High energy scattering and emission in QED&QCD media

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Resumo

O estudo dos fenómenos de coherencia na emisión de cuantos de enerxía nun escenario de múltiples interaccións remóntase aos traballos clásicos de Landau, Pomeranchuk e Migdal sobre a supresión da intensidade de fotóns en medios condensados, un fenómeno amplamente recoñecido como efecto LPM. A altas enerxías, a presenza de múltiples centros de dispersión na dirección de propagación da partícula nunha mesma lonxitude de coherencia, tradúcese en que todos os diagramas de emisión presentes nesa lonxitude actúan de modo coherente entre eles e de xeito incoherente co resto de diagramas. Como resultado, a intensidade de emisión total está fortemente modulada pola fase que regula esta interferencia e, polo tanto, pola enerxía do fotón emitido e polo ángulo de colisión acumulado subtendido pola partícula primixenia con respecto á dirección do fotón. Para altas enerxías do fotón, a intensidade debido ás colisións co medio pódese entender como a suma totalmente incoherente das intensidades individuais, namentres que para baixas enerxías do fotón, pola contra, o medio pódese entender coma unha entidade de radiación por si mesma, na que a estrutura interna das colisións é irrelevante. A saturación, a baixas enerxías do fotón, do efecto LPM para medios condensados finitos, polo tanto, está ditada polas ligazóns impostas polo teorema de fotóns brandos de Weinberg.

O estudo deste fenómeno de coherencia adquiriu recentemente un novo pulo debido a que se pode empregar como mecanismo de predición sobre a creación de estados non confinados de materia hadrónica nas colisións de alta enerxía nos grandes aceleradores de hadróns máis recentes, tanto o RHIC como o LHC. Baixo as condicións extremas dunha colisión de ións pesados de moi alta enerxía, asúmese hoxe en día que a formación do QGP (o plasma de quarks e gluns) é posible e, polo tanto, métodos indirectos de observación e de estudo son estritamente necesarios para achar probas das súa existencia, comprender o seu comportamento e extraer as súas propiedades. Entre algunhas destas probas, como os efectos de anisotropía na distribución de partículas cargadas, relacionadas cun comportamento hidrodinámico do QGP, ou a supresión de mesóns pesados, o estudo da perda de enerxía de quarks e gluóns durante a súa viaxe por materia hadrónica confinada ou deconfinada, resulta fundamental para a entender o comportamento da teoría das interaccións fortes a temperaturas extremadamente

altas, e o devir da materia hadrónica, baixo estas circunstancias, nun estado asintóticamente pouco ligado como pode ser o QGP. O estudo deste novo estado da materia resulta crucial, non só para entender a natureza complexa e a interesante fenomenoloxía das interaccións fortes, senon tamén para establecer predicións sobre os estadíos iniciais do universo xusto tralo Big Bang, na coñecida como época dos quarks.

O obxectivo deste traballo consiste en estabelecer un formalismo para as colisións de alta enerxía con múltiples corpos, e os procesos de emisión que poidan ocorrer neses escenarios, tanto en electrodinámica (QED) como cromodinámica cuántica (QCD), e que permita unha avaliación das intensidades resultantes de emisión sen ter que recorrer á ben coñecida aproximación de Fokker-Planck, que admita o cálculo para medios finitos ou estruturados, e no que a distribución angular das partículas finais poida ser tida en conta.

- Para tal fin no primeiro capítulo introdúcese a integración de Glauber do estado de alta enerxía dun fermión.

$$\frac{\partial \varphi_s^{(n)}(0,\boldsymbol{x})}{\partial x_3} = \left(ip_3 - \frac{i}{\beta} g A_0^{(n)}(\boldsymbol{x})\right)\varphi_s^{(n)}(0,\boldsymbol{x}), \tag{1}$$

  onde p_3 é a compoñente lonxitudinal e dominante do momento inicial da partícula, g a constante de acoplamento co campo externo creado polo medio, $A_0^{(n)}(\boldsymbol{x})$ o campo externo do medio, representado por n fontes de dispersión e $\varphi_s(0,\boldsymbol{x})$ unha solución estacionaria da onda da partícula baixo a ecuación de Schrödinger. En termos de $\varphi_s(0,\boldsymbol{x})$, o estado da partícula lese,

$$\psi^{(n)}(0,\boldsymbol{x}) = \left(1 + \frac{i}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{1}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)\varphi_s^{(n)}(0,\boldsymbol{x}), \tag{2}$$

  onde p_0 é a enerxía e $\alpha_i = \gamma_i\gamma_0$ as matrices de Dirac. Estas solución para as ondas das partículas de alta enerxía serán empregadas nos seguintes capítulos para avaliar as amplitudes de colisión e emisión tanto en QED como en QCD. A necesidade dunha integración do estado da partícula a altas enerxías é clara, xa que procesos que atinxen un número arbitrario de colisións requiren un tratamento non perturbativo. As amplitudes expresadas dun xeito non perturbativo permiten definir, a posteriori, os observables e as intensidades en función da propia flutuación cuántica no número de colisións cos constituíntes do medio. Existen exemplos ben coñecidos deste tipo de proceder, debendo mencionar como tal o traballo de Bethe e Maximon para a avaliación da intensidade de emisión, usando a aproximación de Bess e Furry para o estado do fermión baixo o efecto do campo estático e clásico do núcleo. A simplificación gañada tras tomar o límite de altas enerxías fai posible transformar o problema en termos dunha ecuación de Dirac nun problema de tipo Schrödinger. Os nosos resultados neste senso para partículas de Dirac reproducen, excepto por correccións de cambio de espín, suprimidas de feito a alta enerxía, os ben coñecidos resultados de Glauber para partículas de tipo Schrödinger.

- No segundo capítulo preséntase unha caracterización dos procesos elásticos de colisión con múltiples fontes de dispersión, definindo a amplitude de colisión asociada á integración de altas enerxías da onda do fermión,

$$M^{(n)}_{s_f s_i}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\beta \sqrt{\frac{m}{p_f^0}} \bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i) \sqrt{\frac{m}{p_i^0}} \\ \times \int d^2\boldsymbol{y}_t \, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left(\exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3 \, A_0^{(n)}(\boldsymbol{y})\right] - 1\right). \tag{3}$$

onde β é a velocidade da partícula, p_f e p_i, s_f e s_i, os 4-momentos e espíns finais e iniciais, respectivamente, $u_s(p)$ un espinor de Dirac e m a súa masa. Observaremos que, só a primeira orde perturbativa, a amplitude resultante para n corpos se pode escribir como unha suma das amplitudes individuais. A orde arbitraria na constante de acoplamento, poderemos definir o o cadrado das amplitudes para un medio promediado sobre unha xeometría determinada. Atoparemos que o promedio das amplitudes ao cadrado se pode dividir en dúas contribucións cunha clara interpretación,

$$\left\langle \left| M^{(n)}_{s_f s_i}(p_f, p_i) \right|^2 \right\rangle = \Pi_2^{(n)}(p_f, p_i) + \Sigma_2^{(n)}(p_f, p_i). \tag{4}$$

Por unha banda, a amplitude promediada ao cadrado, contendo os efectos cuánticos e difractivos de borde e que se corresponde cos termos non diagonais dunha expansión no número de colisións, mide as interferencias transversas de colisión con centros distintos. Este término será entendido, entón, coma unha contribución coherente, e está dado por

$$\Pi_2^{(n)}(p_f, p_i) \equiv \left| \left\langle S^{(n)}_{s_f s_i}(p_f, p_i) - 1 \right\rangle \right|^2 = \left| \left\langle M^{(n)}_{s_f s_i}(p_f, p_i) \right\rangle \right|^2. \tag{5}$$

Pola outra banda, a contribución asociada aos elementos diagonais dunha expansión no número de colisións, que representa polo tanto a suma incoherente de todas as n intensidades de colisión ao cadrado cos diferentes centros, estará dada por

$$\Sigma_2^{(n)}(p_f, p_i) \equiv \left\langle S^{(n)}_{s_f s_i}(p_f, p_i) S^{(n)*}_{s_f s_i}(p_f, p_i) \right\rangle - \left\langle S^{(n)}_{s_f s_i}(p_f, p_i) \right\rangle \left\langle \left(S^{(n)}_{s_f s_i}(p_f, p_i) \right)^* \right\rangle . \tag{6}$$

Veremos que no límite macroscópico a contribución coherente redúcese a unha contribución colimada na dirección asintótica inicial do fermión, non contribuíndo polo tanto a cambios do seu momento, namentres que a contribución incoherente reproduce o ben coñecido resultado de Moliere obtido baixo argumentos *markovianos* de homoxeneidade, é dicir, a través dunha ecuación de transporte,

$$\frac{\partial\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)}{\partial l} = -n_0\sigma_t^{(1)}\Sigma_2^{(n)}(\boldsymbol{q},l) + n_0\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q}-\boldsymbol{k},l). \quad (7)$$

onde $\boldsymbol{q}$ é o cambio de momento, l a profundidade no medio na dirección inicial de propagación, n_0 a densidade do medio e $F_{el}^{(1)}(\boldsymbol{k})$ a amplitude debida a unha fonte de dispersión illada ($n = 1$). Obteremos unha corrección para a definición de Moliere para medios de lonxitude finita e definiremos a aproximación de Fokker-Planck para a distribución resultante de momentos. Obteremos tamén o valor promedio do cambio de momento baixo unha interacción xeral co medio, e finalmente avaliaremos as amplitudes e as intensidades de colisión máis aló da aproximación puramente de alta enerxía de Glauber. As correccións inseridas pola existencia de cambios de momento lonxitudinais creará unha ordenación das colisións na dirección de propagación dominante do fermión que levará, en caso de emisión de partículas durante as colisións, a fenómenos de interferencia como o xa mencionado efecto LPM.

- No terceiro capítulo abordaremos o problema da emisión durante escenarios de múltiples colisións en QED recuperando o traballo do anterior capítulo e tentando sentar un procedemento convencional a través de amplitudes en teorías cuánticas de campos. Ao nivel da amplitude a estructura en diagramas de Feynman tórnase máis cristalina e entender os procesos de coherencia entre os diferentes procesos individuais resulta máis sinxelo. Neste proceder, polo tanto, atoparemos que o efecto LPM satura, a moi baixas enerxías do fotón, no plateau de coherencia dictaminado polo teorema do fotón brando de Weinberg. Esta particularidade fará que para circunstancias de total supresión e para medios finitos, a corrección aos resultados de Migdal e Landau sexa substancial. Definiremos a intensidade de emisión a través da nosa amplitude, e coma no caso puramente elástico atoparemos dúas contribucións, unha relacionada coa contribución coherente das colisións, e outra relacionada coa contribución incoherente. Para a maioría das situacións realistas en QED os medios pódense considerar macroscópicos, de xeito que a contribución incoherente é a única relevante. Para medios microscópicos, sen embargo, os efectos coherentes de colisión son os dominantes e a distribución de fotóns hase ver severamente afectada polas resonancias cos patróns difractivos nese réxime de colisións. Acharemos unha expresión para a intensidade de emisión que se pode avaliar para unha interacción calquera co medio, que permite obter o espectro angular das partículas e que no seu límite de medio semi-infinito e baixo a aproximación de Fokker-Planck recupera o caso particular da expresión clásica de Migdal,

$$\omega\frac{dI_{inc}}{d\omega d\Omega} = \frac{e^2}{(2\pi)^2}\left(\prod_{i=1}^{n}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left\{\left(\prod_{i=1}^{n}\phi_{inc}(\delta p_i)\right)-\left(\prod_{i=1}^{n}\phi^{(0)}_{inc}(\delta p_i)\right)\right\}$$
$$\times\left(h^n(y)\left|\sum_{k=1}^{n}\boldsymbol{\delta}^n_k\exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p^\mu_i}{p^0_0-\omega}\delta z\right)\right|^2\right. \tag{8}$$
$$\left.+h^s(y)\left|\sum_{k=1}^{n}\delta^s_k\exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p^\mu_i}{p^0_0-\omega}\delta z\right)\right|^2\right),$$

onde ω é a enerxía do fotón, e^2 a carga do electrón, $\phi_{inc}(\delta p_i)$ as distribucións elásticas para un cambio de 4-momento δp_i nunha lámina de medio de espesura δz, $h^n(y)$ e $h^s(y)$ os pesos cinemáticos das intensidades que involucran conservación ou cambio de espín, respectivamente, $\boldsymbol{\delta}^n_k$ e δ^s_k as correntes de emisión en semellanza á definición da amplitude de Bethe-Heitler, e k e p o 4-momento do fotón e o electrón. Demostraremos que os formalismos de Zakharov e BDMPS están tamén contidos como casos particulares e acharemos que a nosa expresión para a intensidade no límite continuo xeneraliza os resultados de Wiedemann e Gyulassy, producindo pola contra unha correcta expansión en opacidades no límite de pouca espesura, admitindo ademais unha avaliación non *gaussiana* baixo a interacción de Debye. Os nosos resultados serán comparados con datos experimentais dos aceleradores de SLAC e do CERN, atopando un acordo moi grande. En particular, evidencian a existencia dun plateau de coherencia ditado polo teorema do fotón brando de Weinberg, importante para medios de non moi grande lonxitude, e a existencia de fenómenos de transición relacionados coa consideración dos efectos de dispersión, na relación de enerxía momento do fotón, debido á presenza do medio.

- No cuarto capítulo procedemos paralelamente ao traballo feito no segundo capítulo definindo un formalismo de múltiples colisións para as interaccións fortes, e supoñendo un medio en QCD non moi fortemente ligado como pode ser o QGP. Os resultados, salvo efectos de carga relacionados coa estrutura non abeliana de QCD, reproducen os mesmos patróns nas distribucións de momento que os obtidos en QED. En particular, avaliamos a contribución transversa e coherente de colisión que domina a baixo cambio de momento sobre a distribución incoherente, e polo tanto debe ser tida en conta para medios microscópicos, como é o caso da materia hadrónica confinada ou mesmo non confinada, nas distribucións elásticas de colisión para avaliar os procesos de emisión. Do mesmo xeito que no caso de QED, as contribucións relacionadas cos momentos lonxitudinais levarán a efectos de interferencia substanciais no cómputo dos procesos de emisión.

- No quinto capítulo estudamos ditos procesos de interferencia, o efecto LPM, na emisión de gluóns baixo múltiples colisións con materia hadrónica confinada ou deconfinada. O espírito do capítulo coincide co caso de QED, tentaremos sentar as bases a través da definición dunha expresión para a amplitude na que os fenómenos de coherencia sexan entendibles dun xeito doado, e na que a emerxencia de todos os casos dispoñibles de emisión permita identificar o proceso a través de diagramas de Feynman. A consideración de momentos lonxitudinais, como xa diximos, leva a unha ordenación do proceso na dirección de propagación do gluón, o cal en QCD como demostraremos, e debido a que a estrutura non abeliana da teoría permite interaccións do gluón co propio medio, adquire o rol do electrón cando se asume o seu límite de baixa enerxía. A intensidade de emisión que atoparemos no límite de gluóns brandos,

$$\omega\frac{dI}{d\omega d\Omega_k} = \frac{g_s^2 C_f}{(2\pi)^2}\left(\prod_{i=1}^{n-1}\frac{d^3\boldsymbol{k}_i}{(2\pi)^3}\right)\left\{\left(\prod_{i=1}^{n}\phi_{g\bar{g}}^{(n_i)}(\delta\boldsymbol{k}_i)\right)-\left(\prod_{i=1}^{n}\phi_{g\bar{g}}^{(0)}(\delta\boldsymbol{k}_i)\right)\right\} \times\left|\sum_{k=1}^{n}\exp\left(-i\sum_{i=k}^{n-1}\delta z_i\frac{k_\mu^i p_0^\mu}{p_0^0-\omega}\right)\boldsymbol{\delta}_k^n\right|^2, \quad (9)$$

terá polo tanto unha forma moi semellante á de QED. Na anterior expresión ω é a enerxía do gluón, g_s é a constante de acoplamento da interacción forte, $C_f = (N_c^2-1)/2N_c$ un Casimir de SU(N$_c$), N$_c$=3 o número de cores, $\phi_{g\bar{g}}(\delta\boldsymbol{k}_i)$ a distribución elástica do gluón após atravesar unha lámina do medio de espesura δz_i e p e k os 4-momentos do quark e o gluón. A emisión está caracterizada por un réxime de emisión coherente, que será o dominante a baixas enerxías, correspondente aos gluóns emitidos dende as patas externas do medio, que se pode considerar como unha entidade de emisión en si mesmo, namentres que a enerxías máis baixas a emerxencia da estrutura interna de colisións e o desacoplamento dos diagramas internos favorecerá un aumento da intensidade de emisión. Este aumento, como veremos, será maior a maior lonxitude do medio, a maior coeficiente de transporte e a maior masa de apantallamento. Calcularemos tamén a perda de enerxía baixo un escenario de múltiples interaccións de pouca deflexión, e presentaremos unha aproximación para o caso de que queiramos avaliar a intensidade despois dunha colisión que produza un severo cambio de momento no partón. Así mesmo, presentaremos unha ecuación sinxela que axude a entender cualitativamente o efecto LPM en QCD e produce resultados moi razoables, en particular para os límites de total coherencia e total incoherencia, nos que se volve exacta. Acharemos que baixo as aproximacións axeitadas atopamos os resultados de Migdal/Zakharov e os do grupo BDMPS como casos particulares, dentro da aproximación de Fokker-Planck, e os resultados de Wiedemann e Gyulassy e Salgado como o caso xeral dentro da mesma aproximación *gaussiana*. Xa que o noso formalismo permite unha avaliación máis aló da aproximación de Fokker-Planck, o cálculo do espectro de emisión en QCD, baixo o réxime de múltiples interaccións, é presentado por primeira vez baixo unha interacción realista entre a partícula que colisiona e o medio de cor, producindo un aumento substancial da radiación e da perda de enerxía. As diferenzas atopadas, polo tanto, entre o caso apantallado real e as amplamente empregadas aproximacións de Fokker-Planck, suxerirá que as interaccións que ocorren entre o partón e a materia hadrónica observada no RHIC e o LHC non precisarían estar tan fortemente acopladas, como normalmente se asume, para producir as perdas de enerxía observadas indirectamente a través dos factores de modificación nuclear.

