and c_3 of the 4-dimensional metric are related to the d functions as:

$$c_1^2 \,=\, d_3\, d_1^2\ , \qquad c_2^2 \,=\, d_3\, d_2^2\ , \qquad c_3^2 \,=\, d_3\, d_r^2\ . \tag{3.99}$$

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3.5.1 Stress-energy tensor

Let us now construct boundary counterterms which regularize the on-shell effective action and allow to implement the holographic renormalization formalism and compute the VEV of the stress-energy tensor. First of all we obtain a superpotential W_{5d} for the potential U written in (3.93). This superpotential must satisfy the equation:

$$U = \frac{1}{2}\Big[\frac{3}{80}\big(\partial_\gamma W_{5d}\big)^2 + \frac{1}{40}\big(\partial_\gamma W_{5d}\big)^2 + \big(\partial_\phi W_{5d}\big)^2\Big] - \frac{1}{3}W_{5d}^2 \,, \tag{3.100}$$

which is solved by the function:

$$W_{5d} = -6\,e^{\frac{8\gamma}{3}-4\lambda} - 4\,e^{\frac{8\gamma}{3}+6\lambda} + Q_c\,e^{\frac{20}{3}\gamma} \,. \tag{3.101}$$

Notice that the three terms on the right-hand side of (3.101) are in one-to-one correspondence with the terms in the 4-dimensional superpotential W_{4d} which do not contain Q_f (see (3.85)). Let us next define a new function W_{flavor}, related to the last term in (3.85), as:

$$W_{flavor} = 2\,Q_f\,e^{2\lambda+2\gamma+\frac{\phi}{2}} \,. \tag{3.102}$$

The counterterms needed to renormalize the action (3.92) will have the same structure as S_{eff}. First of all, we will have a 5-dimensional part, containing the metric γ_{ab} induced on constant r slices, as well as the Gibbons-Hawking term and the 5-dimensional superpotential (3.101). In addition, we will have a 4-dimensional part corresponding to the smeared sources, which contains the determinant of the metric $\hat{\gamma}_{ab}$ induced on constant r and constant x^3 slices. We construct this term by using the flavor function defined in (3.102). The total boundary action is:

$$S_{boundary} = \frac{V_5}{2\,\kappa_{10}^2}\int_{r\to\infty} d^4x\,\sqrt{\gamma}\,\Big(2\,K + W_{5d}\Big) + \frac{V_5}{2\,\kappa_{10}^2}\int_{r\to\infty} d^4x\,\sqrt{\hat{\gamma}}\,W_{flavor} \,. \tag{3.103}$$

One can check that the addition of $S_{boundary}$ makes the total on-shell action (divided by $V_3 V_{x^3}$) finite. Actually, one has:

$$\frac{S_{renormalized}}{V_3 V_{x^3}} = \frac{S_{eff,on-shell} + S_{boundary,on-shell}}{V_3 V_{x^3}} = \frac{3}{7}\,\beta_s\,Q_f^{\frac{2}{3}}\,Q_c^{\frac{5}{3}}\,T^{\frac{10}{3}} \,. \tag{3.104}$$

Notice that $S_{renormalized}/V_3 V_{x^3}$ coincides with minus the free energy density f in the 10-dimensional approach (see (3.35)), as it should.

The VEV of the stress-energy tensor of the dual theory can be obtained by taking the functional derivative of $S_{renormalized}$ with respect to the boundary metric. As a result of this calculation we get contributions from the two types of terms in (3.103):

$$\langle T^{\mu}{}_{\nu}\rangle = \frac{V_5}{2\,\kappa_{10}^2}\,\sqrt{\gamma}\,\Big[-2\,K^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu}\,\big(2\,K + W_{5d}\big)\Big]_{r\to\infty} + \langle T^{\mu}{}_{\nu}\rangle_{flavor} \,, \tag{3.105}$$

where $\langle T^{\mu}{}_{\nu}\rangle_{flavor}$ is only non-vanishing if both indices μ and ν take values 0, 1, 2 and, in this case, is given by:

$$\langle T^{\mu}{}_{\nu}\rangle_{flavor} \,=\, Q_f\,\frac{V_5}{\kappa_{10}^2}\,\sqrt{\hat{\gamma}}\,e^{2\lambda+2\gamma+\frac{\phi}{2}}\,\delta^{\mu}{}_{\nu}\Big|_{r\to\infty}\,, \qquad\qquad \mu,\nu = 0,1,2\;. \tag{3.106}$$

One can easily verify that $\langle T^{\mu}{}_{\nu}\rangle$ is given by the same expression as in the 10-dimensional analysis, namely by (3.69) with ϵ, p_{xy} and p_z equal to the values written in (3.31) and (3.43).

3.5.2 Holographic dictionary

Clearly, the contribution (3.106) is essential to reproduce the different values of the two pressures p_{xy} and p_z, *i.e.*, to correctly represent the anisotropic behaviour of the model. As argued in section 3.3, this anisotropy is characterized by the D5-brane chemical potential Φ. It is therefore very important to find a dictionary allowing us to read the value of Φ from the value of some supergravity field at the UV boundary. This is the purpose of this subsection.

In our holographic setup Φ should be related to the value of the potential under which the D5-branes are electrically charged. Notice that the D5-branes in our reduced theory extend along $x^0\,x^1\,x^2$ and are smeared along x^3. Therefore, we expect Φ to be extracted from the components of a 3-form $\mathcal{C}_3$ along $x^0\,x^1\,x^2$. One can find $\mathcal{C}_3$ by the following argument. First of all, we write the equation of motion of $\mathcal{F}_1$ (equation (3.267)) as:

$$d\Big(e^{4\gamma+4\lambda+\phi}\,*\,\mathcal{F}_1\Big) = 0\,, \tag{3.107}$$

where $*$ denotes the Hodge dual of the 5-dimensional theory. Next, we interpret (3.107) as a Bianchi identity, *i.e.*, as the closure of the 4-form $\mathcal{F}_4$ defined as:

$$\mathcal{F}_4 \,=\, e^{4\gamma+4\lambda+\phi}\,*\,\mathcal{F}_1\;. \tag{3.108}$$

It follows that $\mathcal{F}_4$ can be represented in terms of a 3-form $\mathcal{C}_3$:

$$\mathcal{F}_4 \,=\, d\,\mathcal{C}_3\,, \tag{3.109}$$

and we will soon verify that $\mathcal{C}_3$ is the 3-form we are seeking. To check this statement we will find the form for our solution. First of all we notice that:

$$*\,\mathcal{F}_1 \,=\, \sqrt{2}\,Q_f\,\frac{d_1\,d_2^2\,d_r}{d_3}\,dx^0\wedge dx^1\wedge dx^2\wedge dr\;. \tag{3.110}$$

From (3.110) we can readily verify that $\mathcal{C}_3$ can be taken as:

$$\mathcal{C}_3 \,=\, \mathcal{A}\left(Q_f^{-\frac{1}{3}}\,r^{\frac{10}{3}}\,+\,\mathcal{C}\right)dx^0\wedge dx^1\wedge dx^2\,, \tag{3.111}$$

where $\mathcal{A}$ is a known numerical constant (independent of Q_f and Q_c) and $\mathcal{C}$ is another constant which we will fix by requiring regularity at the horizon or, equivalently by demanding the vanishing of $\mathcal{C}_3$ at $r = r_h$. This condition leads to the following value of $\mathcal{C}$:

$$\mathcal{C} \,=\, -Q_f^{-\frac{1}{3}}\, r_h^{\frac{10}{3}} \;. \tag{3.112}$$

Taking into account that $r_h^{\frac{10}{3}} \propto Q_c^{\frac{5}{3}}\, T^{\frac{10}{3}}$, we find:

$$\mathcal{C} \,\propto\, Q_c^{\frac{5}{3}}\, Q_f^{-\frac{1}{3}}\, T^{\frac{10}{3}} \;. \tag{3.113}$$

By comparing (3.113) and (3.39) we conclude that the chemical potential Φ and the constant $\mathcal{C}$ are proportional:

$$\Phi \propto \mathcal{C} \;. \tag{3.114}$$

Notice also that $\mathcal{C}$ is (proportional to) the subleading term in the expansion of the $x^0x^1x^2$ component of $\mathcal{C}_3$ near the boundary. This identification of Φ is similar to the one obtained in [143] for the case of an anisotropic background generated by D7-branes. The fact that it is the subleading term that is being identified with Φ, and not the leading term as in other holographic setups, can be traced back to the Hodge duality that we are doing when passing from $\mathcal{F}_1$ to $\mathcal{F}_4$.

3.6 Hydrodynamics

We will now explore the hydrodynamic properties of our system. In particular we will compute the transport coefficients for perturbations propagating along the x^1x^2 plane. The purpose of this calculation is to characterize the effects of flavors, and of the corresponding induced anisotropy, on the transport properties of our system. As already mentioned in the introduction, our main result is that the transport coefficients in the x^1x^2 plane are the same as those of a D2-brane. This result confirms the conclusions of our static thermodynamic analysis and implies that, in our model, the dynamics of the excitations within a layer is governed by an effective strongly coupled SYM theory in 2+1 dimensions.

Following the standard procedure [163], we have to study the fluctuations of the 4-dimensional metric and scalar fields around their background values (3.79) and (3.80). In order to do this, we will perform the following substitution in the equations of motion:

$$g_{mn} \to g_{mn} + h_{mn} \;, \qquad\qquad \Psi \to \Psi \,+\, \delta\Psi \;, \tag{3.115}$$

for $\Psi = (\phi, \gamma, \lambda, \beta)$ and we will keep only the first-order terms in h_{mn} and $\delta\Psi$. Moreover, we will work in the radial gauge for the metric, in which:

$$h_{mr} = 0 \;, \qquad\qquad (m = t, x^1, x^2, r) \;. \tag{3.116}$$

Let us start by computing the variation of the scalar equation (3.77). One can easily check that, at first order, we have:

$$\delta\,\Box\Psi \;=\; \Box\delta\Psi \;+\; \frac{1}{2}\,g^{mn}\,\partial_m\Psi\,\partial_n(h^p{}_p) \;-\; \frac{1}{\sqrt{-g_4}}\,\partial_m\Big(\sqrt{-g_4}\,h^{mn}\,\partial_n\Psi\Big)\;. \tag{3.117}$$

The last term in (3.117) is always zero in the radial gauge when the scalar fields of the background only depend on the radial variable. For a metric of the type (3.78), $\Box\delta\Psi$ becomes:

$$\Box\delta\Psi = \frac{1}{c_3^2}\left[\partial_r^2\,(\delta\Psi) + \partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right)\partial_r\,(\delta\Psi)\right] - \frac{\partial_t^2\,(\delta\Psi)}{c_1^2} \;+\; \frac{\partial_{x^1}^2\,(\delta\Psi) + \partial_{x^2}^2\,(\delta\Psi)}{c_2^2}\;, \tag{3.118}$$

and the first-order equation for $\delta\Psi$ is:

$$\partial_r^2\,(\delta\Psi) + \partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right)\partial_r\,(\delta\Psi) - \frac{c_3^2}{c_1^2}\,\partial_t^2\,(\delta\Psi) \;+\; \frac{c_3^2}{c_2^2}\Big(\partial_{x^1}^2\,(\delta\Psi) + \partial_{x^2}^2\,(\delta\Psi)\Big) \;+$$

$$+\frac{\partial_r\Psi}{2}\,\partial_r\left(\frac{h_{x^1x^1} + h_{x^2x^2}}{c_2^2} - \frac{h_{tt}}{c_1^2}\right) \;=\; c_3^2\;\alpha_\Psi\,\delta[\partial_\Psi V]\;. \tag{3.119}$$

The first-order variation of the Einstein equation (3.75) is:

$$\delta\,R_{mn} \;=\; \sum_\Psi \frac{1}{2\alpha_\Psi}\Big(\partial_m\,(\delta\Psi)\,\partial_n\,\Psi \;+\; \partial_m\,\Psi\,\partial_n(\delta\Psi)\Big) + \frac{1}{2}\,h_{mn}\,V \;+\; \frac{1}{2}\,g_{mn}\,\delta V\;, \tag{3.120}$$

where $\delta\,R_{mn}$ can be written in terms of covariant derivatives of the metric perturbation h_{mn} as:

$$\delta\,R_{mn} \;=\; \frac{1}{2}\Big[D_p\,D_m\,h^p{}_n \;+\; D_p\,D_n\,h^p{}_m \;-\; D_p\,D^p\,h_{mn} \;-\; D_m\,D^n\,h^p{}_p\Big]\;. \tag{3.121}$$

By plugging (3.121) into (3.120), we arrive at the following equation for the metric fluctuations:

$$D_p\,D_m\,h^p{}_n \;+\; D_p\,D_n\,h^p{}_m \;-\; D_p\,D^p\,h_{mn} \;-\; D_m\,D_n\,h^p{}_p \;=$$

$$=\sum_\Psi \frac{1}{\alpha_\Psi}\Big(\partial_m\,(\delta\Psi)\,\partial_n\,\Psi \;+\; \partial_m\,\Psi\,\partial_n(\delta\Psi)\Big) \;+\; h_{mn}\,V \;+\; g_{mn}\,\delta V\;. \tag{3.122}$$

3.6.1 The shear channel

The fluctuation equations (3.119) and (3.122) are highly coupled. However, one can identify several consistent truncations in which only few fluctuations are non-zero. Without loss of generality, let us consider a perturbation propagating along the x^2 direction. The first of the consistent truncations that we will analyze is the so-called shear channel, in

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which only the metric fluctuations $h_{t\,x^1}$ and $h_{x^1\,x^2}$ are excited. Let us assume that these fluctuations have frequency ω and momentum q and, accordingly, let us parameterize them as:

$$h_{t\,x^1} \,=\, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_2^2(r)\,H_{tx}(r)\;,$$

$$h_{x^1\,x^2} \,=\, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_2^2(r)\,H_{xy}(r)\;, \tag{3.123}$$

where $c_2(r)$ is the function written in (3.80) and has been included in the ansatz (3.123) for convenience. The equations of motion of H_{tx} and H_{xy} are studied in detail in appendix 3.C. It turns out that they can be reduced to a single second-order differential equation for a gauge invariant combination X, defined as:

$$X \,\equiv\, q\,H_{tx} \,+\, \omega\,H_{xy}\;. \tag{3.124}$$

The equation satisfied by X is:

$$X'' \,+\, \frac{(10+3\,b(r))\omega^2 \,-\, 13\,b^2(r)\,q^2}{3\,b(r)\,r\,(\omega^2\,-\,b(r)\,q^2)}\,X' \,+\, \frac{R^4}{r^4\,b^2(r)}\,(\omega^2\,-\,b(r)\,q^2)\,X \,=\, 0\;. \tag{3.125}$$

Let us now work in a new radial variable x, related to r as:

$$x \,=\, \big[b(r)\big]^{\frac{1}{2}}\;. \tag{3.126}$$

In this new variable the horizon is located at $x=0$, whereas the boundary is at $x=1$. We will consider the gauge-invariant combination X as a function of x. Moreover, it is quite convenient to introduce the dimensionless momentum and frequency $\hat{q}$ and $\hat{\omega}$, defined as:

$$\hat{q} \,=\, \frac{q}{2\pi\,T}\;, \qquad\qquad \hat{\omega} \,=\, \frac{\omega}{2\pi T}\;. \tag{3.127}$$

Then, if the prime now denotes derivatives with respect to x, equation (3.125) takes the form:

$$X'' \,-\, \frac{1}{x}\,\frac{\hat{q}^2\,x^2\,+\,\hat{\omega}^2}{\hat{q}^2\,x^2\,-\,\hat{\omega}^2}\,X' \,-\, \frac{\hat{q}^2\,x^2\,-\,\hat{\omega}^2}{x^2(1-x^2)^{\frac{7}{5}}}\,X \,=\, 0\;. \tag{3.128}$$

We want to solve (3.128) by imposing infalling boundary conditions at the horizon $x=0$, as well as Dirichlet boundary conditions at the boundary $x=1$. These solutions only exist when the frequency ω and the momentum q are related in a particular way, which determines the dispersion relation $\omega=\omega(q)$ of our modes. In the hydrodynamic regime the momentum q is small and one can expand ω in a power series in q. In the shear channel we are studying this relation takes the form:

$$\omega \,=\, -i\,D_\eta\,q^2\left(1+\tau_s\,D_\eta\,q^2\right)\;, \tag{3.129}$$

where we are keeping terms up to quartic power of q. The dispersion relation (3.129) depends on two transport coefficients D_η and τ_s, which we will calculate for our system in this section. We will work in the dimensionless variables defined in (3.127). Moreover, we define the rescaled coefficients $\hat{D}_\eta$ and $\hat{\tau}_s$ as:

$$\hat{D}_\eta \,=\, 2\pi\, T\, D_\eta \;, \qquad\qquad \hat{\tau}_s \,=\, 2\pi\, T\, \tau_s \;. \tag{3.130}$$

In terms of the rescaled quantities, the dispersion relation (3.129) takes the form:

$$\hat{\omega} \,=\, -i\, \hat{D}_\eta\, \hat{q}^2 \left(1 + \hat{\tau}_s\, \hat{D}_\eta\, \hat{q}^2\right) \;. \tag{3.131}$$

The coefficient $\hat{D}_\eta$ determines the ratio of the shear viscosity η to the entropy density s, namely:

$$\frac{\eta}{s} \,=\, \frac{\hat{D}_\eta}{2\pi} \;. \tag{3.132}$$

Below we will find that, for our system, $\hat{D}_\eta = 1/2$, which is equivalent to having $\eta/s = 1/(4\pi)$. In what follows we compute τ_s explicitly for our system and it turns out that τ_s is the same as the one found in [164] for the geometry of the D2-brane.

Let us come back to the integration of the differential equation (3.128). In order to impose infalling boundary conditions at the horizon $x = 0$, we will adopt the ansatz:

$$X(x) \,=\, x^{-i\hat{\omega}}\, S(x) \;, \tag{3.133}$$

where $S(x)$ must be regular at $x = 0$. Let us expand $S(x)$ in powers of $\hat{q}$ as:

$$S(x) \,=\, S_0(x) \,+\, \hat{q}^2\, S_2(x) \,+\, \cdots \;. \tag{3.134}$$

Plugging the expansions (3.134) and (3.129) into (3.128) and separating the different orders in $\hat{q}$, we get the following system of equations:

$$\begin{aligned} &S_0'' \,-\, \frac{1}{x}\, S_0' \,=\, 0 \;, \\ &S_2'' \,-\, \frac{1}{x}\, S_2' \,=\, \left(\frac{1}{(1-x^2)^{\frac{7}{5}}} \,-\, \frac{2\hat{D}_\eta}{x^2}\right) S_0 \,+\, \frac{2\hat{D}_\eta}{x}\left(1 - \frac{\hat{D}_\eta}{x^2}\right) S_0' \;. \end{aligned} \tag{3.135}$$

We can also expand $S(x)$ in powers of x near $x = 0$:

$$S(x) \,=\, 1 \,+\, \sigma_2\, x^2 \,+\, \sigma_4\, x^4 \,+\, \cdots \;, \tag{3.136}$$

where the coefficients σ_2 and σ_4 are easy to obtain by substituting this expansion into (3.128). They are given by:

$$\begin{aligned} \sigma_2 \,&=\, \frac{5i\, \hat{q}^2 (2i + \hat{\omega}) \,-\, 7i\, \hat{\omega}^3}{20\, \hat{\omega}\, (i + \hat{\omega})} \;, \\ \sigma_4 \,&=\, \frac{-25\, \hat{q}^4\, (4i + \hat{\omega}) \,+\, 70\, \hat{q}^2\, \hat{\omega} (2i + \hat{\omega})^2 \,+\, 7\hat{\omega}^3 (24 - 24i\hat{\omega} - 7\hat{\omega}^2)}{800\, \hat{\omega}\, (i + \hat{\omega})\, (2i + \hat{\omega})} \;. \end{aligned} \tag{3.137}$$

By expanding σ_2 and σ_4 in powers of $\hat{q}$ using the dispersion relation (3.129), we arrive at the following expression of $S(x)$, valid for low x and low $\hat{q}$:

$$S(x) \,=\, 1 \,-\, \frac{x^2}{2\hat{D}_\eta} \,+\, \frac{\hat{q}^2\, x^2}{80\, \hat{D}_\eta} \Big(20\, \hat{D}_\eta \,(2\, \hat{\tau}_s \,-\, 1) \,+\, (14\, \hat{D}_\eta \,-\, 5)\, x^2 \Big) \,+\, \mathcal{O}(\hat{q}^3) \;. \tag{3.138}$$

We will next compare (3.138) with the result of integrating the system (3.135) and expanding the result of this integration in powers of x near $x = 0$. The integration of the first equation in (3.135) is straightforward and yields the result:

$$S_0(x) \,=\, A \,+\, B\, x^2 \;, \tag{3.139}$$

where A and B are integration constants. By comparing (3.139) with the first two terms in (3.138) we conclude that $A = 1$ and $B = -1/(2\hat{D}_\eta)$ and, therefore, $S_0(x)$ is given by:

$$S_0(x) \,=\, 1 \,-\, \frac{x^2}{2\, \hat{D}_\eta} \;. \tag{3.140}$$

By imposing the Dirichlet condition $S_0(x = 1) = 0$ at the boundary, we obtain that, as already announced, $\hat{D}_\eta$ must be:

$$\hat{D}_\eta \,=\, \frac{1}{2} \;, \tag{3.141}$$

and S_0 takes the form:

$$S_0(x) \,=\, 1 \,-\, x^2 \;. \tag{3.142}$$

Using these values of $S_0(x)$ and $\hat{D}_\eta$ on the right-hand side of the second equation of the system (3.135) we arrive at the equation:

$$S_2'' \,-\, \frac{1}{x}\, S_2' \,=\, \frac{1}{(1-x^2)^{\frac{2}{5}}} \,-\, 1 \;, \tag{3.143}$$

whose general solution is:

$$S_2(x) \,=\, C \,+\, (1 + 2D - 2\log x)\, \frac{x^2}{4} \,-\, \frac{25}{24}\, e^{\frac{2\pi i}{5}}\, x^{\frac{6}{5}}\, F\Big(-\frac{3}{5}, \frac{2}{5}; \frac{7}{5}; \frac{1}{x^2} \Big) \;. \tag{3.144}$$

In (3.144) C and D are integration constants which can be determined by expanding the result near $x \approx 0$ and comparing it with the terms proportional to $\hat{q}^2$ of (3.138). The expansion of (3.144) near $x \approx 0$ is:

$$S_2(x) \,=\, C \,+\, \frac{5}{12} \,+\, \frac{x^2}{2} \left[D \,+\, \frac{1}{2} \Big(\gamma - i\pi + \psi\Big(\frac{2}{5}\Big) \Big) \right] \,+\, \frac{x^4}{20} \,+\, \cdots \;, \tag{3.145}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant and $\psi(z)$ is the logarithmic derivative of the Euler gamma function $\Gamma(z)$. This result coincides with (3.138) if the constants C

and D are:

$$C = -\frac{5}{12} \, ,$$
$$D = -\frac{1}{2}\Big[1 \, - \, 2\,\hat{\tau}_s \, + \, \gamma - i\pi + \psi\Big(\frac{2}{5}\Big)\Big] \, . \tag{3.146}$$

Substituting (3.146) in (3.144) we get the function $S_2(x)$, namely:

$$S_2(x) = -\frac{5}{12} + \Big(2\,\hat{\tau}_s - \gamma + i\pi - \psi\Big(\frac{2}{5}\Big) - 2\log x\Big)\frac{x^2}{4} - \frac{25}{24}e^{\frac{2\pi i}{5}}\,x^{\frac{6}{5}}\,F\Big(-\frac{3}{5}, \frac{2}{5}; \frac{7}{5}; \frac{1}{x^2}\Big) \, , \tag{3.147}$$

which only contains $\hat{\tau}_s$ as unknown parameter. By imposing that $S_2(x=1) = 0$, $\hat{\tau}_s$ is fixed to be:

$$\hat{\tau}_s \, = \, \frac{1}{2}\Big[\gamma + \psi\Big(\frac{8}{5}\Big)\Big] \, . \tag{3.148}$$

Equivalently, the unrescaled parameter is:

$$\tau_s \, = \, \frac{1}{4\,\pi\,T}\Big[\gamma + \psi\Big(\frac{8}{5}\Big)\Big] \, . \tag{3.149}$$

This value of τ_s coincides with the one found in the literature for the D2-brane [164].

3.6.2 The sound channel

In the so-called sound channel, the following set of metric fluctuations, propagating along x^2, are decoupled from the others:

$$(h_{tt}, h_{tx^2}, h_{x^1x^1}, h_{x^2x^2}) \, , \tag{3.150}$$

and are coupled to the fluctuations of the scalar fields. Let us parameterize these metric fluctuations as:

$$h_{tt} \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_1^2(r)\,H_{tt}(r) \, , \qquad h_{tx^2} \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_2^2(r)\,H_{ty}(r) \, ,$$
$$h_{x^1x^1} \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_2^2(r)\,H_{xx}(r) \, , \qquad h_{x^2x^2} \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,c_2^2(r)\,H_{yy}(r) \, , \tag{3.151}$$

where $c_1(r)$ and $c_2(r)$ are the functions written in (3.79). Similarly, we represent the scalar fluctuations as:

$$\delta\phi \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,\Phi(r) \, , \qquad \delta\gamma \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,\Gamma(r) \, ,$$
$$\delta\lambda \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,\Lambda(r) \, , \qquad \delta\beta \, = \, e^{-i(\omega\,t\,-\,q\,x^2)}\,B(r) \, . \tag{3.152}$$

Let us now introduce a compact notation for the scalar fluctuations. We denote by $\hat{\Psi}(r)$ the radial part of the fluctuation $\Psi = (\phi, \gamma, \lambda, \beta)$, namely:

$$\hat{\Psi}(r) \,=\, (\Phi(r), \Gamma(r), \Lambda(r), B(r)) \,. \tag{3.153}$$

Then, (3.152) can be rewritten simply as:

$$\delta\Psi \,=\, e^{-i(\omega\, t \,-\, q\, x^2)}\hat{\Psi}(r) \,. \tag{3.154}$$

The full set of equations for the fields of (3.151) and (3.152) is written in appendix 3.C. As usual, these equations are highly redundant due to the diffeomorphism gauge invariance. This redundancy can be reduced by defining new fields. Accordingly, let us define new scalar fluctuation fields Z_Φ, Z_Γ, Z_Λ and Z_B, denoted collectively by $Z_{\hat{\Psi}}$, as the following combination of $\hat{\Psi}$ and H_{xx}:

$$Z_{\hat{\Psi}} \,=\, \hat{\Psi} \,-\, \frac{\Psi'}{\partial_r\, \log c_2^2}\, H_{xx} \,. \tag{3.155}$$

As argued in [160,165,166], these are the gauge invariant combinations of the scalar fields and the metric. It is proved in appendix 3.C that the equations for the Z's close among themselves (see the system (3.289)). Moreover, there is a particular combination Z_S of these fields which can be decoupled from the other scalars. This combination is:

$$Z_S(r) \,\equiv\, 3\, Z_B(r) \,+\, 2\, Z_\Phi(r) \,. \tag{3.156}$$

The equation satisfied by $Z_S(r)$ has been written in (3.291). Following [163], we now define the gauge invariant metric fluctuation Z_H as:

$$Z_H \,=\, H_{yy} \,+\, \frac{2q}{\omega}\, H_{ty} \,+\, \frac{q^2}{\omega^2}\,\frac{c_1^2}{c_2^2}\, H_{tt} \,+\, \Big(\frac{q^2}{\omega^2}\,\frac{c_1^2\partial_r \log c_1}{c_2^2\partial_r \log c_2} \,-\, 1\Big)\, H_{xx} \,. \tag{3.157}$$

The equation satisfied by Z_H has been written in (3.292). This equation shows that Z_H is only coupled to Z_S. Since Z_S does not couple to any other scalar, we can start our analysis by finding Z_S and then using this result in the equation for Z_H. It is shown in appendix 3.C that the only acceptable solution for Z_S is the trivial one $Z_S = 0$. Thus, we are left with a single equation for the gauge invariant metric fluctuation Z_H. Let us adopt for Z_H an ansatz similar to the one used for the fluctuations in the shear channel, namely:

$$Z_H(r) \,=\, \big[b(r)\big]^{-\frac{i\hat{\omega}}{2}}\; Y(r) \,. \tag{3.158}$$

Furthermore, we will work in the x variable defined in (3.126). After some work one can verify that the equation satisfied by $Y(x)$ is:

$$\begin{aligned} Y'' \,+\, &\frac{\big[5 - 2(3+2i\hat{\omega})x^2 - 10 i\,\hat{\omega}\big]\hat{q}^2 \,+\, 7(2i\,\hat{\omega} \,-\, 1)\hat{\omega}^2}{x\big[(5+2x^2)\hat{q}^2 \,-\, 7\,\hat{\omega}^2\big]}\, Y' \,+\, \Big[-\,\frac{\hat{q}^2}{(1-x^2)^{\frac{7}{5}}} \,+ \\ &+\, \frac{1}{x^2}\Big(\frac{1}{(1-x^2)^{\frac{7}{5}}} \,-\, 1\Big)\,\hat{\omega}^2 \,+\, \frac{8(1+i\hat{\omega})\,\hat{q}^2}{(5+2x^2)\hat{q}^2 \,-\, 7\,\hat{\omega}^2}\Big]\, Y \,=\, 0 \,, \end{aligned} \tag{3.159}$$

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where the primes denote derivative with respect to the new variable x.

We want to integrate the differential equations for $Y(x)$ in the hydrodynamic limit of low momentum. We will impose infalling boundary conditions at the horizon for $Z_H(r)$ and we will demand that the fluctuations vanish at the boundary. The infalling boundary condition at the horizon $x = 0$ is equivalent to the regularity of $Y(x)$ at this point. These conditions would require a specific dispersion relation $\omega = \omega(q)$, which at low momentum can be expanded as:

$$\omega \,=\, v_s\, q \,-\, i\Gamma\, q^2 \,+\, \mathcal{T}\, q^3 \;, \tag{3.160}$$

where we have only kept terms up to third order in q. The coefficient v_s of the linear term in (3.160) is the speed of sound and the quadratic coefficient Γ is the attenuation which, in p spatial dimensions, is related to the shear viscosity η and the bulk viscosity ζ as:

$$\Gamma \,=\, \frac{1}{T\, s}\Big[\frac{p-1}{p}\,\eta \,+\, \frac{\zeta}{2}\Big] \;, \tag{3.161}$$

where s is the entropy density. In our $p = 2$ case this expression becomes:

$$\Gamma \,=\, \frac{1}{2\,T\, s}\,(\eta \,+\, \zeta) \;. \tag{3.162}$$

The cubic coefficient $\mathcal{T}$ is usually [167] parameterized as:

$$\mathcal{T} \,=\, \frac{\Gamma}{v_s}\,\Big[v_s^2\,\tau_{eff} - \frac{\Gamma}{2}\Big] \;, \tag{3.163}$$

where τ_{eff} is an effective equilibration time which, in p spatial dimensions, is related to the second-order transport coefficients τ_π and τ_Π of the Israel-Stewart theory as:

$$\tau_{eff} \,=\, \frac{\tau_\pi \,+\, \frac{p}{2(p-1)}\,\frac{\zeta}{\eta}\,\tau_\Pi}{1 + \frac{p}{2(p-1)}\,\frac{\zeta}{\eta}} \;. \tag{3.164}$$

In our $p = 2$ model we have:

$$\tau_{eff} \,=\, \frac{\tau_\pi \,+\, \frac{\zeta}{\eta}\,\tau_\Pi}{1 + \frac{\zeta}{\eta}} \;. \tag{3.165}$$

In what follows it is quite convenient to work with the dimensionless momentum and frequency $\hat{q}$ and $\hat{\omega}$ defined in (3.127). In terms of these rescaled quantities, the dispersion relation (3.160) takes the form:

$$\hat{\omega} \,=\, v_s\, \hat{q} \,-\, i\hat{\Gamma}\, \hat{q}^2 \,+\, \hat{\mathcal{T}}\, \hat{q}^3 \;, \tag{3.166}$$

where $\hat{\Gamma}$ and $\hat{\mathcal{T}}$ are related to Γ and $\mathcal{T}$ as:

$$\hat{\Gamma} \,=\, 2\,\pi\, T\, \Gamma \;, \qquad\qquad \hat{\mathcal{T}} \,=\, (2\pi T)^2\, \mathcal{T} \;. \tag{3.167}$$

Let us now analyze (3.159) in the hydrodynamic approximation. We first expand $Y(x)$ in powers of $\hat{q}$ (up to second order) as:

$$Y(x) \,=\, Y_0(x) \,+\, i\hat{q}\, Y_1(x) \,+\, \hat{q}^2\, Y_2(x) \;. \tag{3.168}$$

By using (3.168) and (3.160) in (3.159), one can readily show that $Y_0(x)$ satisfies the equation:

$$Y_0'' \,+\, \frac{5-7\,v_s^2 \,-\, 6\,x^2}{x(5 \,-\, 7v_s^2 \,+\, 2\,x^2)}\, Y_0' \,+\, \frac{8}{5-7\,v_s^2 \,+2\,x^2}\, Y_0 \,=\, 0 \;, \tag{3.169}$$

whose general solution is:

$$Y_0(x) \,=\, C_1 \Big(1+\frac{2x^2}{7v_s^2-5}\Big) \,+\, C_2 \Big(2(7v_s^2-5) \,+\, (2x^2-5+7v_s^2)\,\log x\Big) \;, \tag{3.170}$$

where C_1 and C_2 are integration constants. Regularity at the horizon ($x=0$) requires that $C_2=0$. By imposing that

$$Y_0(x=1) \,=\, 0 \;, \tag{3.171}$$

we get the speed of sound v_s, namely:

$$v_s \,=\, \sqrt{\frac{3}{7}} \;, \tag{3.172}$$

which coincides with the value we found in our static analysis for the propagation in the x^1x^2 plane, *i.e.*, it is the same as the speed of sound propagating along the gauge theory directions of a D2-brane.

Without loss of generality we can take $C_1=1$ (or, equivalently, $Y_0(x=0)=1$) and, therefore, $Y_0(x)$ becomes:

$$Y_0(x) \,=\, 1-x^2 \;. \tag{3.173}$$

The equation for Y_1 is:

$$Y_1'' \,+\, \frac{5-7\,v_s^2-6\,x^2}{x(5-7\,v_s^2+2\,x^2)}\, Y_1' \,+\, \frac{8}{5-7\,v_s^2+2\,x^2}\, Y_1 = \frac{8v_s}{5-7\,v_s^2+2\,x^2}\,\Big[\frac{14\,\hat{\Gamma}}{5-7\,v_s^2+2\,x^2} \,- \\ -1\Big]\, Y_0 \,+\, \frac{v_s}{x}\Big[2-\frac{112\,x^2\,\hat{\Gamma}}{(5-7\,v_s^2+2\,x^2)^2}\Big]\, Y_0' \;. \tag{3.174}$$

Using the values of Y_0 and v_s written in (3.173) and (3.172) this equation becomes:

$$Y_1''+\frac{1-3x^2}{x(1+x^2)}\, Y_1' \,+\, \frac{4}{1+x^2}\, Y_1 \,=\, 4\sqrt{\frac{3}{7}}\,\frac{7\,\hat{\Gamma}-2}{1+x^2} \;. \tag{3.175}$$

The general solution of (3.175) is:

$$Y_1(x) \,=\, \sqrt{\frac{3}{7}}\,(7\,\hat{\Gamma}-2) \,+\, C_1\,(1-x^2) \,+\, C_2\,\big(4+(1-x^2)\log x^2\big) \;, \tag{3.176}$$

where, again, C_1 and C_2 are integration constants. The regularity requirement at the horizon $x=0$ implies that $C_2=0$. Moreover, the UV condition $Y_1(x=1)=0$ fixes the rescaled attenuation to be:

$$\hat{\Gamma} \,=\, \frac{2}{7}\,, \tag{3.177}$$

which, according to (3.167), is equivalent to the following value of Γ:

$$\Gamma \,=\, \frac{1}{7\pi\,T}\,. \tag{3.178}$$

Taking into account that $\eta/s = 1/4\pi$, it follows from (3.178) and (3.162) that the ratio of the bulk and shear viscosities for our model is:

$$\frac{\zeta}{\eta} \,=\, \frac{1}{7}\,, \tag{3.179}$$

This value for ζ/η is exactly the same as the one corresponding to a D2-brane [160], which saturates Buchel's bound [168]:

$$\frac{\zeta}{\eta} \,=\, 2\Big(\frac{1}{2}\,-\,v_s^2\Big)\,. \tag{3.180}$$

Let us next look at the equation for $Y_2(x)$. Using the values of v_s and $\hat{\Gamma}$ already determined, this equation reduces to:

$$Y_2''+\frac{1-3x^2}{x(1+x^2)}\,Y_2'\,+\,\frac{4}{1+x^2}\,Y_2\,=\,g(x)\,, \tag{3.181}$$

where $g(x)$ is the following function:

$$g(x)\,=\,\frac{1}{(1-x^2)^{\frac{2}{5}}}\,-\,\frac{3}{7}\Big(1\,-\,\frac{1}{x^2}\,+\,\frac{1}{x^2(1-x^2)^{\frac{2}{5}}}\Big)\,-\,\frac{4}{7}\,\frac{7\sqrt{21}\,\hat{\mathcal{T}}\,-\,4}{1+x^2}\,. \tag{3.182}$$

The homogeneous equation in (3.181) is just the same as in (3.175). We already found two independent solutions in (3.176), which we now denote by $y_1(x)$ and $y_2(x)$:

$$y_1(x)\,=\,1\,-\,x^2\,, \qquad\qquad y_2(x)\,=\,(1-x^2)\,\log x^2\,+\,4\,. \tag{3.183}$$

Then, the general solution of (3.181) can be written as:

$$Y_2(x)\,=\,D_1\,y_1(x)\,+\,D_2\,y_2(x)\,+\,y_p(x)\,, \tag{3.184}$$

where D_1 and D_2 are constants and $y_p(x)$ is a particular solution of the full inhomogeneous equation. We will use the method of variation of constants to find $y_p(x)$. The result can be written as:

$$y_p(x)\,=\,y_2(x)\,\int dx\,\frac{y_1(x)\,g(x)}{W(x)}\,-\,y_1(x)\,\int dx\,\frac{y_2(x)\,g(x)}{W(x)}\,, \tag{3.185}$$

where $W(x)$ is the Wronskian:

$$W(x) \,=\, y_1(x)\, y_2'(x) \,-\, y_1'(x)\, y_2(x) \;. \tag{3.186}$$

Let us rewrite (3.185) in a more convenient way following [169,170]. First of all, we define $h(x)$ as the ratio between the two solutions of the homogeneous equation:

$$h(x) \,=\, \frac{y_2(x)}{y_1(x)} \;. \tag{3.187}$$

The Wronskian $W(x)$ is related to the derivative of $h(x)$ as:

$$W(x) \,=\, h'(x)\, y_1^2(x) \;, \tag{3.188}$$

and, therefore, we can rewrite (3.185) as:

$$y_p(x) \,=\, y_1(x) \left[h(x) \int dx \frac{g(x)}{y_1(x)\, h'(x)} \,-\, \int dx \frac{h(x)\, g(x)}{h'(x)\, y_1(x)}\right] \;. \tag{3.189}$$

After an integration by parts, this equation can be recast as:

$$y_p(x) \,=\, y_1(x) \int dx\, h'(x) \int^x \frac{g(z)}{y_1(z)\, h'(z)}\, dz \;. \tag{3.190}$$

We now impose the regularity condition at the horizon $x = 0$. Using the integral expression (3.190) one can show that near $x \to 0$ the solution behaves as:

$$Y_2(x) \approx \mathcal{A} \,+\, \mathcal{B} \log x \,+\, \cdots \tag{3.191}$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants and the dots represent terms that vanish at $x = 0$. Our regularity condition demands that the term with the logarithm be absent in (3.191). Then, we require:

$$\mathcal{B} \,=\, 0 \;. \tag{3.192}$$

This determines the constant D_2 in (3.184) to be:

$$D_2 \,=\, \frac{1}{14} \left[\frac{3}{2}\left(\gamma \,-\, i\pi \,+\, \psi\Big(\frac{2}{5}\Big)\right) \,-\, 1\right] \;, \tag{3.193}$$

where $\gamma = 0.577$ is the Euler-Mascheroni constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. We next impose the UV boundary condition at $x = 1$:

$$Y_2(x \to 1) \,=\, 0 \;, \tag{3.194}$$

which determines the value of $\hat{\mathcal{T}}$ as:

$$\hat{\mathcal{T}} \,=\, \frac{1}{7}\sqrt{\frac{3}{7}}\left(1 + \gamma + \psi\Big(\frac{8}{5}\Big)\right) \;. \tag{3.195}$$

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Numerically, $\hat{\mathcal{T}} \approx 0.1592$. Using (3.167) we find the following value of $\mathcal{T}$:

$$\mathcal{T} \;=\; \frac{\sqrt{3}}{28\sqrt{7}\,(\pi\,T)^2}\,\Big(1+\gamma+\psi\Big(\frac{8}{5}\Big)\Big)\;. \tag{3.196}$$

Taking into account the value of Γ we found (equation (3.178)), this result corresponds to having an equilibration time τ_{eff} equal to:

$$\tau_{eff} \;=\; \frac{1}{4\pi\,T}\Big[\frac{5}{3}\,+\,\gamma\,+\psi\Big(\frac{8}{5}\Big)\Big]\;, \tag{3.197}$$

which again coincides with the one found for the geometry of a D2-brane [169,170]. From this value of τ_{eff} we get the following relation between the two Israel-Stewart coefficients, namely:

$$7\,\tau_\pi\,+\,\tau_\Pi\;=\;\frac{2}{\pi T}\,\Big[\frac{5}{3}\,+\,\gamma\,+\,\psi\Big(\frac{8}{5}\Big)\Big]\;. \tag{3.198}$$

3.7 Discussion

Let us summarize the main results of this chapter. We have succeeded in generalizing the D3-D5 geometry of chapter 2 to include an event horizon. Our solution is analytic and simple and is the gravity dual of the defect theory introduced in [59] at non-zero temperature in the approximation in which the massless flavors are smeared. The geometry found is homogeneous but anisotropic in the gauge theory directions: it preserves translational invariance but breaks rotational symmetry.

We have studied the thermodynamics and hydrodynamics of the model. We have checked several thermodynamic relations and found that the results are consistent with the laws of anisotropic thermodynamics. We also obtained dimensionally reduced gravitational actions for our system in 4 and 5 dimensions. In both dimensionalities we managed to construct boundary terms to renormalize the on-shell action and find the stress-energy tensor. Moreover, we obtained the hydrodynamic transport coefficients (up to second order) for perturbations propagating in the x^1x^2 plane. These transport coefficients are exactly the same as those of the D2-brane, a result which is not obvious despite the 2+1 dimensionality of our defect theory.

It follows from our results that the energy of our system scales with Q_c and Q_f as $Q_c^{\frac{5}{3}}\,Q_f^{\frac{2}{3}}$, which determines the dependence of the effective number of degrees of freedom on the number of colors and flavors. This type of dependence with Q_c and Q_f shows up in our thermodynamic results of section 3.3, as well as in the dependence of the entanglement entropy $S_\parallel$ (see equation (3.14)). The non-integer powers of Q_c and Q_f in this scaling are reflecting the strong coupling regime of the dynamics of the layers. The main result of our thermodynamic and hydrodynamic analysis is that this layer behaviour can be reproduced by an effective D2-brane or, equivalently, by (2+1)-dimensional SYM in the strong coupling regime.

One important feature of this geometry is that it does not have a weak anisotropy limit and, in fact, it is non-analytic in Q_f when $Q_f \to 0$. This is due to the fact that the flavors introduced are massless. It was shown in chapter 2 how to generalize the supersymmetric ($T = 0$) solution to the case in which the flavors are massive. In this case the flavor branes do not reach the origin and there is a cavity around $r = 0$ in which the D5-brane charge is zero and the equations of motion are those of the unflavored system. The radius of the cavity is related to the mass of the quarks. The massive solutions found in chapter 2 interpolate between the unflavored metric in the IR and the massless flavored geometry in the UV. By sending the quark mass to infinity the size of the cavity increases and the geometry becomes $AdS_5 \times \mathbb{S}^5$. This is quite natural from the point of view of field theory since in this infinite mass limit we are making the flavors non-dynamical. From the holographic point of view, the quark mass is an external parameter which allows to modify the degree of anisotropy. It would be very interesting to generalize some of the results found here to this massive case and to explore the development of anisotropy and their effects on the physical observables. In the next chapter, we will construct a fully backreacted massive background and study its physical properties.

3.A Wilson loops and entanglement entropies

In this appendix we calculate the potential energy for static quark-antiquark pairs, as well as the entanglement entropy for slab regions and their complements.

3.A.1 Quark-antiquark potentials

To calculate the potential energy between a "quark" and an "antiquark" we will follow the holographic prescription to compute the Wilson loops developed in [154, 155]. In this method one has to solve the equations of motion of a fundamental string with its two ends lying at the UV boundary. These equations are obtained by extremizing the Nambu-Goto action:

$$S = \frac{1}{2\pi} \int d\tau d\sigma \, e^{\frac{\phi}{2}} \sqrt{-\det g_2} \,, \tag{3.199}$$

where g_2 is the Einstein frame induced metric on the worldvolume of the string. We consider separately the cases in which the quark and the antiquark are in the same layer (*i.e.*, with the same value of the coordinate x^3) and the configuration in which they have the same value of (x^1, x^2) and different values of x^3.

Intra-layer potential

Let us first consider a fundamental string hanging from the UV boundary $r \to \infty$ and extended along one of the layer directions (say along $x^1 \equiv x$) with the other two cartesian coordinates being constant. We parameterize the worldvolume of such a string by means

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of the coordinates $(\tau, \sigma) = (x^0, x^1)$. The Nambu-Goto action (3.199) takes the form:

$$\frac{S}{T} = \int dx\, e^{\frac{\phi}{2}} \sqrt{(r')^2 + \frac{r^4}{R^4}} \equiv \int dx L \,, \tag{3.200}$$

where the prime denotes derivative with respect to x, $T = \int dx^0$ and we have defined an effective lagrangian function L. Since L does not depend explicitly on x, the Euler-Lagrange equation of motion has the following first integral:

$$r' \frac{\partial L}{\partial r'} - L = \text{constant} \,, \tag{3.201}$$

or, more explicitly:

$$\frac{r^4\, e^{\frac{\phi}{2}}}{\sqrt{(r')^2 + \frac{r^4}{R^4}}} = r_0^2\, R^2\, e^{\frac{\phi_0}{2}} \,, \tag{3.202}$$

where r_0 is the turning point, *i.e.*, the minimal value of the coordinate r, and $\phi_0 = \phi(r = r_0)$. It is now straightforward to use (3.202) to obtain r':

$$r' = \pm \frac{r^2}{R^2} \sqrt{\left(\frac{r}{r_0}\right)^4 e^{\phi - \phi_0} - 1} \,, \tag{3.203}$$

from which we we easily get the parallel cartesian coordinate x as a function of the holographic coordinate r:

$$x(r) = \pm \frac{R^2}{r_0} \int_1^{\frac{r}{r_0}} \frac{dy}{y^2 \sqrt{y^{\frac{14}{3}} - 1}} \,. \tag{3.204}$$

It follows that the quark-antiquark distance $d_\parallel$ at the boundary is:

$$d_\parallel = \frac{2R^2}{r_0} \int_1^\infty \frac{dy}{y^2 \sqrt{y^{\frac{14}{3}} - 1}} = \frac{2R^2 \sqrt{\pi}}{r_0} \frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)} \,. \tag{3.205}$$

Let us now use (3.202) to compute the on-shell action for this configuration of the fundamental string.. After some calculation we get:

$$\frac{S_{on-shell}}{T} = 2\, \frac{1}{2\pi}\, \frac{e^{\frac{\phi_0}{2}}}{r_0^2} \int_{r_0}^{r_{\max}} \frac{r^2\, e^{\phi - \phi_0}\, dr}{\sqrt{\left(\frac{r}{r_0}\right)^4 e^{\phi - \phi_0} - 1}} \,. \tag{3.206}$$

Using the value of the dilaton for our background, we obtain:

$$\frac{S_{on-shell}}{T} = \frac{1}{\pi} \left(\frac{3}{4\, Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}} \int_1^{\frac{r_{max}}{r_0}} \frac{y^{\frac{8}{3}}\, dy}{\sqrt{y^{\frac{14}{3}} - 1}} \,, \tag{3.207}$$

which is a divergent integral when $r_{\rm max} \to \infty$. We regularize this divergence by subtracting the action of two fundamental strings going straight from $r = 0$ to the boundary at $r = r_{rmax}$. The resulting finite action dividen by T is identified with the $q\bar{q}$ potential:

$$V_{q\bar{q}} = \frac{S^{reg}_{on-shell}}{T} = \frac{S_{on-shell}}{T} - \frac{2}{2\pi}\int_0^{r_{max}} dr\, e^{\frac{\phi}{2}} = \frac{S_{on-shell}}{T} - \frac{3}{4\pi}\left(\frac{3}{4\,Q_f}\right)^{\frac{1}{3}} r_{max}^{\frac{4}{3}} \ . \tag{3.208}$$

One can easily show that $V_{q\bar{q}}$ can be rewritten as:

$$V_{q\bar{q}} = -\frac{1}{\pi}\left(\frac{3}{4\,Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}}\left[\frac{3}{4} - \int_1^{\infty} dy\, y^{\frac{1}{3}}\left(\frac{y^{\frac{7}{3}}}{\sqrt{y^{\frac{14}{3}}-1}} - 1\right)\right] . \tag{3.209}$$

The integral inside the brackets in this last expression can be computed analytically. We get:

$$V_{q\bar{q}} = -\frac{3}{4\sqrt{\pi}}\left(\frac{3}{4\,Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}}\,\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)} \ . \tag{3.210}$$

By using the relation (3.205) we can eliminate r_0 in favor of the $q\bar{q}$ distance $d_{\|}$. After some calculation we get:

$$V_{q\bar{q}} = -\beta_{\|}\,\frac{Q_c^{\frac{2}{3}}}{Q_f^{\frac{1}{3}}}\,\frac{1}{d_{\|}^{\frac{4}{3}}} \ , \qquad \beta_{\|} = \frac{16\pi^{\frac{1}{6}}}{9\cdot 5^{\frac{2}{3}}}\left(\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)}\right)^{\frac{7}{3}} \ . \tag{3.211}$$

Inter-layer potential

Let us now repeat the analysis of the previous section for the case in which the fundamentals are separated at the boundary in the transverse direction $x^3 \equiv z$ to the layers. We now take $\tau = x^0$ and $\sigma = z$ and consider an ansatz of the form $r = r(z)$. The corresponding Nambu-Goto action becomes:

$$\frac{S}{T} = \frac{1}{2\pi}\int dz\, e^{\frac{\phi}{2}}\sqrt{(r')^2 + e^{-2\phi}\,\frac{r^4}{R^4}} \equiv \int dz L \ , \tag{3.212}$$

where now $r' = dr/dz$. Proceeding as in section 3.A.1, we get:

$$r' = e^{-\phi}\,\frac{r^2}{R^2}\sqrt{\left(\frac{r}{r_0}\right)^4 e^{\phi_0-\phi} - 1} \ , \tag{3.213}$$

which yields the following function $z = z(r)$:

$$z(r) = \pm\frac{R^2}{r_0^{\frac{1}{3}}}\left(\frac{3}{4Q_f}\right)^{\frac{2}{3}}\int_1^{\frac{r}{r_0}} \frac{dy}{y^{\frac{4}{3}}\sqrt{y^{\frac{10}{3}}-1}} \ , \tag{3.214}$$

as well as the following transverse distance:

$$d_\perp = 2\,\frac{R^2}{r_0^{\frac{1}{3}}}\left(\frac{3}{4Q_f}\right)^{\frac{2}{3}}\int_1^\infty \frac{dy}{y^{\frac{4}{3}}\sqrt{y^{\frac{10}{3}}-1}} = 6\sqrt{\pi}\left(\frac{3}{4Q_f}\right)^{\frac{2}{3}}\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)}\frac{R^2}{r_0^{\frac{1}{3}}}\,. \tag{3.215}$$

The unregulated on-shell action in this case is given by:

$$\frac{S_{on-shell}}{T} = \frac{e^{\frac{\phi_0}{2}}}{\pi\,r_0^2}\int_{r_0}^{r_{\max}}\frac{r^2\,dr}{\sqrt{\left(\frac{r}{r_0^2}\right)^4 e^{\phi_0-\phi}-1}} = \frac{1}{\pi}\left(\frac{3}{4Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}}\int_1^{\frac{r_{max}}{r_0}}\frac{y^2\,dy}{\sqrt{y^{\frac{10}{3}}-1}}\,, \tag{3.216}$$

whereas the $q\bar{q}$ potential is:

$$V_{q\bar{q}} = -\frac{1}{\pi}\left(\frac{3}{4\,Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}}\left[\frac{3}{4} - \int_1^\infty dy\,y^{\frac{1}{3}}\left(\frac{y^{\frac{5}{3}}}{\sqrt{y^{\frac{10}{3}}-1}}-1\right)\right]. \tag{3.217}$$

By performing the integral in this last expression we arrive at:

$$V_{q\bar{q}} = -\frac{3}{4\sqrt{\pi}}\left(\frac{3}{4\,Q_f}\right)^{\frac{1}{3}} r_0^{\frac{4}{3}}\,\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)}\,. \tag{3.218}$$

Finally, we can rewrite this intra-layer potential in terms of $d_\perp$ as:

$$V_{q\bar{q}} = -\beta_\perp\,\frac{Q_c^2}{Q_f^3}\,\frac{1}{d_\perp^4}\,, \qquad \beta_\perp = \frac{2^{12}\,\pi^{\frac{3}{2}}}{3^2\cdot 5^2}\left(\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)}\right)^5\,. \tag{3.219}$$

3.A.2 Entanglement entropy

Let A be a spatial region in the gauge theory. The holographic entanglement entropy between A and its complement is obtained by finding the 8-dimensional spatial surface Σ whose boundary coincides with the boundary of A and minimizes the functional [52,171]:

$$S_A = \frac{1}{4G_{10}}\int_\Sigma d^8\xi\,\sqrt{\det g_8}\,, \tag{3.220}$$

where G_{10} is the 10-dimensional Newton constant ($G_{10} = 8\pi^6$ in our units) and g_8 is the induced metric on Σ in the Einstein frame. The entanglement entropy between A and its complement is given by S_A evaluated on the minimal surface Σ. We will obtain S_A when A is a slab extended infinitely in two spatial cartesian directions and having a finite width in the third one. We will consider separately the two cases corresponding to the two possible orientations of the slab.

Parallel slab

Let us consider first the case in which A is the region $\{-\frac{l_\parallel}{2} \leq x^1 \leq \frac{l_\parallel}{2}, -\infty < x^2, x^3 < +\infty\}$, *i.e.*, when the slab has a finite width in the direction parallel to the layers. We will characterize the surface Σ by a function $r = r(x)$, where $x \equiv x^1$. After integrating over all coordinates except x, we get:

$$\frac{S_\parallel}{L_2\,L_3} = \frac{R^4}{32\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 \int e^{-\phi}\, r\,\sqrt{(r')^2 + \frac{r^4}{R^4}}\; dx\;, \tag{3.221}$$

where $L_{2,3} = \int dx^{2,3}$ and $r' = dr/dx$. The Euler-Lagrange equations which minimize $S_\parallel$ admit the following first integral:

$$\frac{r^5\, e^{-\phi}}{\sqrt{(r')^2 + \frac{r^4}{R^4}}} = R^2\, r_0^3\, e^{-\phi_0}\;, \tag{3.222}$$

where r_0 is the minimal value of r and $\phi_0 = \phi(r = r_0)$. It follows that r' is given by:

$$r' = \pm\frac{r^2}{R^2}\sqrt{\left(\frac{r}{r_0}\right)^6 e^{2(\phi_0-\phi)} - 1} = \pm\frac{r^2}{R^2}\sqrt{\left(\frac{r}{r_0}\right)^{\frac{14}{3}} - 1}\;, \tag{3.223}$$

and, therefore:

$$x(r) = \pm\frac{R^2}{r_0}\int_1^{\frac{r}{r_0}} \frac{dy}{y^2\sqrt{y^{\frac{14}{3}} - 1}}\;. \tag{3.224}$$

Then, the length $l_\parallel$ in the direction parallel to the layers is:

$$l_\parallel = \frac{2\,R^2}{r_0}\int_1^\infty \frac{dy}{y^2\sqrt{y^{\frac{14}{3}} - 1}} = \frac{2R^2\sqrt{\pi}}{r_0}\,\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)}\;. \tag{3.225}$$

One can now evaluate the entropy for this configuration. We get:

$$\frac{S_\parallel}{L_2\,L_3} = \frac{R^4}{16\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 r_0^2\, e^{-\phi_0}\int_1^{\frac{r_{max}}{r_0}} \frac{y^{\frac{8}{3}}\, dy}{\sqrt{y^{\frac{14}{3}} - 1}}\;. \tag{3.226}$$

The integral (3.226) is divergent at the UV and has been regulated by introducing a maximal radial coordinate $r_{\max}$. The divergent part of $S_\parallel$ can be obtained by computing the contribution of the upper limit to the integral (3.226) and gives:

$$\frac{S_\parallel^{div}}{L_2\,L_3} = \frac{3\,R^4}{64\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6\left(\frac{4\,Q_f}{3}\right)^{\frac{2}{3}} r_{max}^{\frac{4}{3}}\;. \tag{3.227}$$

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We now define $S_{\parallel}^{finite}$ as:

$$\frac{S_{\parallel}^{finite}}{L_2\, L_3} \;=\; \frac{S_{\parallel}\;-\;S_{\parallel}^{div}}{L_2\, L_3}\;. \tag{3.228}$$

One can readily demonstrate that:

$$\frac{S_{\parallel}^{finite}}{L_2\, L_3} \;=\; -\frac{R^4}{16\,\pi^3}\Big(\frac{3}{2\sqrt{2}}\Big)^6\, r_0^2\, e^{-\phi_0}\left[\frac{3}{4}\;-\;\int_1^{\infty} dy\, y^{\frac{1}{3}}\left(\frac{y^{\frac{7}{3}}}{\sqrt{y^{\frac{14}{3}}-1}}\;-\;1\right)\right]\,, \tag{3.229}$$

which, after performing the integration, gives:

$$\frac{S_{\parallel}^{finite}}{L_2\, L_3} \;=\; -\frac{3\sqrt{\pi}}{64\pi^3}\Big(\frac{3}{2\sqrt{2}}\Big)^6\Big(\frac{4\,Q_f}{3}\Big)^{\frac{2}{3}}\frac{\Gamma\Big(\frac{5}{7}\Big)}{\Gamma\Big(\frac{3}{14}\Big)}R^4\, r_0^{\frac{4}{3}}\;. \tag{3.230}$$

By using the relation (3.225) between r_0 and $l_{\parallel}$, we can rewrite $S_{\parallel}^{finite}$ as:

$$\frac{S_{\parallel}^{finite}}{L_2\, L_3} \;=\; -\gamma_{\parallel}\,\frac{Q_f^{\frac{2}{3}}\, Q_c^{\frac{5}{3}}}{l_{\parallel}^{\frac{4}{3}}}\,, \qquad\qquad \gamma_{\parallel} \;=\; \frac{2}{45\cdot 5^{\frac{2}{3}}\,\pi^{\frac{11}{6}}}\left(\frac{\Gamma\Big(\frac{5}{7}\Big)}{\Gamma\Big(\frac{3}{14}\Big)}\right)^{\frac{7}{3}}\;. \tag{3.231}$$

Transverse slab

We now take A to be $\{-\infty < x^1, x^2 < +\infty\,,\, -\frac{l_{\perp}}{2}\;\le\; x^3\;\le\;\frac{l_{\perp}}{2}\}$, *i.e.*, a slab with finite width in the direction x^3 transverse to the layers. If $z\equiv x^3$, the surface Σ is parameterized by a function $r=r(z)$ and the functional to be minimized is:

$$\frac{S_{\perp}}{L_1\, L_2} \;=\; \frac{R^4}{32\pi^3}\Big(\frac{3}{2\sqrt{2}}\Big)^6\int r\,\sqrt{(r')^2\;+\;\frac{r^4}{R^4}\,e^{-2\phi}}\;dz\;. \tag{3.232}$$

The corresponding first integral is now:

$$\frac{r^5\, e^{-2\phi}}{\sqrt{(r')^2\;+\;\frac{r^4}{R^4}\,e^{-2\phi}}} \;=\; R^2\, r_0^3\, e^{-\phi_0}\;, \tag{3.233}$$

and, as a consequence, r' is given by:

$$r' \;=\; \pm\,\frac{r^2}{R^2}\,e^{-\phi}\sqrt{\Big(\frac{r}{r_0}\Big)^6 e^{2(\phi_0-\phi)}\;-\;1} \;=\; \pm\,\frac{r^2}{R^2}\,e^{-\phi}\sqrt{\Big(\frac{r}{r_0}\Big)^{\frac{14}{3}}\;-\;1}\;. \tag{3.234}$$

Therefore $z(r)$ is the following integral:

$$z(r) \;=\; \pm\Big(\frac{3}{4Q_f}\Big)^{\frac{2}{3}}\,\frac{R^2}{r_0^{\frac{1}{3}}}\int_1^{\frac{r}{r_0}}\frac{dy}{y^{\frac{4}{3}}\,\sqrt{y^{\frac{14}{3}}\;-\;1}}\;, \tag{3.235}$$

and the transverse length $l_\perp$ is related to r_0 as:

$$l_\perp = 6\sqrt{\pi}\left(\frac{3}{4Q_f}\right)^{\frac{2}{3}} \frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)} \frac{R^2}{r_0^{\frac{1}{3}}} \,. \tag{3.236}$$

The functional $S_\perp$ evaluated on the minimal surface is given by:

$$\frac{S_\perp}{L_1\,L_2} = \frac{R^4}{16\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 r_0^2 \int_1^{\frac{r_{max}}{r_0}} \frac{y^{\frac{10}{3}}\,dy}{\sqrt{y^{\frac{14}{3}}-1}} \,, \tag{3.237}$$

and its divergent part is:

$$\frac{S_\perp^{div}}{L_1\,L_2} = \frac{R^4}{32\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 r_{max}^2 \,. \tag{3.238}$$

Defining $S_\perp^{finite}$ by subtracting $S_\perp^{div}$ from $S_\perp$:

$$\frac{S_\perp^{finite}}{L_1\,L_2} = \frac{S_\perp - S_\perp^{div}}{L_1\,L_2} \,, \tag{3.239}$$

we get:

$$\frac{S_\perp^{finite}}{L_1\,L_2} = -\frac{R^4}{16\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 r_0^2 \left[\frac{1}{2} - \int_1^\infty dy\,y\left(\frac{y^{\frac{7}{3}}}{\sqrt{y^{\frac{14}{3}}-1}} - 1\right)\right] \,, \tag{3.240}$$

which, after computing the integral, becomes:

$$\frac{S_\perp^{finite}}{L_1\,L_2} = -\frac{1}{32\,\pi^3}\left(\frac{3}{2\sqrt{2}}\right)^6 \sqrt{\pi}\,\frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)} R^4\,r_0^2 \,. \tag{3.241}$$

Finally, using the relation (3.236), we arrive at:

$$\frac{S_\perp^{finite}}{L_1\,L_2} = -\gamma_\perp\,\frac{Q_c^4}{Q_f^4}\,\frac{1}{l_\perp^6} \,, \qquad\qquad \gamma_\perp = \left(\frac{16}{15}\right)^4 \sqrt{\pi}\left(\frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)}\right)^7 \,. \tag{3.242}$$

3.B Dimensional reduction

In this appendix we give details on the dimensional reduction of our setup. We first consider the reduction to 4 dimensions.

3.B.1 4-dimensional reduction

Let us consider the reduction ansatz of the 10-dimensional metric written in (3.70). For this ansatz, the determinant of the 10-dimensional and 4-dimensional metrics are related as:

$$\sqrt{-G_{10}} \,=\, e^{\frac{10}{3}\gamma-\beta}\,\sqrt{G_5}\,\sqrt{-g_4}\;, \tag{3.243}$$

where G_5 is the determinant of the 5-dimensional compact internal manifold. Moreover, the relation between the Ricci scalars in ten and in 4 dimensions is:

$$R_{10} \,=\, e^{-\frac{10}{3}\gamma+\beta}\Big[R_4-\frac{40}{3}\,(\partial\gamma)^2-20\,(\partial\lambda)^2-\frac{3}{2}\,(\partial\beta)^2+24\,e^{\frac{16}{3}\gamma+2\lambda-\beta}-4\,e^{\frac{16}{3}\gamma+12\lambda-\beta}+\Lambda\Big]\;, \tag{3.244}$$

where Λ is given by:

$$\Lambda \,=\, \frac{1}{\sqrt{-g_4}}\partial_m\Big[\sqrt{-g_4}\,g^{mn}\,\partial_n\Big(\beta-\frac{10}{3}\gamma\Big)\Big]\;. \tag{3.245}$$

As Λ leads to a total derivative in the 4-dimensional Einstein-Hilbert action and, thus, it does not contribute to the equations of motion and we simply drop it from our equations. The Einstein-Hilbert action in 10 dimensions can be written as:

$$\begin{aligned}\int d^{10}X\,\sqrt{-G_{10}}\,R_{10} \,=\, V_5\,V_{x^3}\int d^4z\sqrt{-g_4}\,\Big[R_4\,-\,\frac{40}{3}\,(\partial\gamma)^2\,-\,20\,(\partial\lambda)^2\,-\,\frac{3}{2}\,(\partial\beta)^2\,+\\ +\,24\,e^{\frac{16}{3}\gamma+2\lambda-\beta}\,-\,4\,e^{\frac{16}{3}\gamma+12\lambda-\beta}\Big]\;,\end{aligned} \tag{3.246}$$

where V_5 is the volume of the 5-dimensional compact space and $V_{x^3}\equiv\int dx^3$. Let us now write the contribution of the remaining fields of type IIB supergravity to the reduced action. We start with the contribution of the dilaton ϕ, which is proportional to:

$$\int d^{10}X\,\sqrt{-G_{10}}\,\frac{1}{2}\,G^{MN}\,\partial_M\,\phi\,\partial_N\,\phi \,=\, V_5\,V_{x^3}\int d^4z\,\sqrt{-g_4}\;\frac{1}{2}\,g^{mn}\,\partial_m\,\phi\,\partial_n\phi\;. \tag{3.247}$$

Moreover, the RR 5-form F_5 in these new variables is:

$$F_5 \,=\, Q_c\,e^{\frac{40}{3}\gamma\,-\,\beta}\,\sqrt{-g_4}\;d^4z\,\wedge\,dx^3\;, \tag{3.248}$$

and its contribution to the effective action is proportional to:

$$\int\frac{1}{2}\,F_5\wedge{}^*F_5 \,=\, V_5\,V_{x^3}\int d^4z\sqrt{-g_4}\;\frac{Q_c^2}{2}\,e^{\frac{40}{3}\gamma\,-\,\beta}\;. \tag{3.249}$$

Similarly, the RR 3-form F_3 contributes as:

$$\int\frac{1}{2}\,e^{\phi}\,F_3\wedge{}^*F_3 \,=\, V_5\,V_{x^3}\int d^4z\sqrt{-g_4}\;Q_f^2\,e^{\phi+4\gamma+4\lambda-3\beta}\;. \tag{3.250}$$

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It remains to calculate the contribution of the DBI action of the flavor D5-branes, which is given by:

$$-\frac{3V_5\,V_{x^3}\,Q_f}{\kappa_{10}^2}\int d^4z\,\sqrt{-g_4}\;e^{\frac{14}{3}\gamma-2\beta-2\lambda+\frac{\phi}{2}}\;. \tag{3.251}$$

Putting everything together, we can write the effective action as in (3.71), where V is the potential for the scalar fields ϕ, γ, λ and β written in (3.72).

Let us now write down the equations of motion derived from the action (3.71). First of all, the equation of motion for the 4-dimensional metric is:

$$R_{mn}\;=\;\frac{1}{2}\,\partial_m\phi\,\partial_n\phi\;+\;\frac{40}{3}\,\partial_m\gamma\,\partial_n\gamma\;+\;20\,\partial_m\lambda\,\partial_n\lambda\;+\;\frac{3}{2}\partial_m\beta\,\partial_n\beta\;+\;\frac{1}{2}\,g_{mn}\,V\;, \tag{3.252}$$

where R_{mn} is the Ricci tensor for g_{mn}. These equations are equivalent to the ones written in (3.75). Moreover, if we define the d'Alembertian of any scalar field Ψ as in (3.76), the equations for ϕ, γ, λ and β are:

$$\begin{aligned}
\Box\,\phi\;&=\;\partial_\phi V\;, &\qquad \Box\,\gamma\;&=\;\frac{3}{80}\,\partial_\gamma V\;,\\
\Box\,\lambda\;&=\;\frac{1}{40}\,\partial_\lambda V\;, &\qquad \Box\,\beta\;&=\;\frac{1}{3}\,\partial_\beta V\;,
\end{aligned} \tag{3.253}$$

where we have denoted $\frac{\partial V}{\partial\phi}=\partial_\phi V$ and similarly for the other scalar fields. Notice that the four equations in (3.253) can be written more compactly as in (3.77). Let us now write the equations (3.253) for the scalars more explicitly:

$$\begin{aligned}
\Box\,\phi\;&=\;Q_f^2\,e^{4\gamma+4\lambda-3\beta+\phi}\;+\;3Q_f\,e^{\frac{14}{3}\gamma-2\lambda-2\beta+\frac{\phi}{2}}\,,\\
\Box\,\gamma\;&=\;-\frac{24}{5}\,e^{\frac{16}{3}\gamma+2\lambda-\beta}\;+\;\frac{4}{5}\,e^{\frac{16}{3}\,\gamma+12\lambda-\beta}\;+\;\frac{3\,Q_f^2}{20}\,e^{4\gamma+4\lambda-3\beta+\phi}\;+\;\frac{Q_c^2}{4}\,e^{\frac{40}{3}\gamma-\beta}\;+\\
&+\frac{21Q_f}{20}\,e^{\frac{14}{3}\gamma-2\lambda-2\beta+\frac{\phi}{2}}\;,\\
\Box\,\lambda\;&=\;-\frac{6}{5}\,e^{\frac{16}{3}\gamma+2\lambda-\beta}\;+\;\frac{6}{5}\,e^{\frac{16}{3}\,\gamma+12\lambda-\beta}\;+\;\frac{Q_f^2}{10}\,e^{4\gamma+4\lambda-3\beta+\phi}\;-\;\frac{3\,Q_f}{10}\,e^{\frac{14}{3}\gamma-2\lambda-2\beta+\frac{\phi}{2}}\;,\\
\Box\,\beta\;&=\;8\,e^{\frac{16}{3}\gamma+2\lambda-\beta}\;-\;\frac{4}{3}e^{\frac{16}{3}\,\gamma+12\lambda-\beta}\;-\;Q_f^2e^{4\gamma+4\lambda-3\beta+\phi}\;-\;\frac{Q_c^2}{6}e^{\frac{40}{3}\gamma-\beta}\;-\\
&-4\,Q_f e^{\frac{14}{3}\gamma-2\lambda-2\beta+\frac{\phi}{2}}\;.
\end{aligned} \tag{3.254}$$

3.B.2 5-dimensional reduction

Let us now consider a reduction of the 10-dimensional metric to a 5-dimensional metric according to the ansatz (3.90). The determinants of the 10-dimensional and 5-dimensional

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metrics are related as:

$$\sqrt{-G_{10}} \,=\, e^{\frac{10}{3}\gamma}\,\sqrt{G_5}\,\sqrt{-g_5}\;, \tag{3.255}$$

where G_5 is the determinant of the 5-dimensional compact internal manifold. Up to terms which give a total derivative in the Einstein-Hilbert action, the Ricci scalars in ten and in 5 dimensions are related as:

$$R_{10} \,=\, e^{-\frac{10}{3}\gamma}\Big[R_5 \,-\, \frac{40}{3}\,(\partial\gamma)^2 \,-\, 20\,(\partial\lambda)^2 \,+\, 24\,e^{\frac{16}{3}\gamma+2\lambda} \,-\, 4\,e^{\frac{16}{3}\gamma+12\lambda}\Big]\;. \tag{3.256}$$

Then, the Einstein-Hilbert action in ten dimensions can be written as:

$$\int d^{10}X\,\sqrt{-G_{10}}\,R_{10} = V_5\int d^5z\sqrt{-g_5}\,\Big[R_5 - \frac{40}{3}\,(\partial\gamma)^2 - 20\,(\partial\lambda)^2 + 24\,e^{\frac{16}{3}\gamma+2\lambda} - 4\,e^{\frac{16}{3}\gamma+12\lambda}\Big]\;, \tag{3.257}$$

where V_5 is the volume of the 5-dimensional compact space. Let us write the contribution of the remaining fields of type IIB supergravity to the effective action. The dilaton contributes as:

$$\int d^{10}X\,\sqrt{-G_{10}}\,\frac{1}{2}\,G^{MN}\,\partial_M\,\phi\,\partial_N\,\phi \,=\, V_5\int d^5z\,\sqrt{-g_5}\,\frac{1}{2}\,g^{pq}\,\partial_p\,\phi\,\partial_q\phi\;. \tag{3.258}$$

The RR 5-form is:

$$F_5 \,=\, Q_c\,e^{\frac{40}{3}\gamma}\,\sqrt{-g_5}\,d^5z\,, \tag{3.259}$$

and contributes to the effective action as:

$$\int\frac{1}{2}\,F_5\wedge{}^*F_5 \,=\, V_5\int d^5z\sqrt{-g_5}\,\frac{Q_c^2}{2}\,e^{\frac{40}{3}\gamma}\;. \tag{3.260}$$

Let us consider the following ansatz for the RR 3-form F_3:

$$F_3 \,=\, \frac{1}{\sqrt{2}}\,\mathcal{F}_1\wedge\mathrm{Im}\,\hat{\Omega}_2\;, \tag{3.261}$$

where $\hat{\Omega}_2$ is the 2-form (2.17) and $\mathcal{F}_1$ has only components along the 5-dimensional space. We will represent $\mathcal{F}_1$ in terms of a scalar potential $\mathcal{V}$ as in (3.91). Then, the contribution of $\mathcal{V}$ to the action is:

$$\frac{1}{2}\int_{\mathcal{M}_{10}} F_3\wedge{}^*F_3 \,=\, \frac{V_5}{2}\int d^5z\,\sqrt{-g_5}\,e^{4\gamma+4\lambda+\phi}\,(\partial\mathcal{V})^2\;. \tag{3.262}$$

The DBI action of the flavor D5-branes is:

$$S_{DBI} \,=\, -T_5\sum_{N_f}\int d^6\xi\,e^{\frac{\phi}{2}}\,\sqrt{-\hat{g}_6} \tag{3.263}$$

After smearing and integration over the internal manifold, the DBI action becomes:

$$S_{DBI} \,=\, -\frac{6\,Q_f\,V_5}{2\kappa_{10}^2}\int d^5z\,\sqrt{-\hat{g}_4}\,e^{\frac{\phi}{2}+\frac{14}{3}\gamma-2\lambda}\;, \tag{3.264}$$

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where $\hat{g}_4$ is the determinant of the metric obtained by taking the pullback of the 5-dimensional metric on a surface with constant x^3. Putting everything together we arrive at the effective action (3.92). The equations of motion for the scalars ϕ, γ and λ derived from (3.92) are:

$$
\begin{aligned}
\Box\,\phi &= \partial_\phi U + \frac{1}{2}\,e^{4\lambda+4\gamma+\phi}\,\big(\partial\mathcal{V}\big)^2 + 3\,Q_f\,\frac{\sqrt{-\hat{g}_4}}{\sqrt{-g_5}}\,e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\;, \\
\Box\,\gamma &= \frac{3}{80}\,\partial_\gamma U + \frac{3}{40}\,e^{4\lambda+4\gamma+\phi}\,\big(\partial\mathcal{V}\big)^2 + \frac{21}{20}\,Q_f\,\frac{\sqrt{-\hat{g}_4}}{\sqrt{-g_5}}\,e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\;, \\
\Box\,\lambda &= \frac{1}{40}\,\partial_\lambda U + \frac{1}{20}\,e^{4\lambda+4\gamma+\phi}\,\big(\partial\mathcal{V}\big)^2 - \frac{3}{10}\,Q_f\,\frac{\sqrt{-\hat{g}_4}}{\sqrt{-g_5}}\,e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\;,
\end{aligned}
\tag{3.265}
$$

where $\Box$ is the laplacian operator for the 5-dimensional metric. Let us group the 5-dimensional scalars into a single 3-component field $\Psi = (\phi, \gamma, \lambda)$. Then, the three scalar equations of (3.265) can be compactly written as:

$$
\Box\Psi = \alpha_\Psi\,\partial_\Psi\,U + \frac{1}{2}\,\alpha_\Psi\,\big(\partial\mathcal{V}\big)^2\,\partial_\Psi\Big(e^{4\lambda+4\gamma+\phi}\Big) + 6\,Q_f\,\frac{\sqrt{-\hat{g}_4}}{\sqrt{-g_5}}\,\alpha_\Psi\,\partial_\Psi\Big(e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\Big)\;, \tag{3.266}
$$

where the coefficients α_Ψ are those written in (3.74) for the three scalars (ϕ, γ, λ). The equation of $\mathcal{V}$ is:

$$
\partial_p\Big[\sqrt{-g_5}\,e^{4\lambda+4\gamma+\phi}\,g^{pq}\,\partial_q\,\mathcal{V}\Big] = 0\;, \tag{3.267}
$$

while the Einstein equations are:

$$
\begin{aligned}
R_{pq} = \sum_\Psi \frac{1}{2\alpha_\Psi}\,\partial_p\,\Psi\,\partial_q\,\Psi + \frac{1}{2}\,e^{4\lambda+4\gamma+\phi}\,\partial_p\mathcal{V}\,\partial_q\mathcal{V} + \frac{1}{3}\,g_{pq}\,U\, + \\
+\,3Q_f\,\frac{\sqrt{-g_4}}{\sqrt{-g_5}}\,e^{\frac{14}{3}\gamma-2\lambda+\frac{\phi}{2}}\left(\frac{4}{3}\,g_{pq} - \hat{g}^{(4)}_{pq}\right)\;.
\end{aligned}
\tag{3.268}
$$

It is straightforward to demonstrate that the metric written in (3.94) and (3.95), together with the scalars displayed in (3.96) and (3.97), satisfy (3.265), (3.267) and (3.268).

