ZDM – Mathematics Education

HOW VISUALIZATION AND ARGUMENTATION ARE ARTICULATED IN RESEARCH ON TEACHING AND LEARNING GEOMETRY

--Manuscript Draft--

Manuscript Number:	ZDMI-D-23-00085R2
Full Title:	HOW VISUALIZATION AND ARGUMENTATION ARE ARTICULATED IN RESEARCH ON TEACHING AND LEARNING GEOMETRY
Article Type:	Issue 6 -2024
Keywords:	visualization; argumentation; teaching and learning geometry; Configural Reasoning; Onto-semiotic Approach
Corresponding Author:	Teresa Fernandez Blanco University of Santiago de Compostela: Universidade de Santiago de Compostela Santiago de Compotela, A Coruña SPAIN
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	University of Santiago de Compostela: Universidade de Santiago de Compostela
Corresponding Author's Secondary Institution:	
First Author:	Teresa F. Blanco
First Author Secondary Information:	
Order of Authors:	Teresa F. Blanco
	Leonor Camargo
	Pablo G. Sequeiros
Order of Authors Secondary Information:	
Funding Information:	Agencia Estatal de Investigación (PID2021-122326OB-I00) — Dr. Teresa F. Blanco
Abstract:	Visualization and argumentation are central processes in the learning of geometry, giving rise to specific lines of research in mathematics education. However, it is worth asking whether they should be developed independently or if it is possible and in what way an articulation between them can be promoted. The objective of this paper is to search for this articulation within the most recent advances in empirical research on the teaching and learning of geometry in Spain. The analysis is approached through the tasks used in research employing two theoretical frameworks which have been developed in Spain, the Onto-Semiotic Approach and Configurational Reasoning. Two main types of task were found in the selected studies: visual tasks and conjecture-and-proof tasks. The results show that in visual tasks the articulation between visualization and argumentation is only necessary in some tasks, in which argumentation aims to justify or to describe the external representation provided as a solution. However, in conjecture-and-proof tasks the articulation between visualization and argumentation must always be present to resolve the task. Analyzing this articulation helps to detect and gain a better understanding of the difficulties that students encounter in solving these types of task. In visual tasks, argumentation helps to understand difficulties associated with the use of visualization. In conjecture-and-proof tasks, visualization determines the argumentation followed, in such a way that many of the difficulties originate more from the visualization than from the process of argumentation.

Click here to access/download
Author's Response - Commentary on the reviews
Review table ZDMI-D-23-00085_2.docx

HOW VISUALIZATION AND ARGUMENTATION ARE ARTICULATED IN RESEARCH ON TEACHING AND LEARNING GEOMETRY

Teresa F. Blanco

University of Santiago de Compostela (Spain)

teref.blanco@usc.es

Leonor Camargo Uribe

Universidad Pedagógica Nacional (Colombia)

lcamargo@pedagogica.edu.co

Pablo G. Sequeiros

University of Santiago de Compostela (Spain)

pablo.gonzalez.sequeiros@usc.es

Abstract

Visualization and argumentation are central processes in the learning of geometry, giving rise to specific lines of research in mathematics education. However, it is worth asking whether they should be developed independently or if it is possible and in what way an articulation between them can be promoted. The objective of this paper is to search for this articulation within the most recent advances in empirical research on the teaching and learning of geometry in Spain. The analysis is approached through the tasks used in research employing two theoretical frameworks which have been developed in Spain, the Onto-Semiotic Approach and Configurational Reasoning. Two main types of task were found in the selected studies: visual tasks and conjecture-and-proof tasks. The results show that in visual tasks the articulation between visualization and argumentation is only necessary in some tasks, in which argumentation aims to justify or to describe the external representation provided as a solution. However, in conjecture-and-proof tasks the articulation between visualization and argumentation must always be present to resolve the task. Analyzing this articulation helps to detect and gain a better understanding of the difficulties that students encounter in solving these types of task. In visual tasks, argumentation helps to understand difficulties associated with the use of visualization. In conjecture-and-proof tasks,

visualization determines the argumentation followed, in such a way that many of the difficulties originate more from the visualization than from the process of argumentation.

Keywords: *visualization, argumentation, teaching and learning geometry, Configural Reasoning, Onto-semiotic Approach.*

1. Introduction

In recent years, visualization linked to geometry has been one of the main lines of research carried out by Spanish researchers in the field of the teaching and learning of geometry (Barquero et al., 2022). This research has mainly been approached from the perspective of two theoretical frameworks developed in Spain: The Onto-Semiotic Approach to the Knowledge and Teaching of Mathematics (OSA) (Godino et al., 2007; Godino et al., 2019) and Configurational Reasoning (CR) (Prior & Torregrosa, 2013, 2020; Torregrosa et al., 2010). These two theoretical frameworks pay particular attention to argumentation in solving tasks in which visualization comes into play. Thus, a fundamental question arises: To what extent does Spanish research on the teaching and learning of geometry integrate both processes and promote their articulation with educational implications? This can be considered an important issue in terms of the teaching and learning of geometry since the analysis of the articulation of these two processes in task resolution can facilitate the identification and understanding of difficulties encountered by students when attempting to solve them.

Along the same lines as Gutiérrez (1996), visualization in mathematics will be considered ‘‘as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties’’ (p. 99). This author points out four key elements of visualization: mental images, taken to be any kind of representation of a concept or mathematical property by way of visual or spatial configurations (Presmeg, 2006); external representations, be they verbal or graphic, of concepts or properties, including images, drawings, diagrams, etc.; visual processes, understood as mental or physical actions in which mental images or external spatial-graphic representations intervene (Bishop, 1989); and visualization skills (Del Grande, 1990), stable capacities necessary for the resolution of mathematical tasks, such as visual memory, the identification of spatial relationships, figure-ground perception, etc. The construction of meaning, problem solving and the communication of ideas are heavily influenced by visualization (Arcavi, 2003; Fischbein, 1982; Godino et al., 2012; Gutiérrez, 1996; Presmeg, 2006; Prior & Torregrosa, 2013), particularly in school geometry. However, students tend to shy away from visualization due to the challenges involved in its implementation and the perception that it demands a high cognitive level (Gutiérrez et al., 2018). Ramírez-Uclés & Flores (2017) highlight various factors influencing the utilization of visualization in task resolution, including the contextualization of the teaching process, the task format, the approach adopted, and the strategies employed by students in tackling the task.

On the other hand, argumentation is conceptualized as a collective or individual process, which, in accordance with shared rules, points towards a mutually acceptable conclusion about the truth or falsity of a statement or action (Molina et al., 2019, Camargo et al., 2024). Argumentation is a relevant process in school mathematics education, not only as it contributes to giving meaning and depth to mathematical learning, but also because it helps to generate an environment of inquiry in the classroom where ideas are justified collectively and not only by the teacher’s authority (Boero at al., 2018). Argumentation has been strongly linked to the teaching of geometry due to the possibility of students inferring properties through the exploration of geometrical representations and concluding, from certain propositions already known or assumed to be true, new true propositions following on from the previous ones (Baker, 2009; Douek, 1998; Durand-Guerrier et al., 2012). Depending on the mathematical task being carried out, students can develop different types of arguments (Reid & Knniping, 2010). For instance, during the exploration of a representation and the formulation of conjecture, students may articulate arguments of an inductive, abductive, or analogical nature. These arguments serve to empirically support the certainty of their discoveries or the acceptability of their positions or actions. Empirical arguments are those in which statements are supported with reasons that come from information given or obtained by exploration. The argument will be visual when the reasons are based on information extracted from a figure or visual representation. Furthermore, when validating results in the framework of theoretical systems of reference, the main arguments may be deductive, in which case we refer to them as proofs, since they lead to logically necessary conclusions (Durand-Guerrier et al., 2012). In any case, argumentation has the aim of justifying assertions when seeking to convince, persuade, or validate what is being stated.

The general aim of this research, oriented towards answering the initial question posed in the first paragraph of the Introduction, is to examine how the articulation between visualization and argumentation is manifested in the most recent research carried out in Spain on the teaching and learning of geometry. As specific objectives, the role of argumentation in tasks promoting visualization and the relevance of visualization in conjecture-and-proof tasks shall be examined.

This paper consists of six sections. The first section is aligned with the Introduction, outlining the issue that prompted this study. The second section briefly presents the two theoretical frameworks, which support the most relevant studies in which the links between visualization and argumentation in the teaching and learning of geometry will be identified. The third section details the description and selection of the sample. The following two sections present the analysis carried out based on the types of task proposed in the selected studies. Thus, the fourth section presents the role of argumentation in tasks promoting visualization, while the fifth section shows how visualization works in conjecture-and-proof tasks focusing on argumentation. The final section highlights the conclusions.

2. Visualization and argumentation in the OSA and CR frameworks

The Onto-Semiotic Approach is a modular and inclusive theoretical system that has developed various tools to address the research problems posed by the teaching and learning of mathematics (Godino et al., 2007; Godino et al., 2019). This framework examines the correspondences and synergies between the OSA and other theoretical frameworks used internationally in mathematics education (Drijvers et al., 2013; Font et al., 2015). The OSA postulates that the exploration of mathematical activity initially focuses on the practices carried out by people involved in solving mathematical task. It considers *mathematical practice* to be any action or manifestation (linguistic or not) carried out to solve mathematical task. This includes efforts to communicate the solution to others and to validate and generalize that solution to other contexts and problems (Godino et al., 2007).

The OSA considers six *objects* that intervene in mathematical practices: language, tasks, concepts, propositions, procedures, and arguments (Godino et al., 2007). These objects are related to each other forming *ontosemiotics configurations*, defined as the networks of intervening objects emerging from mathematical practices and relationships established among them. This way, language (terms, expressions, notations, graphics) constitutes the means of expression of all the other objects, helping to point out the remaining objects at play in the resolution of the task. Tasks provide the objects that intervene in the mathematical practices from which the solution emerges. Concepts are introduced through their definitions or descriptions. Propositions are statements about the concepts involved in the mathematical practice. Arguments are used to validate and support propositions and procedures put into play and are the product of an argumentation process that gives meaning, organizes, and relates the other objects and whose organization constitutes a solution to a task. The scheme of an ontosemiotic configuration is shown in Figure 1. A configuration is called *cognitive configuration* when it comes from the student and represents the path followed by the student in the process of solving mathematical tasks. Some examples of cognitive configurations can be seen in Blanco et al. (2012). Following Gutiérrez's (1996) definition of visualization, within this theoretical framework visualization is analyzed from the point of view of these objects, in which it is said that visualization is involved (Godino et al., 2012).

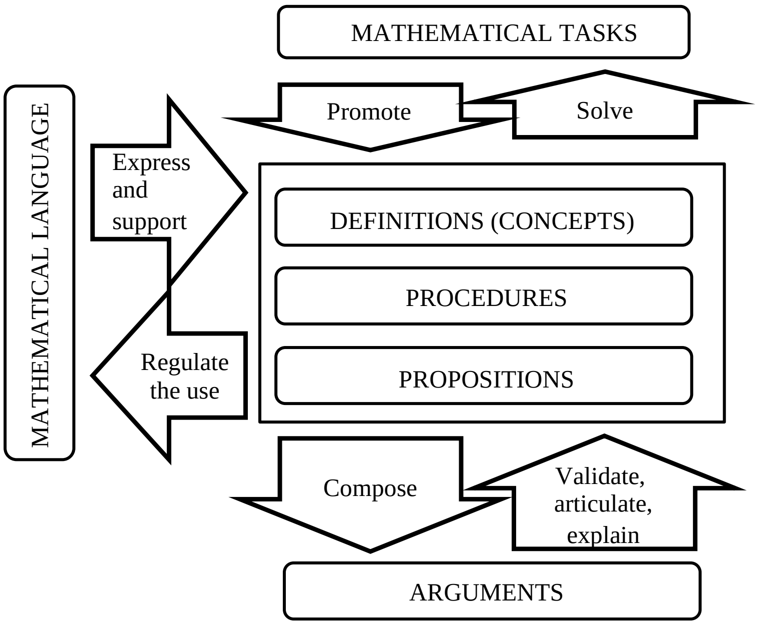


Figure 1 *Scheme of an ontosemiotic configuration*. Adapted from Molina et al. (2019, p.98)

The students' cognitive configurations show that all mathematical practices normally consist of two components, one visual and the other analytical. These two components support each other synergistically in solving the corresponding task. The visual component plays a key role in understanding the nature of the practices and formulating conjectures, whereas the analytical component becomes important when generalizing and justifying solutions via arguments. The degree of visualization put into play in the solution of a task will depend on whether the task is visual or not, and on the cognitive styles of the person who solves it (Godino et al., 2012).

Godino et al. (2012) consider that, in the case of geometry, visualization mainly comes into play in three types of task, which they call visual tasks: (1) Tasks that involve the communication of shapes, their components and structures, whether real or imaginary, using visual language (representations in the form of photographs, drawings, or diagrams); (2) The communication of the relative position of spatial shapes via the use of deictic language (terms such as, up, down, in front of, behind, right, left, near, far, north, south, east, west), which can also be carried out via the use of mediators such as models, plans, or maps; (3) The recognition of invariants in shapes or in their representations when faced with specific transformations, which is supported by the ability of visual discrimination: comparing different spatial shapes, drawings, or images and identifying similarities or differences among them (Del Grande, 1990). Although Blanco et al. (2012) have showed considerable difficulties in eliciting students' argumentation in these three types of task, since most students do not consider it necessary to provide justifications, the OSA emphasizes the importance of connecting visualization with argumentation in mathematical practices. To support an argument based on properties and procedures of an empirical nature, the external representation of those properties, procedures and concepts is sufficient. In other cases, the construction of deductive arguments based on previously accepted rules is required.

