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**REGULARITY
IN LOGARITHMIC
ALGEBRAIC GEOMETRY:
A DIFFERENT VIEWPOINT**

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Regularity in Logarithmic Algebraic Geometry: a different viewpoint

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Regularity in Logarithmic Algebraic Geometry: a different viewpoint

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INFORMA:

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Introduction

We study regularity in logarithmic algebraic geometry from a different viewpoint, which allows us to obtain new results and also to remove unnecessary hypotheses from some known results.

Logarithmic algebraic geometry was introduced independently by P. Deligne [5], G. Faltings [7], and J. M. Fontaine and L. Illusie (developed by K. Kato in [13]). The definition of Fontaine-Illusie is the one usually used. A prelogarithmic scheme is a scheme X with an extra structure: a morphism of sheaves of commutative monoids $\alpha: \mathcal{M} \rightarrow (\mathcal{O}_X, \cdot)$ for some sheaf of monoids \mathcal{M} . If in addition α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$, where \mathcal{O}_X^* is the sheaf of units, we say that it is a logarithmic scheme. Any prelogarithmic scheme induces a logarithmic scheme (replacing \mathcal{M} by the pushout of $\alpha^{-1}(\mathcal{O}_X^*) \hookrightarrow \mathcal{M}$ and $\alpha^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^*$). For instance, if $i: U \hookrightarrow X$ is an open inclusion, $\mathcal{M} = \varphi^{-1}(i_*\mathcal{O}_U^*)$ where $\varphi: \mathcal{O}_X \rightarrow i_*(\mathcal{O}_U)$ is the canonical map, then the inclusion $\mathcal{M} \rightarrow (\mathcal{O}_X, \cdot)$ gives a prelogarithmic structure on X and thus there is an associated logarithmic structure.

Probably the most important concept in this field is log smoothness. It is defined for a morphism of logarithmic schemes in a completely similar way as smoothness is defined for a morphism of schemes, and has good functorial properties as usual smoothness. Log smoothness morphisms work well in logarithmic algebraic geometry as smooth morphisms work well in algebraic geometry. The important fact is that certain morphisms of schemes which are not smooth can be considered as morphisms of logarithmic schemes for some logarithmic structure making them log smooth. For example, if S is the spectrum of a discrete valuation ring, X is a regular scheme and $f: X \rightarrow S$ is a flat morphism with semi-stable reduction (i.e. the fiber over the closed point of S is a divisor D with normal crossings in X), then the logarithmic structures associated as above to the open immersions $i: U := X - D \hookrightarrow X$ and $j: \{\xi\} \rightarrow S$, with ξ the generic point of S , makes f log smooth. So, in some sense, logarithmic geometry is the right context to work with semi-stability, since log smoothness has nice functorial properties (for instance, semi-stability is not preserved under base change, unlike log smoothness).

A related concept, the one of log regularity, was introduced by K. Kato a few years later [14]. The first half of his paper is devoted to generalize results on (classical) regularity to the logarithmic setting. However, the results obtained by Kato usually need some undesirable hypotheses (sometimes finiteness hypotheses, other times log regularity instead of log complete intersection since the concept of log complete intersection or log regular ideal is not defined, etc.). Our first goal is to avoid these unwanted hypotheses. We even avoid the (logarithmic) integrity hypothesis in many of our results. Our second goal is to complement the study of log regularity by Kato with new results. For instance, we obtain the following results (since all the questions are local, we will use the language of logarithmic rings):

Theorem 6.15. *Let $((B, \mathfrak{n}, l), N)$ be a Noetherian local prelog ring with N integral and saturated, and let $k \subset B$ be a field.*

- (i) *If (B, N) is a log formally smooth over $(k, 1)$ for the \mathfrak{n} -adic topology, then (B, N) is log regular.*
- (ii) *Assume that $l|k$ is separable. If (B, N) is log regular, then (B, N) is a log formally smooth over $(k, 1)$ for the \mathfrak{n} -adic topology.*

This result was obtained by Kato when B is finitely generated over k .

Theorem 6.16. *Let (B, N) be a Noetherian local prelog ring with N integral and saturated containing a perfect field k . If (B, N) is log formally smooth over $(k, 1)$ for the topology of the maximal ideal, then $\Omega_{(B, N)|(k, 1)}$ is a flat B -module.*

Here $\Omega_{(B, N)|(k, 1)}$ is the logarithmic module of differentials, which is a generalization of the concept of module of differentials with logarithmic poles along a divisor with normal crossings.

Theorem 6.12. *Let (A, M) be a Noetherian local prelog ring, J a proper ideal of A , $B = A/J$ and $N = M/\alpha_M^{-1}(J)$, where $\alpha_M : M \rightarrow A$ is the structural map.*

- (i) *If (A, M) is log regular and (B, N) is log complete intersection, then J is a log regular ideal.*
- (ii) *If (B, N) is log regular and J is a log regular ideal, then (A, M) is a log regular local ring.*
- (iii) *Assume that M is integral and M^{gp} free (for instance, if M is saturated and sharp). If (A, M) is log regular (or log complete intersection) and J is log regular, then (B, N) is log complete intersection.*

A variant of (i), assuming moreover that (B, N) is log regular and M integral, was obtained by Kato: if (A, M) and (B, N) are log regular with $N = M$ integral, then J is a regular ideal.

In the non logarithmic setting, an important tool to deal with these questions is the cotangent complex introduced by M. André [1] and D. Quillen [20]. The utility of this (co)homology theory comes from two facts: on the one hand, it has a very good behaviour (base change, localization, very important exact sequences, etc.); on the other hand, its vanishing characterizes smoothness, regularity and complete intersection:

Theorem. ([2], [20])

- (i) *Let B be a Noetherian A -algebra and J an ideal of B . Then B is formally smooth over A for the J -adic topology if and only if $H^1(A, B, W) = 0$ for all B/J -modules W .*
- (ii) *Let (A, k) be a Noetherian local ring. Then*
 - (a) *(A, k) is regular if and only if $H_2(A, k, k) = 0$.*
 - (b) *(A, k) is complete intersection if and only if $H_3(A, k, k) = 0$.*

In the logarithmic setting, we also have a “logarithmic cotangent complex” introduced by O. Gabber (details can be seen in [19], where M. Olsson also presents and studies a variant of this complex). This logarithmic cotangent complex will be our main tool to deal with our questions.

Now we will describe the contents of this dissertation. In chapters 1 and 2 we give a quick recall of the definitions and notation of monoids and logarithmic structures used in the following chapters. In Chapter 3 we study log differentials and log derivations. There is nothing essentially new in this part, but we have not found references in the literature for some of the results, so we give a somewhat complete exposition.

In Chapter 4, Section 4.1 is devoted to develop the logarithmic cotangent complex of Gabber. The treatment done in [19] is not enough for our purposes since sometimes it lacks some results that we need, and sometimes it includes the results (base change, localization) but with stronger hypotheses than the ones we can allow. This section contains also one of the two main technical results, that we call the fundamental triangle (Theorem 4.3): if $(A, M) \rightarrow (B, N)$ is a homomorphism of prelog rings and W is a B -module, we have a distinguished triangle in the derived category of \mathbb{Z} -modules

$$\mathbb{L}_{\mathbb{Z}[N]|\mathbb{Z}[M] \otimes_{\mathbb{Z}[N]} W} \longrightarrow (\mathbb{L}_{B|A} \otimes_B W) \oplus (X_{N|M} \otimes_{\mathbb{Z}} W) \longrightarrow \mathbb{L}_{(B,N)|(A,M)} \otimes_B W \longrightarrow$$

where the two left \mathbb{L} are the “usual” cotangent complex and the right \mathbb{L} is the logarithmic cotangent complex, and where $X_{N|M}$ is a complex appearing in a distinguished triangle

$$M^{\text{gp}} \longrightarrow N^{\text{gp}} \longrightarrow X_{N|M} \longrightarrow \quad .$$

This result is fundamental and will be used throughout, since it allows us to relate properties of the logarithmic homomorphism $(A, M) \rightarrow (B, N)$ to properties of the homomorphisms of rings $A \rightarrow B$ and $\mathbb{Z}[M] \rightarrow \mathbb{Z}[N]$.

We include also a proof of the fact that logification does not change the (co)homology of the logarithmic cotangent complex when the structures are integral. This result appears in [19]. We do not know if the hypothesis of integrality are necessary. At some points of Chapter 6, we were not able to go further because of this integrality hypothesis (even for integral structures, since non integral quotients appear in the proofs). That is the reason why we include an alternative proof of this theorem by Olsson: we have also failed to avoid the integrality hypothesis, but we think our proof gives a little more hope to avoid them.

Section 4.2 is not necessary for the sequel, but it is of independent interest. This section is devoted to give a logarithmic analogue of Quillen’s fundamental spectral sequence. If $C \rightarrow B$ is a surjective homomorphism of rings, Quillen gives a convergent spectral sequence [20, 22]

$$E_{p,q}^2 = H_{p+q}(S_q \mathbb{L}_{B|A}) \Rightarrow \text{Tor}_{p+q}^C(B, B)$$

where $S_q \mathbb{L}_{B|A}$ is the q -th symmetric power of the “usual” André-Quillen cotangent complex. If $C = B \otimes_A B$, where B is a flat A -algebra, this spectral sequence takes the form

$$E_{p,q}^2 = H_p(\Lambda^q \mathbb{L}_{B|A}) \Rightarrow \text{HH}_{p+q}(B|A)$$

where Λ^q denotes the q -th exterior power and HH the Hochschild homology. We give a similar spectral sequence in the logarithmic context (Theorem 4.21). Although the non logarithmic spectral sequence is fundamental in the works of Quillen [20, 22], our unique interest in obtaining the logarithmic version is that its abutment (which we define explicitly) gives another candidate for a logarithmic Hochschild (co)homology.

Chapter 4 ends with Section 4.3, where we adapt the methods of [15] and [2, Proposition 15.12] in order to give a smaller complex useful to compute the second (co)homology groups of the logarithmic cotangent complex.

Chapter 5 is devoted to the (co)homological study of log smoothness. We do not want to impose finiteness hypotheses to the morphisms and we also want to avoid integrality hypotheses as much as possible, so the first steps made in [19] and [12] do not help us, and we need to start from scratch. We proceed in the usual way. First we characterize elements of H^1 as extensions of log algebras. In the integral

case this was done in [19], but in general the very definition of extension is a little more involved, and our proof is different and substantially longer than in the integral case. Then we deduce some results characterizing logarithmic smoothness by the vanishing of the logarithmic cotangent complex. This process is standard for logarithmic smoothness, but we need to take into account topologies, that is, to characterize logarithmic formal smoothness (since it is needed when the homomorphisms are not of finite type), making again the proofs longer. For instance (and assuming now the discrete topology in order to simplify; see Theorem 5.14 for the topological case), we obtain:

Theorem. *Let $f: (A, M) \rightarrow (B, N)$ be a homomorphism of (pre)log rings. Then f is log smooth if and only if $H^1((A, M), (B, N), W) = 0$ for all B -modules W .*

We end this chapter with some minor applications giving in particular (Proposition 5.19 and Corollary 5.20) a generalization of a part of [11, Theorem 0.1].

Finally, in Chapter 6, we are ready to study log regularity. We start with the characterization of log regularity in terms of the logarithmic cotangent complex, which is our second main technical result (Theorem 6.3 to Theorem 6.9):

Theorem. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local (pre)log ring and \mathfrak{m}_M the set of non-units of M .*

- (i) *(A, M) is a log regular local ring if and only if the cokernel of the canonical injective homomorphism*

$$\mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{gp} \rightarrow (M/\mathfrak{m}_M)^{gp}), k) \longrightarrow H_2((A, M), (k, M/\mathfrak{m}_M), k)$$

vanishes.

- (ii) *There exists a prelogarithmic structure (A, P) on A inducing the same logarithmic structure as (A, M) such that (A, P) is log regular if and only if $H_2((A, P), (k, P/\mathfrak{m}_P), k) = 0$.*
- (iii) *If M is integral, the monoid P in (ii) can be chosen integral, and therefore (A, P) is log regular if and only if (A, M) is log regular.*
- (iv) *If M is integral, (A, M) log regular implies $H_n((A, M), (k, M/\mathfrak{m}_M), k) = 0$ for any $n \geq 3$.*

In fact, we deduce part (i) of this theorem of a more general result, valid not only for the maximal ideal \mathfrak{m} , but for any ideal of A , which allows to introduce the concept of log regular ideal. This concept plays a role similar to regularity of ideals in the non logarithmic case. For instance, we show that if (A, M) is a log regular local ring

and J a proper ideal of A , then J is a log regular ideal if $(A/J, M/\alpha^{-1}(J))$ is log complete intersection (where $\alpha: M \rightarrow A$ is the canonical map), and the converse is true under some mild hypotheses.

With these results at hand, we are now ready to study log regularity. For instance, the fundamental triangle gives us immediately a relation between log regularity and regularity of a logarithmic ring:

Theorem. *Let (A, M) be a log regular ring. Then A is regular if and only if $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[M/\mathfrak{m}_M])$ is generated by a regular sequence.*

We end this chapter with theorems 6.15 and 6.16 mentioned at the beginning of this introduction.



Chapter 1

Monoids

We will review the definitions and basic properties of monoids that we are going to use. For more details, we can consult [18, Chapter I].

Definition 1.1. A monoid is a set M with an associative and commutative operation that has identity element. All monoids we consider will be commutative, and we will usually use multiplicative notation. Thus the identity element will be denoted by 1 . For example, if A is a commutative ring then (A, \cdot) is a monoid, or $(\mathbb{N}, +)$ is an additive monoid where 0 is the identity element.

A zero element in a monoid M is an element $0 \in M$ such that $0 \cdot x = 0$ for any $x \in M$. If M is a monoid, then the set $M \cup \{\star\}$ with an operation extending the one of M , via $\star \cdot \star = \star$ and $x \cdot \star = \star$ for all $x \in M$, is a monoid whose zero element is \star . If M already had a zero element, this one is no longer a zero element in $M \cup \{\star\}$ and the new one would be \star .

Let M be a monoid. A submonoid M' of M is a subset that contains the identity element of M and that is closed for the multiplication. The intersection of submonoids of M is also a submonoid of M . Consider now a subset X of M , we define the submonoid $\langle X \rangle$ generated by X as the smallest submonoid (in the inclusion sense) of M that contains X . The elements of $\langle X \rangle$ are the (finite) products of elements of X , including the product of an empty set of elements of X (i.e. 1). If there exists a finite subset $X \subset M$ such that $\langle X \rangle = M$, we will say that M is a finitely generated monoid.

Definition 1.2. A homomorphism of monoids $\varphi: M' \rightarrow M$ is a map such that $\varphi(1_{M'}) = 1_M$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in M'$. We denote by $\text{Hom}(M', M)$ the set of homomorphism of monoids from M' to M . It has monoid structure, since $(\varphi_1 \varphi_2)(x) = \varphi_1(x) \cdot \varphi_2(x)$ and the identity element $\underline{1}$ is defined by $\underline{1}(x) := 1_M$ for any $x \in M'$. Note that it is necessary that M would be commutative so that $\varphi_1 \varphi_2 \in \text{Hom}(M', M)$. We will use **Mon** to denote the category of (commutative) monoids.

Let $\varphi: M' \rightarrow M$ be a homomorphism of monoids. We define the submonoids

$$\text{KER } \varphi := \{(x, y) \in M' \times M' \mid \varphi(x) = \varphi(y)\},$$

$$\ker \varphi := \{x \in M' \mid (x, 1) \in \text{KER } \varphi\} = \varphi^{-1}(1).$$

The trivial monoid $M = \{1\}$ is clearly a zero object in **Mon**. Moreover, this category has kernels, since $\ker \varphi$ will be the kernel of φ .

Definition 1.3. A congruence R on a monoid M is an equivalence relation on M compatible with the multiplication of M , that is,

$$(x, x'), (y, y') \in R \Rightarrow (xy, x'y') \in R.$$

The quotient set M/R is a monoid with the multiplication induced by the one on M through the canonical map $p: M \rightarrow M/R$, with $\ker p = R$.

If $\varphi: M' \rightarrow M$ is a homomorphism of monoids, $\ker \varphi$ is a congruence in M' . Therefore we have an injective homomorphism of monoids $\varphi': M'/\ker \varphi \rightarrow M$ that composed with the above canonical homomorphism $M' \rightarrow M/\ker \varphi$ gives us φ .

Example 1.4. Let M be a monoid and H a subgroup of M (i.e. a submonoid that is a group). The set $R = \{(x, y) \in M \times M \mid \exists h \in H \text{ such that } x = yh\}$ is a congruence in M . If M is also a (commutative) group, then the quotient monoid of M by this congruence is just the usual (group) quotient of groups M/H .

If M is again a monoid but H is only a monoid now, the set of invertible elements H^* of H is a subgroup and we can define M/H' . When $H = M$, the unique invertible element of M/M^* is the class of the identity element of M .

Example 1.5. An ideal I of a monoid M is a non empty set of M that verifies $xa \in I$ for all $x \in M, a \in I$. If M has zero element, it belongs to any ideal of M . While if $1 \in I$, then $I = M$.

The intersection of ideals is an ideal as long as it would not be empty, and the union of ideals is always an ideal. If Y is a non empty subset of a monoid M , we call ideal generated by Y to the smallest ideal of M that contains Y and that is $MY := \{xy \mid x \in M, y \in Y\}$.

An ideal I of M is proper if $I \neq M$ (equivalently if $1 \notin I$). An ideal I is prime if it is proper and furthermore $xy \in I$ implies that $x \in I$ or $y \in I$ for every $x, y \in M$. It means that an ideal I is prime if and only if $M - I$ is a submonoid of M .

We define the maximal ideal of a monoid M as $\mathfrak{m}_M := M - M^*$. We will say that a homomorphism of monoids $f: M \rightarrow M'$ is local if $f(\mathfrak{m}_M) \subset \mathfrak{m}_{M'}$.

Example 1.6. Let A be a commutative ring, I an ideal of A , M a monoid and $\alpha: M \rightarrow (A, \cdot)$ a homomorphism of monoids. The set $\alpha^{-1}(I)$ is an ideal of M . Now, if I is an proper ideal of A , $\alpha^{-1}(I)$ is a proper ideal of M . And similarly, in the case I is a prime ideal of A , then $\alpha^{-1}(I)$ is also prime.

Example 1.7. The Rees congruence of the ideal I is the congruence in M

$$R = (I \times I) \cup \Delta(M)$$

where $\Delta(M) = \{(x, x) \in M \times M\}$ is the diagonal. That is, $x \sim y$ if $x, y \in I$ or $x = y$. We will call Rees quotient of M by I and denote M/I to the quotient monoid.

Note that

$$M/I = (M - I) \cup \{\star\}$$

with product

$$xy = \begin{cases} xy & \text{if } x, y, xy \in M - I \quad (\text{i.e. } xy \in M - I), \\ \star & \text{if in any other case,} \end{cases}$$

where \star is the zero element of M/I .

Definition 1.8 (Products). Let $\{M_i\}_{i \in I}$ be a family of monoids. The cartesian product of monoids

$$M = \prod_{i \in I} M_i$$

has monoid structure (component to component) and it is easy to check that it is a product in **Mon**. Note that if every M_i is a group, then M is a group; and therefore it is also the product in the category of groups.

Definition 1.9 (Coproducts). Let $\{M_i\}_{i \in I}$ be a family of monoids. We define its coproduct as the monoid

$$\bigoplus_{i \in I} M_i := \{(x_i) \in \prod_{i \in I} M_i \mid x_i = 1 \text{ for almost all } i \in I\}$$

with the operation component to component. It is the coproduct in **Mon** and coincides with the product when I is finite.

Definition 1.10 (Equalizers). Let $f, g: M_1 \rightarrow M_2$ be homomorphisms of monoids. Then $M_0 = \{x \in M_1 \mid f(x) = g(x)\}$ is a submonoid of M_1 and the inclusion $M_0 \hookrightarrow M_1$ is the equalizer of f and g .

Definition 1.11 (Coequalizers). Let $f, g: M_1 \rightarrow M_2$ be homomorphisms of monoids and we define $M_3 = M_2/R$, where R is the congruence generated by the set $\{(f(x), g(x)) \mid x \in M_1\}$ (i.e. the intersection of all congruences that contain $\{(f(x), g(x)) \mid x \in M_1\}$). The canonical homomorphism $M_2 \rightarrow M_3$ is the coequalizer of f and g .

Corollary 1.12. *The category of monoids has limits and colimits. Moreover, since it has zero object (see Definition 1.2), it has also kernels and cokernels.*

Definition 1.13. *If X is a non empty set, we define free monoid over X as the monoid*

$$\mathbb{N}^X := \bigoplus_{x \in X} N_x$$

where $N_x := (\mathbb{N}, +)$ for every $x \in X$.

The free monoid functor from **Set** to **Mon** is left adjoint to the forgetful functor. From this, for example, we deduce that the forgetful functor preserves limits.

Therefore, in particular, if

$$\begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ \downarrow & & \downarrow p \\ M_2 & \xrightarrow{q} & M_3 \end{array}$$

is a cartesian square (pullback) of monoids, we know that it is also a cartesian square of the underlying sets, and so $M_0 = \{(x_1, x_2) \in M_1 \times M_2 \mid p(x_1) = q(x_2)\}$.

Now we will see with more detail the cocartesian squares (pushouts).

Proposition 1.14. *Let $h_1: P \rightarrow Q_1$, $h_2: P \rightarrow Q_2$ be two homomorphisms of monoids, we denote their pushout by $Q_1 \oplus_P Q_2$. Let E be the congruence on $Q_1 \oplus Q_2$ which define $Q_1 \oplus_P Q_2$ as a quotient of $Q_1 \oplus Q_2$.*

1. *If any of P , Q_1 or Q_2 is a group, then E is the set of pairs $((q_1, q_2), (q'_1, q'_2))$ of $Q_1 \oplus Q_2$ such that there exist $p, p' \in P$ with $q_1 + h_1(p') = q'_1 + h_1(p)$ and $q_2 + h_2(p) = q'_2 + h_2(p')$.*
2. *If P is a group, then $((q_1, q_2), (q'_1, q'_2)) \in E$ if and only if there exists $p \in P$ such that $q'_1 = q_1 + h_1(p)$ and $q'_2 = q_2 - h_2(p)$.*
3. *If P , Q_1 or Q_2 are groups, then $Q_1 \oplus_P Q_2$ is also a group, which is the pushout in the category of Abelian groups.*

Proof. See [18, Proposition I.1.1.5] and also [13, Remark in 1.3]. □

Definition 1.15. *Let M be a monoid and T a set. An action of M on T is a homomorphism of monoids*

$$\theta: M \longrightarrow \text{End}(T)$$

where $\text{End}(T)$ is the monoid not (necessarily) commutative of endomorphisms (of sets) of T (Definition 1.2). That is, θ verifies that $\theta(xy) = \theta(x) \cdot \theta(y)$ for any $x, y \in M$ and that $\theta(1) = \text{id}_T$. We will say that T is an M -set and denote $x \cdot t := \theta(x)(t)$ for $x \in M, t \in T$.

A homomorphism of M -sets is an application φ that makes commutative the diagram

$$\begin{array}{ccc} M \times T_1 & \longrightarrow & T_1 \\ \downarrow (\text{id}_M, \varphi) & & \downarrow \varphi \\ M \times T_2 & \longrightarrow & T_2 \end{array}$$

Example 1.16. If $f: M \rightarrow N$ is a homomorphism of monoids, then N is an M -set with

$$x \cdot y := f(x) \cdot y$$

for all $x \in M, y \in N$.

Remark 1.17. The forgetful functor from the category of M -sets to **Set** has a left adjoint functor: if B is a set, $M \times B$ is an M -set via $x(y, b) = (xy, b)$ (it is said that $M \times B$ is the free M -set with basis B). Therefore the category of M -sets has limits, and they commute with the forgetful functor.

Definition 1.18. Let M be a monoid and $S \subset M$ a submonoid. Consider the direct product $M \times S$ and the congruence

$$(x, s) \sim (y, t) :\Leftrightarrow \exists u \in S \mid syu = xtu.$$

The quotient monoid is denoted by $S^{-1}M$ and is called monoid of fractions. We will denote the class of (x, s) in $S^{-1}M$ by $\frac{x}{s}$, so that $\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st}$ and $1 = \frac{s}{s}$.

We have a canonical homomorphism $\gamma: M \rightarrow S^{-1}M$ defined as the composition

$$\begin{aligned} M &\longrightarrow M \oplus S = M \times S \longrightarrow (M \times S)/\sim = S^{-1}M, \\ x &\longmapsto (x, 1) \end{aligned}$$

that is, $x \mapsto x/1$.

Proposition 1.19. Let M be a monoid, S a submonoid and $\varphi: M \rightarrow N$ a homomorphism of monoids. Then there exists a homomorphism of monoids $h: S^{-1}M \rightarrow N$

making commutative the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\gamma} & S^{-1}M \\
 & \searrow \varphi & \downarrow h \\
 & & N
 \end{array}$$

if and only if $\varphi(s)$ is invertible in N for all $s \in S$. In this case, h is unique and it is define as follows: $h(x/s) := h(x)h(s)^{-1}$.

Proposition 1.20. *Let M be a monoid. Then $M^{-1}M$ is an Abelian group that we denote by M^{sp} .*

Moreover, it verifies the following universal property: Given a group G and a homomorphism of monoids $\varphi: M \rightarrow G$, there exists a unique homomorphism of groups $h: M^{sp} \rightarrow G$ making commutative the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\gamma} & M^{sp} \\
 & \searrow \varphi & \downarrow h \\
 & & G
 \end{array}$$

It is called group completion of M .

Proof. It is clear that $M^{-1}M$ is a group, since $(\frac{x}{y})^{-1} = \frac{y}{x}$, and the universal property follows from Proposition 1.19. \square

Remark 1.21. From Proposition 1.20, we can deduce that $(-)^{sp}$ is a functor from \mathbf{Mon} to \mathbf{Ab} that is left adjoint of the forgetful adjoint. In particular, it preserves coproducts.

Definition 1.22. *Let M be a monoid. An element $t \in M$ is integral if for any $x, y \in M$ such that $tx = ty$, then $x = y$. We say that a monoid M is integral if every element of M is integral.*

Definition 1.23. *Let M be a monoid. If $M^* = \{1\}$, M is said to be sharp.*

Proposition 1.24. *Let M be a monoid and S a submonoid. The canonical homomorphism $\gamma: M \rightarrow S^{-1}M$ is injective if and only if the elements of S are integral in M .*

Therefore, in the case S is integral, we will have $x/s = y/t \Leftrightarrow xt = ys$ in $S^{-1}M$. And moreover, if M is integral, we can think M as a subset of M^{gp} .

Remark 1.25. Let M be a monoid with zero element 0 and S a submonoid with $0 \in S$. Then $S^{-1}M = \{1\}$. If $\varphi: M \rightarrow N$ is a homomorphism of monoids such that $\varphi(s)$ is invertible for all $s \in S$, then $\varphi(0)$ is invertible and so $\varphi(x) = 1$ for all $x \in M$ (since $\varphi(0) = \varphi(0 \cdot x) = \varphi(0)\varphi(x)$). This means that φ is the trivial morphism. For example, under this hypothesis M^{gp} is the (Abelian) trivial group. For example, $(\mathbb{N}, \cdot)^{\text{gp}}$ is the trivial group. However, if $S = \mathbb{N} - \{0\}$, we obtain $S^{-1}(\mathbb{N}, \cdot) = (\mathbb{Q}^{\geq 0}, \cdot)$.

Definition 1.26. If M is a monoid and R is a ring, $R[M]$ will denote the usual monoid R -algebra. In particular, $R[M]$ is a free R -module with basis M .

Remark 1.27. The ring monoid functor $M \mapsto \mathbb{Z}[M]$ from the category of monoids to the category of commutative rings is left adjoint to the functor $A \mapsto (A, \cdot)$ and so it preserves colimits. In particular, if

$$\begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M \end{array}$$

is a cocartesian square of monoids, then

$$\mathbb{Z}[M] = \mathbb{Z}[M_1] \otimes_{\mathbb{Z}[M_0]} \mathbb{Z}[M_2].$$

In other words, we can say that

$$\mathbb{Z}[M_1 \oplus_{M_0} M_2] = \mathbb{Z}[M_1] \otimes_{\mathbb{Z}[M_0]} \mathbb{Z}[M_2].$$

Proposition 1.28. Let M be a monoid. The ring $\mathbb{Z}[M]$ is Noetherian if and only if M is finitely generated.

Proof. See [8, Theorem 7.7]. □

Definition 1.29. Let M be a monoid. M is fine if it is finitely generated and integral. M is said saturated if it is integral and verifies that $x \in M^{\text{gp}}$ and $x^n \in M$ for some $n \geq 1$ implies $x \in M$.

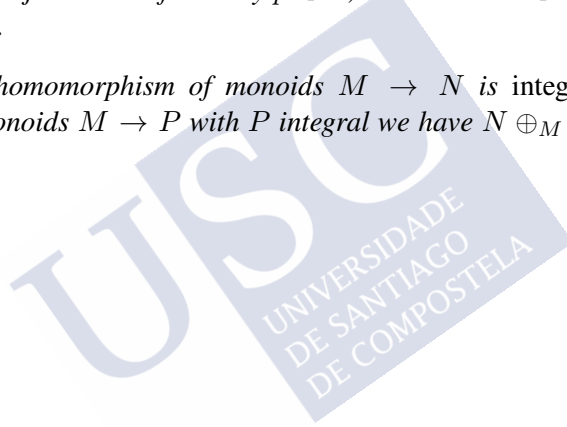
Definition 1.30. A homomorphism of monoids $h: M \rightarrow N$ is exact if the square

$$\begin{array}{ccc} M & \longrightarrow & M^{gp} \\ \downarrow h & & \downarrow h^{gp} \\ N & \longrightarrow & N^{gp} \end{array}$$

is cartesian (i.e. a pullback).

Definition 1.31. Let $f: M \rightarrow P$ be a homomorphism of integral monoids. f is said to be Kummer if it is injective and for every $p \in P$, there exists $n \in \mathbb{N}$ and $m \in M$ such that $f(m) = p^n$.

Definition 1.32. A homomorphism of monoids $M \rightarrow N$ is integral if for any homomorphism of monoids $M \rightarrow P$ with P integral we have $N \oplus_M P$ integral [9, Definition 6.2.1].



Chapter 2

Logarithmic rings

Definition 2.1. A prelog ring (A, M, α) (simply (A, M) or A if there is no confusion) consists of a commutative ring A , a commutative monoid M and a multiplicative homomorphism of monoids $\alpha: M \rightarrow A$. In general, we will denote this structural map by α for any prelog ring (if it is necessary, to avoid confusion, we will use α_A or α_M). A homomorphism of log rings

$$f = (f^\sharp, f^\flat): (A, M, \alpha_A) \longrightarrow (B, N, \alpha_B)$$

is a homomorphism of rings $f^\sharp: A \rightarrow B$ together with a homomorphism of monoids $f^\flat: M \rightarrow N$ such that $\alpha_B f^\flat = f^\sharp \alpha_A$. A log ring is a prelog ring (A, M, α) such that the homomorphism $\alpha^{-1}(A^*) \rightarrow A^*$ induced by α (where A^* is the group of units of A) is an isomorphism.

Definition 2.2. If (A, M, α) is a prelog ring, define M^{\log} as the pushout

$$\begin{array}{ccc} \alpha^{-1}(M) & \xrightarrow{\alpha} & A^* \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\log} \end{array}$$

and let $\alpha^{\log}: M^{\log} \rightarrow A$ be the monoid homomorphism induced by the homomorphisms $\alpha: M \rightarrow A$ and $A^* \hookrightarrow A$. Then $(A, M, \alpha)^{\log} := (A, M^{\log}, \alpha^{\log})$ is a log ring. The functor $(-)^{\log}$ from the category of prelog rings to the category of log rings is left adjoint to the forgetful functor.

Definition 2.3. If (A, M, α) is a prelog ring and $f^\sharp: A \rightarrow B$ is a ring homomorphism, we have a homomorphism of prelog rings

$$f = (f^\sharp, f^\flat): (A, M, \alpha) \longrightarrow (B, f^*(M), \alpha^*)$$

where $(B, f^*(M), \alpha^*) := (B, M, f \circ \alpha)^{\log}$. A homomorphism of log rings

$$f: (A, M) \longrightarrow (B, N)$$

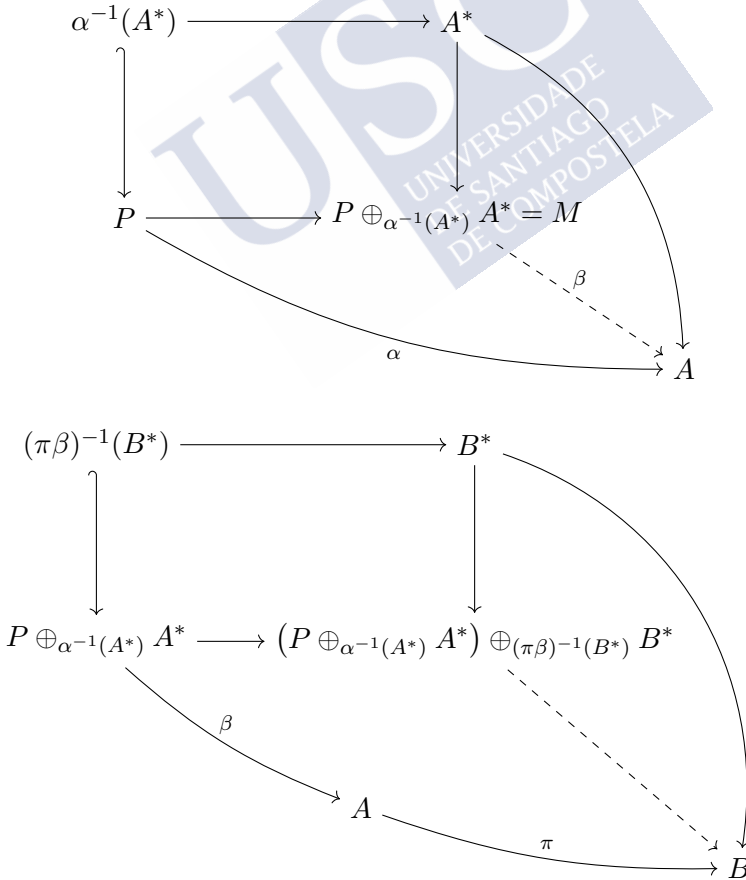
is called strict if the canonical map $f^*(M) \rightarrow N$ is an isomorphism.