3.B.3 5-dimensional $\rightarrow$ 4-dimensional reduction

Let us now perform an additional reduction of the 5-dimensional action to 4 dimensions. We will reduce along the coordinate x^3 and we will adopt the following ansatz for the 5-dimensional metric:

$$
ds_5^2 = e^{-\beta}\,ds_4^2 + e^{2\beta}\,(dx^3)^2\;, \tag{3.269}
$$

where β is a new scalar which depends on the 4-dimensional coordinates. The determinant of the 5-dimensional metric and the one corresponding to the pullback to the surface $x^3 =$ constant are related to the determinant g_4 of the reduced 4-dimensional metric as:

$$
\sqrt{-g_5} = e^{-\beta}\,\sqrt{-g_4}\;, \qquad\qquad \sqrt{-\hat{g}_4} = e^{-2\beta}\,\sqrt{-g_4}\;. \tag{3.270}
$$

Moreover, after neglecting a total derivative, we can relate the Einstein-Hilbert term of the action (3.257) to the one corresponding to the reduced 4-dimensional action as:

$$\int d^5z\,\sqrt{-g_5}\,R_5 \;=\; \int d^4z\,\sqrt{-g_4}\,\big(R_4\,-\,\frac{3}{2}\,(\partial\beta)^2\big)\;. \tag{3.271}$$

Let us now split the 1-form $\mathcal{F}_1$ as:

$$\mathcal{F}_1 \;=\; \chi\, dx^3 \,+\, f_1\;, \tag{3.272}$$

where f_1 is a closed 1-form that has legs only in the 4-dimensional space. Using that:

$$\mathcal{F}_1^2 \;=\; \big(\partial\mathcal{V}\big)^2 \;=\; e^{-2\beta}\,\chi^2\,+\,e^{\beta}\,f_1^2\;, \tag{3.273}$$

we can write the term containing $\mathcal{V}$ in (3.92) as:

$$\sqrt{-g_5}\,\Big[-\frac{1}{2}\,e^{4\gamma+4\lambda+\phi}\,(\partial\mathcal{V})^2\Big] \;=\; \sqrt{-g_4}\,\Big[-\frac{1}{2}\,e^{4\gamma+4\lambda+\phi}\,\Big(e^{-3\beta}\,\chi^2\,+\,f_1^2\Big)\Big]\;. \tag{3.274}$$

Collecting all these results, we can write the effective action as:

$$\begin{aligned} S_{eff} \;=\; \frac{V_5\,V_{x^3}}{2\,\kappa_{10}^2}\int d^4z\,\sqrt{-g_4}\,\Big[R_4\,-\,\frac{40}{3}\,(\partial\gamma)^2-20\,(\partial\lambda)^2-\frac{3}{2}\,(\partial\beta)^2\,-\\ -\frac{1}{2}\,(\partial\phi)^2-\frac{1}{2}\,e^{4\gamma+4\lambda+\phi}\,f_1^2\,-V\Big]\;, \end{aligned} \tag{3.275}$$

where the 4-dimensional potential V is related to the 5-dimensional one U in (3.93) by the relation:

$$V \;=\; e^{-\beta}\,U\,+\,\frac{1}{2}\,e^{4\gamma+4\lambda+\phi-3\beta}\,\chi^2\,+\,6\,Q_f\,e^{\frac{14}{3}\gamma-2\lambda-2\beta+\frac{\phi}{2}}\;. \tag{3.276}$$

It is now straightforward to verify that the action (3.275) reduces to the one written in (3.71) when we truncate the former in such a way that $f_1=0$ and the scalar χ takes the following constant value:

$$\chi \;=\; \sqrt{2}\,Q_f\;. \tag{3.277}$$

3.C Hydrodynamic fluctuations

In this appendix we provide details of the analysis of the hydrodynamic fluctuations, which complement the presentation given in section 3.6 on the main text. We will consider separately the two channels.

3.C.1 Shear channel

One can show that the fluctuation equations (3.119) and (3.122) for the ansatz (3.123) reduce to:

$$H''_{tx} + \partial_r \log\left(\frac{c_2^4}{c_1\,c_3}\right) H'_{tx} + W\,H_{tx} - q\,\frac{c_3^2}{c_2^2}\left(q\,H_{tx} + \omega\,H_{xy}\right) = 0\,,$$

$$H''_{xy} + \partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right) H'_{xy} + W\,H_{xy} + \omega\,\frac{c_3^2}{c_1^2}\left(q\,H_{tx} + \omega\,H_{xy}\right) = 0\,,$$

$$q\,c_1^2\,H'_{xy} + \omega\,c_2^2\,H'_{tx} = 0\,, \tag{3.278}$$

where W is the function:

$$W = c_3^2\,V + 2\,\partial_r^2 \log c_2 + 2\,\partial_r \log c_2\,\partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right)\,. \tag{3.279}$$

One can verify easily that W vanishes for our background. Therefore, we will omit it in the equations that follow in this section. Notice that the last equation in (3.278) is first-order in the radial derivative and it can be used to reduce the number of equations of the system. Actually, if we define the gauge invariant combination X as in (3.124) and combine this definition and the last equation in (3.278) to express the first derivatives of H_{tx} and H_{xy} in terms of X', we get:

$$H'_{tx} = \frac{q\,c_1^2}{q^2\,c_1^2 - \omega^2\,c_2^2}\,X'\,, \qquad H'_{xy} = -\frac{\omega\,c_2^2}{q^2\,c_1^2 - \omega^2\,c_2^2}\,X'\,. \tag{3.280}$$

Moreover, one can show that the system (3.278) reduces to the following second-order differential equation for X:

$$X'' + \frac{q^2\,c_1^2\,\partial_r \log\left(\frac{c_2^4}{c_1\,c_3}\right) - \omega^2\,c_2^2\,\partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right)}{q^2\,c_1^2 - \omega^2\,c_2^2}\,X' - \frac{c_3^2}{c_2^2\,c_1^2}\,(q^2\,c_1^2 - \omega^2\,c_2^2)\,X = 0\,. \tag{3.281}$$

By using the values of c_1, c_2 and c_3 written in (3.79), one can easily demonstrate that (3.281) can be converted into (3.125).

3.C.2 Sound channel

Plugging (3.154) and (3.151) into (3.119) we get the following second-order equation for $\hat{\Psi}(r)$:

$$\hat{\Psi}'' + \partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right)\hat{\Psi}' + \left[\frac{c_3^2}{c_1^2}\,\omega^2 - \frac{c_3^2}{c_2^2}\,q^2\right]\hat{\Psi} +$$

$$+\frac{1}{2}\Psi'\,(H'_{xx} + H'_{yy} - H'_{tt}) = c_3^2\,\alpha_\Psi\,\hat{\delta}\big[\partial_\Psi V\big] \tag{3.282}$$

where, for every $\Psi = (\phi, \gamma, \lambda, \beta)$, we define:

$$\hat{\delta}\big[\partial_\Psi V\big] \;=\; \partial_\phi\partial_\Psi V\,\Phi(r) \,+\, \partial_\gamma\partial_\Psi V\,\Gamma(r) \,+\, \partial_\lambda\partial_\Psi V\,\Lambda(r) \,+\, \partial_\beta\partial_\Psi V\,B(r)\;. \tag{3.283}$$

Let us now write the equations for the metric fluctuations, which are obtained by taking different values for the (m,n) indices in (3.122). To write these equations compactly, let us denote by $\hat{\delta}V$ the following radial function:

$$\hat{\delta}V \;=\; \partial_\phi V\,\Phi(r) \,+\, \partial_\gamma V\,\Gamma(r) \,+\, \partial_\lambda V\,\Lambda(r) \,+\, \partial_\beta V\,B(r)\;. \tag{3.284}$$

Then, one can check that (3.122) is equivalent to the following second-order equations:

$$\begin{aligned}
&H_{tt}'' \,+\, \partial_r \log\Big(\frac{c_1^2 c_2^2}{c_3}\Big)\, H_{tt}' \,-\, \omega^2\,\frac{c_3^2}{c_1^2}\,(H_{xx}+H_{yy}) -\, q^2\,\frac{c_3^2}{c_2^2} H_{tt} - 2\omega\, q\,\frac{c_3^2}{c_1^2} H_{ty} \,- \\
&\qquad\qquad -\partial_r\,\log c_1\,(H_{xx}'+H_{yy}') \,-\, c_3^2\,\hat{\delta}V + H_{tt}\,\tilde{W} = 0\;, \\
&H_{ty}'' \,+\, \partial_r \log\Big(\frac{c_2^4}{c_1\,c_3}\Big)\, H_{ty}' \,+\, \omega\, q\,\frac{c_3^2}{c_2^2}\, H_{xx} \,+\, H_{ty}\, W \;=\; 0\,, \\
&H_{xx}'' + \partial_r \log\Big(\frac{c_1 c_2^3}{c_3}\Big)\, H_{xx}' + \Big(\omega^2\frac{c_3^2}{c_1^2} - q^2\,\frac{c_3^2}{c_2^2}\Big)\, H_{xx} - \partial_r \log c_2 (H_{tt}' - H_{yy}') + c_3^2\,\hat{\delta}V + \\
&+H_{xx}\, W = 0\;, \\
&H_{yy}'' + \partial_r \log\Big(\frac{c_1 c_2^3}{c_3}\Big)\, H_{yy}' + \frac{c_3^2}{c_1^2}\,(\omega^2 H_{yy} + 2q\omega H_{ty}) + q^2\,\frac{c_3^2}{c_2^2}\,(H_{tt} - H_{xx}) \,+ \\
&\qquad\qquad +\partial_r\,\log c_2\,(H_{xx}' - H_{tt}') \,+\, c_3^2\,\hat{\delta}V + H_{yy}\, W = 0\;,
\end{aligned} \tag{3.285}$$

together with three first-order constraints associated to the gauge fixing condition (3.116):

$$\begin{aligned}
&q\, H_{ty}' \,+\, \omega\,(H_{xx}'+H_{yy}') \;=\; \partial_r \log\frac{c_1}{c_2}\,\big(2q\, H_{ty} \,+\, \omega(H_{xx}+H_{yy})\big) \,-\, \omega\,\sum_\Psi \frac{\Psi'}{\alpha_\Psi}\,\hat{\Psi}\;, \\
&\omega\,\frac{c_2^2}{c_1^2}\, H_{ty}' \,+\, q(H_{tt}' - H_{xx}') \;=\; -q\,\partial_r \log\frac{c_1}{c_2}\, H_{tt} \,+\, q\,\sum_\Psi \frac{\Psi'}{\alpha_\Psi}\,\hat{\Psi}\;, \\
&\partial_r \log c_2^2\, H_{tt}' \,-\, \partial_r \log(c_1 c_2)(H_{xx}'+H_{yy}') \;=\; \frac{c_3^2}{c_1^2}\,\Big(\omega^2(H_{xx}+H_{yy}) \,+\, 2\omega q\, H_{ty}\Big) \,+ \\
&\qquad\qquad +q^2\,\frac{c_3^2}{c_2^2}\,(H_{tt} - H_{xx}) + c_3^2\,\hat{\delta}V \,-\, \sum_\Psi \frac{\Psi'}{\alpha_\Psi}\,\hat{\Psi}\;.
\end{aligned} \tag{3.286}$$

In (3.285) W is the function defined in (3.279), which vanishes in our background and, therefore, will be omitted from now on. The function $\tilde{W}$ appearing in the first equation in (3.285) is defined as:

$$\tilde{W} = c_3^2\, V \,+\, 2\,\partial_r^2 \log c_1 \,+\, 2\,\partial_r \log c_1\, \partial_r \log\left(\frac{c_1\, c_2^2}{c_3}\right) . \tag{3.287}$$

This function also vanishes in our background and will also be omitted in the equations that follow.

We now write the equations for the scalar fluctuations in terms of the new fields $Z_{\hat{\Psi}}$ defined in (3.155). With this aim, let us define $\mathcal{W}_\phi$, $\mathcal{W}_\gamma$, $\mathcal{W}_\lambda$ and $\mathcal{W}_\beta$ as the following linear combinations of the $Z_{\hat{\Psi}}$'s:

$$\mathcal{W}_\Psi \,=\, \alpha_\Psi \sum_{\Psi'} \frac{\partial^2 V}{\partial \Psi\, \partial \Psi'}\, Z_{\hat{\Psi}'} \,. \tag{3.288}$$

It turns out that the equations of motion of the scalar fluctuations can be written as:

$$\begin{aligned}
&Z_\Phi'' \,+\, \partial_r\, \log\left(\frac{c_1 c_2^2}{c_3}\right) Z_\Phi' \,+\, c_3^2\left(\frac{\omega^2}{c_1^2} \,-\, \frac{q^2}{c_2^2}\right) Z_\Phi \,-\, \frac{3\, c_3^2}{7}\left(\mathcal{W}_\phi - 2\mathcal{W}_\beta\right) \,=\, 0 \,,\\
&Z_\Gamma'' \,+\, \partial_r\, \log\left(\frac{c_1 c_2^2}{c_3}\right) Z_\Gamma' \,+\, c_3^2\left(\frac{\omega^2}{c_1^2} \,-\, \frac{q^2}{c_2^2}\right) Z_\Gamma \,-\, c_3^2\, \mathcal{W}_\gamma \,=\, 0 \,,\\
&Z_\Lambda'' \,+\, \partial_r\, \log\left(\frac{c_1 c_2^2}{c_3}\right) Z_\Lambda' \,+\, c_3^2\left(\frac{\omega^2}{c_1^2} \,-\, \frac{q^2}{c_2^2}\right) Z_\Lambda \,-\, c_3^2\, \mathcal{W}_\lambda \,=\, 0 \,,\\
&Z_B'' \,+\, \partial_r\, \log\left(\frac{c_1 c_2^2}{c_3}\right) Z_B' \,+\, c_3^2\left(\frac{\omega^2}{c_1^2} \,-\, \frac{q^2}{c_2^2}\right) Z_B \,-\, \frac{2\, c_3^2}{7}\left(2\mathcal{W}_\beta - \mathcal{W}_\phi\right) \,=\, 0 \,.
\end{aligned} \tag{3.289}$$

By combining the first and last equations in (3.289), one can immediately show that the scalar Z_S defined in (3.156) satisfies the simple equation:

$$Z_S'' \,+\, \partial_r\, \log\left(\frac{c_1 c_2^2}{c_3}\right) Z_S' \,+\, c_3^2\left(\frac{\omega^2}{c_1^2} \,-\, \frac{q^2}{c_2^2}\right) Z_S \,=\, 0 \,. \tag{3.290}$$

More explicitly, this equation can be written as:

$$Z_S'' \,+\, \partial_r\, \log\left(r^{\frac{13}{3}}\, b(r)\right) Z_S' \,+\, \frac{R^4}{r^4\, b^2(r)}\left(\omega^2 \,-\, b(r)\, q^2\right) Z_S \,=\, 0 \,. \tag{3.291}$$

One can demonstrate that the equation for the gauge invariant metric fluctuation Z_H takes the form:

$$Z_H'' \,+\, \mathcal{F}(r)\, Z_H' \,+\, \mathcal{G}(r)\, Z_H \,+\mathcal{H}(r)\, Z_S \,=\, 0 \,, \tag{3.292}$$

where the functions $\mathcal{F}(r)$, $\mathcal{G}(r)$ and $\mathcal{H}(r)$ are given by:

$$
\begin{aligned}
\mathcal{F}(r) &= \partial_r \log\left(\frac{c_1\,c_2^2}{c_3}\right) - 4\,\partial_r \log\left(\frac{c_1}{c_2}\right) + \xi_1(r)\;, \\
\mathcal{G}(r) &= c_3^2\Big(\frac{\omega^2}{c_1^2} - \frac{q^2}{c_2^2}\Big) + 4\Big[\partial_r \log\left(\frac{c_1}{c_2}\right)\Big]^2 - \partial_r \log\left(\frac{c_1}{c_2}\right)\xi_1(r)\;, \\
\mathcal{H}(r) &= -\frac{q^2}{\omega^2}\,\frac{c_1^2}{c_2^2}\left[\partial_\beta V\left(1-\frac{\partial_r \log c_1}{\partial_r \log c_2}\right)c_3^2 + 6\,\beta'\,\xi_2(r)\right]\;,
\end{aligned}
\tag{3.293}
$$

with:

$$
\xi_1(r) = \frac{q^2\partial_r c_1^2\,\frac{\partial_r^2 \log c_2}{(\partial_r \log c_2)^2}\left(1-\frac{\partial_r^2 \log c_1\,\partial_r \log c_2}{\partial_r^2 \log c_2\,\partial_r \log c_1}\right) + 4\,\omega^2\,\partial_r c_2^2\,\left(1-\frac{\partial_r \log c_1}{\partial_r \log c_2}\right)}{q^2\,c_1^2\left(\frac{\partial_r \log c_1}{\partial_r \log c_2}+1\right) - 2\omega^2\,c_2^2}\;,
$$

$$
\xi_2(r) = \frac{(q^2c_1^2-\omega^2c_2^2)\frac{\partial_r^2 \log c_2\partial_r \log c_1-\partial_r^2 \log c_1\partial_r \log c_2}{\partial_r \log c_2} + 2\omega^2c_2^2(\partial_r \log c_1 - \partial_r \log c_2)^2}{q^2\,c_1^2\partial_r \log(c_1c_2) - 2\omega^2c_2^2\,\partial_r \log c_2}\;. \tag{3.294}
$$

Notice that Z_H only couples to the scalar field Z_S. Thus, we are left with (3.290) and (3.292) to be solved in the hydrodynamic approximation.

The scalar fluctuation equation (3.291) only involves the function Z_S and, therefore, can be studied independently of Z_H. Let us adopt the following ansatz for $Z_S(r)$:

$$
Z_S(r) = \big[b(r)\big]^{-\frac{i\hat{\omega}}{2}}\,K(r)\;, \tag{3.295}
$$

for which the infalling boundary conditions at the horizon are satisfied if $K(r)$ is regular at the horizon. Actually, it is much more convenient to change variables and work in the variable x defined in (3.126). Recall that the horizon is located at $x=0$, whereas the boundary is at $x=1$. The fluctuation equation (3.291) is equivalent to the following equation for $K(x)$:

$$
K'' + \frac{1-2i\,\hat{\omega}}{x}\,K' + \frac{\big[1-(1-x^2)^{\frac{7}{5}}\big]\hat{\omega}^2 - x^2\,\hat{q}^2}{x^2\,(1-x^2)^{\frac{7}{5}}}\,K = 0\;, \tag{3.296}
$$

where now the primes denote derivatives with respect to the new variable x. We want to solve (3.296) for low $\hat{q}$. Accordingly, we expand $K(x)$ as:

$$
K(x) = K_0(x) + i\hat{q}\,K_1(x) + \hat{q}^2\,K_2(x)\;. \tag{3.297}
$$

Plugging (3.297) and (3.160) into (3.296) and separating the different orders in $\hat{q}$, we find the following systems of equations:

$$
\begin{aligned}
K_0'' + \frac{K_0'}{x} \;&=\; 0 \;, \\
K_1'' + \frac{K_1'}{x} \;&=\; \frac{2v_s}{x}\, K_0' \;, \\
K_2'' \;+\; \frac{K_2'}{x} \;&=\; \frac{2\,\Gamma}{x}\, K_0' \;+\; \frac{1}{(1-x^2)^{\frac{7}{5}}}\Big(1-\frac{v_s^2}{x^2}\Big) K_0 \;+\; \frac{v_s^2}{x^2}\, K_0 \;-\; \frac{2v_s}{x}\, K_1' \;.
\end{aligned}
\tag{3.298}
$$

The equation for $K_0(x)$ can be straightforwardly integrated in general:

$$
K_0(x) \;=\; c_1 + c_2\,\log x \;, \tag{3.299}
$$

where c_1 and c_2 are constants. The regularity requirement of $K_0(x)$ at $x=0$ implies that $c_2=0$, while the condition $K_0(x=1)=0$ imposes that c_1 vanishes and, thus $K_0(x)=0$. For this value of $K_0(x)$ the equation for $K_1(x)$ in (3.298) is the same as the one for $K_0(x)$. Therefore, the only valid solution for our boundary conditions is $K_1(x)=0$. Furthermore, the same happens for $K_2(x)$ and, thus, we finally have that the solution for $K(x)$ satisfying the boundary conditions is the trivial one, namely:

$$
K(x) \;=\; 0 \;. \tag{3.300}
$$

Therefore, it follows that $Z_S(r)=0$, as claimed in the main text.

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4

Backreacted massive flavored D3-D5 intersection

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4.1 Introduction

In this chapter, we generalize the results of chapter 2 to compute the backreacted D3-D5 intersection to the case of having massive fundamentals and explore the properties of the subsequent geometry. Considering the array (2.1), we see that the D3- and D5-branes can be separated in the directions 789 transverse to both types of branes. When this separation is zero the mass of the hypermultiplets living on the defect vanishes, *i.e.*, we have massless flavors. The D3-D5 massless flavor case was considered in chapter 2, where an analytic supersymmetric solution was found that displays a Lifshitz-like anisotropic scaling invariance. The solutions we present here will preserve the same supersymmetry as the massless flavor case, but will not possess the scaling invariance of the latter.

Considering a continuous distribution of D5-branes along the third direction in (2.1), as we already did in the previous chapters, amounts to construct a multilayer structure of the type shown in figure 4.1. In the absence of D5-branes, the geometry generated by the D3-branes is $AdS_5 \times \mathbb{S}^5$. These backgrounds are dual to anisotropic systems, since there is a distinct field theory direction, the direction orthogonal to the defect.

The gravity duals of theories with massive flavors are running solutions which naturally represent a renormalization group flow. This flow is generated by changing the quark mass (see [172] for an example of these massive flavored backgrounds in the ABJM theory). When the mass of the quarks is very large, the flavors decouple and we expect to recover the unflavored solution ($AdS_5 \times \mathbb{S}^5$ in our case). On the other hand, when the

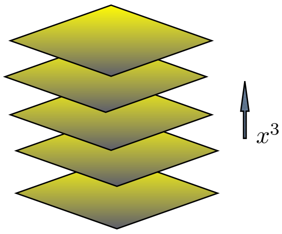

Figure 4.1: Our system has a bulk (3+1)-dimensional theory together with multiple (2+1)-dimensional layers. The direction of the coordinate x^3 is perpendicular to the layers.

flavors are massless we obtain the anisotropic scaling solution of the second chapter. For a finite non-vanishing value of the quark mass we expect to get a background interpolating between these two solutions: unflavored in the IR and massless flavored in the UV. Once we have the background at our disposal we can study the effects of the flow on different observables. In general, we expect to obtain the results corresponding to the isotropic $AdS_5 \times \mathbb{S}^5$ solution in the IR and to be able to tune the amount of anisotropy by changing some of the parameters of our solutions. In our analysis of several observables we will find that, indeed, the UV behaviour is determined by the scaling solution obtained in the second chapter, whereas the long distance IR behaviour depends on a free parameter of our geometry.

The rest of this chapter is organized as follows. In section 4.2 we present our ansatz for the metric, for the dilaton, and for the forms. All functions of the model depend on a master function, which in turn satisfies a second-order differential equation. This equation follows from supersymmetry analysis and is detailed in appendix 4.A. A crucial ingredient entering the equations is the so-called profile function, which encodes the distribution of sources along the holographic coordinate. To determine this function for D5-brane sources one needs to analyze in detail the embeddings of the D5-branes and their kappa symmetry. This analysis is deferred to appendix 4.B.

In section 4.3 we tackle the problem of integrating the master equation. This task requires the redefinition of some of the functions and a change of variables. In its final form our solution depends on a constant parameter which characterizes the IR deformation of the metric. In section 4.4 we begin our study of the observables in our background. In this section we study the Wilson loops and the potentials for quark-antiquark pairs, when these particles are in the same layer or separated in the direction orthogonal to the layers. In section 4.5 we do a similar analysis for the entanglement entropy of slabs.

In section 4.6 we explore our supergravity solution with a probe D5-brane with a worldvolume gauge field dual to a chemical potential. We analyze the zero temperature thermodynamics of the probe and, in particular, the UV-IR flow of the speed of sound. Finally, in section 4.7 we summarize our results.

4.2 Ansatz for the massive geometry

In this section we review the supergravity setup of chapter 2, corresponding to the array (2.1) of D3- and D5-branes. More details are given in appendix 4.A. The D3-branes are color branes which generate an $AdS_5 \times \mathbb{S}^5$ space, whereas the flavor D5-branes create a codimension-1 defect in the (3+1)-dimensional gauge theory and, when the backreaction is included, the original $AdS_5 \times \mathbb{S}^5$ metric gets deformed. The D5-branes are homogeneously distributed in the internal space in such a way that some amount of supersymmetry is preserved. When the $\mathbb{S}^5$ space is represented as a $U(1)$ bundle over $\mathbb{CP}^2$, the deformation of the 5-sphere depends on a single radial function, which measures the relative squashing between the fiber and the base of the deformed $\mathbb{S}^5$. The starting point to our ansatz is similar to the one used in chapter 2, although we will use a different holographic coordinate, ζ, throughout this chapter to construct the massive solutions. This coordinate proves to be more convenient and allows for the construction of a suitable master equation whose solutions completely specify the background. In terms of this radial coordinate ζ (with boundary corresponding to $\zeta = \infty$), the 10-dimensional Einstein frame metric takes the form:

$$ds^2_{10} \,=\, h^{-\frac{1}{2}} \left[-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 \,+\, e^{-2\phi}\,(dx^3)^2 \right] \\ + h^{\frac{1}{2}} \left[\zeta^2 e^{-2f}\, d\zeta^2 + \zeta^2\, ds^2_{\mathbb{CP}^2} + e^{2f}\,(d\tau + A)^2 \right] , \tag{4.1}$$

where ϕ is the dilaton of type IIB supergravity, h is the warp factor, and f is the squashing function. These functions are assumed to depend only on ζ. Moreover, A is the 1-form on $\mathbb{CP}^2$ which implements the non-trivial $U(1)$ bundle, whose expression was given in (2.135). The Minkowski directions x^1 and x^2 are parallel to the defect, whereas x^3 is orthogonal to it.

In this coordinates, the self-dual F_5 flux takes the form:

$$F_5 \,=\, \partial_\zeta \left(e^{-\phi}\, h^{-1} \right) \left(1 + * \right) d^4x \wedge d\zeta \ . \tag{4.2}$$

And as already known, $d\,F_5 = 0$, since the D3-branes have been replaced by a flux in the supergravity solution.

On the contrary, the D5-branes are dynamical and are governed by the standard DBI+WZ action which, in particular, contains the term (2.7), with T_5 given by (2.8), constructed with the 6-form potential C_6 for which $F_7 = -e^{\phi} * F_3$. We will again replace the WZ action by a 10-dimensional integral as in (2.9). This substitution originates the smearing 4-form Ξ, with support on the worldvolume of the D5's and with legs along the directions orthogonal to $\mathcal{M}_6$, which gives the RR D5-brane charge distribution. The equation of motion for C_6 is given by (2.12), where $2\,\kappa^2_{10} \,=\, (2\pi)^7\, g_s^2 \alpha'^4$. In the smearing approach Ξ does not contain Dirac δ-function singularities. Its form can be obtained once the ansatz of F_3 compatible with supersymmetry is fixed. This has been done in chapter 2,

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a result which we now briefly review and adapt to our coordinates. Recalling that the $\mathbb{CP}^2$ manifold is a Kähler-Einstein space endowed with a Kähler 2-form $J = dA/2$ canonically written as in (2.15), where $e^1, \ldots, e^4$ are vielbein 1-forms of $\mathbb{CP}^2$, whose explicit coordinate expression is given by (2.134), we introduce the complex 2-form $\hat{\Omega}_2$ as in (2.17). Then, F_3 is given in our new coordinates by:

$$F_3 \,=\, Q_f \, p(\zeta) \, dx^3 \wedge \mathrm{Im}\, \hat{\Omega}_2 \;, \tag{4.3}$$

where Q_f is a constant and $p(\zeta)$ is an arbitrary function of the holographic coordinate ζ. By computing the exterior derivative of F_3 we get its modified Bianchi identity:

$$dF_3 \,=\, -Q_f \left[3\, p(\zeta) \, dx^3 \wedge \mathrm{Re}\, \hat{\Omega}_2 \wedge (d\tau + A) \,+\, p'(\zeta) \, dx^3 \, \wedge d\zeta \wedge \mathrm{Im}\, \hat{\Omega}_2 \right] . \tag{4.4}$$

Comparing (4.4) and (2.12) we can extract the D5-brane charge distribution Ξ which, in what follows, we will refer to as the smearing form. Clearly, Ξ does not depend on the x^3 coordinate, although it contains dx^3 in its expression. This means that we are continuously distributing our D5-branes along x^3, giving rise to a system of multiple (2+1)-dimensional parallel layers. Moreover, the function $p(\zeta)$ introduces a profile of the charge distribution in the holographic coordinate. Notice that the D3- and D5-branes in the array (2.1) can be separated in the 789 directions. In principle, we could have an arbitrary distribution of D5-branes in these transverse coordinates, which is reflected in the fact that the profile function $p(\zeta)$ is arbitrary. However, for a stack of flavor D5-branes with the same quark mass the function $p(\zeta)$ has a well-defined form (see appendix 4.B) and Q_f is related to the density of smeared branes along the direction x^3. As shown in appendix 4.B, if we distribute N_f D5-branes along a distance L_3 in the third cartesian direction, then $Q_f = \frac{4\pi \, g_s \, \alpha' \, N_f}{9\sqrt{3}\, L_3}$ (see (4.233)).

The preservation of two supercharges for our ansatz imposes a system of first-order differential equations in the radial variable. These BPS equations are reviewed in appendix 4.A, where it is shown that they can be reduced to a single second-order differential equation for a master function $W(\zeta)$. This equation is:

$$\frac{d}{d\zeta}\Big(\zeta \, \frac{dW}{d\zeta}\Big) \,+\, 6\, \frac{dW}{d\zeta} \,=\, -\frac{6\, Q_f \, p(\zeta)}{\zeta^2 \sqrt{W}} \;. \tag{4.5}$$

From W we can reconstruct the full solution. The squashing function $f(\zeta)$ is given by:

$$e^{2f} \,=\, \frac{6\, \zeta^2 \, W}{6\, W \,+\, \zeta \, \frac{dW}{d\zeta}} \;, \tag{4.6}$$

while the dilaton is:

$$e^{-\phi} \,=\, W \,+\, \frac{1}{6}\, \zeta \, \frac{dW}{d\zeta} \;. \tag{4.7}$$

A nice way of measuring the deformation of the metric (4.1) with respect to the $AdS_5 \times \mathbb{S}^5$ geometry is obtained by considering a squashing factor, which we define as follows

$$q \,=\, \frac{e^f}{\zeta} \;. \tag{4.8}$$

It is clear from our ansatz (4.1) that q represents the relative size of the $U(1)$ fiber with respect to the $\mathbb{CP}^2$ base. In terms of the master function W, q is given by:

$$q = \frac{\sqrt{6\,W}}{\sqrt{6\,W + \zeta\,\frac{dW}{d\zeta}}} \ . \tag{4.9}$$

Once f and ϕ are known, the warp factor h can be obtained by integrating the following first-order differential equation:

$$\frac{dh}{d\zeta} + Q_f\,\frac{e^{\frac{3\phi}{2}-f}\,p}{\zeta}\,h = -\frac{Q_c}{\zeta^3}\,e^{-2f} \ , \tag{4.10}$$

where Q_c is related to the number N_c of D3-branes as in (2.136). We can provide a simpler expression to get the warp factor. We will show its derivation in the appendix 4.A. We can show that $h(\zeta)$ is given by:

$$h(\zeta) = C\left[W + \frac{1}{6}\zeta\frac{dW}{d\zeta}\right], \qquad (Q_c = 0), \tag{4.11}$$

in the case $Q_c = 0$ and by:

$$h(\zeta) = Q_c e^{-\phi(\zeta)} \int_\zeta^\infty \frac{d\bar{\zeta}}{\bar{\zeta}^5 W(\bar{\zeta})}, \tag{4.12}$$

in the case in which $Q_c \neq 0$.

In what follows we will study several solutions of the master equation (4.5).

4.2.1 Unflavored solution

Let us consider the master equation in the case in which there are no flavor brane sources. It turns out that the general solution of (4.5) can be analytically found in this case. Indeed, when $Q_f = 0$ the master equation (4.5) can be trivially integrated once as:

$$\zeta\,\frac{dW}{d\zeta} + 6\,W = \text{constant} \ . \tag{4.13}$$

A further integration gives:

$$W = C\left(1 - \frac{b^6}{\zeta^6}\right) \ , \tag{4.14}$$

where C and b are constants. Plugging this expression of W into the right-hand side of (4.7) one readily verifies that the dilaton is constant and given by:

$$e^{-\phi} = C \ . \tag{4.15}$$

Moreover, the function f and the squashing function q are given by:

$$e^{2f} = \zeta^2 \left(1 - \frac{b^6}{\zeta^6}\right) , \qquad\qquad q = \left(1 - \frac{b^6}{\zeta^6}\right)^{\frac{1}{2}} , \tag{4.16}$$

and the warp factor h can be obtained by integrating (4.10). This solution coincides with the general unflavored one found in the second chapter. For $b = 0$ this geometry is just $AdS_5 \times \mathbb{S}^5$. When $b \neq 0$ the solution approaches $AdS_5 \times \mathbb{S}^5$ in the UV. If we take b to be real and positive, then the minimal value of ζ is $\zeta = b$ and the metric has a blown-up $\mathbb{CP}^2$ cycle at $\zeta = b$. It was argued in [137] that this $b \neq 0$ background is dual to the superconformal $\mathcal{N} = 4$ field theory deformed by the VEV of a dimension 6 operator.

4.2.2 Massless flavored solution

Let us now consider the massless flavor case with $Q_f \neq 0$ and $p = 1$. In this case it is possible to find a special solution of (4.5):

$$W = A\,\zeta^{\alpha} . \tag{4.17}$$

Indeed, by plugging this ansatz into the master equation we readily get that the exponent α is given by:

$$\alpha = -\frac{2}{3} . \tag{4.18}$$

Similarly, we can obtain the value of the constant A. The final formula for W is:

$$W = \frac{9}{8} \left(\sqrt{2}\,Q_f\right)^{\frac{2}{3}} \zeta^{-\frac{2}{3}} . \tag{4.19}$$

Using this expression in (4.6) and (4.7) we arrive at the following values of f and ϕ:

$$e^{2f} = \frac{9}{8}\zeta^2 , \qquad\qquad e^{-\phi} = \left(\sqrt{2}\,Q_f\right)^{\frac{2}{3}} \zeta^{-\frac{2}{3}} . \tag{4.20}$$

It is straightforward to verify that this solution coincides with the one found in the second chapter for massless flavors.[1] The warp factor for this solution is:

$$h = \frac{\bar{R}^4}{\zeta^4} , \tag{4.21}$$

where $\bar{R}$ is the same as in the second chapter,

$$\bar{R}^4 = \frac{4}{15}\,Q_c . \tag{4.22}$$

[1] The relation between ζ and the radial variable r used in the second chapter and in the ansatz (4.142) is $\zeta = \frac{3}{2\sqrt{2}}\,r$.

Since the profile function p is constant, this solution represents massless smeared flavors extending all the way down to $\zeta = 0$. As shown in the second chapter, the background corresponding to the master function (4.19) is invariant under a set of anisotropic scale transformations in which the x^3 coordinate transforms with an anomalous scaling dimension. Moreover, the squashing factor (4.8) q is constant and equal to $\frac{3}{2\sqrt{2}} \approx 1.06$, see (4.20). The purpose of this chapter is to find solutions corresponding to massive flavors, which should interpolate between the unflavored solution of subsection 4.2.1 at the IR and the scaling solution studied in this subsection at the UV. We start to discuss these solutions in the next subsection.

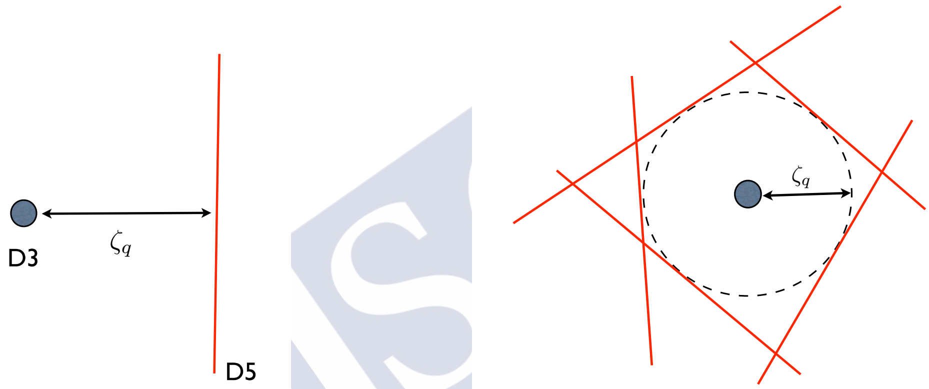

Figure 4.2: On the left we depict a localized embedding of a D5-brane with a separation ζ_q from the stack of color D3-branes. On the right, several D5-branes with different orientations and the same distance ζ_q generate a cavity $\zeta \leq \zeta_q$ which does not contain flavor sources.

4.2.3 Massive flavors

In the holographic approach the mass of the fundamentals is related to the distance between the color and flavor branes (in our case D3's and D5's, respectively). When these two sets of branes are separated, the fundamentals are massive and there is an IR region of the geometry which is not occupied by the flavor brane and, as a consequence, the D5-brane charge density vanishes there. In the smearing setup we have many D5-branes with different orientations in the internal space which, if they correspond to flavors with the same mass, should have the same separation from the D3-branes. As illustrated in figure 4.2, the sourceless region has a ζ coordinate less or equal to some value ζ_q, a region which we will call cavity in the following. The profile function $p(\zeta)$ vanishes inside the cavity and should approach the value appropriate for massless flavors, *i.e.*, $p \to 1$ as

$\zeta \to \infty$. To determine the explicit form of the function $p(\zeta)$ one has to specify the set of source D5-branes of our smeared distribution. This is done in detail in appendix 4.B. The final result for the profile function is:

$$p(\zeta) \,=\, \Big[1 - \Big(\frac{\zeta_q}{\zeta}\Big)^2\Big]^{\frac{1}{2}} \Big[1 \,+\, \frac{1}{2}\Big(\frac{\zeta_q}{\zeta}\Big)^2\Big] \; \Theta(\zeta - \zeta_q) \;. \tag{4.23}$$

Notice that $p(\zeta)$ is continuous at $\zeta = \zeta_q$ and asymptotically $p(\zeta \to \infty) = 1$. The master function W for the profile (4.23) must be obtained numerically. However, we can expand (4.5) in a series expansion about any radial coordinate ζ. Let us start in the neighborhood of $\zeta = \infty$. Indeed, the profile function (4.23) can be expanded in powers of $\frac{\zeta_q}{\zeta}$ as:

$$p(\zeta) = 1 \,-\, \frac{3}{8}\,\frac{\zeta_q^4}{\zeta^4} \,-\, \frac{1}{8}\,\frac{\zeta_q^6}{\zeta^6} \,-\, \frac{9}{128}\,\frac{\zeta_q^8}{\zeta^8} \,+\, \mathcal{O}\Big(\frac{\zeta_q^{10}}{\zeta^{10}}\Big) \;. \tag{4.24}$$

Plugging this expansion into the master equation (4.5) and integrating order by order, we get the following solution for the master function $W(\zeta)$:

$$W = \frac{9}{8}\Big[\frac{\sqrt{2}\,Q_f}{\zeta}\Big]^{\frac{2}{3}} \bigg(1 \,-\, \frac{1}{6}\,\frac{\zeta_q^4}{\zeta^4} \,+\, \frac{1}{6}\,\frac{\zeta_q^6}{\zeta^6} \,+\, \frac{35}{2304}\,\frac{\zeta_q^8}{\zeta^8} \,+\, \mathcal{O}\Big(\frac{\zeta_q^{10}}{\zeta^{10}}\Big)\bigg) \;. \tag{4.25}$$

The asymptotic UV expansions of the different functions of the background are easily obtained from (4.25). We have collected these expansions in appendix 4.A.

Another useful expansion is the neighborhood where the sources set in, *i.e.*, at the edge of the cavity. We have hence perturbatively solved the master equation for $\zeta \geq \zeta_q$ and $\zeta - \zeta_q$ small; details are relegated in appendix 4.A.

Recall that inside the cavity, *i.e.*, for $\zeta \leq \zeta_q$, we have the analytic solution for the master function. This solution was written in (4.14) and depends on two parameters C and b. We want to extend this solution for $\zeta \geq \zeta_q$ and determine the values of C and b for which the function W behaves as in (4.25) for $\zeta \to \infty$. This matching has to be done numerically, for which we implemented the shooting technique accompanied with a convenient change of variables. We will discuss this in the next section.

4.3 Integration of the master equation

Let us introduce a new holographic variable x, related to ζ as follows:

$$x \,\equiv\, \frac{\zeta - b}{\zeta_q} \;. \tag{4.26}$$

Clearly, for our interpolating solutions $x \geq 0$. In this new variable the cavity (*i.e.*, the region without flavor branes) corresponds to $0 \leq x \leq x_q$, where x_q is the edge of the cavity given by:

$$x_q \,=\, 1 \,-\, \frac{b}{\zeta_q} \;. \tag{4.27}$$

Notice that x_q depends on the ratio between the deformation parameter b in the sourceless region and the location of the boundary of this unflavored region in the ζ variable. Moreover, as $b \leq \zeta_q$, we have that:

$$x_q \leq 1 \, . \tag{4.28}$$

Let us write the background in terms of the new variables. It is convenient to absorb the constant C appearing in the master function (4.14) inside the cavity. Accordingly, we define $\hat{W}$ as:

$$\hat{W} \,=\, \frac{W}{C} \, . \tag{4.29}$$

In the region without flavor sources this function is given by:

$$\hat{W}(x) \,=\, 1 \,-\, \frac{(1-x_q)^6}{(x+1-x_q)^6} \,, \qquad\qquad 0 \,\leq\, x \,\leq\, x_q \, . \tag{4.30}$$

This function vanishes at the origin $x = 0$ and $\hat{W}$ and its derivative take the following

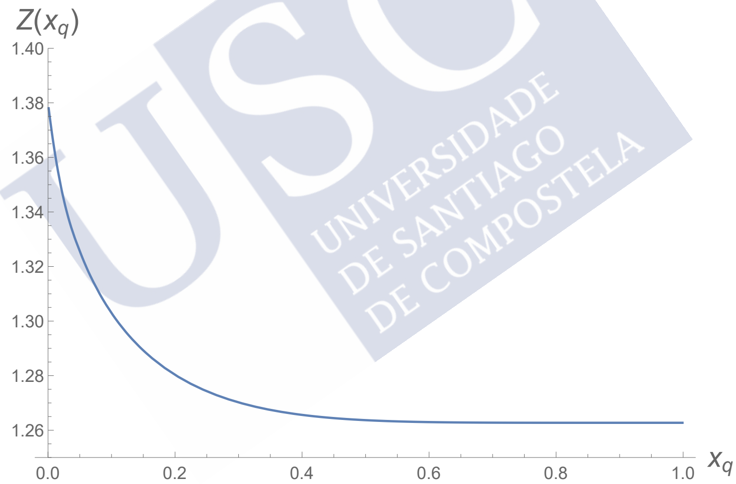

Figure 4.3: Plot of the function $Z(x_q) = C^{\frac{3}{2}}\,\zeta_q/Q_f$, obtained by the shooting method of the master equation (4.33). This function has as asymptotics $Z(x_q = 0) \approx 1.38$ and $Z(x_q = 1) \approx 1.26$. Notice that $Z(x_q)$ is almost constant in the interval $0.5 \leq x_q \leq 1$.

values at the edge of the cavity:

$$\hat{W}(x = x_q) \,=\, 1 \,-\, (1-x_q)^6 \,, \qquad\qquad \frac{d\hat{W}}{dx}\Big|_{x=x_q} \,=\, 6(1-x_q)^6 \, . \tag{4.31}$$

Notice that, for $x_q = 1$, *i.e.*, in the case in which the cavity is largest, $\hat{W}(x) = 1$ inside the cavity and the solution becomes (isotropic) AdS in this region. This implies that x_q measures the amount of anisotropy of our solution.

The profile function in terms of the x variable is:

$$p(x) \;=\; \theta(x-x_q)\,\frac{\sqrt{x-x_q}\,\sqrt{x-x_q+2}}{2\,(x+1-x_q)^3}\,\Big[1\;+\;2(x+1-x_q)^2\Big]\;. \tag{4.32}$$

Moreover, the master equation for $\hat{W}$ becomes:

$$\frac{d}{dx}\Big[(x+1-x_q)\,\frac{d\hat{W}}{dx}\Big]\;+\;6\,\frac{d\hat{W}}{dx}\;=\;-\frac{6\,Q_f}{C^{\frac{3}{2}}\,\zeta_q}\,\frac{p(x)}{\sqrt{\hat{W}}}\;. \tag{4.33}$$

This equation is solved analytically by (4.30) inside the cavity, where $0\le x\le x_q$. For $x\ge x_q$ we can solve (4.33) numerically by imposing the initial conditions (4.31) and by requiring that as $x\to\infty$ it asymptotically behaves as the massless flavored solution,

$$\hat{W}\;\approx\;\frac{9}{8}\Big[\frac{\sqrt{2}\,Q_f}{C^{\frac{3}{2}}\,\zeta_q}\Big]^{\frac{2}{3}}\,x^{-\frac{2}{3}}\;,\qquad\qquad (x\to\infty)\;. \tag{4.34}$$

Notice that (4.33) depends on two parameters x_q and $C^{\frac{3}{2}}\,\zeta_q/Q_f$. In order to have a solution with the asymptotic behaviour (4.34) these two parameters must satisfy a relation, which can be determined numerically. This relation is

$$\zeta_q\;=\;\frac{Q_f}{C^{\frac{3}{2}}}\,Z(x_q)\;, \tag{4.35}$$

where the function $Z(x_q)$ is determinend numerically for $0\le x_q\le 1$. Our results for this function have been plotted in figure 4.3. We observe that $Z(x_q)$ is a decreasing function of x_q and, for given values of Q_f and the constant C, ζ_q reaches its maximum at $x_q=0$. Notice also that x_q gives the size of the cavity in the x variable. This size should be related to the quark mass m_q. In our holographic setup m_q can be determined by evaluating the Nambu-Goto action of a fundamental string extended along the holographic direction, from the origin of the space to the tip of the flavor brane. In the ζ variable we have:

$$m_q\;=\;\frac{1}{2\pi\,\alpha'}\int_b^{\zeta_q}d\zeta\,e^{\frac{\phi}{2}}\,\sqrt{-\det g_2}\;, \tag{4.36}$$

where the $e^{\frac{\phi}{2}}$ factor is due to the fact that we are working in the Einstein frame. By using the metric ansatz (4.1), as well as (4.6) and (4.7), we obtain m_q in terms of an integral involving the master function W:

$$m_q\;=\;\frac{1}{2\pi\,\alpha'}\int_b^{\zeta_q}\frac{d\zeta}{\sqrt{W}}\;=\;\frac{1}{2\pi\,\alpha'\,\sqrt{C}}\int_b^{\zeta_q}\frac{d\zeta}{\sqrt{1-\frac{b^6}{\zeta^6}}}\;. \tag{4.37}$$

Putting $\alpha'=1$ from now on and writing the result in terms of x_q, we get:

$$m_q\;=\;\frac{Q_f}{2\pi\,C^2}\,Z(x_q)\left[(x_q-1)\,\frac{\sqrt{\pi}\,\Gamma\big(\frac{5}{6}\big)}{\Gamma\big(\frac{1}{3}\big)}\;+\;F\big(-\frac{1}{6}\,,\,\frac{1}{2}\,;\,\frac{5}{6}\,;\,(1-x_q)^6\big)\right]\;. \tag{4.38}$$

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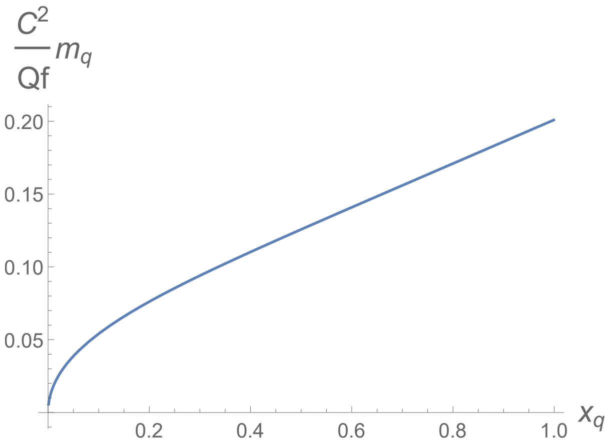

Figure 4.4: Quark mass m_q, rescaled by C^2/Q_f, as a function of x_q.

In figure 4.4 we plot $C^2\, m_q/Q_f$ as a function of x_q. We observe in this plot that m_q grows with x_q and that $m_q = 0$ for $x_q = 0$, *i.e.*, when the cavity has zero size. Moreover, we can increase the quark mass by decreasing C ($C \to 0$ corresponds to $m_q \to \infty$ for $x_q \neq 0$). The precise relation between m_q and the constant C depends on the value of x_q. For $x_q \sim 0, 1$ one can expand the right-hand side of (4.38) and find:

$$m_q \approx \frac{Q_f}{2\pi C^2}\sqrt{\frac{2}{3}}\, Z(0)\,\sqrt{x_q}\;, \qquad (x_q \to 0)\;, \qquad m_q \approx \frac{Q_f}{2\pi C^2}\, Z(1)\;, \qquad (x_q \to 1)\;. \tag{4.39}$$

All the functions of the background can be written in terms of Z and $\hat{W}$. For example, the squashing factor q is:

$$q = \frac{\sqrt{\hat{W}}}{\sqrt{\hat{W} + \frac{1+x-x_q}{6}\frac{d\hat{W}}{dx}}}\;, \tag{4.40}$$

whereas f, g, and the dilaton ϕ are:

$$\begin{aligned}
e^{2g} &= \frac{Q_f^2}{C^3}\, Z^2(x_q)\,(1+x-x_q)^2 \\
e^{2f} &= \frac{Q_f^2}{C^3}\, Z^2(x_q)\,\frac{(1+x-x_q)^2\,\hat{W}}{\hat{W} + \frac{1+x-x_q}{6}\frac{d\hat{W}}{dx}} = \frac{Q_f^2}{C^3}\, Z^2(x_q)\,(1+x-x_q)^2\, q^2(x) \\
e^{-\phi} &= C\left[\hat{W} + \frac{1+x-x_q}{6}\frac{d\hat{W}}{dx}\right] = C\,\frac{\hat{W}}{q^2(x)}\;,
\end{aligned} \tag{4.41}$$

where, in the second step, we have written these functions in terms of $q(x)$. In figure 4.5 we plot the function $\hat{W}$ for different values of x_q. We notice that, as $x_q \to 1$, $\hat{W}$ is almost constant and equal to 1 in the interior of the cavity, where it is given by (4.30). Moreover, outside the cavity, *i.e.*, for $x \geq x_q$ it rapidly approaches the asymptotic expression (4.25),

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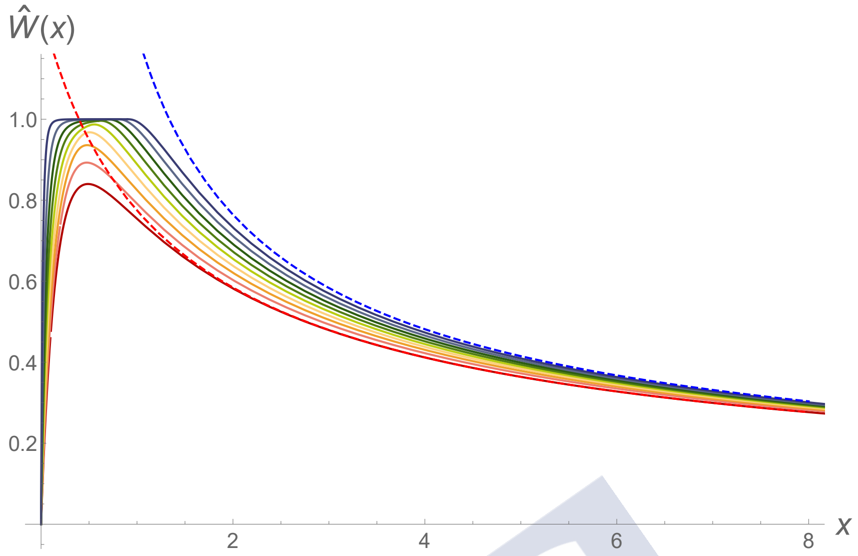

Figure 4.5: Plot of $\hat{W}$ versus the holographic variable x for different values of x_q in the range $0.1 \leq x_q \leq 0.9$. The continuous bottom dark red (top dark blue) curve corresponds to $x_q = 0.1$ ($x_q = 0.9$). The continuous curves between them correspond to $x_q = i/10$ for $i = 2, \ldots, 8$ (bottom-up). The dashed curves are the leading terms in the UV expansion (4.42) for $x_q = 0.1$ (red) and $x_q = 0.9$ (blue).

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which can be rewritten in terms of x and x_q as:

$$\hat{W} = \frac{9}{8}\left[\frac{\sqrt{2}}{Z(x_q)\,(x+1-x_q)}\right]^{\frac{2}{3}}\left(1 - \frac{1}{6}\frac{1}{(x+1-x_q)^4} + \frac{1}{6}\frac{1}{(x+1-x_q)^6} + \frac{35}{2304}\frac{1}{(x+1-x_q)^8} + \ldots\right) . \tag{4.42}$$

The UV expansion of f, ϕ, h, and q in the x variable can be obtained from (4.164)-(4.167) by substituting $\zeta = \zeta_q(x+1-x_q)$.

The comparison of the numerical values for $x \geq x_q$ and those given by (4.42) shows that the agreement with the UV expansion is better if x_q is small, since in this case the background is closer to the massless scaling solutions (for $x_q = 0$ there is no cavity). On the other hand, for x_q close to one the flavor effects are smaller in the IR, since $\hat{W}$ is almost constant and equal to one inside the cavity. Indeed, from (4.27) we get that $x_q \sim 1$ implies that the IR deformation parameter b is small. This is consistent with the fact that, for a given value of C, the quark mass is maximal in this case (see figure 4.4).

Let us now write the metric in the x variable. First of all, we define the rescaled warp factor and cartesian coordinates as:

$$\hat{h} = \frac{Q_f^4}{C^6}\,h\,, \qquad \hat{x}^{\mu} = \frac{Q_f}{C^{\frac{3}{2}}}\,x^{\mu}\,, \qquad (\mu = 0,1,2)\,, \qquad \hat{x}^3 = \frac{Q_f}{C^{\frac{1}{2}}}\,x^3\,. \tag{4.43}$$

Then, in terms of these hatted variables, we have:

$$ds_{10}^2 = \hat{h}^{-\frac{1}{2}} \Big[-(d\hat{x}^0)^2 + (d\hat{x}^1)^2 + (d\hat{x}^2)^2 + \frac{\hat{W}^2}{q^4}(d\hat{x}^3)^2 \Big]$$
$$+ Z^2(x_q)\,\hat{h}^{\frac{1}{2}} \Big[\frac{(dx)^2}{q^2} + (1 + x - x_q)^2 \Big(ds^2_{\mathbb{CP}^2} + q^2\,(d\tau + A)^2 \Big) \Big] \ . \quad (4.44)$$

We have only one free parameter, x_q, corresponding to a family of geometries. Actually, in our unquenched flavored background one would expect the geometry to depend on two quantities, the amount of flavors and their mass. We have rescaled out the quark mass with the definition of the hatted quantities in (4.43) and therefore x_q will serve us to parameterize the amount of flavors.

It is interesting to write down the relation between the hatted and unhatted cartesian coordinates in terms of the quark mass m_q. Taking into account that $C \sim m_q^{-\frac{1}{2}}$, we get for the longitudinal $(x^1\, x^2)$ and transverse (x^3) directions:

$$\hat{x}_{\parallel} \sim m_q^{\frac{3}{4}}\, x_{\parallel} \ , \qquad\qquad \hat{x}_{\perp} \sim m_q^{\frac{1}{4}}\, x_{\perp} \ . \quad (4.45)$$

Therefore, for fixed distances $x_{\parallel}$ and $x_{\perp}$, taking $m_q \to 0$ is equivalent to considering the UV $\hat{x}_{\parallel}, \hat{x}_{\perp} \to 0$ region in the hatted variables, whereas taking large m_q amounts to zooming in the IR region of large $x_{\parallel}$ and $x_{\perp}$. For fixed quark mass, x_q is the parameter

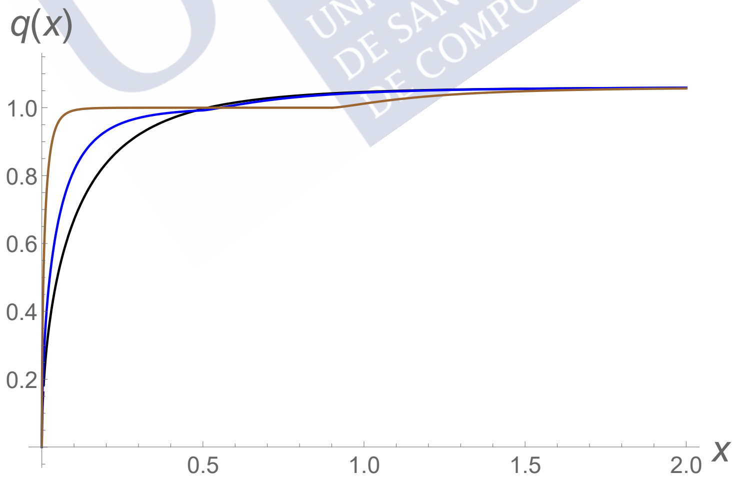

Figure 4.6: The squashing function $q(x)$ for $x_q = 0.005$ (black), $x_q = 0.5$ (blue) and $x_q = 0.9$ (brown). As $x \to 0$ the squashing factor goes to zero for all the values of x_q, except for the case $x_q = 1$. In that particular case $q(x)$ remains finite and equal to one inside the cavity.

that controls the flavor effects in the IR. To illustrate how the flavor branes deform the metric as we move in the holographic direction, we have plotted in figure 4.6 the squashing function $q(x)$ for different values of x_q. For $x_q \sim 1$ the function $q(x)$ is nearly constant and equal to one in the sourceless region $x \leq x_q$ and grows monotonically for $x \geq x_q$, until it reaches its asymptotic UV value $q = 3/2\sqrt{2}$. It follows from these results and those represented in figure 4.5 that the geometry becomes more and more isotropic inside the cavity as we increase the parameter x_q.

In the following sections we will study our gravity dual by computing several observables with the purpose of exploring their change as we move from the UV to the IR as we vary the size of the cavity x_q.

4.4 Wilson loops

In holography, the potential energy between a "quark" and an "antiquark" is obtained from the solution of the equation of motion of a fundamental string hanging from the UV boundary [154, 155]. These equations are derived from the Nambu-Goto action:

$$S = \frac{1}{2\pi} \int d\tau d\sigma \, e^{\frac{\phi}{2}} \sqrt{-\det g_2} \; , \tag{4.46}$$

where g_2 is the induced metric on the worldsheet of the string. We will calculate the $q\bar{q}$ potential in two cases. First we will consider the intralayer case, in which the quark and the antiquark are in the same layer and have the same value of the coordinate x^3. After this we will consider the interlayer configuration, where the quarks are separated in the anisotropic direction.

4.4.1 Intralayer potential

We take $(t, \hat{x}^1)$ as worldvolume coordinates of a fundamental string and we will consider an ansatz with $x = x(\hat{x}^1)$ with the other cartesian coordinates being constant. The induced metric on the 2-dimensional worldsheet is:

$$ds_2^2 = -\hat{h}^{-\frac{1}{2}} (d\hat{x}^0)^2 + \hat{h}^{-\frac{1}{2}} \Big(1 + \frac{\hat{h}\, Z_q^2}{q^2} (x')^2\Big) (d\hat{x}^1)^2 \; , \tag{4.47}$$

where the prime denotes derivative with respect to $\hat{x}^1$ and we have denoted $Z_q \equiv Z(x_q)$. The Nambu-Goto action of the string is:

$$\frac{S}{\hat{T}} = \frac{1}{2\pi} \int d\hat{x}^1 \, e^{\frac{\phi}{2}} \sqrt{-g_2} \equiv \int d\hat{x}^1 \, L \; , \tag{4.48}$$

where $\hat{T} = \int d\hat{x}^0$ and the Lagrangian density L is:

$$L = \frac{1}{2\pi\sqrt{C}} \frac{\hat{h}^{-\frac{1}{2}}\, q}{\sqrt{\hat{W}}} \sqrt{1 + Z_q^2 \, \frac{\hat{h}}{q^2} (x')^2} \; . \tag{4.49}$$

As L does not depend explicitly on the holographic coordinate x, one has the following first integral:

$$x' \, \frac{\partial L}{\partial x'} \,-\, L \,=\, \text{constant} \; , \tag{4.50}$$

from which we get:

$$\frac{q}{\sqrt{\hat{h}\,\hat{W}}} \, \frac{1}{\sqrt{1 \,+\, Z_q^2 \, \frac{\hat{h}}{q^2} \, (x')^2}} \,=\, \frac{q_0}{\sqrt{\hat{h}_0 \, \hat{W}_0}} \; , \tag{4.51}$$

where q_0, $\hat{h}_0$, and $\hat{W}_0$ are the values of the functions q, $\hat{h}$, and $\hat{W}$, respectively, at the turning point $x = x_0$. From this relation we obtain x' as:

$$x' \,=\, \pm \, \frac{q}{Z_q \, \sqrt{\hat{h}}} \, \sqrt{\frac{\hat{h}_0 \, \hat{W}_0 \, q^2}{\hat{h} \, \hat{W} \, q_0^2} \,-\, 1} \; , \tag{4.52}$$

which can be straightforwardly integrated to give:

$$\hat{x}^1(x) \,=\, \pm Z_q \int_{x_0}^{x} \frac{\sqrt{\hat{h}(\bar{x})}}{q(\bar{x})} \, \frac{d\bar{x}}{\sqrt{\frac{\hat{h}_0 \, \hat{W}_0 \, q^2(\bar{x})}{\hat{h}(\bar{x}) \, \hat{W}(\bar{x}) \, q_0^2} \,-\, 1}} \; . \tag{4.53}$$

The (hatted) quark-antiquark distance at the boundary is:

$$\hat{d}_{\parallel} \,=\, 2 \, Z_q \int_{x_0}^{\infty} \frac{\sqrt{\hat{h}(x)}}{q(x)} \, \frac{dx}{\sqrt{\frac{\hat{h}_0 \, \hat{W}_0 \, q^2(x)}{\hat{h}(x) \, \hat{W}(x) \, q_0^2} \,-\, 1}} \; . \tag{4.54}$$

Let us now calculate the on-shell action of the fundamental string. Plugging the solution into the action, we get:

$$\frac{S_{on-shell}}{\hat{T}} \,=\, \frac{Z_q}{\pi \sqrt{C}} \int_{x_0}^{x_{max}} \frac{dx}{\sqrt{\hat{W}} \, \sqrt{1 \,-\, \frac{q_0^2 \, \hat{h} \, \hat{W}}{q^2 \, \hat{h}_0 \, \hat{W}_0}}} \; . \tag{4.55}$$

As usual, this on-shell action is divergent and must be regulated. We do it by subtracting the action of two straight fundamental strings stretched from the origin $x = 0$ to $x = x_{max}$:

$$\frac{S^{reg}_{on-shell}}{\hat{T}} \,=\, \frac{S_{on-shell}}{\hat{T}} \,-\, 2 \, \frac{Z_q}{2\pi} \int_0^{x_{max}} dx \, \frac{e^{\frac{\phi}{2}}}{q} \,=\, \frac{S_{on-shell}}{\hat{T}} \,-\, \frac{Z_q}{\pi \sqrt{C}} \int_0^{x_{max}} \frac{dx}{\sqrt{\hat{W}}} \; . \tag{4.56}$$

The quark-antiquark potential is then given by:

$$V_{q\bar{q}} \,=\, \frac{S^{reg}_{on-shell}}{T} \,=\, \frac{Q_f}{C^{\frac{3}{2}}} \, \frac{S^{reg}_{on-shell}}{\hat{T}} \; , \tag{4.57}$$

where we have used the relation between T and $\hat{T}$:

$$\hat{T} = \frac{Q_f}{C^{\frac{3}{2}}} T \,. \tag{4.58}$$

More explicitly:

$$V_{q\bar{q}} = \frac{Z_q\, Q_f}{\pi\, C^2} \left[\int_{x_0}^{\infty} \frac{dx}{\sqrt{\hat{W}}} \left(\frac{1}{\sqrt{1 - \frac{q_0^2\, \hat{h}\, \hat{W}}{q^2\, \hat{h}_0\, \hat{W}_0}}} - 1 \right) - \int_0^{x_0} \frac{dx}{\sqrt{\hat{W}}} \right] . \tag{4.59}$$

We have numerically evaluated the potential $V_{q\bar{q}}$ as a function of $\hat{d}_{\|}$. The results are pre-

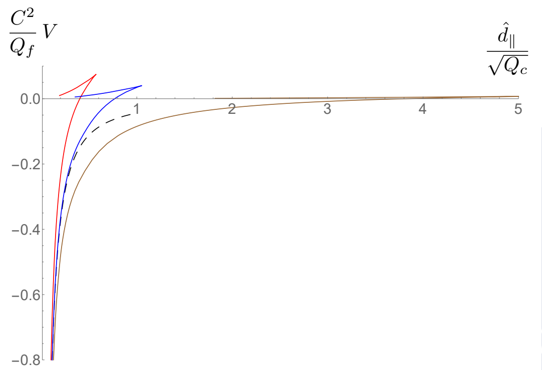

Figure 4.7: We depict the $\bar{q}q$ intralayer potential (4.59) versus $\hat{d}_{\|} \propto m_q^{\frac{3}{4}}\, d_{\|}$ for $x_q = 0.1$ (red), $x_q = 0.5$ (blue), and $x_q = 0.9$ (brown). The dashed curve is the UV limit (4.63). Notice that the maximal separation for $\bar{q}q$ increases with x_q.

sented in figure 4.7 for different values of x_q. We notice that all curves become coincident for small $\hat{d}_{\|}$. As $\hat{d}_{\|} \sim m_q^{\frac{3}{4}}\, d_{\|}$ (see equation (4.45)), one expects to recover the massless scaling solution in the UV domain $\hat{d}_{\|} \to 0$. Indeed, we prove below that $V_{q\bar{q}} \sim \hat{d}_{\|}^{-\frac{4}{3}}$ in the UV region of small $\hat{d}_{\|}$. This behaviour matches the one obtained numerically, as shown in figure 4.7. As we move towards the IR by decreasing the turning point coordinate x_0 and increasing $\hat{d}_{\|}$, we obtain that there is a maximal value of $\hat{d}_{\|}$ (corresponding to a minimal value $x_0^{\min} < x_q$ of the turning point coordinate). For $x_0 < x_0^{\min}$ the dominant configuration is a disconnected one, in which the two ends of the string go straight from the boundary to the origin. This behaviour has been obtained previously in backgrounds dual to unquenched flavors [139, 173–175] (in other types of backgrounds, see also [176–180]). Indeed, dynamical quarks produce string breaking and a maximal length due to pair creation. In our case this breaking is not produced in the scaling solution with $m_q = 0$. Moreover, the critical distance at which the string breaks grows with x_q, as is also evident in figure 4.7, and becomes very large when $x_q \sim 1$. This is easy to understand since the breaking occurs when the string penetrates deeply in the cavity, whose size is maximal when $x_q \sim 1$, and the integrals (4.54) and (4.59) get their main contribution from the sourceless region inside the cavity. For large enough values of $\hat{d}_{\parallel}$ the dominant configuration is the disconnected one with zero energy, which means that the external quarks are completely screened by dynamical quarks popping out from the vacuum.

A recent interesting work [153], in a seemingly unrelated context of holographic QCD, parallels our findings. The authors of [153] demonstrated that large amounts of anisotropy will completely screen the interactions between quarks and anti-quarks, while in the absence of anisotropy the model would otherwise be confining.

UV limit

Let us now evaluate the potential in the UV limit in which $d_{\parallel}$ is small (large x_0) and our embedding is close to the boundary. In this limit we can use that, at leading order in the UV, the squashing function q is constant and that $\hat{W} \sim x^{-\frac{1}{3}}$ and $\hat{h} \sim x^{-4}$ (see equations (4.42), (4.166), and (4.167)). We will use these values to calculate the integrals for $\hat{d}_{\parallel}$ and $V_{q\bar{q}}$ in (4.54) and (4.59). At leading order we get the following relation between $\hat{d}_{\parallel}$ and x_0:

$$\hat{d}_{\parallel} \approx \frac{8\sqrt{2}}{3\sqrt{15}} \frac{\sqrt{Q_c}}{Z_q} \sqrt{\pi} \frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)} \frac{1}{x_0} \ . \tag{4.60}$$

Similarly, we approximate the potential $V_{q\bar{q}}$ by the following integral:

$$V_{q\bar{q}} \approx \frac{2\sqrt{2}}{3 \cdot 2^{\frac{1}{6}}\pi} \frac{Z_q^{\frac{4}{3}} Q_f}{C^2} x_0^{\frac{4}{3}} \left[\int_1^{\infty} dz\, z^{\frac{1}{3}} \left(\frac{z^{\frac{7}{3}}}{\sqrt{z^{\frac{14}{3}} - 1}} - 1 \right) - \frac{3}{4} \right] . \tag{4.61}$$

Performing the integral, we get:

$$V_{q\bar{q}} \approx -\frac{1}{2^{\frac{1}{6}}\sqrt{2\pi}} \frac{Z_q^{\frac{4}{3}} Q_f}{C^2} \frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)} x_0^{\frac{4}{3}} \ . \tag{4.62}$$

By using the relation (4.60) we can eliminate x_0 in favor of the $q\bar{q}$ distance $\hat{d}_{\parallel}$. After some calculation we get:

$$\frac{C^2}{Q_f} V_{q\bar{q}} \approx -\beta_{\parallel} \left(\frac{\sqrt{Q_c}}{\hat{d}_{\parallel}} \right)^{\frac{4}{3}} , \qquad \beta_{\parallel} = \frac{16\pi^{\frac{1}{6}}}{9 \cdot 5^{\frac{2}{3}}} \left(\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)} \right)^{\frac{7}{3}} , \tag{4.63}$$

which coincides with the result found in the previous chapter for the massless scaling background. In figure 4.7 we show that the potential (4.63) does indeed coincide with the numerical results in the UV domain $\hat{d}_{\parallel} \to 0$ for all values of the parameter x_q.