Contents

Preface

We certify that all the possible mistakes have been made, without exception, in the course of this little work. The reader will be surprised to find that some of them have been randomly corrected in the final version. The resulting work is a mess. Put the book it in a bookshelf. Keep in a cold and dry place and please do not touch it anymore. Keep out of reach of researchers and then try to be happy.

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I am grateful also to Carlos Salgado and Néstor Armesto for making us to address the question, long time ago, if there was a way of evaluating bremsstrahlung spectra, whether for photons or gluons, circumventing some limitations widely present in the classic works. Indeed it was possible, and from that anecdote arised this little work. Together with Carlos Pajares, their explanations and discussions were very helpful. I thank also Joaquín Sánchez-Guillen and Jaime Álvarez-Muñiz for their suggestions after carefully reading the papers.

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Introduction

The study of coherence phenomena for the emission of quanta in a multiple interaction scenario dates back to the seminal works of Ter-Mikaelian [1], Landau and Pomeranchuk [2, 3] and Migdal [4] on the bremsstrahlung suppression due to condensed media, the well known Landau-Pomeranchuk-Migdal (LPM) effect. The existence of phases in the elastic amplitudes, which keep track of the placement and momentum change of each of the interactions, lead to interference effects in the squared sum of all the involved amplitudes. At high energies these phases set a coherence length, in the dominant direction of the traveling particle, in which the individual amplitudes can be considered to coherently add in the squared amplitude, but incoherently interfere with the rest of the processes. A manifestation of the coherence between Feynmann diagrams could acquire a more familiar form in analogy with the soft photon theorem [5]. For sufficiently low energy of the emitted photon, the diagrams representing emissions from the internal legs cancel and the total amplitude restricts to the photons coming from the first and last legs. Thus in that limit radiation can be understood as if all the internal interactions coherently emit as a single entity. This immediately addresses the question on how the intensity of photons behaves for larger photon energies and how this energy and the medium modulate the coherence. In other words, when the medium still emits as a single entity, and when, if it occurs, a regime of maximal incoherence between its constituents may be found, in which the total intensity is just the sum of the single [6] intensities.

Predictions and observations of this coherence effect have been extensively carried in the past for quantum electrodynamic interactions (QED) and they have suscited a renewed interest in the era of the hadron colliders as a way to indirectly observe the hadronic matter produced in heavy ion collisions. Soon after the birth of quantum chromodynamics (QCD) as the theory of the strong interactions [7–9], it was assumed that a new state of the hadronic matter, consisting in a deconfined phase of quarks and gluons at extremely high temperatures, should exist in nature [10–16]. At temperatures well above the QCD critical temperature ~ 170 MeV and/or at densities well above the ordinary nuclear density ~ 1 GeV/fm^3, collisions involving large momentum change are required in order to consider the interactions between particles significative. The weak value of the coupling at such high gluon momenta leads, however, to a relatively small amplitude of these kind of hard interactions to occur. This asymptotically free behavior of QCD at high energies suggests, then, a picture of quasi free, softly interacting quarks and gluons under extreme conditions. In analogy with the weakly coupled QED plasmas, this new state of the matter at such extreme conditions was quickly coined [17] with the name of Quark Gluon Plasma (QGP). The study of the QGP properties results of interest not only as a way of understanding the complex nature and rich phenomenology of QCD, but it also becomes crucial for the understanding of the initial stages of the universe in the quark epoch just after the Big Bang, for example. In the search of evidences of QGP formation at the RHIC and LHC hadron colliders several indirect probes can be considered. Among these QGP signatures, like the observation of anisotropies in the charged particle distributions, related to an hydrodynamical behavior of the QGP, or the suppression of heavy quark mesons, a quantitative prediction of the radiative energy loss of the underlying jet particles, while traveling through the formed QCD medium, should provide an indirect evidence of the QGP formation and its characteristics. For that purpose an accurate study of the LPM effect for the gluon bremsstrahlung spectrum in a multiple collision scenario with the QCD medium becomes, then, indispensable.

The structure of this work is approximately guided by the historical developments. We also put some focus in understanding the coherence effects, both in the elastic and the emission processes, following a conventional quantum field theory (QFT) description with amplitudes. In Chapter 1 we lay the foundations for the treatment of high energy scattering under multiple interactions by defining a high energy integration of the fermion state under a general, static and classical external field [18]. In Chapter 2 we present a formalism for multiple scattering at arbitrary perturbative order, and define the medium averaged intensities and observables leading to transverse and longitudinal coherency effects. In Chapter 3 we simply use the former multiple scattering results to build a QFT formalism for high energy emission which can be evaluated for general interactions with mediums of finite size. Under the Fokker-Planck approximation and the semi-infinite medium limit we are able to recover the Migdal result [4] as a particular case. In Chapter 4 we extend the QED multiple scattering results for the QCD scenario and finally, in Chapter 5, we evaluate the gluon bremsstrahlung intensity with these tools. Under the Fokker-Planck approximation, Wiedemann result [19] is found and within some length or mass approximations the well known results of Migdal/Zakharov [4, 20] or the BDMPS group [21, 22] are also recovered.

Part I

QED

1

High energy fermions in an external field

The purpose of this chapter is to introduce the Glauber high energy integration of the wave [18] for particles obeying the Dirac equation. These wave solutions will be used in the following chapters to evaluate the QED and QCD scattering and emission amplitudes in a multiple collision scenario. Consider a high energy particle traveling and interacting with a medium, classically characterized as a set of sources distributed in some region of the space. Since the state of the particle is expected to contain an arbitrary number of interactions with the medium, an exact/non-perturbative solution for the traveling wave is required. Processes typically involving the creation or annihilation of other particles will emerge as a result of the interaction with the medium. Their amplitudes, when expressed in terms of the exact wave integrations, correspond to a sum of all the perturbative orders in g, the medium coupling parameter, or, in other words, all the ways in which the process can be represented by means of elementary interactions with medium constituents. This superposition of Feynman diagrams will lead to a quantum fluctuation of the observables in the number of collisions and they may be expressed, on average, as functions of the number of particles, the geometry and the nature and strength of the interaction with the medium.

Early efforts in computing amplitudes in this exact approach exist [23]. For photon bremsstrahlung due to a single Coulomb field, for example, an exact Dirac or Klein-Fock-Gordon solution for the electron can be obtained in polar coordinates [24, 25], but at the expense of making the emission intensity calculation an almost impossible task [23]. It is obvious then, that if we try to extend the problem to multiple external fields randomly distributed, some kind of approximations have to be done. Fortunately, as we will see later, two approximations will prove to simplify our problem. First, as already mentioned under the conditions of our interest the medium is considered a classical and static source. The validity of this approximation and the form of the interaction is briefly explained in Section 1.1. Second, the high energy solution of the Dirac equation can be given as an operator acting on a Schrodinger like state which is easier to solve in the high energy approximation [18]. A derivation of this result and a discussion of the conditions for the approximation are given in Section 1.2. In Section 1.3 we prove that the perturbative expansion of the found wave agrees with the standard Born approximation, and in Section 1.4 we compute the propagator in momentum and light cone space, showing that except for a negligible spinorial correction, the well known Glauber scattering amplitude [18] is found. This correspondence is a direct consequence of the Schrodinger behavior, except for the spin structure, of the Dirac solution at high energies.

1.1 The medium as a classical source

We consider a medium of ordinary solid matter composed, then, of n static nuclei of charge Ze and mass m_n. When a high energy electron of mass m and charge e passes through this medium a photon field emanates from the induced current, which is going to be considered classical and static. By classical we mean that the nuclei or sources originating the field are completely localized in coordinate space, and by static that these sources have a negligible recoil. Both conditions read for the 4-current

$$J_0^{(n)}(x) = Ze \sum_{i=1}^{n} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{r}_i), \quad \boldsymbol{J}^{(n)}(x) = 0, \tag{1.1}$$

where $\boldsymbol{r}_i$ is the position of each source. In order to take into account screening effects of the electron shell, we will consider that the photon field originated in the interaction has an effective mass μ_d canceling photon propagation to distances much larger than $r_d = 1/\mu_d$. This screening mass of the nuclei can be estimated as $\mu_d = \alpha^2 m Z^{1/3}$, being α the fine structure constant. In the Lorenz gauge [26], now required for current conservation, the Maxwell equation for the massive photon $A_\mu^{(n)}(x)$ due to the n sources reads

$$\left(\partial_\nu \partial^\nu + \mu_d^2\right) A_\mu^{(n)}(x) = 4\pi J_\mu^{(n)}(x). \tag{1.2}$$

To get the form of $A_\mu^{(n)}(x)$ we simply propagate the current (1.1) as

$$A_\mu^{(n)}(x) = \int d^4y D^F_{\mu\nu}(x-y) J_\nu^{(n)}(y), \tag{1.3}$$

by means of a Feynman-Stueckelberg propagator. By direct inspection of equations (1.2) and (1.3), the propagator, then, has to satisfy the equation

$$\left(\partial_\eta \partial^\eta + \mu_d^2\right) D^F_{\mu\nu}(x-y) = 4\pi \delta^{(4}(x-y) g_{\mu\nu}. \tag{1.4}$$

By Fourier transforming equation (1.4) we easily find

$$\hat{D}^F_{\mu\nu}(q^2) = g_{\mu\nu}\frac{4\pi}{-q^2 + \mu_d^2}, \tag{1.5}$$

With the above equation, and using (1.1) and (1.3) it is straightforward to show that the field does not carry energy, as result of time translation invariance, and it has only one non-vanishing component, namely $A_0^{(n)}(x)$, given by

$$A_\mu^{(n)}(x) = g_{\mu 0} Ze \sum_{i=1}^{n} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{4\pi}{\boldsymbol{q}^2 + \mu_d^2} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{r}_i)}. \tag{1.6}$$

Here we notice that the inverse screening radius μ_d is related to the typical momentum change $\boldsymbol{q}$ in a single collision with one nuclei. Finally, the last integral can be accomplished by using Cauchy's theorem,

$$\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{4\pi}{\boldsymbol{q}^2 + \mu_d^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} = \frac{1}{\pi|\boldsymbol{x}|}\,\mathrm{Im}\int_{-\infty}^{+\infty} dq \frac{q}{q^2+\mu_d^2} e^{iq|\boldsymbol{x}|} = \frac{1}{|\boldsymbol{x}|} e^{-\mu_d|\boldsymbol{x}|}, \tag{1.7}$$

so we find the interaction

$$A_0^{(n)}(x) = Ze\sum_{i=1}^{n} D^F_{00}(\boldsymbol{x}-\boldsymbol{r}_i) = \sum_{i=1}^{n}\frac{Ze}{|\boldsymbol{x}-\boldsymbol{r}_i|} e^{-\mu_d|\boldsymbol{x}-\boldsymbol{r}_i|}. \tag{1.8}$$

This is known as a Yukawa/Debye interaction with screening μ_d. In the limit $\mu_d \to 0$ the field equations transform into the gauge covariant Maxwell equations and correspondingly we find the Coulomb interaction. This simple result derived under the classicity of the current can be justified by doing the pertinent approximations in the quantum current. For a single nuclei we would have,

$$J_\mu(x) = \sqrt{\frac{m_n}{p_b^0}}\,\bar{u}_{s_b}(p_b)\gamma_\mu u_{s_a}(p_a)\sqrt{\frac{m_n}{p_a^0}}\, e^{i(p_b-p_a)\cdot x}, \tag{1.9}$$

where we used spinor conventions given in Appendix A. The amplitude of finding the photon carrying momentum $q = p_a - p_b$ is, using (1.3), (1.4) and (1.9)

$$A_\mu(x) = \frac{4\pi Ze}{-(p_a-p_b)^2+\mu_d^2}\sqrt{\frac{m_n}{p_a^0}}\,\bar{u}_{s_a}(p_a)\gamma_\mu u_{s_b}(p_b)\sqrt{\frac{m_n}{p_b^0}}\, e^{+i(p_a-p_b)x}. \tag{1.10}$$

As usual, this photon has to be joined with the electron current and then the resulting quantity integrated in x. Due to both external constraints of the other nuclei reducing its recoil and because it is much heavier than the typical momentum change produced into a single collision μ_d, we have $\boldsymbol{p}_a = 0$ and $\boldsymbol{p}_b \approx \boldsymbol{p}_a$, so

$p_b^0 \approx m_n$,

$$J_0 = \sqrt{\frac{m_n}{p_a^0}}\bar{u}_{s_a}(p_a)\gamma_0 u_{s_b}(p_b)\sqrt{\frac{m_n}{p_b^0}} \approx 1$$
$$J_k = \sqrt{\frac{m_n}{p_a^0}}\bar{u}_{s_a}(p_a)\gamma_k u_{s_b}(p_b)\sqrt{\frac{m_n}{p_b^0}} \approx \frac{(p_b)_k}{m_n} \ll 1. \quad (1.11)$$

Similar conclusions can be placed for the electron in the high energy limit. Using then these slow dependences of the nuclei and electron currents on the momentum, using (1.11) integration in time and momentum can be carried independently at (1.10) and we obtain

$$\langle A_\mu(x)\rangle_{q,t} \equiv \delta_{\mu 0} Ze \int \frac{d^4q}{(2\pi)^4} \int dt \frac{4\pi}{-q^2+\mu_d^2} e^{+iq\cdot x} = \delta_{\mu 0}\frac{Ze}{|\boldsymbol{x}|}e^{-\mu_d|\boldsymbol{x}|}, \quad (1.12)$$

in such a way that the coherent superposition of photon amplitudes recovers the classical field limit. The classical approximation is allowed, then, if the inertias of the nuclei and the electron (the rest mass and the energy, respectively) greatly exceed the typical momentum change μ_d of a single interaction. For moving constituents, like the ones in an ideal gas, it is also sufficient to guarantee that the typical energy of each constituent is much larger than μ_d, so the medium can be considered classic. And if the traveling fermion energy greatly exceeds also the typical energy of the constituents, the medium can be considered static.

1.2 High energy limit of the Dirac equation

We depart, then, from a medium characterized by n sources of the form

$$A_0^{(n)}(x) = \sum_{i=1}^{n} A_0^{(1)}(\boldsymbol{x}-\boldsymbol{r}_i), \quad \partial_0 A_0^{(n)}(x) = 0. \quad (1.13)$$

The state $\psi^{(n)}(x)$ of the electron is coupled with strength $g = e$, its charge, to these n fields. Since these fields carry themselves the charge of each target Ze, let us relabel the overall coupling to $g = Ze^2$ and leave the interaction $A_0^{(n)}(x)$ coupling independent. The electron obeys the Dirac equation under this field, so it has also to obey the squared Dirac equation, and we write

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m - g\gamma^0 A_0^{(n)}(x)\right)\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m - g\gamma^0 A_0^{(n)}(x)\right)\psi^{(n)}(x) = 0. \quad (1.14)$$

Energy operator represents the x_0 evolution of the states , $i\partial_0 \equiv \pm p_0$. Accordingly energy positive solutions are the ones represented by

$$\psi^{(n)}(x) = \psi^{(n)}(0,\boldsymbol{x})e^{-ip_0x_0}. \quad (1.15)$$

Using the relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\alpha^i = \gamma^i\gamma^0$ and taking into account the condition $\partial^0 A_0^{(n)}(x) = 0$, we find an eigenvalue equation in p_0

$$\left(p_0^2 - m^2 + \boldsymbol{\nabla}^2 - 2gp_0A_0^{(n)}(\boldsymbol{x})\right)\psi^{(n)}(0,\boldsymbol{x}) \\ = \left(ig\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}A_0^{(n)}(\boldsymbol{x}) - \left(gA_0^{(n)}\right)^2(\boldsymbol{x})\right)\psi^{(n)}(0,\boldsymbol{x}). \tag{1.16}$$

We take first the infinite momentum limit of (1.16). To do that, we will assume that the energy p_0 of the fermion greatly exceeds the average magnitude of the interaction $\langle gA_0^{(n)}(x)\rangle$, a condition which reads

$$g\,\langle A_0^{(n)}(x)\rangle \equiv g\int d^3\boldsymbol{x}\bar{\psi}^{(n)}(0,\boldsymbol{x})\gamma_0A_0^{(n)}(\boldsymbol{x})\psi^{(n)}(0,\boldsymbol{x}) \ll p_0. \tag{1.17}$$

Under this condition, in (1.16) we can drop off the terms linear in g and g^2 but not the term in gp_0, leading to

$$\left(p_0^2 - m^2 + \boldsymbol{\nabla}^2 - 2gp_0A_0^{(n)}(\boldsymbol{x})\right)\varphi_s^{(n)}(0,\boldsymbol{x}) = 0, \tag{1.18}$$

where we denoted the solution in this limit $\varphi_s^{(n)}(x)$. We state that a free solution with 4-momentum p must be a good approximation up to a modified wave, which we call $\phi_s^{(n)}(\boldsymbol{x})$. Keeping this in mind we write down the ansatz

$$\varphi_s^{(n)}(0,\boldsymbol{x}) = \sqrt{\frac{m}{p_0}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}u(p)\phi_s^{(n)}(\boldsymbol{x}). \tag{1.19}$$

