SPACELIKE ISOPARAMETRIC HYPERSURFACES

VÍCTOR SANMARTÍN-LÓPEZ

Abstract. We generalize Ferus' work to study isoparametric hypersurfaces in semi-Riemannian space forms focusing, in this particular case, on anti-De Sitter spaces. We will show that two is an upper bound for the number of principal curvatures in a spacelike isoparametric hypersurface in the anti-De Sitter space. This fact will lead us to deduce a partial classification of isoparametric hypersurfaces in anti-De Sitter spaces.

1. Introduction

The origin of isoparametric hypersurfaces traces back to the work of Somogliana [19], who studied this kind of objects in a Euclidean space of dimension three, motivated by a problem dealing with geometric optics. Since then, many famous mathematicians became interested in isoparametric hypersurfaces, a fact that is evidenced by the classifications of this type of submanifolds in Euclidean spaces by Levi-Civita [11] (in the three dimensional case) and Segre [17] (in arbitrary dimension), and the classification in real hyperbolic spaces by Cartan [3], together with the remarkable progress he made in spheres [2]. Moreover, recently, the problem has also been solved in spheres [20, 4, 10, 5, 7, 14, 18] and in complex hyperbolic spaces [6].

All these ambient spaces presented so far are Riemannian manifolds. However, it also makes sense to study isoparametric hypersurfaces in the semi-Riemannian case, and more precisely, in Lorentzian space forms, where the breadth of examples is much richer than in the Riemannian case. First of all, the definition of isoparametric hypersurface in this setting can be generalized in a natural way. Following Hahn [9], a hypersurface in a Lorentzian space form is said to be isoparametric if it has constant principal curvatures with constant algebraic multiplicities. In this context, some remarkable progress has been made as well. For instance, these objects are supposed to be classified in the Minkowski space by Magid [13], although in [1], Burth pointed out some gaps in Magid's arguments. There are also partial classifications in the De Sitter space. In this space, Nomizu [15] proved, using the fact that the number of principal curvatures is bounded from above by two, that spacelike hypersurfaces with constant principal curvatures are tubes around totally geodesic submanifolds. He also conjectured in the same paper [15] that examples of spacelike isoparametric hypersurfaces with more than two principal curvatures would appear in the anti-De Sitter space AdS^n. In this paper we answer this question negatively, proving that the number of principal curvatures of a spacelike isoparametric hypersurface

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in AdS^n is less or equal than two, using a different technique than that in [12], where the same question is addressed. Moreover, with the method presented below, we can also deduce some interesting results about Lorentzian isoparametric hypersurfaces in De Sitter spaces, where the problem is still open. These results will be published elsewhere.

This paper is organised as follows. In Section 2 we present the notations and conventions that will be used in this work. In Section 3 we show a general procedure on semi-Riemannian space forms, focused on anti-De Sitter spaces. This process will allow us to conclude, in Section 4, a partial classification of isoparametric hypersurfaces in those spaces.

2. Preliminaries and conventions

From now on, if $\bar{M}$ is a semi-Riemannian manifold we will be using the following notations. If p is a point in $\bar{M}$, we will write $T_p\bar{M}$ for the tangent space of $\bar{M}$ at p, $T\bar{M}$ for the tangent bundle of $\bar{M}$, and $\Gamma(T\bar{M})$ will denote the module of smooth vector fields on $\bar{M}$.

Although the procedure that is presented here may be adapted for a general semi-Riemannian space form, in these notes we will focus on anti-De Sitter spaces as ambient manifolds. Therefore, let $\mathbb{R}^{n+1}_2$, $n \geq 3$, denote the $(n+1)$-dimensional real vector space provided with the semi-riemannian metric $\langle x, y\rangle = -x_1y_1 - x_2y_2 + \sum_{i=3}^{n+1} x_iy_i$. We define the anti-De Sitter space of radio r, $AdS^n(r)$, as

$$AdS^n(r) = \{x \in \mathbb{R}^{n+1}_2 \mid \langle x, x\rangle = -r^2\}.$$

The anti-De Sitter space is a Lorentzian manifold with negative constant (sectional) curvature $c = -\frac{1}{r^2}$, whose curvature tensor $\bar{R}$ then reads $\bar{R}(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$, for $X, Y, Z \in \Gamma(AdS^n(r))$. We will write $\bar{\nabla}$ for the Levi-Civita connection of the anti-De Sitter space.