Configural Reasoning, the other theoretical framework used in this paper, explains the coordination of visualization processes during students' resolution of geometric tasks in a pencil-and-paper context (Clemente et al., 2017, Clemente & Llinares, 2015, Llinares & Clemente, 2014, 2018, Prior & Torregrosa, 2013, 2020, Saorín et al., 2019, Torregrosa, 2017 and Torregrosa et al., 2010). Here visualization is understood as "the transfer of objects, concepts, phenomena, processes and their representations to some type of visual representation and vice versa'' (Prior & Torregrosa, 2013, pp. 342-343). This framework has its origins in Duval's (1998) study on perceptual, discursive, and operative apprehensions. *Perceptual apprehension* refers to the visual recognition of a shape, including the ability to recognize several sub-figures in the perceived figure. *Discursive apprehension* is related to the control of statements over any perceptual recognitions of geometrical properties in geometrical representations. *Operative apprehension* is related with the various ways in which a given figure can be modified: the division of the whole figure into different parts and the combination of these parts into another figure or sub-figures, when the figure is made larger, narrower or slanting, or when the position or orientation of a figure is changed. Torregrosa et al. (2010) call the coordination of discursive and operative apprehensions Configural Reasoning, defining *configuration*[1] as any planar representation of a geometrical shape.

The strength of an expert solver lies in his/her ability to move from the given configuration (through visualization) to discourse and to recognize configurations linked to propositions for constructing deductive arguments based on theoretical knowledge. This practice is not spontaneous for inexpert students. However, Saorín et al. (2019) have stated that students should progressively develop argumentation skills, from empirical to deductive means of arguing, to justify the plausibility of their choices. Passing from argumentation based on intuitive and visual justifications to logical-deductive considerations is no easy task for students (Prusak et al., 2012). Thus, it is important to characterize the conditions which facilitate the transformation of visualization into configural reasoning, enabling a deductive process to be generated. As this coordination does not always arise, studies have analyzed, among other factors, the causes which explain the absence or interruption of coordination. For example, difficulties may arise in relation to the characteristics of the figural representations which can inhibit rather than highlight the identification of geometrical properties present in the representation.

When attempting to compare these two frameworks, it should be taken into consideration that the OSA is an inclusive theoretical system which seeks to articulate different approaches and theoretical models used in research on mathematics education (particularly on visualization), while CR is a specific model which seeks to explain the coordination of operative and discursive apprehension when students resolve proof tasks in a paper-and-pencil geometrical context. In any case, the similarities between the articulation made by these two frameworks of both processes (visualization and argumentation) can be appreciated by contrasting the notions of cognitive configuration and configural reasoning. The notion of cognitive configuration includes the relationships between all the objects which intervene in mathematical practices (tasks, linguistic and material elements, concepts, propositions, procedures, and arguments) and how these relationships are

[1] Note that the notion of *configuration* has different meanings depending on whether it is employed in the OSA or the CR framework.

coordinated. The categorizations made of students' cognitive configurations is based on their arguments about the other objects involved in the practice, with these arguments depending largely on the visualization skills and processes that students bring into play (Blanco et al., 2019). Configural reasoning places visualization as a central element of deductive argumentation in geometry since it involves the coordination of discursive and operative apprehensions. If there is only perceptual apprehension and no arguments are put into play, configural reasoning does not arise. In this case, visualization is an element of configural reasoning that is identifying to perceptual apprehension, while Gutierrez's (1996) definition of visualization would correspond to configural reasoning.

3. Sample description

The starting point of this study is provided by the 95 empirical studies reviewed in Barquero et al. (2022), which were selected from an exhaustive review of publications indexed in high-impact journals over the last 15 years. Those that specifically address visualization and conjecture-and-proof tasks were included in the sample for this paper.

In terms of educational level, production is most extensive at the secondary level, with research here being centered on conjecture-and-proof tasks. However, some authors have focused on higher education, particularly on teacher training programs. Other recent contributions have focused on early years education, particularly in terms of spatial orientation tasks. A qualitative methodology is employed in most of the publications, although some quantitative and mixed-method studies have been found. Some studies address other important processes in the teaching and learning of geometry, such as describing, defining, classifying, and generalizing. The discussion of how visualization and argumentation are articulated in these studies has not been included here since the link between argumentation and visualization in the papers reviewed is not explicit.

The studies underpinned by the OSA which will be addressed in the present paper, emphasize tasks focused on the promotion of visualization. The studies supported by CR address conjecture-and-proof tasks, focusing on the argumentation process. Taking the above into consideration, the articulation will be presented in accordance with the types of task contemplated in the studies: visual tasks on the one hand, and conjecture-and-proof tasks on the other.

4. Argumentative support in visual tasks

This section analyzes the intervention of argumentation in the three types of visual tasks defined in section 2, following the theoretical framework of the OSA. These tasks mainly seek to promote visualization to apply it in other geometrical processes. The interest is focused on determining which elements of visualization (processes, mental images, external representations, or visualization skills) students mobilize in solving tasks and how they do it, with the aim of identifying possible difficulties associated with their use. Since inferring these elements solely from the graphic representations provided by students is often quite difficult, in some tasks students are explicitly asked to justify their answers. It is in this sense that argumentation can be considered to act as a support for visualization.

4.1 Tasks that involve the communication of shapes, their components and structure

Research such as that carried out by Beltrán-Pellicer et al. (2020), Blanco et al. (2012), and Blanco et al. (2019) focus on this first type of visual task. Blanco et al. (2019) analyze the visualization skills and processes involved in a task for pre-service teachers requiring the generation and representation of solids of revolution (Figure 2). A categorization of students' answers is presented by using the cognitive configuration notion from the OSA. In the answers shown in Figure 3, students had difficulties in generating the external representation. In the answer on the left side, the graphic representations could be interpreted as 3D solids due to the succession of point shapes drawn by the student, which seem to infer movement around the axis of rotation. Regarding the answer on the right side, the graphical representation could be interpreted as the symmetrical shape of the given initial shape. In either answer, the visual argument is not enough. The text accompanying the graphic representations describes the mental images generated by the students and allows us to conclude that they generate 3D solids. Despite this, the solutions are not correct in all cases nor are they sufficiently detailed (gaps or holes are not specified). In these two resolutions of the task, the argument has the function of describing the graphic representation provided by the students. In general, most of the students considered the task in 2D, producing an axial symmetry. Most of the errors of those who situated the task in a 3D context were due to difficulties in mobilizing the recognition of spatial relationships and positions. Few students provided a solution based solely on a graphical representation using the conventional codes of planar representation of 3D shapes or perspective representation. In this case, said representation would be considered visual argumentation and would be sufficient to conclude if the solution is correct.

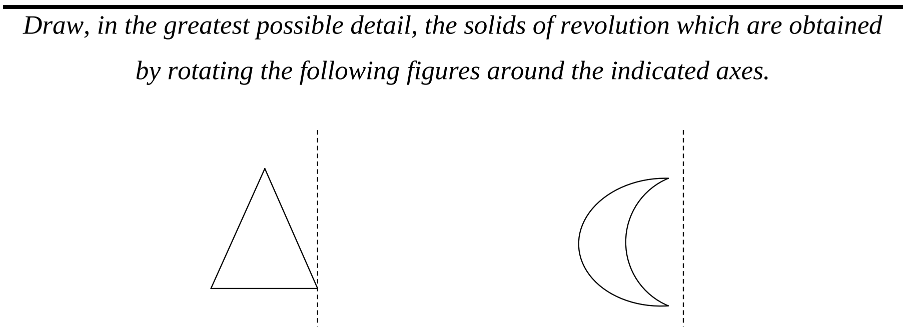


Figure 2 *Visual task of mental rotation* (Blanco et al., 2019, p.772)

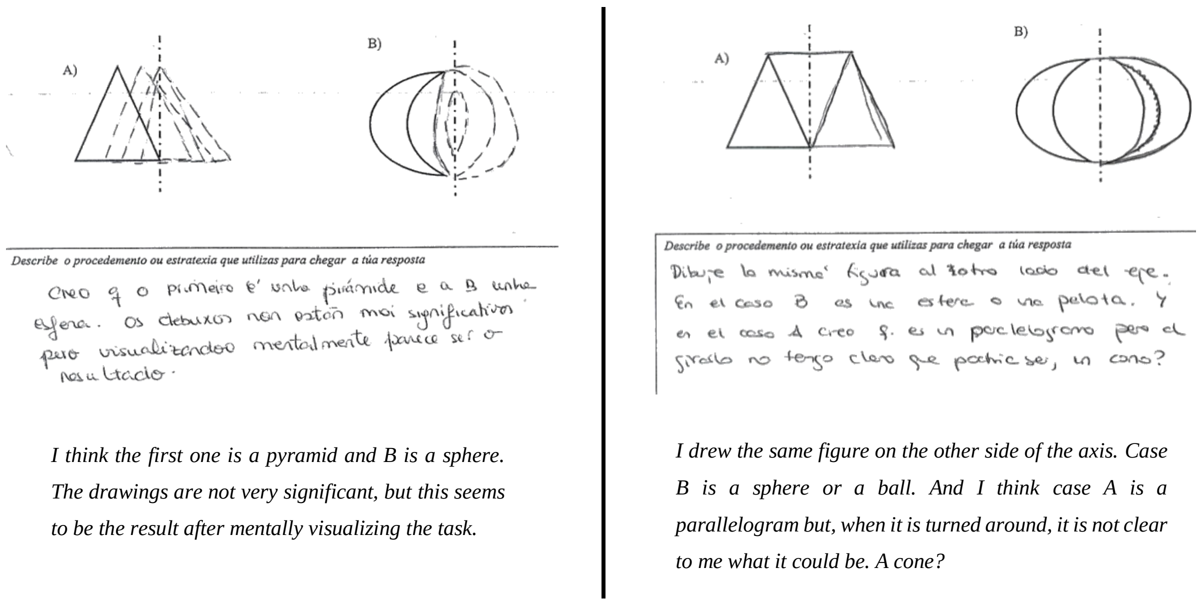


Figure 3 *Two students' resolution of a visual task* (Blanco et al., 2019, pp.778-779)

This first type of task also appears in studies which incorporate technology as a teaching resource for secondary students. Beltrán-Pellicer et al. (2020) use BlocksCAD to model solids, thus connecting visualization with computational thinking, which acts as a mediator. For example, they propose the task of creating a model of a solid (Figure 4). In doing so, different visualization skills come into play, depending on the different ways in which the solid can be built, as these lead to the use of different concepts or mathematical processes and, thus, to the elaboration of different algorithms. Inductive and abductive argumentations arise when students try to persuade their peers which algorithm is the correct or most appropriate one. For example, the solid in Figure 4 can be thought of as a cube with square-based pyramids in the center of each face or as the combination of a cube and an octahedron, in which case the elaboration of the algorithm is considerably shorter. According to these authors, the interest of this resource lies in the fact that, when faced with possible complications of the algorithm arising from the visualization of the shape, the context itself encourages students to look for another visualization that facilitates programming. In this way, the articulation between visualization and argumentation takes place throughout the process of creating the algorithm.

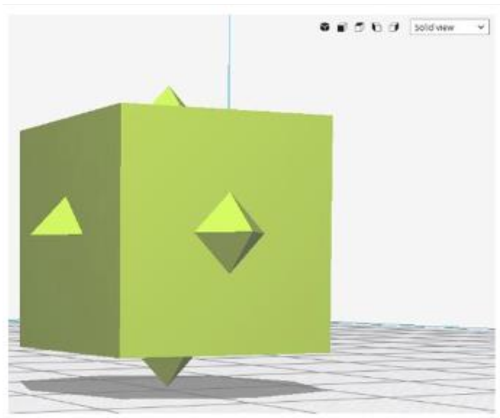


Figure 4 *Solid for the BlocksCAD activity* (Beltrán-Pellicer et al., 2020, p. 45)

4.2 Task of communicating positions

Other studies focus on the second type of task (Bellas et al., 2019; Diago et al., 2021; Gutiérrez et al., 2018; Ramírez-Uclés & Flores, 2017; Ramírez-Uclés et al., 2018). Gutiérrez et al. (2018) analyze the use of visualization skills of secondary students with mathematical talent in a collaborative task requiring them to construct a set of buildings from their orthogonal projections. The dialogues among the students show the discursive arguments they build up as they integrate different views to come to an agreement and achieve the final construction, which implies the empirical argumentation of their conjecture. The results show that there is a relationship between the objective of the student's actions and the type of visualization skills used.

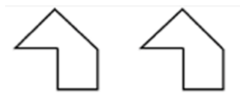
At an early age, tasks are more oriented toward visualizing space through reading maps and describing, constructing, and planning routes. Students justify their decisions via argumentation based on graphic elements (pictograms). Examples of such tasks can be found in the studies by Berciano et al. (2017) and Jiménez-Gestal et al. (2022). The latter proposes a task in which children (3-5 years of age) must find some treasure in their school and draw a map of the route taken to find it. The authors analyze the use of different spatial skills and the children's argumentation when they try to convince their peers about the correct route. Their empirical arguments evolve from representations in which the treasure is the key element (3 years of age) to representations in which all points of reference (tree, stairs, and door) appear and a relationship of order is maintained corresponding to the real location of the child (5 years of age).

These types of task can also be approached employing robotics (Bellas et al., 2019; Diago et al., 2021). For instance, Diago et al. (2021) investigate the effects of an educational intervention with the Bee-bot robot to evaluate the skill of mental rotation and computational thinking in a primary class (8 years of age). Although their research does not provide conclusive results on mental rotation, the authors believe that the Bee-bot can be a useful tool to encourage spatial skills as it lies between the visuospatial, computational, and mathematical domains. Children articulated empirical argumentation with visualization skills by verbally justifying route planning or reviewing the outcome of the instructions given to the robot.

4.3 Tasks of recognition of invariants in shapes or in their representations

The research carried out by Blanco et al. (2018) and Ramírez-Uclés et al. (2017, 2018) concerns the third group of visual tasks. Blanco et al. (2018) present a 2D task for pre-service teachers in which five figures built from two identical pieces are shown. Without picking up either of the pieces, the students are required to identify which figures cannot be built and to justify their answer (Figure 5). In Figure 6, the answer on the left side shows student has made their choice arguing that the exterior right angle of the pieces is not preserved when forming the figure. On the right side, the student’s attention is focused on the visual perception of the concavity. Most of the arguments given by the pre-service teachers were based more on their visual perceptions than on properties related to rigid motions of the plane, thus showing a predominance of visual argumentation in this task. Furthermore, the choice of options A and B as the most widely selected answers shows the students’ difficulties in putting into action the ability to preserve perception.

There are two identical pieces which can be moved but not picked up.



Which of the figures below CANNOT be formed with these two pieces?