Solutions in the pure high energy limit $\varphi_s(x)$ and $\phi_s(x)$ have been labeled with a (s) because they are solutions to an effective Schrodinger equation, as we will see below. Here $u(p)$ stands for a free spinor and $\mathcal{N}(p) = \sqrt{m/p_0}$ the normalization (see Appendix A). With this ansatz we find the equation for $\phi_s^{(n)}(\boldsymbol{x})$

$$\boldsymbol{\nabla}^2\phi_s^{(n)}(\boldsymbol{x}) - 2i\boldsymbol{p}\cdot\boldsymbol{\nabla}\phi_s^{(n)}(\boldsymbol{x}) - 2gp_0A_0^{(n)}(\boldsymbol{x})\phi_s^{(n)}(\boldsymbol{x}) = 0. \tag{1.20}$$

For multiple external fields the above equation does not have a closed exact solution. We investigate, by now, the momentum change induced by $A_0^{(n)}(\boldsymbol{x})$. The modified wave $\phi_s(\boldsymbol{x})$ changes asymptotic momentum, which was $\boldsymbol{p}$, to

$$\langle\varphi_s^{(n)}|\tilde{\boldsymbol{p}}\varphi_s^{(n)}\rangle = \boldsymbol{p} + \int d^3\boldsymbol{x}\,\phi_s^{(n),*}(\boldsymbol{x})\left(+i\boldsymbol{\nabla}\phi_s^{(n)}(\boldsymbol{x})\right) = \boldsymbol{p} + \delta\boldsymbol{p}. \tag{1.21}$$

In order to see how the modification $\delta\boldsymbol{p}$ compares to $\boldsymbol{p}$, one multiplies the last equation by $\boldsymbol{p}$ and uses (1.20), finding

$$\boldsymbol{p}\,\langle\varphi_s^{(n)}|\tilde{\boldsymbol{p}}\varphi_s^{(n)}\rangle = \boldsymbol{p^2} + \frac{1}{2}\int d^3\boldsymbol{x}\,\phi_s^{(n),*}(\boldsymbol{x})\boldsymbol{\nabla}^2\phi_s^{(n)}(\boldsymbol{x}) \\ - gp_0\int d^3\boldsymbol{x}\,\phi_s^{(n),*}(\boldsymbol{x})A_0^{(n)}(\boldsymbol{x})\phi_s^{(n)}(\boldsymbol{x}), \tag{1.22}$$

and, by using the energy momentum relation, we see that, in order to have $\delta \boldsymbol{p} \ll \boldsymbol{p}$, we find the condition

$$g \langle A_0^{(n)}(\boldsymbol{x})\rangle \equiv g \int d^3\boldsymbol{x} \phi_s^{(n),*}(\boldsymbol{x}) A_0^{(n)}(\boldsymbol{x}) \phi_s^{(n)}(\boldsymbol{x}) \ll p_0, \tag{1.23}$$

which guarantees that the initial energy greatly exceeds the averaged potential, together with

$$\boldsymbol{\nabla}^2 \phi_s^{(n)}(\boldsymbol{x}) \sim 0, \tag{1.24}$$

which guarantees that the wave function varies slowly in a wavelength, as expected if the particle wavelength is much smaller than the spread of the interaction. Conditions (1.23) and (1.24) are essentially the same as the ones required for the Glauber high energy approximation of Schrodinger particles [18]. They guarantee that the quotient $\delta p/p$ is very small. If we place the initial particle direction along $\boldsymbol{z}$ the transverse $\boldsymbol{q}_t$ and longitudinal $\boldsymbol{q}_z$ momentum changes verify

$$|\boldsymbol{q}_t| = \beta p_0 \sin\theta \sim \beta p_0 \theta, \quad |\boldsymbol{q}_z| = \beta p_0 (1 - \cos\theta) \sim \beta p_0 \theta^2, \tag{1.25}$$

since θ can be made arbitrarily small. Under these conditions the longitudinal momentum change can be neglected and the paraxial limit for the electron propagation is achieved.

We now introduce the corrections to $\varphi_s^{(n)}(x)$ resulting of not neglecting the g and g^2 terms in (1.16). The full solution $\psi^{(n)}(x)$ to (1.16) can be given with a modified ansatz [27]

$$\psi^{(n)}(0, \boldsymbol{x}) = \mathcal{N}(p) e^{-ip\cdot\boldsymbol{x}} (1 + \Lambda(p)) u(p) \phi_s^{(n)}(\boldsymbol{x}), \tag{1.26}$$

where $\Lambda(p)$ is defined as an operator commuting with $\boldsymbol{\nabla}^2$, $\boldsymbol{\nabla}$ and thus $\boldsymbol{p}$, and $\phi_s(\boldsymbol{x})$ is given by (1.20). Inserting (1.26) in (1.16) we get

$$\begin{aligned} &2gp_0\Big(\Lambda(p) A_0^{(n)}(\boldsymbol{x})\Big) u(p) \phi_s^{(n)}(\boldsymbol{x}) \\ &\qquad = \Big\{ ig\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} A_0^{(n)}(\boldsymbol{x}) - \Big(gA_0^{(n)}\Big)^2(\boldsymbol{x}) \Big\} (1 + \Lambda(p)) u(p) \phi_s^{(n)}(\boldsymbol{x}). \end{aligned} \tag{1.27}$$

Inspecting this equation it's clear that, for positive energy solutions, the operator in the ansatz has to be defined as[1]

$$\Lambda(p) = \frac{i}{2p_0} \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}, \tag{1.28}$$

[1] The wave in the form (1.26) with the operator defined as (1.28) corresponds to a g^2/l^2 truncation of Darwin's solution [24] of the Dirac equation under a Coulomb field, given as an infinite series in spherical harmonics, so it remains also valid at low energies. Further truncations [25, 28] neglecting also terms of order $1/p^0$ in Darwin's solution had already been made, a restriction later shown [23] unnecessary.

so the first term on the right hand side is canceled. The approximation is good up to the neglection of the remaining term, of squared order $(gA_0^{(n)})^2$. Its contribution to certain amplitudes, like the bremsstrahlung one, however, is negligible [23]. The correction introduced by the operator seems of order $1/p_0$, but its matrix structure may introduce terms of order p_0 coming from the spinors in the amplitudes. It results convenient to move the operator completely to the left, acting over $\varphi_s^{(n)}(x)$ instead of over $\phi_s^{(n)}(x)$,

$$\psi^{(n)}(0,\boldsymbol{x}) = \left(1 + \frac{i}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{1}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)\varphi_s^{(n)}(0,\boldsymbol{x}). \tag{1.29}$$

Same equation can be put in terms of the propagators. By noticing

$$\psi^{(n)}(x) = i\int d^3\boldsymbol{y}\, S_F^{(n)}(x,y)\gamma^0\psi^{(n)}(y), \quad \varphi_s^{(n)}(x) = \int d^3\boldsymbol{y}\, G_s^{(n)}(x,y)\varphi_s^{(n)}(y) \tag{1.30}$$

and using (1.29), the propagator at high energies for $\psi^{(n)}(x)$ is related with the propagator for $\varphi_s^{(n)}(x)$ as

$$S_F^{(n)}(x,y) = \left(1 + \frac{i}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{1}{2p_0}\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)\gamma_0 G_s^{(n)}(x,y). \tag{1.31}$$

We are now in position to solve for $\varphi_s^{(n)}(x)$. To do that we notice that equation (1.16) can be arranged in a Schrodinger form by solving for the eigenvalue p_0. In this way the time evolution operator $H_s^{(n)}$ is found to be

$$H_s^{(n)} = -\frac{1}{2(p_0/2)}\boldsymbol{\nabla}^2 + 2gA_0^{(n)}(\boldsymbol{x}), \tag{1.32}$$

so the solution at arbitrary time x_0 reads

$$\varphi_s^{(n)}(x_0,\boldsymbol{x}) = \exp\left(-ix_0 H_s^{(n)}\right)\varphi_s^{(n)}(0,\boldsymbol{x}). \tag{1.33}$$

Notice that $\varphi_s^{(n)}(x)$ has spinorial structure but it evolves like a scalar with a Hamiltonian where the mass role is played by the energy, meaning that in the infinite momentum limit energy becomes the only inertia. In order to solve it, we can write the Schrodinger propagator [29] as a path integral

$$G_s^{(n)}(x_f,x_i) \equiv \int \mathcal{D}^3\boldsymbol{x}(t)\,\exp\left[i\int_{t_i}^{t_f} dt\,\left(\frac{p_0}{4}\dot{\boldsymbol{x}}^2(t) - 2gA_0^{(n)}(\boldsymbol{x}(t))\right)\right], \tag{1.34}$$

and take the limit $t_i \to -\infty$ and $p_0 \gg 1$. In order to do that we go to the reference frame of the asymptotic particle, of initial momentum p, given by the coordinate transformation

$$\boldsymbol{w}(t) = \boldsymbol{x}(t) - 2\frac{\boldsymbol{p}}{p_0}t = \boldsymbol{w}(t) + 2\beta t. \tag{1.35}$$

In this Galilean transformation the role of mass is played by the energy and the Lagrangian transforms to

$$\frac{p_0}{4}\dot{\boldsymbol{x}}^2(t) - 2gA_0^{(n)}(\boldsymbol{x}(t)) := \frac{p_0}{4}\dot{\boldsymbol{w}}^2(t) + \frac{\boldsymbol{p}^2}{p_0} + \dot{\boldsymbol{w}}(t)\cdot\boldsymbol{p} - 2gA_0^{(n)}(\boldsymbol{w}(t) + 2\beta t), \tag{1.36}$$

where the two new non path-dependent terms can be directly integrated out, yielding

$$G_s^{(n)}(x_f, x_i) = \exp\left[+i\boldsymbol{p}\cdot(\boldsymbol{w}(t_f) - \boldsymbol{w}(t_i)) + i\frac{\boldsymbol{p}^2}{p_0}(t_f - t_i)\right] \tag{1.37}$$

$$\cdot \int \mathcal{D}^3\boldsymbol{w}(t)\exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_0}{4}\dot{\boldsymbol{w}}^2(t) - 2gA_0^{(n)}(\boldsymbol{w}(t) + 2\frac{\boldsymbol{p}}{p_0}t)\right)\right]. \tag{1.38}$$

We can see that the interaction is evaluated over the deformed paths (1.35). Path fluctuations weighted by the kinetic term, however, affect only to $\boldsymbol{w}(t)$, so paths in the interaction term start at $\boldsymbol{x}(t_i) + 2\beta t_i$ and end at $\boldsymbol{x}(t_f) + 2\beta t_f$. When the large limit of β is taken and $t_i \to -\infty$, the deformed paths in the interaction term are almost straight lines parallel to $\boldsymbol{p}$, ending at a transverse distance $\boldsymbol{w}_t(t_f)$ with respect to $\boldsymbol{p}$ and at longitudinal distance $\boldsymbol{w}_l(t_f) = 2\beta t_f$. By replacing them by straight lines the path integral in the interaction term can be taken approximately constant for all the fluctuations so it factorizes as

$$\int \mathcal{D}^3\boldsymbol{w}(t)\exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_0}{4}\dot{\boldsymbol{w}}^2(t) - 2gA_0^{(n)}(\boldsymbol{w}(t) + 2\beta t)\right)\right] \tag{1.39}$$

$$\approx \exp\left[-2ig\int_{-\infty}^{t_f} dt\, A_0^{(n)}(\boldsymbol{w}_t(t_f) + 2\beta t)\right]\left(\int \mathcal{D}^3\boldsymbol{w}(t)\exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_0}{4}\dot{\boldsymbol{w}}^2(t)\right)\right]\right).$$

and, by adding the phase terms, defining $\boldsymbol{s} = 2\beta t$, placing the $\boldsymbol{p}$ along the z-axis and undoing the change, one finds

$$G_s^{(n)}(x_f, x_i) = G_s^{(0)}(x_f, x_i)\exp\left[-i\frac{g}{\beta}\chi_0^{(n)}(\boldsymbol{x}_f)\right] \equiv G_s^{(0)}(x_f, x_i)W^{(n)}(\boldsymbol{x}_f, p). \tag{1.40}$$

where we have introduced the free propagator $G_s^{(0)}(x_f, x_i)$ for (1.32) and defined the shorthand notation

$$\chi_0^{(n)}(\boldsymbol{x}) \equiv \int_{-\infty}^{\hat{\boldsymbol{p}}\cdot\boldsymbol{x}} ds\, A_0^{(n)}(\hat{\boldsymbol{p}}\times(\boldsymbol{x}\times\hat{\boldsymbol{p}}) + s\hat{\boldsymbol{p}}). \tag{1.41}$$

In terms of an asymptotic free wave $\psi^{(0)}(x)$ of momentum p we finally find, using equations (1.30), (1.31) and (1.40),

$$\psi^{(n)}(x) = \left\{\left(1 + i\frac{\gamma_k\gamma_0}{2p_0^0}\partial_k\right)W^{(n)}(\boldsymbol{x}, p)\right\}\psi^{(0)}(x), \tag{1.42}$$

sum in repeated indices assumed. We see that, except for the operator correction, as expected, the above result is the generalization for spin-1/2 particles of the Glauber's high energy waves [18]. Finally, an equation satisfied by the high energy limit of the wave can be written. Indeed, by simple derivation we find

$$\frac{\partial \varphi_s^{(n)}(0, \boldsymbol{x})}{\partial x_3} = \left(i p_3 - \frac{i}{\beta} g A_0^{(n)}(\boldsymbol{x}) \right) \varphi_s^{(n)}(0, \boldsymbol{x}). \tag{1.43}$$

This result, or alternatively (1.40), says us that the medium, in the high energy limit, only adds a phase to a free propagation, the phase being just the integration of the static component of the external field along the direction of propagation of the fermion till the position x_3. Furthermore, second derivatives can be neglected in virtue of (1.24).

1.3 Perturbative expansion

Glauber's integration [18] of the wave at high energies do reproduce the Born approximation at high energies of a Schrodinger wave. Therefore, equation (1.42) must agree with the series expansion in g of a spin-1/2 particle. When expanded in the external field $A_0^{(n)}(\boldsymbol{x})$ we get

$$\psi^{(n)}(x) = \psi^{(0)}(x) - i\frac{g}{\beta}\left(1 + \frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2p_0} + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{2p_0}\right)\chi_0^{(n)}(\boldsymbol{x}, p)\psi^{(0)}(x) + (...). \tag{1.44}$$

This series expansion must agree with the standard perturbative series for $\psi(x)$, this is, the Born series given by

$$\psi^{(n)}(x) = \psi^{(0)}(x) + g\int dy\, S_F^{(0)}(x-y)\gamma^0 A_0^{(n)}(y)\psi^{(0)}(y) + \dots. \tag{1.45}$$

In order to check if the eikonal integration in (1.44) equals the leading order term of (1.45), we write the external field as the Fourier transform of some $A_0^{(n)}(\boldsymbol{q})$,

$$A_0^{(n)}(y) = \int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q_0) e^{-iq\cdot y} A_0^{(n)}(q), \tag{1.46}$$

and use the Feynman-Stueckelberg propagator for the Dirac equation

$$S_F^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2} e^{-ik\cdot(x-y)}. \tag{1.47}$$

By inserting these two expressions in equation (1.45) and integrating in y and k one arrives to

$$\psi^{(n)}(x) = \psi^{(0)}(x) + g\int \frac{d^4q}{(2\pi)^4}\, 2\pi\delta(q_0)\frac{\not{q}+\not{p}+m}{(q+p)^2-m^2}\gamma^0 A_0^{(n)}(\boldsymbol{q}) e^{-iq\cdot x}\psi^{(0)}(x). \tag{1.48}$$

Here we have to express the numerator as

$$\begin{aligned}(\not{q} + \not{p} + m)\,\gamma_0\psi^{(0)}(x) &= -q_i\gamma_i\gamma_0\psi^{(0)}(x) + (i\gamma_u\partial^u + m)\gamma_0\psi^{(0)}(x)\\ &= -(\boldsymbol{q}\cdot\boldsymbol{\alpha})\psi^{(0)}(x) + 2i\partial_0\psi^{(0)}(x) - \gamma_0(\not{p} - m)\psi^{(0)}(x)\\ &= -(\boldsymbol{q}\cdot\boldsymbol{\alpha})\psi^{(0)}(x) + 2p_0\psi^{(0)}(x),\end{aligned} \tag{1.49}$$

and replace the term $\boldsymbol{q}$ with a gradient in order to move it out of the integral sign

$$-(\boldsymbol{q}\cdot\boldsymbol{\alpha})e^{-iq\cdot x}\psi^{(0)}(x) = (i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \boldsymbol{\alpha}\cdot\boldsymbol{p})\,e^{-iq\cdot x}\psi^{(0)}(x). \tag{1.50}$$

Since the denominator becomes

$$(p+q)^2 - m^2 = m^2 + q^2 + 2p\cdot q - m^2 = -\boldsymbol{q}^2 - 2\boldsymbol{q}\cdot\boldsymbol{p}, \tag{1.51}$$

one finds

$$\psi^{(n)}(x) = \psi^{(0)}(x) - g\left(1 + \frac{i\boldsymbol{\alpha}\cdot\nabla}{2p_0} + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{2p_0}\right)\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{2p_0}{\boldsymbol{q}^2 + 2\boldsymbol{q}\cdot\boldsymbol{p}}A_0^{(n)}(\boldsymbol{q})\psi^{(0)}(x). \tag{1.52}$$

When $\boldsymbol{p} \gg 1$, the integral is dominated by the term $\boldsymbol{q}\cdot\boldsymbol{p}$ and we can neglect $\boldsymbol{q}^2$. Notice that $\boldsymbol{q}\cdot\boldsymbol{p}$ is the momentum change in the initial direction, which we will take as the z direction. Now, using the decomposition $\boldsymbol{x} = \hat{\boldsymbol{p}}\times(\boldsymbol{x}\times\hat{\boldsymbol{p}}) + \hat{\boldsymbol{p}}(\boldsymbol{x}\cdot\hat{\boldsymbol{p}})$ we write the following trick

$$\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{\boldsymbol{q}\cdot\boldsymbol{p}} \equiv \frac{i}{|\boldsymbol{p}|}\int_{-\infty}^{\boldsymbol{x}\cdot\hat{\boldsymbol{p}}} ds\ \exp\Big[i\boldsymbol{q}\cdot(\hat{\boldsymbol{p}}\times(\boldsymbol{x}\times\hat{\boldsymbol{p}}) + s\hat{\boldsymbol{p}})\Big]. \tag{1.53}$$

By using the Fourier transform of the field we identify now

$$\begin{aligned}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{p_0}{\boldsymbol{q}\cdot\boldsymbol{p}}A_0^{(n)}(\boldsymbol{q}) &= \frac{i}{\beta}\int_{-\infty}^{\boldsymbol{x}\cdot\hat{\boldsymbol{p}}} ds\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}e^{i\boldsymbol{q}\cdot(\hat{\boldsymbol{p}}\times(\boldsymbol{x}\times\hat{\boldsymbol{p}})+s\hat{\boldsymbol{p}})}A_0^{(n)}(\boldsymbol{q})\\ &= \frac{i}{\beta}\int_{-\infty}^{\boldsymbol{x}\cdot\hat{\boldsymbol{p}}} ds\, A_0^{(n)}(\hat{\boldsymbol{p}}\times(\boldsymbol{x}\times\hat{\boldsymbol{p}}) + s\hat{\boldsymbol{p}}),\end{aligned} \tag{1.54}$$

which enables us to recover the first term of the series in (1.44)

$$\psi^{(n)}(x) = \psi^{(0)}(x) - i\frac{g}{\beta}\left(1 + \frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2p_0} + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{2p_0}\right)\chi^{(n)}(\boldsymbol{x}, p)\psi^{(0)}(x) + \ldots. \tag{1.55}$$

The high energy integration of the wave, thus, reproduces the perturbative expansion of the scattered wave.