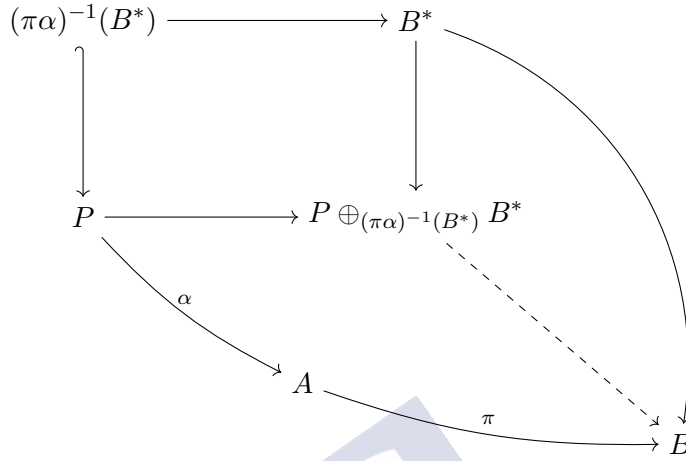
Definition 2.4. A Noetherian local prelog ring $((A, \mathfrak{m}, k), M)$ is a Noetherian local ring (A, \mathfrak{m}, k) , a monoid M such that the monoid ring $\mathbb{Z}[M]$ is Noetherian (i.e. M is finitely generated by Proposition 1.28) and a local homomorphism of monoids $\alpha: M \rightarrow (A, \cdot)$.

We say that $(A, M) \rightarrow (B, N)$ is a homomorphism (essentially) of finite type of Noetherian prelog rings if M, N are finitely generated monoids, A is a Noetherian ring, and $A \rightarrow B$ is a homomorphism (essentially) of finite type. We have then that $\mathbb{Z}[M] \rightarrow \mathbb{Z}[N]$ is a homomorphism of finite type of Noetherian rings, and M^{gp}, N^{gp} are \mathbb{Z} -modules of finite type.

Lemma 2.5. Let (A, M) be a log ring, $P \rightarrow M$ a homomorphism of monoids and $\pi: A \rightarrow B$ a homomorphism of rings. If $(A, P)^{log} = (A, M)$, then $(B, P)^{log} = (B, M)^{log}$.

Proof. Consider the pushouts of monoids





and the maps

$$(P \oplus_{\alpha^{-1}(A^*)} A^*) \oplus_{(\pi\beta)^{-1}(B^*)} B^* \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} P \oplus_{(\pi\alpha)^{-1}(B^*)} B^*$$

defined as follows for any $p \in P$, $a \in A^*$ and $b \in B^*$:

$$\psi(p, b) := ((p, 1), b), \quad \varphi((p, a), b) := (p, \pi(a)b).$$

Let us see that both homomorphisms are well defined. We will use Proposition 1.14. Assume that $((p, a), b) = ((p', a'), b')$ in $(P \oplus_{\alpha^{-1}(A^*)} A^*) \oplus_{(\pi\beta)^{-1}(B^*)} B^*$. Then by Proposition 1.14 there exists a pair $(p_1, a_1), (p_2, a_2) \in (\pi\beta)^{-1}(B^*) \subset P \oplus_{\alpha^{-1}(A^*)} A^*$ such that

$$(i) \quad (p, a) \cdot (p_2, a_2) = (p', a') \cdot (p_1, a_1),$$

$$(ii) \quad b\pi\beta(p_1, a_1) = b'\pi\beta(p_2, a_2).$$

These conditions are equivalent to

(i') There exists $s_1, s_2 \in \alpha^{-1}(A^*) \subset P$ such that

$$(i'_1) \quad pp_2s_2 = p'p_1s_1,$$

$$(i'_2) \quad aa_2\alpha(s_1) = a'a_1\alpha(s_2);$$

(ii') $b\pi\alpha(p_1)\pi(a_1) = b'\pi\alpha(p_2)\pi(a_2)$.

Let $q_1 = p_1s_1$ and $q_2 = p_2s_2$ in P . Clearly, $q_1, q_2 \in (\pi\alpha)^{-1}(B^*)$. In order to see that $(p, \pi(a)b) = (p', \pi(a')b')$ we will see that

$$(I) \quad pq_2 = p'q_1$$

$$(II) \quad \pi(a)b\pi\alpha(q_1) = \pi(a')b'\pi\alpha(q_2)$$

Condition (I) follows from (i'_1) . So as to see (II), it suffices to show the equality after multiplication by the unit $\pi(a_2) \in B^*$. But

$$\begin{aligned} \pi(a)b\pi\alpha(q_1)\pi(a_2) &= \pi(a)b\pi\alpha(p_1)\pi\alpha(s_1)\pi(a_2) \\ &\stackrel{(i'_2)}{=} \pi(a')b\pi\alpha(p_1)\pi\alpha(s_2)\pi(a_1) \\ &\stackrel{(ii')}{=} \pi(a')b'\pi\alpha(p_2)\pi\alpha(s_2)\pi(a_2) \\ &= \pi(a')b'\pi\alpha(q_2)\pi(a_2). \end{aligned}$$

So φ is well defined.

Now assume that $(p, b) = (p', b')$. Then there exist $r_1, r_2 \in (\pi\alpha)^{-1}(B^*) \subset P$ such that

$$(i) \quad pr_2 = p'r_1$$

$$(ii) \quad b\pi\alpha(r_1) = b'\pi\alpha(r_2)$$

We have elements $(r_1, 1), (r_2, 1) \in (\pi\beta)^{-1}(B^*) \subset P \oplus_{\alpha^{-1}(A^*)} A^*$ verifying

$$(p, 1) \cdot (r_2, 1) = (p', 1) \cdot (r_1, 1),$$

$$b\pi\beta(r_1, 1) = b\pi\alpha(r_1) = b'\pi\alpha(r_2) = b'\pi\beta(r_2, 1).$$

Clearly φ and ψ are homomorphisms of monoids and $\varphi\psi = \text{id}$. In order to check that $\psi\varphi = \text{id}$, we have to prove that $((p, a), b) = ((p, 1), \pi(a)b)$. But $(1, a), (1, 1) \in (\pi\alpha)^{-1}(B^*)$ since $a \in A^*$, and

- $(p, a) \cdot (1, 1) = (p, 1) \cdot (1, a),$
- $b\pi\beta(1, a) = b\pi(a) = \pi(a)b\pi\beta(1, 1).$

□

Chapter 3

Derivations and differentials

This chapter contains definitions, notation and known results on derivations and differentials. Though there is nothing new, we give proofs since in a few points we were not able to find exact references.

3.1 The conormal module

Proposition 3.1. *Let $g: L \rightarrow N$ be a surjective homomorphism of monoids and define $J = \ker(\mathbb{Z}[L] \rightarrow \mathbb{Z}[N])$, $W = \ker(L^{sp} \rightarrow N^{sp})$ (note that $\mathbb{Z}[-]$ and $(-)^{sp}$ preserve surjectivity). There exists a natural homomorphism of $\mathbb{Z}[N]$ -modules*

$$\nu_g: J/J^2 \longrightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}} W$$

defined by

$$\nu_g\left(\overline{\sum_{i \in I} \lambda_i l_i}\right) := \sum_{i \in I} \lambda_i g(l_i) \otimes l_i \in \mathbb{Z}[N] \otimes_{\mathbb{Z}} L^{sp},$$

where $\lambda_i \in \mathbb{Z}$ and $l_i \in L$ for all $i \in I$, and we are also denoting the image of l_i in L^{sp} by l_i .

Proof. We define $\tilde{\nu}_g: J \rightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}} L^{sp}$ as $\tilde{\nu}_g(\sum_{i \in I} \lambda_i l_i) := \sum_{i \in I} \lambda_i g(l_i) \otimes l_i$. First, we will show that $\tilde{\nu}_g(x) \in \mathbb{Z}[N] \otimes_{\mathbb{Z}} W$ if $x = \sum_{i \in I} \lambda_i l_i \in J$. For each $n \in N$, we consider $I_n = \{i \in I \mid g(l_i) = n\}$, so that $\sum_{i \in I_n} \lambda_i = 0$ for all $n \in N$ (given that $g(\sum_{i \in I} \lambda_i l_i) = \sum_{n \in N} (\sum_{i \in I_n} \lambda_i) n = 0$ and $\mathbb{Z}[N]$ is a free \mathbb{Z} -module with basis N). We have

$$\begin{aligned} \sum_{i \in I} \lambda_i g(l_i) \otimes l_i &= \sum_{i \in I} g(l_i) \otimes l_i^{\lambda_i} = \sum_{n \in N} \sum_{i \in I_n} n \otimes l_i^{\lambda_i} \\ &= \sum_{n \in N} (n \otimes \prod_{i \in I_n} l_i^{\lambda_i}) \end{aligned}$$

and $\prod_{i \in I_n} l_i^{\lambda_i} \in W$ since if g denotes also the map $L^{sp} \rightarrow N^{sp}$, then

$$g\left(\prod_{i \in I_n} l_i^{\lambda_i}\right) = \prod_{i \in I_n} g(l_i)^{\lambda_i} = \prod_{i \in I_n} n^{\lambda_i} = n^{\sum_{i \in I_n} \lambda_i} = n^0 = 1.$$

The map $\tilde{\nu}_g$ is (clearly) additive, so in order to see that it induces a homomorphism

$$\nu_g: J/J^2 \longrightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}} W$$

we have to show that $\tilde{\nu}_g(x) = 0$ for $x \in J^2$. The elements of J are of the form

$$\sum_{i \in I} (\lambda_i l_{i,1} - \lambda_i l_{i,2})$$

with $\lambda_i \in \mathbb{Z}$, $l_{i,1}, l_{i,2} \in L$ and $g(l_{i,1}) = g(l_{i,2})$ for all $i \in I$ (since $\sum_{i \in I_n} \lambda_i = 0$ for every $n \in N$, as we saw before, and writing each λ_i as a sum of ones or minus ones, there are as many ones as minus ones). Therefore let $\lambda_1, \lambda_2 \in \mathbb{Z}$ and $l_{i,j} \in L$ for $i, j \in \{1, 2\}$ such that $g(l_{1,1}) = g(l_{1,2}) =: n_1$ and $g(l_{2,1}) = g(l_{2,2}) =: n_2$. We obtain

$$\begin{aligned} \tilde{\nu}_g((\lambda_1 l_{1,1} - \lambda_1 l_{1,2})(\lambda_2 l_{2,1} - \lambda_2 l_{2,2})) &= \lambda_1 \lambda_2 g(l_{1,1}) g(l_{2,1}) \otimes l_{1,1} l_{2,1} \\ &\quad - \lambda_1 \lambda_2 g(l_{1,2}) g(l_{2,1}) \otimes l_{1,2} l_{2,1} \\ &\quad - \lambda_1 \lambda_2 g(l_{1,1}) g(l_{2,2}) \otimes l_{1,1} l_{2,2} \\ &\quad + \lambda_1 \lambda_2 g(l_{1,2}) g(l_{2,2}) \otimes l_{1,2} l_{2,2} \\ &= \lambda_1 \lambda_2 n_1 n_2 \otimes l_{1,1} l_{2,1} \\ &\quad - \lambda_1 \lambda_2 n_1 n_2 \otimes l_{1,2} l_{2,1} \\ &\quad - \lambda_1 \lambda_2 n_1 n_2 \otimes l_{1,1} l_{2,2} \\ &\quad + \lambda_1 \lambda_2 n_1 n_2 \otimes l_{1,2} l_{2,2} \\ &= \lambda_1 \lambda_2 n_1 n_2 \otimes l_{1,1} l_{2,1} l_{1,2}^{-1} l_{2,1}^{-1} l_{1,1}^{-1} l_{2,2}^{-1} l_{1,2} l_{2,2} \\ &= \lambda_1 \lambda_2 n_1 n_2 \otimes 1 = 0. \end{aligned}$$

We now see that ν_g is a homomorphism of $\mathbb{Z}[N]$ -modules. We already know that it is an additive homomorphism. Let $m \in N$ be and $l \in L$ such that $g(l) = m$. With the above definition for I_n ,

$$\begin{aligned} \nu_g(m \cdot \overline{\sum_{i \in I_n} \lambda_i l_i}) &= \tilde{\nu}_g(\sum_{i \in I_n} \lambda_i l l_i) = \sum_{i \in I_n} \lambda_i g(l l_i) \otimes l l_i = \sum_{i \in I_n} \lambda_i m n \otimes l l_i \\ &= m n \otimes \prod_{i \in I_n} l^{\lambda_i} l_i^{\lambda_i} = m n \otimes (l^{\sum_{i \in I_n} \lambda_i} \prod_{i \in I_n} l_i^{\lambda_i}) \\ &= m n \otimes (l^0 \prod_{i \in I_n} l_i^{\lambda_i}) = m n \otimes \prod_{i \in I_n} l_i^{\lambda_i} = m (\sum_{i \in I_n} \lambda_i n \otimes l_i) \\ &= m \cdot \tilde{\nu}_g(\sum_{i \in I_n} \lambda_i l_i) = m \cdot \nu_g(\overline{\sum_{i \in I_n} \lambda_i l_i}). \end{aligned}$$

Finally, naturality means that if

$$\begin{array}{ccc} L_1 & \xrightarrow{g_1} & N_1 \\ \downarrow \gamma & & \downarrow \rho \\ L_2 & \xrightarrow{g_2} & N_2 \end{array}$$

is a commutative square of monoids with g_1, g_2 surjective homomorphisms and J_1, W_1, J_2, W_2 are defined in the obvious way, then the diagram

$$\begin{array}{ccc} J_1/J_1^2 & \xrightarrow{\nu_{g_1}} & \mathbb{Z}[N_1] \otimes_{\mathbb{Z}} W_1 \\ \downarrow \tilde{\gamma} & & \downarrow \tilde{\rho} \\ J_2/J_2^2 & \xrightarrow{\nu_{g_2}} & \mathbb{Z}[N_2] \otimes_{\mathbb{Z}} W_2 \end{array}$$

is commutative, and this fact is obvious. \square

Remark 3.2. We have seen in the proof of Proposition 3.1 that the elements of J are $\sum_{i \in I} (\lambda_i l_{i,1} - \lambda_i l_{i,2})$ with $\lambda_i \in \mathbb{Z}$ and $l_{i,1}, l_{i,2} \in L$ verifying $g(l_{i,1}) = g(l_{i,2})$ for all $i \in I$, and the image of one such element by $\tilde{\nu}_g$ is $\sum_{i \in I} \lambda_i g(l_{i,2}) \otimes l_{i,1} l_{i,2}^{-1}$.

Example 3.3. Let $h: M \rightarrow N$ be a homomorphism of monoids and consider the induced surjective homomorphism $g: N \oplus_M N \rightarrow N$. We have $\mathbb{Z}[N \oplus_M N] = \mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N]$ by Remark 1.27 and then $J := \ker(\mathbb{Z}[N \oplus_M N] \rightarrow \mathbb{Z}[N])$ is the kernel of the multiplication $\mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N] \rightarrow \mathbb{Z}[N]$. Therefore we have an isomorphism of $\mathbb{Z}[N]$ -modules

$$J/J^2 = \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]},$$

where $\Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}$ is the usual module of differentials [6, 0IV, Définition 20.4.3].

On the other hand, by Remark 1.21, we obtain an isomorphism of Abelian groups

$$\begin{aligned} W &:= \ker((N \oplus_M N)^{sp} \rightarrow N^{sp}) = \ker(N^{sp} \oplus_{M^{sp}} N^{sp} \rightarrow N^{sp}) \\ &= \text{coker}(M^{sp} \rightarrow N^{sp}). \end{aligned}$$

Then the composition

$$\Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} = J/J^2 \xrightarrow{\nu_g} \mathbb{Z}[N] \otimes_{\mathbb{Z}} W = \mathbb{Z}[N] \otimes_{\mathbb{Z}} N^{sp} / \text{Im}(M^{sp})$$

is the homomorphism of $\mathbb{Z}[N]$ -modules defined by

$$\begin{aligned} dn \longmapsto (n, 1) - (1, n) &\longmapsto n \otimes (n, 1) - n \otimes (1, n) = n \otimes (n, 1)(1, n)^{-1} \\ &= n \otimes (n, n^{-1}) \longmapsto n \otimes \bar{n}, \end{aligned}$$

where \bar{n} is the class of $n \in N$ in $N^{gp} / \text{Im}(M^{gp})$.

Definition 3.4. Let $(f, g): (C, L) \rightarrow (B, N)$ a surjective homomorphism of prelog rings (i.e. $f: C \rightarrow B$ and $g: L \rightarrow N$ are surjective). Let us consider the ideals $J = \ker(\mathbb{Z}[L] \rightarrow \mathbb{Z}[N])$, $W = \ker(L^{gp} \rightarrow N^{gp})$ and $I = \ker(C \rightarrow B)$. We define the (logarithmic) conormal module of $(C, L) \rightarrow (B, N)$ as the pushout $N_{(B, N)|(C, L)}$ of the diagram of B -modules

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} J/J^2 & \xrightarrow{\text{id}_B \otimes_{\mathbb{Z}[N]} \nu_g} & B \otimes_{\mathbb{Z}} W \\ \downarrow u & & \\ I/I^2 & & \end{array}$$

where ν_g is the homomorphism of Proposition 3.1 and u is the homomorphism of B -modules defined by $u(1 \otimes \sum_{i \in I} \lambda_i l_i) := \sum_{i \in I} \lambda_i \alpha(l_i)$, being $\alpha: L \rightarrow C$ is the structural homomorphism and $\lambda_i \in \mathbb{Z}$, $l_i \in L$ for all $i \in I$.

Remark 3.5. The definition of [12, Exposé II, Proposition 4.11] coincides with this one.

Proposition 3.6. Let

$$\begin{array}{ccc} (C_1, L_1) & \longrightarrow & (B_1, N_1) \\ \downarrow & & \downarrow \\ (C_2, L_2) & \longrightarrow & (B_2, N_2) \end{array}$$

be a commutative square of prelog rings where the horizontal homomorphisms are surjective. Then there exists a natural homomorphism

$$B_2 \otimes_{B_1} N_{(B_1, N_1)|(C_1, L_1)} \longrightarrow N_{(B_2, N_2)|(C_2, L_2)}.$$

Proof. It is consequence of the naturality of the homomorphisms ν_g and u in Definition 3.4. \square

Example 3.7. If $L \rightarrow N$ is an isomorphism, then $J = 0$ and $W = \{1\}$. Therefore the conormal module of $(C, L) \rightarrow (B, N)$ coincides with the (usual) conormal module I/I^2 of $C \rightarrow B$. This takes place, for example, when $L = N = \{1\}$.

Remark 3.8. If C is Noetherian and L^{sp} is a finitely generated Abelian group, then $N_{(B,N)|(C,L)}$ is a B -module of finite type.

3.2 Module of differentials

Definition 3.9. Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings. The logarithmic module of differentials $\Omega_{(B,N)|(A,M)}$ of $(A, M) \rightarrow (B, N)$ is defined as the B -module $N_{(B,N)|(C,L)}$ where $(C, L) = (B \otimes_A B, N \oplus_M N)$. By the Example 3.3, the square

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \xrightarrow{v} & B \otimes_{\mathbb{Z}} N^{sp} / \text{Im}(M^{sp}) \\ \downarrow u & & \downarrow \\ \Omega_{B|A} & \longrightarrow & \Omega_{(B,N)|(A,M)} \end{array}$$

is cocartesian, where v is the homomorphism of B -modules verifying $v(1 \otimes dn) = n \otimes \bar{n}$ with $n \in N$ and \bar{n} its class in $N^{sp} / \text{Im}(M^{sp})$. If A is Noetherian, B an A -algebra essentially of finite type and $N^{sp} / \text{Im}(M^{sp})$ a finitely generated Abelian group, then $\Omega_{(B,N)|(A,M)}$ is a B -module of finite type.

By Proposition 3.6, if

$$\begin{array}{ccc} (A_1, M_1) & \longrightarrow & (B_1, N_1) \\ \downarrow & & \downarrow \\ (A_2, M_2) & \longrightarrow & (B_2, N_2) \end{array}$$

is a commutative square of homomorphisms of prelog rings, then there exists a natural homomorphism of B_2 -modules

$$B_2 \otimes_{B_1} \Omega_{(B_1, N_1)|(A_1, M_1)} \longrightarrow \Omega_{(B_2, N_2)|(A_2, M_2)}.$$

Proposition 3.10 (Base Change). *We consider a diagram of log rings*

$$\begin{array}{ccc} (A_1, M_1) & \longrightarrow & (B_1, N_1) \\ & & \downarrow \\ & & (A_2, M_2) \end{array}$$

and let (B_2, N_2) be its pushout, i.e. $B_2 = A_2 \otimes_{A_1} B_1$, $N_2 = M_2 \oplus_{M_1} N_1$. Then the canonical homomorphism

$$B_2 \otimes_{B_1} \Omega_{(B_1, N_1)|(A_1, M_1)} \longrightarrow \Omega_{(B_2, N_2)|(A_2, M_2)}$$

is an isomorphism.

Proof. Since $B_2 \otimes_{B_1} -$ is right-exact, by the cocartesian square in Definition 3.9

$$\begin{array}{ccc} B_1 \otimes_{\mathbb{Z}[N_1]} \Omega_{\mathbb{Z}[N_1]|\mathbb{Z}[M_1]} & \longrightarrow & B_1 \otimes_{\mathbb{Z}} N_1^{\text{gp}} / \text{Im}(M_1^{\text{gp}}) \\ \downarrow & & \downarrow \\ \Omega_{B_1|A_1} & \longrightarrow & \Omega_{(B_1, N_1)|(A_1, M_1)} \end{array}$$

we obtain another cocartesian square

$$\begin{array}{ccc} B_2 \otimes_{\mathbb{Z}[N_1]} \Omega_{\mathbb{Z}[N_1]|\mathbb{Z}[M_1]} & \longrightarrow & B_2 \otimes_{\mathbb{Z}} N_1^{\text{gp}} / \text{Im}(M_1^{\text{gp}}) \\ \downarrow & & \downarrow \\ B_2 \otimes_{B_1} \Omega_{B_1|A_1} & \longrightarrow & B_2 \otimes_{B_1} \Omega_{(B_1, N_1)|(A_1, M_1)} \end{array}$$

and then a commutative diagram where both front and rear squares are cocartesian:

$$\begin{array}{ccccc} B_2 \otimes_{\mathbb{Z}[N_1]} \Omega_{\mathbb{Z}[N_1]|\mathbb{Z}[M_1]} & \longrightarrow & B_2 \otimes_{\mathbb{Z}} N_1^{\text{gp}} / \text{Im}(M_1^{\text{gp}}) & & \\ \downarrow & \searrow \alpha & \downarrow & \searrow \gamma & \\ B_2 \otimes_{\mathbb{Z}[N_2]} \Omega_{\mathbb{Z}[N_2]|\mathbb{Z}[M_2]} & \longrightarrow & B_2 \otimes_{\mathbb{Z}} N_2^{\text{gp}} / \text{Im}(M_2^{\text{gp}}) & & \\ \downarrow & & \downarrow & & \downarrow \\ B_2 \otimes_{B_1} \Omega_{B_1|A_1} & \longrightarrow & B_2 \otimes_{B_1} \Omega_{(B_1, N_1)|(A_1, M_1)} & & \\ \downarrow \beta & & \downarrow & & \downarrow \\ \Omega_{B_2|A_2} & \longrightarrow & \Omega_{(B_2, N_2)|(A_2, M_2)} & & \end{array}$$

It is enough to prove that α, β, γ are isomorphisms. Since

$$\mathbb{Z}[N_2] = \mathbb{Z}[M_2 \oplus_{M_1} N_1] = \mathbb{Z}[M_2] \otimes_{\mathbb{Z}[M_1]} \mathbb{Z}[N_1]$$

(Remark 1.27) we have

$$\Omega_{\mathbb{Z}[N_2]|\mathbb{Z}[M_2]} = \mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_1]} \Omega_{\mathbb{Z}[N_1]|\mathbb{Z}[M_1]}$$

by [2, Lemme 1.13], and therefore α is an isomorphism. Clearly β is an isomorphism too. Finally, from the exact sequence

$$\begin{array}{ccccccc} 1 & \longrightarrow & M_2^{\text{gp}} & \longrightarrow & M_2^{\text{gp}} \oplus_{M_1^{\text{gp}}} N_1^{\text{gp}} & \longrightarrow & N_1^{\text{gp}} / \text{Im}(M_1^{\text{gp}}) \longrightarrow 1 \\ & & a & \longmapsto & (a, 1) & & \\ & & & & (a, b) & \longmapsto & \bar{b} \end{array}$$

we obtain an isomorphism

$$\frac{N_2^{\text{gp}}}{\text{Im}(M_2^{\text{gp}})} = \frac{M_2^{\text{gp}} \oplus_{M_1^{\text{gp}}} N_1^{\text{gp}}}{\text{Im}(M_2^{\text{gp}})} = \frac{N_1^{\text{gp}}}{\text{Im}(M_1^{\text{gp}})}$$

and so γ is an isomorphism. □

Corollary 3.11 (Localization). *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and $S \subset N, T \subset (B, \cdot)$ submonoids such that $\alpha(S) \subset T$ where $\alpha: N \rightarrow B$ is the structural homomorphism. Then*

$$T^{-1}\Omega_{(B,N)|(A,M)} = \Omega_{(T^{-1}B, S^{-1}N)|(A,M)}.$$

Proof. As in the previous proposition, applying the exact functor $T^{-1}(-)$ to the cocartesian square in Definition 3.9 we obtain a cocartesian square

$$\begin{array}{ccc} T^{-1}(B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}) & \longrightarrow & T^{-1}(B \otimes_{\mathbb{Z}} N^{\text{gp}} / \text{Im}(M^{\text{gp}})) \\ \downarrow & & \downarrow \\ T^{-1}(\Omega_{B|A}) & \longrightarrow & T^{-1}(\Omega_{(B,N)|(A,M)}) \end{array}$$

and consequently a commutative diagram where both front and rear squares are cocartesian:

$$\begin{array}{ccc}
 T^{-1}(B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}) & \longrightarrow & T^{-1}(B \otimes_{\mathbb{Z}} N^{\text{gp}} / \text{Im}(M^{\text{gp}})) \\
 \downarrow & \searrow \alpha & \downarrow & \searrow \gamma \\
 T^{-1}B \otimes_{\mathbb{Z}[S^{-1}N]} \Omega_{\mathbb{Z}[S^{-1}N]|\mathbb{Z}[M]} & \longrightarrow & T^{-1}B \otimes_{\mathbb{Z}} (S^{-1}N)^{\text{gp}} / \text{Im}(M^{\text{gp}}) \\
 \downarrow & & \downarrow & \\
 T^{-1}(\Omega_{B|A}) & \longrightarrow & T^{-1}(\Omega_{(B,N)|(A,M)}) \\
 \searrow \beta & & \searrow & \\
 \Omega_{T^{-1}B|A} & \longrightarrow & \Omega_{(T^{-1}B, S^{-1}N)|(A,M)}
 \end{array}$$

The homomorphism γ is an isomorphism because $(S^{-1}N)^{\text{gp}} = N^{\text{gp}}$, and α and β are isomorphisms too. Hence the desired isomorphism follows. \square

Lemma 3.12. *The pushout of right-exact sequences of modules is also a right-exact sequence.*

Proof. Let us consider the commutative diagram

$$\begin{array}{ccccccc}
 & & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & \nearrow & \vdots & & \nearrow & & \vdots & & \\
 A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \nearrow & D & \dashrightarrow & D' & \dashrightarrow & D'' & \dashrightarrow & 0 \\
 & \nearrow & \vdots & & \nearrow & & \vdots & & \\
 B & \longrightarrow & B' & \longrightarrow & B'' & \longrightarrow & 0 & &
 \end{array}$$

where the rows $A \rightarrow A' \rightarrow A'' \rightarrow 0$, $B \rightarrow B' \rightarrow B'' \rightarrow 0$ and $C \rightarrow C' \rightarrow C'' \rightarrow 0$ are right-exact sequence of modules, and the commutative diagrams

$$\begin{array}{ccc}
 A \longrightarrow C & A' \longrightarrow C' & A'' \longrightarrow C'' \\
 \downarrow & \downarrow & \downarrow \\
 B \dashrightarrow D & B' \dashrightarrow D' & B'' \dashrightarrow D''
 \end{array}$$

are cocartesian. We will show that the sequence $D \rightarrow D' \rightarrow D'' \rightarrow 0$ of the pushouts is also a right-exact sequence.

The surjectivity of $D' \rightarrow D''$ and $\text{Im}(D \rightarrow D') \subset \ker(D' \rightarrow D'')$ are obvious, while $\ker(D' \rightarrow D'') \subset \text{Im}(D \rightarrow D')$ can be seen by diagram chasing: Let $b' \in B'$, $c' \in C'$ such that $d' = \overline{b' + c'}$, and let b'', c'', d'' be their images in B'', C'', D'' respectively. Since $d'' = 0$, there exists $a'' \in A''$ such b'' and $-c''$ are both the image of a'' . Let $a' \in A'$ be such that its image in A'' is a'' , and let $\tilde{b}', -\tilde{c}'$ their images in B' and C' respectively. Then \tilde{b}' and \tilde{c}' go to b'' and c'' in C'', D'' respectively, so there exist $b \in B, c \in C$ such that b and c go to $b' - \tilde{b}', c' - \tilde{c}'$ respectively. Then the image of $\overline{b + c} \in D$ in D' is $b' - \tilde{b}' + c' - \tilde{c}' = d' - (\tilde{b}' + \tilde{c}') = d'$ since \tilde{b}' and $-\tilde{c}'$ are the images of $a' \in A'$. Thus it is done. \square

Proposition 3.13. *Let $(A, M) \rightarrow (B, N) \rightarrow (C, P)$ be homomorphisms of prelog rings. There exists a natural exact sequence of C -modules*

$$C \otimes_B \Omega_{(B,N)|(A,M)} \longrightarrow \Omega_{(C,P)|(A,M)} \longrightarrow \Omega_{(C,P)|(B,N)} \longrightarrow 0.$$

Proof. Using Lemma 3.12, we deduce the result from the definition of the module of differentials of prelog rings and from the exact sequences

$$\begin{aligned} C \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} &\longrightarrow C \otimes_{\mathbb{Z}[P]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[M]} \longrightarrow C \otimes_{\mathbb{Z}[P]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[N]} \longrightarrow 0, \\ C \otimes_B \Omega_{B|A} &\longrightarrow \Omega_{C|A} \longrightarrow \Omega_{C|B} \longrightarrow 0 \end{aligned}$$

and

$$N^{\text{gp}} / \text{Im}_{N^{\text{gp}}}(M^{\text{gp}}) \longrightarrow P^{\text{gp}} / \text{Im}_{P^{\text{gp}}}(M^{\text{gp}}) \longrightarrow P^{\text{gp}} / \text{Im}_{P^{\text{gp}}}(N^{\text{gp}}) \longrightarrow 1,$$

which is exact since

$$N^{\text{gp}} / \text{Im}_{N^{\text{gp}}}(M^{\text{gp}}) \longrightarrow \text{Im}_{P^{\text{gp}}}(N^{\text{gp}}) / \text{Im}_{P^{\text{gp}}}(M^{\text{gp}})$$

is surjective and (clearly)

$$\text{Im}_{P^{\text{gp}}}(N^{\text{gp}}) / \text{Im}_{P^{\text{gp}}}(M^{\text{gp}}) \longrightarrow P^{\text{gp}} / \text{Im}_{P^{\text{gp}}}(M^{\text{gp}}) \longrightarrow P^{\text{gp}} / \text{Im}_{P^{\text{gp}}}(N^{\text{gp}}) \longrightarrow 1$$

is exact. \square

Proposition 3.14. *Let $(A, M) \rightarrow (C, L) \rightarrow (B, N)$ be homomorphisms of prelog rings with $C \rightarrow B$ and $L \rightarrow N$ surjective. There exists a natural exact sequence of B -modules*

$$N_{(B,N)|(C,L)} \longrightarrow B \otimes_C \Omega_{(C,L)|(A,M)} \longrightarrow \Omega_{(B,N)|(A,M)} \longrightarrow 0.$$

Proof. In the same way that the proof of the Proposition 3.13. It is deduced from the following exact sequences:

$$B \otimes_{\mathbb{Z}[N]} J/J^2 \longrightarrow B \otimes_{\mathbb{Z}[L]} \Omega_{\mathbb{Z}[L]|\mathbb{Z}[M]} \longrightarrow B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} \longrightarrow 0,$$

$$I/I^2 \longrightarrow B \otimes_C \Omega_{C|A} \longrightarrow \Omega_{B|A} \longrightarrow 0$$

and

$$B \otimes_{\mathbb{Z}} W \longrightarrow B \otimes_{\mathbb{Z}} L^{\text{gp}}/\text{Im}_{L^{\text{gp}}}(M^{\text{gp}}) \longrightarrow B \otimes_{\mathbb{Z}} N^{\text{gp}}/\text{Im}_{N^{\text{gp}}}(M^{\text{gp}}) \longrightarrow 0,$$

where $I = \ker(C \rightarrow B)$, $J = \ker(\mathbb{Z}[L] \rightarrow \mathbb{Z}[N])$ and $W = \ker(L^{\text{gp}} \rightarrow N^{\text{gp}})$. Note that the third sequence is exact because

$$W = \ker(L^{\text{gp}}/\text{Im}_{L^{\text{gp}}}(M^{\text{gp}}) \rightarrow N^{\text{gp}}/\text{Im}_{N^{\text{gp}}}(M^{\text{gp}})),$$

as we deduce, for example, applying the Ker-Coker Lemma to the next diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M^{\text{gp}} & \longrightarrow & L^{\text{gp}} & \longrightarrow & L^{\text{gp}}/\text{Im}_{L^{\text{gp}}}(M^{\text{gp}}) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & M^{\text{gp}} & \longrightarrow & N^{\text{gp}} & \longrightarrow & N^{\text{gp}}/\text{Im}_{N^{\text{gp}}}(M^{\text{gp}}) & \longrightarrow & 1 \quad \square \end{array}$$

Proposition 3.15 (Künneth). *Let $(A, M) \rightarrow (B, N)$, $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings. There exists a natural isomorphism of $B \otimes_A C$ -modules*

$$\Omega_{(B \otimes_A C, N \oplus_M P)|(A, M)} = (\Omega_{(B, N)|(A, M)} \otimes_A C) \oplus (B \otimes_A \Omega_{(C, P)|(A, M)}).$$

Proof. Applying the right-exact functors $B \otimes_A -$ and $C \otimes_A -$ to the cocartesian diagram that define $\Omega_{(B, N)|(A, M)}$ (as in the proof of Proposition 3.10) and taking the direct sum of them, we obtain a pushout

$$\begin{array}{ccc} (D \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}) \oplus (D \otimes_{\mathbb{Z}[P]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[M]}) & \longrightarrow & (D \otimes_{\mathbb{Z}} N^{\text{gp}}/\text{Im}(M^{\text{gp}})) \oplus (D \otimes_{\mathbb{Z}} P^{\text{gp}}/\text{Im}(M^{\text{gp}})) \\ \downarrow & & \downarrow \\ (C \otimes_A \Omega_{B|A}) \oplus (B \otimes_A \Omega_{C|A}) & \longrightarrow & (C \otimes_A \Omega_{(B, N)|(A, M)}) \oplus (B \otimes_A \Omega_{(C, P)|(A, M)}) \end{array}$$

where $D = B \otimes_A C$.