Let $M \subset AdS^n(r)$ be an isoparametric hypersurface, that is, a hypersurface with constant principal curvatures, $\lambda_1, \dots, \lambda_{n-1}$, and whose corresponding algebraic multiplicities, $m_{\lambda_1}, \dots, m_{\lambda_{n-1}}$, are constant along M [9]. Notice that, implicitly, we are assuming that M is a non-degenerate hypersurface of $AdS^n(r)$. Let us denote by ∇ and R the Levi-Civita connection and the curvature tensor of M, respectively. The *second fundamental form II* of M is defined by the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + II(X, Y)$$

for any X, $Y \in \Gamma(TM)$ and relates the connections of the ambient manifold and the hypersurface. Now, locally and up to sign, we can take a unique unit normal vector field $\xi \in \Gamma(\nu M)$. We write $\varepsilon = \langle \xi, \xi\rangle \in \{-1, 1\}$. Hence, the second fundamental form II is a multiple of ξ. The shape operator $\mathcal{S}$ of M with respect to ξ is the self-adjoint operator on M defined by $\langle \mathcal{S}X, Y\rangle = \langle II(X, Y), \xi\rangle$, where X, $Y \in \Gamma(TM)$. Consequently, we may write

$$\bar{\nabla}_X \xi = -\mathcal{S}X.$$

We also have the Gauss and Codazzi equations

$$\langle \bar{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle - \varepsilon\,\langle \mathcal{S}Y,Z\rangle\langle \mathcal{S}X,W\rangle + \varepsilon\,\langle \mathcal{S}X,Z\rangle\langle \mathcal{S}Y,W\rangle,$$
$$0 = \langle \bar{R}(X,Y)Z,\xi\rangle = \langle (\nabla_X\mathcal{S})Y - (\nabla_Y\mathcal{S})X, Z\rangle.$$

3. General procedure

Now, let λ be a real principal curvature of M with constant geometric multiplicity. Under these assumptions, it is easy to check that $T_\lambda = \ker(\mathcal{S}-\lambda I)$ constitutes a distribution on M. In fact, T_λ defines an autoparallel and, consequently, integrable distribution. In order to prove this last claim, let $X, Y \in \Gamma(T_\lambda)$. Then, for each vector field $Z \in \Gamma(TM)$, we obtain

$$\begin{aligned}\langle (\mathcal{S}-\lambda I)(\nabla_X Y), Z\rangle &= \langle \nabla_X(\mathcal{S}-\lambda I)Y - (\nabla_X(\mathcal{S}-\lambda I))Y, Z\rangle = -\langle (\nabla_X(\mathcal{S}-\lambda I))Y, Z\rangle\\ &= -\langle Y, (\nabla_X(\mathcal{S}-\lambda I))Z\rangle = -\langle Y, (\nabla_Z(\mathcal{S}-\lambda I))X\rangle\\ &= -\langle Y, \nabla_Z(\mathcal{S}-\lambda I)X\rangle + \langle Y, (\mathcal{S}-\lambda I)\nabla_Z X\rangle = 0,\end{aligned}$$

where in the third equality we have used the symmetry of $\nabla_X(\mathcal{S}-\lambda I)$ and in the fourth one the Codazzi equation. Therefore, we can construct L_λ, the integral submanifolds of the distribution T_λ through a point $p \in M$. Assume, in what follows, that the geometric and algebraic multiplicities of λ coincide. In this case, for each $p \in M$ and each $\mu \in \mathrm{Spec}(\mathcal{S})\backslash\{\lambda\}$, it is possible to select $r(\mu,p)$ big enough in such a way that, if $T_\mu(p) = \ker(\mathcal{S}-\mu I)_p^{r(\mu,p)}$, we obtain the orthogonal decomposition

$$T_pM = T_\lambda(p) \oplus \bigoplus_{\mu\neq\lambda} T_\mu(p).$$

Take now an element X in $T_\lambda(p)$ orthogonal to all the elements of $T_\lambda(p)$. Since $T_\lambda(p)$ is orthogonal to all the generalized eigenspaces $T_\mu(p)$ with $\mu \neq \lambda$ [16], then we deduce that X is orthogonal to all the elements in T_pM. But taking into account that M is non-degenerate, we can conclude that $X = 0$. Therefore, L_λ is a non-degenerate submanifold of M. In fact, L_λ is totally geodesic as a submanifold of M and totally umbilical as a submanifold of $AdS^n(r)$.