1.4 Propagators in momentum and light cone spaces

Computing the propagator for the Schrodinger wave in momentum space follows the same procedure taken to evaluate the propagator in position space. As we will see in the next chapter, the scattering amplitude for $\psi^{(n)}(x)$, omitting the spin and the operator correction, is essentially given by the following Schrodinger propagator in momentum space. In order to find it we simply Fourier transform

$$G_s^{(n)}(p_f, p_i) \equiv \int d^3\boldsymbol{x}_f \int d^3\boldsymbol{x}_i \, e^{-i\boldsymbol{p}_f\cdot\boldsymbol{x}_f} G_s^{(n)}(x_f, x_i) e^{i\boldsymbol{p}_i\cdot\boldsymbol{x}_i}. \tag{1.56}$$

Now, define the momentum change $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ and rewrite the above expression as

$$\begin{aligned} G_s^{(n)}(p_f, p_i) = &\int d^3\boldsymbol{x}_f \int d^3\boldsymbol{x}_i \, e^{-i\boldsymbol{q}_f\cdot\boldsymbol{x}_f} \\ &\cdot \int \mathcal{D}^3\boldsymbol{x}(t) \exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_0}{4}\dot{\boldsymbol{x}}^2(t) - \boldsymbol{p}_i\cdot\dot{\boldsymbol{x}}(t) - 2gA_0^{(n)}(\boldsymbol{x}(t))\right)\right]. \end{aligned} \tag{1.57}$$

Here, in the last step, the term $i\boldsymbol{p}_i\cdot(\boldsymbol{x}_i - \boldsymbol{x}_f)$ has been introduced back in the integral as a total derivative. Now, as before, let us introduce the change (1.35) leading to

$$\begin{aligned} G_s^{(n)}(p_f, p_i) = &\exp\left[-i\frac{\boldsymbol{p}_i^2}{p_i^0}(t_f - t_i) - 2i\frac{\boldsymbol{q}\cdot\boldsymbol{p}_i}{p_i^0}t_f\right]\int d^3\boldsymbol{w}_f \int d^3\boldsymbol{w}_i \, e^{-i\boldsymbol{q}_f\cdot\boldsymbol{w}_f} \\ &\cdot \int \mathcal{D}^3\boldsymbol{w}(t) \exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_i^0}{4}\dot{\boldsymbol{w}}^2(t) - 2gA_0^{(n)}(\boldsymbol{w}(t) + 2\beta t)\right)\right], \end{aligned} \tag{1.58}$$

so

$$\begin{aligned} G_s^{(n)}(p_f, p_i) \approx &\exp\left[-i\frac{\boldsymbol{p}_i^2}{p_i^0}(t_f - t_i) - 2i\frac{\boldsymbol{q}\cdot\boldsymbol{p}_i}{p_i^0}t_f\right]\int d^3\boldsymbol{w}_f \int d^3\boldsymbol{w}_i \, e^{-i\boldsymbol{q}_f\cdot\boldsymbol{w}_f} \\ &\cdot\left(\int \mathcal{D}^3\boldsymbol{w}(t) \exp\left[i\int_{t_i}^{t_f} dt\left(\frac{p_i^0}{4}\dot{\boldsymbol{w}}^2(t)\right)\right]\right) \\ &\cdot \exp\left[-2ig\int_{t_i}^{t_f} dt\, A_0^{(n)}\left(\boldsymbol{w}_t(t_f) + 2\beta t\right)\right], \end{aligned} \tag{1.59}$$

where the path integration in the interaction term has been factorized out of all the path fluctuations due to the limit $t_i \to -\infty$ and $\beta \to 1$, and the longitudinal and perpendicular coordinates refer to $\boldsymbol{p}_i$. The integral in $\boldsymbol{w}(t_i)$ only affects to the free propagator and consequently, using the normalization condition

$$\int d^3\boldsymbol{w}_i \, G_s^{(0)}(w_f, w_i) = 1, \tag{1.60}$$

and transforming back to the original system and denoting $\hat{\boldsymbol{p}}_i s = 2\beta t$ one finds

$$G_s^{(n)}(p_f, p_i) = \int d^3\boldsymbol{x}_f\, e^{-iq\cdot\boldsymbol{x}_f} \exp\left[-i\frac{g}{\beta}\int_{-\infty}^{x_f^l} ds\, A_0^{(n)}\left(\boldsymbol{x}_f^t + \hat{\boldsymbol{p}}_i s\right)\right]. \tag{1.61}$$

The particular limit $t_f \to \infty$ corresponds to the scattering amplitude if the spinorial part and the low energy corrections of the state are omitted. In this case one finds a conservation of longitudinal momentum change $2\pi\delta(q_l)$ and the integral restricts to the transverse plane to $\boldsymbol{p}_i$. For completeness we may compute the propagator in light-cone variables too. We define light cone variables as customary

$$x_+ = \frac{x_0 + x_3}{\sqrt{2}}, \quad x_- = \frac{x_0 - x_3}{\sqrt{2}}, \quad \boldsymbol{x}_t = (x_1, x_2), \tag{1.62}$$

in such way that the scalar product of two 4-vectors reads

$$x_\mu y^\mu = x_0 y_0 - x_3 y_3 - x_1 x_1 - x_2 x_2 = x_+ y_- + x_- y_+ - \boldsymbol{x}_t \cdot \boldsymbol{y}_t. \tag{1.63}$$

We depart from a free fermion $\psi^{(0)}(x)$ satisfying the Dirac equation and, thus, the squared Dirac equation

$$(\not{p} - m)\psi^{(0)}(x) = 0 \to (\not{p} + m)(\not{p} - m)\psi^{(0)}(x) = (i\partial_\mu\partial_\nu - m^2)\psi^{(0)}(x) = 0. \tag{1.64}$$

Now, since the spinorial structure in the squared equation is factorizable, by using $\psi^{(0)}(x) = \sqrt{m/p_0}\,u(p)\varphi^{(0)}(x)$ and the identity

$$\partial_0\partial_0 - \partial_3\partial_3 = \frac{1}{2}\left(\frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-}\right)^2 = 2\frac{\partial}{\partial x_+}\frac{\partial}{\partial x_-}, \tag{1.65}$$

then the squared Dirac equation for $\varphi^{(0)}(x)$ can be rewritten in the light cone variables as

$$\left(2\frac{\partial}{\partial x_+}\frac{\partial}{\partial x_-} - \frac{\partial^2}{\partial^2\boldsymbol{x}_t} - m^2\right)\varphi^{(0)}(x) = 0. \tag{1.66}$$

Let us use p_- as the generator of the evolution in x_+ of the states and define the Fourier transform of $\varphi^{(0)}(x)$ at time point x_+

$$\varphi^{(0)}(x_+, x_-, \boldsymbol{x}_t) = \int \frac{dp_+}{2\pi}\int \frac{d^2\boldsymbol{p}_t}{(2\pi)^2} e^{-ip_+x_- + i\boldsymbol{p}_t\cdot\boldsymbol{x}_t}\tilde{\varphi}^{(0)}(x_+, p_+, \boldsymbol{p}_t). \tag{1.67}$$

By inserting this expression in equation (1.66) one finds

$$\int \frac{dp_+}{2\pi}\int \frac{d^2\boldsymbol{p}_t}{(2\pi)^2} e^{-ip_+x_- + i\boldsymbol{p}_t\cdot\boldsymbol{x}_t}\left(-2ip_+\frac{\partial}{\partial x_+} + \boldsymbol{p}_t^2 - m^2\right)\tilde{\varphi}^{(0)}(x_+, p_+, \boldsymbol{p}_t) = 0, \tag{1.68}$$

from which it derives the equation

$$\left(-2ip_+\frac{\partial}{\partial x_+}+\boldsymbol{p}_t^2-m^2\right)\tilde{\varphi}^{(0)}(x_+,p_+,\boldsymbol{p}_t)=0, \tag{1.69}$$

or

$$i\frac{\partial}{\partial x_+}\tilde{\varphi}^{(0)}(x_+,p_+,\boldsymbol{p}_t)=\left(\frac{\boldsymbol{p}_t^2}{2p_+}-\frac{m^2}{2p_+}\right)\tilde{\varphi}^{(0)}(x_+,p_+,\boldsymbol{p}_t), \tag{1.70}$$

which is a Schrodinger like equation for a particle of fictitious mass p_+ moving in the transverse plane to x_3. Correspondingly its free Green function is given by

$$\begin{aligned}G_s^{(0)}(x_+^1,\boldsymbol{x}_t^1;x_+^0,\boldsymbol{x}_t^0)=\exp&\left[+i\frac{m^2}{2p_+}(x_+^1-x_+^0)\right]\\ &\left\{\frac{p_+}{2\pi(x_+^1-x_+^0)}\exp\left[+i\frac{p_+}{2(x_+^1-x_+^0)}(\boldsymbol{x}_t^1-\boldsymbol{x}_t^0)^2\right]\right\}.\end{aligned} \tag{1.71}$$

This function propagates the free solutions between two space-time points in light cone variables. In order to add an interaction, we simply use the perturbative definition of the interacting Green function and take into account that the propagation is not constrained, in general, to a plane, if the interaction term is not, so

$$\begin{aligned}&G_s^{(n)}(x_+^1,x_-^0,\boldsymbol{x}_t^1;x_+^0,x_-^0,\boldsymbol{x}_t^0)=G_s^{(0)}(x_+^1,\boldsymbol{x}_t^1;x_+^0,\boldsymbol{x}_t^0)+\\ &-ig\int_{x_+^0}^{x_+^1}dx_+\int d^3x G_s^{(0)}(x_+^1,\boldsymbol{x}_t^1;x_+,\boldsymbol{x}_t)A_0^{(n)}(x)G_s^{(0)}(x_+,\boldsymbol{x}_t;x_+^0,\boldsymbol{x}_t^0)+(...)\end{aligned} \tag{1.72}$$

and, since the free propagation does not depend on the x_- variable, one finds

$$\begin{aligned}&G_s^{(n)}(x_+^1,x_-^0,\boldsymbol{x}_t^1;x_+^0,x_-^0,\boldsymbol{x}_t^0)=G_s^{(0)}(x_+^1,\boldsymbol{x}_t^1;x_+^0,\boldsymbol{x}_t^0)\\ &-ig\int_{x_+^0}^{x_+^1}dx_+\int d^2\boldsymbol{x}_t\,G_s^{(0)}(x_+^1,\boldsymbol{x}_t^1;x_+,\boldsymbol{x}_t)\\ &\qquad\qquad\cdot\left\{\int dx_-A_0^{(n)}(x)\right\}G_s^{(0)}(x_+,\boldsymbol{x}_t;x_+^0,\boldsymbol{x}_t^0)+(...)\\ &\equiv\int\mathcal{D}^2\boldsymbol{r}_t\exp\left[i\int_{x_-^0}^{x_+^1}dx_+\left(\frac{p_+}{2}\dot{\boldsymbol{r}}_t^2-g\int_{-\infty}^{+\infty}dx_-A_0^{(n)}(x)\right)\right]\end{aligned} \tag{1.73}$$

which is the perturbative expansion of a path integral in the effective potential

$$-ig\int dx_-A_0^{(n)}(\boldsymbol{x}):=-ig\int dx_-A_0^{(n)}\left(\boldsymbol{x}_t,\frac{x_+-x_-}{\sqrt{2}}\right). \tag{1.74}$$

Since the interaction terms stops being a function of the x_+ variable once integrated in x_-, consequently, the path integration affects only to the free propagator yielding

$$G_s^{(n)}(x_1, x_0) = G_s^{(0)}(x_1, x_0) \exp\left[-ig \int_{-\infty}^{+\infty} dx_- A_0^{(n)}(x)\right]. \tag{1.75}$$

The above propagator has lost the local dependence of the field with coordinate x_3 as a result of the time evaluation of x_- at $x_0 = \pm\infty$.

2

High energy multiple scattering

With the development of the wave function in the previous chapter we are now in position to define the scattering amplitude, that is, the amplitude of finding the particle with some momentum and spin due to the effect of the medium [30–35]. We will find that the obtained result preserves the unitarity of the emerging wave by proving the optical theorem, and that the series expansion in the coupling $g = Ze^2$ reproduces the perturbative description of the same problem. As previously stated, the evaluation of the wave or the scattering amplitudes for a particular configuration of the medium constituents is not interesting and we will instead look for averaged squared amplitudes and observables. In this way we will find that these averaged quantities for n scattering sources are always expressible as exponentiations of the single n=1 case. In the process of averaging we will discover also the emergence of interference phenomena related to multiple scattering events which suggests a splitting of the cross sections into two contributions. In a certain limit, namely for macroscopic mediums, the resulting cross sections become totally incoherent and therefore the scattering process will acquire an statistical interpretation in terms of probability distributions, whereas for mediums of microscopic size diffractive phenomena and thus, medium coherence, have a prominent role over the incoherent limit. We will also demonstrate that the incoherent contribution leads to the well-known Moliere's derivation [36] based on markovian arguments [37] and we will compute the averaged momentum transfer $\langle \boldsymbol{q}^2 \rangle$. Finally we will introduce a beyond eikonal evaluation by considering a non vanishing longitudinal momentum change. We will show that the beyond eikonal squared amplitude, when averaged over mediums of macroscopic dimensions, reduces to the pure eikonal scattering for on-shell particles. However, longitudinal momentum changes at different energy states can produce interferences in the squared amplitudes, ultimately leading to incoherence phenomena when creation or annihilation of particles is also considered, as we will see in the next chapter.