Now, we note that

$$\begin{aligned}
& ((C \otimes_A B) \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}) \oplus_{\mathbb{Z}[M]} ((B \otimes_A C) \otimes_{\mathbb{Z}[P]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[M]}) \\
&= (C \otimes_A B) \otimes_{\mathbb{Z}[N \oplus_M P]} (\Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} \oplus_{\mathbb{Z}[M]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[M]}) \\
&= (C \otimes_A B) \otimes_{\mathbb{Z}[N \oplus_M P]} (\Omega_{\mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P]|\mathbb{Z}[M]}) \\
&= (B \otimes_A C) \otimes_{\mathbb{Z}[N \oplus_M P]} \Omega_{\mathbb{Z}[N \oplus_M P]|\mathbb{Z}[M]}
\end{aligned}$$

(using Remark 1.27 and [2, Lemme 1.14]).

Moreover, using the fact that the exact sequence

$$\begin{aligned}
1 &\longrightarrow M^{\text{gp}} \longrightarrow N^{\text{gp}} \oplus_{M^{\text{gp}}} P^{\text{gp}} \longrightarrow N^{\text{gp}}/\text{Im}(M^{\text{gp}}) \oplus P^{\text{gp}}/\text{Im}(M^{\text{gp}}) \longrightarrow 1 \\
a &\longmapsto (a, 1) \\
(a, b) &\longmapsto (\bar{a}, \bar{b})
\end{aligned}$$

is exact and Proposition 1.21, we have

$$\begin{aligned}
& (C \otimes_A B \otimes_{\mathbb{Z}} N^{\text{gp}}/\text{Im}(M^{\text{gp}})) \oplus (B \otimes_A C \otimes_{\mathbb{Z}} P^{\text{gp}}/\text{Im}(M^{\text{gp}})) = \\
&= (C \otimes_A B) \otimes_{\mathbb{Z}} (N^{\text{gp}}/\text{Im}(M^{\text{gp}}) \oplus P^{\text{gp}}/\text{Im}(M^{\text{gp}})) \\
&= (C \otimes_A B) \otimes_{\mathbb{Z}} (N^{\text{gp}} \oplus_{M^{\text{gp}}} P^{\text{gp}}/M^{\text{gp}}) \\
&= (C \otimes_A B) \otimes_{\mathbb{Z}} ((N \oplus_M P)^{\text{gp}}/M^{\text{gp}})
\end{aligned}$$

Finally, using [2, Lemme 1.14],

$$(C \otimes_A \Omega_{B|A}) \oplus (B \otimes_A \Omega_{C|A}) = (\Omega_{C|A}) \oplus (\Omega_{B|A}) = \Omega_{B \otimes_A C|A}$$

and we obtain a diagram of pushouts

$$\begin{array}{ccc}
(D \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]}) \oplus (D \otimes_{\mathbb{Z}[P]} \Omega_{\mathbb{Z}[P]|\mathbb{Z}[M]}) & \rightarrow & (D \otimes_{\mathbb{Z}} N^{\text{gp}}/M^{\text{gp}}) \oplus (D \otimes_{\mathbb{Z}} P^{\text{gp}}/M^{\text{gp}}) \\
\downarrow & \searrow = & \downarrow \\
& & D \otimes_{\mathbb{Z}[M \oplus_M P]} \Omega_{\mathbb{Z}[M \oplus_M P]|\mathbb{Z}[M]} \longrightarrow D \otimes_{\mathbb{Z}} ((N \oplus_M P)^{\text{gp}}/M^{\text{gp}}) \\
\downarrow & & \downarrow \\
(C \otimes_A \Omega_{B|A}) \oplus (B \otimes_A \Omega_{C|A}) & \longrightarrow & C \otimes_A \Omega_{(B,N)|(A,M)} \oplus (B \otimes_A \Omega_{(C,P)|(A,M)}) \\
\downarrow = & & \downarrow \\
& & \Omega_{B \otimes_A C|A} \longrightarrow \Omega_{(B \otimes_A C, N \oplus_M P)|(A,M)}
\end{array}$$

□

Examples 3.16. (i) *Free prelog algebras.* Let (A, M) be a prelog ring, X and Y sets, $N = M \oplus \mathbb{N}^Y$ where \mathbb{N}^Y is a direct sum of copies of the (additive) monoid of natural numbers indexed by Y , $B = A[X, Y]$ the polynomial A -algebra on $X \cup Y$, $N \rightarrow B$ the homomorphism which extends $M \rightarrow A$ in the obvious way. We will say that (B, N) is a free (A, M) -prelog algebra. Then $\Omega_{(B,N)|(A,M)}$ is a free B -module with basis $X \cup Y$.

It follows from the pushout that defines $\Omega_{(B,N)|(A,M)}$,

$$\begin{array}{ccc} B^X = B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \xrightarrow{v} & B \otimes_{\mathbb{Z}} N^{gp}/M^{gp} = B^X \\ \downarrow u & & \downarrow w \\ B^{X \cup Y} = \Omega_{B|A} & \longrightarrow & \Omega_{(B,N)|(A,M)} \end{array}$$

(note that v is not an isomorphism). We know u is injective (it can be seen as the canonical inclusion) and so $\text{coker}(u) = B^Y$. Therefore w is also injective and $\text{coker}(w) = B^Y$ again. Thus, we obtain an exact sequence of B -modules

$$0 \longrightarrow B^X \xrightarrow{w} \Omega_{(B,N)|(A,M)} \longrightarrow B^Y \longrightarrow 0$$

that gives us the desired isomorphism, since it is split because B^Y is a free B -module.

(ii) Let $(A, M) \rightarrow (B, N)$ a homomorphism essentially of finite type of Noetherian prelog rings (see Definition 2.4). Then $\Omega_{(B,N)|(A,M)}$ is a B -module of finite type. It follows from the above Example 3.16.(i) and Proposition 3.14.

(iii) Let $M \rightarrow N$ be a homomorphism of monoids, then

$$\Omega_{(\mathbb{Z}[N],N)|(\mathbb{Z}[M],M)} = \mathbb{Z}[N] \otimes_{\mathbb{Z}} N^{gp} / \text{Im}(M^{gp})$$

since, by definition, $\Omega_{(\mathbb{Z}[N],N)|(\mathbb{Z}[M],M)}$ is the pushout of

$$\begin{array}{ccc} \mathbb{Z}[N] \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} N^{gp} / \text{Im}(M^{gp}) \\ \downarrow = & & \\ \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} & & \end{array}$$

- (iv) Let (B, N) be a prelog ring where N is a group. Then the homomorphism $\nu: B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}} \rightarrow B \otimes_{\mathbb{Z}} N^{gp}$ is surjective since $\nu(bn^{-1} \otimes dn) = b \otimes n$ for $n \in N^{gp} = N$ and $b \in B$. Therefore, from the cocartesian square

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}} & \longrightarrow & B \otimes_{\mathbb{Z}} N^{gp} \\ \downarrow & & \downarrow \\ 0 = \Omega_{B|B} & \longrightarrow & \Omega_{(B,N)|(B,1)} \end{array}$$

we deduce that $\Omega_{(B,N)|(B,1)} = 0$.

- (v) Let (B, N) be a prelog ring and M a subgroup of N . From the exact sequence in Proposition 3.13 applied to $(B, 1) \rightarrow (B, M) \rightarrow (B, N)$ and the Example 3.16.(iv), we obtain an isomorphism of B -modules

$$\Omega_{(B,N)|(B,M)} = \Omega_{(B,N)|(B,1)}.$$

- (vi) Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings with $M \rightarrow N$ surjective, then $\Omega_{(B,N)|(A,M)} = \Omega_{B|A}$. Since $N \oplus_M N \rightarrow N$ is an isomorphism, the result follows from Example 3.7.
- (vii) Let $(A, A^*) \rightarrow (B, B^*)$ be the obvious homomorphism of prelog rings where A^* and B^* are the subgroups of units of A and B respectively. Then

$$\Omega_{(B,B^*)|(A,A^*)} = \Omega_{B|A}.$$

By Example 3.16.(iv) we have $\Omega_{(A,A^*)|(A,1)} = 0$ and so, from the exact sequence in Proposition 3.13 applied to $(A, 1) \rightarrow (A, A^*) \rightarrow (B, B^*)$ we obtain

$$\Omega_{(B,B^*)|(A,A^*)} = \Omega_{(B,B^*)|(A,1)}.$$

Using now $\Omega_{(B,B^*)|(B,1)} = 0$ and the exact sequence in Proposition 3.13 applied to $(A, 1) \rightarrow (B, 1) \rightarrow (B, B^*)$ we obtain

$$\Omega_{(B,B^*)|(A,1)} = \Omega_{(B,1)|(A,1)},$$

which is isomorphic to $\Omega_{B|A}$ by Example 3.16.(vi).

3.3 Derivations

Definition 3.17. Let $(A, M) \rightarrow (B, N)$ a homomorphism of prelog rings, T a B -module. An (A, M) -derivation of (B, N) into T is a pair (D, \tilde{D}) , where D is an A -derivation $D: B \rightarrow T$ and \tilde{D} is a homomorphism of groups

$$\tilde{D}: N^{sp}/\text{Im}(M^{sp}) \longrightarrow (T, +)$$

such that $D(n) = n\tilde{D}(\bar{n})$ for all $n \in N$ (where \bar{n} is its image in $N^{sp}/\text{Im}(M^{sp})$) and we denote the image of $n \in N$ in B again by n . We denote by

$$\text{Der}_{(A,M)}((B, N), T)$$

the set of (A, M) -derivations of (B, N) into T , which is a B -module with the induced structure by the structure of B -module of T .

Proposition 3.18. With the above notation, there exists an isomorphism of B -modules

$$\text{Hom}_B(\Omega_{(B,N)|(A,M)}, T) = \text{Der}_{(A,M)}((B, N), T).$$

Proof. Let $(D, \tilde{D}) \in \text{Der}_{(A,M)}((B, N), T)$. From the diagram

$$\begin{array}{ccc}
 B \otimes_{\mathbb{Z}[N]} \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \xrightarrow{i} & B \otimes_{\mathbb{Z}} N^{sp}/\text{Im}(M^{sp}) \\
 \downarrow j & & \downarrow u \\
 \Omega_{B|A} & \xrightarrow{v} & \Omega_{(B,N)|(A,M)} \\
 & \searrow f & \swarrow h \\
 & & T
 \end{array}$$

(Note: A curved arrow g also points from $B \otimes_{\mathbb{Z}} N^{sp}/\text{Im}(M^{sp})$ to T , and a dashed arrow h points from $\Omega_{(B,N)|(A,M)}$ to T .)

where f is the homomorphism of B -modules induced by $D: B \rightarrow T$ and g is the homomorphism of B -modules induced by $\tilde{D}: N^{sp}/\text{Im}(M^{sp}) \rightarrow T$, we obtain a homomorphism $h: \Omega_{(B,N)|(A,M)} \rightarrow T$ making commutative the corresponding triangles (since $fj = gi$ and the square is a pushout).

Now, consider $h \in \text{Hom}_B(\Omega_{(B,N)|(A,M)}, T)$. We define $D \in \text{Der}_A(B, T)$ by the composition

$$B \xrightarrow{d_{B|A}} \Omega_{B|A} \xrightarrow{v} \Omega_{(B,N)|(A,M)} \xrightarrow{h} T$$

and \tilde{D} by the composition

$$N^{\text{gp}} / \text{Im}(M^{\text{gp}}) \longrightarrow B \otimes_{\mathbb{Z}} N^{\text{gp}} / \text{Im}(M^{\text{gp}}) \xrightarrow{u} \Omega_{(B,N)|(A,M)} \xrightarrow{h} T.$$

Since v and h are homomorphism of B -modules, D is a derivation. Similarly, \tilde{D} is a homomorphism of groups. We also have

$$hvi(1 \otimes dn) = hu(n \otimes \bar{n}) = n \cdot hu(1 \otimes \bar{n}) = n\tilde{D}(\bar{n})$$

and $hvj(1 \otimes dn) = D(n)$. Since $ui = vj$, we obtain $D(n) = n\tilde{D}(n)$. \square

Now, we are able to state the version for derivations of Proposition 3.10, Corollary 3.11, Proposition 3.13 and Proposition 3.14.

Corollary 3.19. (i) *We consider a diagram of log rings*

$$\begin{array}{ccc} (A_1, M_1) & \longrightarrow & (B_1, N_1) \\ & & \downarrow \\ & & (A_2, M_2) \end{array}$$

and let (B_2, N_2) be its pushout, i.e. $B_2 = A_2 \otimes_{A_1} B_1$ and $N_2 = M_2 \oplus_{M_1} N_1$. Then the canonical homomorphism

$$\text{Der}_{(A_2, M_2)}((B_2, N_2), T) \longrightarrow \text{Der}_{(A_1, M_1)}((B_1, N_1), T)$$

is an isomorphism for any B_2 -module T .

(ii) *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and $S \subset N$, $T \subset (B, \cdot)$ submonoids such that $\alpha(S) \subset T$ where $\alpha: N \rightarrow B$ is the structural homomorphism. Then for any $T^{-1}B$ -module W*

$$\text{Der}_{(A, M)}((B, N), W) = \text{Der}_{(A, M)}((T^{-1}B, S^{-1}N), W).$$

(iii) *Let $(A, M) \rightarrow (B, N) \rightarrow (C, P)$ be homomorphisms of prelog rings. There exists a natural exact sequence of C -modules for any C -module W*

$$0 \rightarrow \text{Der}_{(B, N)}((C, P), W) \rightarrow \text{Der}_{(A, M)}((C, P), W) \rightarrow \text{Der}_{(A, M)}((B, N), W).$$

(iv) *Let $(A, M) \rightarrow (C, L) \rightarrow (B, N)$ be homomorphisms of prelog rings with $C \rightarrow B$ and $L \rightarrow N$ surjective. There exists a natural exact sequence of B -modules for any B -module W*

$$0 \rightarrow \text{Der}_{(A, M)}((B, N), W) \rightarrow \text{Der}_{(A, M)}((C, L), W) \rightarrow \text{Hom}_B(N_{(B, N)|(C, L)}, W).$$

Proof. All of them are deduced from the respective properties for the module of differentials using Proposition 3.18. \square



Logarithmic André-Quillen (co)homology

In Section 4.1, we develop the theory of the logarithmic cotangent complex introduced by Gabber [19,23]. We obtain many new results that we are going to need in chapters 5 and 6, mainly the fundamental triangle of Theorem 4.3. Sections 4.2 and 4.3 continue this study with interesting results, although these are not going to be used in the following chapters.

4.1 The logarithmic cotangent complex

The forgetful functor \mathcal{U} from the category of prelog rings to the category $\text{Set} \times \text{Set}$ sending (A, M) to their subjacent sets $(\mathcal{U}A, \mathcal{U}M)$ has a left adjoint defined as

$$F(X, Y) := (\mathbb{Z}[X, Y], \mathbb{N}^Y),$$

as in Example 3.16.(i).

The category of simplicial prelog rings (i.e. simplicial objects in the category of prelog rings) has a model structure where (trivial) fibrations are those maps which go to (trivial) fibrations by the functor $s\mathcal{U}$ (the simplicial extension of the above functor \mathcal{U}). With this structure, (trivial) fibrations are those maps $(A_*, M_*) \rightarrow (B_*, N_*)$ such that $A_* \rightarrow B_*$ and $M_* \rightarrow N_*$ are (trivial) fibrations in the usual model structure of the categories of simplicial commutative rings and simplicial commutative monoids. If (A, M) is a (constant) prelog ring, and $(A, M) \xrightarrow{\varphi} (B_*, N_*)$ is a homomorphism of simplicial prelog rings with the property that for each n , (B_n, N_n) is a free (A, M) -prelog algebra, $B_n = A[X_n, Y_n]$, $N_n = M \oplus \mathbb{N}^{Y_n}$ for some sets X_n, Y_n , and all degeneracies $B_n = A[X_n, Y_n] \rightarrow B_{n+1} = A[X_{n+1}, Y_{n+1}]$, $N_n \rightarrow N_{n+1}$ send the basis X_n to X_{n+1} and Y_n to Y_{n+1} , then φ is a cofibration called *free cofibration* (arbitrary cofibrations are retracts of these ones). Details can be seen in [21, §4] or more precisely for prelog rings in [23] and in [3, §5].

Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings. Choose a factorization $(A, M) \rightarrow (F, R) \rightarrow (B, N)$ as a cofibration followed by a trivial fibration (in the category of simplicial prelog rings). We have a well defined object up to homotopy in the category of simplicial B -modules

$$\mathbb{L}_{(B,N)|(A,M)} := \Omega_{(F,R)|(A,M)} \otimes_F B,$$

that we call *logarithmic cotangent complex*.

Definition 4.1. *If W is a B -module, we define for each $n \geq 0$,*

$$\begin{aligned} H_n((A, M), (B, N), W) &:= H_n(\mathbb{L}_{(B,N)|(A,M)} \otimes_B W), \\ H^n((A, M), (B, N), W) &:= H^n(\mathrm{Hom}_B(\mathbb{L}_{(B,N)|(A,M)}, W)). \end{aligned}$$

Example 4.2. *Let $A \rightarrow B$ be a homomorphism of rings, $(A, \{1\}) \rightarrow (B, \{1\})$ the induced map on prelog rings. If $A \rightarrow F \rightarrow B$ is a factorization cofibration - trivial fibration in the category of simplicial rings, so is $(A, \{1\}) \rightarrow (F, \{1\}) \rightarrow (B, \{1\})$ in the category of simplicial prelog rings. Therefore $H_n((A, \{1\}), (B, \{1\}), W) = H_n(A, B, W) = H_n(\mathbb{L}_{B|A} \otimes_B W)$ are the usual André-Quillen homology modules (see [2] and [20]) by Example 3.16.(iii). Similarly for cohomology.*

More generally, if $A \rightarrow B$ is a ring homomorphism where (A, M) is a prelog ring, then

$$H_n((A, M), (B, M), W) = H_n(A, B, W)$$

and similarly for cohomology. The argument is the same.

Theorem 4.3 (Fundamental triangle). *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and let W be a B -module. There exists a natural distinguished triangle in the derived category of B -modules*

$$\mathbb{L}_{\mathbb{Z}[N]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[N]} W \longrightarrow (\mathbb{L}_{B|A} \otimes_B W) \oplus (X_{N|M} \otimes_{\mathbb{Z}} W) \longrightarrow \mathbb{L}_{(B,N)|(A,M)} \otimes_B W \longrightarrow$$

where $X_{N|M}$ is a complex appearing in a distinguished triangle

$$M^{gp} \longrightarrow N^{gp} \longrightarrow X_{N|M} \longrightarrow \dots$$

We have then a natural exact sequence (that we will call fundamental exact sequence)

$$\begin{aligned} \dots \rightarrow H_n(\mathbb{Z}[M], \mathbb{Z}[N], W) &\rightarrow H_n(A, B, W) \oplus H_n(X_{N|M} \otimes_{\mathbb{Z}} W) \rightarrow H_n((A, M), (B, N), W) \rightarrow \\ &\rightarrow H_{n-1}(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow \dots \rightarrow H_0((A, M), (B, N), W) \rightarrow 0, \end{aligned}$$

where

- $H_n(X_{N|M} \otimes_{\mathbb{Z}} W) = 0$ when $n \geq 3$,
- $H_2(X_{N|M} \otimes_{\mathbb{Z}} W) = \text{Tor}_1^{\mathbb{Z}}(\ker(M^{sp} \rightarrow N^{sp}), W)$,
- $H_1(X_{N|M} \otimes_{\mathbb{Z}} W)$ is a B -module appearing in a (non canonically split) exact sequence

$$0 \rightarrow \ker(M^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} W \rightarrow H_1(X_{N|M} \otimes_{\mathbb{Z}} W) \rightarrow \text{Tor}_1^{\mathbb{Z}}(N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp}), W) \rightarrow 0,$$

- $H_0(X_{N|M} \otimes_{\mathbb{Z}} W) = (N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp})) \otimes_{\mathbb{Z}} W$.

We have also a natural exact sequence

$$0 \rightarrow H^0((A, M), (B, N), W) \longrightarrow \dots \longrightarrow H^{n-1}(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow \\ \rightarrow H^n((A, M), (B, N), W) \rightarrow H^n(A, B, W) \oplus H^n(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W)) \rightarrow H^n(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow \dots$$

where

- $H^0(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W)) = \text{Hom}_{\mathbb{Z}}(N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp}), W)$,
- $H^1(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W))$ appears in a (non canonically split) exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(N^{sp}/\text{Im}(M^{sp}), W) \rightarrow H^1(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W)) \rightarrow \text{Hom}_{\mathbb{Z}}(\ker(M^{sp} \rightarrow N^{sp}), W) \rightarrow 0,$$

- $H^2(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W)) = \text{Ext}_{\mathbb{Z}}^1(\ker(M^{sp} \rightarrow N^{sp}), W)$,
- $H^n(\text{Hom}_{\mathbb{Z}}(X_{N|M}, W)) = 0$ when $n \geq 3$.

Proof. By definition of the logarithmic module of differentials (Definition 3.9), we have a pushout for each $n \geq 0$,

$$\begin{array}{ccc} \Omega_{\mathbb{Z}[R_n]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[R_n]} B & \longrightarrow & B \otimes_{\mathbb{Z}} R_n^{sp}/M^{sp} \\ \downarrow \alpha_n & & \downarrow \gamma_n \\ \Omega_{F_n|A} \otimes_{F_n} B & \longrightarrow & \Omega_{(F_n, R_n)|(A, M)} \otimes_{F_n} B \end{array}$$

where $(A, M) \rightarrow (F_*, R_*) \rightarrow (B, N)$ is a factorization with $(A, M) \rightarrow (F_*, R_*)$ a free cofibration and $(F_*, R_*) \rightarrow (B, N)$ a trivial fibration. Since $A \rightarrow F_* \rightarrow B$ is a factorization cofibration - trivial fibration in the category of simplicial rings, $\Omega_{F_*|A} \otimes_{F_*} B = \mathbb{L}_{B|A}$ is the André-Quillen cotangent complex of the A -algebra B .

But $M \rightarrow R_n \rightarrow N$ is also a cofibration - trivial fibration in the category of simplicial monoids, and so by [10, I.2.2.3] and [20, Proposition 2.5 and (2.6)] $\mathbb{Z}[M] \rightarrow \mathbb{Z}[R_*] \rightarrow \mathbb{Z}[N]$ is a cofibration - trivial fibration in the category of simplicial rings.

The maps α_n are injective and split since if $R_n = M \oplus \mathbb{N}^{Y_n}$ and $F_n = A[X_n, Y_n]$, then they are induced by the canonical injections

$$\Omega_{\mathbb{Z}[R_n]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[R_n]} B = B^{Y_n} \hookrightarrow B^{X_n \cup Y_n} = \Omega_{F_n|A} \otimes_{F_n} B.$$

So we have a distinguished triangle

$$\Omega_{\mathbb{Z}[R]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[R]} W \rightarrow (\Omega_{F|A} \otimes_F W) \oplus (R^{\text{gp}}/M^{\text{gp}} \otimes_{\mathbb{Z}} W) \rightarrow \Omega_{(F,R)|(A,M)} \otimes_F W \rightarrow$$

that is

$$\mathbb{L}_{\mathbb{Z}[N]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[N]} W \rightarrow (\mathbb{L}_{B|A} \otimes_B W) \oplus (R^{\text{gp}}/M^{\text{gp}} \otimes_{\mathbb{Z}} W) \rightarrow \mathbb{L}_{(B,N)|(A,M)} \otimes_B W \rightarrow .$$

It remains to compute $H_*(W \otimes_{\mathbb{Z}} R^{\text{gp}}/M^{\text{gp}})$ and $H^*(\text{Hom}_{\mathbb{Z}}(R^{\text{gp}}/M^{\text{gp}}, W))$. We have an exact sequence, split in each degree

$$0 \longrightarrow M^{\text{gp}} \longrightarrow R^{\text{gp}} \longrightarrow R^{\text{gp}}/M^{\text{gp}} \longrightarrow 0.$$

We also have $H_n(M^{\text{gp}}) = 0$ for $n > 0$ and $H_0(M^{\text{gp}}) = M^{\text{gp}}$, since M^{gp} , as a complex, is concentrated in degree 0, and also $H_n(R^{\text{gp}}) = H_n(N^{\text{gp}}) = 0$ for $n > 0$ and $H_0(R^{\text{gp}}) = H_0(N^{\text{gp}}) = N^{\text{gp}}$ by [19, Appendix A: Theorem A.5 and Remark A.6]. So we have $H_n(R^{\text{gp}}/M^{\text{gp}}) = 0$ for $n \geq 2$ and an exact sequence

$$0 \longrightarrow H_1(R^{\text{gp}}/M^{\text{gp}}) \longrightarrow M^{\text{gp}} \longrightarrow N^{\text{gp}} \longrightarrow H_0(R^{\text{gp}}/M^{\text{gp}}) \longrightarrow 0.$$

Therefore we obtain $H_1(R^{\text{gp}}/M^{\text{gp}}) = \ker(M^{\text{gp}} \rightarrow N^{\text{gp}})$ and $H_0(R^{\text{gp}}/M^{\text{gp}}) = N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}})$.

The result now follows from the universal coefficient exact sequences

$$\begin{aligned} 0 \longrightarrow W \otimes_{\mathbb{Z}} H_n(R^{\text{gp}}/M^{\text{gp}}) \longrightarrow H_n(W \otimes_{\mathbb{Z}} R^{\text{gp}}/M^{\text{gp}}) \longrightarrow \\ \longrightarrow \text{Tor}_1^{\mathbb{Z}}(W, H_{n-1}(R^{\text{gp}}/M^{\text{gp}})) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(R^{\text{gp}}/M^{\text{gp}}), W) \longrightarrow H^n(\text{Hom}_{\mathbb{Z}}(R^{\text{gp}}/M^{\text{gp}}, W)) \longrightarrow \\ \longrightarrow \text{Hom}_{\mathbb{Z}}(H_n(R^{\text{gp}}/M^{\text{gp}}), W) \longrightarrow 0. \end{aligned}$$

Finally, the exact sequence $0 \rightarrow M^{\text{gp}} \rightarrow R^{\text{gp}} \rightarrow R^{\text{gp}}/M^{\text{gp}} \rightarrow 0$ and the fact that R^{gp} is quasi-isomorphic to N^{gp} gives the desired triangle

$$M^{\text{gp}} \rightarrow N^{\text{gp}} \rightarrow R^{\text{gp}}/M^{\text{gp}} \rightarrow .$$

□

Proposition 4.4. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism essentially of finite type of Noetherian prelog rings (see Definition 2.4), let C be a Noetherian B -algebra and let W be a C -module of finite type. Then the modules $H_n((A, M), (B, N), W)$ and $H^n((A, M), (B, N), W)$ are C -modules of finite type.*

Proof. This proposition is a consequence of its analogue result for André-Quillen (co)homology [2, Proposition 4.55] and Theorem 4.3. \square

Proposition 4.5. *Let $n \geq 0$ be an integer. In the situation of Proposition 4.4, the following are equivalent:*

- (i) $H_n((A, M), (B, N), W) = 0$ for all C -modules W ,
- (ii) $H^n((A, M), (B, N), W) = 0$ for all C -modules W ,
- (iii) $H_n((A, M), (B, N), C/\mathfrak{n}) = 0$ for all maximal ideals \mathfrak{n} of C ,
- (iv) $H^n((A, M), (B, N), C/\mathfrak{n}) = 0$ for all maximal ideals \mathfrak{n} of C .

Proof. Similar to that of [2, Proposition 4.57]. \square

Proposition 4.6. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and let W be a B -module. Then*

- (i) $H_0((A, M), (B, N), W) = \Omega_{(B,N)|(A,M)} \otimes_B W$.
- (ii) $H^0((A, M), (B, N), W) = \text{Der}_{(A,M)}((B, N), B)$.

Moreover, if $A \rightarrow B$ and $M \rightarrow N$ are surjective, then

- (iii) $H_1((A, M), (B, N), W) = N_{(B,N)|(A,M)} \otimes_B W$.
- (iv) $H^1((A, M), (B, N), W) = \text{Hom}_B(N_{(B,N)|(A,M)}, W)$.

Proof. (i) We have $H_0(\mathbb{Z}[M], \mathbb{Z}[N], W) = \Omega_{\mathbb{Z}[N]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[N]} W$ and $H_0(A, B, W) = \Omega_{B|A} \otimes_B W$ by [2, Proposition 6.3]. Therefore the result follows from Theorem 4.3 and the definition of the logarithmic module of differentials.

(iii) By [2, propositions 6.1 and 6.3], we know that $H_0(\mathbb{Z}[M], \mathbb{Z}[N], W) = 0$, $H_1(\mathbb{Z}[M], \mathbb{Z}[N], W) = W \otimes_{\mathbb{Z}[N]} J/J^2$ where $J = \ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[N])$ and also that $H_1(A, B, W) = W \otimes I/I^2$ where $I = \ker(A \rightarrow B)$. With the notation of Theorem 4.3, $H_1(X_{N|M} \otimes_{\mathbb{Z}} W) = W \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}})$. So again we obtain the isomorphism by definition of the conormal module.

(ii) and (iv) are similar. \square

Proposition 4.7. *Let (A, M) be a prelog ring and (B, N) a free (A, M) -prelog algebra (Example 3.16.(i)). Then*

$$H_n((A, M), (B, N), W) = 0 = H^n((A, M), (B, N), W)$$

for all $n > 0$ and all B -modules W .

Proof. $(A, M) \rightarrow (B, N) \rightarrow (B, N)$ is a cofibration - trivial fibration. \square

Proposition 4.8 (Base Change). *Let $(A, M) \rightarrow (B, N)$ and $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings, let $(D, Q) = (B \otimes_A C, N \oplus_M P)$, and let W be a $B \otimes_A C$ -module. Assume that the following conditions hold:*

(i) $\text{Tor}_i^A(B, C) = 0$ for all $i > 0$.

(ii) $\text{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z}[N], \mathbb{Z}[P]) = 0$ for all $i > 0$.

(iii) *The canonical homomorphism $M^{\text{gp}} \rightarrow N^{\text{gp}} \oplus P^{\text{gp}}$ is injective (e.g. when $M^{\text{gp}} \rightarrow N^{\text{gp}}$ or $M^{\text{gp}} \rightarrow P^{\text{gp}}$ is injective).*

Then the canonical homomorphisms

$$\begin{aligned} H_n((A, M), (B, N), W) &\longrightarrow H_n((C, P), (D, Q), W), \\ H^n((C, P), (D, Q), W) &\longrightarrow H^n((A, M), (B, N), W) \end{aligned}$$

are isomorphisms for all $n \geq 0$.

Proof. The canonical homomorphisms

$$\begin{aligned} H_n(A, B, W) &\longrightarrow H_n(C, D, W), \\ H^n(C, D, W) &\longrightarrow H^n(A, B, W), \\ H_n(\mathbb{Z}[M], \mathbb{Z}[N], W) &\longrightarrow H_n(\mathbb{Z}[P], \mathbb{Z}[Q], W), \\ H^n(\mathbb{Z}[P], \mathbb{Z}[Q], W) &\longrightarrow H^n(\mathbb{Z}[M], \mathbb{Z}[N], W) \end{aligned}$$

are isomorphisms for all $n \geq 0$ by [2, Proposition 4.54].

Since $(-)^{\text{gp}}$ is left adjoint, we have a pushout

$$\begin{array}{ccc} M^{\text{gp}} & \xrightarrow{u} & N^{\text{gp}} \\ \downarrow j & & \downarrow v \\ P^{\text{gp}} & \xrightarrow{w} & Q^{\text{gp}} \end{array}$$

Since $u \oplus j: M^{\text{gp}} \rightarrow N^{\text{gp}} \oplus P^{\text{gp}}$ is injective, we have isomorphisms of Abelian groups

$$\begin{aligned}\ker(u) &= \ker(w), \\ \text{coker}(u) &= \text{coker}(w).\end{aligned}$$

The result then follows from the fundamental exact sequences (Theorem 4.3). \square

Proposition 4.9 (Jacobi-Zariski exact sequence). *Let $(A, M) \rightarrow (B, N) \rightarrow (C, L)$ be homomorphisms of prelog rings and W a C -module. There exist natural exact sequences*

$$\begin{aligned}\cdots &\rightarrow H_n((A, M), (B, N), W) \rightarrow H_n((A, M), (C, L), W) \rightarrow H_n((B, N), (C, L), W) \rightarrow \\ \cdots &\rightarrow H_{n-1}((A, M), (B, N), W) \rightarrow \cdots \rightarrow H_0((B, N), (C, L), W) \rightarrow 0\end{aligned}$$

and

$$\begin{aligned}0 &\rightarrow H^0((B, N), (C, L), W) \rightarrow \cdots \rightarrow H^{n-1}((A, M), (C, L), W) \rightarrow \\ &\rightarrow H^n((B, N), (C, L), W) \rightarrow H^n((A, M), (C, L), W) \rightarrow H^n((A, M), (B, N), W) \rightarrow \cdots.\end{aligned}$$

Proof. See [19, Theorem 8.18]. \square

Proposition 4.10 (Localization). *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings W a B -module. Let $S \subset N$, $T \subset (B, \cdot)$ be submonoids such that $\alpha(S) \subset T$ where $\alpha: N \rightarrow B$ is the structural homomorphism of (B, N) . Then, for all $n \geq 0$:*

- (i) $H_n((B, N), (T^{-1}B, S^{-1}N), T^{-1}W) = 0$,
 $H^n((B, N), (T^{-1}B, S^{-1}N), T^{-1}W) = 0$.
- (ii) $H_n((A, M), (B, N), T^{-1}W) = H_n((A, M), (T^{-1}B, S^{-1}N), T^{-1}W)$,
 $H^n((A, M), (B, N), T^{-1}W) = H^n((A, M), (T^{-1}B, S^{-1}N), T^{-1}W)$.
- (iii) $H_n((A, M), (B, N), T^{-1}W) = T^{-1}H_n((A, M), (B, N), W)$.