The next step is trying to understand the behaviour of the generalized eigenspaces T_μ, with $\mu \neq \lambda$, with respect to T_λ. In this sense, following Ferus' ideas [8], we examine the behaviour of $(\mathcal{S}-\lambda I)$ along a geodesic curve in L_λ. In order to do that, we introduce, as in [8], a tensor field $\mathcal{C}$ defined by

$$\mathcal{C}_X(Y) = -\mathcal{V}\nabla_Y\mathcal{H}X,$$

where, for each $p \in M$, $\mathcal{H}_p$ and $\mathcal{V}_p$ denote the orthogonal projections onto $\ker(\mathcal{S}-\lambda I)_p$ and $\mathrm{Im}(\mathcal{S}-\lambda I)_p$, respectively. It is easy to check that both $\mathcal{H}$ and $\mathcal{V}$ are parallel along L_λ.

Ferus' work focused on Riemannian geometry and consequently some of his results must be adapted to the more general setting of the semi-Riemannian case. The next lemma constitutes a generalization of Lemmas 1 and 2 in [8]. Notice that, even though the final claim is exactly the same, the arguments utilised in the proof should be modified slightly.

Lemma 3.1. *Let X be a vector $T_\lambda(p)$ and $Y \in T_pM$. Then:*

(i) $(\nabla_X(\mathcal{S}-\lambda I))Y = (\mathcal{S}-\lambda I)\circ \mathcal{C}_X(Y)$.

(ii) $(\nabla_X \mathcal{C})_X Y = \mathcal{C}_X^2(Y) + R_X Y$*, where* $R_X Y = \mathcal{V}R(\mathcal{V}Y, X)X$.

Let $\gamma\colon I \subset \mathbb{R} \to L_\lambda$ be a unit speed geodesic in L_λ, with $\eta = \langle \dot\gamma, \dot\gamma\rangle \in \{1,-1\}$. Using the Gauss equation and taking into account that $\dot\gamma \in \Gamma(\gamma^* T_\lambda)$, we obtain the Jacobi operator

$$\begin{aligned} R_{\dot\gamma}(X) &= \mathcal{V}\bar R(\mathcal{V}X,\dot\gamma)\dot\gamma + \varepsilon\langle \mathcal{S}\dot\gamma,\dot\gamma\rangle \mathcal{V}\mathcal{S}\mathcal{V}X - \varepsilon\langle \mathcal{S}\mathcal{V}X,\dot\gamma\rangle\mathcal{V}\mathcal{S}\dot\gamma \\ &= \mathcal{V}\bar R(\mathcal{V}X,\dot\gamma)\dot\gamma + \varepsilon\lambda\eta\mathcal{S}\mathcal{V}X - \varepsilon\lambda^2\langle \mathcal{V}X,\dot\gamma\rangle\mathcal{V}\dot\gamma = \mathcal{V}\bar R(\mathcal{V}X,\dot\gamma)\dot\gamma + \varepsilon\lambda\eta\mathcal{S}\mathcal{V}X \end{aligned}$$

and recalling that $AdS^n(r)$ has constant curvature, we substitute $\bar R$ by its value in the above equation to obtain

$$R_{\dot\gamma}X = \eta(c+\varepsilon\lambda\mathcal{S})\mathcal{V}X. \tag{1}$$

For each $t \in I$, we construct an endomorphism $\mathcal{A}(t)$ of the real vector space $T_{\gamma(t)}M = \ker(\mathcal{S}-\lambda I)_{\gamma(t)} \oplus \operatorname{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}$, defined as the inverse of $(\mathcal{S}-\lambda I)_{\gamma(t)}|_{\operatorname{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}}$ when restricted to $\operatorname{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}$, and defined as zero for the elements in $\ker(\mathcal{S}-\lambda I)_{\gamma(t)}$. Thus, $\mathcal{A}(t)$ is a tensor field along the curve γ. It is convenient to remark here that along γ the equality $\mathcal{V} = \mathcal{A}(\mathcal{S}-\lambda I) = (\mathcal{S}-\lambda I)\mathcal{A}$ holds. Taking derivatives along γ in this last equality, and using Lemma 3.1 we may write