2.1 The scattering amplitude

We depart from an initial asymptotic free fermion with momentum p_i and spin s_i, whose state will be denoted $\psi_i^{(0)}(x)$,

$$\psi_i^{(0)}(x) = \sqrt{\frac{m}{p_i^0}} u_{s_i}(p_i) e^{-ip_i \cdot x}. \tag{2.1}$$

Due to the effect of the external field (1.13) produced by the medium, in the asymptotic final state one finds the superposition of two states, the former wave itself and an infinite superposition of states of deflected momentum p_f and spin s_f. This reads

$$\psi_f^{(n)}(x) = \psi_i^{(0)}(x) + \sum_{s_f=1,2} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} M_{s_f s_i}^{(n)}(p_f, p_i) \sqrt{\frac{m}{p_f^0}} u_{s_f}(p_f) e^{-ip_f \cdot x}. \tag{2.2}$$

The weights are referred to as the scattering amplitude or M-matrix, this is, the amplitude to find the wave with deflected momentum p_f and spin s_f. The asymptotic initial state can be absorbed in the scattering amplitude if we define

$$\psi_f^{(n)}(x) = \sum_{s_f=1,2} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} S_{s_f s_i}^{(n)}(p_f, p_i) \sqrt{\frac{m}{p_f^0}} u_{s_f}(p_f) e^{-ip_f \cdot x}, \tag{2.3}$$

where the S-matrix includes the no collision distribution plus the collisional M-matrix distribution, and by direct inspection of (2.2) and (2.3) verifies then

$$S_{s_f s_i}^{(n)}(p_f, p_i) \equiv M_{s_f s_i}^{(n)}(p_f, p_i) + (2\pi)^3 \delta^3(\boldsymbol{p}_f - \boldsymbol{p}_i) \delta_{s_f s_i}. \tag{2.4}$$

In order to find the scattering amplitude we rewrite the emerging wave in (2.2) by using the Lippmann-Schwinger recursive equation,

$$\psi_f^{(n)}(x) = \psi_i^{(0)}(x) + \int d^4y S_F(x-y) g\gamma^0 A_0^{(n)}(\boldsymbol{y}) \psi^{(n)}(y) = \psi_i^{(0)}(x) + \psi_{diff}^{(n)}(x). \tag{2.5}$$

Inserting the result (1.42), valid for the wave in the high energy limit, we find

$$\psi_{diff}^{(n)}(x) = \int d^4y S_F(x-y) g\gamma^0 A_0^{(n)}(\boldsymbol{y}) \cdot \left\{ \left(1 + i\frac{\gamma_k \gamma_0}{2p_i^0} \partial_k \right) W^{(n)}(\boldsymbol{y}, p_i) \right\} \psi_i^{(0)}(y). \tag{2.6}$$

In order to express this diffracted wave as an infinite superposition broadened around $\boldsymbol{p}_i$, we integrate the Feynman-Stueckelberg propagator (1.47) for energy positive solutions getting

$$S_F^{(+)}(x) = \frac{1}{i}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\not p + m}{2p_0}e^{-ip\cdot x} = \frac{1}{i}\sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{m}{p_0}u_s(p)\otimes \bar{u}_s(p)e^{-ip\cdot x}, \quad (2.7)$$

where the energy momentum $p_0(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ relation arising in the p_0 integration over the positive pole has been left implicit and the completeness relation

$$\sum_{s=1,2} u_s(p)\otimes \bar{u}_s(p) = \frac{\not p + m}{2m}, \quad (2.8)$$

has been used. By inserting (2.7) in (2.6) one finds

$$\begin{aligned}\psi^{(n)}_{diff}(x) = &\sum_{s_1=1,2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\sqrt{\frac{m}{p_f^0}}u^{s_f}(p_f)e^{-ip_f\cdot x}\int d^4y e^{i(p_f-p_i)y}\left(-igA_0^{(n)}(\boldsymbol{y})\right)\\ &\cdot \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma^0\left\{\left(1 + i\frac{\gamma_k\gamma_0}{2p_i^0}\partial_k\right)W^{(n)}(\boldsymbol{y}, p_i)\right\}\sqrt{\frac{m}{p_i^0}}u_{s_i}(p_i)\\ \equiv &\sum_{s_f=1,2}\int \frac{d^3\boldsymbol{p_f}}{(2\pi)^3}\sqrt{\frac{m}{p_f^0}}u_{s_f}(p_f)e^{-ip_f\cdot x}M^{(n)}_{s_fs_i}(p_f, p_i),\end{aligned} \quad (2.9)$$

where the integration must be carried on-shell. The expression (2.9) is interpretable as a superposition with amplitude $M^{(n)}_{s_fs_i}(p_f, p_i)$ of being at the state (p_f, s_f) given by

$$\psi_f^{(0)}(x) = \sqrt{\frac{m}{p_f^0}}u_{s_f}(p_f)e^{-ip_fx}. \quad (2.10)$$

Thus we define the amplitude, following (2.2), as

$$\begin{aligned}M^{(n)}_{s_fs_i}(p_f, p_i) = &\int d^4y e^{i(p_f-p_i)\cdot y}\left(-igA_0^{(n)}(\boldsymbol{y})\right)\\ &\cdot \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma^0\left\{\left(1 + i\frac{\gamma_k\gamma_0}{2p_i^0}\partial_k\right)W^{(n)}(\boldsymbol{y}, p_i)\right\}u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}.\end{aligned} \quad (2.11)$$

The quantity in the bottom line of the above expression, which we call J, contains the non perturbative information of the diffracted wave,

$$J = \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0\left\{\left(1 + i\frac{\gamma_k\gamma_0}{2p_i^0}\partial_k\right)W_0^{(n)}(\boldsymbol{y}, p_i)\right\}u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}. \quad (2.12)$$

Had we expanded the phase in the coupling we would have found

$$W^{(n)}(\boldsymbol{y}, p_i) = 1 + O\left(\frac{g}{\beta_i}\right) \rightarrow J = \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}} + O\left(\frac{g}{\beta_i}\right) \quad (2.13)$$

which, together the Fourier transform of of $A_0^{(n)}(\boldsymbol{y})$ would produce the first perturbative term of the scattering matrix, representing all single kicks combinatory with the medium elements. However, as we stated before, we expect that the perturbative processes of high order are the relevant ones, then we have to compute the term non perturbatively. We have

$$J = \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\left(\gamma_0 + \frac{i}{2p_i^0}\gamma_k^\dagger\frac{\partial}{\partial x_k}\right)u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}W^{(n)}(\boldsymbol{y}, p_i). \quad (2.14)$$

Notice that, although the derivative correction is suppressed with an overall factor $1/p_i^0$, the spinorial terms may introduce p_i^0 corrections. It is easy to show that for elastic scattering this is not the case, since the above expression produces in the high energy limit, $p_f \approx p_i$,

$$J \simeq \delta_{s_i}^{s_f}\left(1 - i\frac{\beta}{2p_i^0}\partial_3\right)W^{(n)}(\boldsymbol{y}, p_i) = \delta_{s_i}^{s_f}\left(1 - \frac{gA_0^{(n)}(\boldsymbol{y})}{2p_i^0}\right)W^{(n)}(\boldsymbol{y}, p_i). \quad (2.15)$$

In this limit spin keeps unchanged $s_f = s_i$ since the spin-flip amplitudes are suppressed a factor $1/p_i^0$ with respect to the non-flip case. The term arising in the phase derivative can be neglected so

$$\frac{g\,\langle A_0^{(n)}(\boldsymbol{y})\rangle}{2p_i^0} \ll 1 \rightarrow J \simeq \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}W^{(n)}(\boldsymbol{y}, p_i) \quad (2.16)$$

where the Glauber condition (1.23) has been used. It can be globally neglected, then, provided that the external interaction is sufficiently smooth and $p_i^0 \rightarrow \infty$. By performing the time integral we get the energy conservation so $\beta_i = \beta_f \equiv \beta$ and

$$M_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}} \quad (2.17)$$
$$\times \int d^3\boldsymbol{y}e^{-i\boldsymbol{q}\cdot\boldsymbol{y}}\left(-igA_0^{(n)}(\boldsymbol{y})\right)W^{(n)}(\boldsymbol{y}, p_0),$$

where the momentum transfer with the external field is denoted as $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$. The remaining integral can be performed by placing $\boldsymbol{p}_i$ along the z axis and taking the high energy limit $q_z \ll q_t$. We find the asymptotic value

$$\int d^3\boldsymbol{y}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} \left(-igA_0^{(n)}(\boldsymbol{y})\right) W^{(n)}(\boldsymbol{y}, p)$$
$$\approx \beta \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \int_{-\infty}^{+\infty} dy_3 \left(-i\frac{g}{\beta}A_0^{(n)}(\boldsymbol{y})\right) \exp\left[-i\frac{g}{\beta}\int_{-\infty}^{y_3} dy_3'\, A_0^{(n)}(\boldsymbol{y})\right]$$
$$= \beta \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left(\exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y})\right] - 1\right). \tag{2.18}$$

Finally, the amplitude of a change of 4-momentum $q = p_1 - p_0$ under the field of n sources, at high energy and at all orders in the coupling is given (2.2) by

$$M_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\beta \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i) \sqrt{\frac{m}{p_i^0}} \tag{2.19}$$
$$\times \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left(\exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y})\right] - 1\right).$$

Similarly, by including the no collision amplitude (2.3), we write

$$S_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\beta \sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i) \sqrt{\frac{m}{p_i^0}} \tag{2.20}$$
$$\times \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y})\right].$$

It results convenient to extract the energy and spin conservation delta out of the above amplitudes by defining the quantities

$$M_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_i}^{s_f}\beta F_{el}^{(n)}(\boldsymbol{q}),\;\; S_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_i}^{s_f}\beta S_{el}^{(n)}(\boldsymbol{q}), \tag{2.21}$$

where $F_{el}^{(n)}(\boldsymbol{q}_t)$ denotes the integral part of (2.19)

$$F_{el}^{(n)}(\boldsymbol{q}_t) \equiv \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left(\exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y})\right] - 1\right). \tag{2.22}$$

and $S_{el}^{(n)}(\boldsymbol{q}_t)$ denotes the integral part of (2.20)

$$S_{el}^{(n)}(\boldsymbol{q}_t) \equiv \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \exp\left[-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y})\right]. \tag{2.23}$$

Since $\beta \to 1$ and energy and spin are preserved $F^{(n)}(\boldsymbol{q}_t)$ contains all the relevant physics. In this approximation, as a result of taking the $q_z \to 0$ limit, the scattering amplitude stops being able to resolve any source structure in the z direction, since the eikonal phase in (2.22) appears integrated in $y_3 = \pm\infty$. This approximation then erased the locality in y_3 of the wave (1.42), which satisfied that at point y_3 electron is only affected by sources verifying $z_i < y_3$, a sign of a gradual transformation of time causality to y_3 causality as a result of taking the $\beta \to 1$ limit. Finally, we can connect this result with Glauber's spinless amplitude $f(\theta)$. For a Schrodinger wave the scattered state can be written as

$$\psi_f(x) = \psi_i(x) + f(\theta)\frac{e^{-i|\boldsymbol{p}_i||\boldsymbol{x}|}}{|\boldsymbol{x}|} \tag{2.24}$$

where $\theta = q/\beta p_i^0$ is the scattering angle. The relation between $f(\theta)$ and the M-matrix is geometric. Since $|f(\theta)|^2$ is the intensity in the solid angle interval Ω and $\Omega + d\Omega$, then dividing by time and the incoming flux, from (2.2) and (2.24)

$$\int \left|f(\theta)\right|^2 d\Omega = \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3}\left|M^{(n)}_{s_f s_i}(p_f, p_i)\right|^2 \to f(\theta) = \frac{|\boldsymbol{p}_i|}{2\pi i}F^{(n)}_{el}(\boldsymbol{q}_t). \tag{2.25}$$

Glauber derived the above amplitude for a Schrodinger particle, thus in principle-omitting relativistic and spin effects, something which seems contradictory with the high energy limit. His description is, however, completely accurate in the limit $\beta \to 1$ since the high energy limit of the Dirac equation is well approximated by a Schrodinger like equation.

2.2 Perturbative expansion and strong coupling

The high energy integration of the scattering amplitude can be expanded in the coupling $g = Ze^2$. Order to order it has to agree with the standard perturbative representation of the elastic scattering with the medium. We note first that the integration of the field in the phase produces for a general interaction (1.13)

$$\begin{aligned}\chi_0^{(n)}(\boldsymbol{y}_t) \equiv \int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y}) &= \sum_{i=1}^{n}\int_{-\infty}^{+\infty} dy_3 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} e^{iq_t\cdot(y_t - r_t^i) + iq_3(y_3 - r_3^i)}\hat{A}_0^{(1)}(\boldsymbol{q}) \\ &= \sum_{i=1}^{n}\int \frac{d^2\boldsymbol{q}_t}{(2\pi)^2} e^{iq_t\cdot(y_t - r_t^i)}\hat{A}_0^{(1)}(\boldsymbol{q}_t, 0).\end{aligned} \tag{2.26}$$

The above result means that the y_3 coordinates of each center are lost, the entire medium is seen as an infinitesimal sheet and any of the single collisions are considered totally eikonal $q_z = 0$. For the Debye screened interaction (1.8), in particular, we find

$$\chi_0^{(n)}(\boldsymbol{y}_t) = \int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y}) = 2\sum_{i=1}^{n} K_0(\mu_d|\boldsymbol{y}_t - \boldsymbol{r}_t^i|). \tag{2.27}$$

At leading order in g, using (2.26) we obtain from (2.22)

$$F_{el}^{(n)}(\boldsymbol{q}_t) = \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t}\left(-i\frac{g}{\beta}\sum_{i=1}^{n}\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}e^{i\boldsymbol{k}_t\cdot(\boldsymbol{y}_t-\boldsymbol{r}_t^i)}A_0^{(1)}(\boldsymbol{k}_t)\right)+O\left(\frac{g^2}{\beta^2}\right)$$
$$= -i\frac{g}{\beta}\hat{A}_0^{(1)}(\boldsymbol{q}_t)\sum_{i=1}^{n}e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^i}+O\left(\frac{g^2}{\beta^2}\right) \tag{2.28}$$

So amplitude (2.19) at first order in the coupling simplifies to

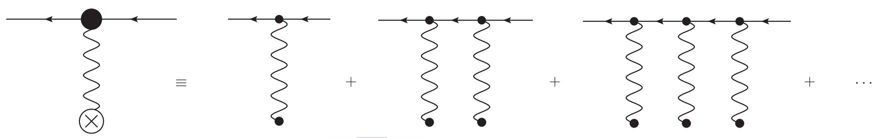


Figure 2.1: Diagrammatic representation of $\left|F_{el}^{(1)}(\boldsymbol{q})\right|^2$ for a single center up to 3rd order in the coupling $g = Ze^2$.

$$M_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\sqrt{\frac{m}{p_f^0}}\bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}$$
$$\times(-ig)\hat{A}_0^{(1)}(\boldsymbol{q}_t)\left(\sum_{i=1}^{n}e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^i}\right)+O\left(\frac{g^2}{\beta^2}\right) \tag{2.29}$$

For the particular case of the Debye screened interaction, using (1.8) or (2.27) we find a superposition of n single Mott amplitudes since

$$-ig\hat{A}_0^{(1)}(\boldsymbol{q}) = -\frac{i4\pi Ze^2}{\boldsymbol{q}^2+\mu_d^2}. \tag{2.30}$$

Only at leading order (l.o.) in g the total amplitude of each center simply adds with a phase related to its position,

$$\left(M_{s_f s_i}^{(n)}(p_f, p_i)\right)_{l.o.} = \left(\sum_{i=1}^{n}e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^i}\right)\left(M_{s_f s_i}^{(1)}(p_f, p_i)\right)_{l.o.} \tag{2.31}$$

For larger values of g a numerical integration of (2.19) has to be carried. In Figure 2.2 various cases of the scattering amplitude (2.19) are shown and compared with the leading order approximation (2.29). We see that for large enough q the $1/q^4$ tail of the Rutherford scattering is found and the arbitrary coupling evaluation matches the leading order approximation. A closed form of the amplitude (2.19) for a single center can be given if $\mu_d \to 0$. In this limit the leading order of

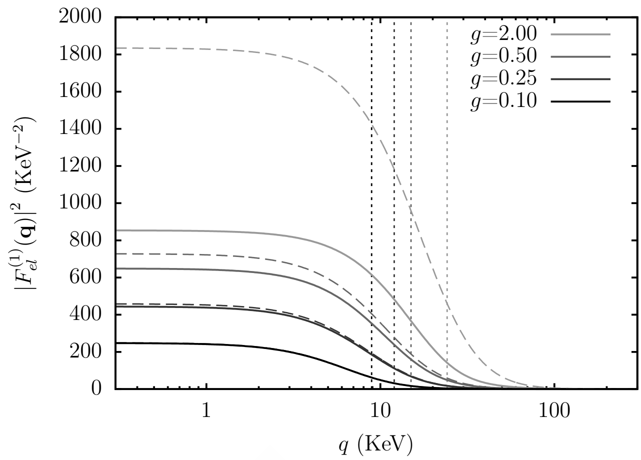


Figure 2.2: Saturation effect of the squared elastic amplitude $\left|F_{el}^{(1)}(\boldsymbol{q})\right|^2$ for a single center at increasing coupling $g = Ze^2$ (continuous lines) compared to the leading order approximation (dashed lines) as a function of the transverse momentum change $\boldsymbol{q}$. Vertical small-dashed lines represent the respective μ_d.

$F^{(1)}(q)$ is well defined but the next orders are not. In order to see it let us rescale expression (2.28), for $n = 1$ and the Debye interaction (2.27), as

$$F_{el}^{(1)}(\boldsymbol{q}) = \frac{2\pi}{q^2}\int dy\, yJ_0(y)\left\{\exp\left[-i\frac{2g}{\beta}K_0(\mu_d y/q)\right] - 1\right\} \tag{2.32}$$

In order to take the limit $\mu_d \to 0$ we use the relations

$$\lim_{\mu_d\to 0} K_0(\mu_d y/q) \approx \lim_{\mu_d\to 0} \log(2q/\mu y) - \gamma,\ J_n(x)e^{1+2ia} = 2^{2ia+1}\frac{\Gamma(1+ig/\beta)}{\Gamma(-ig/\beta)}. \tag{2.33}$$

As a consequence a running coupling is found, of the form

$$\lim_{\mu_d\to 0} F_{el}^{(1)}(\boldsymbol{q}) = \alpha_r(q)\frac{4\pi}{\boldsymbol{q}^2},\ \alpha_r(q) \equiv \frac{1}{2}\frac{1}{q^{2ig\beta}}\frac{\Gamma(1+ig/\beta)}{\Gamma(-ig/\beta)}\cdot \lim_{\mu_d\to 0} \exp\left[\frac{2ig}{\beta}\log(\mu_d)\right] \tag{2.34}$$

The above amplitude has a global divergent phase in the remaining limit which does not affect the first order in g, but it does in the next orders. This term was conjectured by Dalitz as a way of resuming the Coulomb divergencies, and was later proved correct by Weinberg. Equation (2.34) also preserves the quantization levels despite being a high energy approximation, since the same arguments for deriving (2.19) still hold when the small angle limit is taken. Correspondingly, in the forward zone of the amplitude (2.34) the poles of the gamma function, accessible at very small θ are given by $ig/\beta = -n$, where at low energies $\beta^2 \simeq$

$2p_0/m - 2$. We then write

$$\beta^2 = -\frac{g^2}{n^2} \quad \to \quad p_0 = m - \frac{mZ^2e^4}{2n^2} \tag{2.35}$$

which correspond to the quantization levels of a single external Coulomb field.