Proof. (i) It follows from Base Change (Proposition 4.8) applied to the pushout

$$\begin{array}{ccc}(B, N) & \longrightarrow & (T^{-1}B, S^{-1}N) \\ \downarrow & & \downarrow \\ (T^{-1}B, S^{-1}N) & \longrightarrow & (T^{-1}B, S^{-1}N)\end{array}$$

(ii) It follows from (i) and the Jacobi-Zariski exact sequence (Proposition 4.9).

(iii) Property (iii) follows from the universal coefficient spectral sequence

$$\begin{aligned} E_{p,q}^2 &= \mathrm{Tor}_p^B(H_q(\mathbb{L}_{(B,N)|(A,M)} \otimes_B W), T^{-1}B) \\ &= \mathrm{Tor}_p^B(H_q((A, M), (B, N), W), T^{-1}B) \Rightarrow H_n((A, M), (B, N), T^{-1}W). \end{aligned}$$

□

The following two results will not be used in the sequel. We include them for possible future references.

Proposition 4.11. *Let $(R, Q) \rightarrow (A, M)$, $(A, M) \rightarrow (B, N)$, $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings, $D = B \otimes_A C$ and $L = N \oplus_M P$. Assume that the following conditions (the same that we considered in Proposition 4.8) hold:*

(i) $\mathrm{Tor}_i^A(B, C) = 0$ for all $i > 0$,

(ii) $\mathrm{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z}[N], \mathbb{Z}[P]) = 0$ for all $i > 0$,

(iii) *The canonical homomorphism $M^{sp} \rightarrow N^{sp} \oplus P^{sp}$ is injective (e.g. when $M^{sp} \rightarrow N^{sp}$ or $M^{sp} \rightarrow P^{sp}$ is injective).*

Then, for any D -module W we have a natural exact sequence

$$\begin{aligned} \cdots \rightarrow H_n((R, Q), (A, M), W) &\rightarrow H_n((R, Q), (B, N), W) \oplus H_n((R, Q), (C, P), W) \rightarrow \\ &\rightarrow H_n((R, Q), (D, L), W) \rightarrow H_{n-1}((R, Q), (A, M), W) \rightarrow \cdots \\ &\cdots \rightarrow H_0((R, Q), (D, L), W) \rightarrow 0. \end{aligned}$$

Proof. Applying the Jacobi-Zariski exact sequence to the rows of the commutative diagram

$$\begin{array}{ccccc} (R, Q) & \longrightarrow & (A, M) & \longrightarrow & (B, N) \\ & & \downarrow & & \downarrow \\ \parallel & & & & \\ (R, Q) & \longrightarrow & (C, P) & \longrightarrow & (D, L) \end{array}$$

and noting that the right square satisfies the hypotheses of Proposition 4.8, we obtain the result as in [2, Proposition 5.22]. □

Similarly, we also have an analogue to [2, Proposition 5.21]:

Proposition 4.12. *Let $(A, M) \rightarrow (B, N)$, $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings, $D = B \otimes_A C$ and $L = N \oplus_M P$. Consider also a homomorphism of prelog rings $(D, L) \rightarrow (R, Q)$. Assume that the conditions of Proposition 4.8 hold again, as in the previous proposition. Then, for any R -module W we have a natural exact sequence*

$$\begin{aligned} \cdots \rightarrow H_n((A, M), (R, Q), W) \rightarrow H_n((B, N), (R, Q), W) \oplus H_n((C, P), (R, Q), W) \rightarrow \\ \rightarrow H_n((D, L), (R, Q), W) \rightarrow H_{n-1}((A, M), (R, Q), W) \rightarrow \cdots \\ \cdots \rightarrow H_0((D, L), (R, Q), W) \rightarrow 0. \end{aligned}$$

Remark 4.13. Let $H \xrightarrow{i} X \xrightarrow{p} N$ be homomorphisms of simplicial monoids with H, N constants, H a group and pi injective. We suppose that p is a trivial fibration in the category of the simplicial monoids. Since N is a Kan complex (because it is constant) and p is a fibration, X is a Kan complex too (the composition of fibrations is a fibration). We consider the induced actions of H in X and in N and also the quotients X/H , N/H . We will assume that H acts freely on X . Then $X \xrightarrow{\pi} X/H$ is a fibration [17, Lemma 18.2] with fibre H , and then for any basepoint we have an exact sequence

$$\cdots \rightarrow \pi_n(H) \rightarrow \pi_n(X) \rightarrow \pi_n(X/H) \rightarrow \pi_{n-1}(H) \rightarrow \cdots \rightarrow \pi_0(X/H) \rightarrow 1.$$

We have $\pi_n(H) = \pi_n(X) = 1$ for all $n > 0$, and so we obtain $\pi_n(X/H) = 1$ for all $n > 1$ and an exact sequence

$$1 \longrightarrow \pi_1(X/H) \longrightarrow \pi_0(H) \longrightarrow \pi_0(X) \longrightarrow \pi_0(X/H) \longrightarrow 1$$

which takes the form

$$1 \longrightarrow \pi_1(X/H) \longrightarrow H \xrightarrow{pi} N \longrightarrow \pi_0(X/H) \longrightarrow 1.$$

Therefore we deduce

$$\pi_n(X/H) = \begin{cases} N/H & \text{if } n = 0, \\ 1 & \text{if } n > 0. \end{cases}$$

That is, $X/H \rightarrow N/H$ is a weak equivalence, and therefore by [10, I.2.2.3] $\mathbb{Z}[X/H] \rightarrow \mathbb{Z}[N/H]$ is a weak equivalence.

Lemma 4.14. *Let H be an Abelian group and W a $\mathbb{Z}[H]$ -module. Then*

$$\begin{aligned} H_n(\mathbb{Z}, \mathbb{Z}[H], W) &= \mathrm{Tor}_n^{\mathbb{Z}}(H, W), \\ H^n(\mathbb{Z}, \mathbb{Z}[H], W) &= \mathrm{Ext}_n^{\mathbb{Z}}(H, W) \end{aligned}$$

for all $n \geq 0$. In particular, these modules vanish for all $n \geq 2$.

Proof. Let $X \rightarrow H$ be a simplicial resolution of the \mathbb{Z} -module H , that is, $X \rightarrow H$ is surjective, X is a free simplicial \mathbb{Z} -module with $\pi_n(X) = \pi_n(H)$ for all n . Then $\mathbb{Z}[X]$ is a free simplicial \mathbb{Z} -algebra, $\mathbb{Z}[X] \rightarrow \mathbb{Z}[H]$ is surjective and a weak equivalence by [10, I.2.2.3]. So $\mathbb{Z} \rightarrow \mathbb{Z}[X] \rightarrow \mathbb{Z}[H]$ is a free cofibration - trivial fibration factorization of simplicial rings [20, Proposition 2.5]. Therefore $H_n(\mathbb{Z}, \mathbb{Z}[H], W) = H_n(\Omega_{\mathbb{Z}[X]|\mathbb{Z}} \otimes_{\mathbb{Z}[X]} W)$, and similarly for cohomology.

So it suffices to show that $\Omega_{\mathbb{Z}[X]|\mathbb{Z}} = X \otimes_{\mathbb{Z}} \mathbb{Z}[X]$. Each X_n is a free \mathbb{Z} -module. Since Ω , \otimes and $\mathbb{Z}[-]$ commute with colimits, we can assume that X_n is a free \mathbb{Z} -module of finite rank and by Künneth formula for the module of differentials [2, Lemme 1.14] we can assume $X_n = \mathbb{Z}$:

$$\begin{aligned} \Omega_{\mathbb{Z}[H_1 \oplus H_2]|\mathbb{Z}} &= \Omega_{\mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2]|\mathbb{Z}} \\ &= \Omega_{\mathbb{Z}[H_1]|\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \oplus \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \Omega_{\mathbb{Z}[H_2]|\mathbb{Z}} \\ &= H_1 \otimes_{\mathbb{Z}} \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \oplus \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} H_2 \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \\ &= (H_1 \oplus H_2) \otimes_{\mathbb{Z}} (\mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2]) \\ &= (H_1 \oplus H_2) \otimes_{\mathbb{Z}} \mathbb{Z}[H_1 \oplus H_2]. \end{aligned}$$

So we only have to prove $\Omega_{\mathbb{Z}[H]|\mathbb{Z}} = H \otimes_{\mathbb{Z}} \mathbb{Z}[H]$ when $H = \mathbb{Z}$. We have $\mathbb{Z}[H] = \mathbb{Z}[t, \frac{1}{t}] = S^{-1}\mathbb{Z}[t]$ with $S = \{1, t, t^2, \dots\}$. Then

$$\begin{aligned} \Omega_{\mathbb{Z}[H]|\mathbb{Z}} &= S^{-1}\Omega_{\mathbb{Z}[t]|\mathbb{Z}} = S^{-1}\mathbb{Z}[t] = \mathbb{Z} \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z}[t] \\ &= H \otimes_{\mathbb{Z}} \mathbb{Z}[H]. \end{aligned} \quad \square$$

Remark 4.15. Now we will give here a more explicit proof of Lemma 4.14 that can be useful for similar computations to those made in the proofs of some results in Chapter 6.

For the case $n = 0$ we have to show that $\Omega_{\mathbb{Z}[H]|\mathbb{Z}} = H \otimes_{\mathbb{Z}} \mathbb{Z}[H]$. If $H = C_r$ is a finite cyclic group (of order r), then $\mathbb{Z}[H] = \mathbb{Z}[x]/(x^r - 1)$ and the result follows from an immediate computation. If $H = \mathbb{Z}$ is an infinite cyclic group, $\mathbb{Z}[H] = \mathbb{Z}[x, 1/x] = S^{-1}\mathbb{Z}[x]$, where $S = \{1, x, x^2, \dots\}$, and then $\Omega_{\mathbb{Z}[H]|\mathbb{Z}} = S^{-1}\Omega_{\mathbb{Z}[x]|\mathbb{Z}} = S^{-1}\mathbb{Z}[x] = \mathbb{Z} \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z}[x] = H \otimes_{\mathbb{Z}} \mathbb{Z}[H]$.

If H_1 and H_2 are two groups such that $\Omega_{\mathbb{Z}[H_i]|\mathbb{Z}} = H_i \otimes_{\mathbb{Z}} \mathbb{Z}[H_i]$ for $i = 1, 2$, then by the Künneth formula for the module of differentials [2, Lemme 1.14] and Remark 1.27 we have

$$\begin{aligned} \Omega_{\mathbb{Z}[H_1 \oplus H_2]|\mathbb{Z}} &= \Omega_{\mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2]|\mathbb{Z}} \\ &= \Omega_{\mathbb{Z}[H_1]|\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \oplus \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \Omega_{\mathbb{Z}[H_2]|\mathbb{Z}} \\ &= H_1 \otimes_{\mathbb{Z}} \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \oplus \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} H_2 \otimes_{\mathbb{Z}} \mathbb{Z}[H_2] \\ &= (H_1 \oplus H_2) \otimes_{\mathbb{Z}} (\mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2]) \\ &= (H_1 \oplus H_2) \otimes_{\mathbb{Z}} \mathbb{Z}[H_1 \oplus H_2]. \end{aligned}$$

So the result holds for Abelian groups of finite type.

Finally write $H = \varinjlim_i H_i$ with H_i Abelian groups of finite type. Then

$$\begin{aligned} \Omega_{\mathbb{Z}[H]|\mathbb{Z}} &= \Omega_{\mathbb{Z}[\varinjlim_i H_i]|\mathbb{Z}} = \Omega_{\varinjlim_i \mathbb{Z}[H_i]|\mathbb{Z}} = \varinjlim_i \Omega_{\mathbb{Z}[H_i]|\mathbb{Z}} \\ &= \varinjlim_i (H_i \otimes_{\mathbb{Z}} \mathbb{Z}[H_i]) = \varinjlim_i H_i \otimes_{\mathbb{Z}} \varinjlim_i \mathbb{Z}[H_i] \\ &= H \otimes_{\mathbb{Z}} \mathbb{Z}[H], \end{aligned}$$

given that $\mathbb{Z}[-]$, being a left adjoint, commutes with colimits.

Now consider the case $n \geq 2$. If H is cyclic, $\mathbb{Z}[H] = \mathbb{Z}[x]/(x^r - 1)$ or $\mathbb{Z}[H] = \mathbb{Z}[x, 1/x]$, and in both cases it is clear that $H_n(\mathbb{Z}, \mathbb{Z}[H], W) = 0$ for all $n \geq 0$ (it follows for instance from [2, Théorème 6.19] and [2, Corollaire 5.27]). Since $\mathbb{Z}[H_1 \oplus H_2] = \mathbb{Z}[H_1] \otimes_{\mathbb{Z}} \mathbb{Z}[H_2]$, the result also holds for finitely generated Abelian groups by [2, Corollaire 5.23], and since $\mathbb{Z}[-]$ preserves direct limits, the result in general follows from [2, Proposition 3.35].

For the case $n = 1$. Let $H = C_r$ be a finite cyclic group, so that $\mathbb{Z}[H] = \mathbb{Z}[x]/(x^r - 1)$. The the Jacobi-Zariski exact sequence associated to

$$\mathbb{Z} \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]/(x^r - 1)$$

can be written as

$$0 \longrightarrow H_1(\mathbb{Z}, \mathbb{Z}[H], W) \longrightarrow (x^r - 1) \otimes_{\mathbb{Z}[x]} W \xrightarrow{\delta} \mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} W = W \longrightarrow \dots,$$

where the map δ sends $(x^r - 1) \otimes \omega$ to $r\omega$. On the other hand, the exact sequence of Abelian groups $0 \rightarrow r\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow H \rightarrow 0$ induces an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H, W) \longrightarrow r\mathbb{Z} \otimes_{\mathbb{Z}} W \xrightarrow{\delta'} W \longrightarrow \dots$$

where $\delta'(r \otimes w) = rw$. The isomorphism

$$r\mathbb{Z} \otimes_{\mathbb{Z}} W = r\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} W = r\mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} W \xrightarrow{\varphi} (x^r - 1) \otimes_{\mathbb{Z}[x]} W,$$

sending $r \otimes \omega$ into $(x^r - 1) \otimes \omega$ satisfies $\delta\varphi = \delta'$ and so gives

$$H_1(\mathbb{Z}, \mathbb{Z}[H], W) = \mathrm{Tor}_1^{\mathbb{Z}}(H, W).$$

For $H = \mathbb{Z}$ the infinite cyclic group, $H_1(\mathbb{Z}, \mathbb{Z}[H], W) = 0 = \mathrm{Tor}_1^{\mathbb{Z}}(H, W)$. Using again that $\mathbb{Z}[-]$ and Tor preserve coproducts and filtered colimits, the result follows from [2, Lemme 3.35].

Lemma 4.16. *Let $(A, H) \rightarrow (B, G)$ be a homomorphism of prelog rings, where H and G are groups. Then $H_n((A, H), (B, G), W) = H_n(A, B, W)$ for all $n \geq 0$ and all B -modules W , and similarly for cohomology.*

Proof. We have a Jacobi-Zariski exact sequence

$$H_{n+1}((B, H), (B, G), W) \longrightarrow H_n((A, H), (B, H), W)$$

$$\longrightarrow H_n((A, H), (B, G), W) \longrightarrow H_n((B, H), (B, G), W) \longrightarrow \dots$$

Since $H_n((A, H), (B, H), W) = H_n(A, B, W)$ for all $n \geq 0$ by Example 4.2, it suffices to show that $H_n((B, H), (B, G), W) = 0$ for all $n \geq 0$. From the Jacobi-Zariski sequence associated to $(B, \{1\}) \rightarrow (B, H) \rightarrow (B, G)$, we see that it suffices to show that $H_n((B, \{1\}), (B, H), W) = 0$ for any group H and all $n \geq 0$.

Since $H_n(B, B, W) = 0$ for all $n \geq 0$, the fundamental exact sequence gives isomorphisms

$$H_n((B, \{1\}), (B, H), W) = H_{n-1}(\mathbb{Z}, \mathbb{Z}[H], W) = 0$$

for $n \geq 3$ by Lemma 4.14.(ii) and an exact sequence

$$\begin{aligned} 0 &\longrightarrow H_2((B, \{1\}), (B, H), W) \longrightarrow \\ &\longrightarrow H_1(\mathbb{Z}, \mathbb{Z}[H], W) \xrightarrow{\kappa_1} \mathrm{Tor}_1^{\mathbb{Z}}(W, H) \longrightarrow H_1((B, \{1\}), (B, H), W) \longrightarrow \\ &\longrightarrow H_0(\mathbb{Z}, \mathbb{Z}[H], W) \xrightarrow{\kappa_0} W \otimes_{\mathbb{Z}} H \longrightarrow H_0((B, \{1\}), (B, H), W) \longrightarrow 0. \end{aligned}$$

Furthermore, since κ_1 and κ_0 are isomorphisms by Lemma 4.14 we deduce also $H_i((B, \{1\}), (B, H), W) = 0$ for $i = 0, 1, 2$. \square

Proposition 4.17. [19, Theorem 8.20] *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings with M and N integral monoids and W a B -module. Then, for all $n \geq 2$, we have*

$$\begin{aligned} \text{(i)} \quad H_n((A, M), (B, N), W) &= H_n((A, M), (B, N)^{\mathrm{log}}, W) \\ &= H_n((A, M)^{\mathrm{log}}, (B, N)^{\mathrm{log}}, W). \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad H^n((A, M), (B, N), W) &= H^n((A, M), (B, N)^{\mathrm{log}}, W) \\ &= H^n((A, M)^{\mathrm{log}}, (B, N)^{\mathrm{log}}, W). \end{aligned}$$

Proof. We will prove (i). By the Jacobi-Zariski exact sequences, it suffices to show that for a prelog ring (A, M) with M integral, $H_n((A, M), (A, M)^{\mathrm{log}}, W) = 0$ for

any A -module W for all $n \geq 0$. Let $\alpha: M \rightarrow A$ be the structural map. Consider the diagram of pushouts

$$\begin{array}{ccccccc}
 (A, \alpha^{-1}(A^*)) & \longrightarrow & (A, \alpha^{-1}(A^*)^{\text{gp}}) & \longrightarrow & (A, \alpha^{-1}(A^*)^{\text{gp}}/\ker v) & \longrightarrow & (A, A^*) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 (A, M) & \longrightarrow & (A, M_1) & \longrightarrow & (A, M_2) & \longrightarrow & (A, M^{\text{log}})
 \end{array}$$

where $v: \alpha^{-1}(A^*)^{\text{gp}} \rightarrow A^*$ is the canonical map. Applying base change (Proposition 4.8) to the pushout

$$\begin{array}{ccc}
 (A, \alpha^{-1}(A^*)) & \longrightarrow & (A, \alpha^{-1}(A^*)^{\text{gp}}) \\
 \downarrow & & \downarrow \\
 (A, M) & \longrightarrow & (A, M_1)
 \end{array}$$

(since $\text{Tor}_n^{\mathbb{Z}[\alpha^{-1}(A^*)]}(\mathbb{Z}[M], \mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}}]) = 0$ for $n > 0$ because, as we know, $\mathbb{Z}[\alpha^{-1}(A^*)] \rightarrow \mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}}]$ being a localization is flat), we obtain

$$\begin{aligned}
 H_n((A, M), (A, M_1), W) &= H_n((A, \alpha^{-1}(A^*)), (A, \alpha^{-1}(A^*)^{\text{gp}}), W) \\
 &= 0
 \end{aligned}$$

where this last module vanishes by Proposition 4.10.(ii).

Now consider to the pushout

$$\begin{array}{ccc}
 (A, \alpha^{-1}(A^*)^{\text{gp}}) & \longrightarrow & (A, \alpha^{-1}(A^*)^{\text{gp}}/\ker v) \\
 \downarrow & & \downarrow \\
 (A, M_1) & \longrightarrow & (A, M_2)
 \end{array}$$

since M is integral, the map $\alpha^{-1}(A^*)^{\text{gp}} \rightarrow M_1$ is injective, being a localization of the injective map $\alpha^{-1}(A^*) \rightarrow M$. In particular, $\ker(v) \rightarrow M_1$ is injective. If we take a cofibration - trivial fibration factorization

$$\alpha^{-1}(A^*)^{\text{gp}} \longrightarrow X \longrightarrow M_1$$

in the category of simplicial monoids with $X_n = \alpha^{-1}(A_1^*)^{\text{gp}} \oplus \mathbb{N}^{\tilde{X}_n}$ a free $\alpha^{-1}(A^*)^{\text{gp}}$ -monoid, then the action of $\ker(v)$ on X is free, since $\alpha^{-1}(A^*)^{\text{gp}}$ is a group. Therefore from [10, I.2.2.3] and Remark 4.13:

$$\begin{aligned} \text{Tor}_n^{\mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}}]}(\mathbb{Z}[M_1], \mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}} / \ker(v)]) &= \\ &= H_n(\mathbb{Z}[X] \otimes_{\mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}}]} \mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}} / \ker(v)]) \\ &= H_n(\mathbb{Z}[X / \ker(v)]) \\ &= 0 \end{aligned}$$

for $n > 0$.

So base change (Proposition 4.8) gives an isomorphism

$$\begin{aligned} H_n((A, M_1), (A, M_2), W) &= H_n((A, \alpha^{-1}(A^*)^{\text{gp}}), (A, \alpha^{-1}(A^*)^{\text{gp}} / \ker(v)), W) \\ &= 0 \end{aligned}$$

where the last module vanishes by Lemma 4.16 above.

Finally, applying again base change to the pushout

$$\begin{array}{ccc} (A, \alpha^{-1}(A^*)^{\text{gp}} / \ker(v)) & \longrightarrow & (A, A^*) \\ \downarrow & & \downarrow \\ (A, M_2) & \longrightarrow & (A, M^{\text{log}}) \end{array}$$

(since $\alpha^{-1}(A^*)^{\text{gp}} / \ker(v) \rightarrow A^*$ is an injective homomorphism of groups, we know that the homomorphism $\mathbb{Z}[\alpha^{-1}(A^*)^{\text{gp}} / \ker(v)] \rightarrow \mathbb{Z}[A^*]$ is flat) we obtain similarly

$$\begin{aligned} H_n((A, M_2), (A, M^{\text{log}}), W) &= H_n((A, \alpha^{-1}(A^*)^{\text{gp}} / \ker(v)), (A, A^*), W) \\ &= 0 \end{aligned}$$

which vanishes again by Lemma 4.16.

By the Jacobi-Zariski exact sequence associated to the sequence $(A, M) \rightarrow (A, M_1) \rightarrow (A, M_2)$, we obtain $H_n((A, M), (A, M_2), W) = 0$, and then by the Jacobi-Zariski exact sequence associated to $(A, M) \rightarrow (A, M_2) \rightarrow (A, M^{\text{log}})$ we obtain

$$H_n((A, M), (A, M^{\text{log}}), W) = 0$$

as desired. □

4.2 A spectral sequence

Let $C \rightarrow B$ be a surjective homomorphism of rings. We have a convergent spectral sequence [20, Theorem 6.8]

$$E_{p,q}^2 = H_{p+q}(S_q \mathbb{L}_{B|C}) \Rightarrow \mathrm{Tor}_{p+q}^C(B, B),$$

which when $A \rightarrow B$ is a flat homomorphism and $C = B \otimes_A B$ takes the form

$$E_{p,q}^2 = H_p(\wedge^q \mathbb{L}_{B|A}) \Rightarrow \mathrm{HH}_{p+q}(B|A)$$

where $\mathrm{HH}_*(B|A) = \mathrm{Tor}_{p+q}^{B \otimes_A B}(B, B)$ is the Hochschild homology of the A -algebra B and S_q, \wedge^q denote symmetric and exterior powers respectively. We will give here an analogous spectral sequence for the logarithmic cotangent complex.

Let $(C, Q) \rightarrow (B, N)$ be a homomorphism of prelog rings such that $C \rightarrow B$ and $Q \rightarrow N$ are surjective. Let $(C, Q) \rightarrow (R, P) \rightarrow (B, N)$ be a free cofibration - trivial fibration factorization with $R_0 = C$ and $P_0 = Q$. Let $I = \ker(\pi : R \otimes_C B \rightarrow B)$, $J = \ker(\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N] \rightarrow \mathbb{Z}[N])$ and $T = \ker((P \oplus_Q N)^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})$.

Let $D_{(B,N)|(C,Q)}^1$ be the simplicial $R \otimes_C B$ -module defined by the pushout

$$\begin{array}{ccc} U & \xrightarrow{\beta} & B \otimes_{\mathbb{Z}} T \\ \downarrow \alpha & & \downarrow \\ I & \longrightarrow & D_{(B,N)|(C,Q)}^1 \end{array}$$

where $U = (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J$, β is the homomorphism induced by the map $J \rightarrow B \otimes_{\mathbb{Z}} T$ of the proof of Proposition 3.1 and the map $\pi : R \otimes_C B \rightarrow B$, and the homomorphism α is the obvious one.

Proposition 4.18. *In this situation, we have an exact sequence*

$$\begin{aligned} \cdots \longrightarrow H_n(U) \longrightarrow \mathrm{Tor}_n^C(B, B) \oplus W_n \longrightarrow H_n(D_{(B,N)|(C,Q)}^1) \longrightarrow \\ \longrightarrow H_{n-1}(U) \longrightarrow \cdots \longrightarrow H_1(D_{(B,N)|(C,Q)}^1) \longrightarrow 0 \end{aligned}$$

where

$$W_n = \begin{cases} \ker(Q^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \otimes_{\mathbb{Z}} B & \text{if } n = 1, \\ \mathrm{Tor}_1^{\mathbb{Z}}(\ker(Q^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}), B) & \text{if } n = 2, \\ 0 & \text{if } 1 \neq n \neq 2. \end{cases}$$

Proof. If $P = Q \oplus \mathbb{N}^Y$, $R = C[X \cup Y]$, then J is the ideal of $\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N] = \mathbb{Z}[N][Y]$ generated by Y , I is the ideal of $R \otimes_C B = B[X \cup Y]$ generated by $X \cup Y$, and the map α in the diagram is (split) injective. Thus, we have an exact sequence

$$\cdots \rightarrow H_n(U) \rightarrow H_n(I) \oplus H_n(B \otimes_{\mathbb{Z}} T) \rightarrow H_n(D_{(B,N)|(C,Q)}^1) \rightarrow H_{n-1}(U) \rightarrow \cdots .$$

Since $0 \rightarrow I \rightarrow R \otimes_C B \rightarrow B \rightarrow 0$ is exact and $H_n(R \otimes_C B) = \text{Tor}_n^C(B, B)$, we have $H_n(I) = \text{Tor}_n^C(B, B)$ if $n > 0$ and $H_0(I) = 0$ given that $I_0 = 0$. We also have $J_0 = 0$ and then $H_0(U) = 0$.

It remains to compute $H_n(B \otimes_{\mathbb{Z}} T)$. Since

$$P^{\text{gp}} \oplus_{Q^{\text{gp}}} N^{\text{gp}} = P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}),$$

we have an exact sequence

$$0 \rightarrow T \rightarrow P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow N^{\text{gp}} \rightarrow 0 \quad (4.2.1)$$

From the exact sequence

$$0 \rightarrow \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow P^{\text{gp}} \rightarrow P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow 0$$

we deduce $H_n(P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})) = 0$ for $n \geq 2$ and an exact sequence

$$\begin{aligned} 0 &\rightarrow H_1(P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow \\ &\rightarrow \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow N^{\text{gp}} \rightarrow H_0(P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow 0 \end{aligned}$$

showing that

$$H_1(P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})) = \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}),$$

$$H_0(P^{\text{gp}} / \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})) = N^{\text{gp}}.$$

Then, from the exact sequence (4.2.1) we obtain $H_n(T) = 0$ for all $n \geq 2$, $H_1(T) = \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}})$ and an exact sequence

$$0 \rightarrow H_0(T) \rightarrow N^{\text{gp}} \rightarrow N^{\text{gp}} \rightarrow 0$$

showing that $H_0(T) = 0$.

We deduce

$$H_n(B \otimes_{\mathbb{Z}} T) = \begin{cases} H_1(T) \otimes_{\mathbb{Z}} B = \ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \otimes_{\mathbb{Z}} B & \text{if } n = 1, \\ \text{Tor}_1^{\mathbb{Z}}(\ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}), B) & \text{if } n = 2, \\ 0 & \text{if } 1 \neq n \neq 2. \end{cases}$$

□

Example 4.19. We are going to see that $H_1(D_{(B,N)|(C,Q)}^1)$ is the conormal module $N_{(B,N)|(C,Q)}$ defined in Chapter 3.

Let $\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N] \rightarrow X \rightarrow R \otimes_C B$ be a cofibration - trivial fibration factorization, and consider the derived tensor product

$$\tilde{U} = X \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J = (R \otimes_C B) \overset{L}{\otimes}_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J.$$

We have an exact sequence defining Z

$$0 \longrightarrow Z \longrightarrow \tilde{U} \longrightarrow U \longrightarrow 0$$

and $\tilde{U}_0 = 0 = U_0$ since $J_0 = 0$. Therefore $Z_0 = 0$ and then $H_1(\tilde{U}) \rightarrow H_1(U)$ is surjective. So from Proposition 4.18 we have an exact sequence

$$H_1(\tilde{U}) \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \oplus (\ker(Q^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} B) \longrightarrow H_1(D_{(B,N)|(C,Q)}^1) \longrightarrow 0$$

where $\mathfrak{a} = \ker(C \rightarrow B)$.

Let us compute $H_1(\tilde{U})$. By [21, II, Theorem 6] we have a spectral sequence

$$E_{p,q}^2 = \text{Tor}_p^{\text{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} (\text{Tor}_*^C(B, B), H_*(J))_q \Rightarrow H_{p+q}(\tilde{U}).$$

Moreover, from the Tor long exact sequence associated to the exact sequence

$$0 \longrightarrow H_*(J) \longrightarrow \text{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N]) \longrightarrow \mathbb{Z}[N] \longrightarrow 0,$$

we obtain

$$E_{p,q}^2 = \text{Tor}_{p+1}^{\text{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} (\text{Tor}_*^C(B, B), \mathbb{Z}[N])_q$$

for $p > 0$ and an exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Tor}_1^{\text{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} (\text{Tor}_*^C(B, B), \mathbb{Z}[N])_q &\longrightarrow E_{0,q}^2 \longrightarrow \\ &\longrightarrow \text{Tor}_q^C(B, B) \longrightarrow (\text{Tor}_*^C(B, B) \otimes_{\text{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} \mathbb{Z}[N])_q \longrightarrow 0 \end{aligned}$$

for $p = 0$.

Consider now $p > 0$, then

$$\begin{aligned} E_{p,0}^2 &= \text{Tor}_{p+1}^{\text{Tor}_0^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} (\text{Tor}_0^C(B, B), \mathbb{Z}[N]) \\ &= \text{Tor}_{p+1}^{\mathbb{Z}[N]}(B, \mathbb{Z}[N]) \\ &= 0 \end{aligned}$$

In particular, $E_{1,0}^2 = 0$ and $E_{2,0}^2 = 0$. We also have $E_{-2,2}^2 = 0$, since the spectral sequence is located in the first quadrant. Therefore $E_{0,1}^3 = E_{0,1}^2$, and since $E_{0,1}^\infty = E_{0,1}^3$ (again because it is a first quadrant spectral sequence), $E_{0,1}^\infty = E_{0,1}^2$ follows too. Since $E_{1,0}^2 = 0$, we obtain $E_{1,0}^\infty = 0$ and then

$$\begin{aligned} H_1(\tilde{U}) &= E_{0,1}^\infty = E_{0,1}^2 = \mathrm{Tor}_0^{\mathrm{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])}^C(\mathrm{Tor}_*^C(B, B), H_*(J))_1 \\ &= (\mathrm{Tor}_*^C(B, B) \otimes_{\mathrm{Tor}_*^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} H_*(J))_1 \\ &= (\mathrm{Tor}_1^C(B, B) \otimes_{\mathrm{Tor}_0^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} H_0(J)) \\ &\quad \oplus (\mathrm{Tor}_0^C(B, B) \otimes_{\mathrm{Tor}_0^{\mathbb{Z}[Q]}(\mathbb{Z}[N], \mathbb{Z}[N])} H_1(J)) \\ &= (\mathfrak{a}/\mathfrak{a}^2 \otimes_{\mathbb{Z}[N]} 0) \oplus (B \otimes_{\mathbb{Z}[N]} \mathfrak{b}/\mathfrak{b}^2) \\ &= B \otimes_{\mathbb{Z}[N]} \mathfrak{b}/\mathfrak{b}^2 \end{aligned}$$

where $\mathfrak{b} = \ker(\mathbb{Z}[Q] \rightarrow \mathbb{Z}[N])$.

So we have an exact sequence

$$B \otimes_{\mathbb{Z}[N]} \mathfrak{b}/\mathfrak{b}^2 \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \oplus (\ker(Q^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} B) \longrightarrow H_1(D_{(B,N)|(C,Q)}^1) \longrightarrow 0$$

where the first map is the sum of the canonical map $B \otimes_{\mathbb{Z}[N]} \mathfrak{b}/\mathfrak{b}^2 \rightarrow \mathfrak{a}/\mathfrak{a}^2$ and the map $B \otimes_{\mathbb{Z}[N]} \mathfrak{b}/\mathfrak{b}^2 \rightarrow \ker(Q^{sp} \rightarrow N^{sp}) \otimes_{\mathbb{Z}} B$ of Proposition 3.1. By Definition 3.4, we have then

$$H_1(D_{(B,N)|(C,Q)}^1) = N_{(B,N)|(C,Q)}.$$

Note that in the case $C = B \otimes_A B$, $Q = N \oplus_M N$ where $(A, M) \rightarrow (B, N)$ is a homomorphism of prelog rings we have

$$H_1(D_{(B,N)|(C,Q)}^1) = \Omega_{(B,N)|(A,M)}.$$

Example 4.20. If $Q = N$, then $J = T = 0$ and so $D_{(B,N)|(C,Q)}^1$ is the pushout

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ I & \longrightarrow & D_{(B,N)|(C,Q)}^1 \end{array}$$

showing that $H_n(D_{(B,N)|(C,Q)}^1) = \mathrm{Tor}_n^C(B, B)$ for $n > 0$. In the case where $(A, M) \rightarrow (B, N)$ is a homomorphism of prelog rings, with $A \rightarrow B$ flat and $M = N$, taking $(C, Q) = (B \otimes_A B, N \oplus_M N)$, we have $H_n(D_{(B,N)|(C,Q)}^1) = \mathrm{HH}_n(B|A)$ for $n > 0$.