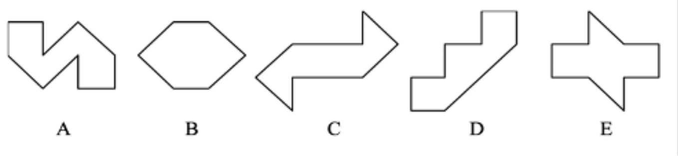
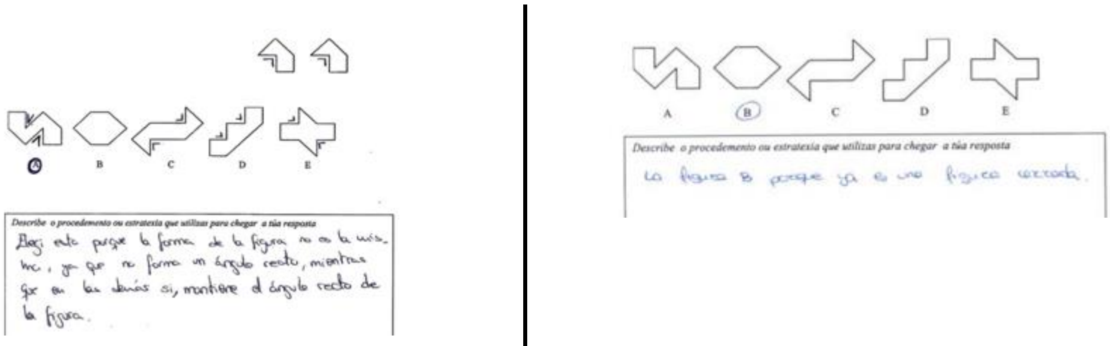


Figure 5 *Visual task of rigid motions of the plane* (Blanco et al., 2018, p. 258)



I chose this one because the shape of the figure is not the same, since it does not form a right angle, while in the others it does, the right angle of the figure is maintained.

Figure B because it is already a closed figure.

Figure 6 *Two students' resolution of the visual task* (Blanco et al., 2018, pp. 266-267)

Also related to motions in the plane, Ramírez-Uclés et al. (2018) present a teaching process for the development of visualization skills in students with mathematical talent, focusing on the evolution of errors over three sessions. They propose a task to configure all the possible structures by putting three model pieces together (Figure 7), seeking the appearance of motions when defining if two structures are equivalent. The arguments provided by the students make it possible to detect three errors related to visualization: establishing a false analogy between the plane and space; drawing conclusions after having examined only some of the possible cases; and producing an argument based on specific limited examples. A decrease in the frequency of errors throughout the experiment is shown, despite the greater complexity of the tasks proposed. In this task, empirical argumentation is necessary but not sufficient since the task implicitly asks to justify that there are no more structures.

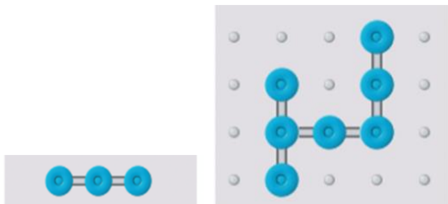


Figure 7 *Model piece and example of a structure with three pieces* (Ramírez-Uclés et al., 2018, p. 41).

The works described suggest that visualization tasks demanding argumentation stimulate better ways of visualizing and allow visualization difficulties to be identified. This is achieved not only by requiring justification of findings and decisions within a task, but also by fostering opportunities for the collective elaboration of constructions, algorithms or structures in which agreements must be reached.

5. Visual support in conjecture-and-proof tasks

This section refers to studies focusing on conjecture-and-proof tasks. Although the OSA also supports research on conjecture-and-proof tasks in geometry, most of the studies reviewed are based on the CR model, which was developed specifically for tasks of this nature (Clemente et al., 2017; Llinares & Clemente, 2018; Prior & Torregrosa, 2020; Saorín et al. 2019; Torregrosa, 2017). Research has mainly been concerned with observing how students use the visualization of a graphic representation in conjecture-and-proof tasks to find logical relationships between geometrical properties with which to argue and create the required conjecture or proof.

5.1 Proof tasks

As stated in section 2, the CR model is oriented to studying the connection between argumentation and visualization through the interaction between discursive and operational apprehensions. The coordination of these two apprehensions (configural reasoning) results from the visualization of a representation which is modified via operative apprehension, adding or removing elements or manipulating parts of the configuration to make it possible to fix attention on a specific subconfiguration (Clemente et al., 2017). Subsequently, mathematical statements are associated with that subconfiguration via discursive apprehension. For example, the task shown in Figure 8 requires proof of the congruence of segments *RC* and *BT*, given the congruence of segments *AB* and *AC* and angles ∢RCB and ∢TBC. The configuration provided in the task (left side of Figure 8) can be examined by paying attention to three different subconfigurations, which are present but overlapping, from which different resolutions can be generated. Thus, it is possible to concentrate attention on two different pairs of triangles (Subconfiguration a and Subconfiguration b) or to focus on two opposite angles and the angles juxtaposed to them (Subconfiguration c). These three possible subconfigurations allow for the recognition of the premises of the triangle congruence criteria. However, the students' success in coordinating the two apprehensions to solve the task depended largely on the subconfiguration considered, with Subconfiguration b having a much higher success rate. As for the visualization skill required, it should be noted that students needed to use the ability of figure-ground perception (Del Grande, 1990; Gutierrez, 1996) to solve this task.

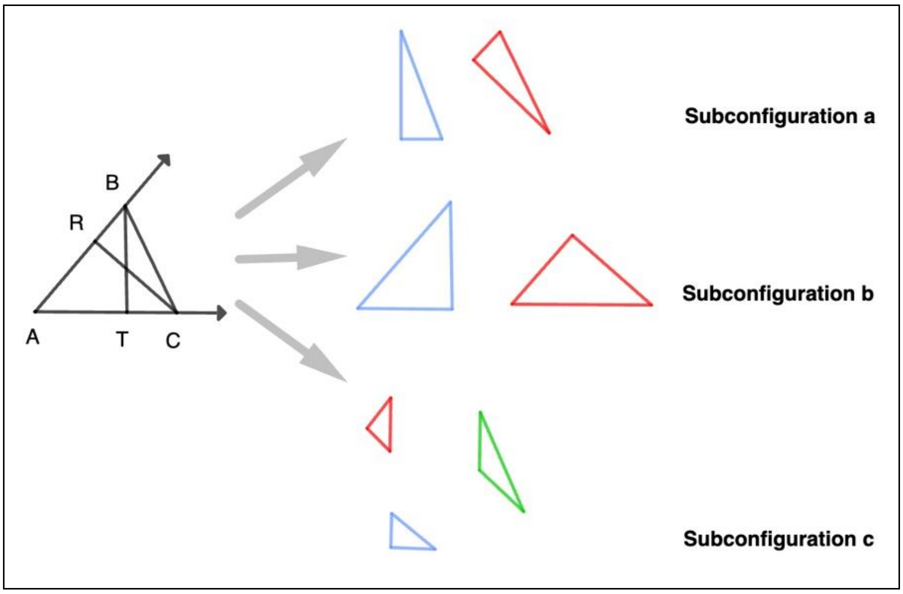


Figure 8 *An example of different subconfigurations.* Adapted from Clemente et al. (2017, p. 505)

The recognition of a relevant subconfiguration in a geometrical configuration is a necessary condition, although it is not sufficient to provoke configural reasoning. For example, Clemente et al. (2017) analyze and explain the relationship that exists between the visualization and identification of prototypical figures and the geometrical knowledge that exists before to the generation of arguments leading to a purely deductive process. These authors conclude that prototypical representations play a relevant role in the dual relationship between visual perception and knowledge linked to changes in visual-discursive anchoring and vice versa. This is because the recognition of a prototypical figure in the initial configuration activates a certain knowledge of geometry. Meanwhile, Torregrosa (2017) and Saorín et al. (2019) focus on the analysis of configural reasoning when tasks in a geometrical context introduce numbers and what is sought is not to prove a general proposition but to find a specific value. In this case, it is concluded that to resolve the task correctly, students must be able to express the geometrical situation posed in the form of relations in the algebraic register. Once this happens, the resolution process becomes independent of any visualization process, since the need to interact with the relevant subconfiguration identified to solve the task ceases.

Furthermore, Llinares & Clemente (2018) focus on factors that may interfere with the productive articulation between the visualization of information present in a representation and the coordination of operative and discursive apprehensions. The success of the process depends on the use of appropriate forms of representation, true statements, and valid forms of reasoning. The results of the study make it possible to achieve a greater understanding of the phenomenon that Duval (1995) calls the double gap between naïve behavior and mathematical behavior in the transition from configural reasoning to the production of proof. The first gap is related to visualization; students need to change the epistemic perspective of the geometrical facts to stop seeing them as mere attributes of visual configurations and begin to see them as assumptions of geometrical propositions that can be linked logically. The second gap is related to argumentation; students must learn to choose which geometrical facts are those that will enable them to link arguments deductively to obtain the required proof.

About the first gap, Prior & Torregrosa (2020) have established links between the CR Model and the Mathematical Working Spaces proposed by Houdement & Kuzniak (2006) to study the influences of the institutional environment in which geometrical activity is developed in configural reasoning. They describe the discursive organization of the responses of secondary education students, who are heavily influenced by their personal geometrical working spaces to highlight the challenges students face when having to move from empirical and intuitive arguments based on visualizations to deductive argumentation.

5.2 Conjecture tasks

In the case of conjecture tasks, argumentation is a justification of the plausibility of what is discovered, while visualization acts as a resource to obtain reasons for the certainty of what is being obtained. In such tasks, students generally explore a visual representation in search of an invariant not mentioned in the statement of the problem which depends on the given conditions. They then formulate a conjecture and justify it. The studies included in this subsection do not follow the CR model. However, it can be considered that their objective is convergent with it since they focus on studying how visualization determines progress towards deductive arguments.

Marrades & Gutiérrez (2000) study possible ways in which students use argumentation to convince themselves or persuade their classmates or teacher of the certainty of a statement. Argumentation arises both in the discovery phase of the conjecture and in its proof. These authors propose a classification of empirical or deductive arguments according to the nature of the justification elaborated and, particularly, to the way in which the students select and employ graphic examples of what they are stating.

Along the same lines as Marrades & Gutiérrez (2000), Spanish researchers, in collaboration with others in Latin America (Araujo et al., 2006; Arce et al., 2014; Camargo, 2010; Fiallo & Gutiérrez, 2017; Ortega & Pecharromán, 2016; Paneque et al., 2017; Sua & Camargo, 2019), have studied advances in argumentation throughout teaching processes, with or without the mediation of dynamic geometry programs and with or without the support of a teacher, in situations of the resolution of conjecture tasks. For example, as can be seen in Figure 9, Sua & Camargo (2019) propose a conjecture task for secondary students (12-16 years old) in which the support that visualization provides to argumentation can be observed. The visualization of the different circumferences obtained lead students to discover and justify a procedure to obtain the geometrical locus of their centers.

Given two points (A and B), is there a circle that contains them? Is there another? Are there more? If so, where are the centers of the circles containing points A and B?

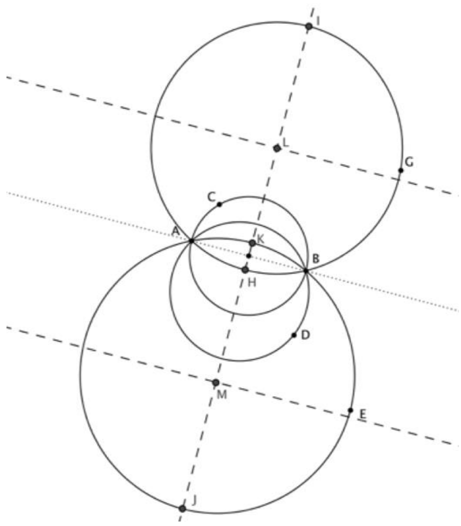


Figure 9 *Conjecture task* (Sua & Camargo, 2019, p.27)

Some studies refer to the evolution of the arguments proposed to justify the conjectures that are obtained. Although slightly different terms are used, the authors of these studies mention visual support for developing empirical arguments, and how visual configurations lose prominence in favor of deductive justifications. This is the case of Araujo et al. (2006), who carried out a study of the long-term individual production and behavior of a group of pre-service teachers who were set geometrical conjecture tasks.

Unlike the previous works, those by Arce et al. (2014), Fiallo & Gutiérrez (2017), Ortega & Pecharromán (2016), and Paneque et al. (2017) propose tasks on elementary geometrical content, such as areas, polygons, and trigonometry. Arce et al. (2014) and Paneque et al. (2017) make use of Harel & Sowder’s (1998) classification to analyze students’ productions when resolving conjecture tasks in which they are required to use or generalize formulas to obtain areas and justify their results. In addition to situating the solutions in a particular type of justification, Arce et al. (2014) classify the pre-service teachers’ errors. They highlight the absence of generic examples to justify the formulas obtained, as many of the participants opted for findings based on a few examples, showing inductive and visual patterns of argumentation.

Meanwhile, Paneque et al. (2017) focus on the development of argumentation among 16-17-year-old students under the influence of an intelligent tutoring system called GeoGebraTutor. The authors characterize GeoGebraTutor-teacher-student interactions at different moments of solving the proof task. They point out that, despite progress in the development of argumentation, and the use of general rather than specific examples, a tendency is perceived among the participants to base themselves on empirical arguments of a visual nature. Ortega & Pecharromán (2016) study the type of argumentation given by pre-service teachers for the accuracy or inaccuracy of several geometrical constructions suggested for the problem of conjecturing how to build a regular pentagon. The results show the difficulties many students have when generating non-empirical argumentation and their problems in terms of the coordination of graphic, algebraic, or arithmetic representations. Fiallo & Gutiérrez (2017) examine the argumentation given by grade 10 students (14-15 years of age) relating to trigonometric reasoning with the support of a dynamic geometry program. The focus of their research is the connection (or lack thereof) between the arguments generated during the establishment of the conjecture and those produced during its validation. They report the role played by the dynamic geometry program in students advancing from visual arguments to those of a deductive nature.