2.3 Total cross section and optical theorem

The optical theorem relates the total cross section with the forward part of the amplitude as a consequence of the unitarity constraint of $S^{(n)}_{s_f s_i}(p_f, p_i)$. We now check that the obtained scattering amplitude (2.19) obeys this relation. The total cross section can be evaluated by taking the square of the amplitude, summing over final states and normalizing by the incoming flux β and time T,

$$\sigma^{(n)}_{tot} \equiv \frac{1}{T\beta}\sum_{s_f}\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3}\left|M^{(n)}_{s_f s_i}(p_f, p_i)\right|^2 . \tag{2.36}$$

At leading order in $\beta \to 1$ we find using (2.19),

$$\begin{aligned}\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3}\left|M^{(n)}_{s_f s_i}(p_f, p_i)\right|^2 &= \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3}\left|2\pi\delta(q^0)\delta_{s_f s_i}\beta\right|^2 \int d^2\boldsymbol{x}_t \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t-\boldsymbol{y}_t)}\\ &\cdot\left(1 - \exp\left[-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right] - \exp\left[+i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{y}_t)\right] + \exp\left[-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t) + i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{y}_t)\right]\right).\end{aligned} \tag{2.37}$$

The momentum integration takes advantage of the slow dependence of the spinorial term with $\boldsymbol{p}_f$ with respect to the term inside the integral. In order to be consistent with the approximation, we apply the same limit in the integration variables at equation (2.36). In the high energy limit we have $\boldsymbol{p}^2 d\Omega \approx d^2\boldsymbol{p}_t$ so

$$\frac{d^3\boldsymbol{p}}{(2\pi)^3} \approx \frac{1}{(2\pi)^3}dp\, d^2\boldsymbol{p}_t = \frac{1}{(2\pi)^3}\frac{p_0}{p}dp_0\, d^2\boldsymbol{p}_t = \frac{1}{(2\pi)^3}\frac{1}{\beta}dp_0\, d^2\boldsymbol{q}_t, \tag{2.38}$$

Using this approximation, the transverse integration of (2.36) is trivial and one easily finds

$$\begin{aligned}\sum_{s_f}\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3}\left|M^{(n)}_{s_f s_i}(p_f, p_i)\right|^2 &= 2\pi\delta(0)\delta_{s_i s_i}\beta \int d^2\boldsymbol{x}_t \left|\exp\left[-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right] - 1\right|^2\\ &= 2\pi\delta(p^0_i - p^0_i)\delta_{s_i s_i}\beta \int d^2\boldsymbol{x}_t \left(2 - 2\,\mathrm{Re}\left(\exp\left[-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right]\right)\right).\end{aligned}$$

We notice the above result is just twice the negative real part of the diagonal values of (2.19), so one can write the relation

$$\sum_{s_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \left| M^{(n)}_{s_f s_i}(p_f, p_i) \right|^2 = -2\,\mathrm{Re}\, M^{(n)}_{s_i s_i}(p_i, p_i), \tag{2.39}$$

which constitutes the well known optical theorem. A more general form of the optical theorem can be derived

$$\sum_{s} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} M^{(n),*}_{s s_i}(p, p_f) M^{(n)}_{s_i s}(p, p_i) = -M^{(n)}_{s_f s_i}(p_f, p_i) - M^{(n),*}_{s_i s_f}(p_i, p_f). \tag{2.40}$$

The proof is very simple, we have using (2.19),

$$\begin{aligned} &\sum_{s} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, M^{(n),*}_{s s_f}(p, p_f) M^{(n)}_{s s_i}(p, p_i) = 2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta \\ &\cdot \int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t} \cdot \left[\exp\left(-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right) - 1\right]\left[\exp\left(+i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right) - 1\right] \\ &= 2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta \int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\left[1 - \exp\left(-i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right)\right. \\ &\left. + 1 - \exp\left(+i\frac{g}{\beta}\chi^{(n)}_0(\boldsymbol{x}_t)\right)\right] \equiv -M^{(n)}_{s_f s_i}(p_f, p_i) - M^{(n),*}_{s_i s_f}(p_i, p_f). \end{aligned} \tag{2.41}$$

The infinite arising in the conservation delta $2\pi\delta(0) \equiv T$, due to an integration of a time-independent quantity, accounts for the uniform rate of scattering in time. On the other hand the β accompanying the amplitude is just the incoming flux,

$$\sqrt{\frac{m}{p^0_i}}\bar{u}_{s_i}(p_i)\gamma_3 u_{s_i}(p_i)\sqrt{\frac{m}{p^0_i}} = \frac{p^3_i}{p^0_i} = \beta. \tag{2.42}$$

Correspondingly, as read from (2.36), the rate of total scattered particles per unit of time and per unit of incoming flux is given by

$$\sigma^{(n)}_{tot} = -2\,\mathrm{Re}\, F^{(n)}_{el}(\boldsymbol{0}) = \frac{4\pi}{|\boldsymbol{p}_i|}\,\mathrm{Im}\, f^{(n)}(0). \tag{2.43}$$

Although scattering amplitude (2.19) only allows changes in perpendicular momentum it results exact if we replace $\boldsymbol{q}_t$ with $\boldsymbol{q}$, since the longitudinal momentum change is implicitly fixed by the energy conservation delta. This can be made manifest for interactions with azimuthal symmetry. We find

$$\chi^{(n)}_0(\boldsymbol{y}_t) = \frac{1}{\beta}\int_{-\infty}^{+\infty} dy_3\, A^{(n)}_0(|\boldsymbol{y}_t|, y_3) \equiv \chi^{(n)}_0(y_t). \tag{2.44}$$

With this extra symmetry, we can integrate in the angular parameter in order to find

$$f^{(n)}(\theta) = \frac{p_i^0}{i} \int_{-\infty}^{+\infty} dy_t y_t J_0(q y_t) \left\{ \exp\left[-i \frac{g}{\beta} \chi_0^{(n)}(y_t) \right] - 1 \right\}, \tag{2.45}$$

where now $\boldsymbol{q}$ is the complete momentum change. Sum in the plane $\boldsymbol{q}_t$ is equivalent to a sum in the sphere. To show that, we check the optical theorem in the sphere. Using

$$\boldsymbol{q}^2 = \boldsymbol{p}_f^2 + \boldsymbol{p}_i^2 - 2\boldsymbol{p}_f \cdot \boldsymbol{p}_i = 2p^2\left(1 - \cos^2\theta\right) = 4p^2 \sin^2\left(\frac{\theta}{2}\right), \tag{2.46}$$

we get

$$d\Omega = \sin\theta d\theta d\varphi = \frac{q dq d\varphi}{p^2}, \tag{2.47}$$

so the total cross section reads

$$\sigma_{tot}^{(n)} = 2\pi \left| \int_0^{+\infty} dx_t\, x_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(n)}(x_t) \right] - 1 \right) \right|^2 \int_0^{2p} dq\, q J_0(q x_t) J_0(q y_t). \tag{2.48}$$

We can now use the orthogonality relation

$$\lim_{p\to\infty} \int_0^{2p} dq\, q J_0(q x_t) J_0(q y_t) = \frac{1}{x_t}\delta(x_t - y_t), \tag{2.49}$$

in order to find

$$\begin{aligned} \sigma_{tot}^{(n)} &= 2\pi \int_0^{+\infty} dx_t\, x_t \left| \exp\left[-i\frac{g}{\beta}\chi_0^{(n)}(x_t) \right] - 1 \right|^2 \\ &= 2\,\mathrm{Re} \int d^2\boldsymbol{x}_t \left(1 - \exp\left[-i\frac{g}{\beta}\chi_0^{(n)}(\boldsymbol{x}_t) \right] \right) = \frac{4\pi}{|\boldsymbol{p}_i|} \mathrm{Im}\, f^{(n)}(0), \end{aligned} \tag{2.50}$$

which agrees with equation (2.43). This implies that only in the infinite momentum frame operations over full momentum change $\boldsymbol{q}^2 d\Omega$ can be replaced with operations over the transverse momentum change $d^2\boldsymbol{q}_t$.

With the above tools we are in position to compute the total elastic cross section for the Debye interaction (1.8) with (2.27). The single case for small values of the coupling g is given, following (2.28), by

$$\begin{aligned} F_{el}^{(1)}(\boldsymbol{q}) &= \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left\{ \exp\Big[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}) \Big] - 1 \right\} \\ &= \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \cdot \left\{ -i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t) - \frac{g^2}{2\beta^2}\left(\chi_0^{(1)}(\boldsymbol{y}_t)\right)^2 + O\left(\frac{g^3}{\beta^3}\right) \right\} \\ &= -\,i\frac{g}{\beta}\left(\frac{4\pi}{\boldsymbol{q}_t^2 + \mu_d^2} \right) - \frac{g^2}{2\beta^2}\left(\frac{16\pi \,\mathrm{arcsinh}(q_t/2\mu_d)}{q_t\sqrt{\boldsymbol{q}_t^2 + 4\mu_d^2}} \right) + O\left(\frac{g^3}{\beta^3}\right). \end{aligned} \tag{2.51}$$

Then we find from equation (2.43)

$$\sigma_{tot}^{(1)} = -2\,\mathrm{Re}\,F_{el}^{(1)}(\mathbf{0}) = \frac{4\pi g^2}{\beta^2\mu^2} + \cdots\,, \tag{2.52}$$

and we define for later uses the oscillatory part also

$$\sigma_{osc}^{(1)} = +2\,\mathrm{Im}\,F^{(1)}(\mathbf{0}) = -\frac{8\pi g}{\beta\mu^2} + \cdots\,. \tag{2.53}$$

For arbitrary larger values of the coupling g the above values are not applicable.

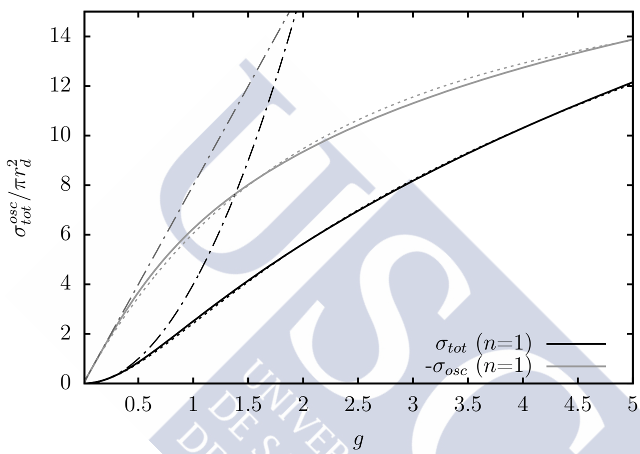


Figure 2.3: Saturation effect of the elastic cross section for a single center at increasing coupling using expressions (2.54) and (2.55) (continuous lines), compared to the leading perturbative order at equations (2.52) and (2.53) (dot dashed lines), together with the approximants given at (2.56) (dot lines), as a function of the coupling $g = Ze^2$ with running $\mu_d = \alpha^2 m_e Z^{1/3}$.

In those cases the single scattering amplitude stops being perturbative in g and we write instead

$$\begin{aligned}\sigma_{tot}^{(1)} &= 4\pi\int_0^\infty dy\, y\left(1-\cos\left[\frac{2g}{\beta}K_0(\mu_d y)\right]\right)\\ &= \frac{4\pi}{\mu_d^2}\int_0^\infty ds\, s\left(1-\cos\left[\frac{2g}{\beta}K_0(s)\right]\right) \equiv \frac{4\pi}{\mu_d^2}\Theta_1\left(\frac{g}{\beta}\right),\end{aligned} \tag{2.54}$$

for the total cross section and

$$\begin{aligned}\sigma_{osc}^{(1)} &= -4\pi\int_0^\infty dy\, y\sin\left[\frac{2g}{\beta}K_0(\mu_d y)\right]\\ &= -\frac{4\pi}{\mu_d^2}\int_0^\infty ds\, s\sin\left[\frac{2g}{\beta}K_0(s)\right] \equiv -\frac{4\pi}{\mu_d^2}\Theta_2\left(\frac{g}{\beta}\right),\end{aligned} \tag{2.55}$$

for the oscillatory part of the cross section. The behavior of $\sigma_{tot}^{(n)}$ and $\sigma_{osc}^{(n)}$ is shown in Figure 2.3. Function $\Theta_1(g/\beta)$ grows as g^2/β^2 for small values and then grows logarithmically at larger values of the parameter, and $\Theta_2(g/\beta)$ grows linearly for small parameters and grows logarithmically at larger values. A good approximation, valid in the range [0, 10] of the parameter $y = g/\beta$ at the 5% level, is given by the expressions

$$\Theta_1(y) = \frac{y^2}{1 + a_1 y^{a_2}}, \qquad \Theta_2(y) = 2\frac{y}{1 + b_1 y^{b_2}}, \tag{2.56}$$

where $a_1 = 0.65$ and $a_2 = 1.5$, and $b_1 = 0.32$ and $b_2 = 1.1$.

2.4 Multiple scattering effects

We have up to now provided some basic properties of the amplitude for a general interaction and presented the main results for the particular n=1 case, but without taking into account the effects of the multiple scattering. In this section we will treat with detail the large n limit of the squared amplitudes and the cross sections. As it is customary we use the shorthands

$$-i\frac{g}{\beta}\int_{-\infty}^{+\infty} dy_3\, A_0^{(n)}(\boldsymbol{y}) = -i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t^i) \equiv -i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^i(\boldsymbol{y}_t). \tag{2.57}$$

The average over medium configurations of the square of (2.19) reads, at leading order in $\beta \to 1$,

$$\left\langle \left| M_{s_f s_i}^{(n)}(p_f, p_i)\right|^2 \right\rangle = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_f s_i}\beta \left\langle \left| F_{el}^{(n)}(\boldsymbol{q}_t)\right|^2 \right\rangle, \tag{2.58}$$

where we erase an overall factor $2\pi\delta(0)$ and β when dividing by time and incoming flux. The relevant quantity in the above expression is $F_{el}^{(n)}(\boldsymbol{q})$, which contains the information of the medium configuration

$$\begin{aligned}\left\langle \left| F_{el}^{(n)}(\boldsymbol{q}_t)\right|^2 \right\rangle = \int d^2\boldsymbol{x}_t \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t - \boldsymbol{y}_t)} \Bigg\langle 1 - \exp\left[-i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^i(\boldsymbol{x}_t)\right] \\ - \exp\left[+i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^i(\boldsymbol{y}_t)\right] + \exp\left[-i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^i(\boldsymbol{x}_t) + i\frac{g}{\beta}\sum_{i=1}^{n}\chi_0^i(\boldsymbol{y}_t)\right]\Bigg\rangle.\end{aligned} \tag{2.59}$$

By taking the average over medium configurations only the multiple scattering effect of the medium geometry is contemplated. Particular configurations of the sources are ignored. We will assume that the target is a cylindrical medium of volume $V = l\Omega$, with l the length and $\Omega = \pi R^2$ the transverse area. Then the above average is understood as

$$\left\langle \left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 \right\rangle = \frac{1}{V^n} \int_V d^3\boldsymbol{r}_1 ... d^3\boldsymbol{r}_n \left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 = \frac{1}{\Omega^n} \int_V d^2\boldsymbol{r}_1^t ... d^2\boldsymbol{r}_n^t \left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 , \tag{2.60}$$

so we find that it reduces to an average over a single center of the form

$$\begin{aligned} &\left\langle \left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 \right\rangle = \int d^2\boldsymbol{x}_t \, d^2\boldsymbol{y}_t \, e^{-i\boldsymbol{q}_t \cdot (\boldsymbol{x}_t - \boldsymbol{y}_t)} \\ &\left[1 - \left(\frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] \right)^n - \left(\frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] \right)^n \right. \\ &\left. + \left(\frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] \exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] \right)^n \right] . \end{aligned} \tag{2.61}$$

The above expression is, however, highly oscillatory and unsuitable for numerical evaluation. One can make a standard approximation, valid for large number of scattering centers $n \gg 1$. By defining the density of sources $n_0 = n/V$ we observe

$$\begin{aligned} &\left(\frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \, \exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] \right)^n \\ &\qquad = \left(1 + \frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \right)^n \\ &\qquad = \left(1 + \frac{n_0 l}{n} \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \right)^n . \end{aligned} \tag{2.62}$$

Since the interaction vanishes at transverse distances larger than μ_d^{-1} then the integral is bounded so, for large enough n we can do

$$\begin{aligned} &\lim_{n\to\infty} \left(\frac{1}{\Omega} \int_\Omega d^2\boldsymbol{r}_t \, \exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] \right)^n \\ &\qquad = \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \right] . \end{aligned} \tag{2.63}$$