Theorem 4.21. *We have a first quadrant convergent spectral sequence*

$$E_{p,q}^2 \Rightarrow H_{p+q}(D_{(B,N)|(C,Q)}^1)$$

where $E_{p,q}^2 = 0$ for $q = 0$ and $E_{p,1}^2 = H_{p+1}(\mathbb{L}_{(B,N)|(C,Q)})$.

Proof. Consider the pushout

$$\begin{array}{ccc} (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J & \xrightarrow{\beta} & B \otimes_{\mathbb{Z}} T \\ \downarrow \alpha & & \downarrow \\ I & \longrightarrow & D_{(B,N)|(C,Q)}^1 \end{array}$$

We have already seen that α is split injective, and similarly, the map

$$(R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J^i \longrightarrow I^i$$

is split injective for all $i > 0$.

For any $i \geq 2$, let D^i be the image of the composition $I^i \hookrightarrow I \rightarrow D_{(B,N)|(C,Q)}^1$. Applying the Ker-Coker Lemma to the following diagram for $i \geq 2$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J^i & \longrightarrow & I^i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J & \longrightarrow & I \oplus (B \otimes_{\mathbb{Z}} T) & \xrightarrow{\beta} & 0 \\ & & & & \downarrow & & \\ & & & & I^i / (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J^i & \longrightarrow & 0 \\ & & & & \vdots & & \\ & & & & \downarrow & & \\ & & & & I^i / (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J^i & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & D_{(B,N)|(C,Q)}^1 & \longrightarrow & 0 \end{array} \quad (4.2.2)$$

(where the vertical morphism to the left is the obvious one, the one in the middle is the inclusion $I^i \hookrightarrow I$ and the zero morphism on $B \otimes_{\mathbb{Z}} T$, and the square to the left is commutative since $i \geq 2$ and the homomorphism $J^2 \rightarrow B \otimes_{\mathbb{Z}} T$ is zero) we obtain an exact sequence

$$(R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J / J^i \rightarrow I / I^i \oplus (B \otimes_{\mathbb{Z}} T) \rightarrow D_{(B,N)|(C,Q)}^1 / D^i \rightarrow 0,$$

which we can reconsider as a pushout:

$$\begin{array}{ccc}
 (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J/J^i & \longrightarrow & B \otimes_{\mathbb{Z}} T \\
 \downarrow & & \downarrow \\
 I/I^i & \longrightarrow & D_{(B,N)|(C,Q)}^1/D^i
 \end{array} \tag{4.2.3}$$

From the commutative diagram (4.2.2) we deduce that $\ker(I^i \rightarrow D_{(B,N)|(C,Q)}^1) = \ker(\beta\alpha) = I^i \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)$, obtaining an exact sequence (by definition of D^i)

$$0 \longrightarrow I^i \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J) \longrightarrow I^i \longrightarrow D^i \longrightarrow 0. \tag{4.2.4}$$

Applying the Ker-Coker Lemma to the diagram of exact sequences for $1 < j < i$,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I^i \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J) & \longrightarrow & I^i & \longrightarrow & D^i \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I^j \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J) & \longrightarrow & I^j & \longrightarrow & D^j \longrightarrow 0
 \end{array}$$

we obtain an exact sequence

$$0 \longrightarrow \frac{I^j \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)}{I^i \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)} \longrightarrow I^j/I^i \longrightarrow D^j/D^i \longrightarrow 0$$

From the exact sequence (4.2.4) we obtain

$$\begin{aligned}
 D^i &= I^i / (I^i \cap ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)) \\
 &= \frac{I^i + ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)}{((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J)}
 \end{aligned}$$

which is the ideal \tilde{I}^i of $(R \otimes_C B) / ((R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} J) = B[X]$ where \tilde{I} is the ideal of the simplicial ring $B[X]$ generated by the variables X . Therefore the hypotheses of [20, Theorem 8.8] are verified (since $C \rightarrow B$ and $P \rightarrow N$ are surjective) and we obtain $H_n(D^i) = H_n(\tilde{I}^i) = 0$ for all $i > n$.

We obtain a convergent spectral sequence [4, XV, Proposition 4.1]

$$E_{p,q}^2 = H_{p+q}(D^q/D^{q+1}) \Rightarrow H_{p+q}(D_{(B,N)|(C,Q)}^1).$$

It remains to show that $D^1/D^2 = \mathbb{L}_{(B,N)|(C,Q)}$. If we consider $R \otimes_C R$ as R -module via $r(a \otimes b) = a \otimes br$, we have a split exact sequence of R -modules (which define I_0)

$$0 \longrightarrow I_0 \longrightarrow R \otimes_C R \longrightarrow R \longrightarrow 0.$$

Applying $- \otimes_R B$, we obtain an split exact sequence

$$0 \longrightarrow I_0 \otimes_R B \longrightarrow R \otimes_C B \longrightarrow B \longrightarrow 0$$

which proves that $I_0 \otimes_R B = I$.

Similarly, considering the ideal $J_0 = \ker(\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \rightarrow \mathbb{Z}[P])$, we obtain $J_0 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N] = J$. Therefore

$$I^i/I^{i+1} = (I_0 \otimes_R B)^i/(I_0 \otimes_R B)^{i+1} = I_0^i \otimes_R B/I_0^{i+1} \otimes_R B = I_0^i/I_0^{i+1} \otimes_R B$$

and similarly

$$J^i/J^{i+1} = J_0^i/J_0^{i+1} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N].$$

Then the pushout (4.2.3) for $i = 2$ takes the form

$$\begin{array}{ccc} (R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]) & \xrightarrow{\gamma} & B \otimes_{\mathbb{Z}} T \\ \downarrow \lambda & & \downarrow \\ B \otimes_R I_0/I_0^2 & \longrightarrow & D_{(B,N)|(C,Q)}^1/D^2 \end{array}$$

Since the homomorphisms λ and γ are factored by the surjective homomorphism

$$(R \otimes_C B) \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]) \rightarrow B \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N])$$

that is obtained by applying $- \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N])$ to the surjective morphism $R \otimes_C B \rightarrow B$, we obtain a pushout

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]) & \longrightarrow & B \otimes_{\mathbb{Z}} T \\ \downarrow & & \downarrow \\ B \otimes_R I_0/I_0^2 & \longrightarrow & D_{(B,N)|(C,Q)}^1/D^2 \end{array}$$

and additionally, since

$$\begin{aligned} B \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[N]} (J_0/J_0^2 \otimes_{\mathbb{Z}[P]} \mathbb{Z}[N]) &= B \otimes_{\mathbb{Z}[N]} \mathbb{Z}[N] \otimes_{\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]} J_0/J_0^2 \\ &= B \otimes_{\mathbb{Z}[N]} J_0/J_0^2 \end{aligned}$$

(because the $\mathbb{Z}[P] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ -module J_0/J_0^2 is already a $\mathbb{Z}[N]$ -module), we obtain a pushout

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} (J_0/J_0^2) & \longrightarrow & B \otimes_{\mathbb{Z}} T \\ \downarrow & & \downarrow \\ B \otimes_R I_0/I_0^2 & \longrightarrow & D_{(B,N)|(C,Q)}^1/D^2 \end{array}$$

Finally, we deduce $T = \ker(P^{\text{gp}}/\ker(Q^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow N^{\text{gp}}) = P^{\text{gp}}/Q^{\text{gp}}$ given that $P_0^{\text{gp}} = Q^{\text{gp}}$. Thus, the previous pushout gives us $\mathbb{L}_{(B,N)|(C,Q)}$ by definition. \square

4.3 A complex to compute H_2

Now we will adapt the definition of [15] to compute $H_2((A, M), (B, N), -)$.

Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings. Let $(A, M) \rightarrow (R, P_0) \rightarrow (B, N)$ be a factorization with $R \rightarrow B$ and $h: P_0 \rightarrow N$ surjective homomorphisms, where $P_0 = M \oplus \mathbb{N}^X$ (with h extending $M \rightarrow N$) and $R = \mathbb{Z}[P_0] \otimes_{\mathbb{Z}[M]} A[Y] = A[X \cup Y]$ (with $R \rightarrow B$ extending $A \rightarrow B$).

We consider $W_0 = \ker(P_0^{\text{gp}} \rightarrow N^{\text{gp}})$, $J = \ker(\mathbb{Z}[P_0] \rightarrow \mathbb{Z}[N])$ and $I = \ker(R \rightarrow B)$. Let

$$0 \longrightarrow V \longrightarrow G \xrightarrow{\pi} J \longrightarrow 0$$

be an exact sequence of $\mathbb{Z}[P_0]$ -modules with G a free \mathbb{Z} -module,

$$0 \longrightarrow U \longrightarrow F \xrightarrow{\tau} I \longrightarrow 0$$

and exact sequence of R -modules with F a free R -module containing $R \otimes_{\mathbb{Z}[P_0]} G$ as a direct summand and τ extending $G \xrightarrow{\pi} J \rightarrow I$, and

$$0 \longrightarrow W_1 \longrightarrow Q_1 \longrightarrow W_0 \longrightarrow 0$$

an exact sequence of \mathbb{Z} -modules with Q_1 a free \mathbb{Z} -module.

Let $U_0 = \{\tau(x)y - \tau(y)x \in U \mid x, y \in F\}$, which is an R -module, and $V_0 = \{\pi(x)y - \pi(y)x \in V \mid x, y \in G\}$ which is a $\mathbb{Z}[P_0]$ -submodule of V .

Consider the following commutative diagram of R -modules

$$\begin{array}{ccccccc}
 B \otimes_{\mathbb{Z}[N]} V/V_0 & \xrightarrow{\beta_2} & B \otimes_{\mathbb{Z}} W_1 & & & & \\
 \searrow d_2 & & \searrow D_2 & & & & \\
 B \otimes_{\mathbb{Z}[P_0]} G = B \otimes_{\mathbb{Z}[N]} G/JG & \xrightarrow{\beta_1} & B \otimes_{\mathbb{Z}} Q_1 & & & & \\
 \searrow d_1 & & \searrow D_1 & & & & \\
 B \otimes_{\mathbb{Z}[P_0]} \Omega_{\mathbb{Z}[P_0]|\mathbb{Z}[M]} & \xrightarrow{\beta_0} & B \otimes_{\mathbb{Z}} P_0^{\text{gp}}/M_0^{\text{gp}} & & & & \\
 \searrow \alpha_1 & & \searrow \alpha_0 & & & & \\
 U/U_0 & \xrightarrow{\delta_2} & F/IF & & & & \\
 \searrow \delta_1 & & \searrow & & & & \\
 & & B \otimes_R \Omega_{R|A} & & & &
 \end{array} \tag{4.3.1}$$

where the homomorphisms are defined as follows:

- δ_2 , d_2 and D_2 are induced by the inclusions $U \rightarrow F$, $V \rightarrow G$ and $W_1 \rightarrow Q_1$ respectively.
- δ_1 is the composition $F/IF \rightarrow I/I^2 \rightarrow B \otimes_R \Omega_{R|A}$, where the first map is the one induced by τ and the second one by the canonical derivation $R \rightarrow \Omega_{R|A}$. The map d_1 is analogously defined while D_1 is induced by the composition $Q_1 \rightarrow W_0 \rightarrow P_0^{\text{gp}} \rightarrow P_0^{\text{gp}}/M_0^{\text{gp}}$.
- The homomorphism β_0 is the one of Definition 3.9: $\beta_0(1 \otimes dp) := h(p) \otimes \bar{p}$.
- The exact sequences

$$0 \longrightarrow V \longrightarrow G \longrightarrow J \longrightarrow 0,$$

$$0 \longrightarrow W_1 \longrightarrow Q_1 \longrightarrow W_0 \longrightarrow 0$$

give a diagram of homomorphisms of $\mathbb{Z}[N]$ -modules

$$\begin{array}{ccccccc}
 V/JV & \longrightarrow & G/JG & \longrightarrow & J/J^2 & \longrightarrow & 0 \\
 & & & & \downarrow v_h & & \\
 0 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 \longrightarrow 0
 \end{array}$$

where v_h , which is defined by

$$\lambda(p - p') \mapsto \lambda h(p') \otimes p(p')^{-1}$$

(where $h(p) = h(p')$), is the map of Definition 3.1 (see Remark 3.2). Since $G/JG = G \otimes_{\mathbb{Z}[P_0]} \mathbb{Z}[N]$ is a projective $\mathbb{Z}[N]$ -module, we have a commutative diagram

$$\begin{array}{ccccccc} V/JV & \xrightarrow{f} & G/JG & \longrightarrow & J/J^2 & \longrightarrow & 0 \\ \downarrow \beta'_2 & & \downarrow \beta'_1 & & \downarrow v_h & & \\ 0 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1 & \xrightarrow{g} & \mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 \longrightarrow 0 \end{array}$$

We define $\beta_1 = B \otimes_{\mathbb{Z}[N]} \beta'_1$. Since $V_0 \subset JG$, $f(V_0/JV) = 0$ and then $\beta'_2(V_0/JV) = 0$ by the injectivity of g . So β'_2 gives a map $\tilde{\beta}'_2: V/V_0 \rightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_1$ and we define $\beta_2 = B \otimes_{\mathbb{Z}[N]} \tilde{\beta}'_2$.

- Finally, $\alpha_2, \alpha_1, \alpha_0$ are the obvious maps.

The commutativity of the square (i.e. $D_1\beta_1 = \beta_0 d_1$) follows from the commutativity of the squares

$$\begin{array}{ccc} G/JG & \longrightarrow & J/J^2 \\ \downarrow \beta'_1 & & \downarrow v_h \\ \mathbb{Z}[N] \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 \end{array}$$

and

$$\begin{array}{ccc} J/J^2 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}[P_0]} \Omega_{\mathbb{Z}[P_0]|\mathbb{Z}[N]} \\ \downarrow v_h & & \downarrow \\ \mathbb{Z}[N] \otimes_{\mathbb{Z}} W_0 & \longrightarrow & \mathbb{Z}[N] \otimes_{\mathbb{Z}} P_0^{\text{gp}}/M^{\text{gp}} \end{array}$$

The commutativity of the remaining squares is clear.

Definition 4.22. We define the complex $(\text{LS}_{(B,N)|(A,M)})_i$, for $i = 0, 1, 2$, as the complex given by the pushout of (α_2, β_2) , (α_1, β_1) and (α_0, β_0) of diagram (4.3.1).

Proposition 4.23. *In this context, we consider the complex*

$$K := (T \otimes_{\mathbb{Z}} W_1 \xrightarrow{D_2} T \otimes_{\mathbb{Z}} Q_1 \xrightarrow{D_1} T \otimes_{\mathbb{Z}} P_0^{sp}/M^{sp})$$

for any \mathbb{Z} -module T . Then:

(i) $H_2(K) = \text{Tor}_1^{\mathbb{Z}}(T, \ker(M^{sp} \rightarrow N^{sp}))$

(ii) We have an exact sequence

$$0 \rightarrow T \otimes_{\mathbb{Z}} \ker(M^{sp} \rightarrow N^{sp}) \rightarrow H_1(K) \rightarrow \text{Tor}_1^{\mathbb{Z}}(T, N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp})) \rightarrow 0.$$

(iii) $H_0(K) = T \otimes_{\mathbb{Z}} N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp})$

Proof. We have $H_2(K) = \ker(B \otimes_{\mathbb{Z}} W_1 \rightarrow B \otimes_{\mathbb{Z}} Q_1)$. Applying $B \otimes_{\mathbb{Z}} -$ to the exact sequence of \mathbb{Z} -modules

$$0 \rightarrow W_1 \rightarrow Q_1 \rightarrow W_0 \rightarrow 0$$

we obtain an exact sequence

$$0 = \text{Tor}_1^{\mathbb{Z}}(B, Q_1) \rightarrow \text{Tor}_1^{\mathbb{Z}}(B, W_0) \rightarrow B \otimes_{\mathbb{Z}} W_1 \rightarrow B \otimes_{\mathbb{Z}} Q_1 \rightarrow B \otimes_{\mathbb{Z}} W_0 \rightarrow 0$$

and then $H_2(K) = \text{Tor}_1^{\mathbb{Z}}(B, W_0)$.

Consider the diagram of exact rows and columns defining the lower row

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker(M^{sp} \rightarrow N^{sp}) & \longrightarrow & M^{sp} & \longrightarrow & \text{Im}(M^{sp} \rightarrow N^{sp}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & W_0 & \longrightarrow & P_0^{sp} = M^{sp} \oplus \mathbb{Z}^X & \longrightarrow & N^{sp} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & W_0/\ker(M^{sp} \rightarrow N^{sp}) & \longrightarrow & \mathbb{Z}^X & \longrightarrow & N^{sp}/\text{Im}(M^{sp} \rightarrow N^{sp}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

(4.3.2)

Since the lower row is exact,

$$\mathrm{Tor}_i^{\mathbb{Z}}(B, W_0/\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})) = \mathrm{Tor}_{i+1}^{\mathbb{Z}}(B, N^{\mathrm{gp}}/\mathrm{Im}(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}))$$

and this last module vanishes for $i + 1 \geq 2$ given that \mathbb{Z} is a principal ideal domain. So, from the left column we deduce then $\mathrm{Tor}_1^{\mathbb{Z}}(B, \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})) = \mathrm{Tor}_1^{\mathbb{Z}}(B, W_0)$, that is,

$$H_2(K) = \mathrm{Tor}_1^{\mathbb{Z}}(B, W_0) = \mathrm{Tor}_1^{\mathbb{Z}}(B, \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})).$$

Consider now the diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) & & & \\
 & & & \downarrow & & & \\
 0 \rightarrow W_1 & \rightarrow & Q_1 & \xrightarrow{\varphi} & W_0 & \rightarrow & 0 \\
 & & \searrow \sigma & & \downarrow & & \\
 & & & & 0 \rightarrow W_0/\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) & \rightarrow & \mathbb{Z}^X \rightarrow N^{\mathrm{gp}}/\mathrm{Im}(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \rightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

We have exact sequences

$$0 \rightarrow \varphi^{-1}(\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})) \rightarrow Q_1 \rightarrow \mathbb{Z}^X \rightarrow N^{\mathrm{gp}}/\mathrm{Im}(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \rightarrow 0, \quad (4.3.3)$$

since $\ker(\sigma) = \varphi^{-1}(\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}))$, and

$$0 \rightarrow W_1 \rightarrow \varphi^{-1}(\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}})) \rightarrow \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \rightarrow 0.$$

Therefore we have a vertical exact sequence of (horizontal) complexes

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 W_1 & \longrightarrow & Q_1 & \longrightarrow & \mathbb{Z}^X & \longrightarrow & N^{\text{gp}} / \text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}) \\
 \downarrow & & \parallel & & \parallel & & \parallel \\
 \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) & \longrightarrow & Q_1 & \longrightarrow & \mathbb{Z}^X & \longrightarrow & N^{\text{gp}} / \text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Applying $B \otimes_{\mathbb{Z}} -$ we obtain a diagram of (horizontal) complexes

$$\begin{array}{ccccccc}
 E' : & B \otimes_{\mathbb{Z}} W_1 & \longrightarrow & B \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & B \otimes_{\mathbb{Z}} \mathbb{Z}^X & \\
 & \downarrow & & \parallel & & \parallel & \\
 E : & B \otimes_{\mathbb{Z}} \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) & \longrightarrow & B \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & B \otimes_{\mathbb{Z}} \mathbb{Z}^X & \\
 & \downarrow & & & & & \\
 E'' : & B \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) & & & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
 \end{array} \tag{4.3.4}$$

where the columns correspond to degrees 2, 1 and 0 respectively (from left to right).

We have

$$H_n(E) = \text{Tor}_n^{\mathbb{Z}}(B, N^{\text{gp}} / \text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}))$$

for $n = 0, 1$ by the exactness of (4.3.3), and the fact that Q_1 and \mathbb{Z}^X are free \mathbb{Z} -modules.

On the other hand,

$$\begin{aligned} \text{Ext}_{\mathbb{Z}}^1(W_0/\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}))) &= \\ &= \text{Ext}_{\mathbb{Z}}^1(N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}), \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}))) \\ &= 0 \end{aligned}$$

by the lower row of (4.3.2), and then the exact sequence

$$0 \longrightarrow \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) \longrightarrow Q_1 \xrightarrow{\sigma} W_0/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}) \longrightarrow 0$$

is split. Therefore

$$0 \longrightarrow B \otimes_{\mathbb{Z}} \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) \longrightarrow B \otimes_{\mathbb{Z}} Q_1$$

is injective and then $H_2(E) = 0$.

Consider the left column of (4.3.4) and let

$$\begin{aligned} \Lambda &:= \ker [B \otimes_{\mathbb{Z}} \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow B \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}})] \\ &= \text{Im} [B \otimes_{\mathbb{Z}} W_1 \rightarrow B \otimes_{\mathbb{Z}} \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}))] \\ &= \text{Im}(B \otimes_{\mathbb{Z}} W_1 \rightarrow B \otimes_{\mathbb{Z}} Q_1). \end{aligned}$$

We have a vertical exact sequence of (horizontal) complexes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \tilde{E}': & & \Lambda & \longrightarrow & B \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & B \otimes_{\mathbb{Z}} \mathbb{Z}^X \\ & & \downarrow & & \parallel & & \parallel \\ E: & B \otimes_{\mathbb{Z}} \varphi^{-1}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}})) & \longrightarrow & B \otimes_{\mathbb{Z}} Q_1 & \longrightarrow & B \otimes_{\mathbb{Z}} \mathbb{Z}^X \\ & & \downarrow & & \downarrow & & \downarrow \\ E'': & B \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

and taking the homology exact sequence

$$\begin{aligned} 0 = H_2(E) \rightarrow H_2(E'') = B \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow H_1(\tilde{E}') = H_1(E') \rightarrow \\ \rightarrow H_1(E) = \text{Tor}_1^{\mathbb{Z}}(B, N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow H_1(E'') = 0 \end{aligned}$$

we obtain the desired exact sequence

$$0 \rightarrow B \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow H_1(K) \rightarrow \text{Tor}_1^{\mathbb{Z}}(B, N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow 0.$$

Finally, it is clear that $H_0(K) = H_0(E) = B \otimes_{\mathbb{Z}} N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}})$. \square

Let $\text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]}$ be the complex of $\mathbb{Z}[N]$ -modules

$$V/V_0 \rightarrow G/JG \rightarrow \mathbb{Z}[N] \otimes_{\mathbb{Z}[P_0]} \Omega_{\mathbb{Z}[P_0]|\mathbb{Z}[M]},$$

$\text{LS}_{B|A}$ the complex

$$U/U_0 \rightarrow F/IF \rightarrow B \otimes_R \Omega_{R|A}$$

and D the complex

$$W_1 \rightarrow Q_1 \rightarrow P_0^{\text{gp}}/M^{\text{gp}}.$$

By [15], for any B -module T , $H_*(\text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]} \otimes_{\mathbb{Z}[N]} T)$ and $H_*(\text{LS}_{B|A} \otimes_B T)$ do not depend on the choices, and similarly $H_*(D \otimes_{\mathbb{Z}} T)$ by Proposition 4.23.

By definition, we have a commutative diagram of complexes

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \xrightarrow{\beta} & B \otimes_{\mathbb{Z}} D \\ \downarrow \alpha & & \downarrow \varepsilon \\ \text{LS}_{B|A} & \xrightarrow{\gamma} & \text{LS}_{(B,N)|(A,M)} \end{array}$$

which is a pushout in each degree, and so $\text{coker}(\alpha_i) = \text{coker}(\varepsilon_i)$.

So we have a commutative diagram of complexes for any B -module T

$$\begin{array}{ccccccc} 0 \rightarrow T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \xrightarrow{T \otimes \alpha} & T \otimes_B \text{LS}_{B|A} & \longrightarrow & \text{coker}(T \otimes \alpha) & \rightarrow & 0 \\ \downarrow T \otimes \beta & & \downarrow T \otimes \gamma & & \parallel & & \\ 0 \longrightarrow T \otimes_{\mathbb{Z}} D & \xrightarrow{T \otimes \varepsilon} & T \otimes_B \text{LS}_{(B,N)|(A,M)} & \longrightarrow & \text{coker}(T \otimes \varepsilon) & \rightarrow & 0 \end{array}$$

Since α_1, α_0 are split injective, so are $\varepsilon_1, \varepsilon_0$, and then both lines are exact except that

$$T \otimes_{\mathbb{Z}[N]} (\text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]})_2 = T \otimes_{\mathbb{Z}[N]} V/V_0 \longrightarrow T \otimes_B U/U_0 = T \otimes_B (\text{LS}_{B|A})_2$$

and

$$T \otimes_{\mathbb{Z}} D_2 = T \otimes_{\mathbb{Z}} W_1 \longrightarrow T \otimes_B (\text{LS}_{(B,N)|(A,M)})_2$$

are not necessarily injective. We deduce a commutative diagram of exact rows

$$\begin{array}{ccccccc} H_2(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]}) & \longrightarrow & H_2(T \otimes_B \text{LS}_{B|A}) & \longrightarrow & H_2(\text{coker}(T \otimes \alpha)) & \longrightarrow & \\ \downarrow & & \downarrow & & \parallel & & \\ H_2(T \otimes_{\mathbb{Z}} D) & \longrightarrow & H_2(T \otimes_B \text{LS}_{(B,N)|(A,M)}) & \longrightarrow & H_2(\text{coker}(T \otimes \varepsilon)) & \longrightarrow & \\ & & \downarrow & & \parallel & & \\ & \longrightarrow & H_1(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]}) & \longrightarrow & \cdots & \longrightarrow & H_0(\text{coker}(T \otimes \alpha)) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \\ & \longrightarrow & H_1(T \otimes_{\mathbb{Z}} D) & \longrightarrow & \cdots & \longrightarrow & H_0(\text{coker}(T \otimes \varepsilon)) \longrightarrow 0 \end{array}$$

By diagram chasing, we deduce an exact sequence

$$\begin{aligned} & H_2(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]}) \rightarrow H_2(T \otimes_B \text{LS}_{B|A}) \oplus H_2(T \otimes_{\mathbb{Z}} D) \rightarrow H_2(T \otimes_B \text{LS}_{(B,N)|(A,M)}) \rightarrow \\ & \rightarrow H_1(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]}) \rightarrow \cdots \rightarrow H_0(T \otimes_B \text{LS}_{(B,N)|(A,M)}) \rightarrow 0 \end{aligned}$$

(if $x \in H_2(T \otimes_B \text{LS}_{B|A})$ and $y \in H_2(T \otimes_{\mathbb{Z}} D)$ are such that their images in $H_2(T \otimes_B \text{LS}_{(B,N)|(A,M)})$ are equal, then $x \in T \otimes_B U/U_0$ is such that $\delta_2(x) = 0$, $y \in B \otimes_{\mathbb{Z}} W$ is such that $D_2(y) = 0$, and $\gamma(x) = \varepsilon(y)$. Therefore there exists $z \in T \otimes_{\mathbb{Z}[N]} V/V_0$ such that $\alpha_2(z) = x$ and $\beta_2(z) = y$. Moreover, $d_2(z) = 0$ given that $\alpha_1 d_2(z) = \delta_2 \alpha_2(z) = \delta_2(x) = 0$ and α_1 is injective, and hence $z \in H_2(T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]})$. This proves exactness at $H_2(T \otimes_B \text{LS}_{B|A}) \oplus H_2(T \otimes_{\mathbb{Z}} D)$. Exactness at the remaining points are clear).

By [2, Proposition 15.12] and Proposition 4.23, this exact sequence takes the form

$$\begin{aligned} & H_2(\mathbb{Z}[M], \mathbb{Z}[N], T) \rightarrow H_2(A, B, T) \oplus H_2(T \otimes_{\mathbb{Z}} D) \rightarrow H_2(T \otimes_B \text{LS}_{(B,N)|(A,M)}) \rightarrow \\ & \rightarrow H_1(\mathbb{Z}[M], \mathbb{Z}[N], T) \rightarrow H_1(A, B, T) \oplus H_1(T \otimes_{\mathbb{Z}} D) \rightarrow H_1(T \otimes_B \text{LS}_{(B,N)|(A,M)}) \rightarrow \\ & \rightarrow H_0(\mathbb{Z}[M], \mathbb{Z}[N], T) \rightarrow H_0(A, B, T) \oplus H_0(T \otimes_{\mathbb{Z}} D) \rightarrow H_0(T \otimes_B \text{LS}_{(B,N)|(A,M)}) \rightarrow 0 \end{aligned}$$

where

- $H_2(D \otimes_{\mathbb{Z}} W) = \text{Tor}_1^{\mathbb{Z}}(T, \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}))$,
- $H_1(D \otimes_{\mathbb{Z}} W)$ appears in an exact sequence

$$0 \rightarrow T \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) \rightarrow H_1(D \otimes_{\mathbb{Z}} W) \rightarrow \text{Tor}_1^{\mathbb{Z}}(T, \text{coker}(M^{\text{gp}} \rightarrow N^{\text{gp}})) \rightarrow 0,$$

- $H_0(D \otimes_{\mathbb{Z}} W) = T \otimes_{\mathbb{Z}} \text{coker}(M^{\text{gp}} \rightarrow N^{\text{gp}})$.

Let $(A, M) \xrightarrow{\varphi} (F, R) \xrightarrow{\psi} (B, N)$ be a factorization in the category of simplicial prelog rings with φ a free cofibration and ψ a trivial fibration. We have a commutative diagram of complexes

$$\begin{array}{ccccc}
 T \otimes_{\mathbb{Z}[R]} \Omega_{\mathbb{Z}[R]|\mathbb{Z}[M]} & \longrightarrow & T \otimes_{\mathbb{Z}} R^{\text{gp}}/M^{\text{gp}} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & T \otimes_{\mathbb{Z}[N]} \text{LS}_{\mathbb{Z}[N]|\mathbb{Z}[M]} & \longrightarrow & T \otimes_{\mathbb{Z}} D \\
 & & \downarrow & & \downarrow \\
 T \otimes_F \Omega_{F|A} & \longrightarrow & T \otimes_B \mathbb{L}_{(B,N)|(A,M)} & & \\
 & \searrow & \downarrow & \searrow & \\
 & & T \otimes_B \text{LS}_{B|A} & \longrightarrow & T \otimes_B \text{LS}_{(B,N)|(A,M)}
 \end{array}$$

inducing a morphism from the exact sequence of Theorem 4.3 into the above exact sequence, and giving isomorphisms in two of each three terms by [2, Proposition 15.12]. Therefore, the remaining homomorphisms

$$H_i(T \otimes_B \mathbb{L}_{(B,N)|(A,M)}) \longrightarrow H_i(T \otimes_B \text{LS}_{(B,N)|(A,M)}), \quad \text{for } i = 0, 1, 2,$$

are also isomorphisms.



Chapter 5

Smoothness

This chapter is devoted to characterize log formal smoothness by the vanishing of cohomology. First we characterize the elements of H^1 as extensions. This is made in [19] and outlined in [12], but we include here a proof since in [19] (see also [23]) it is assumed integrity and in [12] there is no explicit reference to cohomology. Characterization of log smoothness by the vanishing of H^1 is then deduced: theorems 5.14 and 5.15 describe the situation. Finally we give a few examples where this characterization is useful.

5.1 Extensions

Definition 5.1. [12, 18] Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings where (B, N) is a log ring. An (A, M) -extension of (B, N) by I is a strict homomorphism of log rings $\tau: (C, P) \rightarrow (B, N)$ such that $\tau^\sharp: C \rightarrow B$ is surjective with kernel I , verifying $I^2 = 0$ ($\tau^\flat: P \rightarrow N$ is then surjective) and the subgroup $1 + I$ of C^* acts freely on P .

Remark 5.2. Let $\tau: (C, P) \rightarrow (B, N)$ be a homomorphism of log rings with $\tau^\sharp: C \rightarrow B$ surjective and kernel I . If P is integral, then C^* (and so $1 + I$) acts freely on P .

Proposition 5.3. Let $\tau: (C, P) \rightarrow (B, N)$ be an (A, M) -extension of (B, N) by I . Then the map

$$\begin{aligned} (1 + I) \times P &\longrightarrow P \times_N P \\ (u, p) &\longmapsto (p, up) \end{aligned}$$

is an isomorphism.

Proof. [12, Exposé II, Proposition 2.3.(b)] or [18, Proposition IV.2.1.2.3]. □

Definition 5.4. [12] Let $\tau: (C, P) \rightarrow (B, N)$ be an (A, M) -extension of (B, N) by I and $\tau': (C', P') \rightarrow (B, N)$ an (A, M) -extension of (B, N) by I' . A morphism of extensions is a homomorphism of log rings $h: (C, P) \rightarrow (C', P')$ verifying $\tau'h = \tau$.

We have then $h^b(up) = uh^b(p)$ for all $u \in 1 + I$ and $p \in P$ (the action of $1 + I$ on P' is via $1 + I \rightarrow 1 + I'$).

An isomorphism of extensions is a morphism with inverse.

Definition 5.5. Let $\tau : (C, P) \rightarrow (B, N)$ be an (A, M) -extension of (B, N) by W and let $(B', N') \rightarrow (B, N)$ a homomorphism of log rings over (A, M) . Then we have a commutative diagram whose rows are (A, M) -extensions by W

$$\begin{array}{ccc} (C \times_B B', P \times_N N') & \longrightarrow & (B', N') \\ \downarrow & & \downarrow \\ (C, P) & \longrightarrow & (B, N) \end{array}$$

Proposition 5.6. Let $\tau : (C, P) \rightarrow (B, N)$, $\tau' : (C', P') \rightarrow (B, N)$ be (A, M) -extensions of (B, M) by I , and let $h : (C, P) \rightarrow (C', P')$ a morphism of extensions. Then h is an isomorphism of extensions.