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4.4.2 Interlayer potential

We now consider a Wilson loop that extends in the x^3 direction with x^1 and x^2 constant, which corresponds to two fundamentals located at different layers. Accordingly, we take $(t, \hat{x}^3)$ as worldvolume coordinates and consider an ansatz in which $x = x(\hat{x}^3)$. The 2-dimensional induced metric is now:

$$ds_2^2 = -\hat{h}^{-\frac{1}{2}}\,(d\hat{x}^0)^2 \,+\, \frac{\hat{h}^{-\frac{1}{2}}}{q^4}\,\big[\hat{W}^2 \,+\, Z_q^2\,\hat{h}\,q^2\,(x')^2\,\big]\,(d\hat{x}^3)^2\ , \tag{4.64}$$

where the prime now denotes derivative with respect to $\hat{x}^3$. The Nambu-Goto action is:

$$\frac{S}{\hat{T}} \,=\, \frac{1}{2\pi}\int d\hat{x}^3\, e^{\frac{\phi}{2}}\,\sqrt{-g_2} \,\equiv\, \int d\hat{x}^3\, L\ , \tag{4.65}$$

with

$$L \,=\, \frac{1}{2\pi\,\sqrt{C}}\,\frac{\hat{h}^{-\frac{1}{2}}}{q\,\sqrt{\hat{W}}}\,\sqrt{\hat{W}^2 \,+\, Z_q^2\,\hat{h}\,q^2\,(x')^2}\ . \tag{4.66}$$

The first integral derived from L is:

$$\frac{\sqrt{\hat{W}}}{q\,\sqrt{\hat{h}}}\,\frac{1}{\sqrt{1\,+\,\frac{Z_q^2\,q^2\,\hat{h}}{\hat{W}^2}\,(x')^2}} = \frac{\sqrt{\hat{W}_0}}{q_0\,\sqrt{\hat{h}_0}}\ , \tag{4.67}$$

where again q_0, $\hat{h}_0$, and $\hat{W}_0$ are the values of the functions q, $\hat{h}$, and $\hat{W}$, respectively, at the turning point $x = x_0$. Then:

$$x' \,=\, \pm\,\frac{\hat{W}}{Z_q\,q\sqrt{\hat{h}}}\,\sqrt{\frac{\hat{h}_0\,q_0^2\,\hat{W}}{\hat{h}\,q^2\,\hat{W}_0} \,-\, 1}\ . \tag{4.68}$$

Integrating this equation we get:

$$\hat{x}^3(x) \,=\, \pm Z_q\int_{x_0}^{x}\frac{q(\bar{x})\,\sqrt{\hat{h}(\bar{x})}}{\hat{W}(\bar{x})}\,\frac{d\bar{x}}{\sqrt{\frac{\hat{h}_0\,q_0^2\,\hat{W}(\bar{x})}{\hat{h}(\bar{x})\,q^2(\bar{x})\,\hat{W}_0} \,-\, 1}}\ . \tag{4.69}$$

Therefore, the quark-antiquark distance along the direction transverse to the layers is:

$$\hat{d}_\perp \,=\, 2\,Z_q\int_{x_0}^{\infty}\frac{q(x)\,\sqrt{\hat{h}(x)}}{\hat{W}(x)}\,\frac{dx}{\sqrt{\frac{\hat{h}_0\,q_0^2\,\hat{W}(x)}{\hat{h}(x)\,q^2(x)\,\hat{W}_0} \,-\, 1}}\ . \tag{4.70}$$

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The unregulated on-shell action for this case is:

$$\frac{S_{on-shell}}{\hat{T}} = \frac{Z_q}{\pi\sqrt{C}}\int_{x_0}^{x_{max}} \frac{dx}{\sqrt{\hat{W}}\sqrt{1-\frac{q^2\,\hat{h}\,\hat{W}_0}{q_0^2\,\hat{h}_0\,\hat{W}}}}\,. \tag{4.71}$$

Proceeding as in the intralayer case to regulate this action, we arrive at the following quark-antiquark potential:

$$V_{q\bar{q}} = \frac{Z_q\,Q_f}{\pi\,C^2}\left[\int_{x_0}^{\infty}\frac{dx}{\sqrt{\hat{W}}}\left(\frac{1}{\sqrt{1-\frac{q^2\,\hat{h}\,\hat{W}_0}{q_0^2\,\hat{h}_0\,\hat{W}}}}-1\right)-\int_0^{x_0}\frac{dx}{\sqrt{\hat{W}}}\right]\,. \tag{4.72}$$

The numerical results for the interlayer potential have been plotted in figure 4.8. They are

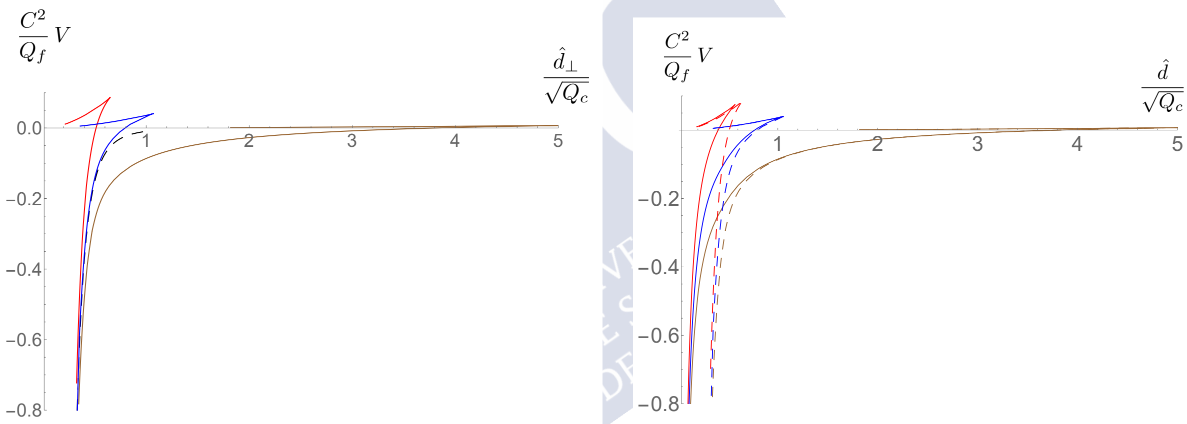

Figure 4.8: On the left we plot the $\bar{q}q$ interlayer potential (4.72) versus $\hat{d}_\perp \propto m_q^{\frac{1}{4}}\,d_\perp$ for $x_q = 0.1$ (red), $x_q = 0.5$ (blue), and $x_q = 0.9$ (brown). The dashed curve is the UV potential (4.76). On the right we compare the intralayer and interlayer potentials for $x_q = 0.1$ (red), $x_q = 0.5$ (blue), and $x_q = 0.9$ (brown). The continuous (dashed) curves correspond to the intralayer (interlayer) potentials.

qualitatively similar to the intralayer case of figure 4.7. In the UV region $\hat{d}_\perp \propto m_q^{\frac{1}{4}}\,d_\perp \to 0$, the potential decays as $V_{q\bar{q}} \sim d_\perp^{-4}$, in agreement with our analytic calculation of section 4.4.2. In this case there is also a maximal length which increases with x_q. We have also compared the intralayer and interlayer potentials for the same value of x_q. These potentials are plotted together on the right panel of figure 4.8, where we notice that they have very different behaviour in the UV but they become very similar in the IR (see also [153]). This IR similarity increases as x_q approaches its maximal value $x_q = 1$, which is consistent with the fact that for large values of x_q the isotropic unflavored limit is rapidly attained in the IR.

UV limit

Let us consider the UV limit in which $\hat{d}_\perp$ is small and we have the following approximate relation between $d_\perp$ and x_0:

$$\hat{d}_\perp \approx \frac{8 \cdot 2^{\frac{1}{6}}}{\sqrt{15}} \frac{\sqrt{Q_c}}{Z_q^{\frac{1}{3}}} \sqrt{\pi} \frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)} \frac{1}{x_0^{\frac{1}{3}}} . \tag{4.73}$$

In this limit the potential can be approximated as:

$$V_{q\bar{q}} \approx \frac{2\sqrt{2}}{3 \cdot 2^{\frac{1}{6}}\pi} \frac{Z_q^{\frac{4}{3}} Q_f}{C^2} x_0^{\frac{4}{3}} \left[\int_1^\infty dz\, z^{\frac{1}{3}} \left(\frac{z^{\frac{5}{3}}}{\sqrt{z^{\frac{10}{3}} - 1}} - 1 \right) - \frac{3}{4} \right] . \tag{4.74}$$

The integral can be computed analytically and yields the following result for the potential:

$$V_{q\bar{q}} \approx -\frac{1}{2^{\frac{1}{6}}\sqrt{2\pi}} \frac{Z_q^{\frac{4}{3}} Q_f}{C^2} \frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)} x_0^{\frac{4}{3}} . \tag{4.75}$$

In terms of $\hat{d}_\perp$ we get:

$$\frac{C^2}{Q_f} V_{q\bar{q}} \approx -\beta_\perp \left(\frac{\sqrt{Q_c}}{\hat{d}_\perp} \right)^4 , \qquad \beta_\perp = \frac{2^{12}\pi^{\frac{3}{2}}}{3^2 \cdot 5^2} \left(\frac{\Gamma\left(\frac{3}{5}\right)}{\Gamma\left(\frac{1}{10}\right)} \right)^5 , \tag{4.76}$$

which is equivalent to the result obtained in the previous chapter for the massless background. As illustrated in figure 4.8, the potential (4.76) nicely matches the numerical results.

4.5 Entanglement entropy

The entanglement entropy of a region and its complement is a good measure of the quantum correlations of the system. In holography the entanglement entropy is obtained by minimizing an area functional for an 8-dimensional surface embedded in 10-dimensional spacetime [52, 171]. Let A be a spatial region in the gauge theory. The holographic entanglement entropy between A and its complement is:

$$S_A = \frac{1}{4G_{10}} \int_\Sigma d^8\xi \sqrt{\det g_8} , \tag{4.77}$$

where Σ is the 8-dimensional spatial surface whose boundary is A and minimizes S_A, G_{10} is the 10-dimensional Newton constant ($G_{10} = 8\pi^6$ in our units) and g_8 is the induced metric on Σ in the Einstein frame. In this section we will apply this prescription when

A is a slab of infinite extent in the two cartesian directions and having a finite width in the remaining cartesian coordinate. Clearly, there are two cases to study, namely parallel and transverse slabs, which we analyze separately. We note that again we find striking similarity with the results in [153].

4.5.1 Parallel slab

First we consider the case in which A is an infinite slab with a finite width parallel to the layers, namely:

$$A = \left\{ -\frac{\hat{l}_{\parallel}}{2} \le \hat{x}^1 \le \frac{\hat{l}_{\parallel}}{2}, -\infty < \hat{x}^2, \hat{x}^3 < +\infty \right\} . \tag{4.78}$$

We will parameterize Σ by a function $x = x(\hat{x}^1)$. The 8-dimensional induced metric on Σ is:

$$ds_8^2 = \hat{h}^{-\frac{1}{2}} \left[1 + Z_q^2 \frac{\hat{h}}{q^2} (x')^2\right] (d\hat{x}^1)^2 + \hat{h}^{-\frac{1}{2}} \left[(d\hat{x}^2)^2 + \frac{\hat{W}^2}{q^4} (d\hat{x}^3)^2\right]$$
$$+ \hat{h}^{\frac{1}{2}} Z_q^2 (1 + x - x_q)^2 \left(ds^2_{\mathbb{CP}^2} + q^2 (d\tau + A)^2\right) , \tag{4.79}$$

where the prime now denotes derivative with respect to $\hat{x}^1$. If we integrate over all the coordinates except x, we get:

$$\frac{S_{\parallel}}{\hat{L}_2 \hat{L}_3} = \frac{Z_q^5}{32\,\pi^3} \int d\hat{x}^1 (1 + x - x_q)^5 \frac{\hat{h}^{\frac{1}{2}} \hat{W}}{q} \sqrt{1 + Z_q^2 \frac{\hat{h}}{q^2} (x')^2} . \tag{4.80}$$

The first integral derived from $S_{\parallel}$ is:

$$\frac{\hat{h}^{\frac{1}{2}} \hat{W}}{q} \frac{(1 + x - x_q)^5}{\sqrt{1 + Z_q^2 \frac{\hat{h}}{q^2} (x')^2}} = \frac{\hat{h}_0^{\frac{1}{2}} \hat{W}_0}{q_0} (1 + x_0 - x_q)^5 , \tag{4.81}$$

where the subscript nought denotes that the corresponding quantity is evaluated at the minimal value x_0 of x. From this last equation we get:

$$x' = \pm \frac{q}{Z_q \sqrt{\hat{h}}} \sqrt{\frac{(1 + x - x_q)^{10}}{(1 + x_0 - x_q)^{10}} \frac{\hat{h}\, \hat{W}^2\, q_0^2}{\hat{h}_0\, \hat{W}_0^2\, q^2} - 1} , \tag{4.82}$$

which can be integrated to give $\hat{l}_{\parallel}$:

$$\hat{l}_{\parallel} = 2 Z_q \int_{x_0}^{\infty} \frac{\sqrt{\hat{h}}}{q} \frac{dx}{\sqrt{\frac{(1+x-x_q)^{10}}{(1+x_0-x_q)^{10}} \frac{\hat{h}\, \hat{W}^2\, q_0^2}{\hat{h}_0\, \hat{W}_0^2\, q^2} - 1}} . \tag{4.83}$$

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The entanglement entropy for this configuration is given by the divergent integral:

$$\frac{S_\parallel}{\hat{L}_2\hat{L}_3} = \frac{Z_q^6}{16\,\pi^3}\int_{x_0}^{x_{max}} \frac{\hat{h}\,\hat{W}}{q^2}\,\frac{(1+x-x_q)^5}{\sqrt{1-\frac{(1+x_0-x_q)^{10}}{(1+x-x_q)^{10}}\,\frac{\hat{h}_0\,\hat{W}_0^2\,q^2}{\hat{h}\,\hat{W}^2\,q_0^2}}}\,dx\ . \tag{4.84}$$

We will regularize $S_\parallel$ by subtracting the entropy of a configuration that we call the flat surface in the following. To conform with the homology constraint, the surface is not disconnected, but it is connected at the bottom $x = 0$. The full flat surface consists of three constant pieces: two straight $\hat{x}^1 = \pm l_\parallel/2$ ones and a horizontal one $x = const. = 0$, each individually being solutions to the equation of motion. It turns out that the contribution of the horizontal surface to the area integral vanishes. The entropy for the flat embedding is thus

$$\frac{S_\parallel^{flat}}{\hat{L}_2\hat{L}_3} = \frac{Z_q^6}{16\,\pi^3}\int_{0}^{x_{max}} \frac{\hat{h}\,\hat{W}}{q^2}\,(1+x-x_q)^5\,dx\ . \tag{4.85}$$

Thus, we define the finite entanglement entropy as:

$$\frac{S_\parallel^{finite}}{\hat{L}_2\hat{L}_3} = \frac{S_\parallel\ -\ S_\parallel^{flat}}{\hat{L}_2\hat{L}_3}\ . \tag{4.86}$$

After some calculations, we get:

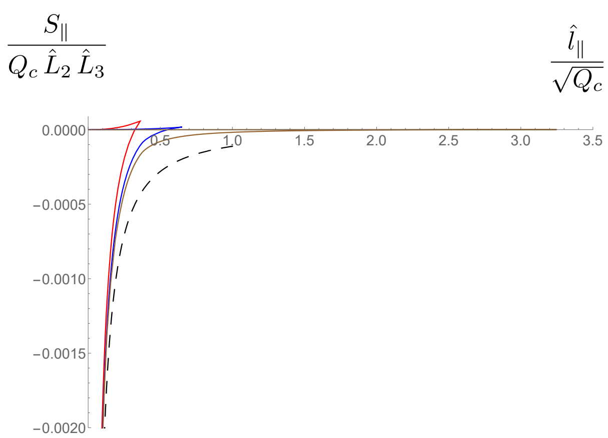

Figure 4.9: Plot of the entanglement entropy for a parallel slab as a function of its rescaled width. The continuous lines are the results of the numerical integration of (4.87) and (4.83) for $x_q = 0.1$ (red), $x_q = 0.5$ (blue) and $x_q = 0.9$ (brown). The dashed curve is the UV result (4.91).

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$$\frac{S_{\parallel}^{finite}}{\hat{L}_2\hat{L}_3} = -\frac{Z_q^6}{16\,\pi^3}\left(\int_{x_0}^{\infty}\frac{\hat{h}\,\hat{W}}{q^2}\,(1+x-x_q)^5\left[1 - \frac{1}{\sqrt{1-\frac{(1+x_0-x_q)^{10}}{(1+x-x_q)^{10}}\,\frac{\hat{h}_0\,\hat{W}_0^2\,q^2}{\hat{h}\,\hat{W}^2\,q_0^2}}}\right]dx\right.$$
$$\left.+\int_0^{x_0}\frac{\hat{h}\,\hat{W}}{q^2}\,(1+x-x_q)^5\,dx\right)\,. \tag{4.87}$$

The numerical results for $S_{\parallel}$ versus $\hat{l}_{\parallel}$ are presented in figure 4.9 for several values of the parameter x_q. In this plot we notice that $S_{\parallel}$ becomes positive when $\hat{l}_{\parallel}$ is large enough. According to our regularization procedure (4.86) this means that the disconnected surface is dominant with respect to the connected one when $\hat{l}_{\parallel} > \hat{l}_{\parallel}^c$, where $\hat{l}_{\parallel}^c$ is a critical length which grows with x_q. On the contrary, for small $\hat{l}_{\parallel}$ the entropy behaves as $S_{\parallel} \sim \hat{l}_{\parallel}^{-\frac{4}{3}}$, with a coefficient independent of x_q. As pointed out in the previous chapter, this universal behaviour is the one corresponding to an effective D2-brane and can be obtained analytically, as we show in the next subsection.

A similar behaviour of the EE has previously been obtained in backgrounds dual to confining theories and unquenched flavors, see [181–183].

UV limit

When $\hat{l}_{\parallel}$ is small, the minimal value x_0 of x is large and we can use the expansion of the functions of the background valid for large x. At leading order, we get:

$$\hat{l}_{\parallel} \approx \frac{8\sqrt{2}}{3\sqrt{15}}\,\frac{\sqrt{Q_c}}{Z_q}\int_{x_0}^{\infty}\frac{dx}{x^2\sqrt{(x/x_0)^{\frac{14}{3}}-1}} = \frac{8\sqrt{2\pi}}{3\sqrt{15}}\,\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)}\,\frac{\sqrt{Q_c}}{Z_q}\,\frac{1}{x_0}\,. \tag{4.88}$$

Similarly, the regulated entanglement entropy for the parallel slab is:

$$\frac{S_{\parallel}^{finite}}{\hat{L}_2\hat{L}_3} \approx -\frac{Z_q^{\frac{4}{3}}\,Q_c}{15\cdot 2^{\frac{5}{3}}\,\pi^3}\,x_0^{\frac{4}{3}}\left[\frac{3}{4} - \int_1^{\infty}dz\,z^{\frac{1}{3}}\left(\frac{z^{\frac{7}{3}}}{\sqrt{z^{\frac{14}{3}}-1}} - 1\right)\right]\,, \tag{4.89}$$

which can be integrated analytically with the result:

$$\frac{S_{\parallel}^{finite}}{\hat{L}_2\hat{L}_3} \approx -\frac{Z_q^{\frac{4}{3}}\,Q_c}{5\cdot 2^{\frac{11}{3}}\,\pi^{\frac{5}{2}}}\,\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)}\,x_0^{\frac{4}{3}}\,. \tag{4.90}$$

If we eliminate x_0 and write the entanglement entropy in terms of $\hat{l}_{\parallel}$, we get:

$$\frac{S_{\parallel}^{finite}}{Q_c\,\hat{L}_2\,\hat{L}_3} \approx -\gamma_{\parallel}\left(\frac{\sqrt{Q_c}}{\hat{l}_{\parallel}}\right)^{\frac{4}{3}}\,, \qquad \gamma_{\parallel} = \frac{2}{45\cdot 5^{\frac{2}{3}}\,\pi^{\frac{11}{6}}}\left(\frac{\Gamma\left(\frac{5}{7}\right)}{\Gamma\left(\frac{3}{14}\right)}\right)^{\frac{7}{3}}\,. \tag{4.91}$$

which is the same result as in the previous chapter and, as shown in figure 4.9, matches perfectly the numerical results when $\hat{l}_{\parallel}$ is small.

4.5.2 Transverse slab

We now consider a slab with finite width in the direction of $\hat{x}^3$. The region A in this case is:

$$A = \Big\{ -\infty < \hat{x}^1, \hat{x}^2 < +\infty \,,\, -\frac{\hat{l}_\perp}{2} \le \hat{x}^3 \le \frac{\hat{l}_\perp}{2} \Big\} \,, \tag{4.92}$$

and the induced metric on Σ becomes:

$$\begin{aligned} ds_8^2 \,=\, & \hat{h}^{-\frac{1}{2}} \Big[(d\hat{x}^1)^2 \,+\, (d\hat{x}^2)^2 \Big] + \hat{h}^{-\frac{1}{2}} \, \frac{\hat{W}^2}{q^4} \Big[1 \,+\, Z_q^2 \, \frac{\hat{h}\, q^2}{\hat{W}^2} \, (x')^2 \Big] \, (d\hat{x}^3)^2 \\ & + \hat{h}^{\frac{1}{2}} \, Z_q^2 \, (1 + x - x_q)^2 \Big(ds^2_{\mathbb{CP}^2} + q^2 \, (d\tau + A)^2 \Big) \,, \end{aligned} \tag{4.93}$$

where now $x = x(\hat{x}^3)$. The transverse entropy functional is:

$$\frac{S_\perp}{\hat{L}_1 \, \hat{L}_2} \,=\, \frac{Z_q^5}{32\, \pi^3} \int d\hat{x}^3 \, (1 + x - x_q)^5 \, \frac{\hat{h}^{\frac{1}{2}} \hat{W}}{q} \sqrt{1 + Z_q^2 \, \frac{\hat{h}\, q^2}{\hat{W}^2} \, (x')^2} \;. \tag{4.94}$$

Now the first integral is:

$$\frac{\hat{h}^{\frac{1}{2}} \hat{W}}{q} \, \frac{(1 + x - x_q)^5}{\sqrt{1 + Z_q^2 \, \frac{\hat{h}\, q^2}{\hat{W}^2} \, (x')^2}} \,=\, \frac{\hat{h}_0^{\frac{1}{2}} \hat{W}_0}{q_0} \, (1 + x_0 - x_q)^5 \,, \tag{4.95}$$

from which it follows that:

$$x' \,=\, \pm \, \frac{\hat{W}}{Z_q \, q \, \sqrt{\hat{h}}} \sqrt{\frac{(1 + x - x_q)^{10}}{(1 + x_0 - x_q)^{10}} \, \frac{\hat{h}\, \hat{W}^2 \, q_0^2}{\hat{h}_0 \, \hat{W}_0^2 \, q^2} \,-\, 1} \;. \tag{4.96}$$

Therefore, the transverse length $\hat{l}_\perp$ is:

$$\hat{l}_\perp \,=\, 2 Z_q \int_{x_0}^{\infty} \frac{q \sqrt{\hat{h}}}{\hat{W}} \, \frac{dx}{\sqrt{\frac{(1+x-x_q)^{10}}{(1+x_0-x_q)^{10}} \, \frac{\hat{h}\, \hat{W}^2 \, q_0^2}{\hat{h}_0 \, \hat{W}_0^2 \, q^2} \,-\, 1}} \;. \tag{4.97}$$

The entropy functional evaluated on the minimal surface for this configuration is:

$$\frac{S_\perp}{\hat{L}_1 \hat{L}_2} \,=\, \frac{Z_q^6}{16\, \pi^3} \int_{x_0}^{x_{max}} \hat{h} \, \frac{(1 + x - x_q)^5}{\sqrt{1 - \frac{(1+x_0-x_q)^{10}}{(1+x-x_q)^{10}} \, \frac{\hat{h}_0 \, \hat{W}_0^2 \, q^2}{\hat{h}\, \hat{W}^2 \, q_0^2}}} \, dx \;, \tag{4.98}$$

whose divergent part is:

$$\frac{S_\perp^{div}}{\hat{L}_1 \hat{L}_2} \,=\, \frac{Z_q^6}{16\, \pi^3} \int_0^{x_{max}} \hat{h} \, (1 + x - x_q)^5 \, dx \;. \tag{4.99}$$

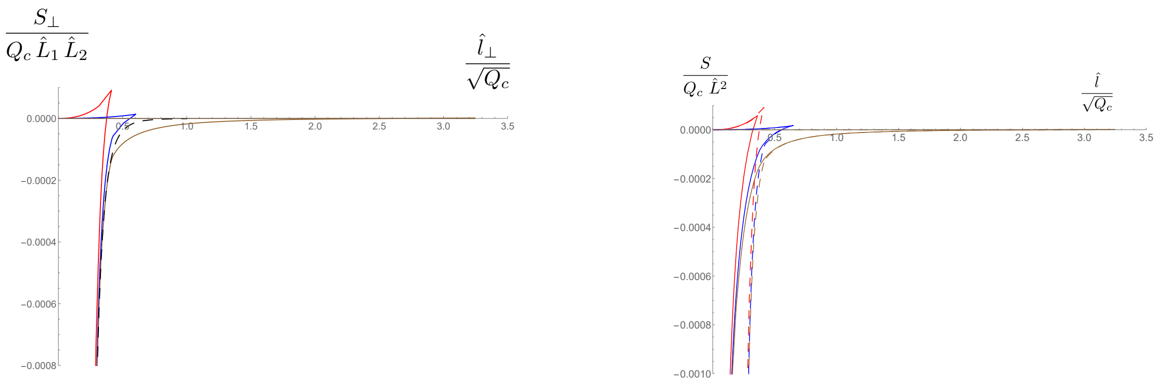

Figure 4.10: On the left we depict the entropy of a transverse slab as a function of $\hat{l}_\perp$. The dashed curve is the UV result (4.105). On the right we compare $\frac{S_\parallel}{\hat{Q}_c\,\hat{L}_2\,\hat{L}_3}$ and $\frac{S_\perp}{\hat{Q}_c\,\hat{L}_1\,\hat{L}_2}$ as functions of their corresponding rescaled width. In both panels curves with the same color correspond to the same value of x_q: red for $x_q = 0.1$, blue for $x_q = 0.5$, and brown for $x_q = 0.9$.

We define $S_\perp^{finite}$ as:

$$\frac{S_\perp^{finite}}{\hat{L}_1\,\hat{L}_2} = \frac{S_\perp\, -\, S_\perp^{div}}{\hat{L}_1\,\hat{L}_2}\,. \tag{4.100}$$

After some calculations one can demonstrate that:

$$\frac{S_\perp^{finite}}{\hat{L}_1\,\hat{L}_2} = -\frac{Z_q^6}{16\,\pi^3}\left(\int_{x_0}^{\infty}\hat{h}\,(1+x-x_q)^5\left[1-\frac{1}{\sqrt{1-\frac{(1+x_0-x_q)^{10}}{(1+x-x_q)^{10}}\,\frac{\hat{h}_0\,\hat{W}_0^2\,q^2}{\hat{h}\,\hat{W}^2\,q_0^2}}}\right]\right.$$
$$\left.+\int_0^{x_0}\hat{h}\,(1+x-x_q)^5\,dx\right)\,. \tag{4.101}$$

In figure 4.10 we plot $S_\perp$ as a function of $\hat{l}_\perp$ obtained by the numerical computation of the integrals in (4.101) and (4.97). For small $\hat{l}_\perp$ the curves for different values of x_q coincide and behave as $S_\perp \sim \hat{l}_\perp^{-6}$. This behaviour is found analytically in the next subsection. For large $\hat{l}_\perp$ the dominant configuration is the disconnected one and $S_\perp$ becomes positive. We have also compared in figure 4.10 the parallel and transverse entanglement entropies. In the UV the difference is significant, but at larger distances the two entanglement entropies become very similar. This similarity is more and more pronounced as $x_q \to 1$.

UV limit

We now evaluate the entropy in the limit in which x_0 is very large and $\hat{l}_\perp$ is very small. Using the UV expansion at leading order of the different functions of the background, we

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get:

$$\hat{l}_{\perp} \approx \frac{8 \cdot 2^{\frac{1}{6}}}{3\sqrt{15}} \frac{\sqrt{Q_c}}{Z_q^{\frac{1}{3}}} \int_{x_0}^{\infty} \frac{dx}{x^{\frac{4}{3}} \sqrt{(x/x_0)^{\frac{14}{3}} - 1}} = \frac{8 \cdot 2^{\frac{1}{6}}}{\sqrt{15}} \sqrt{\pi} \frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)} \frac{\sqrt{Q_c}}{Z_q^{\frac{1}{3}}} \frac{1}{x_0^{\frac{1}{3}}} \,, \tag{4.102}$$

whereas $S_{\perp}^{finite}$ is:

$$\frac{S_{\perp}^{finite}}{\hat{L}_1 \hat{L}_2} \approx -\frac{Z_q^2 Q_c}{60 \pi^3} x_0^2 \left[\frac{1}{2} - \int_1^{\infty} dz \, z \left(\frac{z^{\frac{7}{3}}}{\sqrt{z^{\frac{14}{3}} - 1}} - 1\right)\right] \,, \tag{4.103}$$

which, after performing the integral becomes:

$$\frac{S_{\perp}^{finite}}{\hat{L}_1 \hat{L}_2} \approx -\frac{Z_q^2 Q_c}{120 \pi^{\frac{5}{2}}} \frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)} x_0^2 \,. \tag{4.104}$$

Eliminating x_0 in favor of $\hat{l}_{\perp}$ we reproduce the result of the previous chapter:

$$\frac{S_{\perp}^{finite}}{Q_c \hat{L}_1 \hat{L}_2} \approx -\gamma_{\perp} \left(\frac{\sqrt{Q_c}}{\hat{l}_{\perp}}\right)^6 , \qquad \gamma_{\perp} = \left(\frac{16}{15}\right)^4 \sqrt{\pi} \left(\frac{\Gamma\left(\frac{4}{7}\right)}{\Gamma\left(\frac{1}{14}\right)}\right)^7 \,. \tag{4.105}$$

4.5.3 Flow of mutual information

The mutual information of two entangling regions A_1 and A_2 is a measure of the information shared by these two domains and is defined as:

$$I(A_1, A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2) \,. \tag{4.106}$$

We will analyze the evolution with the intrinsic scale of the background of $I(A_1, A_2)$ when A_1 and A_2 are two slabs of equal length l parallel to each other which are separated by a distance s. In holography there are two possible surfaces contributing to the entanglement entropy of two slab [184, 185]. One of these configurations, which has zero mutual information, dominates when the slab separation is large. Below some critical separation the second configuration is dominant and $I(A_1, A_2)$ becomes positive. In reduced units, the critical separation $\hat{s}$ for two strips of length $\hat{l}$ is determined by the vanishing of the mutual information:

$$2\, S(\hat{l}) = S(2\hat{l} + \hat{s}) + S(\hat{s}) \,. \tag{4.107}$$

Let us analyze (4.107) for our background in the UV region, where both $\hat{l}$ and $\hat{s}$ are small and the entanglement entropy has the scaling behaviour written in (4.91) ((4.105)) for parallel (transverse) slabs. The resulting equation takes the form:

$$\frac{1}{\left(2 + \frac{\hat{s}}{\hat{l}}\right)^a} + \frac{1}{\left(\frac{\hat{s}}{\hat{l}}\right)^a} = 2 \,, \tag{4.108}$$

where $a = \frac{4}{3}$ ($a = 6$) for parallel (transverse) slabs. The solution of (4.108) in these two cases is:

$$\left(\frac{\hat{s}}{\hat{l}}\right)_{\parallel} \approx 0.663 \;, \qquad\qquad \left(\frac{\hat{s}}{\hat{l}}\right)_{\perp} \approx 0.891 \;. \tag{4.109}$$

Let us next study the critical point in the long distance IR region, where we expect to approach the conformal isotropic behaviour of $AdS_5 \times \mathbb{S}^5$. In this last case the critical separation is also determined by (4.108) with $a = 2$ and, thus:

$$\left(\frac{\hat{s}}{\hat{l}}\right)_{AdS_5 \times \mathbb{S}^5} = \sqrt{3} - 1 \approx 0.732 \;. \tag{4.110}$$

Interestingly, viewed as an inverse it equals the metallic mean $\hat{l}/\hat{s} = \sigma_{p=1,q=1/2} = \frac{\sqrt{3}+1}{2}$ [186, 187]. Outside the UV region the critical $\hat{s}/\hat{l}$ depends on $\hat{l}$. We have numerically verified that this flow behaves as expected. For parallel (transverse) slabs $\hat{s}/\hat{l}$ grows (decreases) from its UV value (4.109) as $\hat{l}$ is increased and it actually becomes very close to the conformal isotropic value (4.110) for large $\hat{l}$ and x_q close to one (see [187] for another recent example of a holographic flow of mutual information). We note that for very large distances $\hat{l}$, the dominant phase is that for flat embeddings and the full phase diagram resembles that of [185].

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4.6 Thermodynamics of a massless probe brane

In this section we test our background with a probe D5-brane, embedded as in the array (2.1), in which we switch on a worldvolume gauge field $A = A_0\, dx^0$ dual to a chemical potential μ. Our goal is to study the zero temperature thermodynamics of the probe and its evolution as we move from the UV to the IR. For simplicity we will consider massless embeddings in which the probe reaches the IR end of the space. These type of embeddings have been analyzed in detail in appendix 4.C, where we check that the equations of motion of the probe are satisfied if the worldvolume gauge potential A_0 satisfies (4.264). In this section we will work directly in the radial coordinate ζ, for which the Lagrangian density takes the form:

$$\tilde{\mathcal{L}} = e^{g-f}\,\mathcal{L} = -\mathcal{T}\,\frac{\zeta^2}{\sqrt{W}}\left[\sqrt{1 - W\left(\frac{\partial A_0}{\partial \zeta}\right)^2} - 1\right] , \tag{4.111}$$

where $\mathcal{L}$ is the Lagrangian density (4.262). In (4.111) $\mathcal{T}$ is the constant defined in (4.263) and we have written $\tilde{\mathcal{L}}$ in terms of the master function W. The chemical potential is just the value of A_0 at the UV boundary. In our variables:

$$\mu = d \int_b^\infty \frac{d\zeta}{\sqrt{W}}\,\frac{1}{\sqrt{d^2 + \zeta^4}} \;, \tag{4.112}$$

where d is the integration constant of (4.264) which, as we check below, is proportional to the charge density. The grand potential Ω is given by minus the on-shell action, which in our case is finite and there is no need of regularizing it. This is due to the cancellation of the divergences between the DBI and WZ terms. Removing the Minkowski volume factor, we get:

$$\Omega = \mathcal{N} \int_b^\infty d\zeta \frac{\zeta^2}{\sqrt{W}} \Big[\frac{\zeta^2}{\sqrt{d^2+\zeta^4}} - 1 \Big] , \tag{4.113}$$

where $\mathcal{N} = \frac{16\pi^2}{3\sqrt{3}}$. The charge density ρ can be written as:

$$\rho = -\frac{\partial \Omega}{\partial \mu} = -\frac{\frac{\partial \Omega}{\partial d}}{\frac{\partial \mu}{\partial d}} . \tag{4.114}$$

From (4.112) and (4.113) we can compute the derivatives with respect to d that are needed to calculate ρ:

$$\begin{aligned} \frac{\partial \mu}{\partial d} &= \int_b^\infty d\zeta \frac{\zeta^4}{\sqrt{W}} \frac{1}{(d^2+\zeta^4)^{\frac{3}{2}}} \\ \frac{\partial \Omega}{\partial d} &= -\mathcal{N}\, d \int_b^\infty d\zeta \frac{\zeta^4}{\sqrt{W}} \frac{1}{(d^2+\zeta^4)^{\frac{3}{2}}} . \end{aligned} \tag{4.115}$$

Clearly, one has:

$$\frac{\partial \Omega}{\partial d} = -\mathcal{N}\, d \frac{\partial \mu}{\partial d} , \tag{4.116}$$

and we get that, indeed, ρ is related to d as expected:

$$\rho = \mathcal{N}\, d . \tag{4.117}$$

The energy density ϵ is given by:

$$\epsilon = \Omega + \mu\rho . \tag{4.118}$$

Plugging the values of Ω and ρ into (4.118), we get:

$$\epsilon = \mathcal{N} \int_b^\infty \frac{d\zeta}{\sqrt{W}} \zeta^2 \left[\sqrt{1+\frac{d^2}{\zeta^4}} - 1 \right] . \tag{4.119}$$

Therefore:

$$\frac{\partial \epsilon}{\partial d} = \mathcal{N}\, d \int_b^\infty \frac{d\zeta}{\sqrt{W}} \frac{1}{(d^2+\zeta^4)^{\frac{1}{2}}} = \mathcal{N}\, \mu , \tag{4.120}$$

as expected. Taking into account that $p = -\Omega$, the speed of sound u_s can be obtained as:

$$u_s^2 = -\frac{\frac{\partial \Omega}{\partial d}}{\frac{\partial \epsilon}{\partial d}} = -\frac{1}{\mathcal{N}\, \mu} \frac{\partial \Omega}{\partial d} . \tag{4.121}$$

Thus, we get the following expression of u_s^2:

$$u_s^2 \;=\; \frac{d}{\mu}\int_b^{\infty} d\zeta\,\frac{\zeta^4}{\sqrt{W}}\,\frac{1}{(d^2+\zeta^4)^{\frac{3}{2}}}\;. \tag{4.122}$$

More explicitly, plugging into (4.122) the expression (4.112) of μ, we obtain that u_s^2 is given by the following ratio of two integrals over the holographic coordinate:

$$u_s^2 \;=\; \frac{\int_b^{\infty}\frac{d\zeta}{\sqrt{W}}\,\frac{\zeta^4}{(d^2+\zeta^4)^{\frac{3}{2}}}}{\int_b^{\infty}\frac{d\zeta}{\sqrt{W}}\,\frac{1}{\sqrt{d^2+\zeta^4}}}\;. \tag{4.123}$$

In the unflavored ($W=1$, $b=0$) and massless flavored ($W\propto\zeta^{-\frac{2}{3}}$, $b=0$) cases we get the following d-independent results:

$$u_s^2(\text{unflavored}) \;=\; \frac{1}{2}\;, \qquad\qquad u_s^2(\text{massless flavored}) \;=\; \frac{2}{3}\;. \tag{4.124}$$

The unflavored result $u_s^2=1/2$ is the one expected for a conformal worldvolume theory in $2+1$ dimensions. In general, one should get a value depending on d, which interpolates between these two values. In order to facilitate the numerical calculations, let us rewrite these results in x coordinate. Recall that ζ and x are related as:

$$\zeta \;=\; \frac{Q_f}{C^{\frac{3}{2}}}\,Z_q\,(x+1-x_q)\;. \tag{4.125}$$

It turns out that Q_f and C can be scaled out from our formulas. First of all, we define the rescaled density and chemical potential as:

$$\hat{d} \;\equiv\; \frac{C^3}{Q_f^2}\,d\;, \qquad\qquad \hat{\mu} \;\equiv\; \frac{C^2}{Q_f}\,\mu\;. \tag{4.126}$$

Then, we have:

$$\hat{\mu} \;=\; \hat{d}\,Z_q\int_0^{\infty}\frac{dx}{\sqrt{\hat{W}}}\,\frac{1}{\left[\hat{d}^2\;+\;Z_q^4\,(x+1-x_q)^4\right]^{\frac{1}{2}}}\;, \tag{4.127}$$

where $\hat{W}$ was defined in (4.29). Moreover, Ω and ϵ can be recast as:

$$\Omega \;=\; \hat{\mathcal{N}}\,Z_q^3\int_0^{\infty}\frac{dx}{\sqrt{\hat{W}}}\,(x+1-x_q)^2\left[\frac{(x+1-x_q)^2\,Z_q^2}{\sqrt{\hat{d}^2\;+\;Z_q^4\,(x+1-x_q)^4}}\;-\;1\right]$$

$$\epsilon \;=\; \hat{\mathcal{N}}\,Z_q^2\int_0^{\infty}\frac{dx}{\sqrt{\hat{W}}}\,(x+1-x_q)^2\left[\sqrt{1+\frac{\hat{d}^2}{Z_q^4}\,\frac{1}{(x+1-x_q)^4}}\;-\;1\right]\;, \tag{4.128}$$

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where $\hat{\mathcal{N}}$ is defined as:

$$\hat{\mathcal{N}} \equiv \frac{Q_f^3}{C^5}\mathcal{N} \, . \tag{4.129}$$

The speed of sound can either be written as:

$$u_s^2 \,=\, Z_q^5 \,\frac{\hat{d}}{\hat{\mu}} \int_0^\infty \frac{dx}{\sqrt{\hat{W}}} \frac{(x+1-x_q)^4}{\big[\hat{d}^2 \,+\, Z_q^4\,(x+1-x_q)^4\big]^{\frac{3}{2}}} \, , \tag{4.130}$$

or if we define the integrals I_1 and I_2 as:

$$\begin{aligned} I_1 &= \int_0^\infty \frac{dx}{\sqrt{\hat{W}(x)}} \frac{(x+1-x_q)^4}{\big[\hat{d}^2 \,+\, Z_q^4\,(x+1-x_q)^4\big]^{\frac{3}{2}}} \\ I_2 &= \int_0^\infty \frac{dx}{\sqrt{\hat{W}(x)}} \frac{1}{\big[\hat{d}^2 \,+\, Z_q^4\,(x+1-x_q)^4\big]^{\frac{1}{2}}} \, , \end{aligned} \tag{4.131}$$

then u_s is obtained from the ratio between I_1 and I_2:

$$u_s^2 \,=\, Z_q^4 \,\frac{I_1}{I_2} \, . \tag{4.132}$$

Recall that we can relate the quark mass of the background m_q to the constant C, namely $C \sim \sqrt{Q_f}/\sqrt{m_q}$. Using this relation we get that $\hat{d} \sim d/(\sqrt{Q_f}\, m_q^{\frac{3}{2}})$. Therefore, the UV limit $m_q \to 0$ corresponds to taking $\hat{d} \to \infty$. Accordingly, we should recover in this large $\hat{d}$ limit the massless flavored result of (4.124):

$$u_s^2(\hat{d} \to \infty) \to \frac{2}{3} \, , \qquad \text{(UV limit).} \tag{4.133}$$

The UV limit written above can also be analytically verified. Indeed, let us introduce a new integration variable y, related to x as:

$$x \,=\, x_q \,-\, 1 + \frac{\hat{d}^{\frac{1}{2}}}{Z_q}\, y \,\equiv\, x(y) \, . \tag{4.134}$$

The minimal value of y, corresponding to $x = 0$, is $y = y_q$, where:

$$y_q \,=\, \frac{1 - x_q}{\hat{d}^{\frac{1}{2}}}\, Z_q \, . \tag{4.135}$$

It is now straightforward to write I_1 and I_2 as:

$$\begin{aligned} I_1 &= \frac{1}{Z_q^5\,\hat{d}^{\frac{1}{2}}} \int_{y_q}^\infty \frac{y^4\,dy}{\sqrt{\hat{W}(x(y))}\,(1+y^4)^{\frac{3}{2}}} \\ I_2 &= \frac{1}{Z_q\,\hat{d}^{\frac{1}{2}}} \int_{y_q}^\infty \frac{dy}{\sqrt{\hat{W}(x(y))}\,(1+y^4)^{\frac{3}{2}}} \, . \end{aligned} \tag{4.136}$$

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When $\hat{d} \to \infty$ the lower limit of the integrals becomes $y_q = 0$. Moreover, the argument of the function $\hat{W}$ is large and one can use its UV asymptotic expression:

$$\hat{W}(x(y)) \approx \frac{9}{8}\left[\frac{\sqrt{2}}{Z_q}\right]^{\frac{2}{3}}\left[x_q - 1 + \frac{\hat{d}^{\frac{1}{2}}}{Z_q}\,y\right]^{-\frac{2}{3}} \approx \frac{9}{8}\left[\frac{\sqrt{2}}{\hat{d}^{\frac{1}{2}}}\right]^{\frac{2}{3}} y^{-\frac{2}{3}}\,, \qquad (\hat{d} \to \infty)\,. \tag{4.137}$$

Then for large $\hat{d}$:

$$\begin{aligned} I_1 &\approx \frac{1}{Z_q^5\,\hat{d}^{\frac{1}{2}}}\int_0^\infty dy\,\frac{y^{\frac{13}{3}}}{(1+y^4)^{\frac{3}{2}}} = \frac{1}{Z_q^5\,\hat{d}^{\frac{1}{2}}}\,\frac{\Gamma\!\left(\frac{1}{6}\right)\Gamma\!\left(\frac{1}{3}\right)}{4\sqrt{\pi}} \\ I_2 &\approx \frac{1}{Z_q\,\hat{d}^{\frac{1}{2}}}\int_0^\infty dy\,\frac{y^{\frac{1}{3}}}{(1+y^4)^{\frac{3}{2}}} = \frac{1}{Z_q\,\hat{d}^{\frac{1}{2}}}\,\frac{\Gamma\!\left(\frac{1}{6}\right)\Gamma\!\left(\frac{1}{3}\right)}{6\sqrt{\pi}}\,, \end{aligned} \tag{4.138}$$

and therefore we get the expected UV result:

$$u_s^2 \approx \frac{2}{3}\,, \qquad (\hat{d} \to \infty)\,. \tag{4.139}$$

Let us now explore the small $\hat{d}$ limit. By taking $\hat{d} = 0$ in (4.131) we see that the integrals

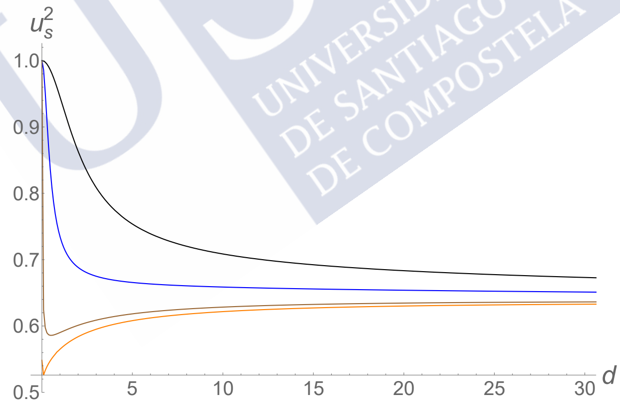

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Figure 4.11: Speed of sound as a function of the density $\hat{d}$ for $x_q = 0.005$ (black,top), $x_q = 0.5$ (blue,second from top), $x_q = 0.9$ (brown,third from top), and $x_q = 0.995$ (orange,bottom).

I_1 and I_2 are proportional $(I_2(\hat{d}=0) = Z_q^4\, I_2(\hat{d}=0))$. Therefore, we get:

$$u_s^2(\hat{d}=0) = 1\,. \tag{4.140}$$

This result is not the one we naively expect since $\hat{d} \to 0$ corresponds to $m_q \to \infty$ and, as the quarks are non-dynamical in this large mass limit, it would seem that we should get the conformal result $u_s^2 = 1/2$ in this IR limit, at least when x_q is close to one. In order to clarify the situation, let us take $\hat{d} = 0$ in the integrals (4.131) and examine the behaviour of the integrand near $x = 0$. For $x_q \neq 1$ we get:

$$\int_0^\infty \frac{dx}{\sqrt{\hat{W}}} \frac{1}{(x+1-x_q)^2} \approx \int_0^\infty dx \left[\frac{1}{\sqrt{6}} \frac{1}{(1-x_q)^{\frac{3}{2}}} \frac{1}{\sqrt{x}} + \frac{1}{4\sqrt{6}} \frac{1}{(1-x_q)^{\frac{5}{2}}} \sqrt{x} + \mathcal{O}(x^{\frac{3}{2}}) \right] , \tag{4.141}$$

where we have used (4.30) to obtain the behaviour of the integrand near $x = 0$. This $x_q \neq 1$ integral is convergent and the two integrals I_1 and I_2 are well defined. On the contrary, for $x_q = 1$ we have $\hat{W} = 1$ inside the cavity and the integrand is divergent when $\hat{d} = 0$ and thus we cannot take directly $\hat{d} = 0$ in the integrals. Actually, in the calculation of u_s, the limits $\hat{d} \to 0$ and $x_q \to 1$ do not commute. By taking $x_q \to 1$ first we indeed get the conformal result $u_s^2 = 1/2$ independently of the value of $\hat{d}$.

The numerical values of u_s^2 obtained from (4.132) as a function of $\hat{d}$ for different values of $\hat{x}_q$ have been plotted in figure 4.11. In this plot we notice that the UV asymptotic result (4.139) is satisfied for all values of x_q and for $x_q \sim 1$ there is a minimum at low $\hat{d}$, in which u_s^2 approaches the conformal value $u_s^2 = 1/2$. This is consistent with the behaviour found in the analysis of other observables: the UV behaviour is independent of x_q and given by the scaling solution, whereas the IR is controlled by x_q. By taking x_q close to one, the long distance behaviour of our system becomes more isotropic.

4.7 Discussion

In this chapter we generalized the scaling solution found in chapter 2 to the case in which the quarks are massive and there is a region in the bulk, which we called the cavity, in which the flavor sources vanish. The background found is supersymmetric and solves the supergravity equations of motion with D5-brane sources. It is given in terms of the master function W, which can be obtained by solving the master equation (4.5). This master equation contains the profile function which can, in principle, be arbitrarily chosen and depends on the particular distribution of the D5-brane charge along the holographic coordinate. The solutions corresponding to a constant profile function p were obtained in chapter 2. In the flavor language, the solutions of chapter 2 are dual to models with massless flavors living on the defect. The corresponding field theories display anisotropy in the third direction which, by construction, is produced by the multiple (2+1)-dimensional layers.

The size x_q of the cavity provides us with a parameter which determines the long distance behaviour of the model. Indeed, as we tuned x_q to its maximal value $x_q = 1$, the IR behaviour of several observables we analyzed approaches the one of the un-layered theory, whereas the short distance behaviour is independent of x_q and given by the scaling solution. Thus, as $x_q \to 1$ the theory in the IR becomes more isotropic and effectively retains its (3+1)-dimensional character.

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4.A Details of the background

The form of the 10-dimensional metric in Einstein frame has been written in (4.1). In this appendix we give further details and an explicit coordinate representation. Let χ be an angular coordinate taking values in the range $0 \le \chi \le \pi$ and let ω^i $(i = 1, 2, 3)$ be a set of left-invariant $SU(2)$ 1-forms satisfying $d\omega^i = \frac{1}{2}\epsilon^{ijk}\omega^j \wedge \omega^k$. Then, we can write ds^2_{10} as:

$$ds^2_{10} = h^{-\frac{1}{2}}\left[-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + e^{-2\phi}(dx^3)^2\right] + h^{\frac{1}{2}}\Big[dr^2 + \frac{e^{2g}}{4}\Big(d\chi^2 + $$

$$+\cos^2\frac{\chi}{2}((\omega^1)^2 + (\omega^2)^2) + \cos^2\frac{\chi}{2}\sin^2\frac{\chi}{2}(\omega^3)^2\Big) + e^{2f}\Big(d\tau + \frac{1}{2}\cos^2\frac{\chi}{2}\omega^3\Big)^2\Big] , \qquad (4.142)$$

where the fiber τ takes values in the range $0 \le \tau \le 2\pi$. Notice that we are using in (4.142) a radial variable r which is different from the one in (4.1) (see below for the relation between r and ζ). The 1-form A in (4.1) is given in (2.135), and the 1-forms ω^i are written in (2.133), with the ranges of the angles specified in that section. The coordinate representation of the canonical vielbein basis of $\mathbb{C}\mathbb{P}^2$ is given by (2.134).

By plugging these expressions into (2.17) and (4.3) we get the value of the RR 3-form F_3, where we have already used the new radial variable ζ defined in (4.146) below.

Our solution preserves some amount of supersymmetry. Indeed, let us choose the following vielbein basis for the 10-dimensional metric (4.142):

$$\begin{aligned}
&E^{x^\mu} = h^{-\frac{1}{4}}\,dx^\mu , \qquad (\mu = 0, 1, 2), \qquad E^{x^3} = h^{-\frac{1}{4}}\,e^m\,dx^3 ,\\
&E^r = h^{\frac{1}{4}}\,dr , \qquad E^i = \frac{1}{2}h^{\frac{1}{4}}\,e^g\cos\frac{\chi}{2}\,\omega^i , \qquad (i = 1, 2) ,\\
&E^3 = \frac{1}{2}h^{\frac{1}{4}}\,e^g\cos\frac{\chi}{2}\sin\frac{\chi}{2}\,\omega^3 , \qquad E^4 = \frac{1}{2}h^{\frac{1}{4}}\,e^g\,d\chi ,\\
&E^5 = h^{\frac{1}{4}}\,e^f\left(d\tau + \frac{1}{2}\cos^2\frac{\chi}{2}\,\omega^3\right) .
\end{aligned} \qquad (4.143)$$

Then, in this basis of 1-forms, the Killing spinor of the background has the form given in (2.25). And if we represent the Killing spinors of type IIB supergravity as a Majorana-Weyl doublet, then η satisfies the projections (2.128) and (2.129), which means that our background preserves two supercharges. Actually, the different functions of our ansatz must satisfy the following system of first-order BPS equations, obtained in the second

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chapter (see (2.24), where here we have already set $m = -\phi$):

$$\begin{aligned}
\phi' &= Q_f\, p(r)\, e^{\frac{3\phi}{2} - 2g} \\
g' &= e^{f-2g} \\
f' &= 3\, e^{-f} \;-\; 2e^{f-2g} \;+\; \frac{Q_f\, p(r)}{2}\, e^{\frac{3\phi}{2} - 2g} \\
h' &= -Q_c\, e^{-4g-f} \;-\; Q_f\, p(r)\, e^{\frac{3\phi}{2} - 2g}\, h \;,
\end{aligned} \tag{4.144}$$

where $p(r)$ is the profile function of (4.3), the prime denotes derivative with respect to the radial coordinate r and Q_f is related to the number of flavors N_f as:

$$Q_f \;=\; \frac{4\pi\, g_s\, \alpha' N_f}{9\sqrt{3}\, L_3} \;, \tag{4.145}$$

where $L_3 \;=\; \int dx^3$, and Q_c is given by (2.136). Let us now write the BPS system in terms of a new radial variable ζ, related to r as:

$$\frac{d\zeta}{dr} \;=\; e^{f-g} \;. \tag{4.146}$$

Then, the equations for ϕ, g, and f take the form:

$$\begin{aligned}
\frac{d\phi}{d\zeta} &= Q_f\, p(\zeta)\, e^{\frac{3\phi}{2} - f - g} \\
\frac{dg}{d\zeta} &= e^{-g} \\
\frac{df}{d\zeta} &= 3\, e^{g-2f} \;-\; 2\, e^{-g} \;+\; \frac{Q_f\, p(\zeta)}{2}\, e^{\frac{3\phi}{2} - f - g} \;.
\end{aligned} \tag{4.147}$$

In this new variable ζ, the BPS equation for g in (4.147) can be immediately integrated:

$$e^{g} = \zeta \;, \tag{4.148}$$

and we can rewrite the remaining BPS equations as:

$$\begin{aligned}
\frac{d\phi}{d\zeta} &= \frac{Q_f\, p(\zeta)}{\zeta}\, e^{\frac{3\phi}{2} - f} \\
\frac{df}{d\zeta} &= 3\,\zeta\, e^{-2f} \;-\; \frac{2}{\zeta} \;+\; \frac{Q_f\, p(\zeta)}{2\,\zeta}\, e^{\frac{3\phi}{2} - f} \;.
\end{aligned} \tag{4.149}$$

Notice that our ansatz for the metric in the ζ variable is precisely the one written in (4.1). We now introduce the new master function W as:

$$W \equiv \frac{e^{2f-\phi}}{\zeta^2} . \tag{4.150}$$

One can easily show that the BPS system (4.149) implies that W satisfies the following first-order differential equation:

$$\frac{dW}{d\zeta} = \frac{6}{\zeta} \left(e^{-\phi} - W\right) . \tag{4.151}$$

Moreover, one can demonstrate that the equation for ϕ in (4.149) can be written as:

$$\frac{d}{d\zeta}\, e^{-\phi} = -\frac{Q_f\, p(\zeta)}{\zeta^2 \sqrt{W}} . \tag{4.152}$$

One can combine these last two equations in the following second-order master equation for W:

$$\frac{d}{d\zeta}\Big(\zeta\, \frac{dW}{d\zeta}\Big) + 6\, \frac{dW}{d\zeta} = -\frac{6\, Q_f\, p(\zeta)}{\zeta^2 \sqrt{W}} , \tag{4.153}$$

which coincides with (4.5). It is also straightforward to verify that the function f and the dilaton ϕ are given in terms of W as in (4.6) and (4.7), respectively.

We can provide a simple expression to get the warp factor. Using (4.6) and (4.7), we get for the coefficient multiplying h in (4.10):

$$Q_f \frac{e^{\frac{3\phi}{2}-f} p}{\zeta} = \frac{1}{W + \frac{1}{6}\zeta \frac{dW}{d\zeta}} \frac{Q_f p(\zeta)}{\zeta^2 \sqrt{W}}. \tag{4.154}$$

Using the master equation (4.5) the second factor on the right-hand side can be written as a total derivative as:

$$\frac{Q_f p(\zeta)}{\zeta^2 \sqrt{W}} = -\frac{d}{d\zeta}\left[W + \frac{1}{6}\zeta \frac{dW}{d\zeta}\right]. \tag{4.155}$$

Therefore we have:

$$Q_f \frac{e^{\frac{3\phi}{2}-f} p}{\zeta} = -\frac{d}{d\zeta} \log\left[W + \frac{1}{6}\zeta \frac{dW}{d\zeta}\right]. \tag{4.156}$$

Moreover, since:

$$\frac{e^{-2f}}{\zeta^3} = \frac{1}{\zeta^5 W}\left[W + \frac{1}{6}\zeta \frac{dW}{d\zeta}\right], \tag{4.157}$$

the equation determining h is:

$$\frac{dh}{d\zeta} - \frac{d}{d\zeta} \log\left[W + \frac{1}{6}\zeta \frac{dW}{d\zeta}\right] h = -Q_c \frac{1}{\zeta^5 W}\left[W + \frac{1}{6}\zeta \frac{dW}{d\zeta}\right]. \tag{4.158}$$

This equation for $Q_c = 0$ is homogeneous and can be immediately solved:

$$h(\zeta) = C\Big[W + \frac{1}{6}\zeta\frac{dW}{d\zeta}\Big], \qquad (Q_c = 0), \tag{4.159}$$

with C being constant. If we allow C to depend on ζ and substitute in the complete equation, we get the following equation for $C(\zeta)$:

$$\frac{dC}{d\zeta} = -\frac{Q_c}{\zeta^5 W(\zeta)}, \tag{4.160}$$

which can be integrated as:

$$C(\zeta) = Q_c \int_\zeta^{\zeta_0} \frac{d\bar\zeta}{\bar\zeta^5 W(\bar\zeta)}, \tag{4.161}$$

where ζ_0 is a constant of integration. Let us choose ζ_0 in such a way that $h(\zeta \to \infty) = 0$. Then:

$$h(\zeta) = Q_c\Big[W(\zeta) + \frac{\zeta}{6}\frac{dW(\zeta)}{d\zeta}\Big] \int_\zeta^\infty \frac{d\bar\zeta}{\bar\zeta^5 W(\bar\zeta)}. \tag{4.162}$$

Alternatively, we can represent $h(\zeta)$ in terms of the dilaton $\phi(\zeta)$:

$$h(\zeta) = Q_c e^{-\phi(\zeta)} \int_\zeta^\infty \frac{d\bar\zeta}{\bar\zeta^5 W(\bar\zeta)}. \tag{4.163}$$

As mentioned in subsection 4.2.3 the master equation can be solved in powers of ζ_q/ζ for large ζ. The result for W was written in (4.25). The function f is readily computed using (4.6):

$$e^f = \frac{3}{2\sqrt{2}}\,\zeta\,\Big[1 - \frac{1}{16}\frac{\zeta_q^4}{\zeta^4} + \frac{3}{32}\frac{\zeta_q^6}{\zeta^6} + \frac{7}{1024}\frac{\zeta_q^8}{\zeta^8} + \mathcal{O}\Big(\frac{\zeta_q^{10}}{\zeta^{10}}\Big)\Big]\,. \tag{4.164}$$

The dilaton in this expansion is:

$$e^\phi = \Big(\frac{\zeta}{\sqrt{2}\,Q_f}\Big)^{\frac{2}{3}}\Big[1 + \frac{1}{24}\frac{\zeta_q^4}{\zeta^4} + \frac{1}{48}\frac{\zeta_q^6}{\zeta^6} + \frac{43}{4608}\frac{\zeta_q^8}{\zeta^8} + \mathcal{O}\Big(\frac{\zeta_q^{10}}{\zeta^{10}}\Big)\Big]\,, \tag{4.165}$$

and the warp factor becomes:

$$\begin{aligned} h \approx \frac{Q_c}{\zeta^4}\Big[\frac{4}{15} + \frac{1}{110}\frac{\zeta_q^4}{\zeta^4} - \frac{3}{140}\frac{\zeta_q^6}{\zeta^6} - \frac{45}{23936}\frac{\zeta_q^8}{\zeta^8}\Big] + \\ + \frac{C_1}{\zeta^{\frac{2}{3}}}\Big[1 - \frac{1}{24}\frac{\zeta_q^4}{\zeta^4} - \frac{1}{48}\frac{\zeta_q^6}{\zeta^6} - \frac{35}{4608}\frac{\zeta_q^8}{\zeta^8}\Big]\,, \end{aligned} \tag{4.166}$$

where C_1 is a constant of integration. The squashing factor $q = e^f/\zeta$ for this solution can be obtained from (4.9) and takes the form:

$$q = \frac{3}{2\sqrt{2}}\Big[1 - \frac{1}{16}\frac{\zeta_q^4}{\zeta^4} + \frac{3}{32}\frac{\zeta_q^6}{\zeta^6} + \frac{7}{1024}\frac{\zeta_q^8}{\zeta^8}\Big] + \mathcal{O}\Big(\frac{\zeta_q^{10}}{\zeta^{10}}\Big)\,. \tag{4.167}$$

We can also solve the master equation (4.5) perturbatively close to the cavity for $\zeta \geq \zeta_q$. In order to do that, let us substitute in the master equation an expansion of the form:

$$W_{IR}^{approx} = C_0 + C_2 \left(\frac{\zeta}{\zeta_q} - 1 \right) + \sum_{i=4} \beta_i \left(\frac{\zeta}{\zeta_q} - 1 \right)^{\frac{i}{2}} . \tag{4.168}$$

Identifying the above approximate solution (4.168) and the analytic solution (4.14) at the edge of the cavity $\zeta = \zeta_q$ we get the following two constraints

$$C_0 \,=\, C \left(1 - \frac{b^6}{\zeta_q^6} \right) \quad , \quad C_2 \,=\, 6\, C\, \frac{b^6}{\zeta_q^6} . \tag{4.169}$$

The first non-vanishing coefficients are

$$\beta_4 = -\frac{7}{2}\, C_2 \ , \qquad \beta_5 = -\frac{12\sqrt{2}}{5\sqrt{C_0}}\, \frac{Q_f}{\zeta_q} \ , \qquad \beta_6 = \frac{28}{3}\, C_2 \ , \tag{4.170}$$

$$\beta_7 = \frac{3\sqrt{2}\,(109\, C_0 \,+\, 6\, C_2)}{35\, C_0^{3/2}}\, \frac{Q_f}{\zeta_q} \ , \qquad \beta_8 = -\,21\, C_2 \ . \tag{4.171}$$

The function f corresponding to (4.168) is:

$$\begin{aligned} \frac{e^f}{\zeta_q \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-\frac{1}{2}}} \;=\; & 1 + \left(1 + \frac{1}{2}\frac{C_2}{C_0} \right) \left(\frac{\zeta}{\zeta_q} - 1 \right) + \frac{1}{\sqrt{2}\, C_0^{\frac{3}{2}}} \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-1} \frac{Q_f}{\zeta_q} \left(\frac{\zeta}{\zeta_q} - 1 \right)^{\frac{3}{2}} \\ & - \frac{5}{4}\frac{C_2}{C_0} \left(1 + \frac{1}{10}\frac{C_2}{C_0} \right) \left(\frac{\zeta}{\zeta_q} - 1 \right)^2 + \ldots \ , \end{aligned} \tag{4.172}$$

and the dilaton can be expanded as:

$$\begin{aligned} e^{\phi} \;=\; & C_0^{-1} \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-1} + \frac{\sqrt{2}}{C_0^{\frac{5}{2}}} \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-2} \frac{Q_f}{\zeta_q} \left(\frac{\zeta}{\zeta_q} - 1 \right)^{\frac{3}{2}} - \frac{41\, C_0 \,+\, 6\, C_2}{10\sqrt{2}\, C_0^{\frac{7}{2}}} \,\cdot \\ & \cdot \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-2} \frac{Q_f}{\zeta_q} \left(\frac{\zeta}{\zeta_q} - 1 \right)^{\frac{5}{2}} + \frac{2}{C_0^4} \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-3} \frac{Q_f^2}{\zeta_q^2} \left(\frac{\zeta}{\zeta_q} - 1 \right)^3 + \ldots \end{aligned} \tag{4.173}$$

Finally, plugging these expansions into (4.10) we can solve for the warp factor h, which is given by

$$\begin{aligned} h \;=\; & C_1 - \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right) \frac{Q_c}{\zeta_q^4} \left(\frac{\zeta}{\zeta_q} - 1 \right) - \frac{\sqrt{2}\, C_1}{C_0^{\frac{3}{2}}} \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right)^{-1} \frac{Q_f}{\zeta_q} \left(\frac{\zeta}{\zeta_q} - 1 \right)^{\frac{3}{2}} \\ & + \frac{5}{2} \left(1 + \frac{1}{5}\frac{C_2}{C_0} \right) \left(1 + \frac{1}{6}\frac{C_2}{C_0} \right) \frac{Q_c}{\zeta_q^4} \left(\frac{\zeta}{\zeta_q} - 1 \right)^2 + \ldots \ . \end{aligned} \tag{4.174}$$

The integration constant C_1 is connected with the integration constant coming from solving the warp factor differential equation inside the cavity.

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4.B Probe embeddings and profile function

In this appendix we study the embeddings of a D5-brane probe which preserves the supersymmetry of a background given by our ansatz. Once these embeddings are characterized we will be able to obtain the corresponding profile function $p(\zeta)$ which, in turn, is a necessary ingredient to solve the BPS equations and determine completely the different functions of the ansatz.

The supersymmetric embeddings we are looking for must satisfy the kappa symmetry condition:

$$\Gamma_\kappa \, \epsilon \,=\, \pm \epsilon \ , \tag{4.175}$$

where ϵ is a Killing spinor of the background. For a D5-brane without any worldvolume gauge field, Γ_κ is given by:

$$\Gamma_\kappa \,=\, \frac{1}{6!\sqrt{-g_6}} \, \epsilon^{\alpha_1 \cdots \alpha_6} \, \sigma_1 \, \gamma_{\alpha_1 \cdots \alpha_6} \ , \tag{4.176}$$

where g_6 is the determinant of the induced worldvolume metric, $\gamma_{\alpha_1 \cdots \alpha_6}$ is the antisymmetrized product of induced Dirac matrices and we are representing the Killing spinors of type IIB supergravity as a Majorana-Weyl doublet. We will take $\xi^\alpha \,=\, (x^0, x^1, x^2, r, \theta, \psi)$ as our set of worldvolume coordinates. In this case the kappa symmetry matrix takes the form:

$$\Gamma_\kappa \,=\, \frac{1}{\sqrt{-g_6}} \, \sigma_1 \, \gamma_{x^0 \, x^1 \, x^2 \, r \, \theta \, \psi} \ . \tag{4.177}$$

Recall that the Killing spinor of the background can be written as in (2.25) in terms of a constant spinor η which satisfies the projections (2.128) and (2.129). We can cast the kappa symmetry condition (4.175) as the following condition on η:

$$\tilde{\Gamma}_\kappa \, \eta \,=\, \pm \eta \ , \tag{4.178}$$

where $\tilde{\Gamma}_\kappa$ is defined as the matrix:

$$\tilde{\Gamma}_\kappa \,\equiv\, e^{-\frac{3}{2}\Gamma_{12}\,\tau} \, \Gamma_\kappa \, e^{\frac{3}{2}\Gamma_{12}\,\tau} \ . \tag{4.179}$$

We will consider an embedding in which x^3 and φ are constant, while

$$\chi = \chi(r) \ , \qquad\qquad \tau = \tau(\psi) \ . \tag{4.180}$$

The induced γ-matrices on the worldvolume for this embedding ansatz are:

$$\begin{aligned}
\gamma_{x^\mu} \,&=\, h^{-\frac{1}{4}} \, \Gamma_{x^\mu} \ , \qquad\qquad (\mu = 0,1,2) \\
\gamma_r \,&=\, h^{\frac{1}{4}} \, \Gamma_r \,+\, \frac{1}{2} \, h^{\frac{1}{4}} \, e^g \, \chi' \, \Gamma_4 \\
\gamma_\theta \,&=\, \frac{1}{2} \, h^{\frac{1}{4}} \, e^g \, \cos\frac{\chi}{2} \, \cos\psi \, \Gamma_1 \,+\, \frac{1}{2} \, h^{\frac{1}{4}} \, e^g \cos\frac{\chi}{2} \, \sin\psi \, \Gamma_2 \\
\gamma_\psi \,&=\, \frac{1}{2} \, h^{\frac{1}{4}} \, e^g \, \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \Gamma_3 \,+\, h^{\frac{1}{4}} \, e^f \left(\frac{1}{2} \cos^2\frac{\chi}{2} \,+\, \dot{\tau} \right) \Gamma_5 \ ,
\end{aligned} \tag{4.181}$$

where we have denoted

$$\chi' = \frac{d\chi}{dr} \, , \qquad \dot{\tau} = \frac{d\tau}{d\psi} \, , \tag{4.182}$$

and the Γ's are constant 10-dimensional Dirac matrices. From these induced matrices we can compute the antisymmetrized product $\gamma_{r\theta\psi}$ and get an expression of the type:

$$\gamma_{r\theta\psi} = \frac{h^{\frac{3}{4}} e^g}{2} \cos\frac{\chi}{2} \Big[c_1 \, \Gamma_{r13} + c_2 \, \Gamma_{r23} + c_3 \, \Gamma_{413} + c_4 \, \Gamma_{423} + c_5 \, \Gamma_{r15} + c_6 \, \Gamma_{r25} + c_7 \Gamma_{415} + c_8 \Gamma_{425} \Big] \, , \tag{4.183}$$

where the different coefficients are given by

$$\begin{aligned}
c_1 &= \frac{e^g}{2} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \cos\psi \, , \; c_2 = \frac{e^g}{2} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \sin\psi \, , \\
c_3 &= \frac{e^{2g}}{4} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \cos\psi \, \chi' \, , \; c_4 = \frac{e^{2g}}{4} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \sin\psi \, \chi' \, , \\
c_5 &= e^f \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) \cos\psi \, , \; c_6 = e^f \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) \sin\psi \, , \\
c_7 &= \frac{e^{f+g}}{2} \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) \cos\psi \, \chi' \, , \; c_8 = \frac{e^{f+g}}{2} \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) \sin\psi \, \chi' \, .
\end{aligned} \tag{4.184}$$

This expression can be rewritten as:

$$\gamma_{r\theta\psi} = \frac{h^{\frac{3}{4}} e^g}{2} \cos\frac{\chi}{2} \, e^{-\psi\Gamma_{12}} \Big[d_1 \, \Gamma_{r13} + d_2 \, \Gamma_{413} + d_3 \, \Gamma_{r15} + d_4 \Gamma_{415} \Big] \, , \tag{4.185}$$

where the coefficients d_i are:

$$\begin{aligned}
d_1 &= \frac{e^g}{2} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, , \qquad & d_2 &= \frac{e^{2g}}{4} \cos\frac{\chi}{2} \, \sin\frac{\chi}{2} \, \chi' \, , \\
d_3 &= e^f \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) , \qquad & d_4 &= \frac{e^{f+g}}{2} \left(\frac{1}{2} \cos^2\frac{\chi}{2} + \dot{\tau} \right) \chi' \, .
\end{aligned} \tag{4.186}$$

In order to compute the form of Γ_κ, we use that

$$\gamma_{x^0 x^1 x^2} = h^{-\frac{3}{4}} \, \Gamma_{x^0 x^1 x^2} \, . \tag{4.187}$$

We find

$$\Gamma_\kappa = \frac{1}{2} \frac{e^g \, \cos\frac{\chi}{2}}{\sqrt{-g_6}} \, e^{-\psi\Gamma_{12}} \, \sigma_1 \, \Gamma_{x^0 x^1 x^2} \Big[d_1 \, \Gamma_{r13} + d_2 \, \Gamma_{413} + d_3 \, \Gamma_{r15} + d_4 \, \Gamma_{415} \Big] \tag{4.188}$$

Let us now calculate the matrix $\tilde{\Gamma}_\kappa$. As

$$\{\Gamma_{12}, \Gamma_{r13}\} = \{\Gamma_{12}, \Gamma_{413}\} = \{\Gamma_{12}, \Gamma_{r15}\} = \{\Gamma_{12}, \Gamma_{415}\} = 0 \, , \tag{4.189}$$

we have that

$$\Gamma_\kappa\, e^{\frac{3}{2}\Gamma_{12}\,\tau} \;=\; e^{-\frac{3}{2}\Gamma_{12}\,\tau}\,\Gamma_\kappa \tag{4.190}$$

and, therefore:

$$\tilde{\Gamma}_\kappa \;=\; \frac{1}{2}\,\frac{e^g\,\cos\frac{\chi}{2}}{\sqrt{-g_6}}\; e^{-(\psi+3\tau)\Gamma_{12}}\,\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\Big[d_1\,\Gamma_{r13} \,+\, d_2\,\Gamma_{413} \,+\, d_3\,\Gamma_{r15} \,+\, d_4\,\Gamma_{415}\Big]\;. \tag{4.191}$$

We now study how the different terms of $\tilde{\Gamma}_\kappa$ act on η. We begin by considering the action of the first term on the right-hand side of (4.191). First of all, we notice that, using the 10-dimensional chirality condition satisfied by ϵ and the projections (2.128) and (2.129), one can show that η satisfies the following expressions, obtained in chapter 2:

$$\Gamma_{x^0\,x^1\,x^2}\,\eta \;=\; i\sigma_2\,\Gamma_{x^3}\,\eta\;. \tag{4.192}$$

Using (4.192) and the projection (2.128), we get:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{r13}\,\eta \;=\; \sigma_1\,\Gamma_{43}\,\Gamma_{14rx^3}\,\eta \;=\; i\sigma_2\,\Gamma_{34}\,\eta\;. \tag{4.193}$$

Finally, the projection (2.129) allows us to write:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{r13}\,\eta \;=\; -\eta\;. \tag{4.194}$$

Let us next consider the second term in $\tilde{\Gamma}_\kappa$. We first use:

$$\Gamma_{413} \;=\; -\Gamma_{r4}\,\Gamma_{r13} \tag{4.195}$$

from which it immediately follows that:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{413}\,\eta \;=\; \Gamma_{r4}\,\eta\;. \tag{4.196}$$

Similarly, since $\Gamma_{r15} = -\Gamma_{35}\,\Gamma_{r13}$, we get that the third term of (4.191) acts on η as:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{r15}\,\eta \;=\; \Gamma_{35}\,\eta\;. \tag{4.197}$$

But, according to the last set of projections in (2.129), we have $\Gamma_{35}\eta \;=\; -\Gamma_{r4}\eta$. Therefore:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{r15}\,\eta \;=\; -\Gamma_{r4}\,\eta\;. \tag{4.198}$$

Finally, as $\Gamma_{415} = -\Gamma_{r4}\,\Gamma_{r15}$, we get:

$$\sigma_1\,\Gamma_{x^0\,x^1\,x^2}\,\Gamma_{415}\,\eta \;=\; -\eta\;. \tag{4.199}$$

Thus, collecting all these results, we have:

$$\tilde{\Gamma}_\kappa\,\eta \;=\; \frac{1}{2}\,\frac{e^g\,\cos\frac{\chi}{2}}{\sqrt{-g_6}}\; e^{-(\psi+3\tau)\Gamma_{12}}\Big[-d_1 - d_4 \,+\, (d_2-d_3)\,\Gamma_{r4}\Big]\,\eta\;. \tag{4.200}$$

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As $\tilde{\Gamma}_\kappa$ should act as (plus or minus) the identity matrix on η, we must have:

$$\sin(\psi + 3\tau) \,=\, 0 \;, \qquad\qquad d_3 \,=\, d_2 \;. \tag{4.201}$$

The first equation fixes that

$$\psi + 3\tau \,=\, n\,\pi \;, \tag{4.202}$$

with $n \in \mathbb{Z}$. This implies that

$$\dot{\tau} \,=\, -\frac{1}{3} \;. \tag{4.203}$$

In order to have τ having values within the range $[0, 2\pi]$, we choose $n = 4$ in (4.202). Therefore, the function $\tau = \tau(\psi)$ is:

$$\tau(\psi) \,=\, \frac{4\pi - \psi}{3} \;, \tag{4.204}$$

and the values of τ for the embedding range from $\tau = 0$ to $\tau = \frac{4\pi}{3}$. Moreover, the second equation in (4.201) implies the following first-order equation for χ:

$$\chi' \,=\, 2\,e^{f-2g}\,\frac{\cos^2\frac{\chi}{2} \,+\, 2\dot{\tau}}{\sin\frac{\chi}{2}\,\cos\frac{\chi}{2}} \;. \tag{4.205}$$

By making use of (4.203), this equation can be written as:

$$\chi' \,=\, \frac{2}{3}\,e^{f-2g}\,\frac{3\,\cos\chi - 1}{\sin\chi} \;. \tag{4.206}$$

The induced metric on the D5-brane worldvolume for our ansatz is:

$$\begin{aligned} ds_6^2 \,=\, & h^{-\frac{1}{2}}\left[\,-\,(dx^0)^2 + (dx^1)^2 + (dx^2)^2\,\right] + h^{\frac{1}{2}}\left[1 \,+\, \frac{e^{2g}}{4}\,(\chi')^2\,\right] dr^2 \,+ \\ & + \frac{h^{\frac{1}{2}}\,e^{2g}}{4}\cos^2\frac{\chi}{2}\,(d\theta)^2 + \frac{h^{\frac{1}{2}}\,e^{2g}}{4}\left[\,\sin^2\frac{\chi}{2}\cos^2\frac{\chi}{2} \,+\, e^{2f-2g}(\cos^2\frac{\chi}{2} + 2\dot{\tau})^2\,\right](d\psi)^2 \;, \end{aligned} \tag{4.207}$$

whose determinant is equal to:

$$\sqrt{-g_6} \,=\, \frac{e^{2g}\,\cos\frac{\chi}{2}}{4}\left[1 \,+\, \frac{e^{2g}}{4}\,(\chi')^2\right]^{\frac{1}{2}}\left[\sin^2\frac{\chi}{2}\,\cos^2\frac{\chi}{2} \,+\, e^{2f-2g}\,(\cos^2\frac{\chi}{2} \,+\, 2\dot{\tau})^2\right]^{\frac{1}{2}} \,. \tag{4.208}$$

Let us compute this determinant for the embeddings satisfying the BPS equations obtained by imposing kappa symmetry. First of all we notice that:

$$\left(1 \,+\, \frac{e^{2g}}{4}\,(\chi')^2\right)_{BPS} \,=\, \frac{\sin^2\frac{\chi}{2}\,\cos^2\frac{\chi}{2} \,+\, e^{2f-2g}\,(\cos^2\frac{\chi}{2} \,+\, 2\dot{\tau})^2}{\sin^2\frac{\chi}{2}\,\cos^2\frac{\chi}{2}} \;, \tag{4.209}$$

from which it follows that:

$$\sqrt{-g_6}\,\Big|_{BPS} \,=\, \frac{e^{2g}}{4\sin\frac{\chi}{2}}\left[\sin^2\frac{\chi}{2}\,\cos^2\frac{\chi}{2} \,+\, e^{2f-2g}\,(\cos^2\frac{\chi}{2} \,+\, 2\dot{\tau})^2\right] \,. \tag{4.210}$$

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Moreover, one can easily verify that:

$$\sqrt{-g_6}\Big|_{BPS} = \frac{e^g \cos\frac{\chi}{2}}{2}\left(d_1 + d_4\right)_{BPS} . \tag{4.211}$$

Plugging this result and the BPS equations (4.201) into (4.200), we get that:

$$\tilde{\Gamma}_\kappa \eta = -\cos(\psi + 3\tau)\,\eta = -\eta , \tag{4.212}$$

which shows that our brane configuration preserves the same amount of SUSY as the background.