The support that visualization provides to argumentation processes is undeniable. In proof tasks, the identification of prototypical configurations and key subconfigurations accompanying the elaboration of proofs indicate ways through which students can proceed. In conjecture tasks, visualization allows to discover geometric relationships and gain confidence in the findings obtained. In some studies on this type of task, a certain preference on the part of students for visual arguments over deductive ones is observed.

6. Conclusions

This article has examined the most recent Spanish research on the teaching and learning of geometry focused on visualization or argumentation to analyze how these two processes are articulated. The analysis has been presented considering the two most frequent types of task in these studies: visual tasks and conjecture-and-proof tasks, which have mainly been supported by the OSA and CR frameworks.

One specific objective has been to examine the role of argumentation in tasks promoting visualization. In some papers on visual tasks, which are focused on the development of visualization processes and skills, the role of argumentation is not as explicit, as seen in section 4. Students' argumentation is inferred from the descriptions they give with the external representations they propose as solutions to the task, the algorithms they elaborate for achieving a construction (inductive, abductive, or analogic argumentation), or the justifications for their solutions (deductive argumentation). In some tasks, like those present in Blanco et al. (2018, 2019), the results of the visual processes alone can lead to a correct solution, providing a valid visual argument to understand the solution offered by the student. However, if the student's graphic representation does not enable a clear interpretation, the argumentation accompanying the representation becomes crucial in determining whether the visual processes have led to a satisfactory resolution or not, as can be seen in Figure 3. In summary, less competence in visualization processes and skills requires greater intervention of different types of argumentation, in addition to visual argumentation, to achieve success in solving the task.

The other specific objective addresses the role of visualization in conjecture-and-proof tasks. In studies on conjecture-and-proof tasks, argumentation is necessary to confer certainty and validity to the properties and procedures involved in task resolution. The type of argumentation (be it visual or analytical) in such tasks is deductive and the role of visualization is clearly defined, as made explicit in the CR framework. Thus, visualization determines the type of subconfiguration the student uses when stating a conjecture or proving a result, as has been illustrated with example of Figure 8 from Clemente et al. (2017). In essence, visualization dictates the argumentation followed, in such a way that many of the difficulties in resolution originate more from the visualization process itself than from the argumentation process. In many cases, these difficulties are related to the use of prototypical images, which do not activate the geometrical properties or relationships that may be considered to solve the task.

It has been showed that the two theoretical frameworks that support most of the studies analyzed show the need for a synergy between visualization and argumentation processes to achieve the desired solution. It is crucial to examine whether this articulation takes place and how, with the aim of gaining an understanding of the cognitive processes involved when students approach such tasks. This approach returns to the premise of the OSA, which sustains the existence of both visual and analytical components in any type of task. Depending on the type of task, this articulation may be stronger or weaker.

Finally, the analysis conducted for this study has highlighted two emerging research directions. One of them is aimed at students with mathematical aptitudes, given the lack of consensus on the relationship between visualization and talent in mathematics. The other promising avenue is to explore the effects of software applications that delve into automated forms of argumentation. Although these studies are in their incipient and experimental phases, it is crucial to highlight the role of these applications in strengthening processes such as justification, argumentation, and validation of theorems.

Acknowledgements

Study carried out within the framework of PID2021-122326OB-I00, Proyectos De Generación de Conocimiento 2021/ MCE-AEI.

All authors declare that they have no conflicts of interest in this article.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241. https://doi.org/10.1023/A:1024312321077

Araujo, J., Giménez, J., & Rosich, N. (2006). Afectos y demostraciones geométricas en la formación inicial docente (Affects and geometrical proof for initial teacher training). *Enseñanza de las Ciencias, 24*(3), 371–386. https://doi.org/10.5565/rev/ensciencias.3788

Arce, M., Conejo, L., & Ortega, T. (2014). Análisis de los procesos de justificación y generalización de la fórmula del área del rectángulo por alumnos del grado de educación primaria (Analysis of justification and generalization processes of the rectangle area formula by primary education students). *Profesorado, Revista de Curriculum y Formación del Profesorado, 18*(2), 209–227. https://revistaseug.ugr.es/index.php/profesorado/article/view/19237

Baker, M. (2009). Argumentative interactions and the social construction of knowledge. In N. Mirza, & A. N. Perret-Clermont (Eds.), *Argumentation and education: Theoretical foundations and practices* (pp. 127–144). Springer.

Bellas, F., Salgado, M., Blanco, T. F., & Duro, R. J. (2019). Robotics in Primary School: A Realistic Mathematics Approach. In L. Daniela (Ed.), *Smart learning with educational robotics*. Springer. https://doi.org/10.1007/978-3-030-19913-5_6

Barquero, B., Batanero, C., Blanco, T. F., Bosch, M., Camacho-Machín, M., Cañadas, M. C., Castro, E., Chamoso, J. M., Contreras, L. C., Gea, M. M., Godino, J. D., Martín-Molina, V., Moreno, M., & Wilhelmi, M. R. (2022). Spanish research on mathematics education. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 223–254). PME.

Beltrán-Pellicer, P., Rodríguez-Jaso, C., & Muñoz-Escolano, J. M. (2020). Introduciendo BlocksCAD como recurso didáctico en matemáticas (Introducing BlocksCAD as an educational resource in mathematics). *Suma, 93*, 39–48.

Berciano, A., Jiménez-Gestal, C., & Salgado, M. (2017). Kindergartners' use of symbols in the semiotic representation of 3-dimensional changes. *International Electronic Journal of Mathematics Education, 12*(3), 311–331. https://doi.org/10.29333/iejme%2F615

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7–16.

Blanco, T. F., Godino, J. D., & Pegito, J. A. C. (2012). Razonamiento geométrico y visualización espacial desde el punto de vista ontosemiótico **(**An onto-Semiotic approach to geometrical reasoning and spatial visualization). *Bolema: Boletim de Educação Matemática, 26*(42a), 39–64. https://doi.org/10.1590/S0103-636X2012000100004

Blanco, T. F., Godino, J. D., & Diego-Mantecón, J. M. (2018). Análisis epistémico y cognitivo de una tarea de visualización en el espacio bidimensional (Epistemic and Cognitive Analysis of a 2D Visualization Task). *Journal of Research in Mathematics Education*, *7*(2), 251–279. https://doi.org/10.17583/redimat.2018.2463

Blanco, T. F., Diego-Mantecón, J. M., & Sequeiros, P. G. (2019). Procesos de visualización en una tarea de generación y representación de cuerpos de revolución (Visualization processes in a task that requires generating and representing solids of revolution). *Bolema: Boletim de Educação Matemática*, *33*(64), 768–789. https://doi.org/10.1590/1980-4415v33n64a16

Boero, P., Fenaroli, G., & Guala, E. (2018). Mathematical argumentation in elementary teacher Education: The key role of the cultural analysis of the content. In A. Stylianides, & G. Harel, *Education research on proof and proving. ICME-13. Monographs* (pp. 49–67). Springer.

Camargo, L. (2010). *Descripción y análisis de un caso de enseñanza y aprendizaje de la demostración en una comunidad de práctica de futuros profesores de educación secundaria* (Description and analysis of a case of teaching and learning demonstration in a community of practice of future secondary education teachers). [Doctoral Thesis, Universidad de Valencia]. https://www.educacion.gob.es/teseo/imprimirFicheroTesis.do?idFichero=0SoxuxZNZIQ%3D

Camargo, L., Perry, P., Molina, Ó., Samper, C., & Vargas, C. (2024). Diversidad de acepciones de argumento: necesidad de la formación de profesores (Diversity of meanings of argument: need the need of teacher training). *PNA*, *18*(3), 313–338. http://doi.org/10.30827/pna.v18i3.26749

Clemente, F., & Llinares, S. (2015). Formas del discurso y razonamiento configural de estudiantes para maestros en la resolución de problemas de geometría (Pre-service primary teachers' ways of discourse and configural reasoning in solving geometrical problems). *Enseñanza de las Ciencias, 33*, 9–27. https://doi.org/10.5565/rev/ensciencias.1332

Clemente, F., Llinares, S., & Torregrosa, G. (2017). Visualización y razonamiento configural (Visualization and Configural Reasoning). *Bolema: Boletim de Educação Matemática*, *31*(57), 497–516. https://doi.org/10.1590/1980-4415v31n57a24

Del Grande, J. (1990). Spatial sense. *The Arithmetic Teacher, 37*(6), 14–20.

Diago, P. D., González-Calero, J. A., & Yáñez, D. F. (2021). Exploring the development of mental rotation and computational skills in elementary students through educational robotics. *International Journal of Child-Computer Interaction*, 100388. https://doi.org/10.1016/j.ijcci.2021.100388

Douek, N. (1998). Some remarks about argumentation and mathematical proof and their educational implications. In I. Schwank (Ed.), *European Research in Mathematics Education I - Proceedings of the First Congress of the European Society for Research in Mathematics Education* (Vol. 1, pp. 125–139). http://www.fmd.uniosnabrueck.de/ebooks/erme/cerme1-proceedings/cerme1-proceedings-1-v1-0-2.pdf

Drijvers, P., Godino, J.D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*, 23–49. https://doi.org/10.1007/s10649-012-9416-8

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna, & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education* (pp. 349–368). Springer.

Duval, R. (1995). *Sémiosis et pensée humaine: registres sémiotiques et apprentissages intellectuels* (Semiosis and human thought: semiotic registers and intellectual learning). Peter Lang.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana, & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–52). Kluwer Academic Publishers.

Fiallo, J., & Gutiérrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*(2), 145–167. https://doi.org/10.3390/educsci10010004

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9–24.

Font, V., Trigueros, M., Badillo, E., & Rubio, N. (2016). Mathematical objects through the lens of two different theoretical perspectives: APOS and OSA. *Educ Stud Math, 91*, 107–122. https://doi.org/10.1007/s10649-015-9639-6

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM – Mathematics Education, 39*, 127–135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., Gonzato, M., Cajaraville, J. A., & Fernandez, T. (2012). Una aproximación ontosemiótica a la visualización en educación matemática (An onto-semiotic approach to visualization in mathematics education). *Enseñanza de las Ciencias, 30*(2), 109–130. https://doi.org/10.5565/rev/ec/v30n2.653

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: Implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 38–43. https://www.jstor.org/stable/2674201

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig, & A. Gutierrez (Eds.), *Proceedings of the 20th P.M.E. Conference* (Vol. 1, pp. 3–19). Universidad de Valencia.

Gutiérrez, A., Ramírez, R., Benedicto, C., Beltrán-Meneu, M. J., & Jaime, A. (2018). Visualization abilities and complexity of reasoning in mathematically gifted students' collaborative solutions to a visualization task. A networked analysis. In K. S. Mix, & M. T. Battista (Eds.), *Visualizing Mathematics* (pp. 309–337). *Research in Mathematics Education*. Springer. https://doi.org/10.1007/978-3-319-98767-5_14

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schonfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (Vol. 7, pp. 234–282). CBMS Issues in Mathematics Education. https://doi.org/10.1090/cbmath/007

Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie (Geometrical Paradigms and Geometry Teaching). *Annales de Didactiques et de Sciences Cognitives, 11*, 175–193. ⟨halshs-00858709⟩

Jiménez-Gestal, C., Berciano, A. & Salgado, M. (2022). Razonamiento y aprehensión ante una tarea geométrica: análisis de la pertinencia didáctica de una trayectoria de aprendizaje en educación infantil (Reasoning and understanding in the resolution of a geometric task: analysis of the didactical pertinence of a learning trajectory in early childhood education). *Bolema: Boletim de Educação Matemática, 36*(72), 332–357. http://dx.doi.org/10.1590/1980-4415v36n72a15

Llinares, S., & Clemente, F. (2014). Characteristics of pre-service primary school teachers' configural reasoning. *Mathematical Thinking and Learning, 16*(3), 234–250. https://doi.org/10.1080/10986065.2014.921133

Llinares, S., & Clemente, F. (2018). Characteristics of the shifts from configural reasoning to deductive reasoning in geometry. *Mathematics Education Research Journal, 31*(3), 259–277. https://doi.org/10.1007/s13394-018-0253-7

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*, 87–125. https://doi.org/10.1023/A:1012785106627

Molina, O., Font, V., & Pino-Fan, L. (2019). Estructura y dinámica de argumentos analógicos, abductivos y deductivos: un curso de geometría del espacio como contexto de reflexión (Structure and dynamic of analogical, abductive and deductive arguments: a course on solid geometry as a context for reflection). *Enseñanza de las Ciencias, 37*(1), 93–116. https://doi.org/10.5565/rev/ensciencias.2484

Ortega, T., & Pecharromán, C. (2016). De la visualización a la demostración (From visualization to demostration). *Contextos Educativos. Revista de Educación*, 45–64. https://doi.org/10.18172/con.2717

Paneque, J. J., Cobo, P., & Fortuny, J. M. (2017). Intelligent tutoring and the development of argumentative competence. *Technology, Knowledge and Learning*, *22*(1), 83–104. https://doi.org/10.1007/s10758-016-9283-y

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205–236). Sense. https://doi.org/10.1163/9789087901127_009

Prior, J., & Torregrosa, G. (2013). Razonamiento configural y procedimientos de verificación en contexto geométrico (Configural reasoning and verification procedures in geometric context). *Revista Latinoamericana de Investigación en Matemática Educativa, 16*(3), 339–368. https://doi.org/10.12802/relime.13.1633

Prior, J., & Torregrosa, G. (2020). Razonamiento configural y espacio de trabajo geométrico en la resolución de problemas de probar (Configural reasoning and geometric workspace in the proof's problem resolution). *Bolema: Boletim de Educação Matemática, 34*(66), 178–198. https://doi.org/10.1590/1980-4415v34n66a09

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics, 79*, 19–40. https://doi.org/10.1007/s10649-011-9335-0

Ramírez-Uclés, R., & Flores, P. (2017). Habilidades de visualización de alumnos con talento matemático: comparativa entre los test psicométricos y las habilidades de visualización manifestadas en tareas geométricas (Visualization abilities of mathematically gifted students:a comparison between the results from psychometric tests and the visualization abilities used for geometrical tasks). *Enseñanza de las Ciencias, 35*(2), 179–196. https://doi.org/10.5565/rev/ensciencias.2152

Ramírez-Uclés, R., Flores, P., & Ramírez-Uclés, I. (2018). Análisis de los errores en tareas geométricas de argumentación visual por estudiantes con talento matemático (Analysis of mistakes in geometric tasks of visual argumentation by students with mathematical talent). *Revista Latinoamericana de Investigación en Matemática Educativa, 21*(1), 29–56. https://doi.org/10.12802/relime.18.2112

Reid, D., & Knniping, C. (2010). Argumentation. In D. A. Reid, & C. Knniping, *Proof in Mathematics Education. Research, Learning and Teaching* (pp. 153–165). Sense. https://doi.org/10.1007/s11858-013-0497-3

Saorín, A., Torregrosa, G., & Quesada, H. (2019). Razonamiento configural y organización discursiva en procesos de prueba en contexto geométrico (Configural reasoning and discursive organization in proof processes in geometrical context). *Revista Latinoamericana de Investigación en Matemática Educativa, 22*(2), 213–244. https://doi.org/10.12802/relime.19.2224

Sua, C., & Camargo, L. (2019) Geometría dinámica y razonamiento científico: Dúo para resolver problemas (Dynamic geometry and scientific reasoning: Duo to solve problems). *Educación Matemática, 31*(1), 7–37. https://doi.org/10.24844/em3101.01.