And finally

$$\begin{aligned} &\left\langle \left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 \right\rangle = \int d^2\boldsymbol{x}_t \, d^2\boldsymbol{y}_t \, e^{-i\boldsymbol{q}_t \cdot (\boldsymbol{x}_t - \boldsymbol{y}_t)} \\ &\qquad \times \left(1 - \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \right] \right. \\ &\qquad\quad - \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) \right] \\ &\qquad\quad \left. + \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) + i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) \right] \right). \end{aligned} \tag{2.64}$$

The above expression is already suitable for numerical evaluation. We notice, however, that it can be split into two parts which admit a clear physical interpretation. One part is related to the coherent scattering which can be interpreted as the scattering in an averaged medium, and the other to an incoherent contribution. To show this fact we use the relation $M = S - 1$ leading to

$$\left\langle M^{(n)}_{s_f s_i}(p_f, p_i)\left(M^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle = \left|\left\langle S^{(n)}_{s_f s_i}(p_f, p_i) - 1\right\rangle\right|^2$$
$$\left(\left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\left(S^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle - \left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\left\langle\left(S^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle\right), \qquad (2.65)$$

where we added and subtracted the term

$$\pm\left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\left\langle\left(S^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle, \qquad (2.66)$$

in order to find the well known result from statistics $\langle x^2\rangle = \langle x\rangle^2 + \sigma^2$. Two contributions appear with a clear physical interpretation. We will call these contributions the coherent and incoherent average, denoted as

$$\left\langle\left|M^{(n)}_{s_f s_i}(p_f, p_i)\right|^2\right\rangle = \Pi^{(n)}_2(p_f, p_i) + \Sigma^{(n)}_2(p_f, p_i), \qquad (2.67)$$

with the coherent contribution given by

$$\Pi^{(n)}_2(p_f, p_i) \equiv \left|\left\langle S^{(n)}_{s_f s_i}(p_f, p_i) - 1\right\rangle\right|^2 = \left|\left\langle M^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\right|^2, \qquad (2.68)$$

and the incoherent contribution being

$$\Sigma^{(n)}_2(p_f, p_i) \equiv \left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\left(S^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle - \left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\left\langle\left(S^{(n)}_{s_f s_i}(p_f, p_i)\right)^*\right\rangle. \qquad (2.69)$$

The first contribution consists in the square of the averaged amplitude. Correspondingly, as usual by dividing by incoming flux and time and factorizing the conservation deltas we define

$$\Pi^{(n)}_2(p_f, p_i) = 2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta\hat{\Pi}^{(n)}_2(\boldsymbol{q}, l), \qquad (2.70)$$

where the relevant coherent average is given by

$$\hat{\Pi}^{(n)}_2(\boldsymbol{q}, l) = \left|\left\langle F^{(n)}_{el}(\boldsymbol{q})\right\rangle\right|^2 \qquad (2.71)$$
$$= \left|\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\left(\exp\left[n_0 l\int_\Omega d^2\boldsymbol{r}_t\left(\exp\left[-i\frac{g}{\beta}\chi^{(1)}_0(\boldsymbol{x}_t - \boldsymbol{r}_t)\right] - 1\right)\right] - 1\right)\right|^2.$$

This contribution consists in a coherent superposition of each center at the level of the amplitude. We can integrate (2.71) for a cylinder whose transverse dimensions greatly exceed the dimensions of a single scatterer, $R \gg 1/\mu_d$. In this case

$$\int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t)\right] - 1\right) = F_{el}^{(1)}(\boldsymbol{0}). \tag{2.72}$$

Inserting this result in (2.71) we find the larger size approximation of the coherent contribution

$$\Pi_2^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_f s_i}\beta\left|(2\pi)^2\delta^2(\boldsymbol{q}_t)\left(\exp\left[n_0 l F_{el}^{(1)}(\boldsymbol{0})\right] - 1\right)\right|^2. \tag{2.73}$$

Due to symmetry arguments, the infiniteness of the medium transverse direction transforms the coherent part into a pure forward contribution. The values of the single scattering amplitude $F_{el}^{(1)}(\boldsymbol{0})$ in the forward direction appearing in the last equation must be related to the single elastic cross section

$$F_{el}^{(1)}(\boldsymbol{0}) = \frac{1}{2}\left(-\sigma_{tot}^{(1)} + i\sigma_{osc}^{(1)}\right), \tag{2.74}$$

and can be read from equations (2.54) and (2.55) for arbitrary coupling or from (2.52) and (2.53) at leading order provided that the single collision case is perturbative in g. In any case, we notice that this does not imply the single scattering regime, since n_0 is arbitrary. For mediums of finite size, however, the coherent term represents the modification of the single scattering matrix due to the border effects. In order to show this fact we write the relation

$$\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t)\right] = 1 + \frac{1}{(2\pi)^2}\int d^2\boldsymbol{k}_t\, e^{+i\boldsymbol{k}_t\cdot(\boldsymbol{x}_t - \boldsymbol{r}_t)} F_{el}^{(1)}(\boldsymbol{k}_t), \tag{2.75}$$

which integrated in a finite section Ω produces an extra window function

$$\int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t)\right] - 1\right) = \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} e^{+i\boldsymbol{k}_t\cdot\boldsymbol{x}_t} F_{el}^{(1)}(\boldsymbol{k}_t)\int_\Omega d^2\boldsymbol{r}_t e^{-i\boldsymbol{k}_t\cdot\boldsymbol{r}_t}. \tag{2.76}$$

The window function for the case of the finite cylinder acquires the form

$$W_\Omega(\boldsymbol{k}_t, R) \equiv \int_\Omega d^2\boldsymbol{r}_t e^{-i\boldsymbol{k}_t\cdot\boldsymbol{r}_t} \to W_{cyl}(\boldsymbol{k}_t, R) = \frac{2\pi R}{|\boldsymbol{k}_t|} J_1(|\boldsymbol{k}_t| R). \tag{2.77}$$

where $J_1(x)$ is the Bessel function of the first kind. For an arbitrary geometry, the window function momentum domain has an oscillatory behavior in $1/R$ modulating the $1/\boldsymbol{q}^4$ fall off. This can be directly observed at low density by expanding (2.71) in n_0 yielding

$$\Pi_2^{(n)}(p_1, p_0) = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_f s_i}\beta\left|n_0 l W_\Omega(\boldsymbol{q}_t, R) F_{el}^{(1)}(\boldsymbol{q}_t)\right|^2. \tag{2.78}$$

As expected for a coherent contribution this term is of order n^2, so at low density the averaged amplitude squared is n^2 times the squared elastic amplitude of a single center, with a window function containing information of the medium geometry. This window can be accounted as a border diffractive effect which factorizes from the single elastic amplitude only at low densities. For a cylinder medium of radius R, in particular, we find

$$\Pi_2^{(n)}(p_1, p_0) = 4n^2 \frac{J_1^2(|\boldsymbol{q}_t|R)}{\boldsymbol{q}_t^2 R^2} \left| M^{(1)}_{s_f s_i}(p_1, p_0) \right|^2 + O(n^4). \tag{2.79}$$

We proceed to compute now the incoherent contribution to the scattering amplitude. Using (2.23), the first term is given by

$$\left\langle S^{(n)}_{s_f s_i}(p_1, p_0) \left(S^{(n)}_{s_f s_i}(p_1, p_0) \right)^* \right\rangle = 2\pi\delta(p_1^0 - p_0^0)\delta_{s_f s_i}\beta \left\langle S^{(n)}_{el}(\boldsymbol{q}) S^{(n),*}_{el}(\boldsymbol{q}) \right\rangle, \tag{2.80}$$

where the incoherent average restricts to the integral parts

$$\begin{aligned} \left\langle S^{(n)}_{el}(\boldsymbol{q})\ S^{(n),*}_{el}(\boldsymbol{q}) \right\rangle = & \int d^2\boldsymbol{x}_t \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t - \boldsymbol{y}_t)} \\ & \left\{ \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) + i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) \right] \right. . \end{aligned} \tag{2.81}$$

It results convenient to rewrite the terms in the exponent as squared single amplitudes. To do that we notice

$$\begin{aligned} & \exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) + i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \\ = & \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) + \left(\exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) \\ & + \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \left(\exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right). \end{aligned} \tag{2.82}$$

By taking the infinite medium limit $R \gg 1/\mu_d$ the first two integrals are just forward single amplitudes,

$$\begin{aligned} n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) &= n_0 l F^{(1)}_{el}(\boldsymbol{0}) \\ n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) &= n_0 l F^{(1),*}_{el}(0), \end{aligned} \tag{2.83}$$

whereas for the mixed term we find, using (2.75)

$$\begin{aligned} & n_0 l \int_\Omega d^2\boldsymbol{r}_t \left(\exp\left[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \right] - 1 \right) \left(\exp\left[+i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t) \right] - 1 \right) \\ = & \, n_0 l \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} e^{+i\boldsymbol{k}_t\cdot(\boldsymbol{x}_t - \boldsymbol{y}_t)} \left| F^{(1)}_{el}(\boldsymbol{k}_t) \right|^2 . \end{aligned} \tag{2.84}$$

The second term of the incoherent contribution is given by

$$\left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\left\langle S^{(n),*}_{s_f s_i}(p_f, p_i)\right\rangle = 2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta\left\langle S^{(n)}_{el}(\boldsymbol{q})\right\rangle\left\langle S^{(n),*}_{el}(\boldsymbol{q})\right\rangle, \quad (2.85)$$

where similarly

$$\begin{aligned}\left\langle S^{(n)}_{el}(\boldsymbol{q})\right\rangle\left\langle S^{(n),*}_{el}(\boldsymbol{q})\right\rangle = &\int d^2\boldsymbol{x}_t \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t-\boldsymbol{y}_t)}\\ &\cdot \exp\Big[n_0 l \int_\Omega d^2\boldsymbol{r}_t\Big(\exp\Big[-i\frac{g}{\beta}\chi^{(1)}_0(\boldsymbol{x}_t-\boldsymbol{r}_t)\Big]-1\Big)\\ &+n_0 l \int_\Omega d^2\boldsymbol{r}_t\Big(\exp\Big[+i\frac{g}{\beta}\chi^{(1)}_0(\boldsymbol{y}_t-\boldsymbol{r}_t)\Big]-1\Big)\Big]. \end{aligned} \quad (2.86)$$

Proceeding in the same way, for a medium satisfying $R \gg \mu_d^{-1}$ we simply have

$$\begin{aligned}\left\langle S^{(n)}_{s_f s_i}(p_f, p_i)\right\rangle\left\langle S^{(n),*}_{s_f s_i}(p_f, p_i)\right\rangle =&2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta\int d^2\boldsymbol{x}_t \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t-\boldsymbol{y}_t)}\\ &\cdot\exp\Big[n_0 l F^{(1)}_{el}(\boldsymbol{0}) + n_0 l F^{(1),*}_{el}(\boldsymbol{0})\Big].\end{aligned} \quad (2.87)$$

Correspondingly after summing these two terms one finds

$$\Sigma^{(n)}_2(p_f, p_i) = 2\pi\delta(q^0)\delta_{s_f s_i}\beta\hat{\Sigma}^{(n)}_2(\boldsymbol{q}, l), \quad (2.88)$$

where the relevant incoherent average is given by

$$\begin{aligned}\hat{\Sigma}^{(n)}_2(\boldsymbol{q}, l) = &\exp\Big[2n_0 l\,\mathrm{Re}\,F^{(1)}_{el}(\boldsymbol{0})\Big]\int d^2\boldsymbol{x}_t d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{x}_t-\boldsymbol{y}_t)}\\ &\times\left\{\exp\Big[n_0 l\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}e^{+i\boldsymbol{k}_t\cdot(\boldsymbol{x}_t-\boldsymbol{y}_t)}\left|F^{(1)}_{el}(\boldsymbol{k}_t)\right|^2\Big]-1\right\}.\end{aligned} \quad (2.89)$$

The above form of the incoherent contribution is related to Moliere's theory of scattering [36, 37] since by using (2.43),

$$\begin{aligned}\hat{\Sigma}^{(n)}_2(\boldsymbol{q}, l) = &\,\Omega\exp\Big[-n_0 l\sigma^{(1)}_{tot}\Big]\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\\ &\cdot\left\{\exp\Big[n_0 l\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}e^{+i\boldsymbol{k}_t\cdot\boldsymbol{x}_t}\left|F^{(1)}_{el}(\boldsymbol{k}_t)\right|^2\Big]-1\right\},\end{aligned} \quad (2.90)$$

where the single cross section can be read from equation (2.52) for small coupling or from equation (2.54) for general coupling, and the trivial integral in one of the impact parameters has produced and overall factor $\Omega = \pi R^2$ accounting for transverse homogeneity when $R \to \infty$. Except for the -1, accounting for boundary effects and allowing the integration for screened interactions, (2.90) constitutes a solution of the Moliere equation. The -1 is strictly necessary to integrate in impact parameter for any screened interaction. The Fourier transform of the single amplitude $F^{(1)}_{el}(\boldsymbol{q})$ in the exponential at (2.90) will be denoted as

$$\sigma^{(1)}_{el}(\boldsymbol{x}) = \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} e^{+i\boldsymbol{k}_t\cdot\boldsymbol{x}_t} \left|F^{(1)}_{el}(\boldsymbol{k}_t)\right|^2 . \tag{2.91}$$

and can be numerically computed for arbitrary g using (2.28) representing the all orders interaction with each single center. The extension to n centers is just its exponentiation. For low coupling we can use

$$F^{(1)}_{el}(\boldsymbol{k}_t) = -\frac{i}{\beta}\frac{4\pi g}{\boldsymbol{k}_t^2+\mu_d^2} + O(g^2) \to \sigma^{(1)}_{el}(\boldsymbol{x}_t) = \frac{4\pi g^2}{\beta^2\mu_d^2}\mu_d|\boldsymbol{x}_t|K_1(\mu_d|\boldsymbol{x}_t|) + O(g^3). \tag{2.92}$$

which after inserted in (2.90) produces a suitable form for fast numerical calculations. Observe that we called this function $\sigma^{(1)}(\boldsymbol{x}_t)$ because it is related to the single cross section as $\sigma^{(1)}(\boldsymbol{0}) \equiv \sigma^{(1)}_{tot}$. An expansion in n_0 produces

$$\begin{aligned} \Sigma^{(n)}_2(p_f,p_i) &= \Omega 2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta \\ &\cdot \int d^2\boldsymbol{x}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\left\{n_0 l \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2}e^{+i\boldsymbol{k}_t\cdot\boldsymbol{x}_t}\left|F^{(1)}_{el}(\boldsymbol{k}_t)\right|^2\right\} + O(n_0^2) \\ &= (n_0\pi R^2 l)2\pi\delta(p^0_f - p^0_i)\delta_{s_f s_i}\beta\left|F^{(1)}_{el}(\boldsymbol{q}_t)\right|^2 + O(n_0^2) \equiv n\left|M^{(1)}_{s_f s_i}(p_f,p_i)\right|^2 + O(n^2). \end{aligned} \tag{2.93}$$

Only at low densities the probability of changing $\boldsymbol{q}_t$ due to the joint effect of n centers is just n times the probability of changing $\boldsymbol{q}_t$ due to a single center. This relation does not hold for higher densities and, in particular, for saturation densities an asymptotic form is going to be later defined. At low n and $R \gg 1/\mu_d$ we can join equations (2.78) and (2.93) leading to

$$\left\langle\left|M^{(n)}_{s_f s_i}(p_f,p_i)\right|^2\right\rangle \simeq \left\{n + 4n^2\frac{J_1^2(|\boldsymbol{q}_t|R)}{\boldsymbol{q}_t^2 R^2}\right\}\left|M^{(1)}_{s_f s_i}(p_f,p_i)\right|^2 . \tag{2.94}$$

The above result may have been obtained from a direct expansion in the number of centers. This approach will give us some insight on the interference pattern and the statistical interpretation of the coherent and incoherent contributions. Let us denote

$$\Gamma_i = \exp\Big(-i\frac{g}{\beta}\chi(\boldsymbol{x}_t - \boldsymbol{r}^i_t)\Big) - 1, \tag{2.95}$$

in such a way that the elastic amplitude can be rewritten as

$$\begin{aligned} F^{(n)}_{el}(\boldsymbol{q}) &= \int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\Big(\prod_{i=1}^n(\Gamma_i+1) - 1\Big) \\ &= \int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\Big(\sum_{i=1}^n\Gamma_i + \sum_{i=1}^n\sum_{j=i+1}^n\Gamma_i\Gamma_j + \cdots\Big) = \sum_{i=1}^n I_i + \sum_{i=1}^n\sum_{j=i+1}^n I_{ij} + \cdots , \end{aligned}$$

which is an expansion in the number of collisions. Consider first the terms where only one (Γ_i) collision appears in the amplitude. We are left with

$$\sum_{i=1}^{n} I_i(\boldsymbol{q}) = \sum_{i=1}^{n} \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \Big(\exp\Big(-i\frac{g}{\beta}\chi_0(\boldsymbol{x}-\boldsymbol{r}_i) \Big) - 1 \Big) = \Big(\sum_{i=1}^{n} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i} \Big) F_{el}^{(1)}(\boldsymbol{q}). \tag{2.96}$$

Any other term of higher order can be always written as a convolution with an additional phase factor of the form

$$I_{ij}(\boldsymbol{q}) = \int \frac{d^2\boldsymbol{q}_i}{(2\pi)^2} \frac{d^2\boldsymbol{q}_j}{(2\pi)^2} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i - i\boldsymbol{q}_j\cdot\boldsymbol{r}_j} F_{el}^{(1)}(\boldsymbol{q}_i) F_{el}^{(1)}(\boldsymbol{q}_j) (2\pi)^2 \delta^2(\boldsymbol{q}-\boldsymbol{q}_i-\boldsymbol{q}_j), \tag{2.97}$$

and so on. Now consider the squared amplitude, in this form it reads

$$\left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} I_i I_j^* + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=j+1}^{n} \Big(I_i I_{jk}^* + I_i^* I_{jk} \Big) + \cdots . \tag{2.98}$$