Proof. We will see that h^b is bijective. The result then follows easily. Let $p_1, p_2 \in P$ be such that $h^b(p_1) = h^b(p_2)$. Then $\tau^b(p_1) = \tau^b(p_2)$ and so $(p_1, p_2) \in P \times_N P$. By Proposition 5.3, there exists $u \in 1 + I$ such that $p_2 = up_1$. Since $h^b(p_1) = h^b(p_2) = h^b(up_1) = uh^b(p_1)$ and $1 + I$ acts freely on P , we deduce $u = 1$ and so $p_2 = p_1$. This proves injectivity. Now let $p' \in P'$. Let $p \in P$ such that $\tau^b(p) = \tau'^b(p')$, and let $p'' = h^b(p)$. We have $(p', p'') \in P' \times_N P'$ and then by Proposition 5.3 there exists $u \in 1 + I$ such that $p' = up''$. Then $h^b(up) = uh^b(p) = up'' = p'$ proving surjectivity. \square

Definition 5.7. Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings with (B, N) a log ring and let W be a B -module. We define the trivial (A, M) -extension of (B, N) by W as $\tau : (C, P) \rightarrow (B, N)$ where $C = B \oplus W$ with the product $(b_1, w_1)(b_2, w_2) := (b_1b_2, b_1w_2 + b_2w_1)$, $P = N \oplus (W, +)$ the product monoid, the structural map $\alpha_C : P \rightarrow C$ given by $\alpha_C(n, w) = (\alpha_B(n), \alpha_B(n)w)$, and τ^\sharp, τ^b are the canonical projections.

We have $C^* = \{(b, w) \mid b \in B^*\}$ since for $b \in B^*$, $(b, w)^{-1} = (b^{-1}, -b^{-2}w)$. Since (B, N) is a log ring, $\alpha_B|_{\alpha_B^{-1}(B^*)} : \alpha_B^{-1}(B^*) \rightarrow B^*$ is an isomorphism, and then $\alpha_C^{-1}(C^*) = \alpha_B^{-1}(B^*) \oplus W$. So $\alpha_C|_{\alpha_C^{-1}(C^*)} : \alpha_C^{-1}(C^*) \rightarrow C^*$ is bijective and (C, P) is a log ring. Moreover, τ is strict and $1 + W$ acts freely on P .

Theorem 5.8. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings with (B, N) a log ring and W a B -module. There exists a bijection, natural on (B, N) in the sense of Definition 5.5, between isomorphisms classes of (A, M) -extensions of (B, N) by W and $H^1((A, M), (B, N), W)$, where the trivial extension goes to zero.*

Proof. Let $\pi: (F, R) \rightarrow (B, N)$ be a homomorphism of prelog rings with π^\sharp, π^\flat surjective and (F, R) a log free (A, M) -algebra (see Example 3.16.(i)). Let $I = \ker(\pi^\sharp: F \rightarrow B)$, $J = \ker(Z[R] \rightarrow Z[N])$ and $T = \ker(R^{\text{gp}} \rightarrow N^{\text{gp}})$. We have an exact sequence of B -modules:

$$\text{Der}_{(A,M)}((F, R), W) \xrightarrow{\theta} \text{Hom}_B(N_{(B,N)|(F,R)}, W) \rightarrow H^1((A, M), (B, N), W) \rightarrow 0$$

by propositions 4.9, 4.7 and 4.6.

Let $(C, P) \xrightarrow{\tau} (B, N)$ be an (A, M) -extension of (B, N) by W . Since (F, R) is log free over (A, M) , we have a commutative diagram of prelog rings

$$\begin{array}{ccc} (F, R) & \xrightarrow{\pi} & (B, N) \\ \downarrow \varphi & & \parallel \\ (C, P) & \xrightarrow{\tau} & (B, N) \end{array}$$

inducing an F -module homomorphism $\varphi^\sharp|_I: I \rightarrow W$ and then a B -module homomorphism $\varphi_1: I/I^2 \rightarrow W$. We define also a homomorphism of B -modules

$$\varphi_2: B \otimes_{\mathbb{Z}} T \rightarrow W$$

as follows. Let $\frac{r_1}{r_2} \in T$ (i.e. $r_1, r_2 \in R$ with $\pi^\flat(r_1) = \pi^\flat(r_2)$). Since $\tau^\flat(\varphi^\flat(r_1)) = \tau^\flat(\varphi^\flat(r_2))$, we have $(\varphi^\flat(r_1), \varphi^\flat(r_2)) \in P \times_N P$ and so by Proposition 5.3 there exists $u \in 1 + W$ such that $\varphi^\flat(r_1) = u\varphi^\flat(r_2)$. This u is unique since the action of $1 + W$ on P is free. We define $\varphi_2(1 \otimes \frac{r_1}{r_2}) := u - 1$.

Consider the square

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} J/J^2 & \xrightarrow{\nu} & B \otimes_{\mathbb{Z}} T \\ \downarrow \eta & & \downarrow \varphi_2 \\ I/I^2 & \xrightarrow{\varphi_1} & W \end{array}$$

where ν, η are the maps defined in Definition 3.4. This square is commutative since if $r_1, r_2 \in R$ and $\pi^b(r_1) = \pi^b(r_2)$, then

$$\begin{aligned} -\varphi_2\nu(1 \otimes (r_1 - r_2)) &= \varphi_2(\alpha_B\pi^b(r_2) \otimes \frac{r_1}{r_2}) = \alpha_B\pi^b(r_2) \cdot (u - 1) \\ &= \alpha_C\varphi^b(r_2) \cdot (u - 1) = u\alpha_C\varphi^b(r_2) - \alpha_C\varphi^b(r_2) \\ &= \alpha_C\varphi^b(r_1) - \alpha_C\varphi^b(r_2) \end{aligned}$$

where u is chosen as above, and

$$-\varphi_1\eta(1 \otimes (r_1 - r_2)) = \varphi^\sharp(\alpha_F(r_1) - \alpha_F(r_2)) = \alpha_C\varphi^b(r_1) - \alpha_C\varphi^b(r_2).$$

Since $B \otimes_{\mathbb{Z}[N]} J/J^2$ is generated as B -module by the elements $1 \otimes (r_1 - r_2)$ as before, we are done.

Then by Definition 3.4, the pair of homomorphisms (φ_1, φ_2) determines an element $\tilde{\varphi} \in \text{Hom}_B(N_{(B,N)|(F,R)}, W)$.

Let now $\psi: (F, R) \rightarrow (B, N)$ be another homomorphism of prelog rings such that $\tau\psi = \pi$. We will see that $\psi - \tilde{\varphi} \in \text{Im}(\theta)$.

We consider the log derivation $(D, \tilde{D}) \in \text{Der}_{(A,M)}((F, R), W)$ given by the A -derivation

$$D = \psi^\sharp - \varphi^\sharp: F \longrightarrow W$$

and the homomorphism of groups

$$\tilde{D}: R^{\text{gp}} / \text{Im}(M^{\text{gp}} \rightarrow R^{\text{gp}}) \longrightarrow (W, +)$$

defined as follows: Let $\frac{r_1}{r_2} \in R^{\text{gp}}$. Since $(\psi^b(r_i), \varphi^b(r_i)) \in P \times_N P$ for $i = 1, 2$, there exists unique elements $u_1, u_2 \in 1 + W$ such that $u_i\varphi^b(r_i) = \psi^b(r_i)$. We define $\tilde{D}(\frac{r_1}{r_2}) = u_1 - u_2 \in W$. If $\frac{r_1}{r_2} \in \text{Im}(M^{\text{gp}} \rightarrow R^{\text{gp}})$, then $u_1 = 1 = u_2$ and so \tilde{D} is well defined. It is a group homomorphism since W is of square zero as ideal of C . If $r \in R$, $\tilde{D}(\bar{r}) = u - 1$ where $u \in 1 + W$ is such that $u\varphi^b(r) = \psi^b(r)$. Then

$$\begin{aligned} \alpha_B\pi^b(r)\tilde{D}(\bar{r}) &= \alpha_B\pi^b(r) \cdot (u - 1) = \alpha_C\varphi^b(r) \cdot (u - 1) = \alpha_C\varphi^b(r)u - \alpha_C\varphi^b(r) \\ &= \alpha_C\psi^b(r) - \alpha_C\varphi^b(r) = \psi^\sharp\alpha_F(r) - \varphi^\sharp\alpha_F(r) = D(\alpha_F(r)) \end{aligned}$$

and so $(D, \tilde{D}) \in \text{Der}_{(A,M)}((F, R), W)$. Finally, $\theta(D, \tilde{D}) = \tilde{\psi} - \tilde{\varphi}$.

Conversely, let $h \in \text{Hom}_B(N_{(B,N)|(F,R)}, W)$, that is, h is defined by a commutative diagram

$$\begin{array}{ccc} B \otimes_{\mathbb{Z}[N]} J/J^2 & \xrightarrow{\nu} & B \otimes_{\mathbb{Z}} T \\ \downarrow \eta & & \downarrow h_2 \\ I/I^2 & \xrightarrow{h_1} & W \end{array}$$

Let $h'_1: I \rightarrow W$ the F -module homomorphism induced by h_1 and $C = F \oplus_I W$ the pushout

$$\begin{array}{ccc} I & \longrightarrow & F \\ \downarrow h'_1 & & \downarrow \\ W & \longrightarrow & C \end{array}$$

with the usual ring structure, that is, $(f_1, w_1) \cdot (f_2, w_2) = (f_1 f_2, f_1 w_1 + f_2 w_2)$. Let $h'_2: T \rightarrow W$ be the homomorphism of Abelian groups induced by h_2 . The pushout

$$\begin{array}{ccc} T & \longrightarrow & R^{\text{gp}} \\ \downarrow h'_2 & & \downarrow \\ W & \longrightarrow & R^{\text{gp}} \oplus_T W \end{array}$$

gives a commutative diagram of exact sequences of Abelian groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & T & \longrightarrow & R^{\text{gp}} & \longrightarrow & N^{\text{gp}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & W & \longrightarrow & R^{\text{gp}} \oplus_T W & \xrightarrow{\varepsilon} & N^{\text{gp}} \longrightarrow 1 \end{array}$$

Let P be the pullback of monoids

$$\begin{array}{ccc} P & \longrightarrow & R^{\text{gp}} \oplus_T W \\ \downarrow & & \downarrow \varepsilon \\ N & \longrightarrow & N^{\text{gp}} \end{array}$$

We define $\alpha_C: P \rightarrow C = F \oplus_I W$ as follows: Let $(n, (\frac{r_1}{r_2}, w)) \in N \times_{N^{\text{gp}}} (R^{\text{gp}} \oplus_T W) = P$. Then $\varepsilon(\frac{r_1}{r_2}, w) = \frac{\pi^b(r_1)}{\pi^b(r_2)} = \frac{n}{1}$. Let $r \in R$ be such that $\pi^b(r) = n$. Then (denoting the images of $r \in R$ and $n \in N$ in B also by r and n) we define

$$\alpha_C(n, (\frac{r_1}{r_2}, w)) := (\alpha_F(r), h_2(r \otimes \frac{rr_2}{r_1}) - nw).$$

We will see first that this definition is independent of the choice of r such that $\pi^b(r) = n$. Let $r, r' \in R$ be such that $\pi^b(r) = \pi^b(r') = n$. Then

$$\begin{aligned}
 & (\alpha_F(r'), h_2(r' \otimes \frac{r'r_2}{r_1}) - nw) - (\alpha_F(r), h_2(r \otimes \frac{rr_2}{r_1}) - nw) = \\
 &= (\alpha_F(r') - \alpha_F(r), nh_2(1 \otimes \frac{r'r_2}{r_1} \cdot \frac{r_1}{rr_2})) \\
 &= (\alpha_F(r') - \alpha_F(r), nh_2(1 \otimes \frac{r'}{r})) \\
 &= (0, nh_2(1 \otimes \frac{r'}{r}) - h'_1(\alpha_F(r') - \alpha_F(r))) \\
 &= (0, h_2\nu(r' - r) - h_1\eta(r' - r)) = 0
 \end{aligned}$$

since $h_2\nu = h_1\eta$.

Now assume that $(n, (\frac{r_1}{r_2}, w)) = (n, (\frac{r'_1}{r'_2}, w')) \in N \times_{N^{\text{gp}}} (R^{\text{gp}} \oplus_T W)$ and we will see that

$$(\alpha_F(r), h_2(r \otimes \frac{rr_2}{r_1}) - nw) = (\alpha_F(r), h_2(r \otimes \frac{rr'_2}{r'_1}) - nw').$$

Let $\frac{s_1}{s_2} \in T$ be such that $\frac{r'_1}{r'_2} = \frac{r_1 s_1}{r_2 s_2}$, $w' = h_2(1 \otimes \frac{s_2}{s_1}) + w$. Then

$$\begin{aligned}
 (\alpha_F(r), h_2(r \otimes \frac{rr'_2}{r'_1}) - nw') &= (\alpha_F(r), h_2(r \otimes \frac{rr_2 s_2}{r_1 s_1}) - nh_2(1 \otimes \frac{s_2}{s_1}) - nw) \\
 &= (\alpha_F(r), h_2(r \otimes \frac{rr_2 s_2}{r_1 s_1} + r \otimes \frac{s_1}{s_2}) - nw) \\
 &= (\alpha_F(r), h_2(r \otimes \frac{rr_2}{r_1}) - nw).
 \end{aligned}$$

Finally, it is immediate to check that α_C is a monoid homomorphism.

Now we define

$$\tau: (C, P) \longrightarrow (B, N)$$

by $\tau^\sharp(f, w) := \pi^\sharp(f)$ and $\tau^b(n, (\frac{r_1}{r_2}, w)) := n$.

We will see now that $\tau^{\text{log}}: (C, P^{\text{log}}) \rightarrow (B, N)$ is strict.

Consider the pushout

$$\begin{array}{ccc}
 (\tau\alpha_C)^{-1}(B^*) & \longrightarrow & B^* \\
 \downarrow & & \downarrow \\
 P & \longrightarrow & P \oplus_{(\tau\alpha_C)^{-1}(B^*)} B^*
 \end{array}$$

We will see that the canonical map $\lambda: P \oplus_{(\tau\alpha_C)^{-1}(B^*)} B^* \rightarrow N$ has an inverse $\mu: N \rightarrow P \oplus_{(\tau\alpha_C)^{-1}(B^*)} B^*$. Note first that $((1, (1, w)), 1) = ((1, (1, 0)), 1)$ in $(N \times_{N^{\text{gp}}} (R^{\text{gp}} \oplus_T W)) \oplus_{(\tau\alpha_C)^{-1}(B^*)} B^*$, since $\tau\alpha_C(1, (1, w)) = 1 \in B^*$.

Let $n \in N$. Let $r \in R$ such that $\pi^b(r) = n$. We define

$$\mu(n) = ((n, (r, 0)), 1) \in (N \times_{N^{\text{gp}}} (R^{\text{gp}} \oplus_T W)) \oplus_{(\tau\alpha_C)^{-1}(B^*)} B^*.$$

If $\pi^b(r) = \pi^b(r')$, then $\frac{r}{r'} \in T$ and so $(r', 0) = (r' \frac{r}{r'}, h'_2(\frac{r'}{r})) = (r, h'_2(\frac{r'}{r}))$ in $R^{\text{gp}} \oplus_T W$ and so

$$((n, (r', 0)), 1) = ((n, (r, h'_2(\frac{r'}{r}))), 1) = ((n, (r, 0)), 1).$$

Therefore μ does not depend on the choice of r . It is surjective since

$$\begin{aligned} ((n, (\frac{r_1}{r_2}, w)), b) &= ((bn, (\frac{\tilde{b}r_1}{r_2}, w)), 1) = ((bn, (\frac{\tilde{b}r_1}{r_2}, 0)), 1) \\ &= ((bn, (r', 0)), 1) \end{aligned}$$

where $r' = \frac{br_1}{r_2} \in R$ is such that $\pi^b(r') = bn$, we have identified $b \in B^*$ with its image via $B^* = \alpha_B^{-1}(B^*) \subset N$ and $\tilde{b} \in R$ is such that $\pi^b(\tilde{b}) = b \in N$. It is injective since $\lambda\mu = \text{id}_N$.

Now let us see that $1+W$ acts freely on P^{log} . Since $\tau^\sharp: C \rightarrow B$ is surjective with square zero kernel, we have $C^* = (\tau^\sharp)^{-1}(B^*) = \{(f, x) \in F \oplus_I W \mid \pi^\sharp(f) \in B^*\}$. Let $[(n, (\frac{r_1}{r_2}, w)), (f, x)] \in (N \times_{N^{\text{gp}}} (R^{\text{gp}} \oplus_T W)) \oplus_{\alpha_C^{-1}(C^*)} C^* = P^{\text{log}}$. We have $\frac{\pi^b(r_1)}{\pi^b(r_2)} = \frac{n}{1}$, $\pi^\sharp(f) \in B^*$. Since (B, N) is a log ring, there exists $m \in N$ such that $\alpha_B(m) = \pi^\sharp(f)$. Let $\tilde{m} \in R$ be such that $\pi^b(\tilde{m}) = m$. We have $\alpha_F(\tilde{m}) - f \in I$ and so $(f, x) = (\alpha_F(\tilde{m}), x')$ where $x' = x - h'_1(\alpha_F(\tilde{m}) - f)$. Let $g \in C$ be such that $\pi^\sharp(g) = \tau^\sharp(f)^{-1} \in B^*$. We have

$$\begin{aligned} \alpha_C(m, (\frac{\tilde{m}}{1}, -gx')) &= (\alpha_F(\tilde{m}), h_2(\tilde{m} \otimes \frac{\tilde{m}}{\tilde{m}}) + mgx') \\ &= (\alpha_F(\tilde{m}), x') = (f, x'). \end{aligned}$$

Therefore

$$\begin{aligned} [(n, (\frac{r_1}{r_2}, w)), (f, x)] &= [(n, (\frac{r_1}{r_2}, w)) \cdot (m, (\frac{\tilde{m}}{1}, -gx')), (1, 0)] \\ &= [(nm, (\frac{\tilde{m}r_1}{r_2}, w - gx')), (1, 0)]. \end{aligned}$$

So we can write any element of P^{\log} as $[(n, (\frac{r_1}{r_2}, w)), (1, 0)]$. An element $1 + z \in 1 + W$ acts on P^{\log} as the multiplication by $[(1, (\frac{1}{1}, -z)), (1, 0)]$ and so the action of $1 + W$ on P^{\log} is free, since $(\frac{r_1}{r_2}, w) = (\frac{r_1}{r_2}, w - z) \in R^{\text{gp}} \oplus_T W$ implies $z = 0$, the map $T \rightarrow R^{\text{gp}}$ being injective.

If $h \in \theta(D, \tilde{D})$, then $h'_2: T \rightarrow W$ factorizes as

$$\begin{array}{ccccccc}
 T & \longrightarrow & R^{\text{gp}} & \longrightarrow & R^{\text{gp}} / \text{Im}(M^{\text{gp}} \rightarrow R^{\text{gp}}) & \xrightarrow{\tilde{D}} & W \\
 & & & & \searrow & & \uparrow \\
 & & & & & & h''_2
 \end{array}$$

and then the exact sequence

$$0 \longrightarrow W \longrightarrow R^{\text{gp}} \oplus_T W \xrightarrow{\varepsilon} N^{\text{gp}} \longrightarrow 1$$

splits by $s: R^{\text{gp}} \oplus_T W \rightarrow W$, $s(t, w) = h''_2(t) + w$. Therefore $R^{\text{gp}} \oplus_T W = N^{\text{gp}} \times W$ and so $P = N \times W$. Similarly, we can show that $C = B \oplus W$ and then the extension obtained is the trivial one.

Finally, we can easily see that the two maps defined between classes of extensions and $H^1((A, M), (B, N), W)$ are inverse, and natural on (B, N) . \square

5.2 Formal smoothness

Definition 5.9. Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and J an ideal of B . We say that (B, N) is log formally smooth over (A, M) for the J -adic topology if for any commutative diagram of prelog rings

$$\begin{array}{ccc}
 (A, M) & \longrightarrow & (B, N) \\
 \downarrow & & \downarrow f \\
 (C, P) & \longrightarrow & (C', P')
 \end{array}$$

where $(C, P) \rightarrow (C', P')$ is an (A, M) -extension and $f(J^t) = 0$ for some $t > 0$, there exist a homomorphism of prelog rings $h: (B, N) \rightarrow (C, P)$ making commutative the triangles.

Proposition 5.10. Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and J an ideal of B . The following are equivalent:

- (i) (B, N) is a log formally smooth (A, M) -algebra for the J -adic topology,

- (ii) $(B, N)^{\log}$ is a log formally smooth (A, M) -algebra for the J -adic topology,
 (iii) $(B, N)^{\log}$ is a log formally smooth $(A, M)^{\log}$ -algebra for the J -adic topology.

Proof. It follows easily from the universal property of the functor $(-)^{\log}$ and the diagram

$$\begin{array}{ccc}
 (A, M) & \longrightarrow & (B, N) \\
 \downarrow & & \downarrow \\
 (A, M)^{\log} & \longrightarrow & (B, N)^{\log} \\
 \downarrow & & \downarrow \\
 (C, P) & \longrightarrow & (C', P')
 \end{array}$$

where $(C, P) \rightarrow (C', P')$ is an (A, M) -extension and $f(J^t) = 0$ for some t . \square

Lemma 5.11. *Let B be a ring, J an ideal of B , $t > 0$ an integer and*

$$\{F_n: B/J^t\text{-modules} \rightarrow B/J^t\text{-modules}\}$$

a direct system of half-exact functors (i.e. if $0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0$ is exact, then $F_n(U) \rightarrow F_n(W) \rightarrow F_n(V)$ is exact). The following are equivalent:

- (i) $\varinjlim_n F_n(W) = 0$ for all B/J -modules W (considering W as a B/J^t -module via $B/J^t \rightarrow B/J$),
 (ii) $\varinjlim_n F_n(W) = 0$ for all B/J^t -modules W .

Proof. We have to show (i) \Rightarrow (ii). Induction on t . The case $t = 1$ is trivial. Let $t > 1$, and assume the result valid for $t - 1$. Let W be a B/J^t -module. In the exact sequence

$$0 \rightarrow JW \rightarrow W \rightarrow W/JW \rightarrow 0$$

we have $\varinjlim_n F_n(JW) = 0$ given that JW is a B/J^{t-1} -module, and similarly we deduce $\varinjlim_n F_n(W/JW) = 0$. Since \varinjlim_n and F_n are half-exact, we conclude that $\varinjlim_n F_n(W) = 0$. \square

Theorem 5.12. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and J an ideal of B . The following are equivalent:*

(i) *For each integer $n \geq 1$, let $p_n: B \rightarrow B/J^n$ be the canonical map. Then*

$$\varinjlim_n H^1((A, M), (B/J^n, p_n^*N), W) = 0$$

for all B/J -modules W .

(ii) *(B, N) is log formally smooth over (A, M) for the J -adic topology.*

Proof. (i) \Rightarrow (ii) Consider a diagram as in Definition 5.9.

$$\begin{array}{ccc} (A, M) & \longrightarrow & (B, N) \\ \downarrow & & \downarrow f \\ (C, P) & \xrightarrow{\tau} & (C', P') \end{array}$$

and let $I = \ker \tau^\sharp(C \rightarrow C')$. For all $n \geq t$ we have a commutative square

$$\begin{array}{ccc} (D_n, Q_n) & \xrightarrow{\theta_n} & (B/J^n, p_n^*N) \\ \downarrow g_n & & \downarrow f_n \\ (C, P) & \xrightarrow{\tau} & (C', P') \end{array}$$

where $p_n: B \rightarrow B/J^n$ is the canonical projection, f_n is the map induced by f , $D_n = C \times_{C'} B/J^n$ and $Q_n = P \times_{P'} p_n^*N$.

Both lines are (A, M) -extensions by I (see Definition 5.5). Lemma 5.11.(i) tell us that

$$\varinjlim_n H^1((A, M), (B/J^n, p_n^*N), I) = 0,$$

and so by Theorem 5.8 there exists some $s \geq t$ such that θ_s has a section σ . The map

$$(B, N) \longrightarrow (B/J^s, p_s^*N) \xrightarrow{\sigma} (D_s, Q_s) \xrightarrow{g_s} (C, P)$$

has the required properties of (ii).

(ii) \Rightarrow (i) Let $\varphi_n: (D, Q) \rightarrow (B/J^n, p_n^*N)$ be an (A, M) -extension of $(B/J^n, p_n^*N)$ by W . By assertion (ii), there exists a homomorphism of prelog rings

$\phi_n: (B, N) \rightarrow (D, Q)$ such that $\varphi_n \phi_n = p_n$. Since $W^2 = 0$ as ideal of D , we have $\phi_n(J^{2n}) = 0$ and so ϕ_n factorizes as

$$(B, N) \xrightarrow{p_{2n}} (B/J^{2n}, p_{2n}^* N) \xrightarrow{\xi_{2n}} (D, Q).$$

We have a commutative diagram where the rows are (A, M) -extensions by W

$$\begin{array}{ccc} (D \times_{B/J^n} B/J^{2n}, Q \times_{p_n^* N} p_{2n}^* N) & \xrightarrow{\theta_{2n}} & (B/J^{2n}, p_{2n}^* N) \\ \downarrow & \swarrow \xi_{2n} & \downarrow \\ (D, Q) & \xrightarrow{\quad} & (B/J^n, p_n^* N) \end{array}$$

The homomorphism ξ_{2n} induces a section of θ_{2n} , showing that the element of $H^1((A, M), (B/J^n, p_n^* N), W)$ corresponding to the extension φ_n (see Theorem 5.8) goes to 0 in $H^1((A, M), (B/J^{2n}, p_{2n}^* N), W)$. Note that if $(E, R) \xrightarrow{\theta} (F, S)$ is an (A, M) -extension of (F, S) by W and σ a section of θ , then the map $S \times W \rightarrow R$, $(s, w) \mapsto (1+w)\sigma(s)$ is an isomorphism with inverse $r \mapsto (\theta(r), w')$ where $w' \in W$ is such that $(1+w')\sigma\theta(r) = r$ (there exists by Proposition 5.3). \square

Lemma 5.13. *Let (B, N) be a prelog ring, J an ideal of B , $p_n: B \rightarrow B/J^n$ the canonical homomorphism and W a B/J -module.*

(i) *Assume that $J = 0$ or N is integral, then*

$$\varinjlim_n H^i((B, N), (B/J^n, p_n^* N), W) = 0$$

for $i = 0$ and $i = 1$.

(ii) *If $J = 0$, or B is a Noetherian ring and N integral, then*

$$\varinjlim_n H^i((B, N), (B/J^n, p_n^* N), W) = 0$$

for all $i \geq 0$.

Proof. If $J \neq 0$, since $(B/J^n, p_n^* N) = (B/J^n, N)^{\text{log}}$, we have by Proposition 4.17 and Example 4.2

$$\begin{aligned} H^i((B, N), (B/J^n, p_n^* N), W) &= H^i((B, N), (B/J^n, N)^{\text{log}}, W) \\ &= H^i((B, N), (B/J^n, N), W) \\ &= H^i(B, B/J^n, W) \end{aligned}$$

and so the result follows from [2, proof of Lemme 10.7 and of Théorème 10.14]. \square

Theorem 5.14. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings, J an ideal of B . Assume $J = 0$, or B Noetherian and N integral. The following are equivalent:*

- (i) (B, N) is an (A, M) -algebra log formally smooth for the J -adic topology.
- (ii) $H^1((A, M), (B, N), W) = 0$ for all B/J -modules W .
- (iii) $H_1((A, M), (B, N), B/J) = 0$ and $\Omega_{(B,N)|(A,M)} \otimes_B B/J$ is a projective B/J -module.
- (iv) For any homomorphism $(F, R) \rightarrow (B, N)$ of prelog rings where (F, R) is a free (A, M) -prelog algebra and the homomorphisms $F \rightarrow B$ and $R \rightarrow N$ are surjective, the homomorphism

$$N_{(B,N)|(F,R)} \otimes_B B/J \longrightarrow \Omega_{(F,R)|(A,M)} \otimes_F B/J$$

is split injective.

Proof. (i) \Leftrightarrow (ii) We have an exact sequence induced by Jacobi-Zariski exact sequences (Proposition 4.9)

$$\begin{aligned} \varinjlim_n H^1((B, N), (B/J^n, p_n^* N), W) &\longrightarrow \varinjlim_n H^1((A, M), (B/J^n, p_n^* N), W) \longrightarrow \\ &\longrightarrow \varinjlim_n H^1((A, M), (B, N), W) \longrightarrow \varinjlim_n H^2((B, N), (B/J^n, p_n^* N), W). \end{aligned}$$

Then, Lemma 5.13 gives an isomorphism

$$\varinjlim_n H^1((A, M), (B/J^n, p_n^* N), W) = H^1((A, M), (B, N), W).$$

So the result follows from Theorem 5.12.

(ii) \Leftrightarrow (iii) The universal coefficient spectral sequence

$$\begin{aligned} E_2^{p,q} &\Rightarrow H^{p+q}(\mathrm{Hom}_{B/J}(\mathbb{L}_{(B,N)|(A,M)} \otimes_B B/J, W)) \\ &= H^{p+q}((A, M), (B, N), W), \end{aligned}$$

with

$$E_2^{p,q} = \mathrm{Ext}_{B/J}^p(H_q(\mathbb{L}_{(B,N)|(A,M)} \otimes_B B/J), W),$$

gives an exact sequence

$$0 \rightarrow \text{Ext}_{B/J}^1(\Omega_{(B,N)|(A,M)} \otimes_B B/J, W) \rightarrow H^1((A, M), (B, N), W) \rightarrow \\ \rightarrow \text{Hom}_{B/J}(H_1((A, M), (B, N), B/J), W) \rightarrow \text{Ext}_{B/J}^2(\Omega_{(B,N)|(A,M)} \otimes_B B/J, W),$$

from which we obtain the desired equivalence.

(iii) \Leftrightarrow (iv) We have a Jacobi-Zariski exact sequence, taking by Proposition 4.6 the form

$$H_1((A, M), (F, R), B/J) \rightarrow H_1((A, M), (B, N), B/J) \rightarrow N_{(B,N)|(F,R)} \otimes_B B/J \rightarrow \\ \rightarrow \Omega_{(F,R)|(A,M)} \otimes_F B/J \rightarrow \Omega_{(B,N)|(A,M)} \otimes_B B/J \rightarrow \Omega_{(B,N)|(F,R)} \otimes_B B/J = 0.$$

The first module vanishes by Proposition 4.7 and furthermore $\Omega_{(F,R)|(A,M)} \otimes_F B/J$ is a free B/J module by Example 3.16.(i). Then the result follows. \square

Theorem 5.15. *Let $(A, M) \rightarrow (B, N)$ be a homomorphism essentially of finite type of Noetherian prelog rings (see Definition 2.4) and J an ideal of B . Assume $J = 0$ or N is integral. The following are equivalent:*

- (i) (B, N) is an (A, M) -algebra log formally smooth for the J -adic topology.
- (ii) $H_1((A, M), (B, N), W) = 0$ for all B/J -modules W .

If moreover B is local and J a proper ideal of B , these conditions are also equivalent to

- (iii) (B, N) is an (A, M) -algebra log formally smooth for the 0 -adic topology.

Proof. Conditions (i) and (ii) are equivalent by Theorem 5.14 and Proposition 4.5, and (iii) implies (ii) is clear by Theorem 5.14.(ii) or by definition.

Therefore, let us assume the hypothesis (i). Then by Theorem 5.14.(ii) we have $H^1((A, M), (B, N), B/\mathfrak{n}) = 0$ where \mathfrak{n} is the maximal ideal of B , and so by Proposition 4.5 we deduce $H^1((A, M), (B, N), W) = 0$ for any B -module W . So we get (iii) by Theorem 5.14. \square

We will see some examples where homological methods give short proofs of known results. If C is a finitely generated Abelian group and R is a ring, then C is finite of order invertible in R if and only if $\text{Hom}_{\mathbb{Z}}(C, W) = 0$ for any R -module W , and the torsion part of C is of order invertible in R if and only if $\text{Ext}_{\mathbb{Z}}^1(C, W) = 0$ for any R -module W . So the following two results are included in [18, Theorem IV.3.1.8, Proposition IV.3.1.13] in the case of finitely generated monoids (and taking $J = 0$).

The following result is dual of [16, Lemma 2.6.2].

Lemma 5.16. *Let A be a ring, B and C two A -algebras and W a $B \otimes_A C$ -module. Then the canonical homomorphism*

$$H^1(C, B \otimes_A C, W) \longrightarrow H^1(A, B, W)$$

is injective.

Proof. Let R be a polynomial A -algebra such that $B = R/I$. We have an exact sequence

$$0 \longrightarrow I \otimes_A C \longrightarrow R \otimes_A C \xrightarrow{p} B \otimes_A C \longrightarrow 0.$$

If we denote $J = \ker p$, then $I \otimes_A C \rightarrow J$ is surjective. Consider now the exact sequences $A \rightarrow R \rightarrow B$ and $C \rightarrow R \otimes_A C \rightarrow B \otimes_A C$, whose Jacobi-Zariski exact sequences of cohomology give us a commutative diagram with exact rows

$$\begin{array}{ccccccc} \text{Der}_C(B \otimes_A C, W) & \longrightarrow & \text{Der}_C(R \otimes_A C, W) & \longrightarrow & & & \\ \parallel & & \parallel & & & & \\ \text{Der}_A(B, W) & \longrightarrow & \text{Der}_A(R, W) & \longrightarrow & & & \\ & \longrightarrow & \text{Hom}_{B \otimes_A C}(J/J^2, W) & \longrightarrow & H^1(C, B \otimes_A C, W) & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow & & \\ & \longrightarrow & \text{Hom}_B(I/I^2, W) & \longrightarrow & H^1(A, B, W) & \longrightarrow & 0 \end{array}$$

It is enough to show that $\gamma: \text{Hom}_{B \otimes_A C}(J/J^2, W) \rightarrow \text{Hom}_B(I/I^2, W)$ is injective. But this is clear, since $I \otimes_A C \rightarrow J$ is surjective and we have the isomorphisms $\text{Hom}_{B \otimes_A C}(J/J^2, W) = \text{Hom}_{R \otimes_A C}(J, W)$ and $\text{Hom}_B(I/I^2, W) = \text{Hom}_R(I, W) = \text{Hom}_{R \otimes_A C}(I \otimes_A C, W)$. \square

Proposition 5.17. *Let $M \rightarrow N$ be a homomorphism of monoids with N integral, R a ring, $A = R[M]$, $B = R[N]$ and J an ideal of B . If*

$$\text{Hom}_{\mathbb{Z}}(\ker(M^{sp} \rightarrow N^{sp}), W) = 0 = \text{Ext}_{\mathbb{Z}}^1(\text{coker}(M^{sp} \rightarrow N^{sp}), W)$$

for any B/J -module W , then $(A, M) \rightarrow (B, N)$ is log formally smooth for the J -adic topology.