Torregrosa, G. (2017). Coordinación de procesos cognitivos en la resolución de problemas: relación entre geometría y álgebra (Coordination of cognitive processes in problem solving: relationship between geometry and algebra). *Avances de Investigación en Educación Matemática, 12,* 1–17. https://doi.org/10.35763/aiem.v1i12.198

Torregrosa, G., Quesada, H., & Penalva, M. C. (2010). Razonamiento configural como coordinación de procesos de visualización (Configural reasoning as coordination of visualisation process). *Enseñanza de las Ciencias, 28*(3) 327–340. https://doi.org/10.5565/rev/ec/v28n3.187

HOW VISUALIZATION AND ARGUMENTATION ARE ARTICULATED IN RESEARCH ON TEACHING AND LEARNING GEOMETRY

Teresa F. Blanco

University of Santiago de Compostela (Spain)

teref.blanco@usc.es

Leonor Camargo Uribe

Universidad Pedagógica Nacional (Colombia)

lcamargo@pedagogica.edu.co

Pablo G. Sequeiros

University of Santiago de Compostela (Spain)

pablo.gonzalez.sequeiros@usc.es

Abstract

Visualization and argumentation are central processes in the learning of geometry, giving rise to specific lines of research in mathematics education. However, it is worth asking whether they should be developed independently or if it is possible and in what way an articulation between them can be promoted. The objective of this paper is to search for this articulation within the most recent advances in empirical research on the teaching and learning of geometry in Spain. The analysis is approached through the tasks used in research employing two theoretical frameworks which have been developed in Spain, the Onto-Semiotic Approach and Configurational Reasoning. Two main types of task were found in the selected studies: visual tasks and conjecture-and-proof tasks. The results show that in visual tasks the articulation between visualization and argumentation is only necessary in some tasks, in which argumentation aims to justify or to describe the external representation provided as a solution. However, in conjecture-and-proof tasks the articulation between visualization and argumentation must always be present to resolve the task. Analyzing this articulation helps to detect and gain a better understanding of the difficulties that students encounter in solving these types of task. In visual tasks, argumentation helps to understand difficulties associated with the use of visualization. In conjecture-and-proof tasks,

visualization determines the argumentation followed, in such a way that many of the difficulties originate more from the visualization than from the process of argumentation.

Keywords: *visualization, argumentation, teaching and learning geometry, Configural Reasoning, Onto-semiotic Approach.*

1. Introduction

In recent years, visualization linked to geometry has been one of the main lines of research carried out by Spanish researchers in the field of the teaching and learning of geometry (Barquero et al., 2022). This research has mainly been approached from the perspective of two theoretical frameworks developed in Spain: The Onto-Semiotic Approach to the Knowledge and Teaching of Mathematics (OSA) (Godino et al., 2007; Godino et al., 2019) and Configurational Reasoning (CR) (Prior & Torregrosa, 2013, 2020; Torregrosa et al., 2010). These two theoretical frameworks pay particular attention to argumentation in solving tasks in which visualization comes into play. Thus, a fundamental question arises: To what extent does Spanish research on the teaching and learning of geometry integrate both processes and promote their articulation with educational implications? This can be considered an important issue in terms of the teaching and learning of geometry since the analysis of the articulation of these two processes in task resolution can facilitate the identification and understanding of difficulties encountered by students when attempting to solve them.

Along the same lines as Gutiérrez (1996), visualization in mathematics will be considered ‘‘as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties’’ (p. 99). This author points out four key elements of visualization: mental images, taken to be any kind of representation of a concept or mathematical property by way of visual or spatial configurations (Presmeg, 2006); external representations, be they verbal or graphic, of concepts or properties, including images, drawings, diagrams, etc.; visual processes, understood as mental or physical actions in which mental images or external spatial-graphic representations intervene (Bishop, 1989); and visualization skills (Del Grande, 1990), stable capacities necessary for the resolution of mathematical tasks, such as visual memory, the identification of spatial relationships, figure-ground perception, etc. The construction of meaning, problem solving and the communication of ideas are heavily influenced by visualization (Arcavi, 2003; Fischbein, 1982; Godino et al., 2012; Gutiérrez, 1996; Presmeg, 2006; Prior & Torregrosa, 2013), particularly in school geometry. However, students tend to shy away from visualization due to the challenges involved in its implementation and the perception that it demands a high cognitive level (Gutiérrez et al., 2018). Ramírez-Uclés & Flores (2017) highlight various factors influencing the utilization of visualization in task resolution, including the contextualization of the teaching process, the task format, the approach adopted, and the strategies employed by students in tackling the task.

On the other hand, argumentation is conceptualized as a collective or individual process, which, in accordance with shared rules, points towards a mutually acceptable conclusion about the truth or falsity of a statement or action (Molina et al., 2019, Camargo et al., 2024). Argumentation is a relevant process in school mathematics education, not only as it contributes to giving meaning and depth to mathematical learning, but also because it helps to generate an environment of inquiry in the classroom where ideas are justified collectively and not only by the teacher’s authority (Boero at al., 2018). Argumentation has been strongly linked to the teaching of geometry due to the possibility of students inferring properties through the exploration of geometrical representations and concluding, from certain propositions already known or assumed to be true, new true propositions following on from the previous ones (Baker, 2009; Douek, 1998; Durand-Guerrier et al., 2012). Depending on the mathematical task being carried out, students can develop different types of arguments (Reid & Knniping, 2010). For instance, during the exploration of a representation and the formulation of conjecture, students may articulate arguments of an inductive, abductive, or analogical nature. These arguments serve to empirically support the certainty of their discoveries or the acceptability of their positions or actions. Empirical arguments are those in which statements are supported with reasons that come from information given or obtained by exploration. The argument will be visual when the reasons are based on information extracted from a figure or visual representation. Furthermore, when validating results in the framework of theoretical systems of reference, the main arguments may be deductive, in which case we refer to them as proofs, since they lead to logically necessary conclusions (Durand-Guerrier et al., 2012). In any case, argumentation has the aim of justifying assertions when seeking to convince, persuade, or validate what is being stated.

The general aim of this research, oriented towards answering the initial question posed in the first paragraph of the Introduction, is to examine how the articulation between visualization and argumentation is manifested in the most recent research carried out in Spain on the teaching and learning of geometry. As specific objectives, the role of argumentation in tasks promoting visualization and the relevance of visualization in conjecture-and-proof tasks shall be examined.

This paper consists of six sections. The first section is aligned with the Introduction, outlining the issue that prompted this study. The second section briefly presents the two theoretical frameworks, which support the most relevant studies in which the links between visualization and argumentation in the teaching and learning of geometry will be identified. The third section details the description and selection of the sample. The following two sections present the analysis carried out based on the types of task proposed in the selected studies. Thus, the fourth section presents the role of argumentation in tasks promoting visualization, while the fifth section shows how visualization works in conjecture-and-proof tasks focusing on argumentation. The final section highlights the conclusions.

2. Visualization and argumentation in the OSA and CR frameworks

The Onto-Semiotic Approach is a modular and inclusive theoretical system that has developed various tools to address the research problems posed by the teaching and learning of mathematics (Godino et al., 2007; Godino et al., 2019). This framework examines the correspondences and synergies between the OSA and other theoretical frameworks used internationally in mathematics education (Drijvers et al., 2013; Font et al., 2015). The OSA postulates that the exploration of mathematical activity initially focuses on the practices carried out by people involved in solving mathematical task. It considers *mathematical practice* to be any action or manifestation (linguistic or not) carried out to solve mathematical task. This includes efforts to communicate the solution to others and to validate and generalize that solution to other contexts and problems (Godino et al., 2007).

The OSA considers six *objects* that intervene in mathematical practices: language, tasks, concepts, propositions, procedures, and arguments (Godino et al., 2007). These objects are related to each other forming *ontosemiotics configurations*, defined as the networks of intervening objects emerging from mathematical practices and relationships established among them. This way, language (terms, expressions, notations, graphics) constitutes the means of expression of all the other objects, helping to point out the remaining objects at play in the resolution of the task. Tasks provide the objects that intervene in the mathematical practices from which the solution emerges. Concepts are introduced through their definitions or descriptions. Propositions are statements about the concepts involved in the mathematical practice. Arguments are used to validate and support propositions and procedures put into play and are the product of an argumentation process that gives meaning, organizes, and relates the other objects and whose organization constitutes a solution to a task. The scheme of an ontosemiotic configuration is shown in Figure 1. A configuration is called *cognitive configuration* when it comes from the student and represents the path followed by the student in the process of solving mathematical tasks. Some examples of cognitive configurations can be seen in Blanco et al. (2012). Following Gutiérrez's (1996) definition of visualization, within this theoretical framework visualization is analyzed from the point of view of these objects, in which it is said that visualization is involved (Godino et al., 2012).

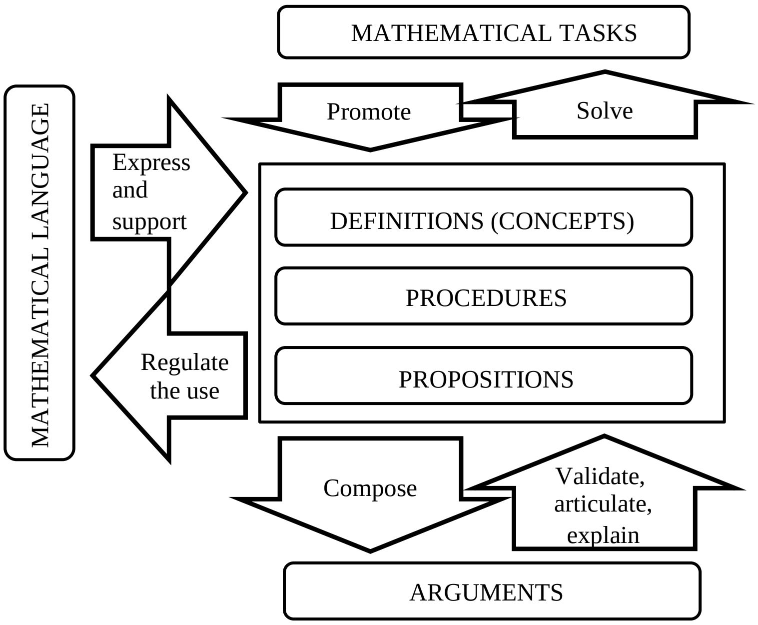


Figure 1 *Scheme of an ontosemiotic configuration*. Adapted from Molina et al. (2019, p.98)

The students' cognitive configurations show that all mathematical practices normally consist of two components, one visual and the other analytical. These two components support each other synergistically in solving the corresponding task. The visual component plays a key role in understanding the nature of the practices and formulating conjectures, whereas the analytical component becomes important when generalizing and justifying solutions via arguments. The degree of visualization put into play in the solution of a task will depend on whether the task is visual or not, and on the cognitive styles of the person who solves it (Godino et al., 2012).

Godino et al. (2012) consider that, in the case of geometry, visualization mainly comes into play in three types of task, which they call visual tasks: (1) Tasks that involve the communication of shapes, their components and structures, whether real or imaginary, using visual language (representations in the form of photographs, drawings, or diagrams); (2) The communication of the relative position of spatial shapes via the use of deictic language (terms such as, up, down, in front of, behind, right, left, near, far, north, south, east, west), which can also be carried out via the use of mediators such as models, plans, or maps; (3) The recognition of invariants in shapes or in their representations when faced with specific transformations, which is supported by the ability of visual discrimination: comparing different spatial shapes, drawings, or images and identifying similarities or differences among them (Del Grande, 1990). Although Blanco et al. (2012) have showed considerable difficulties in eliciting students' argumentation in these three types of task, since most students do not consider it necessary to provide justifications, the OSA emphasizes the importance of connecting visualization with argumentation in mathematical practices. To support an argument based on properties and procedures of an empirical nature, the external representation of those properties, procedures and concepts is sufficient. In other cases, the construction of deductive arguments based on previously accepted rules is required.

Configural Reasoning, the other theoretical framework used in this paper, explains the coordination of visualization processes during students' resolution of geometric tasks in a pencil-and-paper context (Clemente et al., 2017, Clemente & Llinares, 2015, Llinares & Clemente, 2014, 2018, Prior & Torregrosa, 2013, 2020, Saorín et al., 2019, Torregrosa, 2017 and Torregrosa et al., 2010). Here visualization is understood as "the transfer of objects, concepts, phenomena, processes and their representations to some type of visual representation and vice versa'' (Prior & Torregrosa, 2013, pp. 342-343). This framework has its origins in Duval's (1998) study on perceptual, discursive, and operative apprehensions. *Perceptual apprehension* refers to the visual recognition of a shape, including the ability to recognize several sub-figures in the perceived figure. *Discursive apprehension* is related to the control of statements over any perceptual recognitions of geometrical properties in geometrical representations. *Operative apprehension* is related with the various ways in which a given figure can be modified: the division of the whole figure into different parts and the combination of these parts into another figure or sub-figures, when the figure is made larger, narrower or slanting, or when the position or orientation of a figure is changed. Torregrosa et al. (2010) call the coordination of discursive and operative apprehensions Configural Reasoning, defining *configuration*[1] as any planar representation of a geometrical shape.