4.B.1 General integration of the BPS equation

Let us now integrate the BPS equation (4.206) for χ. First of all we perform a change in the holographic variable and write (4.206) in terms of the coordinate ζ defined in (4.148). Using (4.146) we get:

$$\frac{\zeta}{2}\frac{d\chi}{d\zeta} = \frac{\cos\chi - \frac{1}{3}}{\sin\chi} . \tag{4.213}$$

This equation can be integrated as:

$$\cos\chi = \frac{1}{3} + \left(\frac{c}{\zeta}\right)^2 , \tag{4.214}$$

where c is an integration constant. By inspecting (4.214), it is clear that the coordinate ζ of the brane must be greater or equal than some minimal value. Let us denote by ζ_q this minimal value of ζ. Clearly, ζ_q and the constant c are related as:

$$\frac{1}{3} + \frac{c^2}{\zeta_q^2} = 1 , \tag{4.215}$$

or, equivalently:[2]

$$c^2 = \frac{2}{3}\,\zeta_q^2 . \tag{4.217}$$

In terms of ζ_q the embedding angle χ is given by:

$$\cos\chi = \frac{1}{3}\left[1 + 2\left(\frac{\zeta_q}{\zeta}\right)^2\right] . \tag{4.218}$$

[2]We are implicitly assuming that c^2 is positive. If we take $c^2 < 0$ in (4.214) the minimal value ζ_q of the coordinate ζ is achieved when $\cos\chi = -1$ and is given by $\zeta_q^2 = -\frac{3}{4}\,c^2$. Thus, the embedding function in this second branch can be written as:

$$\cos\chi = \frac{1}{3} - \frac{4}{3}\left(\frac{\zeta_q}{\zeta}\right)^2 . \tag{4.216}$$

For these embeddings $\chi = \pi$ at the tip of the brane and decreases when we move towards the UV region $\zeta \to \infty$, reaching the value $\chi = \chi_*$ asymptotically.

It follows from (4.218) that χ vanishes at the tip of the brane $\zeta = \zeta_q$ and grows as we move towards the UV, until it reaches an asymptotic value $\chi = \chi_*$, with χ_* given by:

$$\cos\chi_* \,=\, \frac{1}{3}\ . \tag{4.219}$$

Notice that $\chi = \chi_*$ is a constant angle solution of the BPS equation (4.213). For this $\chi = \chi_*$ embedding, the tip of the brane is at $\zeta_q = 0$ (see (4.218)). Therefore, the brane reaches the origin of the geometry, as it corresponds to having massless quarks.

4.B.2 The profile function

To determine the function $p(r)$ for a set of branes ending at the same distance from the origin, *i.e.*, with the same mass, we compare the smeared and localized brane actions. Let us start with the WZ term, for which the smeared distribution is:

$$S_{WZ}^{smeared} = T_5\int_{\mathcal{M}_{10}} C_6\wedge\Xi\ , \tag{4.220}$$

where C_6 is the RR 6-form potential given by:

$$C_6 \,=\, e^{\frac{\phi}{2}}\,\mathcal{K}_6\ , \tag{4.221}$$

with $\mathcal{K}_6$ being the calibration 6-form written in the second chapter. In terms of the 1-forms E^a of our vielbein basis (4.143), $\mathcal{K}_6$ is given by:

$$\mathcal{K}_6 \,=\, E^{x^0\,x^1\,x^2}\wedge\,\mathrm{Re}\,\Omega\ , \tag{4.222}$$

where Ω is the following complex 3-form:

$$\Omega \,=\, e^{3i\tau}\,\big(E^1+iE^2\big)\wedge\big(E^3+iE^4\big)\wedge\big(E^r+iE^5\big)\ . \tag{4.223}$$

In (4.220) Ξ is the smearing 4-form for the massive case, $dF_3 = 2\kappa_{10}^2\,\Xi$, which, according to (4.4), is

$$2\,\kappa_{10}^2\,T_5\,\Xi \,=\, -Q_f\,\Big[3\,p(r)\,dx^3\wedge\mathrm{Re}\,\hat\Omega_2\wedge(d\tau+A)\,+\,p'(r)\,dx^3\wedge dr\wedge\mathrm{Im}\,\hat\Omega_2\Big]\ . \tag{4.224}$$

Therefore:

$$S_{WZ}^{smeared} \,=\, -\frac{Q_f}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} e^{\frac{\phi}{2}}\,\mathcal{K}_6\,\wedge dx^3\wedge\big[3\,p(r)\wedge\mathrm{Re}\,\hat\Omega_2\wedge(d\tau+A)\,+\,p'(r)\,\wedge dr\wedge\mathrm{Im}\,\hat\Omega_2\big]\ . \tag{4.225}$$

Integrating in (4.225) over the angular variables and x^3 for $-\frac{L_3}{2}\le x^3\le\frac{L_3}{2}$, we get:

$$S_{WZ}^{smeared} \,=\, \int dx^0\,dx^1\,dx^2\;dr\,\mathcal{L}_{WZ}^{smeared}\ , \tag{4.226}$$

where the smeared WZ Lagrangian density is:

$$\mathcal{L}_{WZ}^{smeared} \,=\, \frac{\pi^3\,Q_f\,L_3}{\kappa_{10}^2}\,e^{\frac{\phi}{2}+2g}\,\big(3\,p(r)\,+\,e^f\,p'(r)\big)\ . \tag{4.227}$$

We now compare (4.227) with the WZ action of a localized single D5-brane multiplied by N_f. As all the flavor branes that form the set of our smeared distribution are identical, these two quantities should be equal. The WZ term of the action of a D5-brane without worldvolume gauge field is:

$$S_{WZ} \,=\, T_5\int \hat{C}_6 \,\equiv\, \int d^6\xi\,\mathcal{L}_{WZ}\ , \tag{4.228}$$

where the hat denotes pullback to the worldvolume and C_6 has been written in (4.221) in terms of the calibration form of (4.222). Let us suppose that we take $\zeta^\alpha = (x^0, x^1, x^2, r, \theta, \psi)$ as worldvolume coordinates and consider an embedding of the type (4.180) with x^3 and φ constant. Then, the pullback of C_6 is:

$$\hat{C}_6 \,=\, \frac{e^{\frac{\phi}{2}+2g}}{16}\,\cos\frac{\chi}{2}\,\cos(3\tau+\psi)\,\big(2\sin\chi+e^f\,(1+\cos\chi+4\dot{\tau})\,\chi'\big)\,d^3x\wedge dr\wedge d\theta\wedge d\psi\ , \tag{4.229}$$

where $d^3x \equiv dx^0\wedge dx^1\wedge dx^2$. Let us next assume that $\tau(\psi)$ is given by (4.204) and let us integrate the action over θ and ψ. The resulting Lagrangian density (to be integrated over (x^0, x^1, x^2, r)) multiplied by N_f is:

$$\frac{\pi^2\,N_f\,T_5}{4}\,e^{\frac{\phi}{2}+2g}\,\cos\frac{\chi}{2}\,\Big[2\sin\chi\,+\,e^f\,\big(\cos\chi-\frac{1}{3}\big)\,\chi'\Big]\ . \tag{4.230}$$

Let us now compare the terms in (4.227) and (4.230) without e^f. They are equal provided the profile $p(r)$ is given by:

$$p(r) \,=\, \frac{\kappa_{10}^2\,T_5}{6\pi}\,\frac{N_f}{Q_f\,L_3}\,\cos\frac{\chi(r)}{2}\,\sin\chi(r)\ . \tag{4.231}$$

When $r\to\infty$ the angle $\chi\to\chi_*$, where χ_* is the value written in (4.219). Since $\cos\frac{\chi_*}{2}\,\sin\chi_* = \frac{4}{3\sqrt{3}}$, we have:

$$p(r\to\infty) \,=\, \frac{2\,\kappa_{10}^2\,T_5}{9\sqrt{3}\,\pi}\,\frac{N_f}{Q_f\,L_3}\ . \tag{4.232}$$

As argued on general grounds in section 4.2.3, we should have $p(r\to\infty) = 1$, *i.e.*, the profile should approach the one corresponding to massless quarks in the far UV. This only occurs when Q_f and N_f are related as:

$$Q_f \,=\, \frac{4\pi\,g_s\,\alpha'}{9\sqrt{3}}\,\frac{N_f}{L_3}\ . \tag{4.233}$$

This same conclusion can also be reached by comparing (4.227) for $p = 1$ and (4.230) for $\chi = \chi_*$. It follows that the function $p(r)$ is related to the function $\chi(r)$ for our fiducial embedding as:

$$p(r) \;=\; \frac{3\sqrt{3}}{4}\,\cos\frac{\chi(r)}{2}\,\sin\chi(r)\;. \tag{4.234}$$

Moreover, by equating the terms proportional to e^f in (4.227) and (4.230) we conclude that $p(r)$ should satisfy the differential equation:

$$p'(r) \;=\; \frac{9\sqrt{3}}{4}\,\cos\frac{\chi}{2}\,\Big(\cos^2\frac{\chi}{2} \;-\; \frac{2}{3}\Big)\,\chi'\;. \tag{4.235}$$

By computing the derivative of the right-hand side of (4.234) one can easily show that indeed p' satisfies (4.235).

Let us now check the equality of the DBI terms in the smeared and microscopic approach. The smeared DBI action is:

$$S_{DBI}^{smeared} \;=\; -T_5\int_{\mathcal{M}_{10}} e^{\phi/2}\mathcal{K}_6\wedge\Xi \;=\; \int dx^0\,dx^1\,dx^2\;dr\,\mathcal{L}_{WZ}^{smeared}\;, \tag{4.236}$$

where the smeared DBI Lagrangian density $\mathcal{L}_{DBI}^{smeared}$ is:

$$\mathcal{L}_{DBI}^{smeared} \;=\; -\frac{\pi^3\,Q_f\,L_3}{\kappa_{10}^2}\,e^{\frac{\phi}{2}+2g}\Big[3\,p(r) \;+\; e^f\,p'(r)\Big] \;=\; -\mathcal{L}_{WZ}^{smeared}\;. \tag{4.237}$$

Let us compare the action (4.236) to the DBI action of the fiducial brane multiplied by N_f, which is equal to:

$$-N_f\,T_5\int d^6\xi\,\sqrt{g_6}\Big|_{BPS}\;. \tag{4.238}$$

Taking into account (4.210), we get that (4.237) should be equal to:

$$-\pi^2\,N_f\,T_5\,e^{\frac{\phi}{2}+2g}\;\frac{\sin^2\frac{\chi}{2}\,\cos^2\frac{\chi}{2} \;+\; e^{2f-2g}\,(\cos^2\frac{\chi}{2} \;+\; 2\dot{\tau})^2}{\sin\frac{\chi}{2}}\;. \tag{4.239}$$

By identifying the terms without e^f in (4.237) and (4.239) we get exactly (4.234), whereas the terms with e^f yield:

$$e^{2g-f}\,p'(r) \;=\; \frac{9\sqrt{3}}{2}\,\frac{\Big(\cos^2\frac{\chi}{2} \;-\; \frac{2}{3}\Big)^2}{\sin\frac{\chi}{2}}\;, \tag{4.240}$$

which can be shown to be equivalent to (4.235) after using (4.206).

Let us finally obtain the explicit form of the profile in terms of the radial variable ζ. First of all, we write p as:

$$p \;=\; \frac{3\sqrt{3}}{2}\,\cos^2\frac{\chi}{2}\,\sqrt{1-\cos^2\frac{\chi}{2}}\;. \tag{4.241}$$

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Using (4.218), we get:[3]

$$p(\zeta) \,=\, \Big[1-\big(\frac{\zeta_q}{\zeta}\big)^2\Big]^{\frac{1}{2}}\,\Big[1\,+\,\frac{1}{2}\,\big(\frac{\zeta_q}{\zeta}\big)^2\Big]\;\Theta(\zeta-\zeta_q)\;. \tag{4.243}$$

Notice that $p(\zeta)$ is continuous at $\zeta=\zeta_q$, as it should.

4.C Equations of motion of the probe

The DBI action, in Einstein frame, of a D5-brane with a worldvolume gauge field F turned on is:

$$S_{DBI} \,=\, -T_5\int d^6\xi\, e^{\frac{\phi}{2}}\sqrt{-\det(g_6+e^{-\frac{\phi}{2}}\,F)} \,\equiv\, \int d^6\xi\,\mathcal{L}_{DBI}\;. \tag{4.244}$$

We are interested in having a gauge potential A dual to a $U(1)$ charge density. Therefore, we take

$$A \,=\, A_0(r)\,dt\;, \tag{4.245}$$

for which $F=dA$ is given by:

$$F \,=\, A_0'\,dr\wedge dx^0\;. \tag{4.246}$$

Let us choose $\xi^\alpha \,=\, (x^0,x^1,x^2,r,\theta,\psi)$ as our set of worldvolume coordinates and let us assume an embedding ansatz as in (4.180), *i.e.*, with $\chi\,=\,\chi(r)$ and $\tau\,=\,\tau(\psi)$. The Lagrangian density $\mathcal{L}_{DBI}$ corresponding to the action (4.244) is:

$$\mathcal{L}_{DBI} \,=\, -\frac{T_5}{8}\,e^{\frac{\phi}{2}+2g}\,\cos\frac{\chi}{2}\,\sqrt{\sin^2\chi\,+\,e^{2f-2g}\,(1+\cos\chi+4\dot\tau)^2}\,\sqrt{1\,-\,e^{-\phi}\,(A_0')^2\,+\,\frac{e^{2g}}{4}\,(\chi')^2}\;. \tag{4.247}$$

The WZ term of the action has been written in (4.228). Taking into account (4.229), we can easily demonstrate that $\mathcal{L}_{WZ}$ takes the form:

$$\mathcal{L}_{WZ} \,=\, \frac{T_5}{16}\,e^{\frac{\phi}{2}+2g}\,\cos\frac{\chi}{2}\,\cos(3\tau+\psi)\,\big(2\sin\chi+e^f\,(1+\cos\chi+4\dot\tau)\,\chi'\big)\;. \tag{4.248}$$

Let us now analyze the equations of motion of $\chi(r)$ and $\tau(\psi)$ derived from the total Lagrangian $\mathcal{L}\,=\,\mathcal{L}_{DBI}\,+\,\mathcal{L}_{WZ}$. The equation of motion of $\chi(r)$ derived from the total action is:

$$\frac{\partial}{\partial r}\Big[\frac{\partial\mathcal{L}_{DBI}}{\partial\chi'}\,+\,\frac{\partial\mathcal{L}_{WZ}}{\partial\chi'}\Big]\,-\,\frac{\partial\mathcal{L}_{DBI}}{\partial\chi}\,-\,\frac{\partial\mathcal{L}_{WZ}}{\partial\chi}\,=\,0\;. \tag{4.249}$$

[3] For the second branch of embeddings with fiducial embedding function given by (4.216), the profile $p(\zeta)$ can be obtained by substituting (4.216) into (4.241), with the result:

$$p(\zeta) \,=\, \Big[1\,-\,\Big(\frac{\zeta_q}{\zeta}\Big)^2\Big]\sqrt{1\,+\,2\Big(\frac{\zeta_q}{\zeta}\Big)^2}\;\Theta(\zeta-\zeta_q)\;. \tag{4.242}$$

The derivatives with respect to χ' appearing in (4.249) are:

$$\begin{aligned}
\frac{\partial \mathcal{L}_{DBI}}{\partial \chi'} &= -\frac{T_5}{32}\, e^{\frac{\phi}{2}+4g}\, \cos\frac{\chi}{2}\, \frac{\sqrt{\sin^2\chi \,+\, e^{2f-2g}\,(1+\cos\chi+4\dot{\tau})^2}}{\sqrt{1 \,-\, e^{-\phi}\,(A_0')^2 \,+\, \frac{e^{2g}}{4}\,(\chi')^2}}\, \chi' \\
\frac{\partial \mathcal{L}_{WZ}}{\partial \chi'} &= \frac{T_5}{16}\, e^{\frac{\phi}{2}+2g+f}\, \cos\frac{\chi}{2}\, \cos(3\tau+\psi)\,(1+\cos\chi+4\dot{\tau})\,.
\end{aligned} \tag{4.250}$$

Moreover, if we define the auxiliary functions

$$\begin{aligned}
\mathcal{J}_{DBI}(\chi) &\equiv \cos\frac{\chi}{2}\, \sqrt{\sin^2\chi \,+\, e^{2f-2g}\,(1+\cos\chi+4\dot{\tau})^2} \\
\mathcal{J}_{WZ}(\chi,\chi') &\equiv \cos\frac{\chi}{2}\, \Big(2\sin\chi + e^{f}\,(1+\cos\chi+4\dot{\tau})\,\chi'\Big)\,,
\end{aligned} \tag{4.251}$$

the derivatives with respect to χ we need in (4.249) are:

$$\begin{aligned}
\frac{\partial \mathcal{L}_{DBI}}{\partial \chi} &= -\frac{T_5}{8}\, e^{\frac{\phi}{2}+2g}\, \sqrt{1 \,-\, e^{-\phi}\,(A_0')^2 \,+\, \frac{e^{2g}}{4}\,(\chi')^2}\; \frac{\partial \mathcal{J}_{DBI}(\chi)}{\partial \chi} \\
\frac{\partial \mathcal{L}_{WZ}}{\partial \chi} &= \frac{T_5}{16}\, e^{\frac{\phi}{2}+2g}\, \cos(3\tau+\psi)\, \frac{\partial \mathcal{J}_{WZ}(\chi,\chi')}{\partial \chi}\,.
\end{aligned} \tag{4.252}$$

Let us analyze the dependence on the angular variable ψ of the different terms in (4.249). The only terms that depend on ψ in (4.249) are those that contain $\dot{\tau}$ and $3\tau+\psi$. By inspecting how these terms enter into (4.250) and (4.252), one easily concludes that the equation of motion of χ can only be satisfied if $\dot{\tau}$ and $3\tau+\psi$ are constant. This last condition implies that $\dot{\tau} = -1/3$, as in (4.203). Let us now look at the equation of motion for $\tau(\psi)$:

$$\frac{\partial}{\partial\psi}\Big[\frac{\partial \mathcal{L}_{DBI}}{\partial \dot{\tau}} + \frac{\partial \mathcal{L}_{WZ}}{\partial \dot{\tau}}\Big] - \frac{\partial \mathcal{L}_{DBI}}{\partial \tau} - \frac{\partial \mathcal{L}_{WZ}}{\partial \tau} = 0\,. \tag{4.253}$$

Let us study the different terms in (4.253). When $\dot{\tau}$ is constant, we have

$$\frac{\partial}{\partial\psi}\Big[\frac{\partial \mathcal{L}_{DBI}}{\partial \dot{\tau}}\Big] = 0\,. \tag{4.254}$$

Moreover

$$\frac{\partial}{\partial\psi}\Big[\frac{\partial \mathcal{L}_{WZ}}{\partial \dot{\tau}}\Big] = -\frac{T_5}{4}\, e^{\frac{\phi}{2}+2g}\, \cos\frac{\chi}{2}\, \sin(3\tau+\psi)\,(3\dot{\tau}+1)\,\chi'\,, \tag{4.255}$$

which vanishes if (4.203) holds. The DBI Lagrangian does not depend on τ, thus:

$$\frac{\partial \mathcal{L}_{DBI}}{\partial \tau} = 0\,. \tag{4.256}$$

The remaining τ derivative that we have to compute is:

$$\frac{\partial \mathcal{L}_{WZ}}{\partial \tau} = -\frac{3T_5}{16}\, e^{\frac{\phi}{2}+2g}\, \cos\frac{\chi}{2}\, \sin(3\tau+\psi)\, \Big(2\sin\chi + e^{f}\,(1+\cos\chi+4\dot{\tau})\,\chi'\Big)\,, \tag{4.257}$$

which vanishes independent of χ if $\sin(3\tau + \psi) = 0$, *i.e.*, when τ depends on ψ as in (4.202). As in the supersymmetric solution, we will take $n = 4$ in this equation.

The conclusion of this analysis is that, for the type of ansatz we are considering, the function $\tau(\psi)$ must be given by (4.204) in order to satisfy the equations of motion of the probe D5-brane. We will now study these equations separately in two different cases.

4.C.1 The BPS solution

Let us consider now the BPS configuration in which $\tau(\psi)$ is given by (4.204), A_0 vanishes and $\chi(r)$ satisfies the first-order differential equation (4.206) dictated by kappa symmetry. By using (4.209) to evaluate the different derivatives of (4.250), (4.252), and (4.257) one can easily show that:

$$\frac{\partial \mathcal{L}}{\partial \chi'}\bigg|_{BPS} = \frac{\partial \mathcal{L}}{\partial \chi}\bigg|_{BPS} = 0 \, , \qquad \frac{\partial \mathcal{L}}{\partial \dot{\tau}}\bigg|_{BPS} = \frac{\partial \mathcal{L}}{\partial \tau}\bigg|_{BPS} = 0 \, , \tag{4.258}$$

which implies the fullfilment of the equations of motion for the BPS configuration.

4.C.2 The massless solution

Let us consider a massless embedding of the probe brane with non-zero chemical potential. Therefore, we will try to solve the equations of motion with χ constant. It is clear from the first equation in (4.250) that $\partial\mathcal{L}_{DBI}/\partial\chi'$ vanishes if χ' is zero. Moreover, from the second equation in (4.250) we conclude that $\partial\mathcal{L}_{WZ}/\partial\chi'$ is zero if $\chi = \chi_*$, where χ_* is the angle defined in (4.219). Moreover, since

$$\frac{\partial \mathcal{J}_{DBI}}{\partial \chi}\bigg|_{\chi=\chi_*} = \frac{\partial \mathcal{J}_{WZ}}{\partial \chi}\bigg|_{\chi=\chi_*} = 0 \, , \tag{4.259}$$

it follows that $\chi = \chi_*$ solves the equations of motion of the probe. Let us next define $\mathcal{J}_*$ as:

$$\mathcal{J}_* \equiv \mathcal{J}_{DBI}(\chi = \chi_*) = \frac{1}{2}\, \mathcal{J}_{WZ}(\chi = \chi_*) \, . \tag{4.260}$$

It follows that:

$$\mathcal{J}_* = \frac{4}{3\sqrt{3}} \, . \tag{4.261}$$

It remains to satisfy the equation of motion for A_0. The Lagrangian density for the gauge field is:

$$\mathcal{L} = -\mathcal{T}\, e^{\frac{\phi}{2}+2g} \left[\sqrt{1 - e^{-\phi}\, A_0'^2} - 1 \right] , \tag{4.262}$$

where the effective tension $\mathcal{T}$ is given by:

$$\mathcal{T} = \frac{T_5}{8}\, \mathcal{J}_* = \frac{T_5}{6\sqrt{3}} \, . \tag{4.263}$$

Since A_0 is a cyclic variable in the Lagrangian density (4.262), the equation of motion for A_0 can be integrated once, giving:

$$\frac{e^{2g-\frac{\phi}{2}}\,A_0'}{\sqrt{1-e^{-\phi}\,A_0'^2}} \,=\, d \;, \tag{4.264}$$

where d is a constant proportional to the charge density. From this equation we get:

$$A_0' \,=\, \frac{e^{\frac{\phi}{2}}\,d}{\sqrt{d^2+e^{4g}}} \;. \tag{4.265}$$

The chemical potential μ is just the value of A_0 at the boundary, and is given by the following integral:

$$\mu \,=\, d\int_0^\infty \frac{e^{\frac{\phi}{2}}}{\sqrt{d^2+e^{4g}}}\,dr \;. \tag{4.266}$$

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5

Spin-2 spectrum of Gaiotto-Maldacena geometries

5.1 Introduction

The purpose of this chapter is two-fold. First, we obtain the linearized equations of motion for the fluctuations of the type IIA supergravity fields around an arbitrary background. Second, we use these equations in the class of Gaiotto-Maldacena geometries described in section 1.10, and restrict our attention to solve a consistent truncation of these fluctuations, namely that of the spin-2 excitations of these geometries.

As explained in chapter 1, Gaiotto-Maldacena geometries are characterized by a function that satisfies a Laplace equation. Each geometry (and the corresponding field theory dual) is characterized by a particular solution to this equation. Our analysis will lead us to deal with a second-order differential equation which coincides with the analysis in [188] (see also [98, 99, 127, 189, 190] for similar studies of the spin-2 excitations). We provide a generic expression for the wave operator given in terms of the function that solves the axisymmetric Laplace equation. We use the above operator to study the spectrum of two interesting examples. The first of them is the Abelian (Hopf) T-dual (ATD) of the $AdS_5 \times S^5$ solution [191, 192]. Regardless this solution does not satisfy the appropriate boundary conditions for the axisymmetric Laplacian, it is still a good solution. The second example is the Sfetsos-Thompson solution [15] obtained after applying NATD along the $SU(2)$ isometries inside the S^5 of the maximally supersymmetric solution in the type IIB supergravity. Although this solution defines a singular geometry, it has some interesting properties that make it stand out from others belonging to this class of geometries. For instance, it was shown to be an integrable background [193] as opposed to the generic "smooth" non-singular solutions of the large class of Gaiotto-Maldacena geometries [194, 195]. A detailed study of the field theory dual of the Sfetsos-Thompson solution including a completion to the geometry can be found in [192]. It is worth noticing that the ATD solution can be obtained as a limiting case of the NATD one[1] [192]. We also see this relation at the level of the operators describing the spin-2 excitations. Using holography we can then shed some light towards the understanding of the operator spectrum of the dualized solutions. In the BMN limit this problem was studied in [197].

To analyze the spin-2 spectrum of the solutions, we transformed the equation for the fluctuations into a Schrödinger-like problem of a particle in certain potential. The resulting potential is very similar to the one considered in the study of marginal deformations of supersymmetric backgrounds [198, 199]. In both ATD and NATD examples we were able to find analytic solutions only for the zero value of a quantum number n. For non-zero values of this quantum number we solved the equation numerically using the shooting method. For large masses we compared these solutions with a WKB analysis finding agreement. The analysis of the NATD solution shows a continuous spectrum of masses. This issue is associated with the fact that the "field space" coordinate ρ in the solution is unbounded. A discrete spectrum arises whenever we bound the value of this coordinate. This hard cut-off in the geometry is ascertained by placing D6-branes at a certain position ρ_* [192]. The situation with the ATD solution is very similar, though in this case the "field space" coordinate is periodic giving rise to a discrete spectrum.

The chapter is organized as follows. In section 2 we review the Gaiotto-Maldacena class of geometries. In section 3 we study the spin-2 excitations of these geometries by perturbing the metric components along the AdS_5 sector. We obtain a generic expression for the operator describing the fluctuations in terms of the function that solves the axisymmetric Laplace equation. In section 4 we study the spectrum of the ATD and NATD of $AdS_5 \times S^5$. In both cases we performed analytical as well as numerical methods to obtain the spectrum and its lower bound. In section 5 we summarize our results. We provide detailed appendices with the machinery needed to present the results of this chapter. Appendix A is a compendium of useful formulas to derive the fluctuation equations. Appendix B contains the equations for the fluctuations of all the fields in type IIA supergravity in the Einstein frame. Appendix C presents the ATD and NATD of $AdS_5 \times S^5$. Finally in appendix D we discuss the WKB approach that we used to give support to our numerical study of the spectrum in section 4.

5.2 Gaiotto-Maldacena backgrounds

The theories that we are going to deal with here are $\mathcal{N} = 2$ supersymmetric solutions of the type IIA supergravity, which have been found in [73] and whose metric in the Einstein

[1] Though the dilaton and the RR fields did not match, this issue was solved in [196].

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frame has the following form:

$$ds_E^2 = e^{-\frac{\Phi}{2}} f_0 \left(ds^2_{AdS_5} + f_1 d\Omega_2^2 + f_2 \big(d\eta^2 + d\sigma^2\big) + f_3 d\beta^2 \right), \tag{5.1}$$

with $d\Omega_2^2 = d\chi^2 + \sin^2\chi d\xi^2$ being the line element of a 2-sphere. The dilaton Φ and the functions f_i depend only on the coordinates (η, σ) and they can be expressed in terms of a function $V(\eta, \sigma)$ as:

$$\begin{aligned} &e^{2\Phi} = 2^7 \sigma \frac{\big(\ddot V - 2\dot V\big)^{3/2}}{\sqrt{\ddot V}\dot V \Delta}, \qquad \Delta = \big(\ddot V - 2\dot V\big)\frac{\ddot V}{\sigma^2} + \dot V'^2, \qquad f_0 = \sigma \sqrt{\frac{\ddot V - 2\dot V}{\ddot V}}, \\ &f_1 = -\frac{\ddot V \dot V}{2\sigma^2 \Delta}, \qquad f_2 = -\frac{\ddot V}{2\sigma^2 \dot V}, \qquad f_3 = \frac{\ddot V}{\ddot V - 2\dot V}. \end{aligned} \tag{5.2}$$

Notice that due to the normalizations we adopted here the AdS_5 space has a unit radius. Moreover, primed symbols correspond to derivatives with respect to η while dotted symbols correspond to the action of the operator $\sigma\partial_\sigma$.

The geometry of this class of solutions is supported by a NSNS 2-form and a set of RR potentials:

$$B_2 = \frac{1}{2}\Big(\frac{\dot V \dot V'}{\Delta} - \eta\Big)\mathrm{Vol}_{\Omega_2}, \qquad C_1 = \frac{1}{8}\frac{\dot V \dot V'}{2\dot V - \ddot V} d\beta, \qquad C_3 = \frac{1}{16}\frac{\dot V^2 \ddot V}{\sigma^2 \Delta} d\beta \wedge \mathrm{Vol}_{\Omega_2}, \tag{5.3}$$

where $\mathrm{Vol}_{\Omega_2} = \sin\chi d\chi \wedge d\xi$ is the volume form on the 2-sphere Ω_2 and the RR fields F_2 and F_4 are defined through the potentials C_1 and C_3 as $F_2 = dC_1$ and $F_4 = dC_3 + C_1 \wedge H_3$ with $H_3 = dB_2$.

As it is understood by the previous expressions, any background that fits into the Gaiotto-Maldacena classification is fully determined by the function $V(\eta, \sigma)$. However this function is not arbitrary but instead it has to satisfy the following second-order differential equation:

$$\ddot V + \sigma^2 V'' = 0. \tag{5.4}$$

In the section that follows we will study perturbations of the metric (5.1) along the AdS_5 directions.

5.3 Metric perturbations

In this section we look for a consistent truncation of the equations for the fluctuation of the supergravity fields. Before we start this analysis it is worth to mention some properties of the geometry of the solutions that we are interested in. The first property is that the metric in the Einstein frame is conformal to a direct sum of two 5-dimensional spaces. More specifically it is conformal to the sum of AdS_5 with a 5-dimensional internal space $\mathcal{M}_5$:

$$ds^2 = ds^2(AdS_5) + ds^2(\mathcal{M}_5). \tag{5.5}$$

Thus, it is useful to adopt the following notation for the indices:

$$\begin{aligned}
&M, N, P, K, \Lambda, \Sigma, \ldots : && \text{10-dimensional indices,}\\
&\mu, \nu, \rho, \kappa, \lambda, \sigma, \ldots : && \text{indices in } AdS_5,\\
&m, n, k, p, q, s, \ldots : && \text{indices in } \mathcal{M}_5.
\end{aligned}$$

Moreover we will consider the splitting of the coordinates $X^M = (x^\mu, y^m)$, where x are the coordinates in AdS_5 and y the coordinates in $\mathcal{M}_5$. The 10-dimensional line element (5.5) can be written as:

$$\tilde{g}_{MN}\, dX^M\, dX^N = \tilde{g}_{\mu\nu}\, dx^\mu\, dx^\nu + \tilde{g}_{mn}\, dy^m\, dy^n\,, \tag{5.6}$$

where the metric components $\tilde{g}_{\mu\nu}$ depend only on x and $\tilde{g}_{mn}$ only on y. Using matrix notation the metric reads:

$$\tilde{g}_{MN}(x, y) = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & \tilde{g}_{mn}(y) \end{pmatrix}. \tag{5.7}$$

The second important property of the geometries that we are going to consider is that the metric is diagonal. Next we would like to turn on only the fluctuations of the metric components along the AdS_5 sector, while the fluctuations of the rest of the fields are taken to be zero, i.e. we take into account only the following:

$$\delta g_{\mu\nu} = e^{2A} h_{\mu\nu}\,, \tag{5.8}$$

or

$$ds^2_E = e^{2A}\Big[\big(\tilde{g}_{\mu\nu} + h_{\mu\nu}\big)dx^\mu dx^\nu + \tilde{g}_{mn} dy^m dy^n\Big]. \tag{5.9}$$

In addition we take $h_{\mu\nu}$ to factorize as:

$$h_{\mu\nu}(x, y) = h^{[tt]}_{\mu\nu}(x) Y(y)\,, \tag{5.10}$$

where $h^{[tt]}_{\mu\nu}$ is transverse with respect to $\tilde{\nabla}^\mu$ and traceless, i.e.

$$\tilde{\nabla}^\mu h^{[tt]}_{\mu\nu} = 0\,, \qquad \tilde{g}^{\mu\nu} h^{[tt]}_{\mu\nu} = 0\,. \tag{5.11}$$

Under these considerations we see that the fluctuation of the dilaton equation and also those for the Maxwell equations are trivially satisfied. However from the Einstein equations we see that the only terms that contribute are:

$$\begin{aligned}
0 = {}& \frac{1}{2}\,\tilde{\nabla}^\Sigma \tilde{\nabla}_M h_{\Sigma N} + \frac{1}{2}\,\tilde{\nabla}^\Sigma \tilde{\nabla}_N h_{\Sigma M} - \frac{1}{2}\,\tilde{\nabla}^2 h_{MN} - 4\,\tilde{\nabla}^P A \tilde{\nabla}_P h_{MN} - h_{MN}\,\tilde{\nabla}^2 A \\
& - 8\, h_{MN}\,\big(\tilde{\nabla} A\big)^2 + \frac{1}{2} h_{MN} \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A + \alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\,,
\end{aligned} \tag{5.12}$$

where the notation of $\tilde{\mathcal{A}}_p^2$ is explained in appendix 5.B.2. In order to simplify the above expression we took into account the structure of the background fields (coordinate dependence and index structure) of the Gaiotto-Maldacena solutions. This can be further simplified if we change the order of the covariant derivatives of the first two terms. Using (5.69), we get:

$$\tilde{\nabla}^{\Sigma}\tilde{\nabla}_M h_{\Sigma N} = \tilde{\nabla}_M \tilde{\nabla}^{\Sigma} h_{\Sigma N} + \tilde{g}^{PP'}\, \tilde{R}_{P'M}\; h_{PN} - \tilde{g}^{K\Sigma}\, \tilde{R}^{P}{}_{NKM}\; h_{\Sigma P}\,. \tag{5.13}$$

The first term vanishes due to the transversality condition. Now the Einstein equation becomes:

$$\begin{aligned} 0 = {} & \tilde{g}^{\rho\sigma}\, \tilde{R}_{\sigma\mu}\; h_{\rho\nu} - \tilde{g}^{\kappa\sigma}\, \tilde{R}^{\rho}{}_{\nu\kappa\mu}\; h_{\sigma\rho} + \tilde{g}^{\rho\sigma}\, \tilde{R}_{\sigma\nu}\; h_{\rho\mu} - \tilde{g}^{\kappa\sigma}\, \tilde{R}^{\rho}{}_{\mu\kappa\nu}\; h_{\sigma\rho} - \tilde{\nabla}^2 h_{\mu\nu} \\ & - 8\,\tilde{\nabla}^P A \tilde{\nabla}_P h_{\mu\nu} - 2\, h_{\mu\nu}\, \tilde{\nabla}^2 A - 16\, h_{\mu\nu} \left(\tilde{\nabla} A\right)^2 + h_{\mu\nu} \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A + \alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\,. \end{aligned} \tag{5.14}$$

We recall that the Riemann and Ricci tensors of the AdS_5 of unit radius are:

$$\tilde{R}_{\mu\nu\rho\sigma} = \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho} - \tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} \qquad \Rightarrow \qquad \tilde{R}_{\nu\sigma} = -4\, \tilde{g}_{\nu\sigma}\,. \tag{5.15}$$

So

$$\tilde{g}^{\rho\sigma}\, \tilde{R}_{\sigma\mu}\; h_{\rho\nu} = -4\; h_{\mu\nu}\,, \qquad \tilde{g}^{\kappa\sigma}\, \tilde{R}^{\rho}{}_{\nu\kappa\mu}\; h_{\sigma\rho} = h_{\mu\nu}\,. \tag{5.16}$$

Therefore, the equation that we need to solve is:

$$0 = \tilde{\nabla}^2 h_{\mu\nu} + 10\; h_{\mu\nu} + 8\,\tilde{\nabla}^P A \tilde{\nabla}_P h_{\mu\nu} + h_{\mu\nu} \left[2\,\tilde{\nabla}^2 A + 16\left(\tilde{\nabla} A\right)^2 - \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A + \alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\right]. \tag{5.17}$$

It turns out that, for all the solutions that belong to the Gaiotto-Maldacena class, the term in square brackets equals to -8, thus the equation that we have to solve is:

$$0 = \tilde{\nabla}^{\sigma}\tilde{\nabla}_{\sigma} h_{\mu\nu} + 2\; h_{\mu\nu} + \tilde{\nabla}^m \tilde{\nabla}_m h_{\mu\nu} + 8\,\tilde{\nabla}^m A \tilde{\nabla}_m h_{\mu\nu}\,. \tag{5.18}$$

Notice that the last two terms can be written as:

$$\tilde{\nabla}^m \tilde{\nabla}_m h_{\mu\nu} + 8\,\tilde{\nabla}^m A \tilde{\nabla}_m h_{\mu\nu} = e^{-8A}\tilde{\nabla}^m \left[e^{8A}\tilde{\nabla}_m h_{\mu\nu}\right] := \mathcal{L}(h_{\mu\nu})\,. \tag{5.19}$$

This has exactly the same form as the operator $\mathcal{L}^{(1)}$ in [98]. Also, since the indices of $h_{\mu\nu}$ are along the AdS_5 subspace we understand that $h_{\mu\nu}$ behaves like a scalar for the covariant derivative $\tilde{\nabla}_m$ and the operator $\mathcal{L}$. Now the action of $\mathcal{L}$ on a scalar f can be written as:

$$\mathcal{L}(f) = \tilde{\nabla}^m \tilde{\nabla}_m f + 8\; \tilde{\nabla}^m A\; \tilde{\nabla}_m f = \frac{1}{\sqrt{\tilde{g}_{\mathcal{M}_5}}} \partial_m \left(\sqrt{\tilde{g}_{\mathcal{M}_5}}\; \tilde{g}^{mn}\; \partial_n f\right) + 8\; \tilde{g}^{mn}\; \partial_m A\; \partial_n f\;. \tag{5.20}$$

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Moreover, the equation (5.18) can be recognized as the equation of motion of a massive graviton of mass M propagating in AdS_5 [200, 201]. This is given by the Pauli-Fierz equation:

$$0 = \tilde{\nabla}^{\sigma}\tilde{\nabla}_{\sigma}h_{\mu\nu} + \left(2 - M^2\right)\, h_{\mu\nu}\,. \tag{5.21}$$

Using this, the equation (5.18) reduces to an eigenvalue problem for the operator $\mathcal{L}$:

$$\mathcal{L}(h_{\mu\nu}) = -M^2 h_{\mu\nu}\,. \tag{5.22}$$

In terms of the coordinates of the metric (5.1) the operator $\mathcal{L}$ has the following form:

$$\mathcal{L}(f) = \frac{1}{f_1}\nabla^2_{(2)}f + \frac{1}{f_3}\partial^2_{\beta}f + \frac{1}{\Delta f_0 f_1\sqrt{f_3}}\left[\partial_{\eta}\Big(\frac{\Delta f_0 f_1\sqrt{f_3}}{f_2}\partial_{\eta}f\Big) + \partial_{\sigma}\Big(\frac{\Delta f_0 f_1\sqrt{f_3}}{f_2}\partial_{\sigma}f\Big)\right], \tag{5.23}$$

where $\nabla^2_{(2)}$ is the Laplace operator on the 2-sphere Ω_2. This operator looks quite complicated and for this reason finding its eigenvalues for the general case of an arbitrary solution of (5.4) is a non-trivial task. However, as we will immediately see in the following section, one can focus on specific solutions of (5.4) which lead to a solvable eigenvalue problem for the operator $\mathcal{L}$.

5.4 The spin-2 spectrum

It is known [15, 192] that the ATD of $AdS_5 \times S^5$ as well as its NATD version are both examples of Gaiotto-Maldacena backgrounds. This fact motivates us to look for solutions of the eigenvalue problem (5.22) in the aforementioned two cases. This is feasible because as we will see shortly the operator $\mathcal{L}$ simplifies significantly in both examples. Let us see this in more detail. The details of these two geometries can be found in appendix 5.C.

Operator $\mathcal{L}$ in the ATD

The ATD solution is derived by the Gaiotto-Maldacena ansatz by choosing the potential $V(\eta,\sigma)$ to be [192]:

$$V^{ATD}(\eta,\sigma) = \ln\sigma - \frac{\sigma^2}{2} + \eta^2\,. \tag{5.24}$$

One has also to perform a change of coordinates in the following manner [2]:

$$\eta = 2\psi\,, \qquad \sigma = \sin\alpha\,. \tag{5.25}$$

As a result, we end up with a simple expression for the operator $\mathcal{L}$:

$$\mathcal{L}^{ATD}(f) = \partial^2_{\alpha}f + \big(\cot\alpha - 3\,\tan\alpha\big)\partial_{\alpha}f + \frac{1}{\sin^2\alpha}\partial^2_{\beta}f + \frac{\cos^2\alpha}{4}\,\partial^2_{\psi}f + \frac{4}{\cos^2\alpha}\nabla^2_{(2)}f\,, \tag{5.26}$$

where $\nabla^2_{(2)}$ is the Laplace operator on the 2-sphere $\Omega_2(\chi,\xi)$.

[2] In this notation ψ is the coordinate associated to the T-duality transformation.

Operator $\mathcal{L}$ in the NATD

In the NATD case the potential $V(\eta, \sigma)$ reads [15]:

$$V^{NATD}(\eta, \sigma) = \eta\Big(\ln \sigma - \frac{\sigma^2}{2} \Big) + \frac{\eta^3}{3} \, . \tag{5.27}$$

The change of coordinates that gives the NATD solution is the same as in equation (5.25) where now we are going to rename ψ by ρ [3]. The operator $\mathcal{L}$ in this case is:

$$\begin{aligned} \mathcal{L}^{NATD}(f) = \partial_\alpha^2 f + \ \big(\cot\alpha - 3\ \tan\alpha\big)\partial_\alpha f + \frac{1}{\sin^2\alpha}\partial_\beta^2 f + \\ + \frac{\cos^2\alpha}{4}\ \Big(\partial_\rho^2 f + \frac{2}{\rho}\ \partial_\rho f + \frac{1}{\rho^2}\ \nabla^2_{(2)} f\Big) + \frac{4}{\cos^2\alpha}\nabla^2_{(2)} f \ . \end{aligned} \tag{5.28}$$

This operator organizes nicely in terms of a Laplacian on the 2-sphere $\Omega_2(\chi, \xi)$ and a Laplacian in spherical coordinates (terms in the parenthesis) on the 3-dimensional Euclidean space parameterized by the radial coordinate ρ and the 2-sphere $\Omega_2(\chi, \xi)$. The eigenfunctions of the last are spherical Bessel functions that depend on the eigenvalue ℓ. Notice that at $\rho \gg 1$ we have $\mathcal{L}^{NATD}(f) = \mathcal{L}^{ATD}(f)$ with the identification $\rho = \psi$.

Let us now try to solve the eigenvalue problem for the operators $\mathcal{L}^{ATD}$ and $\mathcal{L}^{NATD}$ starting with the ATD example.

5.4.1 The ATD of $AdS_5 \times S^5$

The form of the operator $\mathcal{L}^{ATD}$ suggests that we should expand $Y(y)$ in eigenfunctions of the operators ∂_β^2, ∂_ψ^2 and $\nabla^2_{(2)}$:

$$\begin{aligned} & Y(y) = \sum_{m,n,\ell,k} f_{m,n,\ell}(\alpha) e^{i(m\beta + n\psi)} \mathcal{Y}_\ell^k(\chi, \xi) \, , \\ & m, n \in \mathbb{Z} \, , \quad \ell = 0, 1, 2, \dots \, , \quad k = -\ell, -\ell+1, \dots, \ell \, , \end{aligned} \tag{5.29}$$

where $\mathcal{Y}_\ell^k$ [4] are the spherical harmonics on the 2-sphere $\Omega_2(\chi, \xi)$ and $f_{m,n,\ell}(\alpha)$ are functions to be determined. Using such an expansion the eigenvalue problem (5.22) translates to a second-order differential equation for the functions $f_{m,n,\ell}(\alpha)$:

$$0 = \partial_\alpha^2 f + \big(\cot\alpha - 3\ \tan\alpha\big)\partial_\alpha f + \left[M^2 - \frac{m^2}{\sin^2\alpha} - \frac{n^2}{4}\cos^2\alpha - \frac{4\ell(\ell+1)}{\cos^2\alpha} \right] f \ , \tag{5.30}$$

[3] Here the coordinate ρ is a radial coordinate that together with the $\Omega_2(\chi, \xi)$ results after non-Abelian T-dualizing a 3-sphere inside the 5-sphere of the $AdS_5 \times S^5$.

[4] Since the index k does not enter anywhere else, we expect that the mass spectrum is $(2\ell + 1)$-times degenerate.

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where for convenience we have suppressed the indices m, n, ℓ in the function $f(\alpha)$. If we perform the following change of variables:

$$z = \sin^2\alpha\,, \qquad z \in [0,1]\,, \tag{5.31}$$

then the differential equation that we have to solve becomes:

$$0 = z(1-z)\,\partial_z^2 f + (1-3z)\,\partial_z f + \left[\frac{M^2}{4} - \frac{\ell(\ell+1)}{1-z} - \frac{m^2}{4\,z} - \frac{n^2}{16}(1-z)\right] f\,. \tag{5.32}$$

For $n = 0$ this equation can be brought into a hypergeometric form and thus it can be solved analytically. For $n \neq 0$ the above equation has two regular singular points at $z = 0, 1$ and one irregular singular point at $z = \infty$. Thus it can be brought to the form of a confluent Heun equation by setting:

$$f(z) = z^{\frac{|m|}{2}}(1-z)^{\ell}\mathfrak{f}(z)\,. \tag{5.33}$$

Indeed, if we do this the function $\mathfrak{f}$ has to satisfy the following differential equation:

$$0 = \partial_z^2\mathfrak{f} + \Big(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\Big)\partial_z\mathfrak{f} + \frac{\alpha z - q}{z(z-1)}\mathfrak{f}\,, \tag{5.34}$$

with

$$\gamma = |m|+1\,, \quad \delta = 2\big(\ell+1\big)\,, \quad \varepsilon = 0\,, \quad \alpha = -\frac{n^2}{16}\,, \quad q = \frac{M^2}{4} - \frac{n^2}{16} - \left(\frac{|m|}{2} + \ell + 1\right)^2 + 1\,, \tag{5.35}$$

which is the confluent Heun equation.

Equation (5.30) can also be put in a form of a Schrödinger-like problem. To do this we redefine $f(\alpha)$ as:

$$f(\alpha) = \frac{\tilde{\mathfrak{f}}(\alpha)}{2\,\cos^{3/2}\alpha\,\sin^{1/2}\alpha}\,. \tag{5.36}$$

Then the function $\tilde{\mathfrak{f}}(\alpha)$ satisfies the Schrödinger equation:

$$-\partial_\alpha^2\tilde{\mathfrak{f}} + V(\alpha)\tilde{\mathfrak{f}} = M^2\tilde{\mathfrak{f}}\,, \tag{5.37}$$

where the potential $V(\alpha)$ is:

$$V(\alpha) = \frac{4m^2-1}{4\sin^2\alpha} + \frac{16\ell(\ell+1)+3}{4\,\cos^2\alpha} + \frac{n^2}{4}\cos^2\alpha - 4\,. \tag{5.38}$$

Potentials of this type were also considered in the study of marginally deformations of supersymmetric backgrounds [198,199]. To the best of our knowledge, it is still not known how to solve analytically eigenvalue problems with potentials like the one above and thus we will use numerical methods.

The analytic case $n = 0$

Let us consider now the case where $n = 0$. It is easy to see that the confluent Heun equation (5.34) reduces to a hypergeometric differential equation:

$$0 = z(1-z)\partial_z^2\mathfrak{f} + \big[c - (a+b+1)z\big]\partial_z\mathfrak{f} - ab\mathfrak{f}\,, \tag{5.39}$$

where the constants a, b, c are given in terms of the eigenvalues ℓ, m and the mass M through the following relations:

$$a = 1 + \ell + \frac{|m|}{2} - \sqrt{1+\frac{M^2}{4}}\,, \quad b = 1 + \ell + \frac{|m|}{2} + \sqrt{1+\frac{M^2}{4}}\,, \quad c = |m| + 1\,. \tag{5.40}$$

The hypergeometric equation above admits two linearly independent solutions. Since in our case c is a non-negative integer one of the two solutions is singular at $z = 0$ and thus we will not consider it. Hence the only permissible solution is:

$$\mathfrak{f}(z) = {}_2F_1(a, b; c; z)\,. \tag{5.41}$$

However in our case $c - a - b = -2\ell - 1 < 0$. For this reason the behaviour of the hypergeometric function near $z = 1$ is given by the formula [202]:

$$\lim_{z\to 1^-} \frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,. \tag{5.42}$$

Thus the only way to make the solution (5.33) regular at $z = 1$ is to require that $a = -\nu$ with $\nu = 0, 1, 2, \ldots$. From this condition we end up with the following tower of masses and conformal dimensions:

$$\begin{aligned} &M^2 = \Big(2(\nu+\ell) + |m|\Big)\Big(2(\nu+\ell+2) + |m|\Big)\,, \qquad \Delta = 2(\nu+\ell+2) + |m|\,, \\ &m \in \mathbb{Z}\,, \qquad \nu, \ell = 0, 1, 2, \ldots\,. \end{aligned} \tag{5.43}$$

Taking $\kappa = 2(\nu + \ell) + |m|$ we can write the above formula as:

$$M^2 = \kappa\big(\kappa + 4\big)\,, \qquad \kappa = 0, 1, 2, \ldots\,. \tag{5.44}$$

The last formula matches the result found in [203] for the excitations of the metric along the AdS_5 directions in the case of $AdS_5 \times S^5$. This is expected as the modes with $n = 0$ that we considered here are inert under the T-duality transformation.

The non-analytic case $n \neq 0$

As it was already mentioned, we are not aware of any method that allows us to solve equation (5.32) analytically for $n \neq 0$. For this reason we restrict ourselves to find an approximate formula for the masses M which is valid for large enough values of M. This can be done using the WKB method of the appendix 5.D.

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In order to be able to apply the WKB method we first have to express equation (5.32) in terms of a suitable variable which we call r and it is related to z as:

$$r = \frac{z}{1-z}\,, \qquad r \in [0, +\infty)\,. \tag{5.45}$$

The next step is to bring equation (5.32) into the form (5.110). As a result, the functions $p(r), w(r)$ and $q(r)$ of equation (5.110) are:

$$p(r) = \frac{r}{1+r}\,, \qquad w(r) = \frac{1}{4\,(1+r)^3}\,, \qquad q(r) = -\frac{m^2}{4r(1+r)^2} - \frac{n^2 + 16\ell(\ell+1)(1+r)^2}{16(1+r)^4}\,. \tag{5.46}$$

Expanding these functions in the vicinity of the two end-points we find:

$$\left.\begin{array}{l} p(r) = r + \mathcal{O}(r^2)\,, \qquad w(r) = \dfrac{1}{4} + \mathcal{O}(r)\,, \\ q(r) = -\dfrac{m^2}{4r} + \dfrac{m^2}{2} - \dfrac{n^2}{16} - \ell(\ell+1) + \dfrac{8\ell(\ell+1) + n^2 - 3m^2}{4}\, r + \mathcal{O}(r^2) \end{array}\right\} \quad \text{at} \quad r \approx 0 \tag{5.47}$$

and

$$\left.\begin{array}{l} p(r) = 1 + \mathcal{O}(r^{-1})\,, \qquad w(r) = \dfrac{1}{4r^3} + \mathcal{O}(r^{-4})\,, \\ q(r) = -\dfrac{\ell(\ell+1)}{r^2} + \dfrac{8\ell(\ell+1) - m^2}{4r^3} + \dfrac{8m^2 - n^2 - 48\ell(\ell+1)}{16r^4} + \mathcal{O}(r^{-5}) \end{array}\right\} \quad \text{at} \quad r \approx +\infty\,. \tag{5.48}$$

The reason for keeping more terms in the expansions of the function $q(r)$ is to exploit all the different possibilities that one can obtain from its asymptotic behaviour. More specifically, one can consider the cases where $(m \neq 0, \ell \neq 0)$ or $(m \neq 0, \ell = 0)$ or $(m = 0, \ell \neq 0)$ or $(m = 0, \ell = 0)$. It turns out that all of them give the same WKB formula for the masses, which is:

$$M^2 = 4\nu(\nu + |m| + 2\ell)\,, \qquad \nu = 1, 2, \ldots\,. \tag{5.49}$$

Notice that the previous formula does not depend on the quantum number n.

As an independent check, we solved equation (5.32) numerically (using the shooting method) for given values of the quantum numbers m, n and ℓ. As it can be seen from the figure below, the values for the masses that are computed numerically are in a good agreement with those computed by the WKB formula (5.49) when the mass is large enough. In order to illustrate that the masses do not depend on the quantum number n as M is getting higher and higher, we plot the tower of masses for fixed values of m and ℓ and different values of n. In the figure below, each line corresponds to a different value of n. It turns out that the lines tend to merge in the sector of large masses. The above feature is not manifest when we fix n and m and vary ℓ or when we vary m while keeping n and ℓ fixed. This can be seen in the following plots where any possible convergence of the different lines seems to happen much slower compared to the figure 5.2.

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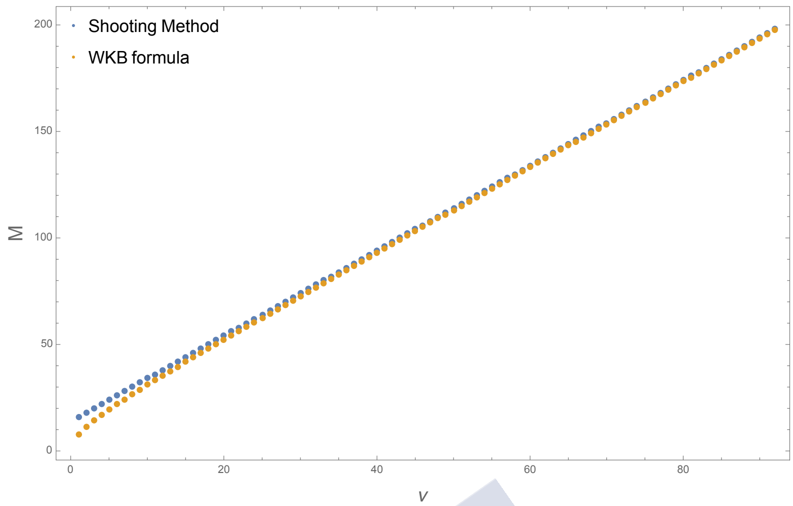

Figure 5.1: Comparison of the masses that are computed numerically with the ones computed using the WKB formula. For the computation we fix the quantum numbers as $(n, m, \ell) = (7, 4, 5)$.

Bound on the spectrum

We will now recast equation (5.34) in the Sturm-Liouville fashion:

$$0 = S\mathfrak{f} + \lambda W(z)\mathfrak{f}\,, \qquad S := \frac{d}{dz}\left(P(z)\frac{d}{dz}\right) + Q(z)\,. \tag{5.50}$$

This is done for:

$$P(z) = z^{|m|+1}(1-z)^{2\ell+2}\,, \qquad Q(z) = -\frac{n^2}{16}\,z^{|m|}(1-z)^{2\ell+2}\,, \qquad W(z) = z^{|m|}(1-z)^{2\ell+1} \tag{5.51}$$

and

$$\lambda = \frac{1}{4}\Big[M^2 - \big(2\ell + |m|\big)\big(2\ell + |m| + 4\big)\Big]\,. \tag{5.52}$$

Let us also introduce the following inner product with respect to the weight function $W(z)$:

$$\big\langle \mathfrak{f}_1, \mathfrak{f}_2 \big\rangle_W := \int_0^1 dz\, W(z)\, \mathfrak{f}_1(z)\, \mathfrak{f}_2(z)\,. \tag{5.53}$$

We will impose boundary conditions such that two eigenfunctions of different eigenvalues are orthogonal. Then we have:

$$\begin{aligned} 0 &= S\mathfrak{f}_1 + \lambda_1 W(z)\mathfrak{f}_1\,, \\ 0 &= S\mathfrak{f}_2 + \lambda_2 W(z)\mathfrak{f}_2\,. \end{aligned} \tag{5.54}$$

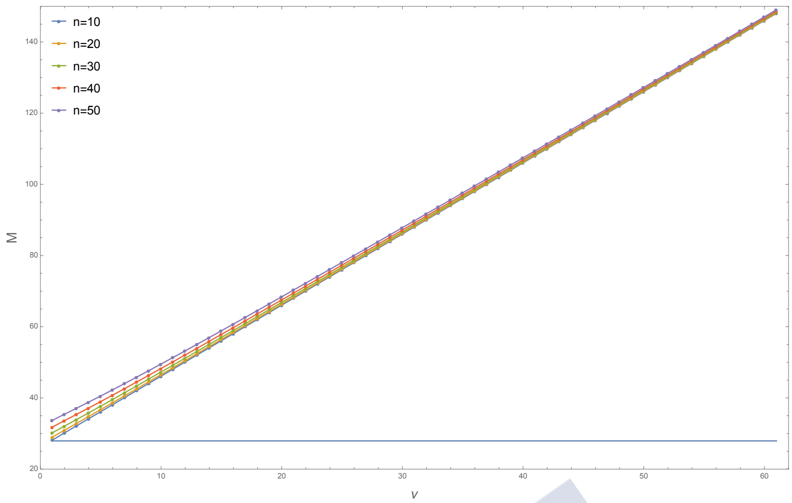

Figure 5.2: Mass spectra for fixed m and ℓ ($m = 10, \ell = 8$) and different values of n. The horizontal line represents the lower bound for the masses given in equation (5.59).

If we multiply the first by $\mathfrak{f}_2$ and the second by $\mathfrak{f}_1$ and then subtract them we find:

$$0 = \mathfrak{f}_2 \frac{d}{dz}\left(P(z)\frac{d\mathfrak{f}_1}{dz}\right) - \mathfrak{f}_1 \frac{d}{dz}\left(P(z)\frac{d\mathfrak{f}_2}{dz}\right) + \big(\lambda_1 - \lambda_2\big) W(z)\, \mathfrak{f}_1\, \mathfrak{f}_2 \,. \tag{5.55}$$

Integrating the last from 0 to 1 we get:

$$\big(\lambda_1 - \lambda_2\big)\big\langle \mathfrak{f}_1, \mathfrak{f}_2 \big\rangle_W = P(z)\left(\mathfrak{f}_1 \frac{d\mathfrak{f}_2}{dz} - \mathfrak{f}_2 \frac{d\mathfrak{f}_1}{dz}\right)\Bigg|_{z=0}^{z=1} . \tag{5.56}$$

Hence we impose:

$$P\,\mathfrak{f}\,\frac{d\mathfrak{f}}{dz}\bigg|_{z=0} = P\,\mathfrak{f}\,\frac{d\mathfrak{f}}{dz}\bigg|_{z=1} = 0\,, \tag{5.57}$$

which in our case is satisfied as long as $\mathfrak{f}$ and $d\mathfrak{f}/dz$ are finite at the endpoints of the interval $[0, 1]$, or if they diverge they do it slow enough.

We can now derive a lower bound for the mass spectrum in the following way. From the Sturm-Liouville equation (5.50) we have:

$$\lambda \left\langle \mathfrak{f}, \mathfrak{f} \right\rangle_W = -\int_0^1 dz\, \mathfrak{f}\, S\mathfrak{f} = \int_0^1 dz\, P \left(\frac{d\mathfrak{f}}{dz}\right)^2 - \int_0^1 dz\, Q\, \mathfrak{f}^2 \geq 0\,, \tag{5.58}$$

which implies that $\lambda \geq 0$ or:

$$M^2 \geq \big(2\ell + |m|\big)\big(2\ell + |m| + 4\big)\,. \tag{5.59}$$

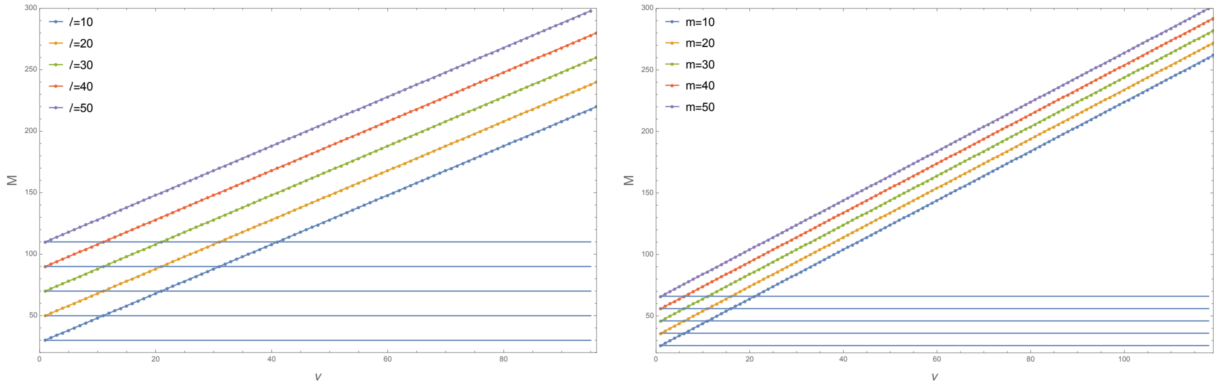

Figure 5.3: The figure on the left represents the mass spectra for $n = 10, m = 8$ and different values of ℓ. The figure on the right shows the mass towers for $n = 4, \ell = 7$ and different values of m. The horizontal lines represent the lower bounds for the masses given in equation (5.59).

Notice that the bound does not depend on the number n.

Obviously the spectrum we found in equation (5.43) satisfies the above bound and it saturates it for $\nu = 0$. From the numerical analysis we get strong evidence that the bound is not violated for non-zero values of n, as it can be seen in the figures 5.2 and 5.3.

5.4.2 The NATD of $AdS_5 \times S^5$

Though in the NATD case things seem to be similar to the Abelian example, one has to be careful with the range of the coordinate ρ. In order to illustrate the possible issues that arise, first we are going to allow ρ to run in the semi-infinite interval $[0, +\infty)$ and then we are going to consider the case where ρ takes values in the interval $[0, \rho_*]$.

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When $\rho \in [0, +\infty)$

The separation of variables scheme for the operator $\mathcal{L}^{NATD}$ when ρ is not bounded from above is:

$$Y(y) = \sum_{m,\ell,k} \int_0^\infty dn\, f_{m,n,\ell}(\alpha) e^{im\beta} j_\ell(n\rho) \mathcal{Y}_\ell^k(\chi, \xi)\,, \qquad (5.60)$$
$$m \in \mathbb{Z}\,, \qquad n \in \mathbb{R}_{\geq 0}\,, \quad \ell = 0, 1, 2, \ldots\,, \quad k = -\ell, -\ell+1, \ldots, \ell\,,$$

where $j_\ell(n\rho)$ are the spherical Bessel functions that are regular at the origin. This scheme leads to the differential equation (5.30) where now the index n is continuous. As a result one expects a non-discrete spectrum of masses with respect to the index n.

Like in the ATD example, when $n = 0$, it is possible to find an analytic solution for the differential equation and following the same reasoning as we did in section 5.4.1 we end up again with the spectrum of (5.43). When $n \neq 0$, in order to determine the mass spectrum one has to resort to numerics. The only difference with respect to the Abelian case is that now the parameter n can take any positive real value. Due to the fact that the functions $f_{m,n,\ell}(\alpha)$ satisfy the differential equation (5.30), a WKB analysis implies that for large quantum numbers the mass spectrum behaves like in equation (5.49).

When $\rho \in [0, \rho_*]$

Let us now restrict the coordinate ρ in the interval $[0, \rho_*]$. This imposes a hard cut-off in the geometry and is along the lines of the CFT completions considered in [192]. Since ρ is now bounded one has to make a choice for the boundary conditions of the function $Y(y)$ at $\rho = \rho_*$. Let us assume for example that $Y|_{\rho=\rho_*} = 0$. This implies that n can take only those values where:

$$n_{\ell s} = \frac{\rho_{\ell s}}{\rho_*}, \qquad s = 1, 2, \dots, \tag{5.61}$$

with $\rho_{\ell s}$ being the roots of $j_\ell(\rho)$. The separation of variables scheme now reads:

$$\begin{aligned} &Y(y) = \sum_{m,\ell,k,s} f_{m,\ell,s}(\alpha) e^{im\beta} j_\ell(n_{\ell s}\rho) \mathcal{Y}_\ell^k(\chi, \xi), \\ &m \in \mathbb{Z}, \quad s = 1, 2, \dots, \quad \ell = 0, 1, 2, \dots, \quad k = -\ell, -\ell+1, \dots, \ell. \end{aligned} \tag{5.62}$$

Again, the functions $f_{m,n,\ell}(\alpha)$ must satisfy the differential equation (5.30), but now the parameter n has to be replaced by $n_{\ell s}$. Consequently, when $n_{\ell s} = 0$, it is possible to find analytic solutions for (5.30). Since we are dealing with the same equation, one expects to find the same mass spectrum as in equation (5.43) where now ℓ can not take the value 0. This is because all the spherical Bessel functions vanish at the origin except j_0 [5]. In the case where $n_{\ell s} \neq 0$ the differential equation (5.30) can only be solved numerically. Since the only difference with the previous examples is the fact that now we have to deal with the values of $n_{\ell s}$ instead of the integer or positive real values of n, we do not attempt such an analysis. For large quantum numbers one can perform a WKB analysis which should give as a result the same behaviour for the masses as in equation (5.49).

Moreover, the mass bound for the aforementioned two examples turns out to be the same as in equation (5.59). This is due to the fact that one has to deal with the same differential equation (5.32) as in the ATD case.

[5] Recall that the asymptotic behaviour of the spherical Bessel functions near the origin is:

$$j_\ell(x) \sim \frac{2^\ell \ell!}{(2\ell+1)!} x^\ell, \qquad x \ll 1.$$

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5.5 Discussion

In this chapter we have written down the whole set of linearized equations of motion for fluctuations of warped geometries in type IIA supergravity with AdS_5 factor. In particular we studied the spin-2 excitations of the Gaiotto-Maldacena class of geometries. We gave a generic expression for the wave operator describing these fluctuations in terms of the solution to the axisymmetric Laplace equation characterizing these geometries. We studied two interesting examples: the Abelian (Hopf) T-dual and the NATD geometries of the maximally supersymmetric solution $AdS_5 \times S^5$. The wave operator for these solutions turned out to be the same when the "field space" coordinate ρ of the NATD solution is large. We were able to find an analytic solution for the spectrum only when the quantum number n vanishes. For the rest of the spectrum we resort on numerical methods. For large masses we showed that our results are in perfect agreement with WKB expectations. Since in the ATD case the "field space" coordinate is compact, we obtained a discrete spectrum of masses. A bound for these masses was found. However, in the NATD case, the "field space" coordinate is unbounded, which originates a continuous spectrum of masses. By imposing a hard cut-off in the geometry that bounds the value of this coordinate a discrete spectrum of masses emerged.

5.A Useful formulas

In this appendix we list some of the most useful formulas and identities that are necessary to derive the linearized equations of motion for the fluctuations of the supergravity fields.

5.A.1 List of formulas in the Riemannian geometry

The Christoffel symbols and the covariant derivatives acting on tensors are given by:

$$\Gamma^P_{MN} = \frac{1}{2}\, g^{P\Sigma} \left(\partial_M g_{\Sigma N} + \partial_N g_{\Sigma M} - \partial_\Sigma g_{MN}\right), \tag{5.63}$$

$$\begin{aligned}\nabla_M T^{A_1\ldots A_n}{}_{B_1\ldots B_m} = \partial_M T^{A_1\ldots A_n}{}_{B_1\ldots B_m} + \sum_{k=1}^{n} \Gamma^{A_k}_{MP}\, T^{A_1\ldots P\ldots A_n}{}_{B_1\ldots B_m} - \\ - \sum_{k=1}^{m} \Gamma^{P}_{MB_k} T^{A_1\ldots A_n}{}_{B_1\ldots P\ldots B_m}\,.\end{aligned} \tag{5.64}$$

Notice that when the upper index of the Christoffel symbols is contracted with one of its lower indices then:

$$\Gamma^M_{MN} := \frac{1}{2}\, g^{M\Sigma} \left(\partial_M g_{\Sigma N} + \partial_N g_{\Sigma M} - \partial_\Sigma g_{MN}\right) = \frac{1}{2}\, g^{M\Sigma}\, \partial_N g_{M\Sigma} = \partial_N \ln\sqrt{g}\,, \tag{5.65}$$

where $g := \det\left(g_{MN}\right)$.