Proof. Let W be a B/J -module and consider the fundamental exact sequence (see Theorem 4.3)

$$\begin{aligned} H^0(R[M], R[N], W) \oplus \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker}(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) &\rightarrow H^0(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow \\ &\rightarrow H^1((A, M), (B, N), W) \rightarrow \\ \rightarrow H^1(R[M], R[N], W) \oplus \operatorname{Ext}_{\mathbb{Z}}^1(\operatorname{coker}(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) \oplus \operatorname{Hom}_{\mathbb{Z}}(\ker(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) &\rightarrow \\ &\rightarrow H^1(\mathbb{Z}[M], \mathbb{Z}[N], W). \end{aligned}$$

The canonical homomorphism $H^0(R[M], R[N], W) \rightarrow H^0(\mathbb{Z}[M], \mathbb{Z}[N], W)$ is an isomorphism by propositions 3.10 and 3.18, and the homomorphism

$$H^1(R[M], R[N], W) \longrightarrow H^1(\mathbb{Z}[M], \mathbb{Z}[N], W)$$

is injective by Lemma 5.16. The result follows from Theorem 5.14. \square

Proposition 5.18. *Let $\varphi: (A, M) \rightarrow (B, N)$ be a homomorphism of prelog rings and J an ideal of B . Assume that N is integral and*

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}}(\ker(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) &= \operatorname{Ext}_{\mathbb{Z}}^1(\operatorname{coker}(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) \\ &= \operatorname{Hom}_{\mathbb{Z}}(\operatorname{coker}(M^{\text{sp}} \rightarrow N^{\text{sp}}), W) = 0 \end{aligned}$$

for any B/J -module W . The following are equivalent:

- (i) φ is log formally smooth for the J -adic topology.
- (ii) B is formally smooth over $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N]$ for the J -adic topology.

Proof. From the fundamental sequence and Theorem 5.14

$$\begin{aligned} H^0(A, B, W) &\xrightarrow{\beta_0} H^0(\mathbb{Z}[M], \mathbb{Z}[N], W) \longrightarrow \\ H^1((A, M), (B, N), W) &\longrightarrow H^1(A, B, W) \xrightarrow{\beta_1} H^1(\mathbb{Z}[M], \mathbb{Z}[N], W) \end{aligned}$$

we see that (i) holds if and only if β_1 is injective and β_0 is surjective for all B/J -modules W .

Now, from the Jacobi-Zariski exact sequence

$$\begin{aligned} H^0(A, B, W) &\xrightarrow{\beta'_0} H^0(A, A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N], W) \longrightarrow \\ H^1(A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N], B, W) &\longrightarrow H^1(A, B, W) \xrightarrow{\beta'_1} H^1(A, A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N], W) \end{aligned}$$

we see that (ii) holds if and only if β'_1 is injective and β'_0 is surjective.

Since in the commutative triangle

$$\begin{array}{ccc}
 H^i(A, B, W) & \xrightarrow{\beta_i} & H^i(\mathbb{Z}[M], \mathbb{Z}[N], W) \\
 & \searrow \beta'_i & \nearrow \gamma_i \\
 & H^i(A, A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N], W) &
 \end{array}$$

γ_1 is injective by Lemma 5.16 and γ_0 is bijective, we are done. \square

Here is a version of [11, Theorem 0.1].

Proposition 5.19. *Let $(A, M) \rightarrow (B, N)$ and $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings. Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi), where:*

- (i) $M \rightarrow P$ is a Kummer homomorphism of integral monoids (see Definition 1.31), M is saturated and $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P] \rightarrow C$ is flat.
- (ii) $M \rightarrow P$ is an exact and injective homomorphism of integral monoids and $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P] \rightarrow C$ is flat.
- (iii) $\mathbb{Z}[M] \rightarrow \mathbb{Z}[P]$ and $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P] \rightarrow C$ are flat, and $M^{sp} \rightarrow P^{sp}$ is injective.
- (iv) $\mathbb{Z}[M] \rightarrow \mathbb{Z}[P]$ and $A \rightarrow C$ are flat, and $M^{sp} \rightarrow P^{sp}$ is injective.
- (v) $\text{Tor}_n^{\mathbb{Z}[M]}(\mathbb{Z}[N], \mathbb{Z}[P]) = 0 = \text{Tor}_n^A(B, C)$ for all $n > 0$, and $M^{sp} \rightarrow P^{sp}$ is injective.
- (vi) For any $B \otimes_A C$ -module W , we have

$$H_n((A, M), (B, N), W) = H_n((C, P), (B \otimes_A C, N \oplus_M P), W)$$

for all $n \geq 0$.

Proof. (i) \Rightarrow (ii) follows from the definition. (ii) \Rightarrow (iii), [11, Lemma 2.1]: Let $m \in M$ and $p \in P$. If $mp \in M$ then $p = (mp)/m \in M^{sp}$ and then $p \in M$ since $M \rightarrow P$ is exact. So the homomorphism of $\mathbb{Z}[M]$ -modules $\mathbb{Z}[M] \rightarrow \mathbb{Z}[P]$ is split by $\mathbb{Z}[P] \rightarrow \mathbb{Z}[M], p \mapsto 0$, if $p \in P - M$.

On the other hand, if $\mathbb{Z}[M] \rightarrow \mathbb{Z}[P]$ is flat then $A \rightarrow A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P]$ is flat. If moreover $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[P] \rightarrow C$ is flat, then by composition $A \rightarrow C$ is flat too. This shows (iii) \Rightarrow (iv).

Finally, (iv) \Rightarrow (v) is trivial and (v) \Rightarrow (vi) is Proposition 4.8. \square

Corollary 5.20. *Let $(A, M) \rightarrow (B, N)$ and $(A, M) \rightarrow (C, P)$ be homomorphisms of prelog rings with $\text{Spec}(C) \rightarrow \text{Spec}(A)$ surjective and assume that condition (iv) of Proposition 5.19 holds. Let J be an ideal of B . Then (B, N) is an (A, M) -algebra formally smooth for the J -adic topology if and only if $(B \otimes_A C, N \oplus_M P)$ is a (C, P) -algebra formally smooth for the $J \otimes_A C$ -adic topology.*

Proof. We know that $A \rightarrow C$ is a faithfully flat homomorphism, so for any B/J -module W we have that $H_1((A, M), (B, N), W) = 0$ if and only if

$$H_1((A, M), (B, N), W) \otimes_A C = H_1((A, M), (B, N), W \otimes_A C) = 0.$$

Then the result follows from Proposition 5.19, (iv) \Rightarrow (vi). □





Chapter 6

Regularity

This chapter contains the main results of the dissertation that will be deduced from the results in chapters 4 and 5.

6.1 Technical results

Theorem 6.1. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring, J a proper ideal of A and $B = A/J$. Let $N = M/\alpha_M^{-1}(J)$ be the quotient of the monoid M by the ideal $\alpha_M^{-1}(J)$, and so (B, N) is a Noetherian local prelog ring with residue field k . Let I be the ideal of A generated by $\alpha_M(M) \cap J$. The following statements are equivalent:*

(i) *For any B -module W , the map*

$$H_2(A, B, W) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{gp} \rightarrow N^{gp}), W) \longrightarrow H_2((A, M), (B, N), W)$$

in the fundamental exact sequence is zero on the first summand and an isomorphism on the second one.

(i') *The map*

$$H_2(A, B, k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{gp} \rightarrow N^{gp}), k) \longrightarrow H_2((A, M), (B, N), k)$$

in the fundamental exact sequence is zero on the first summand and an isomorphism on the second one.

(ii) *$\ker(A/I \rightarrow B)$ is generated by a regular sequence and $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = 0$.*

Proof. First note that $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[N])$ is the ideal \mathfrak{a} of $\mathbb{Z}[M]$ generated by the elements $a-b$ with $a, b \in \alpha_M^{-1}(J)$, and in particular $\mathfrak{a}A \subset I$. If $x \in \alpha_M(M) \cap J \subset \mathfrak{m}$, $1-x$ is a unit in A , and so $x(1-x) = x-x^2 \in \mathfrak{a}A$ implies $x \in \mathfrak{a}A$. Therefore $\mathfrak{a}A = I$. Similarly, $\mathrm{Tor}_*^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = \mathrm{Tor}_*^{\mathbb{Z}[M]}(A, \mathbb{Z}[M]/I')$ where I' is the ideal of $\mathbb{Z}[M]$ generated by $\alpha_M^{-1}(J)$, since for $a \in \alpha_M^{-1}(J)$ the localization

$$\mathbb{Z}[N]_{1-a} = \mathbb{Z}[M]_{1-a}/\mathfrak{a}\mathbb{Z}[M]_{1-a} = \mathbb{Z}[M]_{1-a}/I'\mathbb{Z}[M]_{1-a} = \left(\mathbb{Z}[M]/I' \right)_{1-a}$$

as above, and the image of $1 - a$ in A being a unit,

$$\mathrm{Tor}_*^{\mathbb{Z}[M]}(A, X) = \mathrm{Tor}_*^{\mathbb{Z}[M]}(A, X)_{1-a} = \mathrm{Tor}_*^{\mathbb{Z}[M]}(A, X_{1-a}).$$

(i') \Rightarrow (ii) Consider the diagram on the page 81, in which the rows are exact (Theorem 4.3), the columns are also exact (Jacobi-Zariski exact sequence), $H_2(X) = \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}), k)$ and also $H_1(X) = \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \otimes_{\mathbb{Z}} k$ as in Theorem 4.3. If $H_2(X) \rightarrow H_2((A, M), (B, N), k)$ is an isomorphism, then γ_1 is injective. Moreover, the composition

$$H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\gamma_1} H_1(A, B, k) \oplus H_1(X) \xrightarrow{\mathrm{pr}_2} H_1(X)$$

is zero since it is induced by the map of Proposition 3.1

$$\alpha_M^{-1}(J)\mathbb{Z}[M]/\alpha_J^{-1}(J)^2\mathbb{Z}[M] \otimes_{\mathbb{Z}[N]} k \longrightarrow \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}) \otimes_{\mathbb{Z}} k$$

defined by

$$\overline{\sum_i \lambda_i m_i} \otimes w \longmapsto \sum_i m_i \otimes \lambda_i \tilde{m}_i w = 0,$$

where \tilde{m}_i is the image of $m_i \in B$ which is zero since $m_i \in \alpha_M^{-1}(J)$.

Therefore γ_1 injective and $\mathrm{pr}_2 \gamma_1 = 0$ imply that

$$H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\mathrm{pr}_1 \gamma_1} H_1(\mathbb{Z}[M], \mathbb{Z}[N], k)$$

is injective too. By the commutativity of the diagram, the homomorphism

$$H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\mathrm{pr}_1 \beta_1} H_1(A, A/I, k)$$

is injective. It is also an isomorphism, since it is surjective by [16, Lemma 2.6.2] (since $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[N] = A/\mathfrak{a}A = A/I$), and then, again by the commutativity of the diagram, α_1 is injective. We deduce $\lambda = 0$. Since

$$H_2(A, B, k) \longrightarrow H_2((A, M), (B, N), k)$$

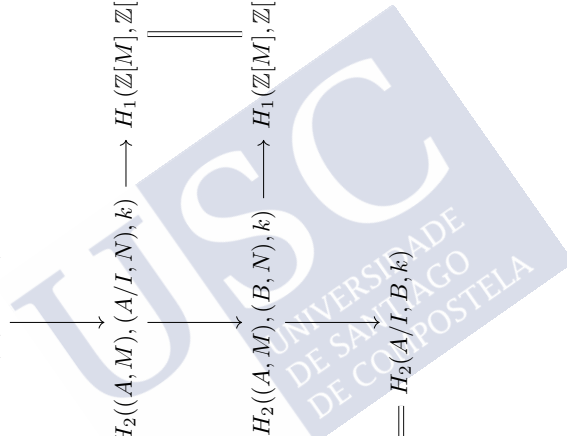
is the zero map, we deduce $\mu = 0$. Therefore

$$H_2(A/I, B, k) = 0$$

which by [2, Théorème 6.25] means that $\ker(A/I \rightarrow B)$ is generated by a regular sequence.

$$\begin{array}{c}
 H_3(A/I, B, k) \xrightarrow{\quad} H_3(A/I, B, k) \\
 \parallel \\
 H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\beta_2} H_2(A, A/I, k) \oplus H_2(X) \longrightarrow H_2((A, M), (A/I, N), k) \longrightarrow H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\beta_1} H_1(A, A/I, k) \oplus H_1(X) \\
 \parallel \\
 H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\gamma_2} H_2(A, B, k) \oplus H_2(X) \longrightarrow H_2((A, M), (B, N), k) \longrightarrow H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\gamma_1} H_1(A, B, k) \oplus H_1(X) \\
 \parallel \\
 H_2(A/I, B, k) \xrightarrow{\quad} H_2(A/I, B, k) \\
 \parallel \\
 H_1(A, A/I, k) \xrightarrow{\quad} H_1(A, A/I, k) \\
 \parallel \\
 H_1(A, B, k) \xrightarrow{\quad} H_1(A, B, k)
 \end{array}$$

$\alpha_2 \oplus \text{id}$ μ λ $\alpha_1 \oplus \text{id}$ α_1



Now we will see that $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = 0$. We have $H_2(A/I, B, k) = 0$ and then $H_3(A/I, B, k) = 0$ by [2, Théorème 6.25]. Therefore in the diagram

$$\begin{array}{ccc} H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) & \xrightarrow{\text{pr}_1 \beta_2} & H_2(A, A/I, k) \\ \parallel & & \downarrow \alpha_2 \\ H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) & \xrightarrow{\text{pr}_1 \gamma_2} & H_2(A, B, k) \end{array}$$

α_2 is an isomorphism. Since the lower map is surjective by hypothesis, the upper one is also surjective. From the exact sequence [2, Proposition 15.18]

$$\begin{aligned} & H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\text{pr}_1 \beta_2} H_2(A, A/I, k) \longrightarrow \text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) \otimes_A k \longrightarrow \\ \longrightarrow & H_1(\mathbb{Z}[M], \mathbb{Z}[N], k) \xrightarrow{\text{pr}_1 \beta_1} H_1(A, A/I, k) \end{aligned}$$

we deduce $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) \otimes_A k = 0$ and then, since A is Noetherian local, $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = 0$.

(ii) \Rightarrow (i) By [2, Proposition 15.18] we have an exact sequence for each B -module W

$$\begin{aligned} & H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) \xrightarrow{\theta_2} H_2(A, A/I, W) \longrightarrow \text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) \otimes_A W \longrightarrow \\ \longrightarrow & H_1(\mathbb{Z}[M], \mathbb{Z}[N], W) \xrightarrow{\theta_1} H_1(A, A/I, W) \longrightarrow 0. \end{aligned}$$

By hypothesis, $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = 0$, so θ_2 is surjective and θ_1 injective. Since $H_n(A/I, B, W) = 0$ for all $n \geq 2$, we deduce that in the diagram

$$\begin{array}{ccc} H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) & \xrightarrow{\theta_2} & H_2(A, A/I, W) \\ \parallel & & \downarrow \alpha_2 \\ H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) & \xrightarrow{\psi_2} & H_2(A, B, W) \end{array}$$

the map α_2 is bijective and so ψ_2 is surjective.

In the diagram

$$\begin{array}{ccc}
 & & H_2(A/I, B, W) = 0 \\
 & & \downarrow \\
 H_1(\mathbb{Z}[M], \mathbb{Z}[N], W) & \xrightarrow{\theta_1} & H_1(A, A/I, W) \\
 \parallel & & \downarrow \alpha_1 \\
 H_1(\mathbb{Z}[M], \mathbb{Z}[N], W) & \xrightarrow{\psi_1} & H_1(A, B, W)
 \end{array}$$

θ_1 is injective and then ψ_1 is also injective.

From the fundamental exact sequence

$$\begin{aligned}
 H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) &\xrightarrow{\psi_2 \oplus \omega_2} H_2(A, B, W) \oplus \text{Tor}_1^{\mathbb{Z}[M]}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W) \longrightarrow \\
 &\longrightarrow H_2((A, M), (B, N), W) \longrightarrow \\
 &\longrightarrow H_1(\mathbb{Z}[M], \mathbb{Z}[N], W) \xrightarrow{\psi_1 \oplus \omega_1} H_1(A, B, W) \oplus \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) \otimes_{\mathbb{Z}} W
 \end{aligned}$$

we see that we are done if we prove that the map

$$H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) \longrightarrow \text{Tor}_1^{\mathbb{Z}[M]}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W)$$

vanishes.

We have a surjective natural homomorphism [20, Theorem 6.16]

$$\text{Tor}_2^{\mathbb{Z}[M]}(\mathbb{Z}[N], W) \longrightarrow H_2(\mathbb{Z}[M], \mathbb{Z}[N], W),$$

so it is sufficient to show that the composition

$$\text{Tor}_2^{\mathbb{Z}[M]}(\mathbb{Z}[N], W) \longrightarrow H_2(\mathbb{Z}[M], \mathbb{Z}[N], W) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W)$$

is zero. We have isomorphisms

$$\text{Tor}_1^{\mathbb{Z}[M]}(\mathfrak{a}, W) = \text{Tor}_2^{\mathbb{Z}[M]}(\mathbb{Z}[N], W),$$

$$\text{Tor}_1^{\mathbb{Z}}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W) = \text{Tor}_1^{\mathbb{Z}[M]}(\mathbb{Z}[M] \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W)$$

where $\mathfrak{a} = \ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[N])$. Since the map

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow \mathbb{Z}[M] \otimes_{\mathbb{Z}} \ker(M^{\text{gp}} \rightarrow N^{\text{gp}})$$

of Proposition 3.1 vanishes after tensoring by $- \otimes_{\mathbb{Z}[M]} B$ and W is a B -module, the map

$$\mathrm{Tor}_1^{\mathbb{Z}[M]}(\mathfrak{a}, W) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}[M]}(\mathbb{Z}[M] \otimes_{\mathbb{Z}} \ker(M^{\mathrm{gp}} \rightarrow N^{\mathrm{gp}}), W)$$

is zero as desired. \square

Definition 6.2. (see [14, Theorem 6.1]) Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring, J a proper ideal of A and $B = A/J$. Let $N = M/\alpha_M^{-1}(J)$ be the quotient of the monoid M by the ideal $\alpha_M^{-1}(J)$ and let I be the ideal of A generated by $\alpha_M(M) \cap J$. We say that J is a log regular ideal if the equivalent conditions of Theorem 6.1 hold.

We say that $((A, \mathfrak{m}, k), M)$ is a log regular local ring if \mathfrak{m} is a log regular ideal.

Theorem 6.3. Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring and consider $((A, \mathfrak{m}, k), M')$ another prelog structure inducing the same log structure. Assume that M and M' are integral. For any proper ideal J of A , J is log regular in $((A, \mathfrak{m}, k), M)$ if and only if it is log regular in $((A, \mathfrak{m}, k), M')$.

Proof. It is sufficient to show the result for $M' = M^{\mathrm{log}}$. With the notation of Definition 6.2, we have $I := \langle \alpha_M(M) \cap J \rangle_A = \langle \alpha_{M^{\mathrm{log}}}(M^{\mathrm{log}}) \cap J \rangle$ since for any element $x = (m, u) \in M \oplus_{\alpha_M^{-1}(A^*)} A^* = M^{\mathrm{log}}$, we have $\alpha_{M^{\mathrm{log}}}(x) = \alpha(m) \cdot u$ where u is a unit in A . In particular, $\ker(A/I \rightarrow B)$ do not change replacing M by M^{log} .

We also have

$$\begin{aligned} \mathbb{Z}[M^{\mathrm{log}}/\alpha_{M^{\mathrm{log}}}^{-1}(J)] &= \mathbb{Z}[M/\alpha_M^{-1}(J)] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{log}}] \\ &= \mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{log}}], \end{aligned}$$

so it is enough to show that

$$\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = \mathrm{Tor}_1^{\mathbb{Z}[M^{\mathrm{log}}]}(A, \mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{log}}]).$$

For simplicity, let us denote $\alpha = \alpha_M: M \rightarrow A$, and factorize $\alpha^{-1}(A^*) \rightarrow A^*$ first as

$$\alpha^{-1}(A^*) \xrightarrow{u} \alpha^{-1}(A^*)^{\mathrm{gp}} \xrightarrow{v} A^*$$

and factorize again the second map as follows:

$$\alpha^{-1}(A^*) \xrightarrow{u} \alpha^{-1}(A^*)^{\mathrm{gp}} \xrightarrow{v_1} \alpha^{-1}(A^*)^{\mathrm{gp}} / \ker v \xrightarrow{v_2} A^*.$$

Therefore, the pushout that defines M^{\log}

$$\begin{array}{ccc} \alpha^{-1}(A^*) & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\log} \end{array}$$

can be decompose in a commutative diagram of pushouts

$$\begin{array}{ccccccc} \alpha^{-1}(A^*) & \xrightarrow{u} & \alpha^{-1}(A^*)^{\text{gp}} & \xrightarrow{v_1} & \alpha^{-1}(A^*)^{\text{gp}}/\ker(v) & \xrightarrow{v_2} & A^* \\ \downarrow r & & \downarrow s & & \downarrow & & \downarrow \\ M & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M^{\log} \end{array}$$

Note that s is injective since it is a localization of the injective homomorphism r and M is integral.

Now, we denote as

$$\begin{array}{ccc} N_1 & \longrightarrow & N_2 \\ \downarrow & & \downarrow \\ N_3 & \longrightarrow & N_4 \end{array}$$

any of previous pushouts and consider W a $\mathbb{Z}[M^{\log}]$ -module and $P = N_3/T$ with T an ideal of N_3 . Let $N_3 \rightarrow X \rightarrow P$ a factorization cofibration-trivial fibration in the category of simplicial monoids. Then $\mathbb{Z}[X]$ is a projective $\mathbb{Z}[N_3]$ -resolution of $\mathbb{Z}[P]$ [10, I.2.2.3] and so

$$\begin{aligned} \text{Tor}_*^{\mathbb{Z}[N_3]}(W, \mathbb{Z}[P]) &= H_*(W \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[X]) = H_*(W \otimes_{\mathbb{Z}[N_4]} \mathbb{Z}[N_4] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[X]) \\ &\stackrel{(1)}{=} \text{Tor}_*^{\mathbb{Z}[N_4]}(W, \mathbb{Z}[N_4] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[P]) \end{aligned}$$

where the equality (1) follows from:

$$\begin{aligned} H_*(\mathbb{Z}[N_4] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[X]) &= H_*(\mathbb{Z}[N_3 \oplus_{N_1} N_2] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[X]) \\ &= H_*(\mathbb{Z}[N_3] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[X]) \\ &= H_*(\mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}[X]) \\ &\stackrel{(2)}{=} \mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}[P] \\ &= \mathbb{Z}[N_4] \otimes_{\mathbb{Z}[N_3]} \mathbb{Z}[P] \end{aligned}$$

where the equality (2) follows (depending of the pushout) from:

- In the first pushout, since $\mathbb{Z}[N_2]$ is a flat $\mathbb{Z}[N_1]$ module (because u is a localization and then $\mathbb{Z}[u]$ is a localization too).
- In the second one, it follows by Remark 4.13, since $\mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}[X] = \mathbb{Z}[X/\ker(v)]$ is a resolution of $\mathbb{Z}[P/\ker(v)] = \mathbb{Z}[N_2] \otimes_{\mathbb{Z}[N_1]} \mathbb{Z}[P]$. Note that P is $M/T \oplus_M M_1 = M_1/\tilde{T}$ in this case (where \tilde{T} denotes the ideal image of T in M_1) and $\ker(v) \rightarrow \alpha^{-1}(A^*)^{\text{gp}} \rightarrow M_1$ is injective. Moreover, since \tilde{T} is an ideal of M_1 , $\ker(v) \rightarrow M_1/\tilde{T}$ is injective. On the other hand, the action of $\ker(v)$ on X is free, since

$$X = M_1 \oplus \mathbb{N}^{\tilde{X}},$$

and M_1 is integral (the homomorphism u is integral by [9, Remark 6.2.5.(i)] and so M_1 is integral because M is also integral).

- In the last pushout, it follows because $\mathbb{Z}[N_2]$ is a free $\mathbb{Z}[N_1]$ module (since v_2 is injective homomorphism of groups).

Finally, applying successively this process to the three pushouts, we obtain as desired

$$\text{Tor}_*^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = \text{Tor}_*^{\mathbb{Z}[M^{\text{log}}]}(A, \mathbb{Z}[N] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\text{log}}]).$$

□

Corollary 6.4. *With the notation of Theorem 6.1, assume that M^{gp} is a free Abelian group. Then J is log regular if and only if $H_2((A, M), (B, N), k) = 0$.*

By Lemma 6.6, we can always work in this case (M^{gp} free).

Lemma 6.5. *Let $p: G_1 \rightarrow G_2$ be a surjective homomorphism of Abelian groups, $i: N \rightarrow G_2$ a homomorphism of monoids. Then $(G_1 \times_{G_2} N)^{\text{gp}} = G_1 \times_{G_2} N^{\text{gp}}$.*

Proof. The homomorphism

$$\begin{aligned} (G_1 \times_{G_2} N)^{\text{gp}} &\longrightarrow G_1 \times_{G_2} N^{\text{gp}} \\ \frac{(g_1, n_1)}{(g_2, n_2)} &\longmapsto \left(\frac{g_1}{g_2}, \frac{n_1}{n_2} \right) \end{aligned}$$

is well defined and has inverse

$$\left(g, \frac{n_1}{n_2} \right) \longmapsto \frac{(gh, n_1)}{(h, n_2)}$$

where $h \in p^{-1}(i(n_2))$.

□

Lemma 6.6. *Let (B, N) be a log ring. There exists a monoid P and a homomorphism of monoids $P \rightarrow N$ such that $(B, N) = (B, P)^{\text{log}}$ and P^{gp} is a free Abelian group.*

Moreover, if N is integral, then P can be chosen integral too.

Proof. [9, Corollary 12.1.35] Let G be a free Abelian group with a surjective homomorphism of groups $\rho: G \rightarrow N^{\text{gp}}$. Let $P := G \times_{N^{\text{gp}}} N$. We have a commutative diagram of pullbacks

$$\begin{array}{ccc}
 H := (\beta\tau)^{-1}(B^*) & \xrightarrow{\pi} & B^* \\
 \downarrow & & \downarrow \\
 P & \xrightarrow{\tau} & N \\
 \downarrow j & & \downarrow i \\
 G & \xrightarrow{\rho} & N^{\text{gp}}
 \end{array}$$

where $\beta: N \rightarrow B$ is the structural homomorphism. So $\ker \pi$ ($:= \pi^{-1}(\{1\})$) = $\ker \tau = \ker \rho$ and $\ker \rho$ is a group. If $P/\ker(\tau)$ denotes the quotient by the action of $\ker(\tau)$ on P , it is easy to check that $P/\ker(\tau) = N$ and $H/\ker(\pi) = B^*$.

Moreover,

$$P \oplus_H B^* = P/\ker(\tau) \oplus_{H/\ker(\pi)} B^* = N \oplus_{B^*} B^* = N$$

and so $(B, P)^{\text{log}} = (B, N)$. Finally, $P^{\text{gp}} = G$ by Lemma 6.5, and so it is a free Abelian group.

Finally, if N is integral, that is, i is injective, then j is injective, and therefore P is integral. \square

Lemma 6.7. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring and the maximal ideal $\mathfrak{m}_M = M - M^*$ of M . Assume that M is integral. If $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/\mathfrak{m}_M]) = 0$, then $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$ for any ideal I of M .*

Proof. Since M is Noetherian, there exists a maximal element of the set of ideals of M such that $\text{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) \neq 0$, assuming this set is not empty. Clearly, $I \subset \mathfrak{m}_M$ (if not $I = M$), and then by hypothesis, $I \subsetneq \mathfrak{m}_M$. Let $a \in \mathfrak{m}_M - I$. Let $T = \{x \in M/x a \in I\}$, so that we have an exact sequence

$$0 \longrightarrow \mathbb{Z}[M/T] \xrightarrow{\cdot a} \mathbb{Z}[M/I] \longrightarrow \mathbb{Z}[M/I \cup Ma] \longrightarrow 0.$$

We have $I \subset T$. If $I \subsetneq T$, $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/T]) = 0 = \mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I \cup Ma])$ by the maximality of I . So $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$.

If $I = T$, we have a surjective homomorphism of A -modules of finite type

$$\tilde{\alpha}: \mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I])$$

where $\tilde{\alpha} := \alpha(a) \in \mathfrak{m} \subset A$. By Nakayama Lemma, $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$. \square

Corollary 6.8. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring and the maximal ideal $\mathfrak{m}_M = M - M^*$ of M . Assume that M is integral. If $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/\mathfrak{m}_M]) = 0$, then*

$$\mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/\mathfrak{m}_M]) = 0$$

for all $n \geq 1$.

Proof. By Lemma 6.7, it is sufficient to show that if $\mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$ for any ideal I of M for some $n \geq 1$, then $\mathrm{Tor}_{n+1}^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$ for any ideal I of M . So assume $\mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$ for any ideal I of M . By [9, 6.1.27.(i)-(iii)], we have then $\mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[T]) = 0$ for any M -set T . Applying Tor to the exact sequence

$$0 \longrightarrow \mathbb{Z}[I] \longrightarrow \mathbb{Z}[M] \longrightarrow \mathbb{Z}[M/I] \longrightarrow 0,$$

we obtain

$$0 = \mathrm{Tor}_{n+1}^{\mathbb{Z}[M]}(A, \mathbb{Z}[M]) \longrightarrow \mathrm{Tor}_{n+1}^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) \longrightarrow \mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[I]) = 0$$

and so $\mathrm{Tor}_{n+1}^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/I]) = 0$. \square

Theorem 6.9. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring and $\mathfrak{m}_M = M - M^*$. The following are equivalent:*

(i) (A, M) is a log regular local ring.

(ii) The map

$$H_2(A, k, k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{sp} \rightarrow (M/\mathfrak{m}_M)^{sp}), k) \longrightarrow H_2((A, M), (k, M/\mathfrak{m}_M), k)$$

is zero on the first summand and an isomorphism on the second one.

If moreover M is integral, then these conditions imply:

(iii) $H_n((A, M), (k, M/\mathfrak{m}_M), k) = 0$ for any $n \geq 3$.

Proof. Equivalence of (i) and (ii) follows from Theorem 6.1. We will see (iii). By Corollary 6.8, $\mathrm{Tor}_n^{\mathbb{Z}[M]}(A, \mathbb{Z}[M/\mathfrak{m}_M]) = 0$ for all $n > 0$ and so by base change [2, Proposition 4.54] we have isomorphisms

$$H_i(\mathbb{Z}[M], \mathbb{Z}[M/\mathfrak{m}_M], k) \longrightarrow H_i(A, A/\alpha(\mathfrak{m}_M)A, k)$$

for all $i \geq 0$. Since $A/\alpha(\mathfrak{m}_M)A$ is a regular ring, we have by [2, Proposition 6.26, Théorème 5.1] the isomorphisms

$$H_i(A, A/\alpha(\mathfrak{m}_M)A, k) \longrightarrow H_i(A, k, k)$$

for all $i \geq 2$. Therefore

$$H_i(\mathbb{Z}[M], \mathbb{Z}[M/\mathfrak{m}_M], k) \longrightarrow H_i(A, k, k)$$

are isomorphisms for all $i \geq 2$, and then from the fundamental exact sequence we deduce

$$H_n((A, M), (k, M/\mathfrak{m}_M), k) = 0$$

for all $n \geq 3$. □

Definition 6.10. *We say that a local prelog ring $((A, \mathfrak{m}, k), M)$ is log complete intersection if $H_3((A, M), (k, M/\mathfrak{m}_M), k) = 0$. In particular, a log regular ring with M integral is log complete intersection.*

6.2 Applications

From the fundamental exact sequence (Theorem 4.3) we obtain the following theorem:

Theorem 6.11. *Let $((A, \mathfrak{m}, k), M)$ be a Noetherian local prelog ring with M an integral monoid.*

- (i) *If (A, M) is a log regular ring, then A is a regular local ring if and only if $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[M/\mathfrak{m}_M])$ is generated by a regular sequence.*
- (ii) *If A is a regular local ring, then (A, M) is log complete intersection if and only if $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[M/\mathfrak{m}_M])$ is generated by a regular sequence.*

Proof. It follows from Theorem 4.3, Theorem 6.9 and [2, Théorème 6.25], having in mind that we have seen at the end of the proof of Theorem 6.1 that the map

$$H_2(\mathbb{Z}[M], \mathbb{Z}[M/\mathfrak{m}_M], k) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}[M]}(\ker(M^{\mathrm{gp}} \rightarrow (M/\mathfrak{m}_M)^{\mathrm{gp}}), k)$$

vanishes. □

Theorem 6.12. *Let (A, M) be a Noetherian local prelog ring, J a proper ideal of A , $B = A/J$ and $N = M/\alpha_M^{-1}(J)$, where $\alpha_M : M \rightarrow A$ is the structural map.*

(i) *If (A, M) is log regular and (B, N) is log complete intersection, then J is a log regular ideal.*

(ii) *If (B, N) is log regular and J is a log regular ideal, then (A, M) is a log regular local ring.*

(iii) *Assume that M is integral and M^{gp} free (for instance if M is saturated and sharp [18, I.1.3.5]). If (A, M) is log regular (or log complete intersection) and J is log regular, then (B, N) is log complete intersection.*

Proof. (i) Assume first that $\alpha_M^{-1}(J) \neq \emptyset$, so $N^{\text{gp}} = \{1\}$.

If (B, N) is log complete intersection, then the map

$$\alpha : H_2((A, M), (B, N), k) \longrightarrow H_2((A, M), (k, M/\mathfrak{m}_M), k)$$

is injective. In the commutative diagram

$$\begin{array}{ccc} H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) & \longrightarrow & H_2(\mathbb{Z}[M], \mathbb{Z}[M/\mathfrak{m}_M], k) \\ \downarrow & & \downarrow \\ H_2(A, B, k) \oplus \text{Tor}_1^{\mathbb{Z}}(M^{\text{gp}}, k) & \xrightarrow{\varepsilon \oplus \text{id}} & H_2(A, k, k) \oplus \text{Tor}_1^{\mathbb{Z}}(M^{\text{gp}}, k) \\ \downarrow \beta_1 \oplus \beta_2 & & \downarrow \gamma_1 \oplus \gamma_2 \\ H_2((A, M), (B, N), k) & \xrightarrow{\alpha} & H_2((A, M), (k, M/\mathfrak{m}_M), k) \end{array}$$

we know that γ_2 is an isomorphism and $\gamma_1 = 0$. Therefore β_2 is surjective and $\beta_1 = 0$. But β_2 is injective (we have seen in the last lines of the proof of Theorem 6.1 that the map $H_2(\mathbb{Z}[M], \mathbb{Z}[N], k) \rightarrow \text{Tor}_1^{\mathbb{Z}}(M^{\text{gp}}, k)$ vanishes), and then J is log regular by Theorem 6.1.