The strength of an expert solver lies in his/her ability to move from the given configuration (through visualization) to discourse and to recognize configurations linked to propositions for constructing deductive arguments based on theoretical knowledge. This practice is not spontaneous for inexpert students. However, Saorín et al. (2019) have stated that students should progressively develop argumentation skills, from empirical to deductive means of arguing, to justify the plausibility of their choices. Passing from argumentation based on intuitive and visual justifications to logical-deductive considerations is no easy task for students (Prusak et al., 2012). Thus, it is important to characterize the conditions which facilitate the transformation of visualization into configural reasoning, enabling a deductive process to be generated. As this coordination does not always arise, studies have analyzed, among other factors, the causes which explain the absence or interruption of coordination. For example, difficulties may arise in relation to the characteristics of the figural representations which can inhibit rather than highlight the identification of geometrical properties present in the representation.

When attempting to compare these two frameworks, it should be taken into consideration that the OSA is an inclusive theoretical system which seeks to articulate different approaches and theoretical models used in research on mathematics education (particularly on visualization), while CR is a specific model which seeks to explain the coordination of operative and discursive apprehension when students resolve proof tasks in a paper-and-pencil geometrical context. In any case, the similarities between the articulation made by these two frameworks of both processes (visualization and argumentation) can be appreciated by contrasting the notions of cognitive configuration and configural reasoning. The notion of cognitive configuration includes the relationships between all the objects which intervene in mathematical practices (tasks, linguistic and material elements, concepts, propositions, procedures, and arguments) and how these relationships are

[1] Note that the notion of *configuration* has different meanings depending on whether it is employed in the OSA or the CR framework.

coordinated. The categorizations made of students' cognitive configurations is based on their arguments about the other objects involved in the practice, with these arguments depending largely on the visualization skills and processes that students bring into play (Blanco et al., 2019). Configural reasoning places visualization as a central element of deductive argumentation in geometry since it involves the coordination of discursive and operative apprehensions. If there is only perceptual apprehension and no arguments are put into play, configural reasoning does not arise. In this case, visualization is an element of configural reasoning that is identifying to perceptual apprehension, while Gutierrez's (1996) definition of visualization would correspond to configural reasoning.

3. Sample description

The starting point of this study is provided by the 95 empirical studies reviewed in Barquero et al. (2022), which were selected from an exhaustive review of publications indexed in high-impact journals over the last 15 years. Those that specifically address visualization and conjecture-and-proof tasks were included in the sample for this paper.

In terms of educational level, production is most extensive at the secondary level, with research here being centered on conjecture-and-proof tasks. However, some authors have focused on higher education, particularly on teacher training programs. Other recent contributions have focused on early years education, particularly in terms of spatial orientation tasks. A qualitative methodology is employed in most of the publications, although some quantitative and mixed-method studies have been found. Some studies address other important processes in the teaching and learning of geometry, such as describing, defining, classifying, and generalizing. The discussion of how visualization and argumentation are articulated in these studies has not been included here since the link between argumentation and visualization in the papers reviewed is not explicit.

The studies underpinned by the OSA which will be addressed in the present paper, emphasize tasks focused on the promotion of visualization. The studies supported by CR address conjecture-and-proof tasks, focusing on the argumentation process. Taking the above into consideration, the articulation will be presented in accordance with the types of task contemplated in the studies: visual tasks on the one hand, and conjecture-and-proof tasks on the other.

4. Argumentative support in visual tasks

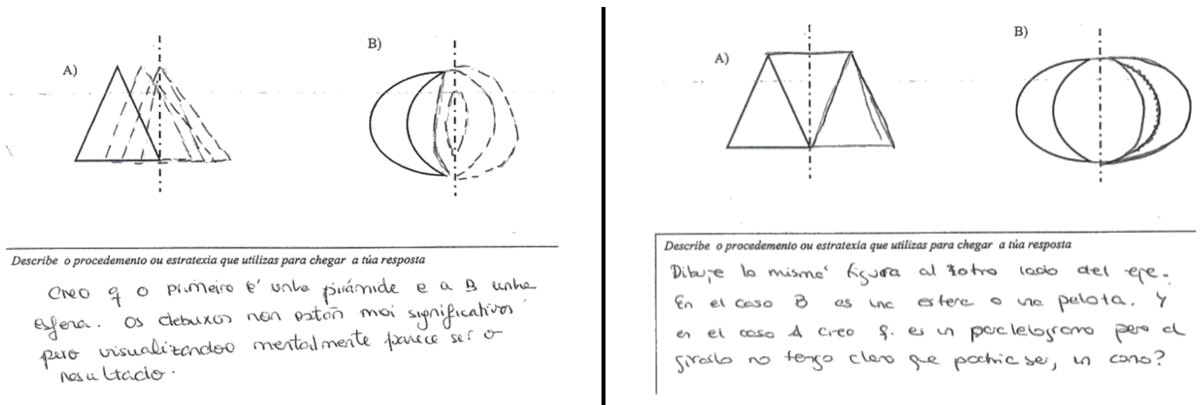
This section analyzes the intervention of argumentation in the three types of visual tasks defined in section 2, following the theoretical framework of the OSA. These tasks mainly seek to promote visualization to apply it in other geometrical processes. The interest is focused on determining which elements of visualization (processes, mental images, external representations, or visualization skills) students mobilize in solving tasks and how they do it, with the aim of identifying possible difficulties associated with their use. Since inferring these elements solely from the graphic representations provided by students is often quite difficult, in some tasks students are explicitly asked to justify their answers. It is in this sense that argumentation can be considered to act as a support for visualization.

4.1 Tasks that involve the communication of shapes, their components and structure

Research such as that carried out by Beltrán-Pellicer et al. (2020), Blanco et al. (2012), and Blanco et al. (2019) focus on this first type of visual task. Blanco et al. (2019) analyze the visualization skills and processes involved in a task for pre-service teachers requiring the generation and representation of solids of revolution (Figure 2). A categorization of students' answers is presented by using the cognitive configuration notion from the OSA. In the answers shown in Figure 3, students had difficulties in generating the external representation. In the answer on the left side, the graphic representations could be interpreted as 3D solids due to the succession of point shapes drawn by the student, which seem to infer movement around the axis of rotation. Regarding the answer on the right side, the graphical representation could be interpreted as the symmetrical shape of the given initial shape. In either answer, the visual argument is not enough. The text accompanying the graphic representations describes the mental images generated by the students and allows us to conclude that they generate 3D solids. Despite this, the solutions are not correct in all cases nor are they sufficiently detailed (gaps or holes are not specified). In these two resolutions of the task, the argument has the function of describing the graphic representation provided by the students. In general, most of the students considered the task in 2D, producing an axial symmetry. Most of the errors of those who situated the task in a 3D context were due to difficulties in mobilizing the recognition of spatial relationships and positions. Few students provided a solution based solely on a graphical representation using the conventional codes of planar representation of 3D shapes or perspective representation. In this case, said representation would be considered visual argumentation and would be sufficient to conclude if the solution is correct.

Draw, in the greatest possible detail, the solids of revolution which are obtained by rotating the following figures around the indicated axes.

Figure 2 *Visual task of mental rotation* (Blanco et al., 2019, p.772)



I think the first one is a pyramid and B is a sphere. The drawings are not very significant, but this seems to be the result after mentally visualizing the task.

I drew the same figure on the other side of the axis. Case B is a sphere or a ball. And I think case A is a parallelogram but, when it is turned around, it is not clear to me what it could be. A cone?

Figure 3 *Two students' resolution of a visual task* (Blanco et al., 2019, pp.778-779)

This first type of task also appears in studies which incorporate technology as a teaching resource for secondary students. Beltrán-Pellicer et al. (2020) use BlocksCAD to model solids, thus connecting visualization with computational thinking, which acts as a mediator. For example, they propose the task of creating a model of a solid (Figure 4). In doing so, different visualization skills come into play, depending on the different ways in which the solid can be built, as these lead to the use of different concepts or mathematical processes and, thus, to the elaboration of different algorithms. Inductive and abductive argumentations arise when students try to persuade their peers which algorithm is the correct or most appropriate one. For example, the solid in Figure 4 can be thought of as a cube with square-based pyramids in the center of each face or as the combination of a cube and an octahedron, in which case the elaboration of the algorithm is considerably shorter. According to these authors, the interest of this resource lies in the fact that, when faced with possible complications of the algorithm arising from the visualization of the shape, the context itself encourages students to look for another visualization that facilitates programming. In this way, the articulation between visualization and argumentation takes place throughout the process of creating the algorithm.

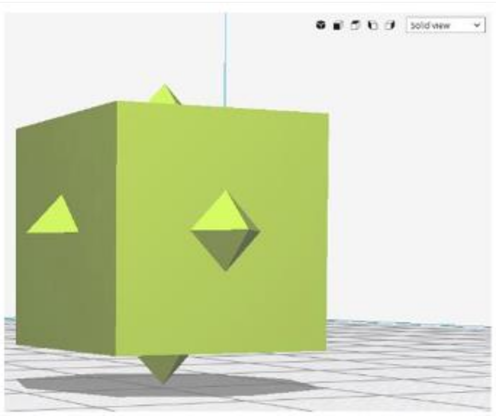


Figure 4 *Solid for the BlocksCAD activity* (Beltrán-Pellicer et al., 2020, p. 45)

4.2 Task of communicating positions

Other studies focus on the second type of task (Bellas et al., 2019; Diago et al., 2021; Gutiérrez et al., 2018; Ramírez-Uclés & Flores, 2017; Ramírez-Uclés et al., 2018). Gutiérrez et al. (2018) analyze the use of visualization skills of secondary students with mathematical talent in a collaborative task requiring them to construct a set of buildings from their orthogonal projections. The dialogues among the students show the discursive arguments they build up as they integrate different views to come to an agreement and achieve the final construction, which implies the empirical argumentation of their conjecture. The results show that there is a relationship between the objective of the student's actions and the type of visualization skills used.

At an early age, tasks are more oriented toward visualizing space through reading maps and describing, constructing, and planning routes. Students justify their decisions via argumentation based on graphic elements (pictograms). Examples of such tasks can be found in the studies by Berciano et al. (2017) and Jiménez-Gestal et al. (2022). The latter proposes a task in which children (3-5 years of age) must find some treasure in their school and draw a map of the route taken to find it. The authors analyze the use of different spatial skills and the children's argumentation when they try to convince their peers about the correct route. Their empirical arguments evolve from representations in which the treasure is the key element (3 years of age) to representations in which all points of reference (tree, stairs, and door) appear and a relationship of order is maintained corresponding to the real location of the child (5 years of age).

These types of task can also be approached employing robotics (Bellas et al., 2019; Diago et al., 2021). For instance, Diago et al. (2021) investigate the effects of an educational intervention with the Bee-bot robot to evaluate the skill of mental rotation and computational thinking in a primary class (8 years of age). Although their research does not provide conclusive results on mental rotation, the authors believe that the Bee-bot can be a useful tool to encourage spatial skills as it lies between the visuospatial, computational, and mathematical domains. Children articulated empirical argumentation with visualization skills by verbally justifying route planning or reviewing the outcome of the instructions given to the robot.

4.3 Tasks of recognition of invariants in shapes or in their representations

The research carried out by Blanco et al. (2018) and Ramírez-Uclés et al. (2017, 2018) concerns the third group of visual tasks. Blanco et al. (2018) present a 2D task for pre-service teachers in which five figures built from two identical pieces are shown. Without picking up either of the pieces, the students are required to identify which figures cannot be built and to justify their answer (Figure 5). In Figure 6, the answer on the left side shows student has made their choice arguing that the exterior right angle of the pieces is not preserved when forming the figure. On the right side, the student's attention is focused on the visual perception of the concavity. Most of the arguments given by the pre-service teachers were based more on their visual perceptions than on properties related to rigid motions of the plane, thus showing a predominance of visual argumentation in this task. Furthermore, the choice of options A and B as the most widely selected answers shows the students' difficulties in putting into action the ability to preserve perception.

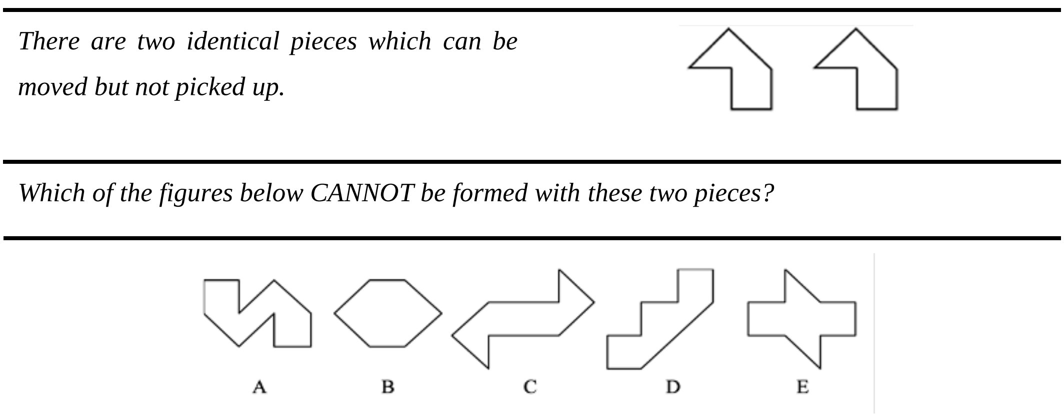
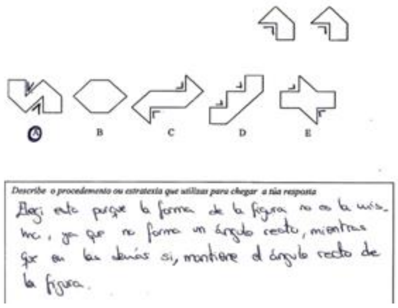


Figure 5 *Visual task of rigid motions of the plane* (Blanco et al., 2018, p. 258)



I chose this one because the shape of the figure is not the same, since it does not form a right angle, while in the others it does, the right angle of the figure is maintained.

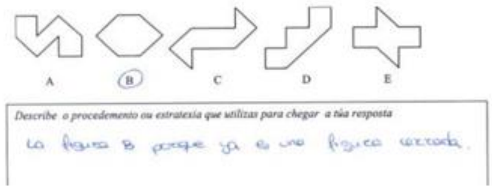


Figure B because it is already a closed figure.