The Riemann tensor is defined through the Christoffel symbols as:

$$R^{\Lambda}{}_{M\Sigma N} := \partial_{\Sigma}\Gamma^{\Lambda}_{MN} - \partial_{N}\Gamma^{\Lambda}_{M\Sigma} + \Gamma^{K}_{MN}\Gamma^{\Lambda}_{K\Sigma} - \Gamma^{K}_{M\Sigma}\Gamma^{\Lambda}_{KN}\,. \tag{5.66}$$

In this form it is clear that the Riemann tensor is antisymmetric under $\Sigma \leftrightarrow N$. The Ricci tensor is defined by contracting the first and third indices of the Riemann tensor, i.e.

$$R_{MN} := R^{\Sigma}{}_{M\Sigma N}\,. \tag{5.67}$$

Obviously the Ricci tensor is symmetric under the exchange of its indices. Finally the Ricci scalar is given by the contraction of the Ricci tensor with the metric:

$$R := g^{MN} R_{MN}\,. \tag{5.68}$$

Another useful object is the commutator of two covariant derivatives acting on a tensor. This can be written in terms of the Riemann tensor as:

$$[\nabla_M, \nabla_N] T^{A_1\ldots A_n}{}_{B_1\ldots B_m} = \sum_{k=1}^{n} R^{A_k}{}_{\Sigma MN}\, T^{A_1\ldots\Sigma\ldots A_n}{}_{B_1\ldots B_m} - \sum_{k=1}^{m} R^{\Sigma}{}_{B_k MN}\, T^{A_1\ldots A_n}{}_{B_1\ldots\Sigma\ldots B_m}\,. \tag{5.69}$$

5.A.2 Metric variations

We consider variations around the background metric $\bar{g}_{MN}$ of the form:

$$g_{MN} = \bar{g}_{MN} + \delta g_{MN}\,, \qquad g^{MN} = \bar{g}^{MN} + \delta g^{MN}\,, \tag{5.70}$$

where:

$$\delta g^{MN} = -\bar{g}^{MP}\, \delta g_{P\Sigma}\, \bar{g}^{\Sigma N}\,. \tag{5.71}$$

We will also take all the geometric quantities (such as the Christoffel symbols, the covariant derivatives, the Riemann and Ricci tensors and also the Ricci scalar) to be constructed with the background metric $\bar{g}_{MN}$. As a consequence:

$$\nabla_P \bar{g}_{MN} = \nabla_P \bar{g}^{MN} = 0\,. \tag{5.72}$$

The variation of the Christoffel symbols reads:

$$\begin{aligned} \delta\Gamma^{P}_{MN} &= \frac{1}{2}\,\bar{g}^{P\Sigma}\left(\nabla_M \delta g_{\Sigma N} + \nabla_N \delta g_{\Sigma M} - \nabla_\Sigma \delta g_{MN}\right), \\ \delta\Gamma^{M}_{MN} &= \frac{1}{2}\,\bar{g}^{M\Sigma}\, \nabla_N \delta g_{\Sigma M}\,. \end{aligned} \tag{5.73}$$

Using the above we can compute the variation of the Riemann tensor which simply becomes:

$$\delta R^{\Lambda}{}_{M\Sigma N} = \nabla_\Sigma \delta\Gamma^{\Lambda}_{MN} - \nabla_N \delta\Gamma^{\Lambda}_{M\Sigma}\,. \tag{5.74}$$

From this we find that the variation of the Ricci tensor is:

$$\begin{aligned}\delta R_{MN} &= \nabla_P \delta\Gamma^P_{MN} - \nabla_N \delta\Gamma^P_{MP} \\ &= \frac{1}{2}\left(\nabla^\Sigma \nabla_M \delta g_{\Sigma N} + \nabla^\Sigma \nabla_N \delta g_{\Sigma M} - \nabla^2 \delta g_{MN} - \bar{g}^{P\Sigma}\, \nabla_N \nabla_M \delta g_{\Sigma P}\right).\end{aligned} \tag{5.75}$$

Finally, for the variation of the Ricci scalar we have:

$$\begin{aligned}\delta R &= R_{MN}\, \delta g^{MN} + \nabla^M \nabla^N \delta g_{MN} - \bar{g}^{MN}\, \nabla^2 \delta g_{MN} \\ &= -R^{MN}\, \delta g_{MN} + \nabla^M \nabla^N \delta g_{MN} - \bar{g}^{MN}\, \nabla^2 \delta g_{MN}\,.\end{aligned} \tag{5.76}$$

5.A.3 Conformal rescalings

Let us now consider the conformal rescaling of the background metric to be:

$$\bar{g}_{MN} = e^{2A} \tilde{g}_{MN}\,. \tag{5.77}$$

Using this, the Christoffel symbols constructed with the background metric $\bar{g}_{MN}$ are given by:

$$\Gamma^P_{MN} = \tilde{\Gamma}^P_{MN} + T^P_{MN}\,, \tag{5.78}$$

where $\tilde{\Gamma}^P_{MN}$ are the Christoffel symbols that are constructed with the metric $\tilde{g}_{MN}$ and we defined [6]

$$T^P_{MN} := \delta^P_N\, \tilde{\nabla}_M A + \delta^P_M\, \tilde{\nabla}_N A - \tilde{g}_{MN}\, \tilde{\nabla}^P A\,. \tag{5.79}$$

Notice that T^P_{MN} is symmetric in its lower indices, i.e. $T^P_{MN} = T^P_{NM}$. Moreover here we raise/lower the indices using $\tilde{g}_{MN}$.

Using the above we can write the Riemann tensor as:

$$\begin{aligned} R^\Lambda{}_{M\Sigma N} = \tilde{R}^\Lambda{}_{M\Sigma N} + \Big[&\delta^\Lambda_N\, \tilde{\nabla}_\Sigma \tilde{\nabla}_M A - \tilde{g}_{MN}\, \tilde{\nabla}_\Sigma \tilde{\nabla}^\Lambda A + \delta^\Lambda_\Sigma\, \tilde{\nabla}_N A\, \tilde{\nabla}_M A - \\ &- \delta^\Lambda_\Sigma\, \tilde{g}_{MN}\, \big(\tilde{\nabla} A\big)^2 + \tilde{g}_{MN}\, \tilde{\nabla}_\Sigma A\, \tilde{\nabla}^\Lambda A - \big(\Sigma \leftrightarrow N\big)\Big]\,.\end{aligned} \tag{5.80}$$

The Ricci tensor is found after contracting Λ with Σ in the expression above giving:

$$\begin{aligned} R_{MN} = \tilde{R}_{MN} &+ \big(2 - \mathcal{D}\big)\, \tilde{\nabla}_M \tilde{\nabla}_N A - \tilde{g}_{MN}\, \tilde{\nabla}^2 A + \\ &+ \big(\mathcal{D} - 2\big)\, \tilde{\nabla}_M A\, \tilde{\nabla}_N A + \big(2 - \mathcal{D}\big)\, \tilde{g}_{MN}\, (\tilde{\nabla} A)^2\,,\end{aligned} \tag{5.81}$$

where $\mathcal{D}$ is the dimension of the spacetime. From this we can obtain the Ricci scalar:

$$R = e^{-2A}\left[\tilde{R} - 2\left(\mathcal{D} - 1\right) \tilde{\nabla}^2 A - \left(\mathcal{D} - 1\right)\left(\mathcal{D} - 2\right)\left(\tilde{\nabla} A\right)^2\right]. \tag{5.82}$$

[6] We use the fact that for any scalar f it is $\partial_M f = \tilde{\partial}_M f = \nabla_M f = \tilde{\nabla}_M f$.

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Another useful quantity is the variation of the Christoffel symbols in terms of the metric $\tilde{g}_{MN}$. From the first equation in (5.73) this is found to be:

$$\delta\Gamma^P_{MN} = \frac{1}{2}\,\tilde{g}^{P\Sigma}\left(\tilde{\nabla}_M h_{\Sigma N} + \tilde{\nabla}_N h_{\Sigma M} - \tilde{\nabla}_\Sigma h_{MN} - 2\,\tilde{g}_{MN}\,h_{\Sigma K}\,\tilde{\nabla}^K A - 2\,h_{MN}\,\tilde{\nabla}_\Sigma A\right), \tag{5.83}$$

where for convenience we set:

$$\delta g_{MN} = e^{2A} h_{MN} \qquad \Rightarrow \qquad \delta g^{MN} = -e^{2A} h^{MN}\,. \tag{5.84}$$

Notice that the indices are raised with $\bar{g}^{MN}$. The expression (5.83) is useful when we want to express variations of covariant derivatives acting on tensors in terms of the metric $\tilde{g}_{MN}$.

5.B The type IIA supergravity equations and their fluctuations

In this appendix we review the equations of motion of the type IIA supergravity in Einstein frame, written in components and present general formulas for their fluctuations.

5.B.1 The equations of motion of the type IIA supergravity

Let us start by writing the equations of motion for the type IIA supergravity fields [98]:

$$\begin{aligned} 0 = {} & R_{MN} - \frac{1}{2}\partial_M\Phi\,\partial_N\Phi - \frac{e^{3\Phi/2}}{2}\Big(F_{MP}F_N{}^P - \frac{1}{16}g_{MN}F_2^2\Big) \\ & - \frac{e^{\Phi/2}}{12}\Big(F_{MPK\Lambda}F_N{}^{PK\Lambda} - \frac{3}{32}g_{MN}F_4^2\Big) - \frac{e^{-\Phi}}{4}\Big(H_{MPK}H_N{}^{PK} - \frac{1}{12}g_{MN}H_3^2\Big)\,, \end{aligned} \tag{5.85}$$

$$0 = \nabla^M\nabla_M\Phi - \frac{3}{8}e^{3\Phi/2}F_2^2 - \frac{e^{\Phi/2}}{96}F_4^2 + \frac{e^{-\Phi}}{12}H_3^2\,, \tag{5.86}$$

$$0 = \nabla^M\big(e^{-\Phi}H_{MNP}\big) - \frac{e^{\Phi/2}}{2}F_{NPK\Lambda}F^{K\Lambda} + \frac{1}{2\cdot 4!\cdot 4!}\varepsilon_{M_1\ldots M_8 NP}F^{M_1\ldots M_4}F^{M_5\ldots M_8}\,, \tag{5.87}$$

$$0 = \nabla^M\big(e^{3\Phi/2}F_{MN}\big) + \frac{e^{\Phi/2}}{6}F_{PK\Lambda N}H^{PK\Lambda}\,, \tag{5.88}$$

$$0 = \nabla^M\big(e^{\Phi/2}F_{MNPK}\big) - \frac{1}{144}\varepsilon_{M_1\ldots M_7 NPK}F^{M_1\ldots M_4}H^{M_5\ldots M_7}\,, \tag{5.89}$$

where $\varepsilon_{M_1\ldots M_{10}}$ is the totally antisymmetric Levi-Civita tensor.

An alternative way to present the dilaton equation (5.86) is by eliminating its dependence in the RR fields. For this reason we first consider the trace of the Einstein equation (5.85):

$$0 = R - \frac{1}{2}\left(\partial\Phi\right)^2 - \frac{3}{16}\,e^{3\Phi/2}\,F_2^2 - \frac{e^{\Phi/2}}{192}\,F_4^2 - \frac{e^{-\Phi}}{24}\,H_3^2\,. \tag{5.90}$$

Using this we can rewrite equation (5.86) as:

$$0 = -2\,R + \nabla^M \nabla_M \Phi + \big(\partial\Phi\big)^2 + \frac{e^{-\Phi}}{6}\,H_3^2\,. \tag{5.91}$$

5.B.2 Fluctuations of the equations of motion

Here we derive the linearized equations of motion for the fluctuations of the supergravity fields. For this purpose we consider the following perturbation scheme:

$$\begin{aligned} &g_{MN} = \bar{g}_{MN} + \delta g_{MN}\,, \qquad \Phi = \bar{\Phi} + \varphi\,, \qquad H_3 = \bar{H}_3 + \delta H_3\,, \\ &F_2 = \bar{F}_2 + \delta F_2\,, \qquad F_4 = \bar{F}_4 + \delta F_4\,, \end{aligned} \tag{5.92}$$

where we use the bar notation for the background values of the various fields. Notice that the background metric $\bar{g}_{MN}$ is the metric in the Einstein frame which is conformally related to the metric $\tilde{g}_{MN}$ through a warp factor as in equation (5.77). Let us now continue with the fluctuations of the equations of motion.

The Einstein equation

Before we start fluctuating the Einstein equation (5.85) we would like to rewrite it in a more uniform way as:

$$0 = R_{MN} - \frac{1}{2}\partial_M\Phi\,\partial_N\Phi - \frac{1}{2}\sum_{p=2}^{4}\gamma_p\,e^{\alpha_p\Phi}\Big[\big(\mathcal{A}_p^2\big)_{MN} - \beta_p\,g_{MN}\mathcal{A}_p^2\Big]\,, \tag{5.93}$$

where we denote $\big(\mathcal{A}_p^2\big)_{MN} := \mathcal{A}_{MM_1\ldots M_{p-1}}\,\mathcal{A}_N{}^{M_1\ldots M_{p-1}}$ and we consider:

$$\mathcal{A}_p := \big(F_2, H_3, F_4\big)\,, \alpha_p := \Big(\frac{3}{2}, -1, \frac{1}{2}\Big)\,, \beta_p := \Big(\frac{1}{16}, \frac{1}{12}, \frac{3}{32}\Big)\,, \gamma_p := \Big(1, \frac{1}{2}, \frac{1}{6}\Big)\,. \tag{5.94}$$

We also introduce the dot product for a tensor $\mathcal{A}_p$ as:

$$\begin{aligned} &\big(\mathcal{A}_p\big)_M \cdot \big(\bar{\mathcal{A}}_p\big)_N = \mathcal{A}_{M\Sigma_1\ldots\Sigma_{p-1}}\,\bar{\mathcal{A}}_N{}^{\Sigma_1\ldots\Sigma_{p-1}} \quad \& \quad \big(\mathcal{A}_p\big)\cdot\big(\bar{\mathcal{A}}_p\big) = \mathcal{A}_{\Sigma_1\ldots\Sigma_p}\,\bar{\mathcal{A}}^{\Sigma_1\ldots\Sigma_p}\,, \\ &\big(\mathcal{A}_p\big)_M \cdot \big(\tilde{\mathcal{A}}_p\big)_N = \mathcal{A}_{M\Sigma_1\ldots\Sigma_{p-1}}\,\tilde{\mathcal{A}}_N{}^{\Sigma_1\ldots\Sigma_{p-1}} \quad \& \quad \big(\mathcal{A}_p\big)\cdot\big(\tilde{\mathcal{A}}_p\big) = \mathcal{A}_{\Sigma_1\ldots\Sigma_p}\,\tilde{\mathcal{A}}^{\Sigma_1\ldots\Sigma_p}\,, \end{aligned} \tag{5.95}$$

where we use the bar and tilde notation in order to stress that the indices are raised with the background metrics $\bar{g}_{MN}$ and $\tilde{g}_{MN}$ respectively. After some algebra one ends up with:

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$$
\begin{aligned}
0 =\ & \frac{1}{2}\,\tilde\nabla^\Sigma\tilde\nabla_M h_{\Sigma N} + \frac{1}{2}\,\tilde\nabla^\Sigma\tilde\nabla_N h_{\Sigma M} - \frac{1}{2}\,\tilde\nabla^2 h_{MN} - \frac{1}{2}\,\tilde\nabla_N\tilde\nabla_M\tilde h + 4\,\tilde\nabla^\Sigma A\,\tilde\nabla_M h_{\Sigma N} + \\
& + 4\,\tilde\nabla^\Sigma A\,\tilde\nabla_N h_{\Sigma M} - h_{MN}\,\tilde\nabla^2 A - 8\,h_{MN}\,\big(\tilde\nabla A\big)^2 - 4\,\tilde\nabla^P A\tilde\nabla_P h_{MN} - \frac{1}{2}\,\partial_M\varphi\,\partial_N\bar\Phi - \\
& - \frac{1}{2}\,\partial_M\bar\Phi\,\partial_N\varphi - \frac{1}{2}\sum_{p=2}^{4}\gamma_p\,e^{2(1-p)A+\alpha_p\bar\Phi}\Big[\big(\delta\mathcal{A}_p\big)_M\cdot\big(\tilde{\mathcal{A}}_p\big)_N + \big(\tilde{\mathcal{A}}_p\big)_M\cdot\big(\delta\mathcal{A}_p\big)_N - \beta_p\,h_{MN}\tilde{\mathcal{A}}_p^2 - \\
& - (p-1)\,h_{PK}\,\tilde{\mathcal{A}}_{M\Sigma_1\ldots\Sigma_{p-2}}{}^{P}\,\tilde{\mathcal{A}}_N{}^{\Sigma_1\ldots\Sigma_{p-2}K}\Big] - \frac{\varphi}{2}\sum_{p=2}^{4}\alpha_p\,\gamma_p\,e^{2(1-p)A+\alpha_p\bar\Phi}\big(\tilde{\mathcal{A}}_p^2\big)_{MN} + \tilde g_{MN}\,t\,.
\end{aligned}
\tag{5.96}
$$

where t is defined as:

$$
\begin{aligned}
t :=\ & \tilde\nabla^\Sigma h_{\Sigma P}\,\tilde\nabla^P A + h_{\Sigma P}\,\tilde\nabla^\Sigma\tilde\nabla^P A - \frac{1}{2}\,\tilde\nabla_\Lambda\tilde h\,\tilde\nabla^\Lambda A + 8\,h_{P\Sigma}\,\tilde\nabla^P A\,\tilde\nabla^\Sigma A + \\
& + \frac{\varphi}{2}\sum_{p=2}^{4}\alpha_p\,\beta_p\,\gamma_p\,e^{2(1-p)A+\alpha_p\bar\Phi}\tilde{\mathcal{A}}_p^2 + \frac{1}{2}\sum_{p=2}^{4}\beta_p\,\gamma_p\,e^{2(1-p)A+\alpha_p\bar\Phi}\Big[2\,\big(\delta\mathcal{A}_p\big)\cdot\big(\tilde{\mathcal{A}}_p\big) - \\
& - p\,h_{PK}\,\tilde{\mathcal{A}}_{\Sigma_1\ldots\Sigma_{p-1}}{}^{P}\,\tilde{\mathcal{A}}^{\Sigma_1\ldots\Sigma_{p-1}K}\Big]\,.
\end{aligned}
\tag{5.97}
$$

Here by $\tilde h$ we mean the contraction $\tilde h := \tilde g^{MN}h_{MN}$.

The dilaton equation

If we consider the fluctuation of the dilaton equation (5.91) we arrive at the following result:

$$
\begin{aligned}
0 =\ & 2\,\tilde R^{MN}\,h_{MN} - 2\,\tilde\nabla^M\tilde\nabla^N h_{MN} + 2\,\tilde\nabla^2\tilde h + \tilde\nabla^2\varphi + 8\,\tilde\nabla^M A\,\tilde\nabla_M\varphi - h_{MN}\,\tilde\nabla^M\tilde\nabla^N\bar\Phi - \\
& - \tilde\nabla^M\bar\Phi\tilde\nabla^N h_{MN} - 8\,h_{MN}\tilde\nabla^M\bar\Phi\tilde\nabla^N A + \frac{1}{2}\tilde\nabla^M\bar\Phi\tilde\nabla_M\tilde h - h_{MN}\tilde\nabla^M\bar\Phi\tilde\nabla^N\bar\Phi + 2\tilde\nabla^M\bar\Phi\tilde\nabla_M\varphi - \\
& - 36\,h_{MN}\,\tilde\nabla^M\tilde\nabla^N A - 144\,h_{MN}\,\tilde\nabla^M A\,\tilde\nabla^N A - 36\,\tilde\nabla^M A\,\tilde\nabla^N h_{MN} + 18\,\tilde\nabla^M A\,\tilde\nabla_M\tilde h - \\
& - 2h_{MN}\tilde\nabla^M\tilde\nabla^N\bar\Phi - \frac{e^{-\bar\Phi-4A}}{6}\varphi\tilde H_3^2 + \frac{e^{-\bar\Phi-4A}}{3}\big(\delta H_3\big)\cdot\big(\tilde H_3\big) - \frac{e^{-\bar\Phi-4A}}{2}\,h_{PK}\tilde H_{MN}{}^{P}\,\tilde H^{MNK}\,.
\end{aligned}
\tag{5.98}
$$

The Maxwell equations

Let us now continue with the variation of the Maxwell equations (5.87), (5.88) and (5.89). The results are summarized in the following three equations:

The equation for the NSNS 3-form:

$$
\begin{aligned}
0 &= 12h_{MN}\tilde{\nabla}^M A\tilde{H}^N{}_{K_1K_2} + h_{MN}\tilde{\nabla}^M\bar{\Phi}\tilde{H}^N{}_{K_1K_2} + \varphi\tilde{\nabla}^M\bar{\Phi}\tilde{H}_{MK_1K_2} - \\
&- 4\varphi\tilde{\nabla}^M A\tilde{H}_{MK_1K_2} - \tilde{\nabla}^M\varphi\tilde{H}_{MK_1K_2} + 4\tilde{\nabla}^M A\delta H_{MK_1K_2} - \tilde{\nabla}^M\bar{\Phi}\delta H_{MK_1K_2} - \\
&- \varphi\tilde{\nabla}^M\tilde{H}_{MK_1K_2} + \tilde{\nabla}^M\delta H_{MK_1K_2} - \frac{1}{2}\big(2\tilde{\nabla}^M h_{\Sigma M} - \tilde{\nabla}_\Sigma\tilde{h}\big)\tilde{H}^\Sigma{}_{K_1K_2} - \\
&- h_{MN}\tilde{\nabla}^N\tilde{H}^M{}_{K_1K_2} - \tilde{\nabla}_N h_{\Sigma K_1}\tilde{H}^{N\Sigma}{}_{K_2} + \tilde{\nabla}_N h_{\Sigma K_2}\tilde{H}^{N\Sigma}{}_{K_1} - \\
&- \frac{e^{\frac{3}{2}\bar{\Phi}-2A}}{4}\varphi\tilde{F}_{K_1K_2K\Lambda}\tilde{F}^{K\Lambda} - \frac{e^{\frac{3}{2}\bar{\Phi}-2A}}{2}\delta\tilde{F}_{K_1K_2K\Lambda}\tilde{F}^{K\Lambda} - \frac{e^{\frac{3}{2}\bar{\Phi}-2A}}{2}\tilde{F}_{K_1K_2}{}^{K\Lambda}\big(\delta\tilde{F}_{K\Lambda} - \\
&- 2h_{KM}\tilde{F}^M{}_\Lambda\big) + \frac{e^{\bar{\Phi}-4A}}{2\cdot 4!\cdot 4!}\Big[\frac{1}{2}\tilde{h}\tilde{\varepsilon}_{M_1\ldots M_8K_1K_2}\tilde{F}^{M_1\ldots M_4}\tilde{F}^{M_5\ldots M_8} + \\
&+ 2\tilde{\varepsilon}^{M_1\ldots M_4}{}_{M_5\ldots M_8K_1K_2}\delta F_{M_1\ldots M_4}\tilde{F}^{M_5\ldots M_8} - 8h_{M_1M_1'}\tilde{\varepsilon}^{M_1}{}_{M_2\ldots M_8K_1K_2}\tilde{F}^{M_1'M_2M_3M_4}\tilde{F}^{M_5\ldots M_8}\Big].
\end{aligned}
\tag{5.99}
$$

The equation for the RR 2-form:

$$
\begin{aligned}
0 &= 12\,h_{MN}\tilde{\nabla}^M A\,\tilde{F}^N{}_{K_1} - \frac{3}{2}\,h_{MN}\tilde{\nabla}^M\bar{\Phi}\,\tilde{F}^N{}_{K_1} + \frac{9}{4}\,\varphi\,\tilde{\nabla}^M\bar{\Phi}\,\tilde{F}_{MK_1} + \\
&+ 9\,\varphi\,\tilde{\nabla}^M A\,\tilde{F}_{MK_1} + \frac{3}{2}\,\tilde{\nabla}^M\varphi\,\tilde{F}_{MK_1} + 6\,\tilde{\nabla}^M A\,\delta F_{MK_1} + \frac{3}{2}\,\tilde{\nabla}^M\bar{\Phi}\,\delta F_{MK_1} + \\
&+ \frac{3}{2}\varphi\,\tilde{\nabla}^M\tilde{F}_{MK_1} + \tilde{\nabla}^M\delta F_{MK_1} - h_{MN}\,\tilde{\nabla}^N\tilde{F}^M{}_{K_1} - \tilde{\nabla}_N h_{\Sigma K_1}\,\tilde{F}^{N\Sigma} - \\
&- \frac{1}{2}\big(2\,\tilde{\nabla}^M h_{\Sigma M} - \tilde{\nabla}_\Sigma\tilde{h}\big)\,\tilde{F}^\Sigma{}_{K_1} + \frac{e^{-4A-\bar{\Phi}}}{12}\varphi\tilde{F}_{PK\Lambda K_1}\tilde{H}^{PK\Lambda} + \\
&+ \frac{e^{-4A-\bar{\Phi}}}{6}\delta F_{PK\Lambda K_1}\tilde{H}^{PK\Lambda} + \frac{e^{-4A-\bar{\Phi}}}{6}\tilde{F}^{PK\Lambda}{}_{K_1}\big(\delta H_{PK\Lambda} - 3\,h_{P\Sigma}\,\tilde{H}^\Sigma{}_{K\Lambda}\big)\,.
\end{aligned}
\tag{5.100}
$$

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The equation for the RR 4-form:

$$
\begin{aligned}
0 = {} & 12\, h_{MN}\tilde{\nabla}^M A\, \tilde{F}^N{}_{K_1K_2K_3} - \frac{1}{2}\, h_{MN}\tilde{\nabla}^M\bar{\Phi}\, \tilde{F}^N{}_{K_1K_2K_3} + \frac{\varphi}{4}\, \tilde{\nabla}^M\bar{\Phi}\, \tilde{F}_{MK_1K_2K_3} + \\
& + \varphi\, \tilde{\nabla}^M A\, \tilde{F}_{MK_1K_2K_3} + \frac{1}{2}\, \tilde{\nabla}^M\varphi\, \tilde{F}_{MK_1K_2K_3} + \frac{\varphi}{2}\, \tilde{\nabla}^M\tilde{F}_{MK_1K_2K_3} + 2\, \tilde{\nabla}^M A\, \delta F_{MK_1K_2K_3} + \\
& + \frac{1}{2}\, \tilde{\nabla}^M\bar{\Phi}\, \delta F_{MK_1K_2K_3} - \frac{1}{2}\big(2\, \tilde{\nabla}^M h_{\Sigma M} - \tilde{\nabla}_\Sigma\tilde{h}\big)\, \tilde{H}^\Sigma{}_{K_1K_2} - h_{MN}\, \tilde{\nabla}^N\tilde{F}^M{}_{K_1K_2K_3} - \\
& - \frac{1}{2}\big(2\, \tilde{\nabla}^M h_{\Sigma M} - \tilde{\nabla}_\Sigma\tilde{h}\big)\, \tilde{F}^\Sigma{}_{K_1K_2K_3} - \tilde{\nabla}_N h_{\Sigma K_1}\, \tilde{F}^{N\Sigma}{}_{K_2K_3} + \\
& + \tilde{\nabla}_N h_{\Sigma K_2}\, \tilde{F}^{N\Sigma}{}_{K_1K_3} - \tilde{\nabla}_N h_{\Sigma K_3}\, \tilde{F}^{N\Sigma}{}_{K_1K_2} - \frac{e^{-2A-\frac{\bar{\Phi}}{2}}}{144}\Big[\frac{1}{2}\,\tilde{h}\,\tilde{\varepsilon}_{M_1\ldots M_7K_1K_2K_3}\, \tilde{F}^{M_1\ldots M_4}\, \tilde{H}^{M_5M_6M_7} + \\
& + \tilde{\varepsilon}^{M_1\ldots M_4}{}_{M_5\ldots M_7K_1K_2K_3}\, \delta F_{M_1\ldots M_4}\, \tilde{H}^{M_5\ldots M_7} - 4\, h_{M_1M_1'}\, \tilde{\varepsilon}^{M_1}{}_{M_2\ldots M_7K_1K_2K_3}\, \tilde{F}^{M_1'M_2\ldots M_4}\, \tilde{H}^{M_5M_6M_7} + \\
& + \tilde{\varepsilon}^{M_1M_2M_3}{}_{M_4\ldots M_7K_1K_2K_3}\, \delta H_{M_1M_2M_3}\, \tilde{F}^{M_4\ldots M_7} - 3\, h_{M_1M_1'}\, \tilde{\varepsilon}^{M_1}{}_{M_2\ldots M_7K_1K_2K_3}\, \tilde{H}^{M_1'M_2M_3}\, \tilde{F}^{M_4\ldots M_7}\Big]\,.
\end{aligned}
\tag{5.101}
$$

5.C The type IIA Gaiotto-Maldacena solutions

The theories that we are going to deal with here are solutions of the type IIA supergravity whose metric in the Einstein frame has the following form:

$$
ds^2 = e^{-\frac{\Phi}{2}}\, ds^2_{(N)ATD} = e^{-\frac{\Phi}{2}}\left(ds^2_{AdS_5} + ds^2_{\mathcal{M}_5}\right), \tag{5.102}
$$

where $ds^2_{(N)ATD}$ are the line elements (in the string frame) of the NATD solutions that we are interested in. Specifically we are going to work with the ones that appear in [197] though we prefer to use more convenient normalizations and also to absorb powers of α' and L. The geometries of those solutions are supported by the corresponding NS and RR fields. Below we present each solution and we make clear the conventions we are using.

5.C.1 The Abelian T-dual solution

This solution arises after performing a Hopf-T-duality transformation along a $U(1)$ direction inside the 5-sphere of the $AdS_5 \times S^5$ solution. The Hopf-T-duality requires a suitable parametrization of the 5-sphere such that a 3-sphere is manifest. After the Hopf-T-duality one finds a solution of the type IIA supergravity with the following geometry:

$$
ds^2_{ATD} = ds^2(AdS_5) + d\Omega_2^2(\alpha,\beta) + \frac{4}{\cos^2\alpha}\, d\psi^2 + \frac{\cos^2\alpha}{4}\, d\Omega_2^2(\chi,\xi)\,, \tag{5.103}
$$

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where

$$d\Omega_2^2(\alpha,\beta) = d\alpha^2 + \sin^2\alpha\, d\beta^2\,, \qquad d\Omega_2^2(\chi,\xi) = d\chi^2 + \sin^2\chi\, d\xi^2\,. \tag{5.104}$$

Also the AdS_5 space is normalized such that $R_{\mu\nu}^{AdS} = -4\, g_{\mu\nu}^{AdS}$ and thus $R^{AdS} = -20$.

The NS sector of this solution consists also of a non-trivial dilaton and a NSNS 2-form:

$$e^{-2\Phi} = \frac{\cos^2\alpha}{4}\,, \qquad B_2 = \psi\, \sin\chi\, d\chi\wedge d\xi\,. \tag{5.105}$$

Finally, the solution is also supported by a RR 4-form whose expression is:

$$F_4 = \frac{1}{2}\cos^3\alpha\, \sin\alpha\, \sin\chi\, d\alpha\wedge d\beta\wedge d\chi\wedge d\xi\,. \tag{5.106}$$

5.C.2 The non-Abelian T-dual solution

The non-Abelian T-dual solution that we are interested in is a result of a non-Abelian T-duality transformation along an $SU(2)$ isometry inside the 5-sphere of the type IIB background with geometry $AdS_5\times S^5$. In order to perform such a non-Abelian transformation one has to parametrize the 5-sphere in such a way that a 3-sphere becomes manifest. Then the transformation takes place in the 3-sphere. The resulting type IIA solution has a geometry of the form:

$$ds^2 = ds^2(AdS_5) + d\Omega_2^2(\alpha,\beta) + \frac{4}{\cos^2\alpha}\, d\rho^2 + \frac{4\,\rho^2\,\cos^2\alpha}{16\,\rho^2 + \cos^4\alpha}\, d\Omega_2^2(\chi,\xi) \tag{5.107}$$

where the line elements for the 2-spheres are given in equation (5.104). Again the AdS_5 space is normalized such that it has a unit radius.

The NSNS sector consists of a non-trivial dilaton and a NSNS 2-form:

$$e^{-2\Phi} = \frac{\cos^2\alpha}{64}\left(16\,\rho^2 + \cos^4\alpha\right), \qquad B_2 = \frac{16\,\rho^3}{16\,\rho^2+\cos^4\alpha}\, \sin\chi\, d\chi\wedge d\xi\,. \tag{5.108}$$

Finally the RR sector consists of a 2- and a 4-form given below:

$$\begin{aligned} F_2 &= \frac{1}{2}\cos^3\alpha\, \sin\alpha\, d\alpha\wedge d\beta\,, \\ F_4 &= \frac{8\,\rho^3\,\cos^3\alpha}{16\,\rho^2+\cos^4\alpha}\, \sin\alpha\, \sin\chi\, d\alpha\wedge d\beta\wedge d\chi\wedge d\xi\,. \end{aligned} \tag{5.109}$$

5.D WKB approximation

In this appendix we review the WKB approximation method following the lines of [204]. The formalism that was developed in [204] applies to eigenvalue problems for second-order differential equations of the following form:

$$\partial_r\big(p(r)\partial_r\Psi\big) + \Big(M^2 w(r) + q(r)\Big)\Psi = 0\,, \tag{5.110}$$

where M represents the eigenvalue (in our case it represents the graviton mass). The functions $p(r), w(r)$ and $q(r)$ are independent of M. When equation (5.110) is written in appropriate variables there is a point r_* where the above functions behave as:

$$p \approx p_1(r-r_*)^{s_1}\,, \qquad w \approx w_1(r-r_*)^{s_2}\,, \qquad q \approx q_1(r-r_*)^{s_3} \qquad \text{as} \quad r \to r_*\,, \tag{5.111}$$

where p_1, w_1, q_1 and s_1, s_2, s_3 are constants. Similarly, we assume

$$p \approx p_2 r^{t_1}\,, \qquad w \approx w_2 r^{t_2}\,, \qquad q \approx q_2 r^{t_3} \qquad \text{as} \quad r \to \infty\,, \tag{5.112}$$

where again p_2, w_2, q_2 and t_1, t_2, t_3 are constants.

Using the above expansions one can derive an approximate formula for the eigenvalue M whose accuracy is good for large enough values of M. This formula reads:

$$M^2 = \frac{\pi^2}{\xi^2}\,\nu\Big(\nu - 1 + \frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\Big) + \mathcal{O}(\nu^0)\,, \qquad \nu = 1, 2, \ldots\,, \tag{5.113}$$

where ξ is given by:

$$\xi := \int_{r_*}^{\infty} dr \sqrt{\frac{w}{p}}\,. \tag{5.114}$$

Also the constants α_1, α_2 and β_1, β_2 are determined by the expansion parameters:

$$\alpha_1 = s_2 - s_1 + 2\,, \qquad \beta_1 = t_1 - t_2 - 2 \tag{5.115}$$

and

$$\begin{aligned} \alpha_2 &= |s_1 - 1| \qquad \text{or} \qquad \alpha_2 = \sqrt{(s_1-1)^2 - 4\frac{q_1}{p_1}} \quad (\text{if} \quad s_3 - s_1 + 2 = 0)\,, \\ \beta_2 &= |t_1 - 1| \qquad \text{or} \qquad \beta_2 = \sqrt{(t_1-1)^2 - 4\frac{q_2}{p_2}} \quad (\text{if} \quad t_1 - t_3 - 2 = 0)\,. \end{aligned} \tag{5.116}$$

6

Supersymmetric probes in the Brandhuber-Oz background

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6.1 Introduction

In this chapter, we add supersymmetric defects to the Brandhuber-Oz (BO) solution described in chapter 1. In this introduction, we will briefly explain how to add defects to a quantum field theory and its meaning from the point of view of the dual gravitational theory.

One natural way to probe a theory is to consider its dynamics in the presence of boundaries. Such boundaries must carry the appropriate degrees of freedom and thus host a lower-dimensional QFT which is coupled with the ambient QFT. These defect QFT's can be very interesting by themselves and lead to new insights in QFT (*e.g.* [205, 206]). The story becomes particularly interesting in the case of ambient CFT's with a gravity dual, where it has been shown that, by considering the appropriate branes in the dual geometry, it is possible to find a gravity dual to the defect QFT [59, 207, 208]. This gravity dual can be either in terms of probe branes in the geometry dual to the ambient QFT or, if the number of probe branes is large (*i.e.* if the number of degrees of freedom of the defect is large), in terms of the fully backreacted geometry corresponding to the brane intersection which gives rise to the defect QFT (see for example [209, 210]). As a by-product of these constructions, since in particular the defect theories typically host degrees of freedom in the fundamental representation of the gauge group, defect QFT's with a gravity dual provide a way to find a holographic description for more realistic

QFT's. This has been explored in the past to study the operator content and particle masses in these theories (see for example [62, 211–215]).

Motivated by these considerations, in this chapter we study, from a holographic point of view, SUSY-preserving defects in 5-dimensional SCFT's. We will concentrate on the higher-rank version of the E_{N_f+1} theories whose gravitational dual is the Brandhuber-Oz solution studied in chapter 1. In view of the String Theory construction one may, in principle, classify all possible supersymmetric defects by adding supersymmetric branes to the D4-O8-D8 configuration. Depending on the number of added branes the most convenient description will be either as probe branes in the BO background or as fully backreacted backgrounds. Indeed, some such defects have been studied in the past. In [216] (see also [217]), codimension-3 defects leading to defect QFT's with conformal invariance were studied upon backreaction from a supergravity point of view, finding the corresponding AdS_3 backgrounds. This was extended in [218–221] to codimension-4 defects, leading to AdS_2 backgrounds. Instead, in this chapter we will study codimension-1 and -2 defects. These correspond, respectively, to D6- and (certain) D4-branes, which we will treat as probes. Hence we will be able to study the open string degrees of freedom associated to the defect theory, allowing us to study the operator content of the defect theories.

Codimension-1 defects corresponding to D6-branes lead to 4-dimensional conformal defect QFT's. In turn, the case of D4-branes is richer in the sense that they can be arranged in two different ways preserving supersymmetry in both cases. One such arrangement engineers codimension-2 defects which correspond to non-conformal 4-dimensional defect theories. The other corresponds to codimension-4 defects, which can actually be regarded as the Wilson loops studied in [94]. Our analysis shows that there is actually a wider class of solutions out of which those in [94] are a particular case.

The structure of this chapter is as follows. In section 6.2 we review the BO background and its dual CFT, introducing appropriate coordinate choices for our purposes. In section 6.3 we study codimension-2 defects engineered as probe D4-branes in the BO background, discussing their supersymmetry and fluctuations and proposing an operator-fluctuation correspondence. In section 6.4 we study another configuration of D4-branes, where they wrap the internal space and correspond to generalizations of the configuration capturing the antisymmetric Wilson loop. In section 6.5 we study codimension-1 defects, constructed through probe D6-branes and discuss their supersymmetry, fluctuations and the operator-fluctuation map. In section 6.6 we summarize our results. The technical aspects of the computations of the fluctuations for the defects are a bit lenghty and will be omitted, since they can be found in the article [21]. In it, the appendices B and C show the computations of the fluctations for the defect D4-branes and wrapped D4-branes respectively. Appendix D of the same paper provides the corresponding computations for the defect D6-branes. Finally, we provide an appendix with a review of the relevant Killing spinors.

6.2 The D4-D8 background

The BO background corresponds to the near-brane geometry of the D4-O8-N_f D8 system. Since the constituents are mutually supersymmetric, it can be constructed modularly starting with the O8-D8's geometry and later adding the D4-branes. Since it will be useful for latter purposes, let us briefly review the construction. Assuming the D8-branes extending along $\{x^1, \cdots, x^8\}$, so that x^9 is the transverse coordinate to the D8-brane stack, the string frame 10-dimensional metric corresponding to the D8-branes is

$$ds^2 = \frac{dx_{1,8}^2}{\sqrt{\frac{x^9}{m}}} + \sqrt{\frac{x^9}{m}}\,(dx^9)^2\,. \tag{6.1}$$

The parameter $m \sim 8 - N_f$ is the Romans' mass. Upon introducing $x^9 \sim z^{\frac{2}{3}}$ the background can be brought to a conformally flat form [222]

$$ds^2 = \Omega(z)^2\left[dx_{1,8}^2 + dz^2\right], \qquad \Omega(z) = H^{-\frac{1}{4}}\,, \qquad H = \left(\frac{2}{3}mz\right)^{\frac{2}{3}}. \tag{6.2}$$

We now add a stack of N D4-branes, which we will assume to be coincident with the D8-branes. In order for them to be SUSY, they must be completely inside the D8-O8. Thus, let us split the $\mathbb{R}^8$ into the $\mathbb{R}^4_{\rm wv}$ wrapped by both the D4-branes and the D8-branes parameterized by $\{x^1, \cdots, x^4\}$, and the $\mathbb{R}^4_X$ transverse to the D4-branes inside the D8-branes parameterized by $\{X^1, \cdots, X^4\}$. The Freund-Rubin ansatz for such a brane system is

$$ds^2 = \Omega(z)^2\left[h^{-\frac{1}{2}}dx_{1,4}^2 + h^{\frac{1}{2}}\left(d\vec{X}^2 + dz^2\right)^2\right], \qquad F_6 = d(h^{-1})\wedge d\mathrm{Vol}_{\mathrm{Min}_{1,4}}\,, \qquad e^{-\Phi} = H^{\frac{5}{4}}\,h^{\frac{1}{4}}\,. \tag{6.3}$$

Using the equations of motion, one easily finds

$$h = \frac{C}{(\vec{X}^2 + z^2)^{\frac{5}{3}}}\,, \tag{6.4}$$

where C a constant, to be fixed by flux quantization, proportional to N. One can now introduce polar coordinates in $\mathbb{R}^4_X$, denoting the radial coordinate by R:

$$R = \sqrt{\vec{X}^{\,2}} \qquad \rightsquigarrow \qquad d\vec{X}^2 = dR^2 + R^2\,d\Omega_3^2\,. \tag{6.5}$$

Upon doing the further change of coordinates

$$R = r^{\frac{3}{2}}\,\sin\alpha\,, \qquad z = r^{\frac{3}{2}}\cos\alpha\,, \tag{6.6}$$

the geometry becomes the Brandhuber-Oz AdS_6 background[1]

$$ds_{10}^2 = \frac{Q^3}{K^8(\alpha)}\left[\frac{9}{4}\,ds_{AdS_6}^2 + d\Omega_4^2\right], \quad e^{-\Phi} = \frac{\left[K(\alpha)\right]^{20}}{Q^6}\,, \quad K(\alpha) \equiv Q^{\frac{5}{16}}\left(3m\,\cos\alpha\right)^{\frac{1}{24}}\,, \tag{6.7}$$

[1]We use Poincaré coordinates for AdS_6 where $ds^2_{AdS_6} = r^2\,dx_{1,4}^2 + \frac{dr^2}{r^2}$.

where $d\Omega_4^2 = d\alpha^2 + \sin^2\alpha\, d\Omega_3^2$. In addition there is a RR 4-form F_4 given by:

$$F_4 = -\frac{10}{3}\, Q^{\frac{9}{10}}\, e^{-\frac{2\Phi}{5}}\, \mathrm{Vol}(\mathbb{S}^4) = -\frac{10}{3}\, Q^{-\frac{3}{2}}\, K^8(\alpha)\, \mathrm{Vol}(\mathbb{S}^4) \quad \rightsquigarrow \quad C_5 = -\Big(\frac{3}{2}\Big)^5 Q^{\frac{3}{2}}\, r^5\, d^5x \ . \tag{6.8}$$

Finally, the charge quantization conditions require

$$m = \frac{8-N_f}{4\pi}\ , \qquad\qquad Q = \left(\frac{2^{11}\,\pi^4}{3^4\,(8-N_f)}\right)^{\frac{1}{3}} N\ . \tag{6.9}$$

The effect of the change of coordinates in (6.6) is two-fold. On the one hand, it produces the AdS_6. On the other hand, it allows to combine the $\mathbb{S}^3$ inside the $\mathbb{R}^4_X$ with the extra angular coordinate into an $\mathbb{S}^4$. Note however that the range of α is $0 \leq \alpha \leq \pi/2$. Thus we really have not a full $\mathbb{S}^4$ but an $\mathbb{S}^4$-hemisphere, at whose $\mathbb{S}^3$ boundary, located at $\alpha = \frac{\pi}{2}$, the dilaton diverges. Such boundary is naturally interpreted as the position of the orientifold [88].

The background, as written in (6.3), keeps track of the $\mathbb{R}^4_X$, which corresponds to the directions inside the D8-branes transverse to the D4-branes; as well as the z, which is essentially the coordinate transverse to all branes. Since it will be very useful for our later purposes, let us explicitly quote the string frame background in these coordinates

$$ds^2_{10} = \frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}}\,\frac{(\vec{X}^2+z^2)^{\frac{1}{6}}}{z^{\frac{1}{3}}}\left[\frac{9}{4}\big(\vec{X}^2+z^2\big)^{\frac{2}{3}}\, dx^2_{1,4} + \frac{1}{\vec{X}^2+z^2}\,(d\vec{X}^2+dz^2)\right]. \tag{6.10}$$

The dilaton is:

$$e^{-\Phi} = \frac{K^{20}}{Q^6} = Q^{\frac{1}{4}}\,(3m)^{\frac{5}{6}}\,\frac{z^{\frac{5}{6}}}{(\vec{X}^2+z^2)^{\frac{5}{12}}}\ . \tag{6.11}$$

This background has a RR 5-form potential C_5 given by:

$$C_5 = -\Big(\frac{3}{2}\Big)^5 Q^{\frac{3}{2}}\,\big(\vec{X}^2+z^2\big)^{\frac{5}{3}}\, d^5x\ . \tag{6.12}$$

6.2.1 The dual CFT

The CFT dual to the BO background above can be read-off from the brane construction as the worldvolume theory on the stack of N D4-branes on top of the O8$^-$+N_f D8-branes. The String Theory construction shows that this is a strongly coupled rank N CFT with global symmetry E_{N_f+1}. This symmetry arises as the gauge symmetry on N_f D8-branes on top of an O8$^-$ upon tuning the dilaton to diverge on the D8-brane stack (for example (6.7), which shows that the dilaton diverges at $\alpha = \frac{\pi}{2}$). These theories are the higher-rank generalization of [77, 78].

The 5-dimensional CFT's admit a mass-deformation by turning on a Yang-Mills coupling. The String Theory counterpart is to turn on an integration constant in the dilaton which makes it not to diverge on top of the orientifold. At any rate, this deformation triggers an RG flow towards a conventional gauge theory, which can be easily read-off from the brane construction: it is a $USp(2N)$ gauge theory with one antisymmetric hypermultiplet and N_f fundamental hypermultiplets. From this perspective, the global $SO(2N_f)$ symmetry of the hypermultiplets combines with the $U(1)_I$ topological symmetry[2] enhancing it to E_{N_f+1} in the UV fixed point. Moreover, the gauge theory provides intuition for the dynamics of the CFT. For instance, the 5-dimensional vector multiplet contains a real scalar which is naturally identified with the direction transverse to all branes, *i.e.* x^9 (or its z avatar). In turn, the antisymmetric hypermultiplet is naturally identified with the $\mathbb{R}^4_X$ [93]. More specifically, denoting, in 4-dimensional $\mathcal{N}=1$ language, by $A_{1,2}$ the two antisymmetric chirals making up for the antisymmetric hypermultiplet, the meson-like operators constructed out of them are in one-to-one correspondence with the holomorphic functions in $\mathbb{C}^2_X \sim \mathbb{R}^4_X$. Consistently, note that $X^i \sim r^{\frac{3}{2}}$, which (since $A_1 \sim X^1 + i\,X^2$, $A_2 \sim X^3 + i\,X^4$) suggests that the corresponding fields have scaling dimension $\frac{3}{2}$, just as expected for scalar fields in 5 dimensions. In turn $x^9 \sim z^{\frac{2}{3}} \sim (r^{\frac{3}{2}})^{\frac{2}{3}} \sim r$, which is consistent with the identification of x^9 with the (real) scalar in the vector multiplet.[3]

6.3 D4-brane defects

The first example of a supersymmetric probe brane configuration in the BO background we are going to consider is the one corresponding to a D4′-brane probe that creates a codimension-2 defect on the worldvolume of the D4-brane of the background. This setup can be represented by the following array:

$$\begin{array}{cccccccccc} & x^1 & x^2 & x^3 & x^4 & X^1 & X^2 & X^3 & X^4 & z \\ D4: & \times & \times & \times & \times & \_ & \_ & \_ & \_ & \_ \\ D4': & \times & \times & \_ & \_ & \times & \times & \_ & \_ & \_ \end{array} \tag{6.13}$$

where the coordinates are those used in (6.10)-(6.12). To analyze the dynamics of the probe we will use the following set of worldvolume coordinates:

$$\zeta^a \;=\; (x^0, x^1, x^2, X^1, X^2) \;, \tag{6.14}$$

and we will treat x^3, x^4, X^3, X^4 and z as scalar fields parameterizing the embedding. Actually, we will denote:

$$W^1 = x^3 \;, \qquad\qquad W^2 = x^4 \;, \tag{6.15}$$

[2] In 5-dimensional any gauge theory automatically contains a topologically conserved $U(1)$ current $j_I = \star \mathrm{Tr} F \wedge F$. The states electrically charged under this symmetry are instantonic particles.

[3] SUSY requires the kinetic term for a scalar in the vector multiplet to be $g_{YM}^{-2}\,\partial\phi^2$. Since in 5 dimensions g_{YM}^2 has mass-dimension 1, ϕ has mass-dimension 1. Note that scalars in a hypermultiplet have a standard kinetic term, and thus they have the expected mass-dimension $\frac{3}{2}$.

and we shall adopt the following embedding ansatz:

$$W^1 \,=\, W^1(X^1,X^2)\;, \qquad W^2 \,=\, W^1(X^1,X^2)\;, \qquad X^3, X^4, z = \text{constant}\;. \tag{6.16}$$

With no loss of generality we can take

$$X^3 \,=\, L\;, \qquad X^4 \,=\, 0\;, \qquad z \,=\, a\;. \tag{6.17}$$

Moreover, we define the variable σ as:

$$\sigma \,=\, \sqrt{(X^1)^2+(X^2)^2}\;. \tag{6.18}$$

For such an ansatz, the induced metric on the worldvolume of the probe D4-brane can be obtained by computing the pullback of the line element (6.10):

$$ds^2_5 \,=\, \frac{Q^{\frac{1}{2}}}{(3\,m\,a)^{\frac{1}{3}}}\left[\frac{9}{4}\,(\sigma^2+L^2+a^2)^{\frac{5}{6}}\,dx^2_{1,2} \,+\, h_{ij}\,dX^i\,dX^j\right]\;, \tag{6.19}$$

where the indices i,j can take the values $1,2$, $\partial_i \equiv \partial_{X^i}$ and h_{ij} is the matrix:

$$h_{ij} \,=\, \frac{\delta_{ij}}{(\sigma^2+L^2+a^2)^{\frac{5}{6}}} \,+\, \frac{9}{4}\,(\sigma^2+L^2+a^2)^{\frac{5}{6}}\big(\,\partial_i W^1\,\partial_j W^1 + \partial_i W^2\,\partial_j W^2\,\big)\;. \tag{6.20}$$

The action of the probe is the sum of a DBI and WZ term:

$$S \,=\, S_{DBI} \,+\, S_{WZ}\;. \tag{6.21}$$

If we do not excite the woldvolume gauge field, the DBI action can be written as:

$$S_{DBI} \,=\, -T_4\int d^5\zeta\; e^{-\Phi}\,\sqrt{-\det g_5}\;, \tag{6.22}$$

where T_4 is the tension of the D4-brane and g_5 is the induced metric (6.19). More explicitly we have:

$$S_{DBI} \,=\, -\Big(\frac{3}{2}\Big)^3\,Q^{\frac{3}{2}}\,T_4\int d^5\zeta\,(\sigma^2+L^2+a^2)^{\frac{5}{6}}\sqrt{\det h_{ij}}\;. \tag{6.23}$$

The WZ term of the action for our setup is given by:

$$S_{WZ} \,=\, -T_4\int \hat{C}_5\;, \tag{6.24}$$

where $\hat{C}_5$ denotes the pullback of the RR 5-form potential to the worlvolume. For our system of coordinates this pullback is given by:

$$\hat{C}_5 \,=\, -\Big(\frac{3}{2}\Big)^5\,Q^{\frac{3}{2}}\,(\sigma^2+L^2+a^2)^{\frac{5}{3}}\,\big(\partial_1\,W^1\partial_2\,W^2-\partial_1\,W^2\partial_2\,W^1\big)d^3x\wedge dX^1\wedge dX^2\;, \tag{6.25}$$

with $d^3x = dx^0\wedge dx^1\wedge dx^2$. It follows from this last expression that S_{WZ} is given by:

$$S_{WZ} \,=\, \Big(\frac{3}{2}\Big)^5\,Q^{\frac{3}{2}}\,T_4\int d^5\zeta\,(\sigma^2+L^2+a^2)^{\frac{5}{3}}\,\big(\partial_1\,W^1\partial_2\,W^2-\partial_1\,W^2\partial_2\,W^1\big)\;. \tag{6.26}$$

6.3.1 Kappa symmetry

We now study SUSY configurations of a D4-brane probe embedded in the warped $AdS_6 \times \mathbb{S}^4$ background as described in the previous section. These configurations should satisfy the kappa symmetry condition:

$$\Gamma_\kappa \, \epsilon \;=\; \pm \epsilon \;, \tag{6.27}$$

where ϵ is a Killing spinor and, if there are no excited gauge fields on the worldvolume, the matrix Γ_κ is:

$$\Gamma_\kappa \;=\; \frac{1}{\sqrt{-\det g_5}} \, \frac{1}{5!} \, \epsilon^{a_1 \cdots a_5} \, \Gamma_{11} \, \gamma_{a_1 \cdots a_5} \;=\; \frac{1}{\sqrt{-\det g_5}} \, \Gamma_{11} \, \gamma_{x^0 \, x^1 \, x^2 \, X^1 \, X^2} \;, \tag{6.28}$$

where, in the second step, we have written the form of Γ_κ for the system (6.14) of worldvolume coordinates. In (6.28) the γ's are induced Dirac matrices along the different worldvolume coordinates and $\gamma_{\alpha_1 \cdots \alpha_5}$ denotes their antisymmetrized product. The γ's are related to the flat Dirac matrices of the background (denoted by Γ) as:

$$\begin{aligned}
\gamma_{x^\mu} \;&=\; \frac{3}{2} \, \frac{Q^{\frac{1}{4}}}{(3\,m\,a)^{\frac{1}{6}}} \big(\sigma^2 \;+\; L^2 \;+ a^2\big)^{\frac{5}{12}} \, \Gamma_{x^\mu} \;, \qquad (\mu = 0, 1, 2) \;, \\
\gamma_{X^i} \;&=\; \frac{Q^{\frac{1}{4}}}{(3\,m\,a)^{\frac{1}{6}}} \Bigg[\frac{3}{2} \big(\sigma^2 \;+\; L^2 \;+ a^2\big)^{\frac{5}{12}} \big(\partial_i \, W^1 \, \Gamma_{x^4} + \partial_i \, W^2 \, \Gamma_{x^5}\big) \;+ \\
&\qquad\qquad + \frac{\Gamma_{X^i}}{\big(\sigma^2 \;+ L^2 \;+ a^2\big)^{\frac{5}{12}}} \Bigg] , \qquad (i = 1, 2) \;.
\end{aligned} \tag{6.29}$$

After a simple calculation one can check that Γ_κ can be written as:

$$\Gamma_\kappa = \Big(\frac{3}{2}\Big)^3 \, \frac{Q^{\frac{3}{4}}}{(3ma)^{\frac{1}{2}}} \, \frac{\big(\sigma^2 \;+\; L^2 \;+ a^2\big)^{\frac{5}{4}}}{\sqrt{-\det g_5}} \, \Gamma_{11} \, \Gamma_{x^0 x^1 x^2} \, \gamma_{X^1 \, X^2} \;, \tag{6.30}$$

where $\gamma_{X^1 \, X^2}$ is given by:

$$\begin{aligned}
\gamma_{X^1 X^2} \;=\; \frac{Q^{\frac{1}{2}}}{(3\,m\,a)^{\frac{1}{3}}} \Bigg[& \frac{\Gamma_{X^1 X^2}}{\big(\sigma^2 + L^2 + a^2\big)^{\frac{5}{6}}} + \frac{9}{4} \big(\sigma^2 + L^2 + a^2\big)^{\frac{5}{6}} (\partial_1 W^1 \partial_2 W^2 \,- \\
& -\partial_1 W^2 \partial_2 W^1) \Gamma_{x^4 x^5} \;+ \frac{3}{2} \Big(\partial_2 W^1 \Gamma_{X^1 x^4} + \partial_2 W^2 \Gamma_{X^1 x^5} \,- \\
& -\partial_1 W^1 \Gamma_{X^2 x^4} - \partial_1 W^2 \Gamma_{X^2 x^5} \Big) \Bigg] \;.
\end{aligned} \tag{6.31}$$

Let us assume that the spinor ϵ satisfies the projection corresponding to the D4-branes of the background, namely:

$$\Gamma_{11} \, \Gamma_{x^0 x^1 x^2 x^3 x^4} \epsilon \;=\; \epsilon \;. \tag{6.32}$$

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Moreover, we impose the following additional projection:

$$\Gamma_{11}\,\Gamma_{x^0x^1x^2X^1X^2}\epsilon\,=\,\epsilon\;, \tag{6.33}$$

which is the one corresponding to having D4′-branes extended as in the array (6.13). Combined together, (6.32) and (6.33) lead to:

$$\Gamma_{X^1X^2}\epsilon\,=\,\Gamma_{x^4x^5}\epsilon\;. \tag{6.34}$$

Then, after a simple calculation one can demonstrate that:

$$\begin{aligned}\frac{(3\,m\,a)^{\frac{1}{3}}}{Q^{\frac{1}{2}}}\gamma_{X^1\,X^2}\,\epsilon\,=\,&\Bigg[\frac{1}{\big(\sigma^2+L^2+a^2\big)^{\frac{5}{6}}}+\\&+\frac{9}{4}\,(\sigma^2+L^2+a^2)^{\frac{5}{6}}\,(\partial_1W^1\partial_2W^2-\partial_1W^2\partial_2W^1)\Bigg]\Gamma_{X^1X^2}\epsilon\,+\\&+\frac{3}{2}\,(\partial_2\,W^1+\partial_1W^2)\,\Gamma_{X^1x^4}\,\epsilon\,+\,\frac{3}{2}\,(\partial_2\,W^2-\partial_1W^1)\,\Gamma_{X^1x^5}\,\epsilon\;.\end{aligned} \tag{6.35}$$

We can now use this result, as well as the projection (6.33), to compute $\Gamma_\kappa\,\epsilon$. One can show that, in order to cancel the terms that do not contain the unit matrix one should impose the following conditions to the embedding functions W^1 and W^2:

$$\partial_1W^1\,=\,\partial_2\,W^2\;,\qquad\qquad \partial_2\,W^1\,=\,-\partial_1W^2\;. \tag{6.36}$$

The two equations in (6.36) are nothing but the Cauchy-Riemann equations. Indeed, if we define the complex variables Z and W as

$$Z\,=\,X^1+i\,X^2\;,\qquad\qquad W\,=\,W^1+i\,W^2\;, \tag{6.37}$$

as well as the holomorphic and antiholomorphic derivatives ∂ and $\bar\partial$ as:

$$\partial\,=\,\frac{1}{2}(\partial_1\,-\,i\partial_2)\;,\qquad\qquad \bar\partial\,=\,\frac{1}{2}(\partial_1\,+\,i\partial_2)\;, \tag{6.38}$$

then, the BPS conditions (6.36) become simply:

$$\bar\partial\,W\,=\,0\;. \tag{6.39}$$

Equation (6.39) is solved by an arbitrary holomorphic function of the type:

$$W\,=\,W(Z)\;, \tag{6.40}$$

i.e., by a function W that depends on Z and not on $\bar Z$. Moreover, if (6.39) holds one can show that the action of Γ_κ on ϵ is:

$$\sqrt{-\det g_5}\,\Gamma_\kappa\epsilon\Big|_{BPS}\,=\,\Big(\frac{3}{2}\Big)^3\frac{Q^{\frac{5}{4}}}{(3ma)^{\frac{5}{6}}}\big(\sigma^2+L^2+a^2\big)^{\frac{5}{12}}\Bigg[1+\frac{9}{4}\big(\sigma^2+L^2+a^2\big)^{\frac{5}{3}}\partial W\bar\partial\bar W\Bigg]\epsilon\;. \tag{6.41}$$

Taking into account that:

$$\sqrt{\det h_{ij}}\Big|_{BPS} = \frac{1}{\left(\sigma^2 + L^2 + a^2\right)^{\frac{5}{6}}} \left[1 + \frac{9}{4}\left(\sigma^2 + L^2 + a^2\right)^{\frac{5}{3}} \partial W \bar{\partial} \bar{W}\right] , \tag{6.42}$$

one can straightforwardly demonstrate that $\sqrt{-\det g_5}$ for a BPS configuration equals the function multiplying the spinor on the right-hand side of (6.41), which proves that $\Gamma_\kappa \, \epsilon = \epsilon$ when the holomorphic condition (6.39) is satisfied.

Let us now study the action of the probe for the holomorphic embeddings. By using (6.36) in (6.23) we get that the DBI term for a BPS configuration is:

$$S_{DBI}\Big|_{BPS} = -\left(\frac{3}{2}\right)^3 Q^{\frac{3}{2}} \, T_4 \int d^5\zeta \left[1 + \frac{9}{4}\left(\sigma^2 + L^2 + a^2\right)^{\frac{5}{3}} \partial W \bar{\partial} \bar{W}\right] . \tag{6.43}$$

Moreover, from (6.26) we get the form of the WZ term:

$$S_{WZ}\Big|_{BPS} = \left(\frac{3}{2}\right)^5 Q^{\frac{3}{2}} \, T_4 \int d^5\zeta \left(\sigma^2 + L^2 + a^2\right)^{\frac{5}{3}} \partial W \bar{\partial} \bar{W} \, . \tag{6.44}$$

We notice that (6.44) cancels against the second term in (6.43), and thus for any holomorphic embedding we find the no-force condition characteristic of supersymmetry.

Let us now consider the induced metric for the BPS embeddings. First of all, it is quite convenient to parameterize the $X^1 \, X^2$ plane by the radial variable σ introduced in (6.18) and by an angle ϑ in such a way that:

$$(dX^1)^2 + (dX^2)^2 = d\sigma^2 + \sigma^2 \, d\vartheta^2 \, . \tag{6.45}$$

Moreover, let us change the radial variable σ and use instead the new coordinate ϱ, related to σ as:

$$\sigma = \varrho^{\frac{3}{2}} \sqrt{1 - \frac{L^2 + a^2}{\varrho^3}} \, . \tag{6.46}$$

Then, for an holomorphic embedding, one can check that the induced metric on the D4-brane worldvolume (6.19) becomes:

$$\begin{aligned} ds_5^2 = \frac{Q^{\frac{1}{2}}}{(3ma)^{\frac{1}{3}}} \, \varrho^{\frac{1}{2}} \Bigg[& \frac{9}{4} \, \varrho^2 \, dx_{1,2}^2 \, + \\ & + \left(1 + \frac{9}{4} \, \varrho^5 \partial W \bar{\partial} \bar{W}\right) \left(\frac{9}{4} \, \frac{d\varrho^2}{\varrho^2 \left(1 - \frac{L^2 + a^2}{\varrho^3}\right)} + \left(1 - \frac{L^2 + a^2}{\varrho^3}\right) d\vartheta^2\right) \Bigg] . \end{aligned} \tag{6.47}$$

Note that in (6.47) $\varrho^3 \geq L^2 + a^2$. Moreover, note that while L may be taken to zero, a must be non-vanishing. Thus, the space naturally ends at $\varrho^3 \geq a^2$ when L is zero.

When the embedding is trivial, *i.e.*, when W is constant, the metric (6.47) in the UV becomes $AdS_4 \times \mathbb{S}^1$, up to a warp factor proportional to $\varrho^{\frac{1}{2}}$. Let us now determine the non-trivial embeddings that preserve this UV limit. Let us suppose that $W(Z)$ behaves in the UV as:

$$W \sim Z^{-\lambda} \,, \tag{6.48}$$

where λ is a real exponent. Taking into account that, for large ϱ, one has

$$\varrho^5 \, \partial W \bar{\partial} \bar{W} \; \sim \; \varrho^{2-3\lambda} \,, \qquad\qquad (\varrho \to \infty) \,, \tag{6.49}$$

it follows that the exponent λ should be bounded from below by:

$$\lambda \geq \frac{2}{3} \,. \tag{6.50}$$

For $\lambda > 2/3$ the term depending on W does not contribute to the UV behavior of the induced metric, whereas in the critical case $\lambda = 2/3$ the non-trivial bending of the probe changes the radius of the warped $AdS_4 \times \mathbb{S}^1$. Notice that these marginal embeddings can also be written as $Z \sim W^{-\frac{3}{2}}$. Note that the transverse scalars, corresponding to scalar fields in hypermultiplets in the dual theory, have precisely mass-dimension $\frac{3}{2}$, thus supporting the identification of this configuration with the Higgs branch of the defect field theory.

6.3.2 Fluctuations

Let us now explore the fluctuations of the probe D4-brane around the simplest of the configurations found above, namely that with $L = W = 0$. The detailed derivation of the equations governing these fluctuations and the analysis of the different modes can be found in appendix B of [21], summarizing and interpreting here the results. In the unperturbed configuration the scalars x^3, x^4, X^3 and X^4 have vanishing value, whereas $z = a$. We will allow the probe D4-brane to oscillate around this configuration and we will denote the fluctuations as:

$$(\delta x^3, \delta x^4) \;=\; (U^1, U^2) \,, \qquad\qquad (\delta X^3, \delta X^4, \delta z) \;=\; (Y^1, Y^2, Y^3) \,. \tag{6.51}$$

Moreover, we switch on a worldvolume gauge field A_μ (the corresponding fluctuation mode is denoted by V_μ). The corresponding second-order action is written in appendix B of [21]. The metric $\mathcal{G}_{ab}$ entering the action has been written in the same appendix and it corresponds to a geometry that approaches a (radially warped) $AdS_4 \times \mathbb{S}^1$ space in the UV to which we can apply holography in the generalized sense of [223]. The radial warping suggests that the dual theory is a 3-dimensional non-conformal defect QFT preserving (3-dimensional) $\mathcal{N} = 2$, containing the restriction of the rank N E_{N_f+1} theory on the color D4-branes down to the (3+1)-dimensional defect, in addition to N_{D4} hypermultiplets. Since $a^{\frac{2}{3}} \sim x_9^{D4}$ is the distance between the two types of D4-branes, it sets the flavor mass

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scale. Thus $x_9^{D4} \sim M$ or, equivalently $a \sim M^{\frac{3}{2}}$. We stress that the brane configuration becomes singular if $a \to 0$ and, therefore, it is not possible to remove the mass scale M.

To further understand the role of M, let us consider the worldvolume action of the probe D4-brane. Let us suppose that we turn on the gauge field $F_{\mu\nu}$ along the Minkowski directions. It is easy to extract the dependence on a of the action for the fluctuations by rescaling the holographic coordinate ϱ as $\varrho = a^{\frac{2}{3}}\,\hat{\varrho}$. Doing this the dependence on a of the action appears as a global coefficient and one schematically gets:

$$S \sim \int d^3x\, \frac{1}{a^{\frac{2}{3}}}\, F^2 \sim \int d^3x\, \frac{1}{M}\, F^2 \ . \tag{6.52}$$

Thus, we see that the Yang-Mills coupling for the flavor symmetry is $e^2 \sim a^{\frac{2}{3}} \sim x_9^{D4}$. Therefore $e \sim M^{\frac{1}{2}}$ and the power of M above is naturally understood as coming from the flavor symmetry gauge coupling.

The dimensions of the operators dual to the U, Y and V_μ can also be found in the appendix. They depend on the number n that fixes their dependence on the angular coordinate ϑ (*i.e.*, the winding number along the $\mathbb{S}^1$ of the metric). This dependence is of the type:

$$\Delta \,=\, \Delta(n=0) + \frac{3n}{2}\ , \tag{6.53}$$

and it is easily understood as due to the insertion of n adjoint scalars represented by the coordinates (X^1, X^2). Focusing on the Y and V_μ modes, the lowest dimension fluctuations have, respectively, $\Delta_Y = \frac{3}{2}$ and $\Delta_V = \frac{3}{2}$. Given the mass dependence of the terms in the fluctuation action, it is quite natural to think that the dual operators are rescaled by a factor of the type M^α for some α, this rescaling being a multiplicative renormalization of the operator. In the cases of the lowest-lying Y and V_μ modes we conjecture that they are dual to the flavor current conserved multiplet, with the rough identification:

Fluctuation	Δ	Dual operator
Y	$\frac{3}{2}$	$M^{-\frac{1}{2}}\,\bar{\psi}\,\psi$
V_μ	$\frac{3}{2}$	$M^{-\frac{1}{2}}\left(\bar{\psi}\gamma_\mu\psi + q^\dagger D_\mu q - \tilde{q}^\dagger D_\mu \tilde{q}\right)$

(6.54)

where q and $\tilde{q}$ are 3-dimensional scalar fields (with canonical dimension 1/2) and ψ and $\bar{\psi}$ are fermionic fields in 3 dimensions with dimension 1. Notice that, in the Y-fluctuations, the distance between color and flavor D4-branes in the space orthogonal to both of them changes. In the holographic setup this distance is related to the quark mass, that sources the meson operator $\bar{\psi}\,\psi$, which we have identified in (6.54) as the dual of the Y-fluctuation. Moreover, it is clear that the fluctuation of the vector field on the probe should couple to a vector current, as in (6.54).

In the U-mode the probe D4-brane expands into the worldvolume directions of the color D4-branes. This kind of brane recombination is naturally associated with the Higgs

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branch of the field theory and the corresponding dual operator is a bilinear in the scalar fields [211, 212]. Thus, we are led to identify the lowest-lying U-mode with an operator of the type $M^3(q^\dagger q - \tilde{q}^\dagger \tilde{q})$, where the power of M has been adjusted to have $\Delta_U = 4$ (for more details, see appendix B of [21]).

6.4 Wrapped D4-branes

In this section we will analyze a second setup with probe D4-branes, in which the probes wrap the 3-sphere of the internal space, extend along the holographic coordinate and intersect the D4-branes of the background at a single point, creating in this way a point-like codimension-4 defect. Let us suppose that we write the BO metric as in (6.7) and let φ, θ and ψ be three angles that parameterize the $\mathbb{S}^3$ inside the half $\mathbb{S}^4$. Then, the configuration we are going to study can be represented by the following array:

$$\begin{array}{cccccccccc} & x^1 & x^2 & x^3 & x^4 & r & \varphi & \theta & \psi & \alpha \\ D4: & \_ & \_ & \_ & \_ & \times & \times & \times & \times & \_ \end{array} \tag{6.55}$$

We will take the following set of worldvolume coordinates:

$$\zeta^a = (x^0, r, \varphi, \theta, \psi) , \tag{6.56}$$

and we will adopt the following embedding ansatz:

$$\alpha = \alpha(r) , \qquad\qquad x^i = \text{constant} , \qquad (i = 1, \cdots, 4) . \tag{6.57}$$

Inspecting the WZ term of the D4-brane worldvolume action one readily concludes that the flux of the RR 4-form F_4 sources a non-trivial worldvolume gauge field F due to the term $\int C_3 \wedge F$, where C_3 is the potential for F_4. Since C_3 has only legs along the $\mathbb{S}^3$, it follows that the F_{0r} component of F is the one that is induced by the RR flux. Accordingly, we will switch on in our ansatz the F_{0r} component of F. The induced metric on the worldvolume in Einstein frame takes the form:

$$ds^2_{5,E} = K^2(\alpha) \left[-\frac{9}{4} r^2 (dx^0)^2 + \frac{9}{4r^2} \left(1 + \frac{4}{9} r^2 \alpha'^2\right) dr^2 + \sin^2\alpha \, d\Omega_3^2 \right] , \tag{6.58}$$

where $d\Omega_3^2$ is the line element of the $\mathbb{S}^3$, which we will represent as in (6.194) in terms of the three $SU(2)$ invariant 3-forms that, in turn, can be parameterized with the three angles φ, θ and ψ and their differentials as in (6.195). Notice that, for fixed α, the induced metric is of the form $AdS_2 \times \mathbb{S}^3$. The worldvolume gauge field F lives in the AdS_2 part of the metric.