$$\begin{aligned} 0 &= (\nabla_{\dot\gamma}\mathcal{V})\mathcal{A} = (\nabla_{\dot\gamma}\mathcal{A}(\mathcal{S}-\lambda I))\mathcal{A} = \{(\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S}-\lambda I) + \mathcal{A}(\nabla_{\dot\gamma}(\mathcal{S}-\lambda I))\}\mathcal{A} \\ &= \{(\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S}-\lambda I) + \mathcal{A}(\mathcal{S}-\lambda I)\mathcal{C}_{\dot\gamma}\}\mathcal{A} = (\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S}-\lambda I)\mathcal{A} + \mathcal{V}\mathcal{C}_{\dot\gamma}\mathcal{A} = \nabla_{\dot\gamma}\mathcal{A} + \mathcal{C}_{\dot\gamma}\mathcal{A}. \end{aligned}$$

Taking derivatives again and using the expression above together with (1) and Lemma 3.1 we obtain

$$\begin{aligned} 0 &= (\nabla^2_{\dot\gamma}\mathcal{A}) + (\nabla_{\dot\gamma}\mathcal{C}_{\dot\gamma}\mathcal{A}) = (\nabla^2_{\dot\gamma}\mathcal{A}) + (\nabla_{\dot\gamma}\mathcal{C}_{\dot\gamma})\mathcal{A} + \mathcal{C}_{\dot\gamma}(\nabla_{\dot\gamma}\mathcal{A}) = (\nabla^2_{\dot\gamma}\mathcal{A}) + \mathcal{C}^2_{\dot\gamma}\mathcal{A} + R_{\dot\gamma}\mathcal{A} - \mathcal{C}^2_{\dot\gamma}\mathcal{A} \\ &= (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda\mathcal{S})\mathcal{V}\mathcal{A} = (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda\mathcal{S} - \varepsilon\lambda^2 I + \varepsilon\lambda^2 I)\mathcal{A} \\ &= (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda^2)\mathcal{A} + \eta\varepsilon\lambda\mathcal{V}. \end{aligned}$$

It is possible to rewrite this differential equation in a simpler way.

$$\nabla^2_{\dot\gamma}\{(c+\varepsilon\lambda^2)\mathcal{A} + \varepsilon\lambda\mathcal{V}\} + \eta(c+\varepsilon\lambda^2)\{(c+\varepsilon\lambda^2)\mathcal{A} + \varepsilon\lambda\mathcal{V}\} = 0. \tag{2}$$

The next step is to solve this equation with the purpose of understanding and extracting all the relevant information codified in it. In fact, in the Riemannian case, it seems that all the information can be summarized in the Cartan formula. However, although it is possible to rewrite a semi-Riemannian version of the Cartan formula using this equation as well, there is some more information which would not remain summarized in it. Actually, this geometric information will lead us to conclude a bound on the number of principal curvatures of a spacelike isoparametric hypersurfaces in the anti-De Sitter space. In other settings, like Lorentzian isoparametric hypersurfaces in De Sitter spaces, this procedure presented so far is still valid and it might be utilised to obtain some results concerning the number of principal curvatures and the relations between them.

4. Spacelike isoparametric hypersurfaces in the Anti-De Sitter space

We will focus now our attention on a spacelike hypersurface with constant principal curvatures in the anti-De Sitter space $AdS^n(r)$. Therefore, this isoparametric hypersurface M has diagonalisable shape operator at each point p in M. Assume that we have more than one constant principal curvature and select, without loss of generality, $\lambda = \lambda_1$. In this particular situation, we could develop the process we have just explained and, moreover, according to the considerations of Section 3, we have that the constant $\eta(c+\varepsilon\lambda^2)$ is strictly less than zero ($c < 0$, $\varepsilon = -1$, $\eta = 1$). Under all these conditions, we can easily integrate equation (2), and writing $F(t) = (c + \varepsilon\lambda^2)\mathcal{A}(t) + \varepsilon\lambda\mathcal{V}(t)$ and $k = \eta(c + \varepsilon\lambda^2)$ for the sake of simplicity, its solution may be written as

$$F(t) = \cosh\left(\sqrt{-k}t\right)\mathcal{P}_{F(0)}(t) + \frac{1}{\sqrt{-k}}\sinh\left(\sqrt{-k}t\right)\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t), \tag{3}$$