Figure 6 *Two students' resolution of the visual task* (Blanco et al., 2018, pp. 266-267)

Also related to motions in the plane, Ramírez-Uclés et al. (2018) present a teaching process for the development of visualization skills in students with mathematical talent, focusing on the evolution of errors over three sessions. They propose a task to configure all the possible structures by putting three model pieces together (Figure 7), seeking the appearance of motions when defining if two structures are equivalent. The arguments provided by the students make it possible to detect three errors related to visualization: establishing a false analogy between the plane and space; drawing conclusions after having examined only some of the possible cases; and producing an argument based on specific limited examples. A decrease in the frequency of errors throughout the experiment is shown, despite the greater complexity of the tasks proposed. In this task, empirical argumentation is necessary but not sufficient since the task implicitly asks to justify that there are no more structures.

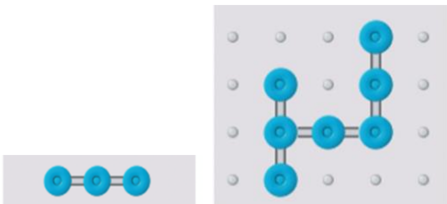


Figure 7 *Model piece and example of a structure with three pieces* (Ramírez-Uclés et al., 2018, p. 41).

The works described suggest that visualization tasks demanding argumentation stimulate better ways of visualizing and allow visualization difficulties to be identified. This is achieved not only by requiring justification of findings and decisions within a task, but also by fostering opportunities for the collective elaboration of constructions, algorithms or structures in which agreements must be reached.

5. Visual support in conjecture-and-proof tasks

This section refers to studies focusing on conjecture-and-proof tasks. Although the OSA also supports research on conjecture-and-proof tasks in geometry, most of the studies reviewed are based on the CR model, which was developed specifically for tasks of this nature (Clemente et al., 2017; Llinares & Clemente, 2018; Prior & Torregrosa, 2020; Saorín et al. 2019; Torregrosa, 2017). Research has mainly been concerned with observing how students use the visualization of a graphic representation in conjecture-and-proof tasks to find logical relationships between geometrical properties with which to argue and create the required conjecture or proof.

5.1 Proof tasks

As stated in section 2, the CR model is oriented to studying the connection between argumentation and visualization through the interaction between discursive and operational apprehensions. The coordination of these two apprehensions (configural reasoning) results from the visualization of a representation which is modified via operative apprehension, adding or removing elements or manipulating parts of the configuration to make it possible to fix attention on a specific subconfiguration (Clemente et al., 2017). Subsequently, mathematical statements are associated with that subconfiguration via discursive apprehension. For example, the task shown in Figure 8 requires proof of the congruence of segments *RC* and *BT*, given the congruence of segments *AB* and *AC* and angles ∢RCB and ∢TBC. The configuration provided in the task (left side of Figure 8) can be examined by paying attention to three different subconfigurations, which are present but overlapping, from which different resolutions can be generated. Thus, it is possible to concentrate attention on two different pairs of triangles (Subconfiguration a and Subconfiguration b) or to focus on two opposite angles and the angles juxtaposed to them (Subconfiguration c). These three possible subconfigurations allow for the recognition of the premises of the triangle congruence criteria. However, the students' success in coordinating the two apprehensions to solve the task depended largely on the subconfiguration considered, with Subconfiguration b having a much higher success rate. As for the visualization skill required, it should be noted that students needed to use the ability of figure-ground perception (Del Grande, 1990; Gutierrez, 1996) to solve this task.

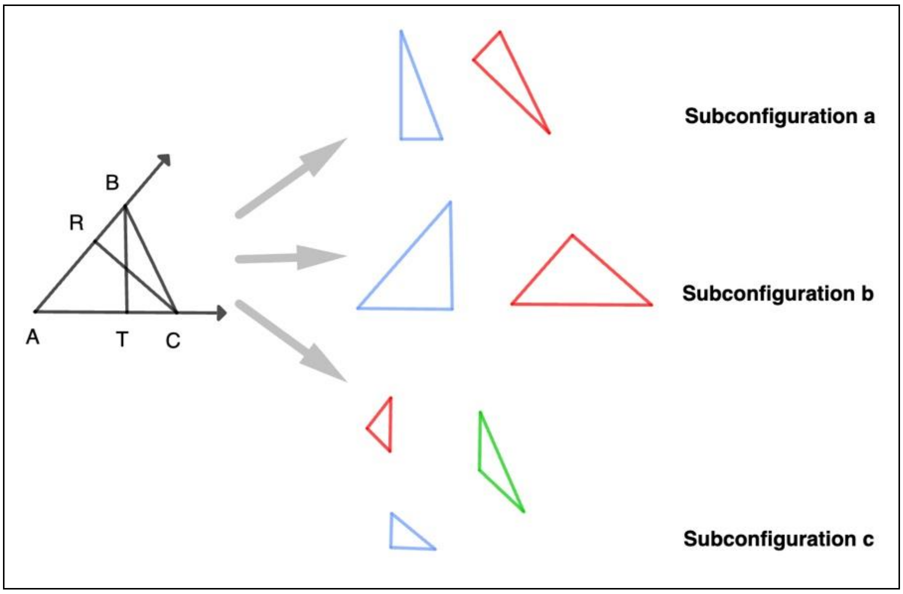


Figure 8 *An example of different subconfigurations.* Adapted from Clemente et al. (2017, p. 505)

The recognition of a relevant subconfiguration in a geometrical configuration is a necessary condition, although it is not sufficient to provoke configural reasoning. For example, Clemente et al. (2017) analyze and explain the relationship that exists between the visualization and identification of prototypical figures and the geometrical knowledge that exists before to the generation of arguments leading to a purely deductive process. These authors conclude that prototypical representations play a relevant role in the dual relationship between visual perception and knowledge linked to changes in visual-discursive anchoring and vice versa. This is because the recognition of a prototypical figure in the initial configuration activates a certain knowledge of geometry. Meanwhile, Torregrosa (2017) and Saorín et al. (2019) focus on the analysis of configural reasoning when tasks in a geometrical context introduce numbers and what is sought is not to prove a general proposition but to find a specific value. In this case, it is concluded that to resolve the task correctly, students must be able to express the geometrical situation posed in the form of relations in the algebraic register. Once this happens, the resolution process becomes independent of any visualization process, since the need to interact with the relevant subconfiguration identified to solve the task ceases.

Furthermore, Llinares & Clemente (2018) focus on factors that may interfere with the productive articulation between the visualization of information present in a representation and the coordination of operative and discursive apprehensions. The success of the process depends on the use of appropriate forms of representation, true statements, and valid forms of reasoning. The results of the study make it possible to achieve a greater understanding of the phenomenon that Duval (1995) calls the double gap between naïve behavior and mathematical behavior in the transition from configural reasoning to the production of proof. The first gap is related to visualization; students need to change the epistemic perspective of the geometrical facts to stop seeing them as mere attributes of visual configurations and begin to see them as assumptions of geometrical propositions that can be linked logically. The second gap is related to argumentation; students must learn to choose which geometrical facts are those that will enable them to link arguments deductively to obtain the required proof.

About the first gap, Prior & Torregrosa (2020) have established links between the CR Model and the Mathematical Working Spaces proposed by Houdement & Kuzniak (2006) to study the influences of the institutional environment in which geometrical activity is developed in configural reasoning. They describe the discursive organization of the responses of secondary education students, who are heavily influenced by their personal geometrical working spaces to highlight the challenges students face when having to move from empirical and intuitive arguments based on visualizations to deductive argumentation.

5.2 Conjecture tasks

In the case of conjecture tasks, argumentation is a justification of the plausibility of what is discovered, while visualization acts as a resource to obtain reasons for the certainty of what is being obtained. In such tasks, students generally explore a visual representation in search of an invariant not mentioned in the statement of the problem which depends on the given conditions. They then formulate a conjecture and justify it. The studies included in this subsection do not follow the CR model. However, it can be considered that their objective is convergent with it since they focus on studying how visualization determines progress towards deductive arguments.

Marrades & Gutiérrez (2000) study possible ways in which students use argumentation to convince themselves or persuade their classmates or teacher of the certainty of a statement. Argumentation arises both in the discovery phase of the conjecture and in its proof. These authors propose a classification of empirical or deductive arguments according to the nature of the justification elaborated and, particularly, to the way in which the students select and employ graphic examples of what they are stating.

Along the same lines as Marrades & Gutiérrez (2000), Spanish researchers, in collaboration with others in Latin America (Araujo et al., 2006; Arce et al., 2014; Camargo, 2010; Fiallo & Gutiérrez, 2017; Ortega & Pecharromán, 2016; Paneque et al., 2017; Sua & Camargo, 2019), have studied advances in argumentation throughout teaching processes, with or without the mediation of dynamic geometry programs and with or without the support of a teacher, in situations of the resolution of conjecture tasks. For example, as can be seen in Figure 9, Sua & Camargo (2019) propose a conjecture task for secondary students (12-16 years old) in which the support that visualization provides to argumentation can be observed. The visualization of the different circumferences obtained lead students to discover and justify a procedure to obtain the geometrical locus of their centers.

Given two points (A and B), is there a circle that contains them? Is there another? Are there more? If so, where are the centers of the circles containing points A and B?

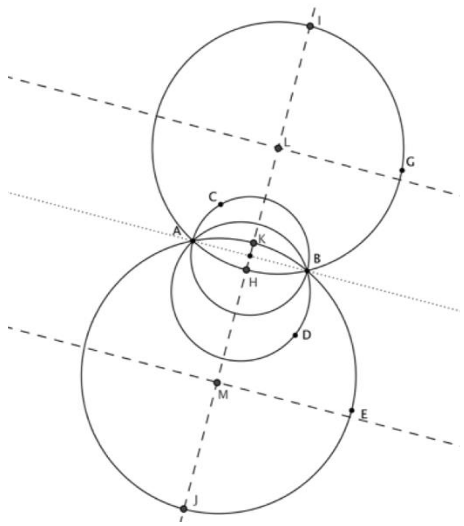


Figure 9 *Conjecture task* (Sua & Camargo, 2019, p.27)

Some studies refer to the evolution of the arguments proposed to justify the conjectures that are obtained. Although slightly different terms are used, the authors of these studies mention visual support for developing empirical arguments, and how visual configurations lose prominence in favor of deductive justifications. This is the case of Araujo et al. (2006), who carried out a study of the long-term individual production and behavior of a group of pre-service teachers who were set geometrical conjecture tasks.

Unlike the previous works, those by Arce et al. (2014), Fiallo & Gutiérrez (2017), Ortega & Pecharromán (2016), and Paneque et al. (2017) propose tasks on elementary geometrical content, such as areas, polygons, and trigonometry. Arce et al. (2014) and Paneque et al. (2017) make use of Harel & Sowder’s (1998) classification to analyze students’ productions when resolving conjecture tasks in which they are required to use or generalize formulas to obtain areas and justify their results. In addition to situating the solutions in a particular type of justification, Arce et al. (2014) classify the pre-service teachers’ errors. They highlight the absence of generic examples to justify the formulas obtained, as many of the participants opted for findings based on a few examples, showing inductive and visual patterns of argumentation.

Meanwhile, Paneque et al. (2017) focus on the development of argumentation among 16-17-year-old students under the influence of an intelligent tutoring system called GeoGebraTutor. The authors characterize GeoGebraTutor-teacher-student interactions at different moments of solving the proof task. They point out that, despite progress in the development of argumentation, and the use of general rather than specific examples, a tendency is perceived among the participants to base themselves on empirical arguments of a visual nature. Ortega & Pecharromán (2016) study the type of argumentation given by pre-service teachers for the accuracy or inaccuracy of several geometrical constructions suggested for the problem of conjecturing how to build a regular pentagon. The results show the difficulties many students have when generating non-empirical argumentation and their problems in terms of the coordination of graphic, algebraic, or arithmetic representations. Fiallo & Gutiérrez (2017) examine the argumentation given by grade 10 students (14-15 years of age) relating to trigonometric reasoning with the support of a dynamic geometry program. The focus of their research is the connection (or lack thereof) between the arguments generated during the establishment of the conjecture and those produced during its validation. They report the role played by the dynamic geometry program in students advancing from visual arguments to those of a deductive nature.

The support that visualization provides to argumentation processes is undeniable. In proof tasks, the identification of prototypical configurations and key subconfigurations accompanying the elaboration of proofs indicate ways through which students can proceed. In conjecture tasks, visualization allows to discover geometric relationships and gain confidence in the findings obtained. In some studies on this type of task, a certain preference on the part of students for visual arguments over deductive ones is observed.

6. Conclusions

This article has examined the most recent Spanish research on the teaching and learning of geometry focused on visualization or argumentation to analyze how these two processes are articulated. The analysis has been presented considering the two most frequent types of task in these studies: visual tasks and conjecture-and-proof tasks, which have mainly been supported by the OSA and CR frameworks.

One specific objective has been to examine the role of argumentation in tasks promoting visualization. In some papers on visual tasks, which are focused on the development of visualization processes and skills, the role of argumentation is not as explicit, as seen in section 4. Students' argumentation is inferred from the descriptions they give with the external representations they propose as solutions to the task, the algorithms they elaborate for achieving a construction (inductive, abductive, or analogic argumentation), or the justifications for their solutions (deductive argumentation). In some tasks, like those present in Blanco et al. (2018, 2019), the results of the visual processes alone can lead to a correct solution, providing a valid visual argument to understand the solution offered by the student. However, if the student's graphic representation does not enable a clear interpretation, the argumentation accompanying the representation becomes crucial in determining whether the visual processes have led to a satisfactory resolution or not, as can be seen in Figure 3. In summary, less competence in visualization processes and skills requires greater intervention of different types of argumentation, in addition to visual argumentation, to achieve success in solving the task.