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6.4.1 Kappa symmetry

We will start our analysis by determining the conditions which make our configuration kappa-symmetric (and therefore SUSY-preserving). Since our ansatz contains a worldvolume gauge field, the kappa symmetry matrix Γ_κ for the D4-brane is now:

$$\Gamma_\kappa \,=\, \frac{1}{\sqrt{-\det(g+\hat{F})}} \sum_{n=0}^{\infty} \frac{1}{2^n\, n!}\, \gamma^{a_1\, b_1}\, \gamma^{a_2\, b_2} \,\cdots \gamma^{a_n\, b_n}\, \hat{F}_{a_1\, b_1}\, \hat{F}_{a_2\, b_2} \cdots \hat{F}_{a_n\, b_n}\, J^{(n)} \,, \tag{6.59}$$

where $\hat{F} \,=\, e^{-\frac{\Phi}{2}}\, F$ and the matrix $J^{(n)}$ is given by:

$$J^{(n)} \,=\, \frac{1}{5!}\, \big(\Gamma_{11}\big)^{n+1}\, \epsilon^{a_1\cdots a_5}\, \gamma_{a_1\cdots a_5} \,. \tag{6.60}$$

The $e^{-\frac{\Phi}{2}}$ factor multiplying F is due to the fact that we are working in the Einstein frame. For our configuration, only the terms with $n = 0, 1$ contribute to Γ_κ:

$$\begin{aligned}
\Gamma_\kappa \,&=\, \frac{1}{5!\,\sqrt{-\det(g+\hat{F})}} \Big[\Gamma_{11} \,+\, \frac{1}{2}\,\gamma^{b_1\, b_2}\, \hat{F}_{b_1\, b_2}\Big] \epsilon^{a_1\,\cdots a_5}\, \gamma_{a_1\,\cdots a_5} \,= \\
&=\, \frac{1}{\sqrt{-\det(g+\hat{F})}} \Big[\Gamma_{11} \,+\, \gamma^{x^0\, r}\, \hat{F}_{x^0\, r}\Big]\, \gamma_{x^0\, r\, \varphi\, \theta\, \psi} \,.
\end{aligned} \tag{6.61}$$

We have to impose (6.27) for the matrix Γ_κ written in (6.61) and ϵ being a Killing spinor of the BO geometry. We will assume that the ϵ are ordinary Killing spinors which, in this AdS_6 coordinate system can be written as in (6.218) in terms of a constant spinor η. Actually, if we define the modified matrix $\tilde{\Gamma}_\kappa$ as:

$$\tilde{\Gamma}_\kappa \,=\, e^{\frac{\alpha}{2}\,\Gamma^{789}}\, \Gamma_\kappa\, e^{-\frac{\alpha}{2}\,\Gamma^{789}} \,, \tag{6.62}$$

then, the kappa symmetry condition (6.27) takes the following form in terms of η:

$$\tilde{\Gamma}_\kappa\, \eta \,=\, \pm \eta \,. \tag{6.63}$$

Let us now obtain $\tilde{\Gamma}_\kappa$ in terms of constant Dirac matrices corresponding to the 1-form basis written in (6.196). The induced γ-matrices for our ansatz are:

$$\begin{aligned}
&\gamma_{x^0} \,=\, \frac{3}{2}\, K(\alpha)\, r\, \Gamma_0 \,, \qquad\qquad \gamma_r \,=\, K(\alpha)\, \Big[\frac{3}{2r}\, \Gamma_5 \,+\, \alpha'\, \Gamma_6\Big] \,, \\
&\gamma_\varphi \,=\, \frac{K(\alpha)\sin\alpha}{2}\, \Big[\cos\theta\, \Gamma_9 \,+\, \sin\theta\big(\cos\psi\, \Gamma_8 + \sin\psi \Gamma_7\big)\Big] \,, \\
&\gamma_\theta \,=\, \frac{K(\alpha)\sin\alpha}{2}\, \big[\cos\psi\, \Gamma_7 \,-\, \sin\psi\, \Gamma_8\big] \,, \qquad\qquad \gamma_\psi \,=\, \frac{K(\alpha)\sin\alpha}{2}\, \Gamma_9 \,.
\end{aligned} \tag{6.64}$$

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Therefore, their antisymmetrized product is:

$$\gamma_{x^0\, r\,\varphi\,\theta\,\psi} \;=\; -\frac{9}{32}\, K^5(\alpha)\, \sin^3\alpha\, \sin\theta \Big[\Gamma_{05} \,+\, \frac{2}{3}\, r\alpha'\,\Gamma_{06}\Big]\,\Gamma_{789}\;. \tag{6.65}$$

The indices of the induced γ-matrices are raised with the inverse of the induced metric. Thus, we have:

$$\gamma^{x^0\, r} \;=\; g^{x^0\, x^0}\, g^{r\,r}\,\gamma_{x^0\, r} \;=\; \frac{K^3(\alpha)}{8}\,\frac{\sin^3\alpha\,\sin\theta}{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}}\,\Big[\Gamma_{05} \,+\, \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\Big]^2\,\Gamma_{789}\;. \tag{6.66}$$

Since $\Gamma_{05}^2 \;=\; \Gamma_{06}^2 \;=\; 1$ and $\{\Gamma_{05}, \Gamma_{06}\} = 0$, it follows that:

$$\Big[\Gamma_{05} \,+\, \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\Big]^2 \;=\; 1 \,+\, \frac{4}{9}\,r^2\,\alpha'^{\,2}\;, \tag{6.67}$$

and then:

$$\gamma^{x^0\, r}\,\gamma_{x^0\, r\,\varphi\,\theta\,\psi} \;=\; \frac{K^3(\alpha)}{8}\,\sin^3\alpha\,\sin\theta\,\Gamma_{789}\;. \tag{6.68}$$

Using (6.65) and (6.68) to evaluate Γ_κ in (6.61), we get:

$$\sqrt{-\det(g+\hat{F})}\,\Gamma_\kappa \;=\; \frac{K^5}{8}\,\sin^3\alpha\,\sin\theta\Big[-\frac{9}{4}\,\Gamma_{11}\,\big(\Gamma_{05} + \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\,\big) \,+\, \frac{\hat{F}_{0r}}{K^2}\Big]\,\Gamma_{789}\;. \tag{6.69}$$

We now compute $\tilde{\Gamma}_\kappa$, as defined in (6.62). We use:

$$e^{\frac{\alpha}{2}\,\Gamma^{789}}\,\Gamma_{11}\,\big(\Gamma_{05} + \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\,\big)\,e^{-\frac{\alpha}{2}\,\Gamma^{789}} \;=\; \Gamma_{11}\,\big(\Gamma_{05} + \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\,\big)(\cos\alpha - \sin\alpha\,\Gamma_{789})\;. \tag{6.70}$$

Taking into account that $(\Gamma_{789})^2 \;=\; -1$, we obtain:

$$\begin{aligned}\sqrt{-\det(g+\hat{F})}\,\tilde{\Gamma}_\kappa \;=\; & \frac{K^5}{8}\,\sin^3\alpha\,\sin\theta\,\Big[-\frac{9}{4}\,\Gamma_{11}\,\big(\Gamma_{05} \,+ \\ & + \frac{2\,r\,\alpha'}{3}\,\Gamma_{06}\,\big)(\cos\alpha\,\Gamma_{789} \,+\, \sin\alpha) \,+\, \frac{\hat{F}_{0r}}{K^2}\,\Gamma_{789}\Big]\;.\end{aligned} \tag{6.71}$$

Let us now study the action of $\tilde{\Gamma}_\kappa$ on the constant spinors η. After using the projections written in (6.218) we get:

$$\begin{aligned}\sqrt{-\det(g+\hat{F})}\,\tilde{\Gamma}_\kappa\,\eta = & \frac{K^5}{8}\,\sin^3\alpha\,\sin\theta\,\Bigg[-\frac{3}{2}\Big(\frac{3}{2}\,\sin\alpha \,+\, \cos\alpha\, r\alpha'\Big)\Gamma_{11}\,\Gamma_{05} \,+ \\ & + \frac{3}{2}\Big(\frac{3}{2}\,\cos\alpha \,-\, \sin\alpha\, r\alpha'\Big)\Gamma_6\Gamma_{11}\,\Gamma_{05} \,+\, \frac{\hat{F}_{0r}}{K^2}\,\Gamma_6\Bigg]\eta\;.\end{aligned} \tag{6.72}$$

We now impose an extra projection which corresponds to having (anti)fundamental strings extended along the radial direction:

$$\Gamma_{11}\,\Gamma_{05}\,\eta \,=\, \sigma\,\eta\;, \qquad\qquad \sigma\,=\,\pm 1\;. \tag{6.73}$$

Using (6.73) we can convert (6.72) into:

$$\sqrt{-\det(g+\hat F)}\,\tilde\Gamma_\kappa\,\eta \,=\, \frac{K^5}{8}\,\sin^3\alpha\,\sin\theta\,\Bigg[-\,\frac{3\sigma}{2}\,\Big(\frac{3}{2}\,\sin\alpha\,+\,\cos\alpha\,r\alpha'\Big)\,+ \\ +\,\Bigg[\frac{3\sigma}{2}\,\Big(\frac{3}{2}\,\cos\alpha\,-\,\sin\alpha\,r\alpha'\Big)\,+\,\frac{\hat F_{0r}}{K^2}\Bigg]\,\Gamma_6\,\Bigg]\,\eta\;. \tag{6.74}$$

As $\tilde\Gamma_\kappa$ should act on η as ± 1, the terms multiplying Γ_6 on the right-hand side of (6.74) should vanish. This condition implies the following BPS relation between the worldvolume gauge field and the embedding function $\alpha(r)$:

$$\hat F_{0r}\,=\,\frac{3\sigma}{2}\,K^2(\alpha)\,\Big(\sin\alpha\,r\alpha'\,-\,\frac{3}{2}\,\cos\alpha\Big)\;. \tag{6.75}$$

If (6.75) holds we have:

$$\sqrt{-\det(g+\hat F)}\,\tilde\Gamma_\kappa\,\eta\,\Big|_{BPS}\,=\,-\frac{9\sigma}{32}\,K^5(\alpha)\,\sin^3\alpha\,\sin\theta\,\Big(\sin\alpha\,+\,\frac{2}{3}\cos\alpha\,r\alpha'\Big)\eta\;. \tag{6.76}$$

In order to determine $\tilde\Gamma_\kappa\,\eta$ for the BPS configurations, let us now compute the DBI determinant for our ansatz. A short calculation shows that:

$$\sqrt{-\det(g+\hat F)}\,=\,\frac{9}{32}\,K^5(\alpha)\,\sin^3\alpha\sin\theta\,\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}\,-\,\frac{16}{81\,K^4(\alpha)}\,\hat F_{0r}^2}\;. \tag{6.77}$$

When the BPS condition (6.75) is satisfied, the square root on the right-hand side of (6.77) becomes simply:

$$\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}\,-\,\frac{16}{81\,K^4(\alpha)}\,\hat F_{0r}^2}\;\Bigg|_{BPS}\,=\,\sin\alpha\,+\,\frac{2}{3}\cos\alpha\,r\alpha'\;, \tag{6.78}$$

and, therefore:

$$\tilde\Gamma_\kappa\,\eta\,\big|_{BPS}\,=\,-\sigma\,\eta\;, \tag{6.79}$$

which implies that our embeddings are kappa symmetric when (6.75) is satisfied.

Let us look in more detail at the value of the gauge field we have found in (6.75) for the BPS configurations. Since:

$$K^2(\alpha)\,e^{\frac{\Phi}{2}}\,=\,\frac{Q^{\frac12}}{(3m)^{\frac13}}\,\big(\cos\alpha\big)^{-\frac13}\;, \tag{6.80}$$

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we can rewrite F_{0r} as:

$$F_{0r} \,=\, \sigma\,\frac{3}{2}\,\frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}}\,\Big[r\,\big(\cos\alpha\big)^{-\frac{1}{3}}\,\sin\alpha\,\alpha' \,-\, \frac{3}{2}\,\big(\cos\alpha\big)^{\frac{2}{3}}\Big]\;. \tag{6.81}$$

In this form F_{0r} can be written as a total radial derivative:

$$F_{0r} \,=\, -\sigma\,\frac{9}{4}\,\frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}}\,\partial_r\,\Big(r\big(\cos\alpha\big)^{\frac{2}{3}}\Big)\;. \tag{6.82}$$

Let us represent $F_{0r} \,=\, -\partial_r\,A_0$ in the $A_r = 0$ gauge. Clearly, the gauge potential A_0 is given by:

$$A_0 \,=\, \sigma\,\frac{9}{4}\,\frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}}\,r\big(\cos\alpha\big)^{\frac{2}{3}}\;. \tag{6.83}$$

6.4.2 Equations of motion

Let us now study the configuration of the wrapped D4-brane from the point of view of its worldvolume action. This action is the sum of a DBI and WZ term. The former is given by:

$$S_{DBI} \,=\, -T_4\int d^5\zeta\,e^{\frac{\Phi}{4}}\,\sqrt{-\det\big(g_{5,E} + e^{-\frac{\Phi}{2}}\,F\big)}\;, \tag{6.84}$$

where T_4 is the tension of the D4-brane and $g_{5,E}$ is the induced metric in Einstein frame written in (6.58). After integrating over the angles of the worldvolume, we get:

$$S_{DBI} \,=\, \int dt\,dr\,\mathcal{L}_{DBI}\;, \tag{6.85}$$

where the DBI lagrangian density $\mathcal{L}_{DBI}$ is given by:

$$\frac{\mathcal{L}_{DBI}}{16\,\pi^2\,T_4} \,=\, -\frac{9}{32}\,Q^{\frac{3}{2}}\,\sin^3\alpha\,\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^2 \,-\, \frac{16}{81\,K^4(\alpha)}\,e^{-\Phi}\,F_{0r}^2}\;. \tag{6.86}$$

Similarly, the WZ action can be written as:

$$S_{WZ} \,=\, \sigma\,T_4\int \hat{C}_3\wedge F\;, \tag{6.87}$$

where $\sigma = \pm 1$ is the same sign as in (6.73) and (6.75) and $\hat{C}_3$ is the pullback of the RR 3-form potential of F_4 (*i.e.*, $F_4 = dC_3$). Taking into account the expression (6.8) of F_4:

$$F_4 \,=\, -\frac{5}{12}\,Q\,(3m)^{\frac{1}{3}}\,\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^3\alpha\,\sin\theta\,d\alpha\wedge d\theta\wedge d\varphi\wedge d\psi\;, \tag{6.88}$$

we write the 3-form potential C_3 as:

$$C_3 \,=\, \frac{Q\,(3m)^{\frac{1}{3}}}{8}\,C(\alpha)\,\sin\theta\,\wedge d\theta\wedge d\varphi\wedge d\psi\;, \tag{6.89}$$

where $C(\alpha)$ is the solution of the differential equation:

$$\frac{dC}{d\alpha}\,=\,-\frac{10}{3}\,\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^3\alpha\;, \qquad\qquad C(\alpha=0)\,=\,0\;. \tag{6.90}$$

Notice that we have fixed our gauge freedom in such a way that we have no sources of worldvolume gauge field at $\alpha = 0$, since the gauge potential vanishes at that point. This prescription is similar to the one used in [224]. The integration of (6.90) is straightforward and gives:

$$C(\alpha)\,=\,\big(\cos\alpha\big)^{\frac{4}{3}}\,\Big(\sin^2\alpha\,+\,\frac{3}{2}\Big)\,-\,\frac{3}{2}\;. \tag{6.91}$$

If we now write the WZ action as:

$$S_{WZ}\,=\,\int dt\,dr\,\mathcal{L}_{WZ}\;, \tag{6.92}$$

then the WZ lagrangian density is:

$$\frac{\mathcal{L}_{WZ}}{16\,\pi^2\,T_4}\,=\,\sigma\,\frac{Q\,(3m)^{\frac{1}{3}}}{8}\,C(\alpha)\,F_{0r}\;. \tag{6.93}$$

Therefore, if $\mathcal{L}\,=\,\mathcal{L}_{DBI}+\mathcal{L}_{WZ}$ is the total lagrangian density:

$$S\,=\,S_{DBI}\,+\,S_{WZ}\,=\,\int dt\,dr\,\mathcal{L}\;, \tag{6.94}$$

we have:

$$\frac{\mathcal{L}}{16\,\pi^2\,T_4}\,=\,-\frac{9}{32}\,Q^{\frac{3}{2}}\,\sin^3\alpha\,\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^2\,-\,\frac{16}{81\,K^4(\alpha)}\,e^{-\Phi}\,F_{0r}^2}\,+\,\sigma\,\frac{Q\,(3m)^{\frac{1}{3}}}{8}\,C(\alpha)\,F_{0r}\;. \tag{6.95}$$

The equation of motion for the gauge field derived from $\mathcal{L}$ implies Gauss' law:

$$\frac{\partial\mathcal{L}}{\partial F_{0r}}\,=\,\text{constant}\;. \tag{6.96}$$

Let us analyze in more detail this Gauss' law. The left-hand side of (6.96) is given by:

$$\frac{\partial\mathcal{L}}{\partial F_{0r}}\,=\,16\pi^2\,T_4\,\left[\frac{Q^{\frac{3}{2}}}{18}\,\frac{\sin^3\alpha}{K^4}\,\frac{e^{-\Phi}\,F_{0r}}{\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^2\,-\,\frac{16}{81\,K^4}\,e^{-\Phi}\,F_{0r}^2}}\,+\,\sigma\,\frac{Q\,(3m)^{\frac{1}{3}}}{8}\,C(\alpha)\right]\;. \tag{6.97}$$

Moreover, the right-hand side of (6.96) should be quantized. Following [224], we impose the following quantization condition:

$$\frac{\partial \mathcal{L}}{\partial F_{0r}} \;=\; -\sigma\, n\, T_f \;, \tag{6.98}$$

where T_f is the tension of the fundamental string and $n \in \mathbb{Z}$. Let us next take into account that:

$$\frac{n\,T_f}{16\,\pi^2\,T_4} \;=\; \frac{n\,\pi}{2} \;, \tag{6.99}$$

and let us define a new function $\mathcal{C}_n(\alpha)$ as:

$$\mathcal{C}_n(\alpha) \;\equiv\; C(\alpha) \;+\; \frac{4\,n\,\pi}{Q\,(3m)^{\frac{1}{3}}} \;. \tag{6.100}$$

Then, the quantization condition (6.98) can be written as

$$Q^{\frac{3}{2}}\,\frac{\sin^3\alpha}{K^4}\,\frac{e^{-\Phi}\,F_{0r}}{\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}\;-\;\frac{16}{81\,K^4}\,e^{-\Phi}\;F_{0r}^2}} \;=\; -\,\frac{9\sigma}{4}\,Q\,(3m)^{\frac{1}{3}}\,\mathcal{C}_n(\alpha) \;. \tag{6.101}$$

Moreover, since:

$$Q\,(3m)^{\frac{1}{3}} \;=\; \frac{8\pi}{3}\,N \;, \tag{6.102}$$

the function $\mathcal{C}_n(\alpha)$ can be written as:

$$\mathcal{C}_n(\alpha) \;=\; C(\alpha) + \frac{3}{2}\,\frac{n}{N} \;=\; \big(\cos\alpha\big)^{\frac{4}{3}}\,\Big(\,\sin^2\alpha\;+\;\frac{3}{2}\Big)\;+\;\frac{3}{2}\,\Big(\frac{n}{N}\,-\,1\Big) \;. \tag{6.103}$$

Let us next find a first-order BPS differential equation for the embedding function $\alpha = \alpha(r)$. Plugging (6.75) on the left-hand side of (6.101), we arrive at:

$$\frac{Q^{\frac{3}{2}}\,e^{-\frac{\Phi}{2}}}{K^2}\,\sin^3\alpha\,\Big(\sin\alpha\,r\alpha'\;-\;\frac{3}{2}\,\cos\alpha\Big)\;=\;-\frac{3}{2}\,Q\,(3m)^{\frac{1}{3}}\,\Big(\sin\alpha\;+\;\frac{2}{3}\cos\alpha\,r\alpha'\Big)\,\mathcal{C}_n(\alpha) \;. \tag{6.104}$$

Next, we use:

$$\frac{Q^{\frac{3}{2}}\,e^{-\frac{\Phi}{2}}}{K^2} \;=\; Q\,(3m)^{\frac{1}{3}}\,\big(\cos\alpha\big)^{\frac{1}{3}} \;, \tag{6.105}$$

and solve for α'. We get:

$$r\,\alpha' \;=\; \frac{3}{2}\,\sin\alpha\,\frac{\big(\cos\alpha\big)^{\frac{4}{3}}\,\sin^2\alpha\;-\;\mathcal{C}_n(\alpha)}{\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^4\alpha\;+\;\cos\alpha\,\mathcal{C}_n(\alpha)} \;. \tag{6.106}$$

Let us now define the function $\Lambda_n(\alpha)$ as:

$$\Lambda_n(\alpha) \;=\; \big(\cos\alpha\big)^{\frac{4}{3}}\,\sin^2\alpha\;-\;\mathcal{C}_n(\alpha) \;. \tag{6.107}$$

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This function appears in the numerator of the BPS equation (6.106) for $\alpha(r)$. As:

$$\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^4\alpha\,+\,\cos\alpha\,\mathcal{C}_n(\alpha)\,=\,\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^2\alpha\,-\,\cos\alpha\,\Lambda_n(\alpha)\;, \tag{6.108}$$

then, the BPS equation (6.106) can be rewritten as:

$$r\,\alpha'\,=\,\frac{3}{2}\,\frac{\sin\alpha\,\Lambda_n(\alpha)}{\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin^2\alpha\,-\,\cos\alpha\,\Lambda_n(\alpha)}\;. \tag{6.109}$$

The explicit form of $\Lambda_n(\alpha)$ is:

$$\Lambda_n(\alpha)\,=\,-\frac{3}{2}\Big[\big(\cos\alpha\big)^{\frac{4}{3}}\,+\,\frac{n}{N}\,-\,1\Big]\;. \tag{6.110}$$

Its derivative is rather simple:

$$\frac{d\,\Lambda_n(\alpha)}{d\alpha}\,=\,2\,\big(\cos\alpha\big)^{\frac{1}{3}}\,\sin\alpha\;. \tag{6.111}$$

In order to study the energy of our configurations let us define the energy density $\mathcal{H}$ by performing the Legendre transform of $\mathcal{L}$:

$$\mathcal{H}\,=\,F_{0r}\,\frac{\partial\,\mathcal{L}}{\partial F_{0r}}\,-\,\mathcal{L}\;. \tag{6.112}$$

Explicitly, we get:

$$\mathcal{H}\,=\,\frac{9\pi^2}{2}\,T_4\,Q^{\frac{3}{2}}\,\sin^3\alpha\,\frac{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}}{\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}\,-\,\frac{16}{81\,K^4}\,e^{-\Phi}\,F_{0r}^2}}\;. \tag{6.113}$$

Let us next eliminate F_{0r} from the right-hand side of (6.113) by using Gauss' law. From (6.101) we obtain the relation between F_{0r} and $\alpha(r)$:

$$F_{0r}\,=\,-\frac{9}{4}\sigma\,\frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}}\,\frac{\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}}}{\big(\cos\alpha\big)^{\frac{2}{3}}\,\sqrt{\sin^6\alpha\,+\,\big(\cos\alpha\big)^{-\frac{2}{3}}\mathcal{C}_n(\alpha)^2}}\,\mathcal{C}_n(\alpha)\;. \tag{6.114}$$

Using this last expression we can easily demonstrate that:

$$\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}\,-\,\frac{16}{81\,K^4}\,e^{-\Phi}\,F_{0r}^2}\,=\,\sin^3\alpha\,\frac{\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}}}{\sqrt{\sin^6\alpha\,+\,\big(\cos\alpha\big)^{-\frac{2}{3}}\mathcal{C}_n(\alpha)^2}}\;. \tag{6.115}$$

Plugging this result on the right-hand side of (6.113), we finally arrive at:

$$\mathcal{H}\,=\,\frac{9}{32\,\pi^2}\,Q^{\frac{3}{2}}\,\sqrt{1+\frac{4}{9}\,r^2\,\alpha'^{\,2}}\,\sqrt{\sin^6\alpha\,+\,\big(\cos\alpha\big)^{-\frac{2}{3}}\mathcal{C}_n(\alpha)^2}\;. \tag{6.116}$$

This function $\mathcal{H}$ is nothing but the Routhian density of the system, obtained after eliminating the worldvolume gauge field, which is a cyclic variable. The equations of motion of the system are equivalent to the Euler-Lagrange equations derived from $\mathcal{H}$. One can directly show that any function $\alpha = \alpha(r)$ satisfying the first-order equation (6.109) also solves the second-order Euler-Lagrange equation. Instead of giving details of this computation, let us demonstrate that the solutions of (6.109) saturate an energy bound and, therefore, minimize $\int dr\,\mathcal{H}$. Indeed, let us consider a general configuration with an arbitrary function $\alpha = \alpha(r)$. One can show that the hamiltonian density (6.116) can written as:

$$\mathcal{H} \,=\, \frac{9}{32\,\pi^2}\, Q^{\frac{3}{2}}\, \sqrt{\mathcal{Z}^2 \,+\, \mathcal{Y}^2}\ , \tag{6.117}$$

where, for any function $\alpha = \alpha(r)$, $\mathcal{Z}$ is a total derivative:

$$\mathcal{Z} \,=\, \frac{d}{dr}\Big[r\Big(\sin^2\alpha \,-\, (\cos\alpha)^{\frac{2}{3}}\,\Lambda_n(\alpha)\Big)\Big]\ , \tag{6.118}$$

and $\mathcal{Y}$ is given by:

$$\mathcal{Y} \,=\, \sin\alpha\,(\cos\alpha)^{-\frac{1}{3}}\,\Lambda_n(\alpha) \,-\, \frac{2}{3}\, r\,\alpha'\,\Big(\sin^2\alpha \,-\, (\cos\alpha)^{\frac{2}{3}}\,\Lambda_n(\alpha)\Big)\ . \tag{6.119}$$

It immediately follows that $\mathcal{H}$ is bounded by:

$$\mathcal{H} \geq \frac{9}{32\pi^2}\, Q^{\frac{3}{2}}\, \big|\mathcal{Z}\big|\ . \tag{6.120}$$

Since $\mathcal{Z}$ is a total derivative, the integrated energy $\int dr\,\mathcal{H}$ is bounded by a quantity that only depends on the boundary values of $\alpha(r)$. This implies that any $\alpha(r)$ saturating the bound also solves the Euler-Lagrange variational problem, *i.e.*, the equations of motion.

The saturation of the bound occurs when $\mathcal{Y} = 0$, which is equivalent to our first-order BPS equation (6.109). This proves that the solutions of (6.109) also solve the second-order equations of motion derived from the Routhian $\mathcal{H}$.

6.4.3 Constant angle solutions

Let us first recover the solutions of [94] by searching for constant angle solutions of the BPS equation (6.109). Clearly, a constant α (with $\alpha \neq 0, \pi$) solves (6.109) if it is a zero of the function $\Lambda_n(\alpha)$. Thus, we define the angle α_n as the root of the equation:

$$\Lambda_n(\alpha_n) \,=\, 0\ . \tag{6.121}$$

The angles α_n are:

$$\cos\alpha_n \,=\, \Big(1 - \frac{n}{N}\Big)^{\frac{3}{4}}\ , \qquad\qquad 0 < n < N\ . \tag{6.122}$$

They are a discrete set of "latitudes" on the hemi-sphere, each of which defines a 3-sphere where we can wrap the D4-brane. We show below that these configurations behave as bound states of n fundamental strings stretched in the radial direction. Equivalently, this configuration should be dual to adding Kondo-like impurities in the dual theory. In order to reach this conclusion, let us obtain the energy of the BPS constant angle solutions (6.122). One can check that they minimize $\mathcal{H}$ for constant α:

$$\frac{\partial\mathcal{H}}{\partial\alpha}\Big|_{\alpha=\alpha_n} \,=\, 0\ . \tag{6.123}$$

Moreover, we can compute the energy density for these configurations. Indeed, as $\Lambda_n(\alpha_n) = 0$, we have:

$$\mathcal{C}_n(\alpha_n) \,=\, (\cos\alpha_n)^{\frac{4}{3}}\,\sin^2\alpha_n\ , \tag{6.124}$$

and we can easily prove that:

$$\mathcal{E}_n \,=\, \mathcal{H}(\alpha=\alpha_n) \,=\, \frac{9}{32\,\pi^2}\,Q^{\frac{3}{2}}\,\sin^2\alpha_n\ . \tag{6.125}$$

Using the value of the angles α_n written in (6.122), we get:

$$\mathcal{E}_n \,=\, \frac{9}{32\,\pi^2}\,Q^{\frac{3}{2}}\left[1\,-\,\Big(1-\frac{n}{N}\Big)^{\frac{3}{2}}\right]\ . \tag{6.126}$$

Let us see that $\mathcal{E}_n$ can be interpreted as the energy density (or tension) of a bound state of n fundamental strings. With this purpose, let us consider the Nambu-Goto action of a fundamental string:

$$S_f \,=\, -T_f\int d^2\zeta\,e^{\frac{\Phi}{2}}\,\sqrt{-\det g_{2,E}}\ , \tag{6.127}$$

where $g_{2,E}$ is the induced metric on the worldsheet and the dilaton factor appears because we are in the Einstein frame. We extend the string in (t,r) at constant angle α. The induced metric is:

$$ds^2_{2,E} \,=\, \frac{9}{4}\,K^2(\alpha)\left[\,-\,r^2\,dt^2\,+\,\frac{dr^2}{r^2}\,\right]\ . \tag{6.128}$$

Therefore, the energy density for the string is:

$$\mathcal{H}_f \,=\, \frac{9}{4}\,T_f\,e^{\frac{\Phi}{2}}\,K^2(\alpha) \,=\, \frac{9}{8\pi}\,\frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}\,\big(\cos\alpha\big)^{\frac{1}{3}}}\ . \tag{6.129}$$

Using (6.102), we get:

$$\mathcal{H}_f \,=\, \frac{27}{64\pi^2}\,\frac{Q^{\frac{3}{2}}}{N}\,\big(\cos\alpha\big)^{-\frac{1}{3}}\ . \tag{6.130}$$

Let us define τ_f as the tension of the fundamental string stretched along the radial direction at $\alpha=0$:

$$\tau_f \,\equiv\, \mathcal{H}_f(\alpha=0)\ , \tag{6.131}$$

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or equivalently:

$$\tau_f \;=\; \frac{27}{64\pi^2}\,\frac{Q^{\frac{3}{2}}}{N}\;. \tag{6.132}$$

It follows that the tension of the string at an angle α_n is related to the one at $\alpha = 0$ as:

$$\mathcal{H}_f \;=\; \frac{\tau_f}{\left(1-\frac{n}{N}\right)^{\frac{1}{4}}}\;. \tag{6.133}$$

We can now write $\mathcal{E}_n$ in terms of τ_f as:

$$\mathcal{E}_n \;=\; \frac{2}{3}\,N\,\tau_f\left[1\;-\;\left(1-\frac{n}{N}\right)^{\frac{3}{2}}\right]\;. \tag{6.134}$$

This energy has the interpretation of the one corresponding to a bound state of n strings, indeed corresponding to a Wilson loop in the antisymmetric representation as described in [94]. To check this interpretation, let us obtain the limit of $\mathcal{E}_n$ as $N \to \infty$:

$$\lim_{N\to\infty}\;\mathcal{E}_n \;=\; n\,\tau_f\;. \tag{6.135}$$

Notice also that $\alpha_n \to 0$ when $N \to \infty$. In this $N \to \infty$ limit the strings become free. One can verify that the formation of the bound state is energetically favored with respect to having n free strings at $\alpha = \alpha_n$. Indeed, the tension of one of such strings is given by (6.133) and one can show that:

$$\mathcal{E}_n \;\le\; n\,\frac{\tau_f}{\left(1-\frac{n}{N}\right)^{\frac{1}{4}}}\;. \tag{6.136}$$

6.4.4 General solution

Let us now find the general solution of the BPS differential equation (6.109). First of all, we use (6.111) to write (6.109) as:

$$r\,\frac{d\alpha}{dr} \;=\; \frac{3\sin\alpha\,\Lambda_n(\alpha)}{\sin\alpha\,\frac{d\Lambda_n(\alpha)}{d\alpha}\;-\;2\,\cos\alpha\,\Lambda_n(\alpha)}\;. \tag{6.137}$$

Written in this form the equation can be immediately integrated:

$$C\,r \;=\; \frac{\Big[\Lambda_n(\alpha)\Big]^{\frac{1}{3}}}{\big(\sin\alpha\big)^{\frac{2}{3}}}\;, \tag{6.138}$$

where C is a constant of integration. We should regard (6.138) as giving implicitly the function $\alpha = \alpha(r)$. The constant angle soutions correspond to take the constant $C = 0$. When $C \neq 0$ and for generic values of $0 < n < N$, the solution reaches $r = 0$ with $\alpha \to \alpha_n$

and $r \to \infty$ at $\alpha = 0$ or $\alpha = \pi$ (depending on the phase chosen for the constant C). Clearly, $n = 0, N$ are special cases. When $n = 0$, after choosing the phase of the constant C appropriately, the BPS solution is:

$$\left(\frac{r}{r_0}\right)^{\frac{3}{2}} = \frac{\left[1 - \left(\cos\alpha\right)^{\frac{4}{3}}\right]^{\frac{1}{2}}}{\sin\alpha} \,, \qquad (n = 0). \tag{6.139}$$

In this $n = 0$ solution the coordinate r remains finite as $\alpha \in [0, \pi/2]$ and

$$\lim_{\alpha \to 0} r = \left(\frac{2}{3}\right)^{\frac{1}{3}} r_0 \qquad \lim_{\alpha \to \pi/2} r = r_0 \,, \qquad (n = 0). \tag{6.140}$$

For $n = N$ the solution is:

$$\left(\frac{r}{r_0}\right)^{\frac{3}{2}} = \frac{\left(\cos\alpha\right)^{\frac{2}{3}}}{\sin\alpha} \,, \qquad (n = N). \tag{6.141}$$

In this case:

$$\lim_{\alpha \to 0} r = \infty \,, \qquad \lim_{\alpha \to \pi/2} r = 0 \,, \qquad (n = N). \tag{6.142}$$

6.4.5 Fluctuations of the constant angle solutions

In order to verify the stability of the constant angle configurations of the wrapped D4-branes we should analyze the fluctuations around the $\alpha = \alpha_n$ solutions of subsection 6.4.3. This study is performed in detail in appendix C of [21], where we consider the most general fluctuation of the type:

$$\delta\alpha = \xi \,, \qquad \delta F_{\mu\nu} = f_{\mu\nu} \,, \qquad \delta x^i = \chi^i \,. \tag{6.143}$$

These fluctuation modes are fields in $AdS_2 \times \mathbb{S}^3$ that are generically coupled. Therefore, in some cases, one needs to redefine the fields in order to diagonalize the fluctuation equations. After separating the angular variables one ends up with reduced equations for massive fields in AdS_2. The masses of these AdS_2 fields depend on the angular momentum quantum number l of the $\mathbb{S}^3$ harmonic. The spectrum of these AdS_2 masses and of the corresponding dual dimensions is worked out in detail in the same appendix and will not be shown here. It suffices to say that the dimensions are generically irrational but positive, which ensures the stability that we wanted to check.

6.5 D6-brane defects

Let us now address the problem of finding supersymmetric D6-brane probes in the BO background. We will consider the configuration in which the probe D6-brane creates a codimension-1 defect in the 5-dimensional gauge theory. In terms of the cartesian coordinates used in (6.10)-(6.12) our setup can be summarized in the array:

$$\begin{array}{cccccccccc} & x^1 & x^2 & x^3 & x^4 & X^1 & X^2 & X^3 & X^4 & z \\ D6: & \times & \times & \times & \_ & \times & \times & \_ & \_ & \times \end{array} \tag{6.144}$$

Actually, it will be more convenient to describe the embedding of the D6-brane to change from the coordinates (X^1, X^2, z) to (ρ, γ, ϕ), related as:

$$X^1 = \rho^{\frac{3}{2}} \sin\gamma \sin\phi \,, \qquad X^2 = \rho^{\frac{3}{2}} \sin\gamma \cos\phi \,, \qquad z = \rho^{\frac{3}{2}} \cos\gamma \,. \tag{6.145}$$

Moreover, we will group the third and fourth components of $\vec{X}$ into a new vector $\vec{y}$:

$$\vec{y} = (y_1, y_2) = (X^3, X^4) \,. \tag{6.146}$$

Let us choose the following set of worldvolume coordinates for the D6-brane probe:

$$\zeta^a = (x^0, x^1, x^2, x^3, \rho, \gamma, \phi) \,. \tag{6.147}$$

We embed the probe in such a way that:

$$x^4 = x^4_0 = \text{constant} \,, \qquad \vec{y} = (y(\rho), 0) \,. \tag{6.148}$$

The induced metric for this type of embeddings can be readily obtained from (6.10), namely:

$$ds^2_7 = \frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}} (\cos\gamma)^{-\frac{1}{3}} \Bigg[\frac{9}{4} \frac{(\rho^3 + \vec{y}^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}} dx^2_{1,3} + \Big(\frac{9}{4}\rho + y'^2\Big) \frac{d\rho^2}{\rho^{\frac{1}{2}}(\rho^3 + \vec{y}^2)^{\frac{5}{6}}} + \frac{\rho^{\frac{5}{2}}}{(\rho^3 + \vec{y}^2)^{\frac{5}{6}}} d\Omega^2_2 \Bigg] \equiv \frac{Q^{\frac{1}{2}}}{(3m)^{\frac{1}{3}}} (\cos\gamma)^{-\frac{1}{3}} \mathcal{G}_{ab}\, \delta\zeta^a\, d\zeta^b \,, \tag{6.149}$$

where $d\Omega^2_2 = d\gamma^2 + \sin^2\gamma\, d\phi^2 \equiv h_{ij}\, d\zeta^i\, d\zeta^j$ is the metric of the 2-sphere and, in the last step, we have defined the effective 7-dimensional metric $\mathcal{G}_{ab}$. Notice that, in the UV region $\rho \to \infty$, the metric $\mathcal{G}_{ab}$ is of the form:

$$\mathcal{G}_{ab}\, \delta\zeta^a\, d\zeta^b \approx \frac{9}{4} \Big(\rho^2\, dx^2_{1,3} + \frac{d\rho^2}{\rho^2}\Big) + d\Omega^2_2 \,, \qquad (\rho \to \infty) \,, \tag{6.150}$$

which corresponds to $AdS_5 \times \mathbb{S}^2$.

Up to a constant, the DBI lagrangian density of the probe is:

$$e^{-\Phi} \sqrt{-\det g_7} = \Big(\frac{9}{4}\Big)^2 \frac{Q^2}{(3m)^{\frac{1}{3}}} \frac{\sin\gamma}{(\cos\gamma)^{\frac{1}{3}}} \rho^{\frac{5}{2}} \sqrt{\frac{9}{4}\rho + y'^2} \,. \tag{6.151}$$

It is trivial to verify that the equation of motion of y is satisfied if y is constant. Let us take:

$$y = L \,, \tag{6.152}$$

where L is proportional to the mass of the quarks (*i.e.*, fields in the fundamental representation) introduced by the D6-brane.

We now consider a more general configuration of the D6-brane with internal flux (corresponding to the Higgs branch of the theory). We will choose worldvolume coordinates as in (6.147) and will turn on a flux along (γ, ϕ). For consistency, the brane must be bent in the x^4 direction (see below). Therefore, our embedding is characterized by:

$$y = L \,, \qquad\qquad x^4 = x(\rho) \,, \qquad\qquad F = F_{\gamma\phi}\, d\gamma \wedge d\phi \,, \tag{6.153}$$

where L is constant. The DBI lagrangian density is now proportional to:

$$e^{-\Phi}\sqrt{-\det(g_7+F)} = \Big(\frac{3}{2}\Big)^5 \frac{Q^2}{(3m)^{\frac{1}{3}}}\,\frac{\sin\gamma}{(\cos\gamma)^{\frac{1}{3}}}\,\rho^3\sqrt{1+\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho}\,(x')^2}\,\times$$

$$\times\sqrt{1+\frac{(3m)^{\frac{2}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^5}\,\frac{(\cos\gamma)^{\frac{2}{3}}}{\sin^2\gamma}\,F_{\gamma\phi}^2}\,. \tag{6.154}$$

In this configuration with worldvolume flux the WZ term of the D6-brane is turned on and given by:

$$S_{WZ} = T_6\int \hat{C}_5 \wedge F \,, \tag{6.155}$$

where $\hat{C}_5$ is the pullback of the RR 5-form potential (6.12), namely:

$$\hat{C}_5 = -\Big(\frac{3}{2}\Big)^5 Q^{\frac{3}{2}}\,(\rho^3+L^2)^{\frac{5}{3}}\,x'\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge d\rho \,. \tag{6.156}$$

Therefore, the WZ lagrangian density is:

$$\mathcal{L}_{WZ} = -T_6\Big(\frac{3}{2}\Big)^5 Q^{\frac{3}{2}}\,(\rho^3+L^2)^{\frac{5}{3}}\,x'\,F_{\gamma\phi} \,. \tag{6.157}$$

Notice that the worldvolume flux sources a non-trivial bending $x = x(\rho)$, as stated above. The total lagrangian density takes the form:

$$\mathcal{L} = -T_6\Big(\frac{3}{2}\Big)^5 Q^2\Bigg[\frac{\sin\gamma}{(3m)^{\frac{1}{3}}(\cos\gamma)^{\frac{1}{3}}}\,\rho^3\sqrt{1+\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho}\,(x')^2}\,\times$$

$$\times\sqrt{1+\frac{(3m)^{\frac{2}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^5}\,\frac{(\cos\gamma)^{\frac{2}{3}}}{\sin^2\gamma}\,F_{\gamma\phi}^2}\; + Q^{-\frac{1}{2}}\,(\rho^3+L^2)^{\frac{5}{3}}\,x'\,F_{\gamma\phi}\Bigg]\,. \tag{6.158}$$

By inspecting $\mathcal{L}$ we notice that x and the worldvolume gauge potential are cyclic coordinates. This means that the corresponding Euler-Lagrange equations can be integrated once as:

$$\frac{\partial \mathcal{L}}{\partial F_{\gamma\phi}} \,=\, c_1 \;, \qquad\qquad \frac{\partial \mathcal{L}}{\partial x\,'} \,=\, c_2 \;, \tag{6.159}$$

with c_1 and c_2 constants. We will show below that the solution with $c_1 = c_2 = 0$ is simple and preserves SUSY. Let us explore the equations for the gauge field. By computing the derivative of $\mathcal{L}$ with respect to $F_{\gamma\phi}$, we get:

$$\frac{\partial \mathcal{L}}{\partial F_{\gamma\phi}} \sim \frac{(3m)^{\frac{1}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^2}\,\frac{(\cos\gamma)^{\frac{1}{3}}}{\sin\gamma}\,\sqrt{1+\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho}\,(x\,')^2}\;\times$$

$$\times\frac{F_{\gamma\phi}}{\sqrt{1+\frac{(3m)^{\frac{2}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^5}\,\frac{(\cos\gamma)^{\frac{2}{3}}}{\sin^2\gamma}\,F_{\gamma\phi}^2}} \,+\, Q^{-\frac{1}{2}}\,(\rho^3+L^2)^{\frac{5}{3}}\,x' \tag{6.160}$$

According to (6.159) the right-hand side of (6.160) has to be constant and, in particular, independent of the angle γ. Apart from the explicit dependence, only $F_{\gamma\phi}$ can contain functions of γ. Actually, in order to cancel the dependence on γ, it is straightforward to demonstrate that $F_{\gamma\phi}$ must be of the form:

$$F_{\gamma\phi} \,=\, f\,\frac{\sin\gamma}{(\cos\gamma)^{\frac{1}{3}}}\;, \tag{6.161}$$

where f is a constant. Notice that (6.161) is rather natural since the volume element of the (γ,ϕ) 2-sphere in the metric (6.149) depends on γ through $(\cos\gamma)^{-\frac{1}{3}}\,\sin\gamma$. Plugging (6.161) into (6.160) we get:

$$\frac{\partial \mathcal{L}}{\partial F_{\gamma\phi}} \sim \frac{(3m)^{\frac{1}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^2}\,\sqrt{1+\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho}\,(x\,')^2}\;\times$$

$$\times\frac{f}{\sqrt{1+\frac{(3m)^{\frac{2}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^5}\,f^2}} \,+\, Q^{-\frac{1}{2}}\,(\rho^3+L^2)^{\frac{5}{3}}\,x'\;. \tag{6.162}$$

Let us now impose the first equation in (6.159) with $c_1 = 0$. It can readily be proved that this equation is equivalent to the following equation for $x\,'$:

$$\frac{\sqrt{Q}}{(3m)^{\frac{1}{3}}}\,\frac{\rho^2\,x\,'}{\sqrt{1+\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho}\,(x\,')^2}} \,=\, -\frac{f}{\sqrt{1+\frac{(3m)^{\frac{2}{3}}}{Q}\,\frac{(\rho^3+L^2)^{\frac{5}{3}}}{\rho^5}\,f^2}}\;. \tag{6.163}$$

After some computation, one can get from (6.163) the following simple expression for x':

$$x' = -\frac{(3m)^{\frac{1}{3}}}{\sqrt{Q}}\,\frac{f}{\rho^2}\,, \tag{6.164}$$

which can be integrated as:

$$x(\rho) = \frac{(3m)^{\frac{1}{3}}}{\sqrt{Q}}\,\frac{f}{\rho}\,. \tag{6.165}$$

One can also show that this function $x(\rho)$ and the flux (6.161) satisfy the equation of motion of x in (6.159) with $c_2 = 0$.

As mentioned above, this configuration is supersymmetric. We will verify this fact explicitly in the next subsection by analyzing the kappa symmetry of the probe D6-brane.

6.5.1 Kappa symmetry

Let us write the 10-dimensional metric in the Einstein frame in the coordinates used to describe the Higgs branch. We have:

$$ds^2_{10,E} = \Omega^2\Big[\frac{9}{4}\,dx^2_{1,4}+\frac{1}{(\rho^3+L^2)^{\frac{5}{3}}}\Big(\frac{9}{4}\rho\,d\rho^2+\rho^3(d\gamma^2+\sin^2\gamma d\phi^2)+dL^2+L^2d\beta^2\Big)\Big]\,, \tag{6.166}$$

where $(dX^3)^2 + (dX^4)^2 = dL^2 + L^2\, d\beta^2$ and Ω is the function:

$$\Omega = (3m)^{\frac{1}{24}}\,Q^{\frac{5}{16}}\,\rho^{\frac{1}{16}}(\rho^3+L^2)^{\frac{5}{16}}\,(\cos\gamma)^{\frac{1}{24}}\,. \tag{6.167}$$

Let us choose the following basis of 1-forms for the metric:

$$\begin{aligned}
e^{\mu} &= \frac{3}{2}\,\Omega\,dx^{\mu}\,, \qquad (\mu=0,1,2,3,4)\,, & e^{\rho} &= \frac{3}{2}\,\Omega\,\frac{\rho^{\frac{1}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}d\rho\,,\\
e^{\gamma} &= \Omega\,\frac{\rho^{\frac{3}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,d\gamma\,, & e^{\phi} &= \Omega\,\frac{\rho^{\frac{3}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,\sin\gamma\,d\phi\,,\\
e^{L} &= \frac{\Omega}{(\rho^3+L^2)^{\frac{5}{6}}}\,dL\,, & e^{\beta} &= \Omega\,\frac{L}{(\rho^3+L^2)^{\frac{5}{6}}}\,d\beta\,,
\end{aligned} \tag{6.168}$$

in terms of which we can write the ordinary (non-conformal) Killing spinors:

$$\epsilon = (\cos\gamma)^{\frac{1}{48}}\,\rho^{\frac{1}{32}}\,(\rho^3+L^2)^{\frac{5}{32}}\,e^{\frac{\gamma}{2}\Gamma_{\rho\gamma}}\,e^{\frac{\beta-\phi}{2}\Gamma_{\phi\gamma}}\,\eta\,, \tag{6.169}$$

where η is a constant spinor satisfying the projections:

$$\Gamma_{11}\,\Gamma_{01234}\,\eta = \eta\,, \qquad\qquad \Gamma_{\rho}\,\eta = \eta\,. \tag{6.170}$$

The kappa symmetry matrix for a D6-brane in the Higgs branch configuration is:

$$\Gamma_\kappa \,=\, \frac{1}{7!}\,\frac{1}{\sqrt{-\det(g_{7,E}+\hat{F})}}\,\Big[1+\frac{1}{2}\,\gamma^{\nu_1\nu_2}\,\hat{F}_{\nu_1\nu_2}\,\Gamma_{11}\Big]\epsilon^{\mu_1\cdots\mu_7}\,\gamma_{\mu_1\cdots\mu_7}\,\,, \tag{6.171}$$

where $\hat{F}\,=\,e^{-\frac{\Phi}{2}}\,F$. The induced γ-matrices appearing in Γ_κ, for the worldvolume coordinates (6.147), are:

$$\begin{aligned}
\gamma_\mu &= \frac{3}{2}\,\Omega\,\Gamma_\mu\,, \qquad (\mu=0,1,2,3)\,, \qquad \gamma_\rho \,=\, \frac{3}{2}\,\Omega\left[\frac{\rho^{\frac{1}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,\Gamma_\rho\,+\,x'\,\Gamma_4\right]\,,\\
\gamma_\gamma &= \Omega\,\frac{\rho^{\frac{3}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,\Gamma_\gamma\,, \qquad \gamma_\phi \,=\, \Omega\,\frac{\rho^{\frac{3}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,\sin\gamma\,\Gamma_\phi\,,
\end{aligned} \tag{6.172}$$

where we have assumed that the D6-brane embedding is such that L and β are constant. From these expressions of the γ's we get:

$$\begin{aligned}
\frac{1}{7!}\,\epsilon^{\mu_1\cdots\mu_7}\,\gamma_{\mu_1\cdots\mu_7} &= \left(\frac{3}{2}\right)^5\Omega^7\,\sin\gamma\,\frac{\rho^3}{(\rho^3+L^2)^{\frac{5}{3}}}\left[\frac{\rho^{\frac{1}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}}\,\Gamma_\rho\,+\,x'\,\Gamma_4\right]\Gamma_{\gamma\phi}\,\Gamma_{0123}\,,\\
\frac{1}{2}\,\gamma^{\nu_1\nu_2}\,\hat{F}_{\nu_1\nu_2} &= \frac{(\rho^3+L^2)^{\frac{5}{3}}}{\Omega^2\,\sin\gamma\,\rho^3}\,\hat{F}_{\gamma\phi}\,\Gamma_{\gamma\phi}\,\equiv\,\mathcal{F}\,\Gamma_{\gamma\phi}\,.
\end{aligned} \tag{6.173}$$

Let us assume that $F_{\gamma\phi}$ has the form (6.161). Then, it is easy to check that the quantity $\mathcal{F}$ defined in the second equation in (6.173) takes the form:

$$\mathcal{F}\,=\,\frac{(3m)^{\frac{1}{3}}}{\sqrt{Q}}\,\frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{5}{2}}}\,f\,. \tag{6.174}$$

Moreover, if we define Λ as:

$$\Lambda\equiv\left(\frac{3}{2}\right)^5\frac{\Omega^7\,\sin\gamma}{\sqrt{-\det(g_{7,E}+\hat{F})}}\,\frac{\rho^{\frac{7}{2}}}{(\rho^3+L^2)^{\frac{5}{2}}}\,, \tag{6.175}$$

then, the kappa symmetry matrix is:

$$\Gamma_\kappa\,=\,\Lambda\Big[1+\mathcal{F}\,\Gamma_{\gamma\phi}\,\Gamma_{11}\Big]\Big[\Gamma_\rho\,+\,\frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}}\,x'\,\Gamma_4\Big]\,\Gamma_{0123}\,\Gamma_{\gamma\phi}\,. \tag{6.176}$$

We have to impose the following condition on the Killing spinor ϵ:

$$\Gamma_\kappa\,\epsilon\,=\,\epsilon\,. \tag{6.177}$$

Given the form of the matrix Γ_κ and the spinor ϵ, it is straightforward to conclude that the above condition is equivalent to the following one on the constant spinor η of (6.169):

$$\tilde{\Gamma}_\kappa \eta \,=\, \eta \;, \tag{6.178}$$

where $\tilde{\Gamma}_\kappa$ is defined as:

$$\tilde{\Gamma}_\kappa \,=\, e^{-\frac{\gamma}{2}\Gamma_{\rho\gamma}}\,\Gamma_\kappa\, e^{\frac{\gamma}{2}\Gamma_{\rho\gamma}} \;. \tag{6.179}$$

To compute $\tilde{\Gamma}_\kappa$ we use that:

$$\begin{aligned}
&[\,\Gamma_{\rho\gamma}\,,\,\Gamma_\rho\,\Gamma_{\gamma\phi}\,\Gamma_{0123}\,] \,=\, [\,\Gamma_{\rho\gamma}\,,\,\Gamma_{11}\,\Gamma_4\,\Gamma_{0123}\,] \,=\, 0 \;,\\
&\{\,\Gamma_{\rho\gamma}\,,\,\Gamma_4\,\Gamma_{\gamma\phi}\,\Gamma_{0123}\,\} \,=\, \{\,\Gamma_{\rho\gamma}\,,\,\Gamma_{11}\,\Gamma_\rho\,\Gamma_{0123}\,\} \,=\, 0 \;.
\end{aligned} \tag{6.180}$$

Then:

$$\begin{aligned}
\tilde{\Gamma}_\kappa \,=\, \Lambda \Bigg[\Gamma_\rho \,-\, \mathcal{F}\,\frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}}\,x'\,\Gamma_4\,\Gamma_{\gamma\phi}\,\Gamma_{11} \,+\\
+e^{-\gamma\Gamma_{\rho\gamma}}\Big(\mathcal{F}\,\Gamma_{\gamma\phi}\,\Gamma_{11}\Gamma_\rho + \frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}}\,x'\,\Gamma_4\Big)\Bigg]\,\Gamma_{0123}\,\Gamma_{\gamma\phi} \;.
\end{aligned} \tag{6.181}$$

Let us now impose the following extra projection to η:

$$\Gamma_{0123}\,\Gamma_{\gamma\phi}\,\eta \,=\, \eta \;, \tag{6.182}$$

which combined with the two conditions (6.170) gives rise to:

$$\Gamma_{\gamma\phi}\,\Gamma_{11}\,\eta \,=\, \Gamma_4\,\eta \;. \tag{6.183}$$

Using these projections, we can write the action of $\tilde{\Gamma}_\kappa$ on η as:

$$\tilde{\Gamma}_\kappa\,\eta \,=\, \Lambda\left[1 \,-\, \mathcal{F}\,\frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}}\,x' \,+\, \left(\mathcal{F} \,+\, \frac{(\rho^3+L^2)^{\frac{5}{6}}}{\rho^{\frac{1}{2}}}\,x'\right)e^{-\gamma\Gamma_{\rho\gamma}}\Gamma_4\right]\eta \;. \tag{6.184}$$

In order to satisfy (6.178), $\tilde{\Gamma}_\kappa$ should acts as the unity on η. Thus, the terms on the right-hand side of (6.184) containing matrices should vanish, which happens when the following BPS condition holds:

$$x' \,=\, -\mathcal{F}\,\frac{\rho^{\frac{1}{2}}}{(\rho^3+L^2)^{\frac{5}{6}}} \;, \tag{6.185}$$

which, after using (6.174), can be shown to be identical to (6.164). We can now evaluate $\tilde{\Gamma}_\kappa\,\eta$ on the BPS configuration:

$$\tilde{\Gamma}_\kappa\,\eta\big|_{BPS} \,=\, \Lambda_{BPS}\,(1+\mathcal{F}^2)\,\eta\big|_{BPS} \;. \tag{6.186}$$

As:

$$\sqrt{-\det(g_{7,E}+\hat{F})}\Big|_{BPS} \;=\; \Big(\frac{3}{2}\Big)^{5}\,\Omega^{7}\,\frac{\rho^{\frac{7}{2}}\,\sin\gamma}{(\rho^{3}+L^{2})^{\frac{5}{2}}}\big(1+\mathcal{F}^{2}\big)\;, \tag{6.187}$$

we have:

$$\Lambda_{BPS} \;=\; \frac{1}{1+\mathcal{F}^{2}}\;, \tag{6.188}$$

and, indeed, one has:

$$\tilde{\Gamma}_{\kappa}\,\eta\big|_{BPS} \;=\; \eta\big|_{BPS}\;. \tag{6.189}$$

6.5.2 Fluctuations

Let us now study the fluctuation spectrum of the D6-brane probe around the fluxless configuration with y constant and $f = x' = 0$. The detailed analysis of these configurations is carried out in appendix D of [21] and summarized in this subsection. First of all, we notice that the unperturbed worldvolume metric in the UV is $AdS_5\times\mathbb{S}^2$, with an angular warping factor. In our perturbed configuration the worldvolume gauge field is non-zero and we have two scalar fields y and η such that:

$$\delta y \;=\; \eta\;, \qquad\qquad \delta x^{4} \;=\; \chi\;. \tag{6.190}$$

Therefore the scalar η (χ) represents the deformation of the D6-branes along the directions transverse (parallel) to the D4-branes of the background. Each of the independent modes is dual to a tower of operators whose conformal dimensions have been determined and have the generic form:

$$\Delta \;=\; \Delta_0 \,+\, \alpha \,+\, \frac{3n}{2} \,+\, 3k\;, \tag{6.191}$$

where Δ_0 is the dimension of the lowest-lying operator in the tower, n and k are non-negative integers and, depending on the particular mode, the number α can take the values $\alpha = 0, 1, 2$ (see appendix D of [21] for details).

The field theory dual to the D6-brane that we are fluctuating is a defect 4-dimensional CFT preserving $\mathcal{N} = 1$ supersymmetry. The fields of this theory are the restriction of the rank N E_{N_f+1} theory on the D4-branes down to the (3+1)-dimensional defect, in addition to N_{D6} hypermultiplets. It is not difficult to match the different fluctuations with composite operators of the theory (see [62] for a similar analysis in the D3-D5 system). Let us denote by $(q\,,\,\tilde{q})$ the scalars of the quark hypermultiplet and by ψ the fermionic components of this hypermultiplet. Let us consider first the fluctuation mode with the lowest dimension, which is the mode denoted by I_+ in appendix D of [21] for $\alpha = k = n = 0$. This mode has dimension $\Delta = 2$ and is naturally identified with the operator $q^\dagger q - \tilde{q}^\dagger\tilde{q}$. On the other hand, the lowest-lying vector mode (denoted by type II and type III in the same appendix) has dimension $\Delta = 3$ and is naturally identified with the vector current $\bar{\psi}\gamma_\mu\psi + q^\dagger D_\mu q - \tilde{q}^\dagger D_\mu \tilde{q}$. Notice that these dimensions agree with the canonical ones since, in 4 dimensions, a scalar field has dimension 1 and a spinor has dimension 3/2. Moreover, these operators precisely correspond to the bosonic content of the $\mathcal{N}=1$ conserved flavor current multiplet. In both towers of states, as it can be read-off from the form of the corresponding warped harmonics, the $\alpha+\frac{3n}{2}+3k$ contribution to Δ correspond to extra insertions of $\mathcal{O}^{\alpha}_{n,\,k}=\phi^{\alpha}\,A_1^n\,\mathcal{P}_k(|A_1|^2,\,\phi^3)$, where $\mathcal{P}_k(x,\,y)$ is a degree k homogeneous polynomial on its variables. The field $A_1\sim X^1+iX^2$ is a complex scalar of the 5-dimensional theory and $\phi\sim x^9\sim z^{\frac{2}{3}}$ is the real scalar of the 5-dimensional vector multiplet. The canonical dimension of A_1 (a scalar field in 5 dimensions) is 3/2 and that of ϕ (a scalar in the 5-dimensional vector multiplet) is 1, which matches with the previous expression of the operators $\mathcal{O}^{\alpha}_{n,\,k}$. The precise expression of these homogeneous polynomials for the different modes can be found in appendix E of [21].

The lowest-lying fluctuation mode of the transverse scalar η is naturally identified with an operator of the type $\bar{\psi}\psi+q^{\dagger}\phi q$. The dimension for this mode is $\Delta=5/2$, which differs from the canonical value $\Delta=3$ for this type of bilinear operators. However, these fluctuations correspond to non-holomorphic unprotected quantities in $\mathcal{N}=1$ for which holography gives a prediction. The same thing happens with the type I_- modes, also described in appendix D of [21]. Summarizing, we have the rough fluctuation/operator dictionary in this case:

$$\begin{array}{c|lc} \text{Fluctuation} & \qquad\Delta & \text{Dual operator} \\ \hline I_+ & 2+\alpha+\frac{3n}{2}+3k & \left(q^{\dagger}q-\tilde{q}^{\dagger}\tilde{q}\right)\mathcal{O}^{\alpha}_{n,\,k} \\ \text{Vector} & 3+\alpha+\frac{3n}{2}+3k & \left(\bar{\psi}\gamma_{\mu}\psi+q^{\dagger}D_{\mu}q-\tilde{q}^{\dagger}D_{\mu}\tilde{q}\right)\mathcal{O}^{\alpha}_{n,\,k} \\ \text{Scalar} & \frac{5}{2}+\alpha+\frac{3n}{2}+3k & \left(\bar{\psi}\psi+q^{\dagger}\phi q\right)\mathcal{O}^{\alpha}_{n,\,k} \end{array} \tag{6.192}$$

6.6 Discussion

In this chapter we have studied 5-dimensional QFT's by coupling them to lower-dimensional defects. On general grounds, such defects host, in addition to the restriction to the defect worldvolume of the ambient 5-dimensional fields, extra degrees of freedom in the fundamental representation of the gauge group, all combining into interesting defect QFT's. In particular, we have concentrated on a specific class of ambient 5-dimensional QFT's; namely high-rank generalizations of the E_{N_f+1} theories of [77]. Besides exhibiting very interesting features, including E_{N_f+1} global symmetries, these theories have a fully explicit gravity dual in type I' String Theory [87] which we have used to study holographically certain classes of defects.

Technically, defects are constructed as probe branes in the (warped) AdS_6 background dual to the ambient QFT's. In this chapter, in particular, we have concentrated on codimension-1 and codimension-2 defects, corresponding in the gravity dual respectively to probe D6- and D4-branes.

Starting with D4-branes, they give rise to codimension-2 defects, *i.e.* 3-dimensional defect QFT's. We have found an infinite family of embeddings for the probe brane characterized by a holomorphic curve which encodes information about the vacuum of the defect

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QFT. Among all profiles, two cases, corresponding to the trivial vacuum and to the Higgs branch, are of special relevance, as the induced worldvolume metric on the defect brane approaches an $AdS_4 \times \mathbb{S}^1$ with a radial-dependent warping. Interestingly, these defects can not be brought on top of the "color branes" and must be separated a distance a. In turn, such distance governs the dynamics of the probe brane, effectively providing a hard-wall for the induced worldvolume metric.

We have also considered a different arrangement of probe D4-branes, this time creating codimension-4 defects in the ambient 5-dimensional QFT. Such defects host an effective AdS_2 worldvolume, and provide a generalization to the antisymmetric Wilson loop configurations in [94].

We have also considered codimension-1 defects, engineered by probe D6-branes. In this case, in the asymptotic region, the worldvolume metric of the probe D6-branes is $AdS_5 \times \mathbb{S}^2$. Moreover, upon setting the separation between the probe D6 and the "color branes" to zero (*i.e.* setting the mass of the 4-dimensional quarks to zero), the full worlvolume of the probe branes becomes AdS_5, thus suggesting that the defect QFT is indeed a 4-dimensional CFT (at least within the supergravity approximation). Note that such CFT would be very interesting, as it would inherit the E_{N_f+1} global symmetry of the ambient 5-dimensional CFT in addition to the extra $U(F)$ associated to the F extra D6-branes (for very small F, of course).

6.A Killing spinors

In this appendix we review and adapt to our notation the determination of the Killing spinors for the BO background [225]. Let us work in the AdS_6 coordinates of (6.7) and (6.8). It is quite convenient to deal with the metric in the Einstein frame, which is given by:

$$ds^2_{10,E} = K^2(\alpha)\left[\frac{9}{4}\,ds^2_{AdS_6} + d\alpha^2 + \frac{\sin^2\alpha}{4}\,d\Omega_3^2\right], \tag{6.193}$$

where $K(\alpha)$ is the function written in (6.7). Let us parameterize $d\Omega_3^2$ in terms of three left-invariant $SU(2)$ 1-forms ω^1, ω^2 and ω^3 satisfying $d\omega^i = -\frac{1}{2}\,\epsilon^{ijk}\,\omega^j \wedge \omega^k$. We have:

$$d\Omega_3^2 = \frac{1}{4}\left[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2\right]. \tag{6.194}$$

The ω^i's can be represented in terms of three angles (φ, θ, ψ) as:

$$\omega^1 = \cos\psi\, d\theta + \sin\psi\,\sin\theta\, d\varphi, \quad \omega^2 = -\sin\psi\, d\theta + \cos\psi\,\sin\theta\, d\varphi, \quad \omega^3 = d\psi + \cos\theta\, d\varphi\,. \tag{6.195}$$

In order to study the supersymmetry variations of the dilatino and gravitino, let us adopt the following basis of frame 1-forms:

$$
\begin{aligned}
e^{\mu} &= \frac{3}{2} K(\alpha)\, r\, dx^{\mu} \,, && (\mu = 0,1,2,3,4) \,, \\
e^{5} &= \frac{3}{2} K(\alpha)\, \frac{dr}{r} \,, && e^{6} = K(\alpha)\, d\alpha \,, \\
e^{6+a} &= \frac{1}{2} \sin\alpha\, K(\alpha)\, \omega^{a} \,, && (a = 1,2,3) \,.
\end{aligned}
\tag{6.196}
$$

The SUSY variations of the dilatino λ and gravitino ψ_M are:

$$
\begin{aligned}
\delta\lambda &= \Big[-\frac{1}{2} \partial_M \Phi\, \Gamma^M - \frac{5}{4} m\, e^{\frac{5\Phi}{4}} - \frac{1}{192} e^{\frac{\Phi}{4}} F_{MNPQ}\, \Gamma^{MNPQ} \Big] \epsilon \,, \\
\delta\psi_M &= \Big[\nabla_M - \frac{m}{16} e^{\frac{5\Phi}{4}}\, \Gamma_M + \frac{e^{\frac{\Phi}{4}}}{256} F_{NPQR} \big(\Gamma_M^{\ \ NPQR} - \frac{20}{3} \delta_M^{N}\, \Gamma^{PQR} \big) \Big] \epsilon \,,
\end{aligned}
\tag{6.197}
$$

where the Romans' mass m for the BO background has been written in (6.9) and the dilaton Φ and 4-form $F^{(4)}$ are those displayed in (6.7) and (6.8) respectively. Let us first impose that the variation of the dilatino is zero. Using that:

$$
m\, e^{\frac{5\Phi}{4}} = \frac{1}{3 \cos\alpha\, K(\alpha)} \,, \qquad e^{\frac{\Phi}{4}} F_4 = -\frac{10}{3} \big[K(\alpha) \big]^{-1} e^{6789} \,,
\tag{6.198}
$$

we readily get that $\delta\lambda = 0$ is equivalent to the following condition satisfied by the spinor ϵ:

$$
\big[1 + \sin\alpha\, \Gamma^{6} - \cos\alpha\, \Gamma^{6789} \big] \epsilon = 0 \,.
\tag{6.199}
$$

We can rewrite this equation as:

$$
\Gamma^{6789}\, e^{\alpha\, \Gamma^{789}}\, \epsilon = \epsilon \,.
\tag{6.200}
$$

Taking into account that $\{ \Gamma^{6789} \,, \Gamma^{789} \} = 0$, we can solve this equation as:

$$
\epsilon = e^{-\frac{\alpha}{2} \Gamma^{789}}\, \hat{\eta} \,,
\tag{6.201}
$$

where the spinor η satisfies the following α-independent projection:

$$
\Gamma^{6789}\, \hat{\eta} = \hat{\eta} \,.
\tag{6.202}
$$

Let us now impose the SUSY preserving conditions $\delta\psi_M = 0$ for the gravitino. We need to compute the covariant derivative acting on spinors, which depends on the spin-connection.

In the frame (6.196) the non-vanishing components of the spin-connection 1-form are:

$$
\begin{aligned}
\omega^{\mu 5} &= \frac{2}{3K(\alpha)}\, e^{\mu} \ , \qquad\qquad (\mu \,=\, 0,1,2,3,4) \ , \\
\omega^{p 6} &= -\frac{1}{20K(\alpha)}\, \Phi'\, e^{p} \ , \qquad\qquad (p \,=\, 0,1,2,3,4,5) \ , \\
\omega^{6 n} &= -\frac{1}{20\, K(\alpha)}\, (20 \cot\alpha \,-\, \Phi')\, e^{n} \ , \qquad\qquad (n \,=\, 7,8,9) \ , \\
\omega^{78} &= -\frac{\csc\alpha}{K(\alpha)}\, e^{9} \ , \qquad \omega^{89} \,=\, -\frac{\csc\alpha}{K(\alpha)}\, e^{7} \ , \qquad \omega^{97} \,=\, -\frac{\csc\alpha}{K(\alpha)}\, e^{8} \ ,
\end{aligned}
\tag{6.203}
$$

where:

$$
\Phi' \,=\, \partial_\alpha \Phi \,=\, \frac{5}{6}\, \tan\alpha \ . \tag{6.204}
$$

When the index M in the second equation in (6.197) is one of the AdS_6 directions, we find the following equations for ϵ:

$$
\begin{aligned}
&\Big(\frac{2}{r}\, \partial_{x^{\mu}} \,+\, \Gamma_{\mu 5} \,-\, \Gamma_{\mu 6789}\Big)\, \epsilon \,=\, 0 \ , \qquad\qquad (\mu = 0,1,2,3,4) \ , \\
&(2r\, \partial_r \,-\, \Gamma_{56789})\, \epsilon \,=\, 0 \ .
\end{aligned}
\tag{6.205}
$$

If we take M to be the polar angle α, the gravitino equation becomes:

$$
\Big(3\, \partial_\alpha \,-\, \frac{1}{16\, \cos\alpha}\, \Gamma^{6} \,+\, \frac{25}{16}\, \Gamma^{789}\Big)\, \epsilon \,=\, 0 \ . \tag{6.206}
$$

Let us solve this last equation to determine the complete α dependence of ϵ. We first write ϵ as in (6.202), with $\hat\eta$ being:

$$
\hat\eta \,=\, f(\alpha)\, \eta \ , \tag{6.207}
$$

where $f(\alpha)$ is a function proportional to the unit matrix to be determined. Plugging our ansatz into (6.206), we get:

$$
\Big(\cos\alpha\, \Gamma^{789} \,-\, \Gamma^{6} \,+\, 48\, \cos\alpha\, \frac{f'}{f}\Big)\, \epsilon \,=\, 0 \ , \tag{6.208}
$$

where $f' = \partial_\alpha f$. Taking into account that ϵ must satisfy (6.199), this last equation reduces to:

$$
\Big(\sin\alpha \,+\, 48\, \cos\alpha\, \frac{f'}{f}\Big)\, \Gamma^{6}\, \epsilon \,=\, 0 \ . \tag{6.209}
$$

As the eigenvalues of Γ^6 are ± 1, Γ^6 can never annihilate ϵ. Therefore, the only way to satisfy (6.209) is by requiring that:

$$
\frac{f'}{f} \,=\, -\frac{1}{48}\, \frac{\sin\alpha}{\cos\alpha} \ . \tag{6.210}
$$

This equation can be readily integrated, yielding:

$$f(\alpha) \,=\, \big[\cos\alpha\big]^{\frac{1}{48}} \,. \tag{6.211}$$

Therefore, the Killing spinors are of the form:

$$\epsilon \,=\, \big[\cos\alpha\big]^{\frac{1}{48}}\, e^{-\frac{\alpha}{2}\,\Gamma^{789}}\,\eta \,, \tag{6.212}$$

where η is independent of α and should satisfy the projection (6.199) (with $\hat{\eta} \to \eta$).

Let us now look at (6.205). These equations can be easily recast in terms of η. Actually, if we define:

$$\Gamma_* \,\equiv\, \Gamma^{56789} \,, \tag{6.213}$$

then, the system (6.205) becomes:

$$\begin{aligned} &\Big(\partial_\mu \,+\, \frac{r}{2}\,\Gamma_\mu\,\Gamma_5\,(1-\Gamma_*)\Big)\eta \,=\, 0 \,, \\ &\Big(\partial_r \,-\, \frac{1}{2r}\,\Gamma_*\Big)\eta \,=\, 0 \,. \end{aligned} \tag{6.214}$$

To solve this system we take into account that $\Gamma_*^2 = 1$ and consider constant spinors of the two posible eigenvalues of Γ_*:

$$\Gamma_*\,\eta_\pm \,=\, \pm\eta_\pm \,. \tag{6.215}$$

There are two classes of solutions of (6.214). The first type are the so-called ordinary spinors, which are independent of the cartesian coordinates x^μ and given by:

$$\eta_1 \,=\, \sqrt{r}\,\eta_+ \,, \tag{6.216}$$

with η_+ being an arbitrary constant spinor with positive Γ_* eigenvalue. The second class of solutions are the so-called conformal spinors, which do depend on x^μ and are of the form:

$$\eta_2 \,=\, \Big(\frac{1}{\sqrt{r}}\,\Gamma_5\,\Gamma_* \,+\, \sqrt{r}\,x^\mu\,\Gamma_\mu\Big)\,\eta_- \,, \tag{6.217}$$

with η_- being constant and satisfying $\Gamma_*\eta_- = -\eta_-$. Notice that the conformal spinors η_2 do not have well-defined Γ_* eigenvalue since $\{\Gamma_*, \Gamma_\mu\} = 0$.

One can check that the equations for the gravitino along the other angular directions are satisfied provided the spinors η do not depend on the coordinates (θ, φ, ψ) used to parametrize the $SU(2)$ 1-forms ω^1, ω^2 and ω^3.

In our analysis of probes we will mostly deal with the ordinary spinors. Let us summarize its form:

$$\epsilon \,=\, \big[\cos\alpha\big]^{\frac{1}{48}}\, e^{-\frac{\alpha}{2}\,\Gamma^{789}}\,\eta \,, \qquad \Gamma^{6789}\,\eta \,=\, \eta \,, \qquad \Gamma^5\,\eta \,=\, \eta \,. \tag{6.218}$$

These two projections, in particular, imply that:

$$\Gamma^{01234}\,\Gamma_{11}\,\epsilon\,=\,\epsilon\;, \tag{6.219}$$

with $\Gamma_{11}\,=\,\Gamma_{01234}\,\Gamma_{56789}$.

In our studies of kappa symmetry for different probe branes we need to know the form of the ordinary Killing spinors in several coordinate systems and 1-form bases. The form of the ordinary ϵ's in these coordinates can be easily obtained by following similar steps as those shown here. For example, in the cartesian coordinate system (6.10)-(6.12), the ϵ's can be written as:

$$\epsilon\,=\,\big(\vec{X}^{\,2}\,+\,z^2\big)^{\frac{5}{32}}\,z^{\frac{1}{48}}\,\eta\;, \tag{6.220}$$

where η are constant spinors satisfying the projections:

$$\Gamma_z\,\eta\,=\,\eta\;,\qquad\qquad \Gamma_{11}\,\Gamma_{x^0\,x^1\,x^2\,x^3\,x^4}\,\eta\,=\,\eta\;, \tag{6.221}$$

with the Γ's being constant Dirac matrices for the different directions along the cartesian coordinates. The representation (6.220) is used in section 6.3.1. Similarly, in our analysis of the SUSY D6-brane configurations we need the form of the spinors in the coordinate system (6.166). This form has been written in section 6.5.1.

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7

Integrability of strings in the Cremonesi-Tomasiello backgrounds

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7.1 Introduction

In this chapter, we study the integrability properties of the 6-dimensional $\mathcal{N} = (1, 0)$ SUSY CFTs described in section 1.12. We discuss the constructions of the dual backgrounds and study some of their properties and observables. After that, we focus on their non-integrability properties. As it was already mentioned in the first chapter, showing the integrability of a system is usually quite a difficult task. In some situations, it may be easier to prove that it is non-integrable. There are different methods developed to this end. We will use those in the papers [226, 227]. The idea is to find a string soliton (in the holographic language, this soliton captures the dynamics of a long, spinning, heavy operator in the dual CFT) and show that the dynamics of such an object is non-integrable in the sense defined by Liouville. Various techniques developed by mathematicians studying this problem are explained and used in this chapter. This can be complemented numerically, by studying the chaos indicators (Poincaré sections, Lyapunov exponent) of the Hamiltonian system in question.

The plan of the chapter is the following: in section 7.2, we will briefly review the formalism developed in the papers [113, 114, 118, 120] for the construction of gravity duals to the $\mathcal{N} = (1, 0)$ SCFT's described in section 1.12. We give new examples of pairs (CFT-backgrounds) and present some developments that might be useful in future studies of these systems. In section 7.3, we discuss the non-integrability of these CFT's. By focusing on the generic examples developed in section 7.2, we analytically show their non-integrable behaviour. In section 7.4, we provide a careful numerical analysis of the material in section 7.3, showing that indeed, these systems are chaotic. We summarise our results in section 7.5. Various interesting appendices complement our technical presentation.