where $\mathcal{P}_{F(0)}(t)$ and $\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t)$ denote the parallel transport of the endomorphisms $F(0)$ and $(\nabla_{\dot\gamma}F)(0)$ of $T_{\gamma(0)}M$ along the curve γ from the point $\gamma(0) = p$ to the point $\gamma(t)$. We will show that $(c + \varepsilon\lambda^2)\mathcal{A}(t) + \varepsilon\lambda\mathcal{V}(t)$ is a self-adjoint endomorphism for each $t \in I$ by checking that both $\mathcal{A}(t)$ and $\mathcal{V}(t)$ are self-adjoint endomorphisms for all $t \in I$. This is clear for $\mathcal{A}$ because it is the inverse of a self-adjoint operator. For $\mathcal{V}$, we can compute

$$\langle\mathcal{V}X, Y\rangle = \langle\mathcal{V}X, \mathcal{H}Y + \mathcal{V}Y\rangle = \langle\mathcal{V}X, \mathcal{V}Y\rangle = \langle\mathcal{H}X + \mathcal{V}X, \mathcal{V}Y\rangle = \langle X, \mathcal{V}Y\rangle.$$

At this moment, we can determine the eigenvalue structure of the endomorphism $F(t)$ of the real vector space $T_{\gamma(t)}M$, for each $t \in I$. Firstly, by hypothesis, we know that the principal curvatures of M and their algebraic and geometric multiplicities are constant along M. But, taking into account that for each $t \in I$ the tensor field $\mathcal{A}(t)$ is zero when restricted to $\ker(\mathcal{S} - \lambda I)_{\gamma(t)}$ and the inverse of $(\mathcal{S} - \lambda I)_{\gamma(t)}|_{\mathrm{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}}$ when restricted to $\mathrm{Im}(\mathcal{S} - \lambda I)_{\gamma(t)}$, we can deduce that the eigenvalues of $\mathcal{A}(t)$ are: zero with algebraic and geometric multiplicity m_λ and $\frac{1}{\lambda_i-\lambda}$ with algebraic and geometric multiplicity m_{λ_i}, for $i = 2, 3, \ldots, n-1$. On the other hand, the spectrum of the endomorphism $\mathcal{V}$ is zero with multiplicity m_λ and one with multiplicity $n - 1 - m_\lambda$. Notice that these eigenvalues together with their algebraic and geometric multiplicities are constant along M precisely because they only depend on the dimension of the subspaces involved in the orthogonal decomposition $\ker(\mathcal{S}-\lambda I)_q \oplus \mathrm{Im}(\mathcal{S}-\lambda I)_q$. The dimensions of these subspaces are constant because M is isoparametric and, thus, the eigenvalues of $\mathcal{V}$ are constant along the curve γ.

Therefore, the tensor field $F(t) = (c + \varepsilon\lambda^2)\mathcal{A}(t) + \varepsilon\lambda\mathcal{V}(t)$ has constant eigenvalues with constant algebraic and geometric multiplicities. These eigenvalues are: zero with geometric and algebraic multiplicity m_λ and $\frac{c+\varepsilon\lambda\lambda_i}{\lambda_i-\lambda}$ with algebraic and geometric multiplicities m_{λ_i}, for $i = 2, \ldots, n-1$.

Note at this point that the parallel transport $\mathcal{P}_{F(0)}(t)$ of the endomorphism $F(0)$ along γ has the same eingenvalues as those of $F(0)$ for all $t \in I$. Furthermore, the eigenvectors are exactly the parallel translation of those of $F(0)$. So parallel translation of endomorphisms also preserves algebraic and geometric multiplicities. In fact, let $\{X_1, \ldots, X_{n-1}\}$ be an orthonormal basis of $T_{\gamma(0)}M$. Then, writing F instead of $F(0)$ for the sake of simplicity, we may deduce

$$\begin{aligned}\nabla_{\dot\gamma}\langle \mathcal{P}_F(t)\mathcal{P}_{X_i}(t), \mathcal{P}_F(t)\mathcal{P}_{X_j}(t)\rangle &= \langle (\nabla_{\dot\gamma}\mathcal{P}_F\mathcal{P}_{X_i})(t), \mathcal{P}_{X_j}(t)\rangle + \langle \mathcal{P}_F(t)\mathcal{P}_{X_i}(t), (\nabla_{\dot\gamma}\mathcal{P}_{X_j})(t)\rangle \\ &= \langle (\nabla_{\dot\gamma}\mathcal{P}_F)(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle + \langle F(\nabla_{\dot\gamma}\mathcal{P}_{X_i})(t), \mathcal{P}_{X_j}(t)\rangle = 0.\end{aligned}$$