The other specific objective addresses the role of visualization in conjecture-and-proof tasks. In studies on conjecture-and-proof tasks, argumentation is necessary to confer certainty and validity to the properties and procedures involved in task resolution. The type of argumentation (be it visual or analytical) in such tasks is deductive and the role of visualization is clearly defined, as made explicit in the CR framework. Thus, visualization determines the type of subconfiguration the student uses when stating a conjecture or proving a result, as has been illustrated with example of Figure 8 from Clemente et al. (2017). In essence, visualization dictates the argumentation followed, in such a way that many of the difficulties in resolution originate more from the visualization process itself than from the argumentation process. In many cases, these difficulties are related to the use of prototypical images, which do not activate the geometrical properties or relationships that may be considered to solve the task.

It has been showed that the two theoretical frameworks that support most of the studies analyzed show the need for a synergy between visualization and argumentation processes to achieve the desired solution. It is crucial to examine whether this articulation takes place and how, with the aim of gaining an understanding of the cognitive processes involved when students approach such tasks. This approach returns to the premise of the OSA, which sustains the existence of both visual and analytical components in any type of task. Depending on the type of task, this articulation may be stronger or weaker.

Finally, the analysis conducted for this study has highlighted two emerging research directions. One of them is aimed at students with mathematical aptitudes, given the lack of consensus on the relationship between visualization and talent in mathematics. The other promising avenue is to explore the effects of software applications that delve into automated forms of argumentation. Although these studies are in their incipient and experimental phases, it is crucial to highlight the role of these applications in strengthening processes such as justification, argumentation, and validation of theorems.

Acknowledgements

Study carried out within the framework of PID2021-122326OB-I00, Proyectos De Generación de Conocimiento 2021/ MCE-AEI.

All authors declare that they have no conflicts of interest in this article.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*, 215–241. https://doi.org/10.1023/A:1024312321077

Araujo, J., Giménez, J., & Rosich, N. (2006). Afectos y demostraciones geométricas en la formación inicial docente (Affects and geometrical proof for initial teacher training). *Enseñanza de las Ciencias*, *24*(3), 371–386. https://doi.org/10.5565/rev/ensciencias.3788

Arce, M., Conejo, L., & Ortega, T. (2014). Análisis de los procesos de justificación y generalización de la fórmula del área del rectángulo por alumnos del grado de educación primaria (Analysis of justification and generalization processes of the rectangle area formula by primary education students). *Profesorado, Revista de Curriculum y Formación del Profesorado*, *18*(2), 209–227. https://revistaseug.ugr.es/index.php/profesorado/article/view/19237

Baker, M. (2009). Argumentative interactions and the social construction of knowledge. In N. Mirza, & A. N. Perret-Clermont (Eds.), *Argumentation and education: Theoretical foundations and practices* (pp. 127–144). Springer.

Bellas, F., Salgado, M., Blanco, T. F., & Duro, R. J. (2019). Robotics in Primary School: A Realistic Mathematics Approach. In L. Daniela (Ed.), *Smart learning with educational robotics*. Springer. https://doi.org/10.1007/978-3-030-19913-5_6

Barquero, B., Batanero, C., Blanco, T. F., Bosch, M., Camacho-Machín, M., Cañadas, M. C., Castro, E., Chamoso, J. M., Contreras, L. C., Gea, M. M., Godino, J. D., Martín-Molina, V., Moreno, M., & Wilhelmi, M. R. (2022). Spanish research on mathematics education. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 223–254). PME.

Beltrán-Pellicer, P., Rodríguez-Jaso, C., & Muñoz-Escolano, J. M. (2020). Introduciendo BlocksCAD como recurso didáctico en matemáticas (Introducing BlocksCAD as an educational resource in mathematics). *Suma*, *93*, 39–48.

Berciano, A., Jiménez-Gestal, C., & Salgado, M. (2017). Kindergartners' use of symbols in the semiotic representation of 3-dimensional changes. *International Electronic Journal of Mathematics Education*, *12*(3), 311–331. https://doi.org/10.29333/iejme%2F615

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7–16.

Blanco, T. F., Godino, J. D., & Pegito, J. A. C. (2012). Razonamiento geométrico y visualización espacial desde el punto de vista ontosemiótico (An onto-Semiotic approach to geometrical reasoning and spatial visualization). *Bolema: Boletim de Educação Matemática, 26*(42a), 39–64. https://doi.org/10.1590/S0103-636X2012000100004

Blanco, T. F., Godino, J. D., & Diego-Mantecón, J. M. (2018). Análisis epistémico y cognitivo de una tarea de visualización en el espacio bidimensional (Epistemic and Cognitive Analysis of a 2D Visualization Task). *Journal of Research in Mathematics Education*, *7*(2), 251–279. https://doi.org/10.17583/redimat.2018.2463

Blanco, T. F., Diego-Mantecón, J. M., & Sequeiros, P. G. (2019). Procesos de visualización en una tarea de generación y representación de cuerpos de revolución (Visualization processes in a task that requires generating and representing solids of revolution). *Bolema: Boletim de Educação Matemática*, *33*(64), 768–789. https://doi.org/10.1590/1980-4415v33n64a16

Boero, P., Fenaroli, G., & Guala, E. (2018). Mathematical argumentation in elementary teacher Education: The key role of the cultural analysis of the content. In A. Stylianides, & G. Harel, *Education research on proof and proving. ICME-13. Monographs* (pp. 49–67). Springer.

Camargo, L. (2010). *Descripción y análisis de un caso de enseñanza y aprendizaje de la demostración en una comunidad de práctica de futuros profesores de educación secundaria* (Description and analysis of a case of teaching and learning demonstration in a community of practice of future secondary education teachers). [Doctoral Thesis, Universidad de Valencia]. https://www.educacion.gob.es/teseo/imprimirFicheroTesis.do?idFichero=0SoxuxZNZIQ%3D

Camargo, L., Perry, P., Molina, Ó., Samper, C., & Vargas, C. (2024). Diversidad de acepciones de argumento: necesidad de la formación de profesores (Diversity of meanings of argument: need the need of teacher training). *PNA, 18*(3), 313–338. http://doi.org/10.30827/pna.v18i3.26749

Clemente, F., & Llinares, S. (2015). Formas del discurso y razonamiento configural de estudiantes para maestros en la resolución de problemas de geometría (Pre-service primary teachers' ways of discourse and configural reasoning in solving geometrical problems). *Enseñanza de las Ciencias, 33*, 9–27. https://doi.org/10.5565/rev/ensciencias.1332

Clemente, F., Llinares, S., & Torregrosa, G. (2017). Visualización y razonamiento configural (Visualization and Configural Reasoning). *Bolema: Boletim de Educação Matemática*, *31*(57), 497–516. https://doi.org/10.1590/1980-4415v31n57a24

Del Grande, J. (1990). Spatial sense. *The Arithmetic Teacher, 37*(6), 14–20.

Diago, P. D., González-Calero, J. A., & Yáñez, D. F. (2021). Exploring the development of mental rotation and computational skills in elementary students through educational robotics. *International Journal of Child-Computer Interaction*, 100388. https://doi.org/10.1016/j.ijcci.2021.100388

Douek, N. (1998). Some remarks about argumentation and mathematical proof and their educational implications. In I. Schwank (Ed.), *European Research in Mathematics Education I - Proceedings of the First Congress of the European Society for Research in Mathematics Education* (Vol. 1, pp. 125–139). http://www.fmd.uniosnabrueck.de/ebooks/erme/cerme1-proceedings/cerme1-proceedings-1-v1-0-2.pdf

Drijvers, P., Godino, J.D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*, 23–49. https://doi.org/10.1007/s10649-012-9416-8

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna, & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education* (pp. 349–368). Springer.

Duval, R. (1995). *Sémiosis et pensée humaine: registres sémiotiques et apprentissages intellectuels* (Semiosis and human thought: semiotic registers and intellectual learning). Peter Lang.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana, & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–52). Kluwer Academic Publishers.

Fiallo, J., & Gutiérrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*(2), 145–167. https://doi.org/10.3390/educsci10010004

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9–24.

Font, V., Trigueros, M., Badillo, E., & Rubio, N. (2016). Mathematical objects through the lens of two different theoretical perspectives: APOS and OSA. *Educ Stud Math, 91*, 107–122. https://doi.org/10.1007/s10649-015-9639-6

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM – Mathematics Education, 39*, 127–135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., Gonzato, M., Cajaraville, J. A., & Fernandez, T. (2012). Una aproximación ontosemiótica a la visualización en educación matemática (An onto-semiotic approach to visualization in mathematics education). *Enseñanza de las Ciencias, 30*(2), 109–130. https://doi.org/10.5565/rev/ec/v30n2.653

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: Implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 38–43. https://www.jstor.org/stable/2674201

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig, & A. Gutierrez (Eds.), *Proceedings of the 20th P.M.E. Conference* (Vol. 1, pp. 3–19). Universidad de Valencia.

Gutiérrez, A., Ramírez, R., Benedicto, C., Beltrán-Meneu, M. J., & Jaime, A. (2018). Visualization abilities and complexity of reasoning in mathematically gifted students' collaborative solutions to a visualization task. A networked analysis. In K. S. Mix, & M. T. Battista (Eds.), *Visualizing Mathematics* (pp. 309–337). *Research in Mathematics Education*. Springer. https://doi.org/10.1007/978-3-319-98767-5_14

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schonfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (Vol. 7, pp. 234–282). CBMS Issues in Mathematics Education. https://doi.org/10.1090/cbmath/007

Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie (Geometrical Paradigms and Geometry Teaching). *Annales de Didactiques et de Sciences Cognitives, 11*, 175–193. ⟨halshs-00858709⟩

Jiménez-Gestal, C., Berciano, A. & Salgado, M. (2022). Razonamiento y aprehensión ante una tarea geométrica: análisis de la pertinencia didáctica de una trayectoria de aprendizaje en educación infantil (Reasoning and understanding in the resolution of a geometric task: analysis of the didactical pertinence of a learning trajectory in early childhood education). *Bolema: Boletim de Educação Matemática, 36*(72), 332–357. http://dx.doi.org/10.1590/1980-4415v36n72a15

Llinares, S., & Clemente, F. (2014). Characteristics of pre-service primary school teachers' configural reasoning. *Mathematical Thinking and Learning, 16*(3), 234–250. https://doi.org/10.1080/10986065.2014.921133

Llinares, S., & Clemente, F. (2018). Characteristics of the shifts from configural reasoning to deductive reasoning in geometry. *Mathematics Education Research Journal, 31*(3), 259–277. https://doi.org/10.1007/s13394-018-0253-7

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*, 87–125. https://doi.org/10.1023/A:1012785106627

Molina, O., Font, V., & Pino-Fan, L. (2019). Estructura y dinámica de argumentos analógicos, abductivos y deductivos: un curso de geometría del espacio como contexto de reflexión (Structure and dynamic of analogical, abductive and deductive arguments: a course on solid geometry as a context for reflection). *Enseñanza de las Ciencias, 37*(1), 93–116. https://doi.org/10.5565/rev/ensciencias.2484

Ortega, T., & Pecharromán, C. (2016). De la visualización a la demostración (From visualization to demostration). *Contextos Educativos. Revista de Educación*, 45–64. https://doi.org/10.18172/con.2717

Paneque, J. J., Cobo, P., & Fortuny, J. M. (2017). Intelligent tutoring and the development of argumentative competence. *Technology, Knowledge and Learning*, *22*(1), 83–104. https://doi.org/10.1007/s10758-016-9283-y

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205–236). Sense. https://doi.org/10.1163/9789087901127_009

Prior, J., & Torregrosa, G. (2013). Razonamiento configural y procedimientos de verificación en contexto geométrico (Configural reasoning and verification procedures in geometric context). *Revista Latinoamericana de Investigación en Matemática Educativa, 16*(3), 339–368. https://doi.org/10.12802/relime.13.1633

Prior, J., & Torregrosa, G. (2020). Razonamiento configural y espacio de trabajo geométrico en la resolución de problemas de probar (Configural reasoning and geometric workspace in the proof's problem resolution). *Bolema: Boletim de Educação Matemática, 34*(66), 178–198. https://doi.org/10.1590/1980-4415v34n66a09

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics, 79*, 19–40. https://doi.org/10.1007/s10649-011-9335-0

Ramírez-Uclés, R., & Flores, P. (2017). Habilidades de visualización de alumnos con talento matemático: comparativa entre los test psicométricos y las habilidades de visualización manifestadas en tareas geométricas (Visualization abilities of mathematically gifted students:a comparison between the results from psychometric tests and the visualization abilities used for geometrical tasks). *Enseñanza de las Ciencias, 35*(2), 179–196. https://doi.org/10.5565/rev/ensciencias.2152

Ramírez-Uclés, R., Flores, P., & Ramírez-Uclés, I. (2018). Análisis de los errores en tareas geométricas de argumentación visual por estudiantes con talento matemático (Analysis of mistakes in geometric tasks of visual argumentation by students with mathematical talent). *Revista Latinoamericana de Investigación en Matemática Educativa, 21*(1), 29–56. https://doi.org/10.12802/relime.18.2112

Reid, D., & Knniping, C. (2010). Argumentation. In D. A. Reid, & C. Knniping, *Proof in Mathematics Education. Research, Learning and Teaching* (pp. 153–165). Sense. https://doi.org/10.1007/s11858-013-0497-3

Saorín, A., Torregrosa, G., & Quesada, H. (2019). Razonamiento configural y organización discursiva en procesos de prueba en contexto geométrico (Configural reasoning and discursive organization in proof processes in geometrical context). *Revista Latinoamericana de Investigación en Matemática Educativa, 22*(2), 213–244. https://doi.org/10.12802/relime.19.2224

Sua, C., & Camargo, L. (2019) Geometría dinámica y razonamiento científico: Dúo para resolver problemas (Dynamic geometry and scientific reasoning: Duo to solve problems). *Educación Matemática, 31*(1), 7–37. https://doi.org/10.24844/em3101.01.

Torregrosa, G. (2017). Coordinación de procesos cognitivos en la resolución de problemas: relación entre geometría y álgebra (Coordination of cognitive processes in problem solving: relationship between geometry and algebra). *Avances de Investigación en Educación Matemática, 12,* 1–17. https://doi.org/10.35763/aiem.v1i12.198

Torregrosa, G., Quesada, H., & Penalva, M. C. (2010). Razonamiento configural como coordinación de procesos de visualización (Configural reasoning as coordination of visualisation process). *Enseñanza de las Ciencias, 28*(3) 327–340. https://doi.org/10.5565/rev/ec/v28n3.187