7.2 Cremonesi-Tomasiello formulation

After various manipulations, Cremonesi and Tomasiello [114] wrote the Massive type IIA backgrounds dual to the 6-dimensional conformal field theories as,

$$\begin{aligned} ds^2 &= f_1(z)ds^2_{AdS_7} + f_2(z)dz^2 + f_3(z)d\Omega^2(\chi,\xi), \\ B_2 &= f_4(z)\mathrm{Vol}_{S^2}, \quad F_2 = f_5(z)\mathrm{Vol}_{S^2}, \quad e^{\phi} = f_6(z). \end{aligned} \tag{7.1}$$

We have defined $d\Omega^2(\chi,\xi) = d\chi^2 + \sin^2\chi\, d\xi^2$ and $\mathrm{Vol}_{S^2} = \sin\chi\, d\chi\wedge d\xi$. The functions $f_i(z)$ are written in terms of another function $\alpha(z)$ and its derivatives,

$$\begin{aligned} &f_1(z) = 8\sqrt{2}\pi\sqrt{-\frac{\alpha}{\alpha''}}, \quad f_2(z) = \sqrt{2}\pi\sqrt{-\frac{\alpha''}{\alpha}}, \quad f_3(z) = \sqrt{2}\pi\sqrt{-\frac{\alpha''}{\alpha}}\left(\frac{\alpha^2}{\alpha'^2 - 2\alpha\alpha''}\right), \\ &f_4(z) = \pi\left(-z + \frac{\alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''}\right), \quad f_5(z) = \left(\frac{\alpha''}{162\pi^2} + \frac{\pi F_0\alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''}\right), \\ &f_6(z) = 2^{\frac{5}{4}}\pi^{\frac{5}{2}}3^4\frac{(-\alpha/\alpha'')^{\frac{3}{4}}}{\sqrt{\alpha'^2 - 2\alpha\alpha''}}. \end{aligned} \tag{7.2}$$

The different geometries specified by $\alpha(z)$ are supersymmetric solutions of the Massive type IIA equations of motion (with mass parameter F_0), if $\alpha(z)$ solves the differential equation

$$\alpha''' = -162\pi^3 F_0. \tag{7.3}$$

Given a 6-dimensional $\mathcal{N} = (1,0)$ SCFT encoded in a quiver diagram, Cremonesi and Tomasiello [114] gave a recipe to find the precise solution to equation (7.3), such that when replaced in equation (7.1) gives the holographic dual to the SCFT. We discuss now some interesting examples.

Examples of SCFT's and their holographic type IIA backgrounds

Given a quiver diagram encoding the dynamics of a 6-dimensional $\mathcal{N} = (1,0)$ SCFT, we revise here the prescription of [114] to construct the function $\alpha(z)$.

It is clear that solutions to the equation (7.3) are of the form:

$$\alpha(z) = c_0 + c_1 z + c_2 z^2 - 27\pi^3 F_0 z^3.$$

In general the solutions we search for, are continuous, differentiable and piecewise defined in the interval $(i, i+1)$ with i an integer. The first region is $(0,1)$, here the function $\alpha(z)$

must satisfy that $\alpha(z=0)=0$. In the last region, the interval $(P, P+1)$, the other boundary condition is $\alpha(z=P+1)=0$. In general, this allows for a solution of the form,

$$\alpha(z) = -81\pi^2 \begin{cases} a_1 z + \frac{a_2}{2}z^2 + \frac{a_3}{6}z^3 & 0 \le z \le 1 \\ b_0 + b_1(z-1) + \frac{b_2}{2}(z-1)^2 + \frac{b_3}{6}(z-1)^3 & 1 \le z \le 2 \\ c_0 + c_1(z-2) + \frac{c_2}{2}(z-2)^2 + \frac{c_3}{6}(z-2)^3 & 2 \le z \le 3 \\ .... & i \le z \le i+1 \\ p_0 + p_1(z-P) + \frac{p_2}{2}(z-P)^2 + \frac{p_3}{6}(z-P)^3 & P \le z \le P+1 \end{cases}$$

where the constants (a_2, a_3), (b_2, b_3),..., (p_2, p_3) are determined by inspecting the function $R(z)$ that describes the ranks of the gauge groups. In fact, this implies that $a_2 = 0$ and a_3 is the slope corresponding to the first node in the quiver. The other coefficients are determined by imposing continuity of the functions $\alpha(z)$, $\alpha'(z)$, and that $\alpha(z=P+1)=0$. The procedure to be followed is better understood by inspecting some examples.

Let us first consider the quiver depicted in figure 7.1, with three gauge groups $SU(N) \times SU(2N) \times SU(3N)$ ending with a flavour group $SU(4N)$. Notice that each node satisfies $N_f = 2N_c$. The function $R(z)$ describing the ranks of this first quiver is

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Figure 7.1: The quiver encoding the dynamics of our first example CFT.

$$R_1(z) = N \begin{cases} z & 0 \le z \le 3 \\ 12 - 3z & 3 \le z \le 4, \end{cases}$$

indicating the presence of gauge groups $SU(N)$ at $z=1$, $SU(2N)$ at $z=2$ and $SU(3N)$ at $z=3$. In this sense, the z-direction of the supergravity background encodes the field theory information. The slope of the first three nodes is $s=N$, which translates to $a_3 = b_3 = c_3 = N$. Similarly $p_3 = -3N$. The change in slope $\Delta s = -4N$ indicates the presence of the $SU(4N)$ flavour group. On the other hand, in the first interval $0 \le z \le 1$, there is no gauge group, hence $a_2 = 0$, while the gauge group in the second interval is $SU(N)$, indicating that $b_2 = 1$. Similarly $c_2 = 2$ and $p_2 = 3$, reflecting the presence of the $SU(2N)$ and $SU(3N)$ gauge groups. With this, we can write,

$$\alpha(z) = -81\pi^2 N \begin{cases} a_1 z + \frac{1}{6}z^3 & 0 \le z \le 1 \\ b_0 + b_1(z-1) + \frac{1}{2}(z-1)^2 + \frac{1}{6}(z-1)^3 & 1 \le z \le 2 \\ c_0 + c_1(z-2) + \frac{2}{2}(z-2)^2 + \frac{1}{6}(z-2)^3 & 2 \le z \le 3 \\ p_0 + p_1(z-3) + \frac{3}{2}(z-3)^2 - \frac{1}{2}(z-3)^3 & 3 \le z \le 4. \end{cases}$$

where the remaining constants are determined by imposing continuity of α, α' and that $\alpha(z=4)=0$. This gives,

$$a_1 = -\frac{5}{2}, \;\; b_1 = -2, \;\; c_1 = -\frac{1}{2}, \;\; p_1 = 2; \quad b_0 = -\frac{7}{3}, \;\; c_0 = -\frac{11}{3}, \;\; p_0 = -3. \tag{7.4}$$

The function $\alpha(z)$ describing the background in equation (7.1), dual to the quiver CFT in figure 7.1 reads

$$\alpha_1(z) = -81\pi^2 N \begin{cases} -\frac{5}{2}z + \frac{1}{6}z^3 & 0 \le z \le 1 \\ -\frac{7}{3} - 2(z-1) + \frac{1}{2}(z-1)^2 + \frac{1}{6}(z-1)^3 & 1 \le z \le 2 \\ -\frac{11}{3} - \frac{1}{2}(z-2) + \frac{2}{2}(z-2)^2 + \frac{1}{6}(z-2)^3 & 2 \le z \le 3 \\ -3 + 2(z-3) + \frac{3}{2}(z-3)^2 - \frac{1}{2}(z-3)^3 & 3 \le z \le 4. \end{cases} \tag{7.5}$$

We have worked with a quiver with three colour nodes and one flavour node. Strictly speaking, the supergravity description is valid if the number of colour nodes is taken to be large [114]. Our example in equation (7.5) illustrates the procedure. In order to have a better holographic description of the CFT, we should work with a quiver with $SU(N) \times SU(2N) \times SU(3N) \times SU(4N) \times .... \times SU(PN)$ closed by an $SU(PN+N)$-flavour group (and taking P to be large). In that case, we write the function, $\gamma(z) \equiv -\frac{\alpha_1(z)}{81\pi^2 N}$ as:

$$\gamma(z) = \begin{cases} a_1 z + \frac{1}{6}z^3 & 0 \le z \le 1 \\ (ka_1 + \frac{k^3}{6}) + (a_1 + \frac{k^2}{2})(z-k) + \frac{k}{2}(z-k)^2 + \frac{1}{6}(z-k)^3 & k \le z \le (k+1) \\ (Pa_1 + \frac{P^3}{6}) + (a_1 + \frac{P^2}{2})(z-P) + \frac{P}{2}(z-P)^2 - \frac{P}{6}(z-P)^3 & P \le z \le P+1 \end{cases}$$

where

$$k = 1, ..., P-1, \qquad -6a_1 = P^2 + 2P. \tag{7.6}$$

The case of a quiver with increasing ranks, not closed by the flavour group, is described holographically by the function $\alpha_1(z) = -81\pi^2 N(a_1 z + \frac{z^3}{6})$, being a_1 a free parameter.

It is instructive to plot the function $-\frac{\alpha_1(z)}{81\pi^2}$ and its derivatives for the background defined by equation (7.5), see figure 7.2. We also plot the fields defining the background and the Ricci scalar, see figures 7.3 and 7.4. None of these functions are divergent for the $\alpha(z)$ in equation (7.5). We shall use this background in the coming sections to study the dynamics of a string configuration that rotates and winds on it. As a second example, we can work out the function $\alpha(z)$ for the quiver in figure 7.5. This quiver starts with an $SU(N)$-flavour node followed by three nodes $SU(N)$-colour, and it is closed by a final $SU(N)$-flavour node. The function describing the ranks is,

$$R_2(z) = N \begin{cases} z & 0 \le z \le 1 \\ 1 & 1 \le z \le 3 \\ 4 - z & 3 \le z \le 4, \end{cases}$$

and the function that determines the holographic description $\alpha_2(z)$ is,

$$\alpha_2(z) = -81\pi^2 N \begin{cases} -\frac{3}{2}z + \frac{1}{6}z^3 & 0 \le z \le 1 \\ -\frac{4}{3} - (z-1) + \frac{1}{2}(z-1)^2 & 1 \le z \le 2 \\ -\frac{11}{6} + \frac{1}{2}(z-2)^2 & 2 \le z \le 3 \\ -\frac{4}{3} + (z-3) + \frac{1}{2}(z-3)^2 - \frac{1}{6}(z-3)^3 & 3 \le z \le 4. \end{cases} \tag{7.7}$$

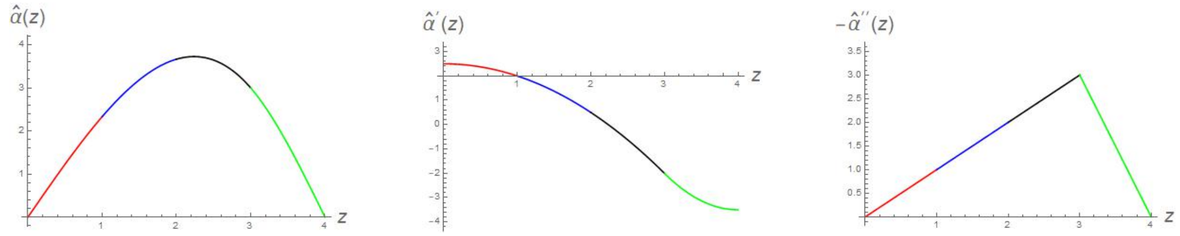

Figure 7.2: The function $\hat{\alpha}(z) \equiv \frac{\alpha(z)}{81\pi^2 N}$ and its derivatives, that describe the CFT associated with the quiver in figure 7.1.

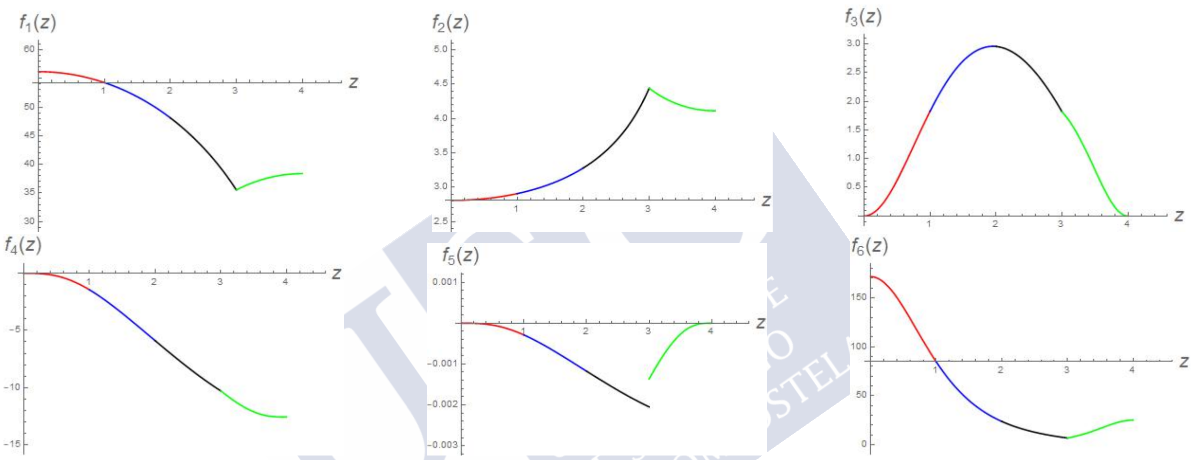

Figure 7.3: From top-left to bottom-right, the functions $f_1(z), ..., f_6(z)$, that describe the CFT associated with the quiver in figure 7.1.

The holographic description of this CFT is trustable when the number of nodes is large. We take the above quiver to be long enough for the illustrative purposes we aim at.

Finally, we shall consider an *endless* quiver. The quiver starts with an $SU(N)$-flavour group and is continued by an infinite tail of $SU(N)$-colour groups. As a consequence, the z-coordinate is unbounded. There is one integration constant that remains undetermined. The function describing the ranks is

$$R_3(z) = N \begin{cases} z & 0 \leq z \leq 1 \\ 1 & 1 \leq z \leq \infty, \end{cases}$$

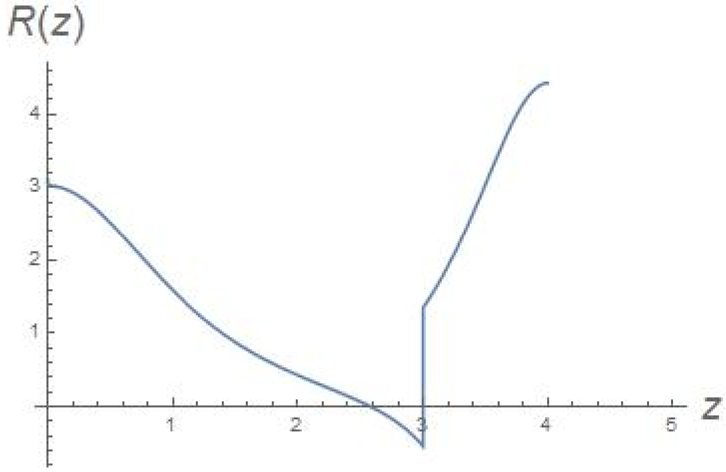

Figure 7.4: The Ricci scalar associated to the quiver in figure 7.1.

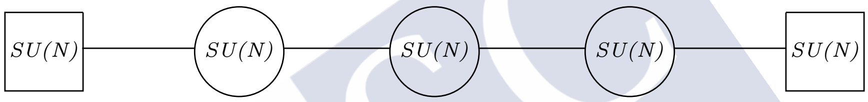

Figure 7.5: The quiver encoding the dynamics of our second example CFT.

The function $\delta(z) \equiv \frac{\alpha_3(z)}{-81\pi^2 N}$ reads,

$$\delta(z) = \begin{cases} a_1 z + \frac{1}{6}z^3 & 0 \le z \le 1 \\ (a_1 + \frac{1}{6}) + (a_1 + \frac{1}{2})(z-1) + \frac{1}{2}(z-1)^2 & 1 \le z \le 2 \\ (2a_1 + \frac{7}{6}) + (a_1 + \frac{3}{2})(z-2) + \frac{1}{2}(z-2)^2 & 2 \le z \le 3 \\ (3a_1 + \frac{19}{6}) + (a_1 + \frac{5}{2})(z-3) + \frac{1}{2}(z-3)^2 & 3 \le z \le 4 \\ (4a_1 + \frac{37}{6}) + (a_1 + \frac{7}{2})(z-4) + \frac{1}{2}(z-4)^2 & 4 \le z \le 5 \\ .... & \\ (Pa_1 + \frac{3P^2-3P+1}{6}) + (a_1 + \frac{2P-1}{2})(z-P) + \frac{1}{2}(z-P)^2 & P \le z \le P+1 \\ ..... & \end{cases} \tag{7.8}$$

Ending the quiver with a flavour $SU(P+1)$ node, reflects in a cap-off of the geometry at $z = P+1$. This is achieved by adding a term $-\frac{1}{6}(z-P)^3$ to the last line and choosing $a_1 = -\frac{P}{2}$. This would correspond to the quiver in figure 7.6. The various background functions associated with the holographic description of the quivers in equations (7.7)-(7.8) are displayed in appendix 7.A. Let us now analyse some observables characterising these CFTs.

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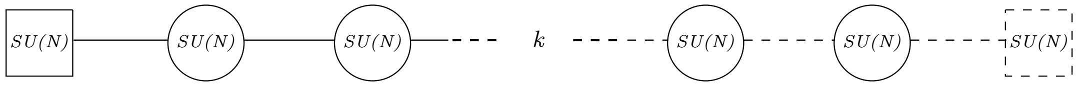

Figure 7.6: The quiver encoding the dynamics of the third example CFT. The long tail of colour $SU(N)$ ends with a flavour group.

7.2.1 Page charges and central charge

In this short section, we will discuss the Page and central charges characterising the backgrounds described in section 7.2. Most of this material, in different notation and for slightly different examples, was discussed in [113–126].

In the backgrounds of equation (7.1), we have that $B_2 \wedge B_2 = 0$. This gives for the Page charges,

$$Q_{Dp} = \frac{1}{2\kappa_{10}^2 T_{Dp}} \int F_{8-p} - B_2 \wedge F_{6-p},$$
$$Q_{NS5} = \frac{1}{2\kappa_{10}^2 T_{NS5}} \int H_3, \quad 2\kappa_{10}^2 T_{Dp} = (2\pi)^{7-p} g_s \alpha'^{\frac{7-p}{2}}, \quad 2\kappa_{10}^2 T_{NS5} = (2\pi)^2 g_s \alpha'.$$

As in the rest of this chapter, we set $\alpha' = g_s = 1$. We start computing the charge of NS5-branes. Using that $H_3 = dB_2$ and integrating on a 3-manifold $\Sigma_3 = [z, \chi, \xi]$ we have,

$$Q_{NS5} = \frac{1}{4\pi^2} \int H_3 = \frac{1}{4\pi^2} \int_{\Omega_2} B_2(z = z_*) - B_2(z = 0) = -(P + 1). \qquad (7.9)$$

We have used the expression for B_2 in equation (7.1), the explicit expression for the function $f_4(z)$ in equation (7.2) and the fact that $\alpha(0) = \alpha(z_* = P + 1) = 0$ in our backgrounds.

The Page charge of D6-branes is,

$$Q_{D6} = \frac{1}{2\pi} \int_{\Omega_2} F_2 - F_0 B_2 = \frac{1}{81\pi^2}(\alpha'' + 162\pi^3 F_0 z) = \frac{1}{81\pi^2}(\alpha'' - \alpha''' z). \quad (7.10)$$

We have used the expression for F_2, B_2 in equations (7.1)-(7.2) and the differential equation (7.3) that guarantees BPS solutions. For the two generic quivers described around equations (7.6) and (7.8) we find

$$Q_{D6,1} = -N \begin{cases} 0 & 0 \le z \le 1 \\ k & k \le z \le (k+1), \quad k = 1, 2, 3, 4...P - 1 \\ P & P \le z \le P + 1 \end{cases} \qquad (7.11)$$

and

$$Q_{D6,2} = -N \begin{cases} 0 & 0 \le z \le 1 \\ 1 & k \le z \le (k+1), \quad k = 1, 2, 3, 4...P - 1 \\ 1 & P \le z \le P + 1 \end{cases} \qquad (7.12)$$

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The negative sign reflects a choice in orientation. The D8-brane charge can be found by studying F_0 from the equation (7.3).

It is also interesting to calculate the central charge of these $\mathcal{N} = (1,0)$ SCFT's. A detailed study is presented in [114]. Here we do a related calculation as described in [181], [228] and [229]. As explained in these papers, in any background (with dilaton Φ) dual to a (d+1)-dimensional field theory,

$$ds^2 = \alpha\ dx_{1,d}^2 + \alpha\beta\ dR^2 + g_{ij}d\theta^i d\theta^j, \quad \Phi(R, \theta^i),$$

we calculate the quantities,

$$V_{int} = \int d\theta_i \sqrt{e^{-4\Phi} \det[g_{int}]\alpha^d}, \quad H = V_{int}^2,$$

and together with the Newton constant $G_N = 8\pi^6$ (in the units we choose here), we calculate a monotonic quantity

$$c = \frac{d^d}{G_N}\frac{\beta^{d/2}H^{(\frac{2d+1}{2})}}{(H')^d}. \tag{7.13}$$

Using the expressions in equations (7.1)-(7.2) and Poincaré coordinates to parameterize the AdS_7 space, we find

$$\begin{aligned}
&\alpha = f_1(z)R^2, \quad \beta = \frac{1}{R^4}, \quad d = 5, \ ds_{int}^2 = f_2 dz^2 + f_3 d\Omega_2,\\
&V_{int} = \mathcal{N}R^5, \quad \mathcal{N} = -2(\frac{2}{3})^8 \int_0^{z_*} \alpha(z)\alpha''(z)dz,\\
&c = \frac{1}{16G_N}(\frac{2}{3})^8 \int_0^{z_*} -\alpha(z)\alpha''(z)dz.
\end{aligned} \tag{7.14}$$

Notice that equation (7.14) is reminiscent of equation (4.10) in the paper [114]. In fact, $\alpha''(z)$ is proportional to the rank-function $R(z)$.

We calculate the integral $\int_0^{z_*} -\alpha(z)\alpha''(z)dz$ for the two generic quivers described around equations (7.6) and (7.8). For the background described around equation (7.6), we find that $I \equiv \int_0^{z_*} \frac{\alpha_1\alpha_1''}{(81\pi^2 N)^2}$ is given by:

$$I = \begin{cases} \frac{10a_1+1}{30} & 0 \le z \le 1 \\ \frac{1}{30}(1+5k+10k^2+10k^3+5k^4+10a_1(1+3k+3k^2)) & k \le z \le (k+1), \\ \frac{P^4+P^3}{12} + \frac{a_1 P^2}{2} + \frac{P^2}{30} + \frac{a_1 P}{6} & P \le z \le P+1. \end{cases}$$

We sum over all the intervals, and use that $k = 1, ..., (P-1)$ and $-6a_1 = P^2 + 2P$. In the holographic limit of large P (when the background above are trustable duals to the corresponding quivers) we obtain,

$$c_1 = \frac{2}{45\pi^2}N^2 P^5 \left(1 + O(1/P)\right). \tag{7.15}$$

For the background described around equation (7.8), we obtain

$$\int_0^{z_*} \alpha_3\alpha_3'' = (81\pi^2 N)^2 \begin{cases} \frac{10a_1+1}{30} & 0 \leq z \leq 1 \\ \frac{1}{12}(1+6k^2+6a_1+12ka_1) & k \leq z \leq (k+1), \\ \frac{P^2}{4}+\frac{a_1 P}{2}-\frac{P}{12}+\frac{a_1}{6}+\frac{1}{30} & P \leq z \leq P+1. \end{cases}$$

Summing over all the intervals $k = 1, ..., (P-1)$, we obtain, in the holographic limit,

$$c_3 = \frac{1}{6\pi^2} N^2 P^3 \left(1 + O(1/P)\right). \tag{7.16}$$

This concludes the analysis of the characteristic charges of the quivers, as calculated from the dual backgrounds.

7.2.2 A Formal Elaboration

In this section we discuss a formal development that might find some applications in further studies of 6-dimensional CFTs and their dual backgrounds.

Consider the differential equation (7.3) defining the function $\alpha(z)$. We explained a way of solving it in section 7.2. Another way may be to extend the function F_0 in an even-periodic way, with period $T = 2(P+1)$ and attempt a solution in terms of a Fourier series. For example, in the case of the background defined around equation (7.6),

$$\alpha_1''' = -162\pi^3 F_0 = -81\pi^2 N \begin{cases} -P & -(P+1) \leq z \leq -P \\ 1 & -P \leq z \leq P \\ -P & P \leq z \leq P+1. \end{cases}$$

We then decompose this even function in terms of a cosine-Fourier series,

$$\alpha_1''' = -162\pi N(P+1) \sum_{n=1}^{\infty} \left[\frac{\sin(\frac{n\pi P}{P+1})}{n}\right] \cos\left(\frac{n\pi}{P+1} z\right). \tag{7.17}$$

This implies that after integrating three times,

$$\alpha_1 = \frac{162 N (P+1)^4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\sin(\frac{n\pi P}{P+1})}{n^4}\right] \sin\left(\frac{n\pi}{P+1} z\right). \tag{7.18}$$

We have chosen the three integration constants to zero. Notice that in this way, we satisfy the boundary conditions $\alpha(z=0) = \alpha(z=P+1) = 0$. Evenly-extending F_0 guarantees that the function α is odd.

For the example of the background with $\alpha(z)$ defined around equation (7.8), we proceed similarly. In this case we have

$$\alpha_3''' = -162\pi^3 F_0 = -81\pi^2 N \begin{cases} -1 & -(P+1) \leq z \leq -P \\ 0 & -P \leq z \leq 1 \\ 1 & -1 \leq z \leq 1 \\ 0 & 1 \leq z \leq P \\ -1 & P \leq z \leq P+1. \end{cases}$$

and by the same reasoning as above we find,

$$\alpha_3 = \frac{162N(P+1)^3}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\sin(\frac{n\pi}{P+1})+\sin(\frac{n\pi P}{P+1})}{n^4}\right]\sin\Big(\frac{n\pi}{P+1}z\Big). \tag{7.19}$$

Some observations are in order. First, if we use the functions $\alpha(z)$ as defined by equations (7.18) and (7.19), their plot looks exactly like those in figures 7.2 and 7.15. Second, studying the Fourier series one can check that it can be written as a sum of poly-logarithmic functions $Li_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}$. Third, the third derivative of $\alpha(z)$ is afflicted by the Gibbs phenomenon at the discontinuity points.

More interestingly is the observation that we are writing $\alpha(z) \sim \sum \sin(\omega z)$, a sum of harmonics. Each of the harmonics solves the equation (7.3), defining a background that has $\alpha'' = -\omega^2\alpha$ and hence has constant warp factors of the AdS_7 and z-direction $f_1(z) \sim f_2(z) \sim 1$. It also has the warp factor of the $S^2(\chi,\xi)$, $f_3(z) \sim \frac{\sin^2(\omega z)}{3-\cos^2(\omega z)}$, again vanishing in the two ends of the space (but depending on the harmonic in some intermediate points too). The background generated by each harmonic is non-singular as can be seen by computing the Ricci scalar and the dilaton, both bounded. Each of these harmonics contributes to the central charge as

$$c \sim -\int_0^{P+1}\alpha\alpha'' \sim \frac{P+1}{2},$$

therefore, the coefficient of the Fourier series is relevant for this counting. Indeed, let us study this in detail. For the expansions in equations (7.18) and (7.19) we compute $-\alpha(z)\alpha''(z)$ and integrate it in $[0, P+1]$. Using orthogonality relations and the definition in equation (7.13), we find

$$\begin{aligned} c_1 &= \frac{1}{16G_N}\Big(\frac{2}{3}\Big)^8\frac{(162)^2}{2\pi^2}N^2(P+1)^7\sum_{n=1}^{\infty}\left[\frac{\sin(\frac{n\pi P}{P+1})}{n^3}\right]^2, \\ c_3 &= \frac{1}{16G_N}\Big(\frac{2}{3}\Big)^8\frac{(162)^2\pi^2}{2\pi^2}N^2(P+1)^5\sum_{n=1}^{\infty}\left[\frac{\sin(\frac{n\pi}{P+1})+\sin(\frac{n\pi P}{P+1})}{n^3}\right]^2. \end{aligned} \tag{7.20}$$

The sums can be explicitly evaluated in terms of poly-logarithmic functions. We can use that these formulas are good approximations in the limit of very long linear quivers and expanding at first order for $P \to \infty$, we find

$$c_1 \sim \frac{2N^2P^5}{45\pi^2}, \qquad c_3 \sim \frac{N^2P^3}{6\pi^2}, \tag{7.21}$$

in coincidence with the leading order result of equations (7.15) and (7.16).

We now move into the study of integrability for these quivers. We will use the holographic perspective as described above.

7.3 Dynamics of strings on $AdS_7 \times M_3$

In this section, we study the dynamics of classical strings moving in backgrounds dual to $\mathcal{N} = (1,0)$ SCFT's. We will apply the formalism developed in the papers [226, 227] to show that a given string soliton is non-integrable in the Liouvillian sense. This in turn translates into the non-integrability of the SCFT, as discussed in the introduction.

We will study the dynamics derived from the Polyakov action,

$$S_P = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma (G_{\mu\rho} h^{\alpha\beta} + B_{\mu\rho}\epsilon^{\alpha\beta})\partial_\alpha X^\mu \partial_\beta X^\rho, \tag{7.22}$$

supplemented by the Virasoro constraint $T_{ab} = 0$, with

$$T_{ab} = \partial_a X^\mu \partial_b X^\rho G_{\mu\rho} - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X^\rho G_{\mu\rho}. \tag{7.23}$$

In the equations above, we choose by convention $-h_{\tau\tau} = h_{\sigma\sigma} = 1$ and $\epsilon^{\tau\sigma} = 1$. We consider a string soliton sitting at the centre of the AdS_7 space given by,

$$t = t(\tau), \quad z = z(\tau), \quad \chi = \chi(\tau), \quad \xi = \nu\sigma, \tag{7.24}$$

where the parameter ν indicates how many times the string winds around the ξ-direction. The equations of motion derived from the action in equation (7.22) are equivalent to those derived from the effective Lagrangian (in equation (7.2), the reader can find the definitions of f_i),

$$\mathcal{L} = f_1(z)\dot{t}^2 - f_2(z)\dot{z}^2 - f_3(z)\dot{\chi}^2 + \nu^2 f_3(z)\sin^2\chi + 2\nu f_4(z)\sin\chi\dot{\chi}. \tag{7.25}$$

These equations of motion read,

$$\begin{aligned}
&2f_1(z)\dot{t} = 2E \\
&2f_3(z)\ddot{\chi} = 2\nu f_4'(z)\dot{z}\sin\chi - 2f_3'(z)\dot{\chi}\dot{z} - 2\nu^2 f_3(z)\sin\chi\cos\chi \\
&2f_2(z)\ddot{z} = -\frac{f_1'(z)}{f_1(z)^2}E^2 - \dot{z}^2 f_2'(z) + f_3'(z)(\dot{\chi}^2 - \nu^2\sin^2\chi) - 2\nu\dot{\chi}\sin\chi f_4'(z).
\end{aligned} \tag{7.26}$$

The dot indicates a derivative with respect to τ and the prime, as above, a derivative with respect to z. Notice also that the t-equation, the first in (7.26) was used in the equation for $z(\tau)$. These equations need to be supplemented by the Virasoro constraint. On this configuration it takes the form,

$$\begin{aligned}
&2T_{\tau\tau} = 2T_{\sigma\sigma} = -f_1(z)\dot{t}^2 + f_2(z)\dot{z}^2 + f_3(z)\dot{\chi}^2 + \nu^2 f_3(z)\sin^2\chi = 0, \\
&T_{\sigma\tau} = 0.
\end{aligned} \tag{7.27}$$

Using the Euler-Lagrange equations (7.26), the reader can check that $\partial_\sigma T_{\sigma\sigma} = \partial_\tau T_{\sigma\sigma} = 0$. Hence the constraint $T_{\sigma\sigma} = T_{\tau\tau} = 0$ can be satisfied by a judicious choice of the integration constant E in the first equation of (7.26).

It is useful to define the conjugate momenta and the effective Hamiltonian,

$$\dot{t} = \frac{p_t}{2f_1(z)}, \quad \dot{\chi} = -\frac{1}{2f_3(z)}(p_\chi - 2\nu f_4(z)\sin\chi), \quad \dot{z} = -\frac{p_z}{2f_2(z)},$$
$$\mathcal{H} = \frac{p_t^2}{4f_1(z)} - \frac{p_z^2}{4f_2(z)} - \frac{1}{4f_3(z)}(p_\chi - 2\nu f_4(z)\sin\chi)^2 - \nu^2 f_3(z)\sin^2\chi. \quad (7.28)$$

The Hamilton equations are,

$$\dot{t} = \frac{p_t}{2f_1(z)}, \quad \dot{z} = -\frac{p_z}{2f_2(z)}, \quad \dot{\chi} = -\frac{1}{2f_3(z)}(p_\chi - 2\nu f_4(z)\sin\chi), \quad \dot{p}_t = 0,$$
$$\dot{p}_\chi = 2\nu^2\left(\frac{f_4(z)^2}{f_3(z)} + f_3(z)\right)\sin\chi\cos\chi - \nu\frac{f_4(z)}{f_3(z)}p_\chi\cos\chi,$$
$$\dot{p}_z = \frac{p_t^2}{4f_1(z)^2}f_1'(z) - \frac{p_z^2}{4f_2(z)^2}f_2'(z) + \nu^2 f_3'(z)\sin^2\chi$$
$$-\frac{f_3'(z)}{4f_3(z)^2}(p_\chi - 2\nu f_4\sin\chi)^2 - \frac{\nu f_4'(z)}{f_3(z)}\sin\chi(p_\chi - 2\nu f_4(z)\sin\chi). \quad (7.29)$$

The reader can check that these equations are equivalent to the Euler-Lagrange equations (7.26).

In what follows, we will use the formalism of [117, 181, 227–239] to analytically study the non-integrability of the string soliton in equation (7.24).

7.3.1 Liouvillian integrability

The strategy we will use to prove non-integrability in the Liouville sense is the one described in [227] and exploited in various papers [230–244].

We inspect the equations in (7.26). In our case, we have only two equations, those for $z(\tau)$ and $\chi(\tau)$. We shall find a simple solution for one of these equations. Then, we study a fluctuation of the remaining equation (evaluated in the solution found above). We call this the Normal Variational Equation (NVE). We apply Kovacic's criterium to this fluctuated NVE equation, in order to determine the classical Liouvillian integrability (or non-integrability), of the system. For a summary of Kovacic's procedure see appendix 7.B.

For the case of equations (7.26), one can check that the choice $\chi(\tau) = \dot{\chi}(\tau) = \ddot{\chi}(\tau) = 0$, reduces the system of equations to,

$$2f_2(z)\ddot{z} = -\frac{f_1'(z)}{f_1(z)^2}E^2 - \dot{z}^2 f_2'(z). \quad (7.30)$$

Using the explicit expression for $f_1(z), f_2(z)$, this equation reads,

$$2\sqrt{-\frac{\alpha''}{\alpha}}\ddot{z} = \left(\frac{\alpha\alpha''' - \alpha'\alpha''}{2\alpha^2}\right)\sqrt{-\frac{\alpha}{\alpha''}}(\dot{z}^2 - \frac{E^2}{16\pi^2}), \quad (7.31)$$

which, after a convenient choice of integration constants, admits a simple solution

$$z_{sol} = \frac{E}{4\pi}\tau. \tag{7.32}$$

We now study the equation for the functions $\chi(\tau) = 0 + \epsilon x(\tau)$ and expand for small values of ϵ. We obtain the NVE,

$$\begin{aligned} &\ddot{x}(\tau) + \mathcal{B}\dot{x}(\tau) + \mathcal{A}x(\tau) = 0, \\ &\mathcal{B} = \frac{Ef_3'(z)}{4\pi f_3(z)}|_{z_{sol}}, \quad \mathcal{A} = (\nu^2 - \nu\frac{Ef_4'(z)}{4\pi f_3(z)})|_{z_{sol}}. \end{aligned} \tag{7.33}$$

More explicitly, using the expressions in equation (7.2) the coefficients $\mathcal{A}$ and $\mathcal{B}$ are

$$\begin{aligned} \mathcal{A} &= \left(\nu^2 - \frac{E\nu}{4\pi\sqrt{2}}\frac{1}{\sqrt{-\alpha\alpha''}}\frac{(-3\alpha'^2\alpha'' + 6\alpha\alpha''^2 - 2\alpha\alpha'\alpha''')}{(-\alpha'^2 + 2\alpha\alpha'')}\right)_{z_{sol}}, \\ \mathcal{B} &= \frac{E}{8\pi}\left(3\frac{\alpha'}{\alpha} + \frac{(\alpha'^2 + 2\alpha\alpha'')}{(\alpha'^2 - 2\alpha\alpha'')}\frac{\alpha'''}{\alpha''}\right)_{z_{sol}}. \end{aligned} \tag{7.34}$$

The Liouvillian integrability of the string soliton depends on the function $\alpha(z)$ defining the background. Below, we shall study this integrability.

7.3.2 Analytical study of the non-integrability of the SCFT's

In this section, we apply the Kovacic algorithm [245] to the equations (7.33)-(7.34). We particularise in the cases studied in section 1.12, more concretely for the quivers in figures 7.1 and 7.5. The whole problem boils down to study the presence (or not) of Liouvillian solutions to equations (7.33), given the different functions $\alpha(z)$.

Consider first the case in which the function $\alpha(z)$ is

$$\alpha(z) = -81\pi^2 k\left(\frac{1}{2}z^2 - \frac{2R_0^2}{81\pi^2k^2}\right). \tag{7.35}$$

This function corresponds to a background in (massless) type IIA (since $\alpha''' = 0$). In this background there are k D6-branes. Once lifted to 11 dimensions the metric is $AdS_7 \times S^4/Z_k$. Notice that the coordinate range is $|z| \leq \frac{2R_0}{9\pi k}$. The background is singular at the ends of the space. We find that the NVE equation (7.33) reads in this case,

$$\ddot{x}(\tau) - \frac{243E^2k^2\pi^2\tau}{16\pi^2\left(4R_0^2 - 81k^2\pi^2(\frac{E\tau}{4\pi})^2\right)}\dot{x}(\tau) + \nu\left[\nu + \frac{27kE\pi}{4\pi\sqrt{4R_0^2 - 81k^2\pi^2(\frac{E\tau}{4\pi})^2}}\right]x(\tau) = 0. \tag{7.36}$$

This equation is hard to solve exactly. We observe that for large values of the parameter R_0 (or for very short times), the equation (7.36) reduces to an oscillator equation. This indicates that in such a regime of parameters, the string soliton is Liouville integrable and possibly the full CFT is integrable in that limit too. This may be reminiscent of the "islands of integrability" discussed in [241]. In fact they appear in the regime in which $E\tau \to 0$. Nevertheless, for finite R_0 (or $E\tau \sim R_0$) we failed to find a Liouvillian solution.

Let us perform a more refined analysis (the details of the logic behind the analysis are in appendix 7.B).

The first step is to write the NVE as a second-order differential equation with rational coefficients. We choose $64R_0^2 = 81k^2E^2 = 1$ to ease the algebra (not loosing generality). The NVE equation reads,

$$\ddot{x} - \frac{3\tau}{1-\tau^2}\dot{x} + \left(1 + \frac{3}{\sqrt{1-\tau^2}}\right)x = 0.$$

We change variables to $\tau = \sqrt{1-v^2}$. The NVE differential equation in this new variable reads,

$$x''(v) + \mathcal{C}(v)x'(v) + \mathcal{D}(v)x(v) = 0, \quad \mathcal{C} = \frac{1}{\frac{dv}{d\tau}}\left(\mathcal{B}(v) + \frac{d}{dv}\left(\frac{dv}{d\tau}\right)\right), \quad \mathcal{D} = \frac{\mathcal{A}(v)}{(\frac{dv}{d\tau})^2}. \tag{7.37}$$

Where, in this particular case we have

$$\begin{aligned} &v = \sqrt{1-\tau^2}, \quad \frac{dv}{d\tau} = -\frac{\sqrt{1-v^2}}{v}, \quad \frac{d}{dv}\left(\frac{dv}{d\tau}\right) = \frac{1}{v^2\sqrt{1-v^2}}, \\ &\mathcal{C} = \frac{3v^2-4}{v-v^3}, \quad \mathcal{D} = \frac{v^2+3v}{1-v^2}. \end{aligned} \tag{7.38}$$

Following the analysis detailed in appendix 7.B, we construct a function $4V(v) = 4\mathcal{D} - \mathcal{C}^2 - 2\mathcal{C}'$,

$$4V = -4 + \frac{3}{4(v-1)^2} - \frac{17}{4(v-1)} - \frac{24}{v^2} + \frac{3}{4(v+1)^2} - \frac{31}{4(v+1)}. \tag{7.39}$$

The pole structure of this function is analysed according to the criteria in appendix 7.B. The existence of poles of order 1 and the fact that the function V is of order 1 at infinity, implies that none of the three possible cases detailed in appendix 7.B can be satisfied. Therefore, the equation has a non-Liouvillian solutions. The string soliton is non-integrable, and also is non-integrable the associated CFT. For a study of the integrability of a membrane equations in the 11-dimensional lift of this solution see appendix 7.C.

7.3.3 Integrability for the quivers of figures 7.1 and 7.5

Consider now the quiver CFT with holographic dual defined by the function $\alpha_1(z)$ in equation (7.5). Finding an exact solution to equations (7.33)-(7.34) is challenging. We may attempt to rewrite equations (7.33) by redefining $x(\tau) = e^{-\frac{1}{2}\int \mathcal{B} d\tau} f(\tau)$, leading to an equation in the Schrödinger form,

$$f''(\tau) + V(\tau) f(\tau) = 0, \quad V(\tau) = \mathcal{A} - \frac{1}{4}\mathcal{B}^2 - \frac{1}{2}\mathcal{B}'|_{z_{sol}}. \tag{7.40}$$

To solve exactly this last equation is a daunting task. Nevertheless, we can simplify matters if we study the problem very close to $z \sim \tau \sim 0$, that is for short times. Indeed, choosing $E = 4\pi$ and $\nu = 1$ to avoid cluttered expressions we find for a series expansion in τ,

$$\mathcal{A} \sim \gamma_1 - \gamma_2 \tau^2, \quad \mathcal{B} \sim \frac{\gamma_4}{\tau} - \gamma_3 \tau. \tag{7.41}$$

The explicit expression of the coefficients $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is not important for this analysis. Using the leading terms in equation (7.41), the differential equations (7.33), (7.40) admit Liouvillian solutions. Nevertheless, when the subleading terms are included, both equations present solutions that contain Hermite polynomials, and hypergeometric functions ${}_1F_1$. This implies the non-Liouvillian character of the solution, indicating non-integrability of the string soliton of equation (7.24).

A more refined study is presented in appendix 7.B. We change variables to have an NVE with rational coefficients. The necessary conditions for this NVE to admit Liouvillian solutions are not satisfied (the details are given below equation (7.51)). This translates into the non-integrability of the $\mathcal{N} = (1, 0)$ SCFT described by the quiver in figure 7.1. The same can be concluded about any background whose defining function $\alpha(z)$ starts as $\alpha(z) \sim a_1 z + a_3 z^3$ close to $z = 0$.

The non-Liouvillian integrability can also be studied numerically. In what follows, we provide a detailed numerical analysis of different observables that suggests that the system of equations (7.26), (7.29) is non-Liouvillian and chaotic.

7.4 Numerical analysis

In this section, we carry out some explicit numerical computations that provide a solid back-up to our findings of *analytic* non-integrability associated with $\mathcal{N} = (1, 0)$ SCFT's in 6 dimensions. We study the dynamics of classical strings on the backgrounds in equations (7.1)-(7.3). We demonstrate that the phase space dynamics of classical strings on these backgrounds is chaotic (and hence non-integrable). Let us mention that non-integrability does not necessarily imply chaos. However, as far as the gauge/gravity duality is concerned, all the examples encountered, that have been found to be non-integrable were also chaotic in general.

The evolution of a dynamical system is given by a set of deterministic differential equations that allow us to calculate the state of a system at a time t, knowing an earlier state of the system at some initial time t_0. A dynamical system is said to be chaotic when it is exponentially sensitive to its initial conditions, making it practically impossible to accurately predict the long term dynamical behaviour. Indeed, when we have two adjacent initial conditions $x_1(t_0)$ and $x_2(t_0) = x_1(t_0) + \epsilon$, we say the system exhibits chaotic dynamics when $|x_1(t) - x_2(t)| \sim e^{\lambda t}$, provided that the trajectory of our system in phase-space remains *bounded.* This boundedness of the trajectories is to rule out the trivial case where the trajectories move off to infinity and only diverge exponentially because they are moving apart [246].

In our case we are studying the motion of classical strings that sit at the centre of AdS_7 spacetime, while moving and rotating in an internal space of the form $\mathbb{R} \times S^2$. This is described by the system in equations (7.26)-(7.27) or their analog (7.29). The coordinates z, χ are bounded and the respective momenta along the p_z and p_χ-directions are bounded also due to the conserved Hamiltonian in equation (7.28).

The trajectory of this string embedding in the phase space will therefore be bounded if the z-coordinate itself is bounded. In this case, the Lyapunov exponent (that measures the exponential divergence of initial conditions) indeed provides a good observable to determine whether the dynamics of this classical string embedding is chaotic or not.

The numerical analysis that follows is quite dense. The plan is the following: first, in section 7.4.1, we examine the motion of classical strings over the background solutions (7.5) and (7.7), by numerically evaluating the equations of motion. We calculate the corresponding power spectra and discuss how this is indicative of chaotic dynamics. In section 7.4.2, we explore the Lyapunov spectrum [248], demonstrating that the dynamical behaviour is indeed chaotic. We end the analysis in section 7.4.3, discussing the Poincaré sections of these solutions and their implications on the chaotic dynamics associated to classical string configurations considered in this chapter. Appendix 7.D complements the numerical analysis with some rigorous definitions.

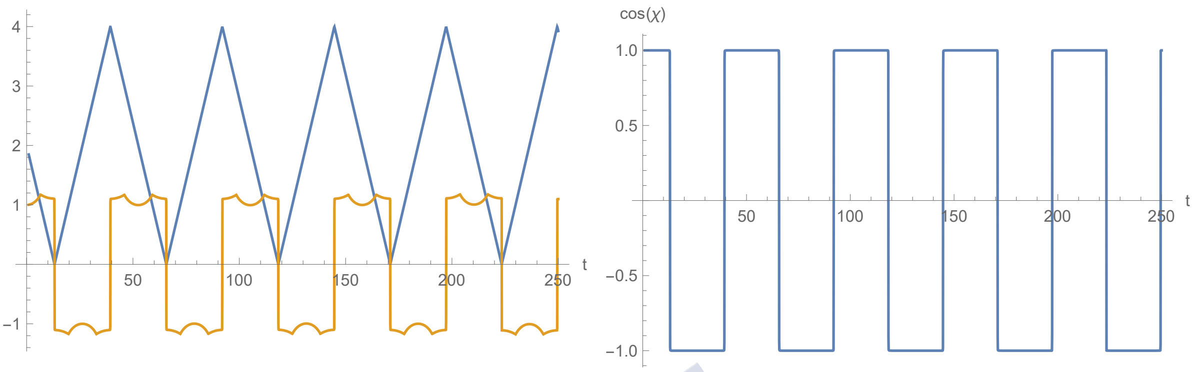
7.4.1 Numerical evolution and power spectra

The equations of motion for the classical strings (7.26)-(7.27) are considerably simplified with the choice $\ddot{\chi} = \dot{\chi} = \chi = 0$ (or $\chi = \pi$), i.e. when the string stays fixed at the north or the south pole of the 2-sphere. With this choice, the remaining equation for the motion along z is,

$$2f_2(z)\ddot{z} + \dot{z}^2 f_2'(z) + \frac{f_1'(z)}{f_1(z)^2}E^2 = 0, \tag{7.42}$$

which is equation (7.30).

Let us study how the classical string dynamics becomes increasingly disorganised as we allow the strings to move further away from the poles of the 2-sphere. First, consider a classical string on the background solutions that were already discussed in equations (7.5) and (7.7). Below, we refer to them as quiver 1 and quiver 3 respectively. In figure 7.7a, we see that when the string stays very close to the poles of the 2-sphere, it moves along the z-direction until it hits the end of the z-domain. Then, it turns around and moves back along the z-direction. On the 2-sphere, the string starts out located near the north pole (at, χ=0.01). However, when the string turns around along the z-direction it moves almost instantaneously from the north-pole to the south-pole (see, figure 7.7b). Now, we



(a) $z(t)$ and $p_z(t)$ in blue and yellow respectively

(b) $\chi(t)$

Figure 7.7: Numerical evolution for a string on the background solution (7.7) (quiver 3) with initial conditions $\chi(0) = 0.001$, $p_\chi(0) = 0$, $z(0) = 2$ and $p_z(0) = 1$, corresponding to an energy $E \approx 3.83$.

move the initial position of the string away from the poles (that are located at $\chi = 0$ or, $\chi = \pi$) to the middle of the 2-sphere, located at $\chi(0) = \pi/2$. We keep all the other initial conditions the same. In figure 7.8b, we consider an initial $\chi(0) = 0.1$, corresponding to $E \approx 6.75$. We see that the square shaped trajectory that the string traces out in the (z, χ)-plane gets deformed. The dynamics of the system becomes more complicated as the additional χ-dependence weighs in equations (7.26). Similar conclusions could be drawn for the background in equation (7.5) (quiver 1). Like in the previous example, the trajectories start looking unstructured as we increase $\chi(0)$. We now discuss the power spectra [246]. By taking the Fourier transform of the numerical evolutions in figure 7.8, we can distinguish whether $z(t)$ and $\chi(t)$ are periodic, quasi-periodic or chaotic. When a signal is perfectly periodic with a frequency ω, its Fourier spectrum will show a vertical line at the characteristic frequency of the system.

Notice that, for very low values of $\chi(0)$, the string moves almost periodically along the z-direction with a jigsaw motion (see figure 7.7a) and along the χ-direction with a square-wave profile (see figure 7.7b). From figure 7.8a, we can see that this motion is *not exactly* periodic, as the path of the string in the (z, χ)-plane does not exactly close on itself. We can see a confirmation of this in the corresponding Fourier spectrum (see figure 7.9a). In fact, we clearly see a *fundamental frequency* of value 0.02 and the corresponding oscillations along the z-axis that have a period of roughly $55t$. In addition to this, we see the higher harmonics of the jigsaw and box shaped waveforms. The finite width of these peaks however suggests that this is not a periodic signal but that there is instead some noise present. Such a noisy power spectrum is a typical characteristic of a deterministic chaotic system.

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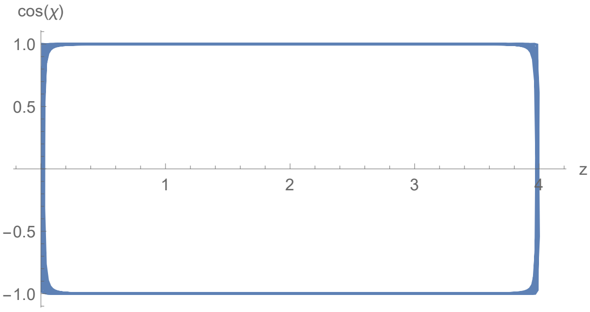
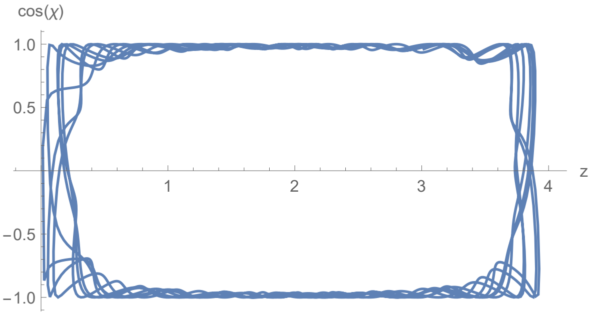
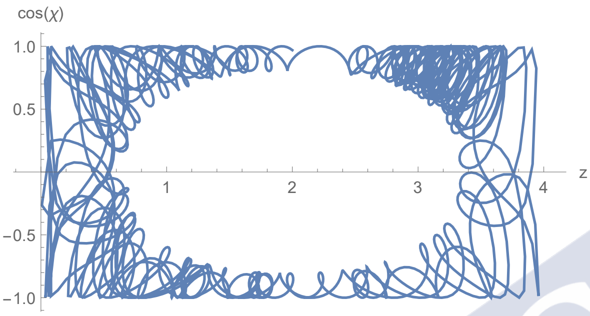


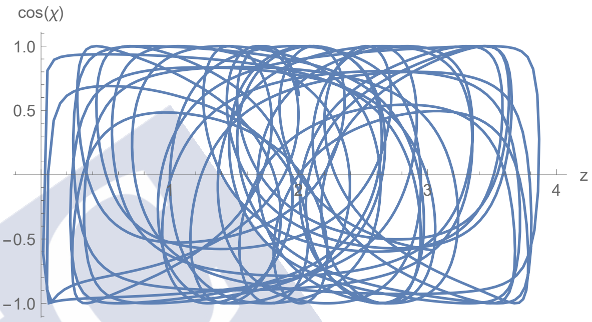
(a) $\chi(0) = 0.01,\ E \approx 3.83,\ t_{\max} = 400$



(b) $\chi(0) = 0.10,\ E \approx 6.75,\ t_{\max} = 400$



(c) $\chi(0) = 0.9,\ E \approx 43.82,\ t_{\max} = 100$



(d) $\chi(0) = 0.9,\ E \approx 43.82,\ t_{\max} = 100$

Figure 7.8: Different trajectories in the (z, χ)-plane for a string-embedding of the form in equation (7.24) on the background in (7.7), we run the evolution up to to $t = t_{\max}$ and only change the initial condition $\chi(0)$.

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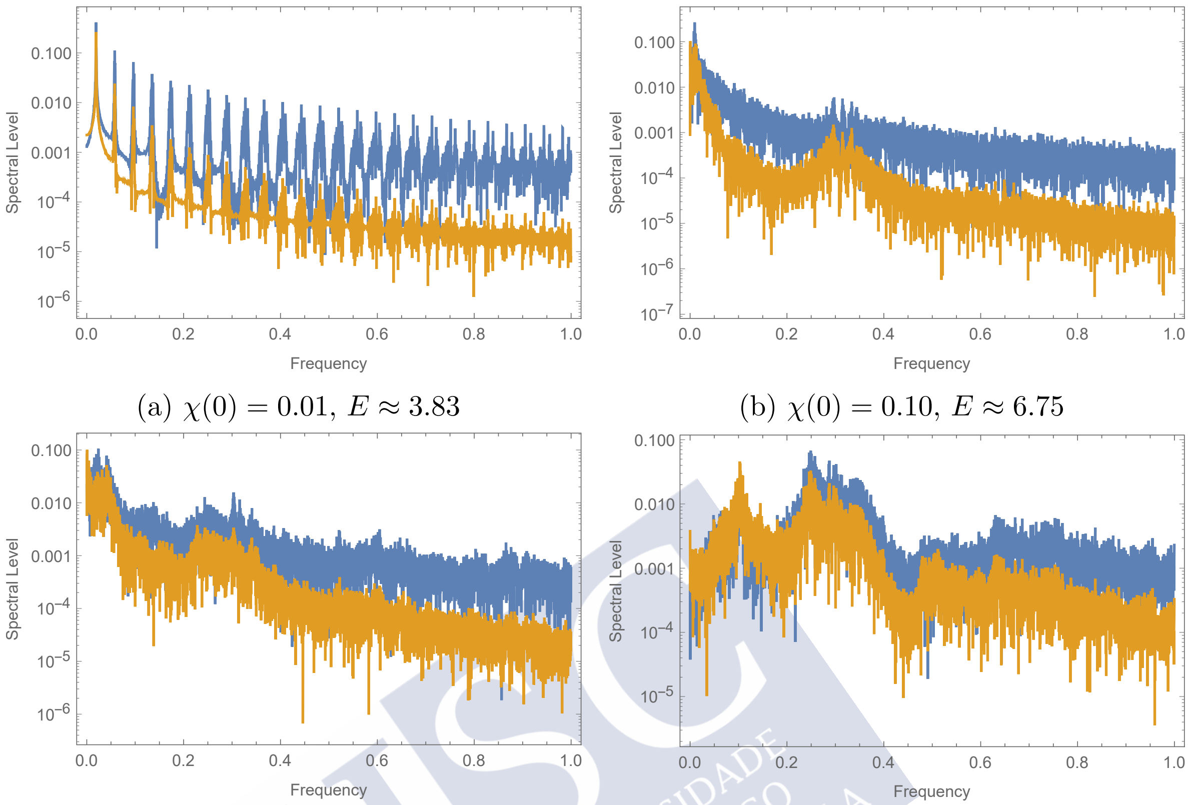
As we increase the value of $\chi(0)$, we first see that the jigsaw and box-shaped waveforms become distorted (see figure 7.8b). This is reflected by the corresponding power spectrum, loosing their higher harmonics as shown in figure 7.9b. Increasing $\chi(0)$ even further, we see that a broad band of noise around a frequency 0.35 starts to overpower the spectrum (see figures 7.9b-7.9c). At even higher values of $\chi(0)$, we even loose the initial peak at frequency 0.02. The spectrum becomes primarily dominated by noise.

These plots and its analysis have been done for the background in equation (7.7) (quiver 3). For the background defined by equation (7.5), we see exactly the same qualitative behaviour, for roughly the same values of $\chi(0)$.

7.4.2 The Lyapunov spectrum

The Lyapunov spectrum is generally introduced as a measure of the dynamical information loss in a chaotic system. This loss of information is what sources the dynamical Kolmogorov-Sinai (KS) entropy production within a chaotic system. Typically, for dynamical systems with a non-zero Lyapunov, the time evolution associated with two nearby trajectories in the phase space turns out to be highly sensitive to a tiny change in the initial conditions that is eventually amplified exponentially at sufficiently late times. In



(a) $\chi(0) = 0.01$, $E \approx 3.83$

(b) $\chi(0) = 0.10$, $E \approx 6.75$

(c) $\chi(0) = 0.25$, $E \approx 14.31$

(d) $\chi(0) = 0.90$, $E \approx 43.82$

Figure 7.9: Power spectra for the trajectories in figures 7.8a-7.8d. Here the spectra for both $z(t)$ and $\chi(t)$ are shown in yellow and blue respectively. To calculate these spectra we ran a numerical evolution up to $t = 5000$ (roughly 100 oscillations along the z-direction), with a resolution of 10 data-points per time-unit.

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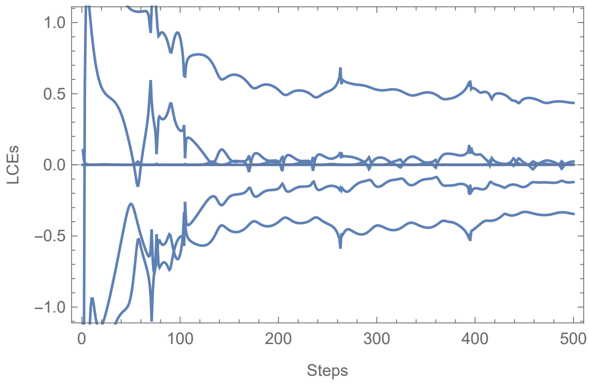
other words, the existence of a non-zero Lyapunov exponent, for a point $X = (q, p)$ in the phase space with initial condition $X_0 = (q(t = 0), p(t = 0))$ is,

$$\lambda = \lim_{\tau\to\infty} \lim_{\Delta X_0 \to 0} \frac{1}{\tau} \log \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)} \tag{7.43}$$

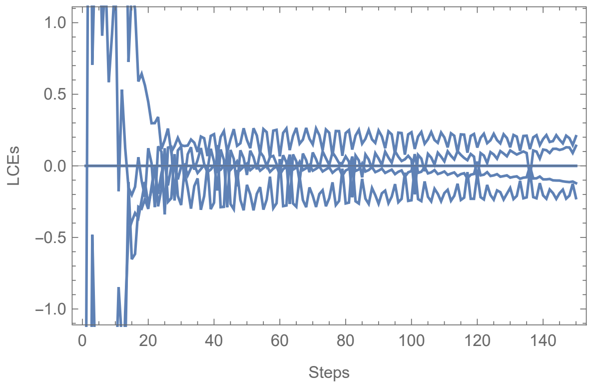
is intimately related to the degree of randomness associated with the dynamical phase space of a Hamiltonian system. It is typically introduced as a quantitative measure of the rate of separation between two infinitesimally close trajectories in the phase space. The function $\Delta X(X_0, \tau)$ measures the separation between two infinitesimally close trajectories (at very late times) as a function of this initial location. Typically, for chaotic systems, one ends up with

$$\Delta X(X_0, \tau) \sim \Delta X(X_0, 0) e^{\lambda\tau}. \tag{7.44}$$

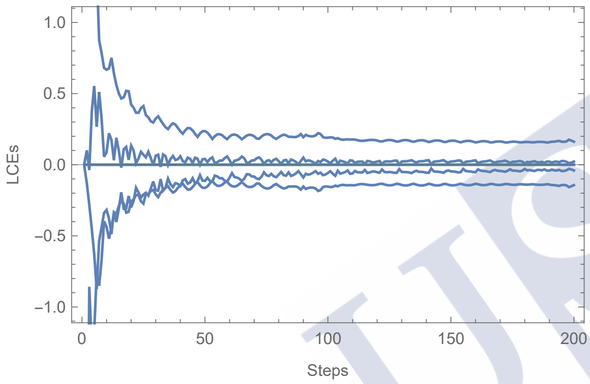
Below, we provide a detailed analysis of the computation of the Lyapunov exponents [248]



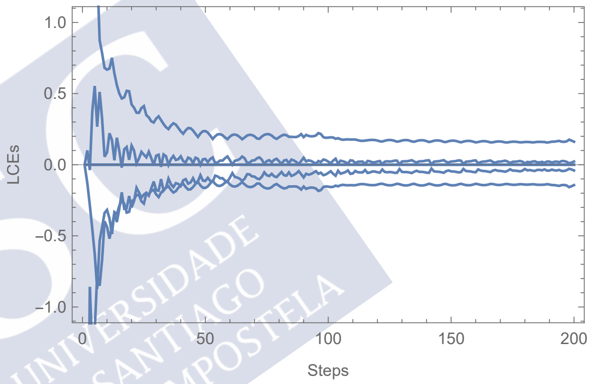
(a) LCE massless: $t = 0, z = 0.05, \chi = 0.05, p_t = 0.5, p_z = 4.60155, p_\chi = 0.01$. Integration: $\tau = 0.1, K = 500, T = 0.009$



(b) LCE quiver 1: $t = 0, z = 0.15, \chi = 0.15, p_t = 15, p_z = 0.845631, p_\chi = 0.01$. Integration: $\tau = 0.1, K = 150, T = 0.01$



(c) LCE quiver 2 $t = 0, z = 0.09, \chi = 0.09, p_t = 15, p_z = 0.491105, p_\chi = 0.22$. Integration: $\tau = 0.5, K = 200, T = 0.02$



(d) LCE quiver 3 $t = 0, z = 0.09, \chi = 0.09, p_t = 9, p_z = 0.821405, p_\chi = 0.22$. Integration: $\tau = 0.5, K = 200, T = 0.02$

Figure 7.10: LCEs for different quivers.

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corresponding to various background solutions, characterised by a function $\alpha(z)$.

We will also be interested in the *massless background solution* described in equation (7.35) and in the solution corresponding to the quiver that *never ends* described by equation (7.8) (without the "closure"). We denote this last one as quiver 2 in the plots below. We set $P = 10$ to be the length of the quiver for the purposes of the numerical analysis.

The computation of the Lyapunov exponents is solely based on the prescription of [248][1]. The initial conditions are fixed to satisfy the Hamiltonian constraint, $H = 0$. This

[1]The details of the numerical techniques and the precise definition of K and T are provided in the appendix 7.D

in turn implies that for a $2N$-dimensional phase space,

$$\sum_{i=1}^{2N} \lambda_i = 0 \tag{7.45}$$

where we denote by λ_i the i-th *Lyapunov Characteristic Exponent* (LCE). It is the measure of the exponential growth associated to the i-th direction in the phase space. For some systems, the sum of all positive Lyapunov exponents measures the KS entropy production during the dynamic evolution in the phase space.

We substitute some appropriate initial conditions into the dynamical equations in order to generate a solution. Choosing these initial conditions for the phase space variables to satisfy the vanishing of the Hamiltonian, we find the corresponding Lyapunov spectrum for each of the quiver configurations, which clearly have non-zero LCEs (see figures 7.10a-7.10d). Notice that, in our analysis, we are eventually computing four LCEs (λ_i) that characterise a 4-dimensional dynamical phase space. The addition of all them gives a vanishing number as it corresponds to a Hamiltonian system. At this stage, it is worthwhile to mention an interesting limit associated with the massless solution mentioned above. This is the limit in which we set the parameter $R_0 \to \infty$, which as can be seen from equation (7.36) yields the NVE,

$$\ddot{x}(\tau) + \nu^2 x(\tau) = 0 \tag{7.46}$$

which is therefore trivially integrable. To perform the corresponding (numerical) compu-

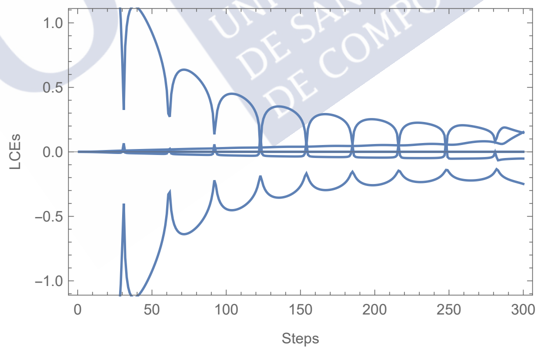


Figure 7.11: LCEs for the massless solution with large R_0.

tation on the LCEs, we choose the parameters

$$k = 1,\ R_0 = 500,\ \nu = 1 \tag{7.47}$$

taking as initial conditions for the phase space variables:

$$t = 0,\ z = 0.05,\ \chi = 0.05,\ p_t = 100,\ p_z = 0.159689,\ p_\chi = 0.01 \tag{7.48}$$

that satisfy the vanishing of the Hamiltonian. From the plot in figure 7.11, it is easy to notice that the largest LCE is substantially reduced for large enough times.

7.4.3 Poincaré sections and non-integrability

An N-dimensional integrable system possesses N independent integrals of motion that are in *involution*, namely, the Poisson bracket of any two of these conserved quantities vanishes. As a consequence of this, the corresponding phase space trajectories are confined to the surface of an N-dimensional KAM torus [246]. When we change our variables to action-angle variables $(q_i, p_i) \to (\phi_i, J_i)$, such that our Hamiltonian only depends on J_i, the corresponding trajectories on this KAM torus are completely specified in terms of N frequencies (ω_i) that specify the velocities along the different angles on this torus. When there is no set of integers n^i such that $\omega_i n^i = 0$, these trajectories are said to be *quasi-periodic*, they do not close on themselves but fill the surface of a KAM torus.

As a consequence of this, we can see whether a system is integrable or not, by taking cross-sections of its phase-space trajectories. When we plot for example (ϕ_1, J_1) every time $\phi_2 = 0$ we will see a large number of foliated circular KAM curves associated with the 2-dimensional cross-sections of these foliated KAM tori. Such a cross-section is known as a Poincaré section [246]. The KAM theorem tells us how these KAM curves will change when we perturb an integrable Hamiltonian with a small deformation $\epsilon H'$, where $\epsilon \ll 1$. The resonant tori - for which these trajectories close on themselves, will be destroyed by this perturbation. However, a large number of these non-resonant KAM tori will survive. As we continue to increase the strength of this perturbation, more and more of these tori are destroyed until the motion becomes seemingly random and we loose all of the KAM curves in our Poincaré section.

In order to generate Poincaré sections for the background solutions in equations (7.5) and (7.7) we first choose a set of different initial conditions, all having the same energy E. We do this by setting, $z(0) = 2$, $p_\chi(0) = 0$, and varying $p_z(0) \in [0, 10]$ and $\chi(0)$ in such a way that the Virasoro constraint in equation (7.27) is always satisfied for a given value of the energy. We then run the numerical evolution for these initial points, and plot the points (z, p_z) every time $\chi(t) = 0$ (see figure 7.12).

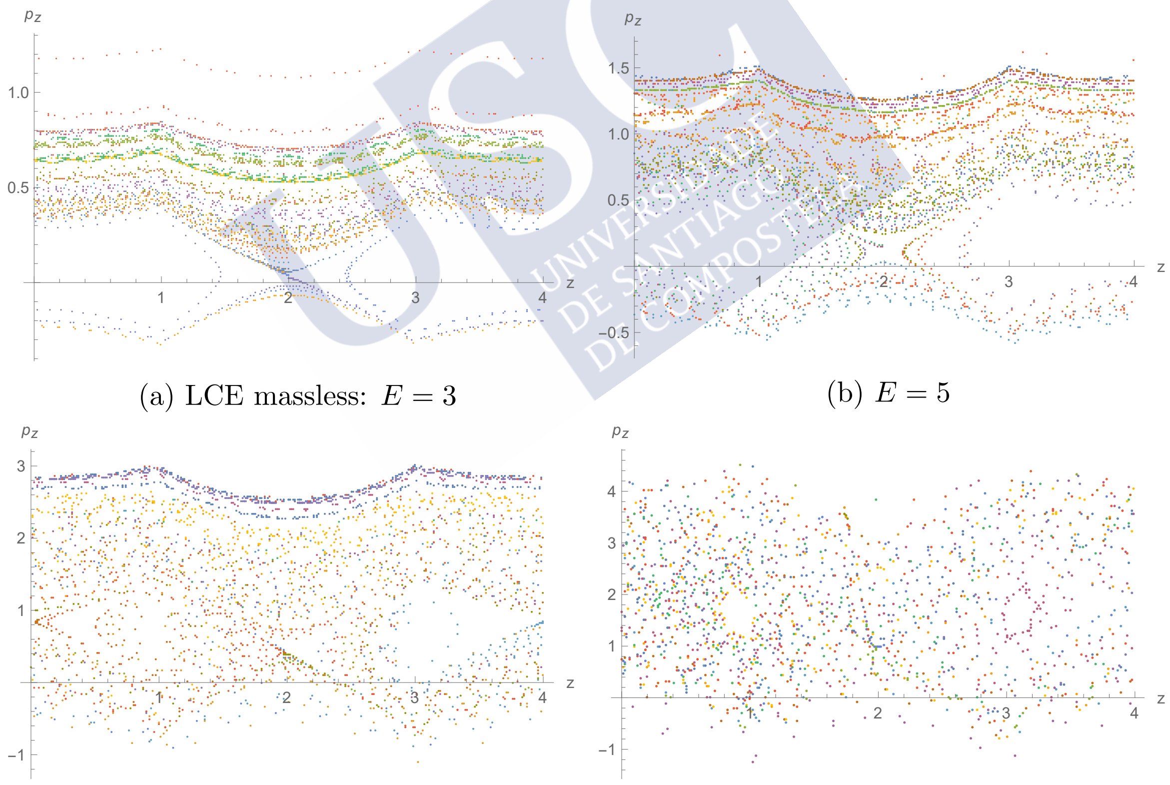
If the string motion were integrable, the corresponding trajectories would have been constrained to a 2-dimensional torus in this (z, p_z, χ, p_χ) phase-space. The Poincaré cross-sections of the phase-space would then show the different resonant tori as embedded circles. The absence of such embedded circular KAM curves (in figures 7.12, 7.13 and 7.14) indicates that we are dealing with a non-integrable system, in agreement with the results we found in the earlier sections.

From our earlier study of the numerical evolution in section 7.4.1, we know that for low energies the string oscillates between the different poles of the two-sphere when turning around along the z-axis. As we explained, the point $\chi(t) = 0$ corresponds to the string being on the north pole of the 2-sphere. We noticed that for low enough energies the string stays localised at this pole while moving along the z-axis. This is clearly seen from the horizontal lines in figure 7.12a. Also, notice that for very low momenta the string does not reach the other side of the z-domain and stays localised around one of the endpoints.

As we increase the energy (and consequently choose a higher value for χ to satisfy the Virasoro constraint) the string is no longer fixed at the pole but starts to oscillate quasi-periodically around the poles as it transverses the z-direction. This is the nature of the circles that we see appearing along the horizontal lines in figures 7.12b and 7.12c. Finally, as we increase the energy even further we see that the Poincaré section looses all of its structure, since the string seems to move randomly along the 2-sphere as the string moves along the z-direction.

Similar Poincaré sections for the background in equation 7.5 (quiver 1), can be seen in figure 7.14. Though this second quiver solution is not left-right symmetric along the z-direction the behaviour of its numerical evolution is in both cases very similar.

Finally, a different Poincaré cross-section for the quiver in equation (7.7) is shown in figure 7.14. Here we choose our initial conditions in a similar manner but we now plot the points (χ, p_χ) every time $z = 2$. We see again that for low energies the string stays located at the poles where $\cos\chi = 1$ or $\cos\chi = -1$. As we go to higher energies the string is located at random points on the 2-sphere every time we cross $z = 2$.



(a) LCE massless: $E = 3$

(b) $E = 5$

(c) $E = 10$

(d) $E = 15$

Figure 7.12: Poincaré sections for the (z, p_z)-plane at $\chi(t) = 0$, for the quiver in equation (7.7) at different energies.