Therefore, the function $t \in I \to \langle \mathcal{P}_{F(0)}(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{F(0)}(t)\mathcal{P}_{X_j}(t)\rangle$ is constant and takes the value $\delta_{ij}\lambda_i = \delta_{ij}\lambda_j$ at zero. Thus, our claim is proved. This might be thought as a particularisation of a more general result which claims that the parallel transport of an endomorphism along a curve preserves eigenvalues together with their algebraic and geometric multiplicities. Moreover, the eigenvectors of the parallel transport of an endomorphism are exactly the parallel transport of the eigenvectors of the initial endomorphism. Thus $\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t)$ has also constant eigenvalues with constant algebraic multiplicities for all $t \in I$.

It is important to remark that $(\nabla_{\dot\gamma}F)(t)$ is a self-adjoint endomorphism for each $t \in I$. Since $F(t)$ is self-adjoint we have the equality $\langle F(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle = \langle \mathcal{P}_{X_i}(t), F(t)\mathcal{P}_{X_j}(t)\rangle$, where $\{X_1, \ldots, X_{n-1}\}$ is again, as above, an orthonormal basis of $T_{\gamma(0)}M$. Taking derivatives in the left hand side we get $\langle (\nabla_{\dot\gamma}F)(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle$. By symmetry, in the right hand side we obtain $\langle \mathcal{P}_{X_i}(t), (\nabla_{\dot\gamma}F)(t)\mathcal{P}_{X_j}(t)\rangle$. Thus, $(\nabla_{\dot\gamma}F)(t)$ is a self-adjoint endomorphism of the real vector space $T_{\gamma(t)}M$ for all $t \in I$.

This means, in particular, that each one of the addends of $F(t)$ in (3) diagonalises with real eigenvalues. Furthermore, taking into account that F has constant eingenvalues along the geodesic curve γ, one may argue that the map $t \in I \to \operatorname{tr}(F(t))$ is a constant function. But it is then clear that $F^2(t)$ diagonalises with real eigenvalues, the square of the eigenvalues of F, for all $t \in I$. Therefore, the function $\operatorname{tr}(F^2(t))$ is again constant and we may write

$$0 = \left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{tr}(F^2(t)) = \operatorname{tr}((\nabla^2_{\dot\gamma}F^2)(0)) = 2|k| \operatorname{tr}(F^2(0)) + 2\operatorname{tr}((\nabla_{\dot\gamma}F)^2(0)) \tag{4}$$

But this last equalitiy clearly implies that both $F(0)$ and $(\nabla_{\dot\gamma}F)(0)$ are the zero endomorphims. Consequently, $F(t) = 0$ for all $t \in I$ by (3) and recalling the definition of F we have just shown that $(c+\varepsilon\lambda^2)\mathcal{A}(t) = -\varepsilon\lambda\mathcal{V}(t)$. If we now decompose $T_{\gamma(t)}M$ into $\ker(\mathcal{S}-\lambda I)_{\gamma(t)}$ and $\operatorname{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}$ as usual, and we express both family of endomorphisms in their matrix form with respect to that decomposition,

$$(c+\varepsilon\lambda^2)\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{\lambda_i-\lambda} \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\varepsilon\lambda \end{array}\right),$$

one can easily deduce that M only has two principal curvatures: λ, the curvature we had assumed at the very beginning to have the same algebraic and geometric multiplicity, and $\frac{-c}{\varepsilon\lambda} = \frac{c}{\lambda}$. According to the bound achieved in [6, Lemma 3.4, Lemma 3.10] on the number of principal curvatures for a Lorentzian isoparametric hypersurface in the anti-De Sitter space, this allows to state the following

Proposition 4.1. *Let $M \subset AdS^n$ be an isoparametric hypersurface. Then, the number of principal curvatures is less or equal than two.*

Moreover, coming back to spacelike isoparametric hypersurfaces in anti-De Sitter spaces, it is easy to check, using Jacobi vector field theory, that the focal submanifold of this isoparametric hypersurface M is a totally geodesic submanifold. Therefore, we have the following

Theorem 4.2. *Spacelike isoparametric hypersurfaces with more than one principal curvature in the anti-De Sitter space AdS^n, $n \geq 3$, are tubes around totally geodesic submanifolds of AdS^n.*

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Department of Mathematics, Universidade de Santiago de Compostela, Spain
Email address: `victor.sanmartin@usc.es`