Now if $\alpha_M^{-1}(J) = \emptyset$, then $M = N$, so $\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) = \{1\}$. Therefore J is a log regular ideal if and only if $H_2((A, M), (B, N), k) = 0$. As in the previous diagram, we obtain that the map

$$\beta_1 : H_2(A, B, k) \longrightarrow H_2((A, M), (B, N), k)$$

vanishes. But since $M = N$, this map is an isomorphism (Example 4.2).

(ii) Assume first that $\alpha_M^{-1}(J) \neq \emptyset$. We have the above commutative diagram again, where now α is surjective (since $\mathrm{Tor}_1^{\mathbb{Z}}(N^{\mathrm{gp}}, k) = H_2((B, N), (k, M/\mathfrak{m}_M), k)$ is zero, $\beta_1 = 0$ and β_2 is an isomorphism. This implies that γ_2 is an isomorphism since the map $H_2(\mathbb{Z}[M], \mathbb{Z}[M/\mathfrak{m}_M], k) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(M^{\mathrm{gp}}, k)$ is zero as above. Also, $\gamma_1 = 0$ by diagram chasing. If $\alpha_M^{-1}(J) = \emptyset$, the proof is similar to the proof of (i) when $\alpha_M^{-1}(J) \neq \emptyset$.

(iii) By Corollary 6.4, J is log regular if and only if $H_2((A, M), (B, N), k) = 0$. Then the result follows from the Jacobi-Zariski exact sequence and Theorem 6.9.(iii). \square

Remark 6.13. Theorem 6.12.(i) together with Definition 6.10 extends to the logarithmic setting the fact that a closed immersion between regular schemes is a regular immersion. Another extension of this fact was already proved by Kato [14, Theorem 4.2]:

Theorem. Let (A, M) be a Noetherian log ring with M integral and J a proper ideal of A . Let $(B, N) = (A/J, p^*M)$ where $p: A \rightarrow B$ is the canonical projection. If (A, M) and (B, N) are log regular, then J is generated by a regular sequence.

We can deduce this theorem easily from our results: Let $P \rightarrow M$ be a homomorphism of monoids with P integral and P^{gp} free such that $(A, P)^{\mathrm{log}} = (A, M)$ (Lemma 6.6). By Lemma 2.5, $(B, N) = (B, P)^{\mathrm{log}}$. Also, by Theorem 6.3, (A, P) and (B, P) are log regular. We have then an exact sequence, where k is the residue field of A and B ,

$$H_3((B, P), (k, P/\mathfrak{m}_P), k) \rightarrow H_2((A, P), (B, P), k) \rightarrow H_2((A, P), (k, P/\mathfrak{m}_P), k)$$

and where the modules to the left and right are zero by Theorem 6.9.(iii) and Corollary 6.4. Therefore, by Example 4.2,

$$H_2(A, B, k) = H_2((A, P), (B, P), k) = 0,$$

and so J is generated by a regular sequence by [2, Théorème 6.25].

Example 6.14. Let (A, \mathfrak{m}, k, M) be a Noetherian local prelog ring with M integral saturated and sharp. Let \hat{A} be the completion of A for the \mathfrak{m} -adic topology and consider the prelog ring (\hat{A}, M) . We have $H_n((A, M), (\hat{A}, M), k) = 0$ for all $n \geq 0$ by Example 4.2 and [2, Remarque 10.22]. Then from the Jacobi-Zariski exact sequence we obtain

$$H_n((A, M), (k, M/\mathfrak{m}_M), k) = H_n((\hat{A}, M), (k, M/\mathfrak{m}_M), k)$$

for all $n \geq 0$. In particular, (A, M) is log regular (resp. log complete intersection) if and only if so is (\hat{A}, M) . Let $R := C[[M]][[x_1, \dots, x_n]] \rightarrow \hat{A}$ be a surjective

homomorphism of rings with C a Cohen ring or a field [18, I.3.6]. Then (R, M) is log regular ([14, Theorem 3.2] or [18, Theorem III.1.11.2]) and by Theorem 6.12 (A, M) is log complete intersection if and only if $\ker(R \rightarrow \hat{A})$ is log regular. So (A, M) is log complete intersection if and only if (\hat{A}, M) is the quotient of a log regular ring by a regular ideal. This fact, together with theorems 6.1 and 6.9, gives a description of log complete intersection rings.

Now we will show how these homological methods allow us to compare log smoothness and log regularity. The following result is due to Kato [14, Proposition 8.3] in the finitely generated case.

Theorem 6.15. *Let $((B, \mathfrak{n}, l), N)$ be a Noetherian local prelog ring with N integral and saturated, and let $k \subset B$ be a field.*

- (i) *If (B, N) is a log formally smooth over $(k, 1)$ for the \mathfrak{n} -adic topology, then (B, N) is log regular.*
- (ii) *Assume that $l|k$ is separable. If (B, N) is log regular, then (B, N) is a log formally smooth over $(k, 1)$ for the \mathfrak{n} -adic topology.*

Proof. By Theorem 6.3, Proposition 5.10 and [9, Corollary 12.1.36] we can assume that N is also sharp, and then that N^{gp} is free [18, I.1.3.5]. Theorem 6.9 says then that (B, N) is log regular if and only if $H_2((B, N), (l, N/\mathfrak{m}_N), l) = 0$, while Theorem 5.14 says that (B, N) is a log formally smooth over $(k, 1)$ if and only if $H_1((k, 1), (B, N), l) = 0$. Since we have a Jacobi-Zariski exact sequence

$$\begin{aligned} H_2((k, 1), (l, N/\mathfrak{m}_N), l) &\longrightarrow H_2((B, N), (l, N/\mathfrak{m}_N), l) \longrightarrow \\ &\longrightarrow H_1((k, 1), (B, N), l) \longrightarrow H_1((k, 1), (l, N/\mathfrak{m}_N), l), \end{aligned}$$

it suffices to show that $H_2((k, 1), (l, N/\mathfrak{m}_N), l) = 0$, and that if $l|k$ is separable then $H_1((k, 1), (l, N/\mathfrak{m}_N), l) = 0$.

Since $N/\mathfrak{m}_N = \{0, 1\}$, we have a fundamental exact sequence

$$\begin{aligned} H_2(k, l, l) &\longrightarrow H_2((k, 1), (l, N/\mathfrak{m}_N), l) \longrightarrow \\ &\longrightarrow H_1(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \longrightarrow H_1(k, l, l) \longrightarrow H_1((k, 1), (l, N/\mathfrak{m}_N), l) \\ &\longrightarrow H_0(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \end{aligned}$$

where $\varepsilon: \mathbb{Z}[x] \rightarrow l$ is the map sending x to zero.

From the results [2, Proposition 7.4, Définition 7.11, Proposition 7.13] we deduce $H_2(k, l, l) = 0$, and if $l|k$ is separable then $H_1(k, l, l) = 0$. Therefore it suffices to show $H_i(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) = 0$ for $i = 0, 1$. We have an exact sequence

$$\begin{aligned} H_1(\mathbb{Z}, \mathbb{Z}[x], l) &\rightarrow H_1(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \rightarrow H_1(\mathbb{Z}[x], \mathbb{Z}[x]/(x - x^2), l) \xrightarrow{\varphi} \\ &\xrightarrow{\varphi} H_0(\mathbb{Z}, \mathbb{Z}[x], l) \rightarrow H_0(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \rightarrow H_0(\mathbb{Z}[x], \mathbb{Z}[x]/(x - x^2), l) \end{aligned}$$

which by [2, Corollaire 3.36, Proposition 6.1, Proposition 6.3] takes the form

$$\begin{aligned} 0 &\longrightarrow H_1(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \longrightarrow (x - x^2) \otimes_{\mathbb{Z}[x]} l \xrightarrow{\varphi} \\ &\xrightarrow{\varphi} \Omega_{\mathbb{Z}[x]|\mathbb{Z}} \otimes_{\mathbb{Z}[x]} l \longrightarrow H_0(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) \longrightarrow 0. \end{aligned}$$

Since

$$(x - x^2) \otimes 1 \xrightarrow{\varphi} (dx - 2xdx) \otimes 1 = dx \otimes 1 - 2dx \otimes \varepsilon(x) = dx \otimes 1 - 2dx \otimes 0 = dx \otimes 1,$$

we see that φ is an isomorphism of rank 1 l -vector spaces, and therefore we obtain $H_i(\mathbb{Z}, \mathbb{Z}[x]/(x - x^2), l) = 0$ for $i = 0, 1$. \square

Theorem 6.16. *Let (B, N) be a Noetherian local prelog ring with N integral and saturated containing a perfect field k . If (B, N) is log formally smooth over $(k, 1)$ for the topology of the maximal ideal, then $\Omega_{(B, N)|(k, 1)}$ is a flat B -module.*

Proof. (B, N) is log regular by Theorem 6.15, and by [9, 12.5.47] for any prime ideal \mathfrak{q} of B , $(B_{\mathfrak{q}}, N_{\mathfrak{q}})$ is log regular, where $N_{\mathfrak{q}} := T^{-1}N$ with $T = N - \alpha_B^{-1}(\mathfrak{q})$.

The monoid $N_{\mathfrak{q}}$ is saturated as follows: Let $\frac{n_1}{n_2} \in N^{\text{gp}}$ be such that $(\frac{n_1}{n_2})^r \in N_{\mathfrak{q}}$, that is, $\frac{n_1^r}{n_2^r} = \frac{n}{t}$ with $t \in T$; then $(\frac{tn_1}{n_2})^r = t^{r-1}n \in N$. Since N is saturated, $\frac{tn_1}{n_2} \in N$, and so $\frac{n_1}{n_2} \in N_{\mathfrak{q}}$.

Therefore we know that $N_{\mathfrak{q}}$ is an integral monoid and also saturated. Again by Theorem 6.15, $(B_{\mathfrak{q}}, N_{\mathfrak{q}})$ is log formally smooth over $(k, 1)$ for the $\mathfrak{q}B_{\mathfrak{q}}$ -adic topology. By Proposition 4.10 and Theorem 5.14 we have

$$H_1((k, 1), (B, N), W) = H_1((k, 1), (B_{\mathfrak{q}}, N_{\mathfrak{q}}), W) = 0$$

for any $k(\mathfrak{q})$ -module W , where $k(\mathfrak{q})$ is the residue field of $B_{\mathfrak{q}}$.

By [9, Corollary 12.1.36] there exists $N' \rightarrow N_{\mathfrak{q}}$, with N' integral saturated and sharp, inducing the same log structure in B . Therefore by propositions 4.10 and 4.17 we have isomorphisms

$$H_n((k, 1), (B, N), W) = H_n((k, 1), (B_{\mathfrak{q}}, N_{\mathfrak{q}}), W) = H_n((k, 1), (B_{\mathfrak{q}}, N'), W)$$

for any $n \geq 0$ and any $k(\mathfrak{q})$ -module W .

Now consider the Jacobi-Zariski exact sequence

$$\begin{aligned} H_{n+1}((k, 1), (k(\mathfrak{q}), N'/\mathfrak{m}_{N'}), W) &\longrightarrow H_{n+1}((B_{\mathfrak{q}}, N'), (k(\mathfrak{q}), N'/\mathfrak{m}_{N'}), W) \longrightarrow \\ &\longrightarrow H_n((k, 1), (B_{\mathfrak{q}}, N'), W) \longrightarrow H_n((k, 1), (B_{\mathfrak{q}}, N'), W). \end{aligned}$$

By Theorem 6.9 we have $H_{n+1}((B_{\mathfrak{q}}, N'), (k(\mathfrak{q}), N'/\mathfrak{m}_{N'}), W) = 0$ for all $n \geq 2$ and any $k(\mathfrak{q})$ -module W . Since $N'/\mathfrak{m}_{N'} = \{0, 1\}$, we saw in the proof of Theorem 6.15 that $H_n((k, 1), (k(\mathfrak{q}), N'/\mathfrak{m}_{N'}), W) = 0$ for all $n \geq 1$. Therefore $H_n((k, 1), (B_{\mathfrak{q}}, N'), W) = 0$ for all $n \geq 2$ and any $k(\mathfrak{q})$ -module W .

We have proved

$$H_n((k, 1), (B, N), W) = 0$$

for all $n \geq 1$, any $k(\mathfrak{q})$ -module W and any prime ideal \mathfrak{q} of B . Now the same proof as in [2, Supplément, Proposition 29] gives

$$H_n((k, 1), (B, N), X) = 0$$

for all $n \geq 1$ and any B -module X .

Finally, the universal coefficient exact sequence

$$\begin{aligned} H_1((k, 1), (B, N), B) \otimes_B X &\longrightarrow H_1((k, 1), (B, N), X) \longrightarrow \\ &\longrightarrow \mathrm{Tor}_1^B(H_0((k, 1), (B, N), B), X) \longrightarrow 0 \end{aligned}$$

gives us that $\Omega_{(B, N)|(k, 1)} = H_0((k, 1), (B, N), B)$ is a flat B -module. \square

Resumo en galego

O obxectivo deste traballo é estudar a regularidade en xeometría alxébrica logarítmica dende un punto de vista diferente, o que nos permitirá obter novos resultados e eliminar certas hipóteses innecesarias noutros resultados xa coñecidos.

A xeometría alxébrica logarítmica foi introducida de xeito independente por P. Deligne [5], por G. Faltings [7], e tamén por J. M. Fontaine e L. Illusie (desenvolvido por K. Kato en [13]), sendo esta última a empregada habitualmente. Un *esquema prelogarítmico* é un esquema X cunha estrutura adicional: un morfismo de feixes de monoides conmutativos $\alpha: \mathcal{M} \rightarrow (\mathcal{O}_X, \cdot)$ para algún feixe de monoides \mathcal{M} . Se ademais α induce un isomorfismo $\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$, onde \mathcal{O}_X^* é o feixe de unidades, dicimos que é un *esquema logarítmico*. Calquera esquema prelogarítmico induce un esquema logarítmico (substituíndo \mathcal{M} polo cadrado cocartesiano de $\alpha^{-1}(\mathcal{O}_X^*) \hookrightarrow \mathcal{M}$ e $\alpha^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^*$). Por exemplo, se $i: U \hookrightarrow X$ é unha inclusión aberta e $\mathcal{M} = \varphi^{-1}(i_*\mathcal{O}_U^*)$, onde $\varphi: \mathcal{O}_X \rightarrow i_*(\mathcal{O}_U)$ é a aplicación canónica, entón a inclusión $\mathcal{M} \rightarrow (\mathcal{O}_X, \cdot)$ dá unha estrutura prelogarítmica en X , obtendo así a estrutura logarítmica asociada.

Un dos conceptos máis importantes neste campo é a lisitude logarítmica.

Definición. *Sexa $f: (X, \mathcal{M}) \rightarrow (B, \mathcal{N})$ un morfismo de esquemas logarítmicos, onde para simplificar suporemos que os monoides son íntegros. Dise que f é log liso se é de tipo finito (como morfismo de esquemas) e verifica a seguinte condición: para todo morfismo de (Y, \mathcal{N}) -esquemas logarítmicos $j: (Z', \mathcal{P}') \rightarrow (Z, \mathcal{P})$, onde $Z \rightarrow Z'$ é un encaixe pechado definido por un ideal nilpotente e onde $(Z', \mathcal{P}') = (Z' = \mathcal{P})^{\log}$ (considerando (Z', \mathcal{P}) como esquema prelogarítmico vía $j^{-1}\mathcal{P} \rightarrow \mathcal{P}' \rightarrow (\mathcal{O}_{Z'}, \cdot)$), e para todo (Y, \mathcal{N}) -morfismo $(Z', \mathcal{P}') \rightarrow (X, \mathcal{M})$, existe localmente en Z un (Y, \mathcal{N}) -morfismo $(Z, \mathcal{P}) \rightarrow (X, \mathcal{M})$ tal que o diagrama*

$$\begin{array}{ccc} (X, \mathcal{M}) & & \\ \uparrow & \swarrow & \\ (Z', \mathcal{P}') & \longrightarrow & (Z, \mathcal{P}) \end{array}$$

é conmutativo.

Do mesmo xeito que os morfismos lisos funcionan ben en xeometría alxébrica, os morfismos log lisos tamén funcionan ben en xeometría alxébrica logarítmica. A clave é que certos morfismos de esquemas que non son lisos poden verse como morfismos de esquemas logarítmicos para certa estrutura logarítmica que fai que sexan logarítmicamente lisos. Por exemplo, se S é o espectro dun anel de valoración discreta, X é un esquema regular e $f: X \rightarrow S$ un morfismo plano cunha redución semi-estable (i.e. a fibra sobre un punto pechado de S é un divisor D con cruzamentos normais en X), entón as estruturas logarítmicas asociadas como anteriormente ás inmersións abertas $i: U := X - D \hookrightarrow X$ e $j: \{\xi\} \rightarrow S$, sendo ξ o punto xenérico de S , fan que f sexa log liso. Deste xeito, en certo sentido, a xeometría logarítmica resulta un contexto adecuado para traballar coa semi-estabilidade, dado que á lisitude logarítmica ten propiedades functoriais boas (por exemplo, a semi-estabilidade non se conserva baixo cambio de base, a diferenza do que sucede coa lisitude logarítmica).

Un concepto relacionado e introducido por K. Kato algúns anos despois [14], é o da regularidade logarítmica. A primeira parte deste traballo de Kato céntrase na xeneralización de resultados sobre regularidade (clásica) ao caso logarítmico. Porén, os resultados que obtivo Kato necesitan na súa maior parte algunhas hipóteses non desexables (nalgúns casos hipóteses de finitude, noutras fan referencia á regularidade logarítmica en lugar de intersección completa logarítmica ao non ter definido este concepto nin o de ideal regular logarítmico etc.). O noso primeiro obxectivo é evitar estas hipóteses indesexables. En moitos dos nosos resultados, mesmo evitamos as hipóteses de integridade (logarítmica). O segundo obxectivo desta memoria é completar con novos resultados o estudo sobre a regularidade logarítmica que realiza Kato. Por exemplo, obtemos os seguintes resultados (por seren todas as cuestións locais, usaremos a linguaxe de aneis logarítmicos):

Teorema 6.15. *Sexa $((B, n, l), N)$ un anel prelog local noetheriano con N íntegro e saturado, e $k \subset B$ un corpo.*

- (i) *Se (B, N) é log formalmente liso sobre $(k, 1)$ para a topoloxía n -ádica, entón (B, N) é log regular.*
- (ii) *Supoñamos que a extensión $l|k$ é separable. Se (B, N) é log regular, entón (B, N) é log formalmente liso sobre $(k, 1)$ para a topoloxía n -ádica.*

O anterior resultado foi obtido por Kato para os casos nos que B é finitamente xerado sobre k .

Teorema 6.16. *Sexa (B, N) un anel prelog local noetheriano con N íntegro e saturado que contén un corpo perfecto k . Se (B, N) é log formalmente liso sobre $(k, 1)$ para a topoloxía do ideal maximal, entón $\Omega_{(B, N)|(k, 1)}$ é un B -módulo plano.*

Aquí $\Omega_{(B,N)|(k,1)}$ é o módulo de diferenciais logarítmico, que é unha xeneralización do concepto de módulo de diferenciais con polos logarítmicos ao longo dun divisor con cruzamentos normais.

Teorema 6.12. *Sexa (A, M) un anel prelog local noetheriano, J un ideal propio de A , $B = A/J$ e $N = M/\alpha_M^{-1}(J)$, onde $\alpha_M : M \rightarrow A$ é a aplicación estrutural.*

- (i) *Se (A, M) é log regular e (B, N) é log intersección completa, entón J é un ideal log regular.*
- (ii) *Se (B, N) é log regular e J é un ideal log regular, entón (A, M) é un anel local log regular.*
- (iii) *Supoñamos que M é íntegro e M^{sp} libre (por exemplo, se M é saturado e sharp). Se (A, M) é log regular (ou log intersección completa) e J é log regular, entón (B, N) é log intersección completa.*

Unha variante do apartado (i), asumindo a maiores que (B, N) é log regular e M íntegro, foi obtida por Kato: se (A, M) e (B, N) son log regular e ademais $N = M$ íntegro, entón J é un ideal regular.

No contexto non logarítmico, unha importante ferramenta para enfrontarse a estas cuestións é o complexo cotanxente introducido por M. André [1] e D. Quillen [20]. A utilidade desta teoría de (co)homoloxía vén dada por dous feitos: por unha banda, o complexo compórtase moi ben (cambio de base, localización, importantes sucesións exactas etc.); mentres que, por outra banda, a súa anulación caracteriza a lisitude, a regularidade e a intersección completa:

Teorema. ([2], [20])

- (i) *Sexa B unha A -álgebra noetheriana e J un ideal de B . Entón, B é formalmente liso sobre A para a topoloxía J -ádica se e só se $H^1(A, B, W) = 0$ para todo B/J -módulo W .*
- (ii) *Sexa (A, k) un anel local noetheriano. Entón:*
 - (a) *O anel (A, k) é regular se e só se $H_2(A, k, k) = 0$.*
 - (b) *O anel (A, k) é intersección completa se e só se $H_3(A, k, k) = 0$.*

No contexto logarítmico tamén temos un “complexo cotanxente logarítmico”, que foi introducido por O. Gabber (os detalles poden consultarse en [19], onde M. Olsson tamén introduce e estuda unha variante deste complexo). Este complexo cotanxente logarítmico será a nosa principal ferramenta para lidar coas cuestións centrais deste traballo.

A continuación describiremos os contidos da tese. Nos capítulos 1 e 2 dáse unha rápida revisión das definicións e notacións de monoides e aneis logarítmicos que serán empregadas durante os seguintes capítulos. No Capítulo 3 estudamos as diferenciais logarítmicas e as derivacións logarítmicas. Non hai nada esencialmente novo nesta parte, pero dado que non atopamos referencias na literatura para algúns destes resultados, polo que preferimos incluír unha exposición dos mesmos.

A Sección 4.1 céntrase no desenvolvemento do complexo logarítmico cotanxente de Gabber. O estudo realizado en [19] non é suficiente para os nosos propósitos, dado que carece de resultados que necesitamos en certos momentos, outros son incluídos pero con hipóteses máis fortes que as que podemos asumir (cambio de base ou da localización). Esta sección contén tamén un dos dous principais resultados técnicos, ao que chamaremos triángulo fundamental (Teorema 4.3): se $(A, M) \rightarrow (B, N)$ é un homomorfismo de aneis prelogarítmicos e W é un B -módulo, temos un triángulo distinguido na categoría derivada de \mathbb{Z} -módulos

$$\mathbb{L}_{\mathbb{Z}[N]|\mathbb{Z}[M] \otimes_{\mathbb{Z}[N]} W} \longrightarrow (\mathbb{L}_{B|A} \otimes_B W) \oplus (X_{N|M} \otimes_{\mathbb{Z}} W) \longrightarrow \mathbb{L}_{(B,N)|(A,M)} \otimes_B W \longrightarrow$$

onde os dous complexos \mathbb{L} da esquerda son os complexos cotanxentes “usuais” e o \mathbb{L} da dereita é o complexo cotanxente logarítmico, mentres que $X_{N|M}$ é un complexo que aparece no triángulo distinguido

$$M^{\text{gp}} \longrightarrow N^{\text{gp}} \longrightarrow X_{N|M} \longrightarrow \dots$$

Este resultado é fundamental e será empregado de comezo a fin, xa que nos permitirá relacionar propiedades do homomorfismo logarítmico $(A, M) \rightarrow (B, N)$ con propiedades dos homomorfismos de aneis $A \rightarrow B$ e $\mathbb{Z}[M] \rightarrow \mathbb{Z}[N]$ a través da sucesión exacta fundamental

$$\begin{aligned} \dots &\rightarrow H_n(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow H_n(A, B, W) \oplus H_n(X_{N|M} \otimes_{\mathbb{Z}} W) \rightarrow H_n((A, M), (B, N), W) \rightarrow \\ &\rightarrow H_{n-1}(\mathbb{Z}[M], \mathbb{Z}[N], W) \rightarrow \dots \rightarrow H_0((A, M), (B, N), W) \rightarrow 0, \end{aligned}$$

onde

- $H_n(X_{N|M} \otimes_{\mathbb{Z}} W) = 0$ para $n \geq 3$,
- $H_2(X_{N|M} \otimes_{\mathbb{Z}} W) = \text{Tor}_1^{\mathbb{Z}}(\ker(M^{\text{gp}} \rightarrow N^{\text{gp}}), W)$,
- $H_1(X_{N|M} \otimes_{\mathbb{Z}} W)$ é un B -módulo que aparece na sucesión exacta (que escinde de xeito non canónico)

$$0 \rightarrow \ker(M^{\text{gp}} \rightarrow N^{\text{gp}}) \otimes_{\mathbb{Z}} W \rightarrow H_1(X_{N|M} \otimes_{\mathbb{Z}} W) \rightarrow \text{Tor}_1^{\mathbb{Z}}(N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}}), W) \rightarrow 0,$$

- $H_0(X_{N|M} \otimes_{\mathbb{Z}} W) = (N^{\text{gp}}/\text{Im}(M^{\text{gp}} \rightarrow N^{\text{gp}})) \otimes_{\mathbb{Z}} W$.

Incluimos tamén unha demostración do feito de que a loguificación non cambia a (co)homoloxía do complexo cotanxente logarítmico cando as estruturas son íntegras. Este resultado aparece en [19]. Non sabemos se as hipóteses de integridade son necesarias. Nalgúns puntos do Capítulo 6 non poderemos ir máis lonxe por mor destas hipóteses de integridade (mesmo para estruturas íntegras, xa que aparecen cocientes non íntegros nas probas). Esta é a razón pola cal incluimos unha demostración alternativa deste teorema de Olsson: non fomos quen de evitar as hipóteses de integridade, pero pensamos que a nosa proba está un pouco máis preto de poder evitalas.

A Sección 4.2 non é necesaria para o resto do contido da memoria, pero ten un interese independente. Nela damos unha versión logarítmica e análoga da sucesión espectral de Quillen. Se $C \rightarrow B$ é un homomorfismo sobrexectivo de aneis logarítmicos, Quillen dá unha sucesión espectral converxente [20, 22]

$$E_{p,q}^2 = H_{p+q}(S_q \mathbb{L}_{B|A}) \Rightarrow \mathrm{Tor}_{p+q}^C(B, B)$$

onde $S_q \mathbb{L}_{B|A}$ é a q -ésima potencia simétrica do complexo cotanxente. Se ademais $C = B \otimes_A B$, onde B é unha A -álgebra plana, esta sucesión espectral toma a forma

$$E_{p,q}^2 = H_p(\Lambda^q \mathbb{L}_{B|A}) \Rightarrow \mathrm{HH}_{p+q}(B|A)$$

onde Λ^q denota a q -ésima potencia exterior e HH é a homoloxía de Hochschild. Nós damos unha sucesión espectral similar no caso logarítmico (Teorema 4.21). Aínda que a sucesión espectral non logarítmica ten un papel fundamental nos traballos de Quillen [20, 22], o noso único interese para obter a versión logarítmica é que o seu *abutment* (que definimos de forma explícita na memoria) dá outro candidato para ser a (co)homoloxía logarítmica de Hochschild.

O Capítulo 4 finaliza coa Sección 4.3, onde adaptamos os métodos de [15] e [2, Proposition 15.12] co obxectivo de obter o complexo máis pequeno que sexa útil para calcular a segunda (co)homoloxía de grupos do complexo cotanxente logarítmico.

O Capítulo 5 dedícase ao estudo (co)homolóxico da lisitude logarítmica. A idea é evitar as hipóteses de finitude nos morfismos e queremos tamén evitar as hipóteses de integridade tanto como sexa posible, polo cal os primeiros pasos feitos en [19] e [12] non nos son de axuda. Necesitamos polo tanto comezar dende o inicio. Procederemos da forma habitual: primeiro caracterizamos os elementos de H^1 como extensións de álgebras logarítmicas. No caso íntegro está feito en [19], pero en xeral a propia definición de extensión está máis involucrada, e así a nosa demostración é diferente e substancialmente máis longa ca no caso íntegro. Posteriormente deducimos algúns resultados que caracterizan a lisitude logarítmica a través da anulación do complexo cotanxente logarítmico. Este proceso é o estándar para a lisitude logarítmica, pero necesitamos ter en conta as topoloxías, é dicir, caracterizar a lisitude logarítmica

formal (dado que é necesaria cando os homomorfismos non son de tipo finito), dando lugar a demostracións máis longas. Por exemplo, obtemos os teoremas 5.14 e 5.15.

Este capítulo remata con algunhas aplicacións de menor importancia: a Proposición 5.18 e a Proposición 5.19. En particular, a Proposición 5.19 é unha ampla xeneralización da parte de lisitude de [11, Theorem 0.1]. Para unha mellor comparación destes resultados, engadimos o seguinte corolario desta proposición:

Corolario 5.20. *Sexan $(A, M) \rightarrow (B, N)$ e $(A, M) \rightarrow (C, P)$ homomorfismos de aneis prelogarítmicos con $\text{Spec}(C) \rightarrow \text{Spec}(A)$ sobrexectivo e supoñamos que $\mathbb{Z}[M] \rightarrow \mathbb{Z}[P]$ e $A \rightarrow C$ son planos, e tamén que $M^{sp} \rightarrow P^{sp}$ é inxectivo. Consideramos un ideal J de B . Entón (B, N) é unha (A, M) -álgebra formalmente lisa para a topoloxía J -ádica se e só se $(B \otimes_A C, N \oplus_M P)$ é unha (C, P) -álgebra formalmente lisa para a topoloxía $J \otimes_A C$ -ádica.*

Finalmente, no Capítulo 6, estamos preparados para estudar a regularidade logarítmica. Comezamos coa caracterización desta en termos do complexo cotanxente logarítmico, que é o noso segundo principal resultado técnico (Teorema 6.3 a Teorema 6.9):

Teorema. *Sexa $((A, \mathfrak{m}, k), M)$ un anel (pre)logarítmico local noetheriano e \mathfrak{m}_M o conxunto de non unidades de M .*

(i) *(A, M) é un anel logarítmico local regular se e só se o conúcleo do homomorfismo canónico inxectivo*

$$\text{Tor}_1^{\mathbb{Z}}(\ker(M^{sp} \rightarrow (M/\mathfrak{m}_M)^{sp}), k) \longrightarrow H_2((A, M), (k, M/\mathfrak{m}_M), k)$$

se anula.

(ii) *Existe unha estrutura prelogarítmica (A, P) en A que induce a mesma estrutura logarítmica que (A, M) e tal que (A, P) é log regular se e só se $H_2((A, P), (k, P/\mathfrak{m}_P), k) = 0$.*

(iii) *Se M é un monoide íntegro, o monoide P en (ii) tamén pode ser escollido íntegro, e polo tanto (A, P) é log regular se e só se (A, M) é log regular.*

(iv) *Se M é un monoide íntegro, entón que (A, M) sexa log regular implica que $H_n((A, M), (k, M/\mathfrak{m}_M), k) = 0$ para todo $n \geq 3$.*

En particular, o apartado (i) do teorema anterior dedúcese dun resultado máis xeral, que non só é válido para o ideal maximal \mathfrak{m} , senón que tamén é certo para calquera ideal de A . Isto permite introducir o concepto de ideal log regular. Este xoga un papel similar ao da regularidade de ideais no caso non logarítmico. Por exemplo,

probamos que se (A, M) é un anel local log regular e J un ideal propio de A , entón J é un ideal log regular se $(A/J, M/\alpha^{-1}(J))$ é log intersección completa (onde $\alpha: M \rightarrow A$ é a aplicación canónica), e a implicación contraria é certa baixo certas hipóteses máis relaxadas. A definición é a seguinte: Sexa $((A, \mathfrak{m}, K), M)$ un anel prelogarítmico local noetheriano, J un ideal propio de A , $\alpha: M \rightarrow A$ o homomorfismo estrutural e $N = M/\alpha^{-1}(J)$ o monoide cociente. Sexa I o ideal de A xerado por $\alpha(M) \cap J$. Dise que J é un ideal *log regular* se se verifican as dúas condicións seguintes:

- (i) O ideal $\ker(A/I \rightarrow A/J)$ de A/I está xerado por unha sucesión regular.
- (ii) $\mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[N]) = 0$.

Esta definición ten unha cómoda caracterización homolóxica (Teorema 6.1):

Teorema. *Baixo as condicións anteriores, son equivalentes:*

- (i) J é log regular.
- (ii) A aplicación

$$H_2(A, A/J, k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{sp} \rightarrow N^{sp}), k) \rightarrow H_2((A, M), (A/J, N), k)$$

na sucesión fundamental exacta é a aplicación nula no primeiro sumando e un isomorfismo no segundo.

- (iii) Para todo A/J -módulo W , a aplicación

$$H_2(A, A/J, W) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\ker(M^{sp} \rightarrow N^{sp}), W) \rightarrow H_2((A, M), (A/J, N), W)$$

na sucesión fundamental exacta é a aplicación nula no primeiro sumando e un isomorfismo no segundo.

Á vista destes resultados, estamos xa en disposición de estudar a regularidade logarítmica. Por exemplo, a partir do triángulo fundamental chegamos de xeito inmediato á relación entre regularidade logarítmica e regularidade dun anel logarítmico:

Teorema 6.11. *Sexa $((A, \mathfrak{m}, k), M)$ un anel prelogarítmico local noetheriano con M un monoide íntegro.*

- (i) *Se (A, M) é un anel log regular, entón temos que A é un anel regular se e só se $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[M/\mathfrak{m}_M])$ está xerado por unha sucesión regular.*
- (ii) *Se A é un anel local regular, entón cúmprese que (A, M) é log intersección completa se e só se $\ker(\mathbb{Z}[M] \rightarrow \mathbb{Z}[M/\mathfrak{m}_M])$ está xerado por unha sucesión regular.*

Este derradeiro capítulo remata cos teoremas 6.15 e 6.16, que xa foron enunciados ao comezo deste resumo.



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