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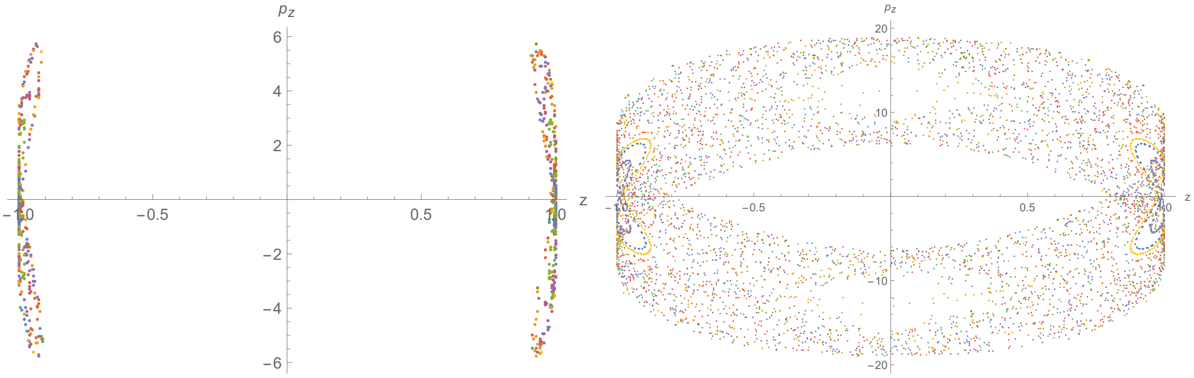
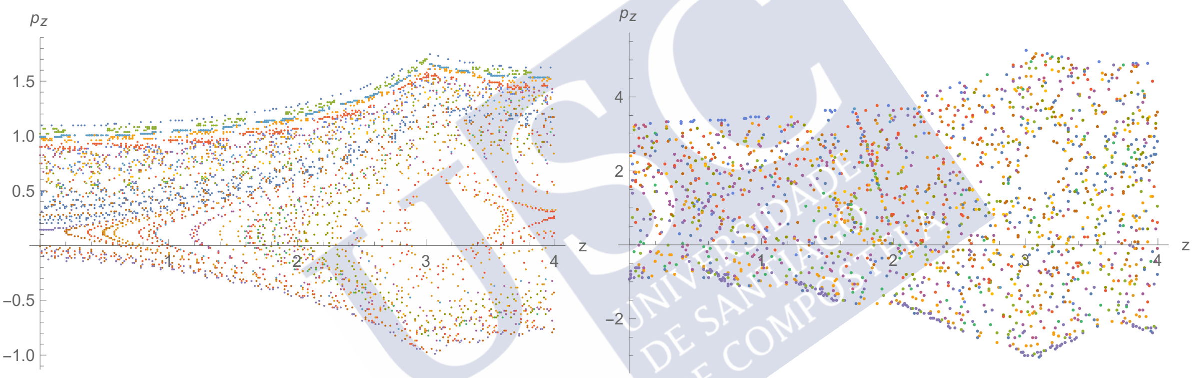
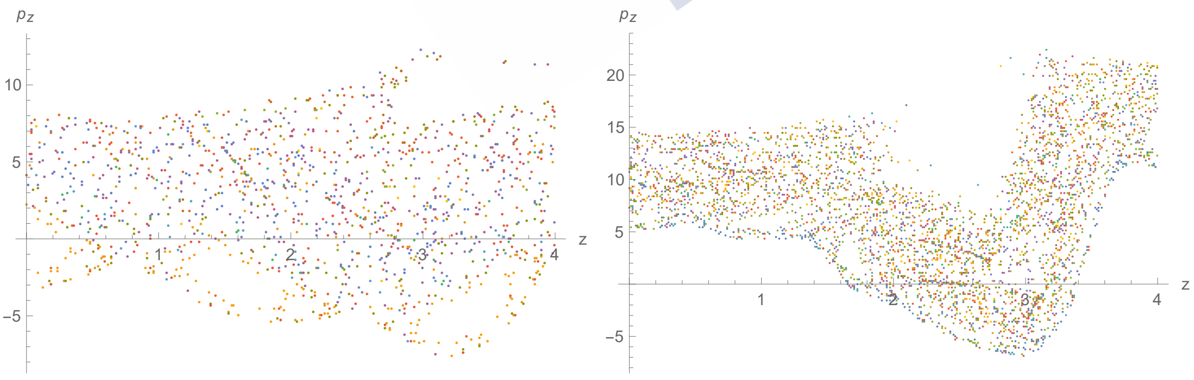


(a) $E = 15$ (b) $E = 35$

Figure 7.13: Poincaré sections for the (χ, p_χ)-plane at $z(t) = 2$, for the quiver in equation (7.7) at different energies.



(a) LCE massless: $E = 5$ (b) $E = 15$



(c) $E = 35$ (d) $E = 65$

Figure 7.14: Poincaré sections for the (z, p_z)-plane at $\chi(t) = 0$, for the quiver in equation (7.6) at different energies.

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7.5 Discussion

We conclude this chapter by summarising the key ideas that drive the present analysis and highlighting the main results obtained. The purpose of this chapter was to explore the dynamics of 6-dimensional SCFT's with $\mathcal{N} = (1, 0)$ SUSY. We discussed the holographic representation of the strongly coupled dynamics and focused on analytically showing the non-integrability of these theories. The fact that the system is strongly coupled at the conformal fixed point, together with the absence of a local Lagrangian, makes it a very difficult task to explore anything using the field theory description itself. However, studying the dual string description is promising. In the Massive type IIA dual background one can perform semi-classical computations that unveil some of the issues related to the non-integrability.

The traditional way of establishing the analytic integrability of a dynamical system is to find the appropriate Lax pair. Unfortunately, there is no general prescription to construct Lax pairs for a given field theory. Therefore, we have chosen a different procedure, namely to consider a solitonic excitation in the dual string background, and studied it as a dynamical system. We used a simple prescription (that goes under the name of Kovacic's algorithm [245]) to show the presence of non-integrability associated with the phase space of our classical string embedded in Massive type IIA. This implies that the eigenvalues of the dilatation operator of the CFT cannot be determined with the usual techniques.

By probing the dual type IIA background with a classical string (our soliton) and studying the Hamiltonian dynamics of this excitation, we gained information on "long" operators of different quivers in the $\mathcal{N} = (1, 0)$ SCFT's. These operators have large quantum numbers (scaling and angular momentum). The classical strings correspond to that specific sector within the dual SCFT's. By virtue of the Maldacena duality, the analysis is equivalent to exploring the non-integrability associated to that specific CFT sector. In fact, if the classical dynamics associated to specific string embeddings in the bulk fails to satisfy the Kovacic's criterium, the corresponding phase space dynamics is non-integrable. This, in turn, implies non-integrability in the full dual $\mathcal{N} = (1, 0)$ SCFT's.

In order to put our analytic findings on a more solid ground, we carried out a numerical analysis where we computed various physical quantities like the Lyapunov coefficients, the power spectra and the Poincaré sections that eventually display the onset of the chaotic behaviour associated with the corresponding phase space dynamics of the classical string embeddings.

7.A The function $\alpha(z)$ for the quivers in figures 7.5 and 7.6

In this appendix we plot the function $\frac{\alpha(z)}{81\pi^2 N}$ and its derivatives for the quivers in figures 7.5 and 7.6.

For the quiver in figure 7.1, we have: With these values of $\alpha(z)$ and its derivatives,

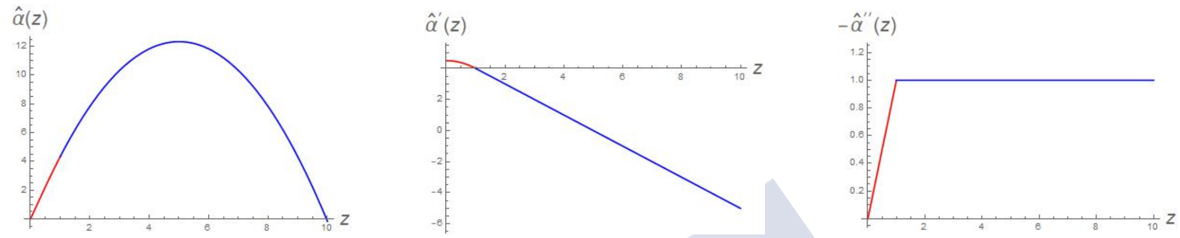

Figure 7.15: $\hat{\alpha}(z)$ and its first and second derivatives, for the dual background of the quiver in figure 7.5.

we can easily construct the functions $f_i(z)$ that describe the gravitational background. We plot, in order, $f_i(z)$ where $i \in \{1, 2, 3, 4, 5, 6\}$. Now we do the same for the quiver

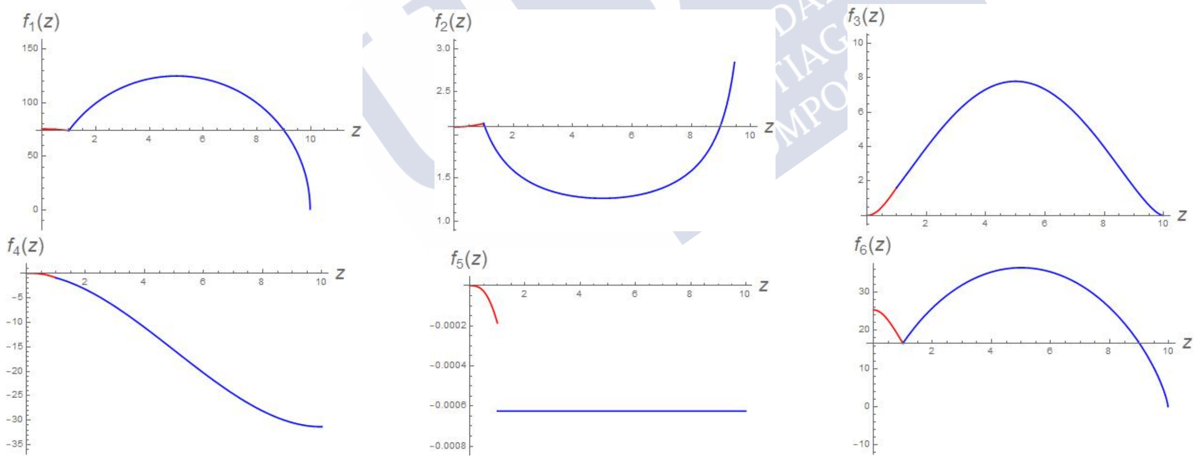

Figure 7.16: $f_i(z)$ for the dual background to the quiver in figure 7.5.

in figure 7.6. We depict the function $\alpha(z)$ and its derivatives in figure 7.17 and $f_i(z)$ in figure 7.18. It is easy to construct the Ricci scalar of these geometries. It can be written in terms of $\alpha(z)$ and its derivatives:

$$R = \frac{1}{4\sqrt{2}\pi\alpha^2\alpha''^2(\alpha'^2 - 2\alpha\alpha'')^2}\sqrt{-\frac{\alpha}{\alpha''}}\big[-21\alpha'^6\alpha''^2 + 42\alpha\alpha'^5\alpha''\alpha^{(3)}$$
$$-252\alpha^2\alpha'^3\alpha''^2\alpha^{(3)} + 336\alpha^3\alpha'\alpha''^2\alpha^{(3)} + 8\alpha^3\alpha'^2\alpha''(13(\alpha^{(3)})^2 - 7\alpha''\alpha^{(4)})$$
$$+12\alpha^3\alpha''^2(-7\alpha''^3 - 15\alpha(\alpha^{(3)})^2 + 6\alpha\alpha''\alpha^{(4)}) + \alpha\alpha'^{(4)}(63(\alpha'')^3 - 21\alpha(\alpha^{(3)})^2 + 10\alpha\alpha''\alpha^{(4)})\big]$$

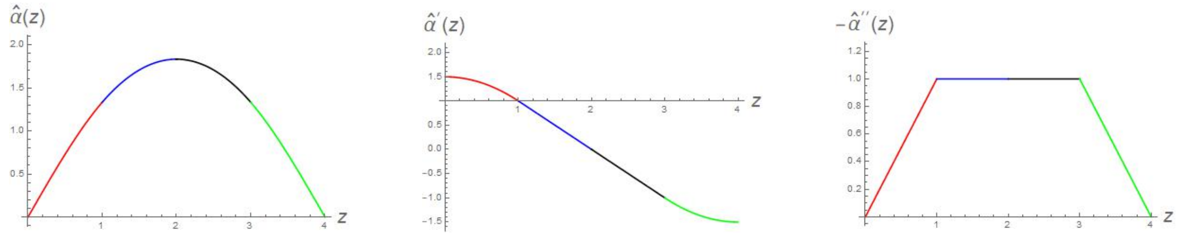

Figure 7.17: $\hat{\alpha}(z)$ and its derivatives, for the dual background of the quiver in figure 7.6.

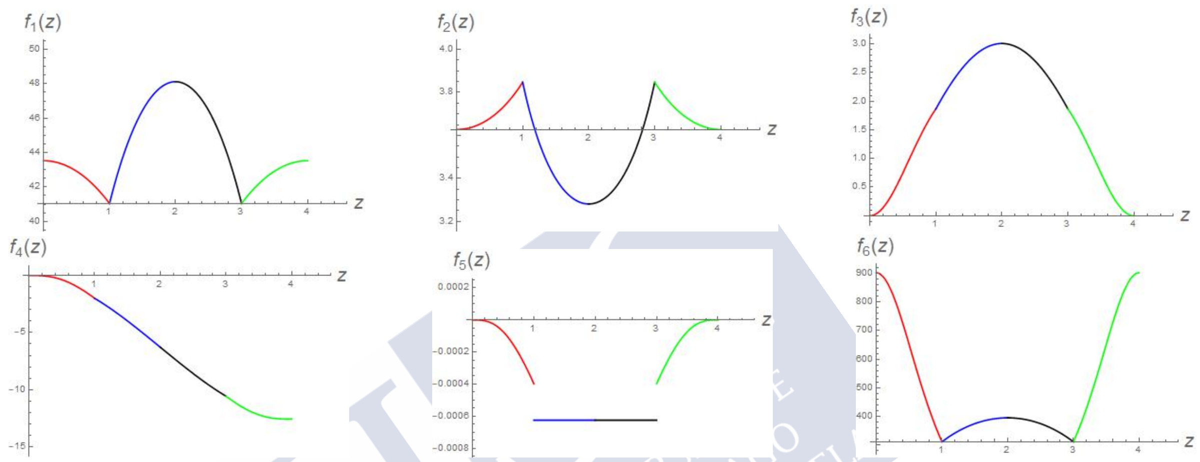

Figure 7.18: $f_i(z)$ for the dual background to the quiver in figure 7.6.

In figure 7.19 we plot the Ricci scalar for both geometries.

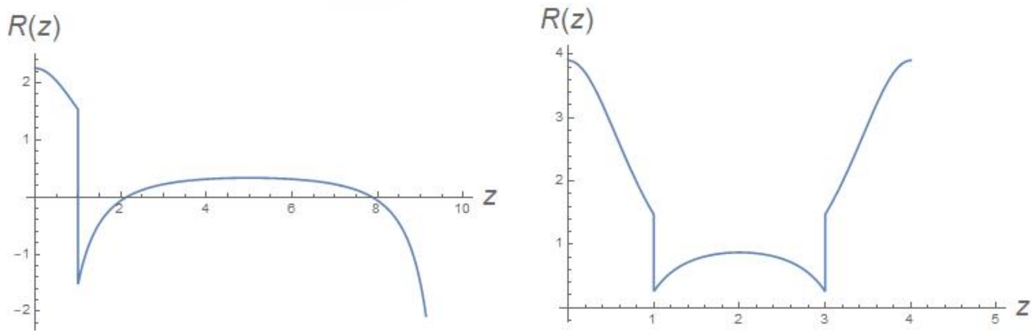

Figure 7.19: Ricci scalars for the backgrounds associated to the quivers in figures 7.5 and 7.6 respectively.

7.B Liouvillian integrability and Kovacic's algorithm

In this appendix, we briefly describe Kovacic's algorithm [245]. Consider a second-order differential equation,

$$y''(x) + B(x)y'(x) + A(x)y(x) = 0, \tag{7.49}$$

where $A(x), B(x)$ are complex rational functions. We are interested in the existence of closed form solutions, namely solutions that can be expressed in terms of algebraic, exponential, and trigonometric functions, and integrals of the previous functions. If this is the case, we call the solution Liouvillian. The algorithm of [245] provides one such solution or shows there is none (in which case we refer to the differential equation (7.49) as non-integrable). We will not describe the algorithm itself (that is efficiently implemented by many different softwares). We will limit us to explain the logic behind Kovacic's algorithm and some *necessary but not-sufficient* conditions that a combination of the functions A, B, B' must satisfy, for the equation (7.49) to be Liouville-integrable.

We start by redefining the function $y(x)$ and rewriting the differential equation as,

$$\begin{aligned} &y(x) = e^{\int w(x) - \frac{B(x)}{2} dx}, \\ &w'(x) + w(x)^2 = V(x), \quad V(x) = \frac{2B' + B^2 - 4A}{4}. \end{aligned} \tag{7.50}$$

It was shown that if the function $w(x)$ is algebraic of degrees 1, 2, 4, 6, or 12, then the equation (7.49) is Liouville integrable [245]. This result comes from the application of Galois theory to differential equations (this is called Picard-Vessiot theory). This formalism studies the most general group of invariances of the differential equation (7.49), that is the transformations that act on the solutions of the equation, that is a subgroup of $SL(2, C)$. Kovacic showed that there are four possible cases of subgroups of $SL(2, C)$ that can arise

- **Case 1:** the subgroup is generated by the matrix of the form

$$G = \begin{bmatrix} a & 0 \\ b & \frac{1}{a} \end{bmatrix},$$

  with a, b complex numbers. In this case $w(x)$ is a rational function of degree 1.

- **Case 2:** the subgroup of $SL(2, C)$ is generated by matrices of the form (here c is a complex number),

$$G = \begin{bmatrix} c & 0 \\ 0 & \frac{1}{c} \end{bmatrix}, \quad G = \begin{bmatrix} 0 & c \\ -\frac{1}{c} & 0 \end{bmatrix},$$

  in this case the function $w(x)$ is rational of degree 2.

- **Case 3:** the situation in which G is a finite group, not included in the two above cases. In this case, the degree of $w(x)$ is either 4, 6 or 12.

- **Case 4:** the group is $SL(2, C)$ and the solutions for $w(x)$, if they exist are non-Liouvillian.

Interestingly, Kovacic provided not only an algorithm to find the solutions in the first three cases above, but also a set of *necessary but not sufficient* conditions that the function $V(x)$ in equation (7.50) must satisfy to be in any of the first three cases detailed above [245]. For each of the cases as ordered above, the conditions are:

- **Case 1:** every pole of $V(x)$ has order 1 or has even order. The order of the function $V(x)$ at infinity is either even or greater than 2.

- **Case 2:** $V(x)$ has either one pole of order 2, or poles of odd-order greater than 2.

- **Case 3:** the order of the poles of V does not exceed 2, and the order of V at infinity is at least 2.

If none of the above is satisfied, the analytic solution (if it exists), is non-Liouvillian.

7.B.1 One example

Let us work out an example to see the criteria at work. We study the NVE in equations (7.33)-(7.34). To simplify matters, we will just study the NVE in the interval $0 \leq z \leq 1$, so that the function $\alpha(z) = -81\pi^2 N(a_1 z + \frac{z^3}{6})$. In this case the coefficients are

$$\mathcal{A} = 1 - \sqrt{3}\frac{(z^4 + 20a_1 z^2 - 60a_1^2)}{\sqrt{-6a_1 - z^2}(z^4 + 12a_1 z^2 - 12a_1^2)},$$
$$\mathcal{B} = \frac{2}{z} + \frac{3z}{(6a_1 + z^2)} - 4\frac{6a_1 z + z^3}{(z^4 + 12a_1 z^2 - 12a_1^2)}. \tag{7.51}$$

To avoid cluttering the expressions, we have chosen the coefficients $E = 4\pi$ (such that $z = \tau$) and $\nu = 1$.

The coefficients of this NVE are not rational functions. To amend this, we change from z to a new variable v,

$$z = \sqrt{-6a_1 - v^2} \tag{7.52}$$

denoting $x' = \frac{dx}{dv}$, the NVE reads

$$x''(v) + \mathcal{C}x'(v) + \mathcal{D}x(v) = 0, \quad \mathcal{C} = \frac{\left(\mathcal{B}(v) + \frac{d}{dv}\frac{dv}{dz}\right)}{\frac{dv}{dz}}, \quad \mathcal{D} = \frac{\mathcal{A}(v)}{(\frac{dv}{dz})^2}. \tag{7.53}$$

According to what was explained around equation (7.50), we now need to analyse the principal part of the potential $4V = 2\frac{d\mathcal{C}}{dv} + \mathcal{C}^2 - 4\mathcal{D}$.

For the particular case of the $\mathcal{A}$ and $\mathcal{B}$, and the change of variables in equations (7.51)-(7.52) we find,

$$
\begin{aligned}
&\mathcal{C} = \frac{v^6 - 12a_1 v^4 - 240a_1^2 v^2 - 576a_1^3}{v(v^2+6a_1)(v^4-48a_1^2)},\\
&\mathcal{D} = -1 + \frac{6a_1 + 5\sqrt{3}v}{v^2+6a_1} - \frac{4\sqrt{3}v(v^2-4a_1)}{v^4-48a_1^2},\\
&4V = 4 + \frac{\gamma_0}{v^2} + \frac{\gamma_1}{(v^2+6a_1)^2} + \frac{\gamma_2+\gamma_3 v}{(v^2+6a_1)} + \frac{\gamma_4 v^2}{(v^4-48a_1^2)^2} + \\
&\frac{\gamma_5+\gamma_6 v+\gamma_7 v^2+\gamma_8 v^3}{(v^4-48a_1^2)}.
\end{aligned} \tag{7.54}
$$

The coefficients γ_i are numerical constants, not very relevant for our analysis below. Notice that the potential has a pole of order 1 at $v = \pm i\sqrt{6a_1}$. The order of the potential (leading power of the degrees of the denominator minus numerator) is one. Hence, $V(x)$ does not fall in any of the three allowed cases. The solution to the equation should then be non-Liouvillian.

Notice that this analysis covers the cases of our quivers in figures 7.1 and 7.5. Indeed, both these quivers start with the function $\alpha(z) = -81\pi^2 N(a_1 z + \frac{z^3}{6})$. Hence, both backgrounds and both quiver CFTs are non-integrable.

7.C Integrability in type IIA and M-theory

In this short appendix (the details of which will be fully worked out elsewhere), we will indicate the steps that lead to the a more detailed study of our string configuration for the case in which the function $\alpha(z) = (R^2 - \mu^2 z^2)$, characterising a background in type IIA (with mass parameter $m = 0$) that lifts to and $AdS_7 \times S^4/Z_k$ in 11-dimensional supergravity. This is the case we studied around equation (7.35). We shall lift the background to 11 dimensions, and then study the dynamics of a membrane that mimics our type IIA string. The lifted background is,

$$
\begin{aligned}
&ds_{11}^2 = f_6^{-2/3}\left[f_1\, ds^2_{AdS_7} + f_2\, dz^2 + f_3\, d\Omega^2(\chi,\xi)\right] + f_6^{4/3}(dy - f_5\cos\chi\, d\xi)^2,\\
&6C_3 = f_4 \sin\chi\, d\chi\wedge d\xi \wedge dy.
\end{aligned} \tag{7.55}
$$

We define $\gamma_{ij} = G_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$, where $i,j = \tau,\sigma,\rho$ are the worldvolume coordinates of the membrane, the functions f_i are all functions of z and are defined in equation (7.2). We use the action and constraints for a membrane (see for example [247]),

$$
\begin{aligned}
&S = \int d\tau d\sigma d\rho \left(\gamma_{\tau\tau} + L^2 \det(\gamma_{\alpha\beta}) + 2L\epsilon^{ijk} C_{\mu\nu\delta}\partial_i X^\mu \partial_j X^\nu \partial_k X^\delta\right).\\
&\gamma_{\tau\alpha} = 0, \quad \gamma_{\tau\tau} + L^2 \det\gamma_{\alpha\beta} = 0. \quad (\alpha,\beta = \sigma,\rho)
\end{aligned} \tag{7.56}
$$

We propose a membrane configuration that is the natural lift of the string soliton we proposed in the main part of this chapter,

$$t = t(\tau), \quad z = z(\tau), \quad \chi = \chi(\tau), \quad \xi = k\sigma, \quad y = \lambda\rho. \tag{7.57}$$

We find an effective Lagrangian and constraint that can be written as,

$$\begin{aligned} L &= f_6^{-2s/3}\left[f_1\dot{t}^2 - f_2\dot{z}^2 - f_3\dot{\chi}^2 + L^2k^2\lambda^2 f_3 f_6^{4s/3}\sin^2\chi + Lk\lambda f_4 f_6^{2s/3}\dot{\chi}\sin\chi\right], \\ 0 &= -f_1\dot{t}^2 + f_2\dot{z}^2 + f_3\dot{\chi}^2 + L^2k^2\lambda^2 f_3 f_6^{4s/3}\sin^2\chi. \end{aligned} \tag{7.58}$$

One can see that choosing $s = 0$ (and identifying $L\lambda k = \nu$), we recover the expressions for strings written below equation (7.24). On the other hand, for $s = 1$, we have the Lagrangian and constraint for the membrane.

We now study the equations of motion derived from equation (7.58). We find,

$$\dot{t} = \frac{E}{f_1} f_6^{2s/3}, \tag{7.59}$$

$$2f_3\ddot{\chi} = -2L^2k^2\lambda^2 f_3\cos\chi\sin\chi f_6^{4s/3} - 2\dot{z}\dot{\chi}f_3' + 2Lk\lambda\sin\chi\dot{z}f_4' f_6^{2s/3} + \frac{4s}{3}\frac{f_3 f_6'}{f_6}\dot{z}\dot{\chi}.$$

$$\ddot{z} + E^2\frac{f_6^{4s/3}}{2f_1f_2}\Big(\frac{f_1'}{f_1} - \frac{2s}{3}\frac{f_6'}{f_6}\Big) + \dot{z}^2\Big(\frac{f_2'}{2f_2} - \frac{s}{3}\frac{f_6'}{f_6}\Big) + \dot{\chi}^2\frac{f_3}{2f_2}\Big(-\frac{f_3'}{f_3} + \frac{2s}{3}\frac{f_6'}{f_6}\Big) +$$
$$+ Lk\lambda\sin\chi\,\dot{\chi}\frac{f_4'}{f_2}f_6^{2s/3} + L^2k^2\lambda^2\frac{f_3f_6^{4s/3}\sin^2\chi}{2f_2}\Big(\frac{f_3'}{f_3} + \frac{2s}{3}\frac{f_6'}{f_6}\Big) = 0.$$

The reader can check that for $s = 0$ the equation of motion of the string are recovered.

We now apply the same algorithmic procedure as in the main body of the chapter. The configuration with $\chi(\tau) = \dot{\chi}(\tau) = \ddot{\chi}(\tau) = 0$ solves the $\chi-$equation of motion and leaves the $z-$equation as,

$$\ddot{z} + E^2\frac{f_6^{4s/3}}{2f_1f_2}\Big(\frac{f_1'}{f_1} - \frac{2s}{3}\frac{f_6'}{f_6}\Big) + \dot{z}^2\Big(\frac{f_2'}{2f_2} - \frac{s}{3}\frac{f_6'}{f_6}\Big) = 0. \tag{7.60}$$

Calculating explicitly for the function $\alpha(z) = \mu(1 - z^2)$ (after choosing constants appropriately), using the explicit expression for $f_i(z)$, we find that $z-$equation is solved by

$$z_s(\tau) = \cosh\tau. \tag{7.61}$$

Fluctuating the $\chi-$equation as $\chi(\tau) = 0 + \epsilon f(\tau)$, we find the NVE,

$$\begin{aligned} &\ddot{f} + \mathcal{B}\dot{f} + \mathcal{A}f = 0, \\ &\mathcal{B} = \dot{z}(\tau)\left(\frac{f_3'}{f_3} - \frac{2s}{3}\frac{f_6'}{f_6}\right)|_{z_s} = 2\coth\tau, \\ &\mathcal{A} = L^2k^2\lambda^2 f_6^{4s/3} - L\lambda k f_6^{2s/3}\frac{f_4'}{f_3}\dot{z}(\tau)|_{z=z_s} = n_1\sinh\tau + n_2\sinh^2\tau. \end{aligned} \tag{7.62}$$

With n_1, n_2 two numbers. In what follows we take $s = 1$ to discuss the case of the membrane only. It is convenient to change the variable $v = e^{-\tau}$, to have an NVE that reads,

$$f'' + \frac{3v^2+1}{v(v^2-1)}f' + \left(\frac{n_1}{2v^3}(1-v^2) + \frac{n_2}{4v^4}(1-v^2)^2\right) f = 0. \tag{7.63}$$

We denoted $f' = \frac{df}{dv}$. We can construct the effective potential of the associated Schrödinger problem, as indicated in equation (7.50),

$$V(v) = \frac{2\mathcal{B}' + \mathcal{B}^2 - 4\mathcal{A}}{4} = \frac{3}{4v^2} - \frac{n_1}{2v^3} + \frac{n_1}{2v} - \frac{n_2}{4v^4} - \frac{n_2}{4} + \frac{n_2}{2v^2}. \tag{7.64}$$

We observe that the first of the necessary conditions discussed in appendix 7.B is satisfied. The Kovacic algorith should produce a Liouvillian solution for the membrane, making the membrane configuration in equation (7.57) Liouville-integrable.

We see that the problem with the string is that it "misses" the effects of the dilaton, represented above by the various powers of $f_6^{2/3}$. It is the presence of the dilaton (that the "classical limit" of the Polyakov action misses), what changes the equation to introduce integrability. Note that the dilaton goes very large at the ends of the interval $z = \pm 1$ (in these units), hence it cannot be neglected.

7.D Computation of the Lyapunov spectrum

In the following we discuss the algorithm used to compute the Lyapunov characteristic exponents (LCEs) for a generic system, in particular for a system of canonical equations such as the ones we have previously studied. We will make use of the prescription described in [248].

Let us consider a generic n-dimensional smooth dynamical system, which can generically be written as:

$$\dot{q} = V(q) \tag{7.65}$$

where $q(\tau)$ is the n-dimensional state vector $q = \left(\vec{X}(\tau), \vec{P}(\tau)\right)$ at time τ, $\dot{q} = \frac{dq}{d\tau}$ and V is a vector field on an open set U of the phase space manifold, which generates a flow f:

$$\dot{f}^\tau(q) = V(f^\tau(q)) \quad \text{for all } q \in U, \tau \in \mathrm{R} \tag{7.66}$$

where $f^\tau(q) = f(q, \tau)$.

Consider the evolution under the flow of two nearby points in the phase space, q_0 and $q_0 + \delta_0$, being δ_0 a small perturbation of the initial point q_0. After a time τ, the perturbation δ_τ will become:

$$\delta_\tau \equiv f^\tau(q_0 + \delta_0) - f^\tau(q_0) \approx D_{q_0} f^\tau(q_0) \cdot \delta_0 \tag{7.67}$$

The average exponential rate of divergence (or convergence) of two trajectories is defined by:

$$\lambda(q_0, \delta_0) = \lim_{\tau\to\infty} \frac{1}{\tau} \log \frac{||\delta_\tau||}{||\delta_0||} = \lim_{\tau\to\infty} \frac{1}{\tau} \log ||D_{q_0} f^\tau(q_0) \cdot \delta_0|| \tag{7.68}$$

being $||\delta||$ the length of the vector δ. If $\lambda(x, u) > 0$, we have exponential divergence of nearby orbits. Under weak conditions on the nature of the dynamical system, the limit (7.68) exists, it is finite and it is equal to the largest LCE λ_1, see [249] for reference.

The LCEs of order p, $1 \leq p \leq n$, are introduced to describe the mean rate of growth of a p-dimensional volume in the tangent space. Considering a parallelepiped U_0 in the tangent space whose edges are the p vectors $\delta_1, ..., \delta_p$, the LCEs of order p are defined by:

$$\lambda^p(q_0, U_0) = \lim_{\tau\to\infty} \frac{1}{\tau} \log[\mathrm{Vol}^p(D_{q_0} f^\tau(U_0))] \tag{7.69}$$

being Vol^p the p-dimensional volume defined in the tangent space. It can be seen [249] that there exist p linearly independent vectors $u_1, ..., u_p$ satisfying:

$$\lambda^p(q_0, U_0) = \lambda_1 + ... + \lambda_p \tag{7.70}$$

The tangent vector δ_τ defined in (7.67) evolves in time satisfying:

$$\dot{\Phi}_\tau(q_0) = D_q V(f^\tau(q_0)) \cdot \Phi_\tau(q_0), \quad \Phi_0(q_0) = I \tag{7.71}$$

where $\Phi_\tau(q_0) = D_{q_0} f^\tau(q_0)$. To calculate the trajectory, we have to integrate the system:

$$\left\{ \begin{array}{c} \dot{q} \\ \dot{\Phi} \end{array} \right\} = \left\{ \begin{array}{c} V(q) \\ D_q V(q) \cdot \Phi \end{array} \right\}, \left\{ \begin{array}{c} q(\tau_0) \\ \Phi(\tau_0) \end{array} \right\} = \left\{ \begin{array}{c} q_0 \\ I \end{array} \right\} \tag{7.72}$$

To compute the spectrum of LCEs, we will use the algorithm discussed in [250], based on the calculation of the order-p LCEs defined in equation (7.70) and on a repeated application of a Gram-Schmidt orthonormalization procedure (which avoids technical difficulties that arise in the implementation of the recipe described in [249]) that we briefly summarize here. Recall that if we compute an orthonormal set of vectors $\{\hat{\delta}_i\}$ out of the original set of vectors $\{\delta_i\}$, by using the Gram-Schmidt orthogonalisation procedure, the volume of the parallelepiped spanned by $\delta_1, ..., \delta_p$ is

$$\mathrm{Vol}\{\delta_1, ..., \delta_p\} = ||\hat{\delta}_1||...||\hat{\delta}_p|| \tag{7.73}$$

Then the algorithm starts by choosing an initial condition q_0 and a $n \times n$ matrix $\Delta_0 = [\delta_1^0, ..., \delta_n^0]$. Using the Gram-Schmidt procedure, we calculate the corresponding matrix of orthonormal vectors $\hat{\Delta}_0 = [\hat{\delta}_1^0, ..., \hat{\delta}_n^0]$ and integrate the equation (7.72) from $\{q_0, \Delta_0\}$ for a short interval T, to obtain $q_1 = f^T(q_0)$ and

$$\Delta_1 \equiv [\delta_1^1, ..., \delta_n^1] = D_{q_0} f^T(\Delta_0) = \Phi_T(q_0) \cdot [\delta_1^0, ..., \delta_n^0] \tag{7.74}$$

The algorithm proceeds by repeating this integration-orthonormalization procedure K times. During the k-th step, the p-dimensional volume Vol^p defined in (7.69) increases by a factor of $||w_1^k||...||w_p^k||$, where $\{w_1^k, ..., w_p^k\}$ is the set of orthogonal vectors calculated from U_k using the Gram-Schmidt technique. Then:

$$\lambda^p(q_0, \Delta_0) = \lim_{k\to\infty} \frac{1}{kT} \sum_{i=1}^{k} \log(||\hat{\delta}_1^i||...||\hat{\delta}_p^i||) \tag{7.75}$$

From which we can derive

$$\lambda_p = \lim_{k\to\infty} \frac{1}{kT} \sum_{i=1}^{k} \log ||\hat{\delta}_p^i|| \tag{7.76}$$

To obtain the Lyapunov spectrum, we continue calculating the quantities:

$$\frac{1}{KT} \sum_{i=1}^{K} \log ||\hat{\delta}_1^i|| \approx \lambda_1, ..., \frac{1}{KT} \sum_{i=1}^{K} \log ||\hat{\delta}_1^n|| \approx \lambda_n \tag{7.77}$$

for a suitable value of T, until they show convergence.

7.E A relation with non-Abelian T-duality?

Let us briefly discuss a possible relation between the metrics in the Cremonesi-Tomasiello backgrounds (see section 7.2) and NATD.

Since the work of Sfetsos and Thompson [15], the non-Abelian version of the usual T-duality has regained interest and played a role as a solution generating technique. Various papers taking a perspective inspired by holography, have made clear that when applied to symmetric enough backgrounds (characteristically $AdS_{p+1} \times M_{9-p}$ backgrounds) the generated solutions correspond to CFTs in p dimensions, realised on a $D_p - NS_5 - D_{p+2}$ system (for a sample of results, see the papers [86, 89, 251–253] for early attempts and [192, 254–256] for more recent and precise connections between NATD and brane setups.

It is natural to ask if the Massive type IIA AdS_7 backgrounds studied in this chapter can be thought of as the NATD of some background, conjecturally in type IIB, with dilaton and F_3 flux.

Let us give some comments in the direction of realising the previous idea. We consider a solution in Massive type IIA that is the simplest possible. Consider, for example, the case in which the function $\alpha(z)$ in Massive type IIA is

$$\alpha(z) = A \sin \omega z.$$

This is a solution to the equation of motion if $\alpha''' = -162\pi^2 F_0 = -A\omega^3 \cos \omega z$, which implies that the mass-parameter is actually position dependent. A possible way to understand this, suggests a position dependent smearing of D8-branes. Since $\alpha'' = -\omega^2 \alpha$, we have that the background and the Ricci scalar read,

$$
\begin{aligned}
ds^2 &= 8\pi\frac{\sqrt{2}}{\omega}AdS_7 + \sqrt{2}\pi\omega dz^2 + \frac{\sqrt{2}\pi}{\omega}\left(\frac{\sin^2\omega z}{1+\sin^2\omega z}\right)d\Omega_2,\\
e^{-2\phi} &= e^{-2\phi_0}(1+\sin^2\omega z), \quad B_2 = \pi\left(-z+\frac{\sin\omega z\cos\omega z}{\omega(1+\sin^2\omega z)}\right)d\Omega_2,\\
F_0 &= -\frac{A\omega^3\cos\omega z}{162\pi^2}, \quad F_2 = -\frac{A\omega^2}{81\pi^2}\left(\frac{\sin^3\omega z}{1+\sin^2\omega z}\right)d\Omega_2,\\
R &= \frac{\omega\sin^4\omega z}{4\sqrt{2}\pi}\left(\frac{12+100\cot^2\omega z+75\cot^4\omega z}{1+\sin^2\omega z}\right)
\end{aligned}
\tag{7.78}
$$

Notice that the background is non-singular. We expand this Massive type IIA solution close to $z \to 0$ and we find,

$$
\begin{aligned}
&ds^2 \sim 8\pi\frac{\sqrt{2}}{\omega}AdS_7 + \sqrt{2}\pi\omega\left(dz^2 + z^2 d\Omega_2\right),\\
&e^{-2\phi} \sim e^{-2\phi_0}(1+\omega^2 z^2), \; B_2 \sim -\frac{5\pi\omega^2}{3}z^3 d\Omega_2,\\
&F_0 \sim -\frac{A\omega^3}{162\pi^2}, \; F_2 \sim -\frac{A\omega^5}{81\pi^2}z^3 d\Omega_2.
\end{aligned}
\tag{7.79}
$$

We want to think about this background as obtained by NATD applied on some seed-solution. We can consider a spacetime of the form $AdS_7 \times S^3$ in type IIB. Notice that this is *not* a solution to the equations of motion. Hence we need to consider a more complicated background, depending on the coordinates of S^3. Importantly, notice that this background needs not be SUSY, it may be the case that the duality creates the supersymmetry. This putative background should have warp factors that can be decomposed in harmonics of S^3. Consider the s-wave and perform NATD on it. We insist, this s-wave should not be a solution of the equations of motion. We shall obtain,

$$
\begin{aligned}
ds^2 &= L_{AdS}^2 AdS_7 + L^2\left(dr^2 + \frac{r^2}{r^2+1}d\Omega_2\right),\\
e^{-2\phi} &= e^{-2\phi_0}(1+r^2), \quad B_2 = \mu_0\frac{r^3}{1+r^2}d\Omega_2,\\
F_0 &= f_0, \quad F_2 = \nu_0\frac{r^3}{1+r^2}z^3 d\Omega_2.
\end{aligned}
\tag{7.80}
$$

Where $\mu_0, \nu_0, L_{AdS}, L, f_0, \phi_0$ are some constants. Consider an expansion of this background near $r \to 0$. We find that after appropriately choosing the constants, it has the form in equation (7.79).

Notice that the same result would be obtained by starting with the background described around equation (7.6) and expanding for $z \to 0$. This is obvious as close to $z = 0$ the function $\alpha = a_1 z + \frac{z^3}{6}$ and $\alpha = A\sin\omega z$ coincide for some choice of A, ω, a_1.

This is reminiscent of the observations in the paper [192]. In fact, there the NATD of $AdS_5 \times S^5$ (the Sfetsos-Thompson solution [15]) was considered. A solution in type IIA with a linear charge density $\lambda = Nz$ is found and a completion of the geometry is proposed. Analogously, for the background defined around equation (7.6), we have linear charge density, and for small values of the z-coordinate, the charge density (that is the rank function $R(z) = -\alpha''$) is also linear for the background in equation (7.78).

In other words, the backgrounds in equation (7.6) or that in equation (7.78) provide a completion to a background like that in equation (7.80), obtained via NATD on a putative type IIB background. This structure, observed in examples with AdS_4, AdS_5, AdS_6 could be repeated for the case of AdS_7 if a true-solution in type IIB could be found, such that its NATD has the form given in equation (7.80).

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Summary

The gauge/gravity duality constitutes one of the cornerstones of theoretical physics since its appearance in 1997, revolutionizing our understanding of gauge theories, theories of gravity and the relations between them. Its importance, both theoretically and phenomenologically, is difficult to underestimate, as evidenced by the large number of articles to which it has given rise. From the theoretical point of view, this correspondence supposes the first explicit realization of the holographic principle, which establishes that the degrees of freedom of a theory of gravity in a volume must be encoded in the boundary that delimits it. One of the consequences of this principle is that it allows a way out to the information paradox and it supposes a great advance towards an understanding of quantum gravity. The formulation of this conjecture is a point in favor of string theories, which would therefore have phenomenological interest even in areas for which they were not designed, through the connection of the gravitational problem with the corresponding dual quantum field theory. In this way, string theories play the role of a tool to study quantum field theories, used as a framework to model certain physical systems. Therefore, the interest of the gauge/gravity duality is also phenomenological.

The main advantage of this correspondence lies in the fact that it relates opposite regimes of both theories. When the gravitational problem in String Theory is tractable, corresponding to the limit of classical gravity, without quantum corrections or coming from the massive states of the strings, the regime in the field theory is the one of a large coupling constant in the planar limit. Given that there are generally no ways to study the theories with a large coupling constant, since the perturbation expansion in Feynman diagrams would not be applicable, the gauge/gravity duality would be the only theoretical tool with which to deal with strong coupling. The main limitation of this duality is the fact that a gravitational dual of the main phenomenological theories, such as QCD, is not available, but only the duals of theories with large symmetry. However, the development and generalization of this correspondence has allowed to find the dual of a large number of field theories, some of them showing similar characteristics to the main phenomenological theories of interest. In this way, the holographic duality is not a precision tool that provides quantitatively good results, but rather qualitative results. It is expected to be able to capture characteristics and behaviors that are common to different theories and try to establish results that are universal in phenomenological theories of greater interest. Therefore, one can understand the duality applied to a field theory that models with greater or lesser accuracy a given physical system as a tool that allows to make predictions in such theory through the corresponding calculation in the gravitational dual. In that sense, finding the duals to interesting theories and understanding the properties of these duals is of great relevance.

This thesis can be divided into four parts. The first part focuses on the generation of supergravity solutions corresponding to the low energy limit of the D3-D5 brane system and the study of its properties and is made up of chapters 2, 3 and 4. The second part includes chapter 5 and focuses on the study of the spectrum of fluctuations of the Gaiotto-

Maldacena geometries. The third part, corresponding to chapter 6, focuses on the addition of defects to the Brandhuber-Oz geometry and on the study of its fluctuations and the identification of its corresponding dual in field theory. In the last part, corresponding to chapter 7, integrability in the theories $\mathcal{N} = (1, 0)$ SCFT's in 6 dimensions is studied, through the study of the integrability in its gravitational dual, constructed through several works and whose final form was provided by Cremonesi and Tomasiello. Below we will briefly explain the main results obtained in this thesis.

In chapter 2 we tried to build a dual supergravity solution to the quantum field theories generated when N_f flavor D5-branes intersect N_c color D3-branes over a (2+1)-dimensional space. Its dual is a quantum field theory with defects in which (2+1)-dimensional matter in the fundamental representation is coupled with a gauge theory in 3+1 dimensions. This system had been studied previously, in the limit in which the flavor branes are introduced in the probe brane approach, that is, without taking into account their effect on the geometry originated by the D3-branes. This means that the fundamental matter introduced (quarks) is infinitely massive, so they are non-dynamic degrees of freedom. This approximation is valid when the number of flavor branes is small compared to the number of color branes. When both are of the same order of magnitude, the approximation is no longer good and the effect of the D5-branes in the geometry must be taken into account (this effect is called backreaction). In this chapter we build a geometry of the D3-D5 system in which the backreaction of the D5-branes is taken into account, in the Veneziano regime, in which $N_c \sim N_f \to \infty$, with $\frac{N_c}{N_f}$ fixed. The case in which the internal space is a 5-dimensional general Sasaki-Einstein space is considered, with the corresponding field theory being a quiver. The solution preserves a certain amount of supersymmetry, for which it is required that the D5-branes are wrapped in a 3-dimensional submanifold of the Calabi-Yau cone built over the Sasaki-Einstein space. To build a solution that incorporates backreaction, the D5-branes must be included as sources in the supergravity equations of motion. These induce a violation of the Bianchi identities for those RR fluxes for which the corresponding potential is coupled with the brane, in this case, the 3-form F_3. This violation of the Bianchi identity for F_3 is a 4-form that encodes the distribution of charge of the flavor branes. This information allows us to write an ansatz for F_3 that preserves some supersymmetry. The presence of supersymmetry allows us to write a system of first-order BPS differential equations for the geometry. This system can be reduced to a single second-order master equation for a new function with which the entire geometry can be specified. This equation can be integrated in general for the case in which N_f is zero. For the case with non-massive flavors, a particular solution can be obtained that provides a metric with anisotropic scale invariance. We also consider the ansatz for the case of massive flavors, for which a master equation can also be found and numerical solutions are constructed that interpolate between the geometry with $N_f = 0$ in the IR and the geometry with massless flavors in the UV.

In chapter 3, we tried to generalize the results of the previous chapter to the case in which the dual quantum field theory is at a non-zero temperature. In the gravitational dual, this is equivalent to finding a black hole geometry, that is, with a horizon. Again, we worked in the Veneziano limit, and we considered massless quarks. The solution found for the black hole is analytical and simple. This solution, as happened in the previous chapter, is anisotropic, and the holographic dual can be understood as a multilayer system, in which each layer is created by the D5-branes distributed in parallel 2-dimensional planes within the 3-dimensional space. For this geometry we studied both physical observables that live in a single layer and observables that connect different layers, and it was found, non-trivially, that the dynamics for the observables of the same layer is the same as that of a stack of effective D2-branes, which means that to describe our system one can use the (2+1)-dimensional Super Yang-Mills theory. In this chapter the thermodynamics of the system is studied, and the VEV of the energy-momentum tensor of the dual theory is computed by different methods. Next, the hydrodynamics of the system is studied. For this, an effective reduced action is obtained in both 4 and 5 dimensions, and the fluctuation of the fields of the reduced supergravity action is carried out, solving the resulting equations to find the transport coefficients, in the so-called shear and sound channels. The effective actions found in 4 and 5 dimensions also allow us to compute the VEV of the energy-momentum tensor and confirm the results obtained in the thermodynamic analysis.

Chapter 4 generalizes the results of chapter 2 for the construction of geometries of D5-branes with D3-branes in the case in which the introduced flavors are massive, taking into account the backreaction. In the configuration of the D3-D5 system studied in chapters 2 and 3, both types of branes can be separated in their three transverse directions. When this separation is null, the mass of the hypermultiplets that live in the defect is zero, that is, they are massless flavors. The massless case was studied in chapter 2 and its generalization to non-zero temperature in chapter 3. The massive case is characterized by the profile function, whose argument is the holographic coordinate, and which describes the charge distribution of the D5-branes in the space. There is a minimum distance for which the value of this function is zero, which can be interpreted geometrically with the appearance of a cavity within which there are no D5-branes. This profile function can be determined if the function that describes how the D5-branes are embedded into the geometry originated by the D3-brane system is known. The tool that allows us to know the profile function is the kappa symmetry. In this chapter the analysis of the kappa symmetry is carried out, to determine the way in which the D5-branes generate the geometry considered in the chapter, and to determine other possible ways to embed the D5-branes that give rise to interesting geometric configurations with a clear interpretation in the field theory dual. To determine the geometry, an equation for a master function is again written, from which the entire geometry can be determined. This equation depends explicitly on the profile function. This equation is integrated numerically, and once the geometry is known, the physical observables of it are studied. The subsequent geometry depends in its final form on a parameter that characterizes the deformation of the metric in the IR. The study of the Wilson loops and the potentials for the quark-antiquark pairs are carried out, when the fundamentals are in the same layer or separated in the direction orthogonal to the layers. The same analysis is carried out with the entanglement entropy.

Next, we test the supergravity solution constructed through the addition of probe D5-branes dual to a chemical potential. The thermodynamics at zero temperature of the probe brane and the flow of the speed of sound from the UV to the IR are analyzed. Finally, the limit behaviors of this geometry are studied. When the mass of the quarks is very large, the flavors are decoupled and the unflavored solution $AdS_5 \times S^5$ is obtained. When the flavors are massless, the solution with anisotropic scale invariance of chapter 2 is obtained. For a non-zero finite value of the mass of the quarks, a solution is obtained that interpolates between these solutions: null flavor in the IR and not null massless flavor in the UV. The different observables analyzed also show a behavior whose limit values correspond to these regimes.

In chapter 5 the spectrum of fluctuations of the Gaiotto-Maldacena geometries is studied. These geometries are described in a generic way by a function that solves the 3-dimensional axisymmetric Laplace equation. Each solution to this equation determines a geometry. The dual field theory of these solutions is known, and can be obtained from the low energy limit of the D4-NS5-D6 brane system. This dual is a quiver, characterized by the number of color and flavor nodes. Each quiver is determined by a concrete solution of the Laplace equation in the gravitational dual. Obtaining the spectrum of fluctuations in the theory of gravity allows us to determine the spectrum of conformal dimensions of the operators of the dual theory. However, the complete determination of the spectrum is often difficult, given that the linearized equations of supergravity that are obtained couple a large number of fields in a nontrivial way. It is usually easier to partially solve the system of equations, that is, to find truncations of equations, which is equivalent to turn on only a small subset of the fluctuations. This is the perspective that was adopted in this chapter. In it, we first obtain the set of linearized equations for a generic type IIA supergravity solution conformal to a product space of the type $AdS_5 \times \mathcal{M}_5$. The Gaiotto-Maldacena geometries fall within this type of spaces. Then, we only turn on the fluctuations of the metric in the Minkowski directions. If such fluctuations are further required to be transverse and traceless, the Maxwell equations for the RR and NSNS fluxes, as well as the equation for the dilaton, are trivially satisfied, leaving the Einstein equation as the only non-trivial equation. This equation is greatly simplified for the Gaiotto-Maldacena geometries and can be written as an eigenvalue equation for an operator $\mathcal{L}$, whose form is determined by the function V that satisfies the axisymmetric Laplace equation. The study of this equation was particularized for the cases in which V corresponds to the geometries resulting from doing an Abelian and non-Abelian T-duality transformation on the $AdS_5 \times S^5$ solution acting on the sphere. For these solutions it was possible to solve the differential equation, analytically for the inert modes under the transformation of T-duality (or its non-Abelian version) and numerically and approximately through the WKB method for the non-inert modes under the transformation. For the analytical case, an already known result in the literature was found. For the non-analytical case, the mass spectrum was obtained numerically, showing great agreement with the results obtained by the WKB approximation in the range in which the approach is valid, that is, for large values of the masses.

In chapter 6 we tried to add defects to the Brandhuber-Oz geometry, through probe D4- and D6-branes in different configurations, and the fluctuations of these configurations and their interpretation in the field theory dual were analyzed. The Brandhuber-Oz solution is the gravitational dual of the higher-rank version of a type of theories E_{N_f+1}. This solution corresponds to the low energy limit of a configuration of D4-branes near an orientifold $O8$-plane with $N_f < 8$ D8-branes. The complete geometry in 10 dimensions is a solution to the massive type IIA supergravity equations, and has the form of a fibered space of the type $AdS_6 \times S^4$, with non-zero mass F_0. The addition of defects to this geometry originates a quantum field theory of lower dimension that is coupled with the ambient quantum field theory, the one generated by the D4-O8-D8 configuration. In this chapter, different defects are added, preserving supersymmetry. First, we study the codimension-2 defects, originated through probe D4-branes. Its supersymmetry and its fluctuations are analyzed, and the correspondence between operators and fluctuations is proposed. Next, another different configuration of D4-branes is studied, in which these are wrapped in the internal space, which corresponds to the generalization of the configuration that captures the antisymmetric Wilson loop. In addition to the discussion of supersymmetry, the fluctuations of this configuration are also analyzed. Finally, we study the codimension-1 defects, constructed through probe D6-branes. Its supersymmetry is described and the fluctuations are calculated, including the map between operators and fluctuations.

In chapter 7, the integrability of $\mathcal{N} = (1,0)$ SCFT's in 6 dimensions is studied, through the study of its holographic dual. These quantum field theories are realized as the low energy limit of the D6-NS5-D8 brane system. Such configurations admit a gravitational dual in Massive type IIA supergravity, whose construction was carried out through several articles that culminated in a formulation due to Cremonesi and Tomasiello, which was taken as the starting point in our study. The integrability of a field theory or the absence of it is an interesting problem, to which many efforts have been devoted after the discovery of integrable structures within the AdS/CFT correspondence. While finding integrability is often a difficult task, finding non-integrability is simpler. In this chapter the non-integrability of the aforementioned theories is tested. The starting point is the geometries of Cremonesi-Tomasiello. These are described through a function of the holographic coordinate that satisfies a third-order differential equation. The form of this function completely determines the dual quantum field theory, this being a quiver whose flavor and color nodes are determined through this function. In this chapter, the formulation of Cremonesi and Tomasiello is reviewed and explicit examples of these geometries and their dual quiver are provided. Next, the integrability of these theories is studied. For this, we study the integrability of the motion of a string propagating in this geometry. The hamiltonian of the string is written and it is proved, through both analytical and numerical methods, that the motion is non-integrable. The analytical method is based on the construction of the Normal Variational Equation (NVE), and the application of the Kovacic's criterion to such equation, which proves its non-integrability. As numerical methods that support this result, the power spectrum is calculated, which serves to characterize a chaotic dynamic, the spectrum of Lyapunov exponents, which characterizes the separation rate of two trajectories that start arbitrarily close in the phase space, and the Poincaré sections are calculated, which describe the shape of the phase space. For integrable systems, such phase space is foliated by the so-called KAM-tori, but this foliation disappears when we are facing a non-integrable perturbation. These numerical methods confirm the analytical results, proving the non-integrability in general of the dual quantum field theory.

The results of this thesis have both theoretical and phenomenological interest. On the one hand, the construction of new geometries for the D3-D5 intersection, corresponding to chapters 2, 3 and 4, is of phenomenological relevance, since this system has been used, among other things, for modelling the quantum Hall effect, as a holographic model of graphene and it appears in the context of bubbling geometries. On the other hand, the results of chapter 5 are of theoretical interest, since they represent an advance in the understanding of the effect of the T-duality transformations and their non-Abelian generalization. In this case we studied the change that these transformations induce in the mass spectrum of the fluctuations of the geometries related by the duality. Also of both theoretical and phenomenological interest are the results of chapter 6, which allow to provide a dual to a quantum field theory with defects and their realization through supersymmetric branes in the dual gravity theory. Finally, chapter 7 is an interesting theoretical advance, given that it allows to generically discard integrability in a large class of quantum field theories.

This thesis opens new lines of research. In particular, a project of interest would be to find the system in condensed matter for which the geometries constructed in chapters 2, 3 and 4 constitute holographic duals. These anisotropic solutions would also be useful for studying the strongly coupled matter of the nuclei of neutron stars. Similarly, the generic equations obtained in chapter 5 for the fluctuations of a type IIA supergravity solution can be used to study the spectrum of fluctuations of other geometries. Another line of research connects with the results of chapter 6, and focuses on the addition of defects to other interesting geometries, the study of their supersymmetry and their fluctuations. Similarly, the methods applied in chapter 7 can be used to characterize integrability in other field theories of interest.

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Resumen

La dualidad gravedad/gauge constituye una de las piedras angulares de la física teórica desde su aparición en 1997, revolucionando nuestra comprensión de las teorías gauge, las teorías de gravedad y las relaciones entre ambas. Su importancia tanto teórica como fenomenológica es difícil de subestimar, como prueba la gran cantidad de artículos a los que ha dado lugar. Desde el punto de vista teórico, esta correspondencia supone la primera realización explícita del principio holográfico, que establece que los grados de libertad de una teoría de gravedad en un volumen deben estar codificados en la frontera que lo delimita. Una de las consecuencias de este principio es que permite una salida a la paradoja de la información y supone un gran avance hacia una comprensión de la gravedad cuántica. La formulación de esta conjetura es un punto a favor de las teorías de cuerdas, que pasan a tener interés fenomenológico incluso en ámbitos para los que no fueron diseñadas, a través de la conexión del problema gravitatorio con la teoría cuántica de campos dual correspondiente. De esta forma, las teorías de cuerdas juegan el papel de herramienta para el estudio de teorías cuánticas de campos, usadas como marco para modelar determinados sistemas físicos. Por lo tanto, el interés de la dualidad gravedad/gauge es también fenomenológico.

La principal ventaja de esta correspondencia reside en el hecho de que relaciona regímenes opuestos de ambas teorías. Cuando el problema gravitatorio en Teoría de Cuerdas es tratable, correspondiente al límite de gravedad clásica, sin correcciones cuánticas ni procedentes de estados masivos de la cuerdas, el régimen en teoría de campos es el de constante de acoplamiento grande en el límite planar. Dado que no se dispone en general de formas de tratar las teorías con constante de acoplamiento grande, pues la expansión perturbativa en diagramas de Feynman no sería aplicable, la dualidad gravedad/gauge constituiría la única herramienta teórica con la que tratar el acoplamiento fuerte. La principal limitación de esta dualidad es el hecho de que no se dispone de un dual gravitatorio de las principales teorías fenomenológicas, como por ejemplo QCD, sino de duales de teorías con gran simetría. No obstante, el desarrollo y generalización de esta correspondencia ha permitido encontrar el dual de gran número de teorías de campos, algunas de ellas mostrando características similares a las principales teorías fenomenológicas. De esta manera, la dualidad holográfica no es una herramienta de precisión que proporcione resultados cuantitivamente buenos, sino más bien resultados cualitativos. Se espera ser capaces de capturar características y comportamientos que sean comunes a diferentes teorías y tratar de establecer resultados que sean universales en teorías fenomenológicas de mayor interés. Por lo tanto, se puede entender la dualidad aplicada a una teoría de campos que modela con mayor o menor exactitud a un sistema físico dado como una herramienta que permite hacer predicciones en dicha teoría a través del cálculo correspondiente en el dual gravitatorio. En ese sentido, encontrar duales a teorías interesantes y entender las propiedades de dichos duales es de gran relevancia.

Esta tesis se puede dividir en cuatro partes. La primera parte se centra en la generación de soluciones de supergravedad correspondientes al límite de baja energía del sistema de branas D3-D5 y el estudio de sus propiedades y la forman los capítulos 2, 3 y 4. La segunda parte comprende el capítulo 5 y se centra en el estudio del espectro de fluctuaciones de las geometrías de Gaiotto-Maldacena. La tercera parte, correspondiente al capítulo 6, se centra en la adición de defectos a la geometría de Brandhuber-Oz y en el estudio de sus fluctuaciones y la identificación de su correspondiente dual en teoría de campos. En la última parte, correspondiente al capítulo 7, se estudia la integrabilidad en las teorías $\mathcal{N} = (1,0)$ SCFT's en 6 dimensiones, a través del estudio de la integrabilidad en su dual gravitatorio, construido a lo largo de varios trabajos y cuya forma final fue proporcionada por Cremonesi y Tomasiello. A continuación explicaremos brevemente los principales resultados obtenidos en esta tesis.

En el capítulo 2 se trató de construir una solución de supergravedad dual a las teorías cuánticas de campos generadas cuando N_f D5-branas de sabor intersecan N_c D3-branas de color a lo largo de un espacio (2+1)-dimensional. Su dual es una teoría cuántica de campos con defectos en la que la materia (2+1)-dimensional en representación fundamental se acopla con una teoría gauge en 3+1 dimensiones. Este sistema había sido estudiado previamente, en límite en el que las branas de sabor se introducen en la aproximación de brana de prueba, es decir, sin tener en cuenta su efecto en la geometría originada por las D3-branas. Esto significa que la materia fundamental introducida (quarks) es infinitamente masiva, por lo que son grados de libertad no dinámicos. Esta aproximación es válida cuando el número de branas de sabor es pequeño en comparación con el número de branas de color. Cuando ambos son del mismo orden de magnitud, la aproximación deja de ser buena y hay que tener en cuenta el efecto de las D5-branas en la geometría (este efecto se denomina backreaction). En este capítulo se construye una geometría del sistema D3-D5 en la que se tiene en cuenta la backreaction de las D5-branas, en el régimen de Veneziano, en el que $N_c \sim N_f \to \infty$, con $\frac{N_c}{N_f}$ fijo. Se considera el caso en el que el espacio interno es un espacio general 5-dimensional Sasaki-Einstein, siendo la correspondiente teoría de campos un quiver. La solución preserva cierta cantidad de supersimetría, para lo que se requiere que las D5-branas estén enrolladas en una subvariedad 3-dimensional del cono de Calabi-Yau construido sobre el espacio de Sasaki-Einstein. Para construir una solución que incorpore la backreaction, han de incluirse las D5-branas como fuentes en las ecuaciones del movimiento de supergravedad. Estas inducen una violación de las identidades de Bianchi para aquellos flujos de RR para los que el potencial correspondiente se acopla con la brana, en este caso, la 3-forma F_3. Esta violación de la identidad de Bianchi para F_3 es una 4-forma que codifica la distribución de la carga de las branas de sabor. Esta información nos permite escribir un ansatz para F_3 que preserva alguna supersimetría. La presencia de supersimetría nos permite escribir un sistema de ecuaciones diferenciales BPS de primer orden para la geometría. Este sistema se puede reducir a una única ecuación maestra de segundo orden para una nueva función con la que se puede especificar toda la geometría. Esta ecuación puede integrarse en general para el caso en el que N_f es cero. Para el caso con sabores no masivos se puede obtener una solución particular que proporciona una métrica con invariancia de escala anisotrópica. También se considera el ansatz para el caso de sabores masivos, para el que también se puede encontrar una ecuación maestra y se construyen soluciones numéricas que interpolan entre la geometría con $N_f = 0$ en el IR y la geometría con sabores no masivos en el UV.

En el capítulo 3 se trataron de generalizar los resultados del capítulo anterior al caso en el que la teoría cuántica de campos dual se halla a temperatura no nula. En el dual gravitatorio, esto es equivalente a encontrar una geometría de agujero negro, es decir, con un horizonte. De nuevo se trabajó en el límite de Veneziano, y se consideraron quarks no masivos. La solución encontrada para el agujero negro es analítica y sencilla. Dicha solución, al igual que ocurría en el capítulo anterior, es anisotrópica, y el dual holográfico puede entenderse como un sistema de multicapas, en el que cada capa está creada por las D5-branas distribuidas en planos paralelos 2-dimensionales dentro del espacio 3-dimensional. Para esta geometría se estudiaron tanto observables físicos que viven en una única capa como observables que conectan diferentes capas, y se encontró, no trivialmente, que la dinámica para los observables de la misma capa es el mismo que la de una pila de D2-branas efectivas, lo que significa que para describir nuestro sistema se puede utilizar la teoría (2+1)-dimensional Super Yang-Mills. En este capítulo se estudia la termodinámica del sistema, y se calcula el VEV del tensor de energía-momento de la teoría dual por diferentes métodos. A continuación se estudia la hidrodinámica del sistema. Para ello, se obtiene una acción reducida efectiva tanto en 4 como en 5 dimensiones, y se lleva a cabo la fluctuación de los campos de la acción de supergravedad reducida, resolviendo a continuación las ecuaciones resultantes para encontrar los coeficientes de transporte, en los denominados canales de cizallamiento y de sonido. Las acciones efectivas encontradas en 4 y en 5 dimensiones nos permiten también calcular el VEV del tensor de energía-momento y confirmar los resultados obtenidos en el análisis termodinámico.

En el capítulo 4 se generalizan los resultados del capítulo 2 para la construcción de geometrías de D5-branas con D3-branas en el caso en el que los sabores introducidos son masivos, teniendo en cuenta la backreaction. En la configuración del sistema D3-D5 estudiada en los capítulos 2 y 3, ambos tipos de branas pueden separarse en sus tres direcciones transversas. Cuando esta separación es nula, la masa de los hipermultipletes que viven en el defecto es nula, es decir, se tienen sabores no masivos. El caso no masivo se estudió en el capítulo 2 y su generalización a temperatura no nula en el capítulo 3. El caso masivo viene caracterizado por la función de perfil, cuyo argumento es la coordenada holográfica, y que describe la distribución de carga de D5-branas en el espacio. Existe una distancia mínima a partir de la cual el valor de esta función es nula, lo que puede interpretarse geométricamente con la aparición de una cavidad dentro de la cual no hay D5-branas. Esta función de perfil puede determinarse si se conoce la función que describe como se encajan las D5-branas en la geometría originada por el sistema de D3-branas. La herramienta que nos permite conocer la función de perfil es la simetría kappa. En este capítulo se lleva a cabo el análisis de la simetría kappa, tanto para determinar la forma en la que las D5-branas generan la geometría considerada en el capítulo, como para determinar otras posibles formas de encajar las D5-branas que den lugar a configuraciones geométricas interesantes con una interpretación clara en la teoría de campos dual. Para determinar la geometría, se escribe de nuevo una ecuación para una función maestra, a partir de la cual se puede determinar toda la geometría. Dicha ecuación depende explícitamente de la función de perfil. Esta ecuación se integra de manera numérica, y una vez conocida la geometría, se estudian los observables físicos de la misma. Dicha geometría depende en su forma final de un parámetro que caracteriza la deformación de la métrica en el IR. Se lleva a cabo el estudio de los lazos de Wilson y los potenciales para los pares quark-antiquark, cuando los fundamentales están en la misma capa o separados en la dirección ortogonal a las capas. El mismo análisis se lleva a cabo con la entropía de entrelazamiento cuántico. A continuación se explora la solución de supergravedad construida a través de la adición de D5-branas de prueba duales a un potencial químico. Se analiza la termodinámica a temperatura cero de la brana de prueba y el flujo del UV al IR de la velocidad del sonido. Para finalizar, se estudian los comportamientos límite de esta geometría. Cuando la masa de los quarks es muy grande, los sabores se desacoplan y se obtiene la solución sin sabor $AdS_5 \times S^5$. Cuando los sabores son no masivos, se obtiene la solución con invariancia de escala anisotrópica del capítulo 2. Para un valor finito no nulo de la masa de los quarks, se obtiene una solución que interpola entre estas soluciones: sabor nulo en el IR y sabor no nulo y no masivo en el UV. Los diferentes observables analizados también muestran un comportamiento cuyos valores límite se corresponden con estos regímenes.

En el capítulo 5 se estudia el espectro de fluctuaciones de las geometrías de Gaiotto-Maldacena. Dichas geometrías están descritas de manera genérica por una función que resuelve la ecuación de Laplace axisimétrica en 3 dimensiones. Cada solución a dicha ecuación determina una geometría. El dual en teoría de campos de estas soluciones es conocido, y puede obtenerse a partir del límite de baja energía del sistema de branas D4-NS5-D6. Dicho dual es un quiver, caracterizado por el número de nodos de color y de sabor. Cada quiver viene determinado por una solución concreta de la ecuación de Laplace en el dual gravitatorio. La obtención del espectro de fluctuaciones en la teoría de gravedad permite determinar el espectro de dimensiones conformes de los operadores de la teoría dual. No obstante, la determinación completa del espectro es a menudo difícil, dado que las ecuaciones linealizadas de la teoría de gravedad que se obtienen acoplan a gran número de campos de manera no trivial. Suele ser más sencillo resolver parcialmente el sistema de ecuaciones, es decir, encontrar truncaciones del mismo, lo que equivale a dar valor no nulo sólo a un pequeño subconjunto de las fluctuaciones. Esta es la perspectiva que se adoptó en este capítulo. En él, se obtienen en primer lugar el conjunto de ecuaciones linealizadas para una solución genérica de supergravedad tipo IIA conforme a un espacio producto del tipo $AdS_5 \times \mathcal{M}_5$. Las geometrías de Gaiotto-Maldacena entran dentro de este tipo de espacios. A continuación, se encienden exclusivamente las fluctuaciones de la métrica en las direcciones de Minkowski. Si se requiere además que dichas fluctuaciones sean transversas y sin traza, las ecuaciones de Maxwell para los flujos de RR y NSNS, así como la ecuación para el dilatón, se satisfacen trivialmente, dejando como única ecuación no trivial la de Einstein. Dicha ecuación se simplifica enormemente para las geometrías de Gaiotto-Maldacena, y se puede escribir como una ecuación de autovalores para un operador $\mathcal{L}$, cuya forma está determinada por la función V que satisface la ecuación de

Laplace axisimétrica. Se particularizó el estudio de esta ecuación para los casos en los que V se corresponde con las geometrías resultado de hacer una transformación de T-dualidad Abeliana y no-Abeliana sobre la solución $AdS_5 \times S^5$ actuando en la esfera. Para dichas soluciones fue posible la resolución de la ecuación diferencial, de manera analítica para los modos inertes bajo la transformación de T-dualidad (o su versión no-Abeliana) y de manera numérica y aproximada a través del método WKB para los modos no inertes bajo la transformación. Para el caso analítico se encontró un resultado ya conocido en la literatura. Para el caso no analítico, se obtuvo de manera numérica el espectro de masas, mostrando gran acuerdo con los resultados obtenidos por la aproximación WKB en el rango en el que esta aproximación es válida, es decir, para valores altos de las masas.

En el capítulo 6 se trató de añadir defectos a la geometría de Brandhuber-Oz, a través de branas de prueba D4 y D6 en diferentes configuraciones, y se analizaron las fluctuaciones de dichas configuraciones y su interpretación en la teoría de campos dual. La solución de Brandhuber-Oz es el dual gravitacional de la versión de mayor rango de un tipo de teorías E_{N_f+1}. Esta solución se corresponde con el límite de baja energía de una configuración de D4-branas cerca de un plano orientifoldio $O8$ con $N_f < 8$ D8-branas. La geometría completa en 10 dimensiones es una solución a las ecuaciones de supergravedad tipo IIA masiva, y tiene la forma de un espacio fibrado del tipo $AdS_6 \times S^4$, con masa F_0 no nula. La adición de defectos a esta geometría origina una teoría cuántica de campos de dimensión menor que está acoplada con la teoría cuántica de campos del ambiente, la generada por la configuración D4-O8-D8. En este capítulo se añaden diferentes defectos preservando supersimetría. En primer lugar, se estudian los defectos de codimension 2, originados a través de D4-branas de prueba. Se analiza su supersimetría y sus fluctuaciones, y se propone la correspondencia entre operadores y fluctuaciones. A continuación se estudia otra configuración diferente de D4-branas, en la que estas están enrolladas en el espacio interno, lo que se corresponde con la generalización de la configuración que captura el lazo de Wilson antisimétrico. Además de la discusión de la supersimetría, se analizan también las fluctuaciones de esta configuración. Por último, se estudian los defectos de codimension 1, construidos a través de D6-branas de prueba. Se describe su supersimetría y se calculan las fluctuaciones, incluyendo el mapa entre operadores y fluctuaciones.

En el capítulo 7, se estudia la integrabilidad de las teorías del tipo $\mathcal{N} = (1, 0)$ SCFT's en 6 dimensiones, a través del estudio de su dual holográfico. Dichas teorías cuánticas de campos se realizan a través del límite de baja energía del sistema de branas D6-NS5-D8. Tales configuraciones admiten un dual gravitatorio en supergravedad tipo IIA masiva, cuya construcción fue llevada a cabo a través de varios artículos que culminaron en una formulación debida a Cremonesi y a Tomasiello, que se tomó como punto de partida en nuestro estudio. Señalar la integrabilidad de una teoría de campos o la ausencia de la misma es un problema interesante, al que se han dedicado muchos esfuerzos tras el descubrimiento de estructuras integrables dentro de la correspondencia AdS/CFT. Si bien encontrar integrabilidad es una labor a menudo difícil, señalar la no-integrabilidad es más sencillo. En este capítulo se prueba la no-integrabilidad de las teorías mencionadas. El punto de partida son las geometrías de Cremonesi-Tomasiello. Estas vienen descritas a través de una función de la coordenada holográfica que satisface una ecuación diferencial de tercer orden. La forma de esta función determina completamente la teoría cuántica de campos dual, siendo esta un quiver cuyos nodos de color y sabor quedan determinados a través de dicha función. En este capítulo, se revisa la formulación de Cremonesi y Tomasiello y se proporcionan ejemplos explícitos de dichas geometrías y su quiver dual. A continuación se estudia la integrabilidad de estas teorías. Para ello, se estudia la integrabilidad del movimiento de una cuerda propagándose en esta geometría. Se escribe el hamiltoniano de la cuerda y se prueba, a través de métodos tanto analíticos como numéricos, que el movimiento es no-integrable. El método analítico se basa en la construcción de la Ecuación Variacional Normal (NVE), y la aplicación del criterio de Kovacic a dicha ecuación, que prueba su no-integrabilidad. Como métodos numéricos que apoyan dicho resultado, se calcula el espectro de potencia, que sirve para caracterizar una dinámica caótica, el espectro de exponentes de Lyapunov, que caracteriza la tasa de separación de dos trayectorias que comienzan arbitrariamente juntas en el espacio de fases, y se calculan las secciones de Poincaré, que describen la forma del espacio de fases. Para sistemas integrables, dicho espacio de fases se halla foliado por los denominados toros de KAM, pero esta foliación desaparece cuando estamos ante una perturbación no integrable. Estos métodos numéricos confirman los resultados analíticos, probando la no-integrabilidad en general de la teoría cuántica de campos dual.

Los resultados de esta tesis tienen interés tanto teórico como fenomenológico. Por un lado, la construcción de nuevas geometrías para la intersección D3-D5, correspondiente a los capítulos 2, 3 y 4, es de relevancia fenomenológica, ya que dicho sistema ha sido utilizado, entre otras cosas, para el modelado del efecto Hall cuántico, como modelo holográfico del grafeno y aparece en el contexto de las geometrías “bubbling”. Por otro lado, los resultados del capítulo 5 tienen interés teórico, pues suponen un avance en la comprensión del efecto de las transformaciones de T-dualidad y de su generalización no-Abeliana. En este caso se estudió el cambio que dichas transformaciones inducen en el espectro de masa de las fluctuaciones de las geometrías relacionadas por la dualidad. También de interés tanto teórico como fenomenológico son los resultados del capítulo 6, que permiten proporcionar un dual a una teoría cuántica de campos con defectos y su realización a través de branas supersimétricas en el dual de gravedad. Por último, el capítulo 7 supone un avance teórico interesante, dado que permite descartar de manera genérica la integrabilidad en una extensa clase de teorías cuánticas de campos.

Esta tesis abre nuevas líneas de investigación. En particular, un proyecto de interés sería encontrar el sistema en materia condensada para el cual las geometrías construidas en los capítulos 2, 3 y 4 constituyen duales holográficos. Estas soluciones anisotrópicas también resultarían útiles para estudiar la materia fuertemente acoplada de los núcleos de las estrellas de neutrones. De igual forma, las ecuaciones genéricas obtenidas en el capítulo 5 para las fluctuaciones de una solución en supergravedad tipo IIA, pueden usarse para estudiar el espectro de fluctuaciones de otras geometrías. Otra línea de investigación conecta con los resultados del capítulo 6, y se centra en la adición de defectos a otras geometrías interesantes, el estudio de su supersimetría y de sus fluctuaciones. De igual modo, los métodos aplicados en el capítulo 7 se pueden usar para caracterizar la integrabilidad en otras teorías de campos de interés.

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