The first contribution is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} I_i I_j^* = \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \left| F_{el}^{(1)}(\boldsymbol{q}) \right|^2 = \Big(n + \sum_{i=1}^{n} \sum_{j\neq i}^{n} e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \Big) \left| F_{el}^{(1)}(\boldsymbol{q}) \right|^2 . \tag{2.99}$$

The interpretation of the two above terms is clear. The first term gives the contribution from n independent collisions, whereas the second term gives the interference, at this order, between them. Notice that at $\boldsymbol{q} = 0$ we have

$$\sum_{i=1}^{n} \sum_{j\neq i}^{n} e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \bigg|_{q=0} = n(n-1) \tag{2.100}$$

which is the constructive interference between the n centers. For $\boldsymbol{q} \neq 0$ this factor is not positive defined, so that one can not interpret it as a probability density. One can average, however, over center configurations obtaining, for a cylinder of section $\Omega = \pi R^2$

$$\left\langle \sum_{i=1}^{n} \sum_{j\neq i}^{n} e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \right\rangle = \frac{n(n-1)}{\Omega^2} \int_\Omega d^2\boldsymbol{r}_i \int_\Omega d^2\boldsymbol{r}_j e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} = 4n(n-1)\frac{J_1^2(qR)}{(qR)^2}. \tag{2.101}$$

In the $R \to \infty$ limit we retrieve the forward constraint since the contribution is given by $n(n-1)/R^2\delta(\boldsymbol{q})$. Thereby, the contribution to the scattering by decomposition in single scatterings is given by

$$\left| F_{el}^{(n)}(\boldsymbol{q}) \right|^2 \simeq \left\{ n + 4n(n-1)\frac{J_1^2(qR)}{(qR)^2} \right\} \left| F_{el}^{(1)}(\boldsymbol{q}) \right|^2 , \tag{2.102}$$

the first term being the independent (incoherent) sum of the n scattering centers and the second term the (coherent) interference between them, which results in the original scattering amplitude squared modulated by the Fourier transform of the shape of the medium, matching the result (2.94). The coherent term has interference peaks at $q = 0$ and $q = 5.14/R$. For large R the second term will be negligible for q larger than about $q = (4\pi nl/R)^{1/3}$. For the terms with two collisions in the amplitude we find

$$\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k=1}^{n}\sum_{l\neq k}^{n} I_{ij}I_{kl}^{*} = \int\frac{d^2\boldsymbol{q}_i}{(2\pi)^2}\int\frac{d^2\boldsymbol{q}_j}{(2\pi)^2}\int\frac{d^2\boldsymbol{q}_k}{(2\pi)^2}\int\frac{d^2\boldsymbol{q}_l}{(2\pi)^2}W_2(\boldsymbol{q}) \cdot F_{el}^{(1)}(\boldsymbol{q}_i)F_{el}^{(1)}(\boldsymbol{q}_j)F_{el}^{(1),*}(\boldsymbol{q}_k)F_{el}^{(1),*}(\boldsymbol{q}_l), \tag{2.103}$$

where the diffractive window function reads now

$$\begin{aligned} W_2(\boldsymbol{q}) &= \sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k=1}^{n}\sum_{l\neq k}^{n} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i-i\boldsymbol{q}_j\cdot\boldsymbol{r}_j+i\boldsymbol{q}_k\cdot\boldsymbol{r}_k+i\boldsymbol{q}_l\cdot\boldsymbol{r}_l} \\ &= \sum_{k=i=1}^{n}\sum_{l=j\neq i}^{n} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i}^{n}\sum_{l\neq k\neq j} e^{-i\boldsymbol{q}_i\cdot\boldsymbol{r}_i-i\boldsymbol{q}_j\cdot\boldsymbol{r}_j+i\boldsymbol{q}_k\cdot\boldsymbol{r}_k+i\boldsymbol{q}_l\cdot\boldsymbol{r}_l}, \end{aligned} \tag{2.104}$$

where again we have split the sum into a diagonal and a non diagonal part. The diagonal part is given by

$$\begin{aligned} W_2^{dg}(\boldsymbol{q}) &= \frac{n(n-1)}{\Omega^2}\int_\Omega d^2\boldsymbol{r}_i\int_\Omega d^2\boldsymbol{r}_j e^{-i\boldsymbol{r}_i\cdot(\boldsymbol{q}_i-\boldsymbol{q}_k)-i\boldsymbol{r}_j\cdot(\boldsymbol{q}_j-\boldsymbol{q}_l)} \\ &= \frac{n(n-1)}{\Omega^2}\frac{2\pi RJ_1(|q_i-q_k|R)}{|q_i-q_k|}\frac{2\pi RJ_1(|q_j-q_l|R)}{|q_j-q_l|}, \end{aligned} \tag{2.105}$$

which simplifies to $\delta(\boldsymbol{q}_i-\boldsymbol{q}_k)\delta(\boldsymbol{q}_j-\boldsymbol{q}_l)$ in the limit $R\to\infty$ producing

$$\left(\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k=1}^{n}\sum_{l\neq k}^{n} I_{ij}I_{kl}^{*}\right)^{dg} = \int\frac{d^2\boldsymbol{q}_i}{(2\pi)^2}\int\frac{d^2\boldsymbol{q}_j}{(2\pi)^2}F_{el}^{(1)}(\boldsymbol{q}_i)F_{el}^{(1)}(\boldsymbol{q}_j)(2\pi)^2\delta^2(\boldsymbol{q}-\boldsymbol{q}_i-\boldsymbol{q}_j), \tag{2.106}$$

that is, a convolution of two independent collisions with different centers leading to a total momentum change $\boldsymbol{q}$. The non diagonal part will contribute through convolutions of the same type to the forward amplitude. Therefore, the structure of the expansion in n is always split into a contribution which in the large medium limit $R \gg \mu_d^{-1}$ can be interpreted in probabilistic terms, corresponding to the expansion in n_0 of equation (2.90), and a contribution accounting for diffractive effects.

For general n and g the probability of finding the fermion with final momentum p_f is given necessarily by a numerical evaluation of the full expression (2.64) or by using the coherent (2.71) and incoherent (2.90) terms, easier to compute than (2.64). In Figure 2.4 the probability of finding an electron with transverse momentum $\boldsymbol{q}$ is shown and compared with the leading order in n approximation together with the incoherent contribution. One can check that the optical theorem

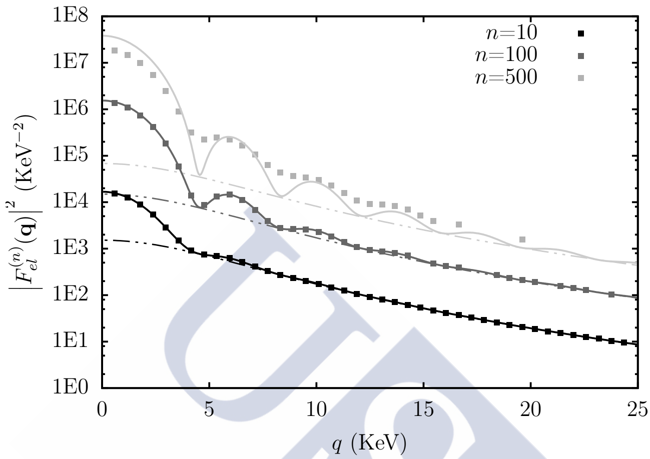


Figure 2.4: Squared elastic amplitude $\left|F_{el}^{(n)}(\boldsymbol{q})\right|^2$ after traversing a medium with $Z = 7$ corresponding to a coupling of $g = 0.05$, Debye mass of μ_d=71 KeV for increasing values of n centers distributed in cylinder of radius $R = 6r_d$ and length $l = R/10$. Dots are direct evaluation of (2.64), whereas continuum lines are the low density result (2.94) and dot-dashed lines are the incoherent contribution (2.90).

is valid also for the medium averaged cross sections. Indeed,

$$\sum_{s_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \left\langle \left| M^{(n)}_{s_f s_i}(p_f, p_i) \right|^2 \right\rangle = 2\pi\delta(p_i^0 - p_i^0)\delta_{s_i s_i}\beta \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \left\langle \left| F^{(n)}_{el}(\boldsymbol{q}) \right|^2 \right\rangle . \tag{2.107}$$

By using (2.64) we find

$$\int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \left\langle \left| F^{(n)}_{el}(\boldsymbol{q}) \right|^2 \right\rangle = \tag{2.108}$$

$$2\,\mathrm{Re} \int d^2\boldsymbol{x}_t \left(1 - \exp\left[n_0 l \int_\Omega d^2\boldsymbol{r}_t \Big(\exp\Big[-i\frac{g}{\beta}\chi_0^{(1)}(\boldsymbol{x}_t - \boldsymbol{r}_t) \Big] - 1 \Big) \right] \right),$$

so finally

$$\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \left\langle \left| M^{(n)}_{s_f s_i}(p_f, p_i) \right|^2 \right\rangle = 2\pi\delta(p_i^0 - p_i^0)\delta_{s_i s_i}\beta 2\,\mathrm{Re} \left\langle F^{(n)}_{el}(\boldsymbol{0}) \right\rangle$$
$$\equiv -2\,\mathrm{Re} \left\langle M^{(n)}_{s_i s_i}(p_i, p_i) \right\rangle . \tag{2.109}$$

This is remarkable, since the averaging at the level of the amplitudes is rather different than the averaging at the level of the cross sections. Similar result can be obtained by direct integration of the coherent and incoherent contributions (2.73) and (2.90) arriving to

$$\begin{aligned}\sigma^{(n)}_{tot} &= \sum_{s_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \left[\Pi^{(n)}_2(p_f, p_i) + \Sigma^{(n)}_2(p_f, p_i)\right] \\ &= 2\pi R^2 \,\mathrm{Re}\left[1 - \exp\left(n_0 l F^{(1)}_{el}(\mathbf{0})\right)\right] = 2\pi R^2 \left[1 - \exp\left(-\frac{n_0 l \sigma^{(1)}_{tot}}{2}\right)\cos\left(\frac{n_0 l \sigma^{(1)}_{osc}}{2}\right)\right].\end{aligned} \tag{2.110}$$

where we used the single cross section definitions (2.54) and (2.55). At small density we simply have

$$\sigma^{(n)}_{tot} = 2\pi R^2 \frac{n_0 l \sigma^{1}_{tot}}{2} + \cdots = n\sigma^{(1)}_{tot}, \tag{2.111}$$

as expected. In the limit of very large density, instead, we get $\sigma^{(n)}_{tot} = 2\pi R^2$, which is the correct high energy limit for the cross section of a black disk of radius R. Oscillations of (2.110) are a diffractive effect due to the coherent scattering of the entire medium. Indeed, for the split contributions to the total cross section we find in the incoherent part

$$\sigma^{(n)}_{inc} \equiv \sum_{s_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \Sigma^{(n)}_2(p_f, p_i) = \pi R^2 \left[1 - \exp\left(-\frac{n_0 l \sigma^{(1)}_{tot}}{2}\right)\right], \tag{2.112}$$

whereas for the coherent contribution

$$\begin{aligned}\sigma^{(n)}_{coh} &\equiv \sum_{s_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \Pi^{(n)}_2(p_f, p_i) \\ &= \pi R^2 \left[1 + \exp\left(-n_0 l \sigma^{(1)}_t\right) - 2\exp\left(-\frac{n_0 l \sigma^{(1)}_{tot}}{2}\right)\cos\left(\frac{n_0 l \sigma^{(1)}_i}{2}\right)\right].\end{aligned} \tag{2.113}$$

Oscillations in (2.110) with surface density $n_0 l$ due to a coherently acting medium have a period $n_0 l \sigma^{(1)}_{osc}/2$. Since $\sigma^{(1)}_{osc}$ is of order $O(g)$ whereas $\sigma^{(1)}_{tot}$ is of order $O(g^2)$ for small coupling the oscillations will be clearly seen in the total cross section. We see that for $n_0 l \approx 2/\sigma^{(1)}_{tot}$ the total cross section saturates. This defines a saturation scale given by

$$n_0 l = \frac{2}{\sigma^{(1)}_t} \approx \frac{\mu_d^2}{2\pi g^2}, \tag{2.114}$$

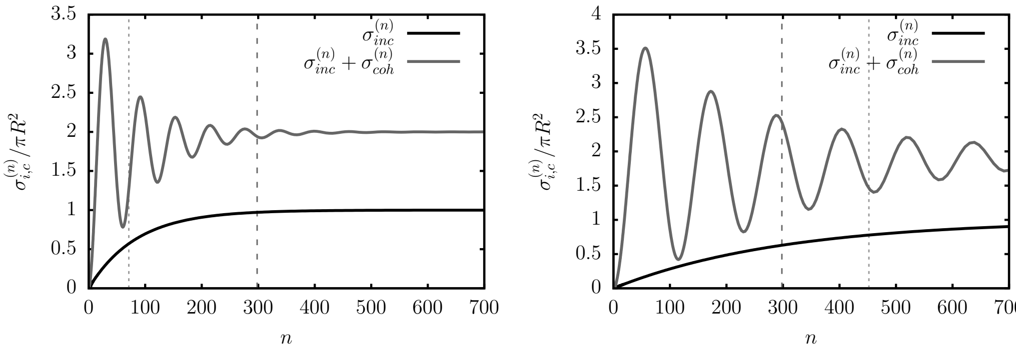


Figure 2.5: Total cross section in terms of the incoherent and coherent contributions as a function of the number of centers n for a medium with $g = 0.5$ (top) and $g = 0.25$ (bottom). Radius and length are chosen as $R = 6r_d$ and $l = 3r_d$, $\mu_d = \alpha_e m Z^{1/3}$. Short-dashed vertical lines correspond to the saturation scale n_{sat} whereas long-dashed vertical lines correspond to the degenerated limit n_0/r_d^3=1 in which doubled sources should be absorbed into a coupling redefinition.

where the last equality holds only for small coupling, or, using $n_0 = n/\pi R^2 l$,

$$n_{sat} = \frac{R^2 \mu_d^2}{2g^2}. \tag{2.115}$$

In Figure 2.5 we show the fully integrated total cross section as a function of the number of centers n, for a couple of values of the coupling g. For smaller values the oscillation is larger and lasts longer. The saturation value may be achieved independently of the degenerate scale, that is, the density at which centers start to overlap. This occurs more easily in mediums of small thickness l and coupling g, as expected.

2.5 Relation to Moliere's theory and Fokker-Planck approximation.

When the medium transverse dimension extends far beyond the dimensions of a single scatterer and, thus, can be considered infinite, and the coherent diffractive part can be neglected, the emergence of a infinite transverse symmetry has to lead to a diffusive behavior of the momentum distributions. Notice that since the oscillations are of width $1/R$, in the infinite transverse limit $R \gg \mu_d^{-1}$ the coherent term stops being observable and transforms into a pure forward component

$$\lim_{R\to\infty} \left\langle \left| M^{(n)}_{s_f s_i}(p_f, p_i) \right|^2 \right\rangle = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_f s_i}\beta\pi R^2 (2\pi)^2 \delta(\boldsymbol{q}_t^2) + \Sigma_2^{(n)}(p_f, p_i), \tag{2.116}$$

and the process is dominated by incoherent scattering. In order to find the cor-

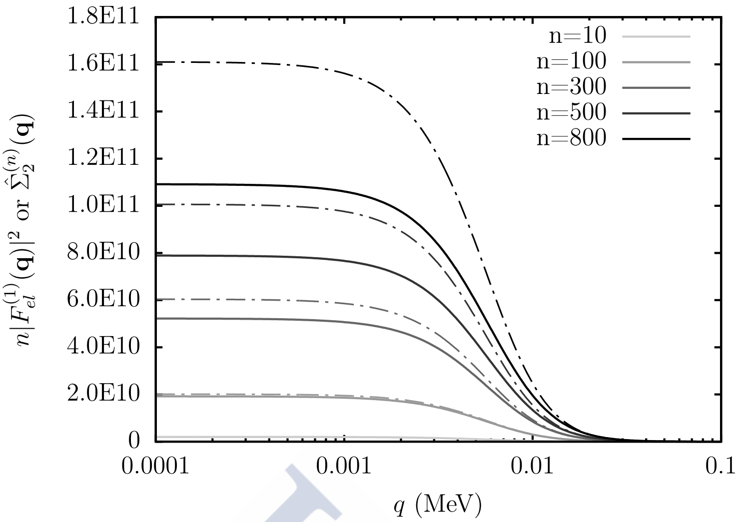


Figure 2.6: Incoherent (solid lines) and incoherent n_0 expansions (dot-dashed lines) contribution to the squared elastic amplitude as a function of $\boldsymbol{q}$ for various number of centers n for a medium with $Z = 10$. Radius and length are chosen as $R = 6r_d$ and $l = 3r_d$, $\mu_d = \alpha_e m Z^{1/3}$.

responding transport equation we take the derivative of (2.90) with respect to l obtaining

$$\frac{\partial}{\partial l}\Sigma_2^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\delta_{s_f s_i}\beta\left(\frac{\partial}{\partial l}\hat{\Sigma}_2^{(n)}(\boldsymbol{q}, l)\right), \tag{2.117}$$

By deriving we find

$$\begin{aligned}\frac{\partial}{\partial l}\hat{\Sigma}_2^{(n)}(\boldsymbol{q}, l) &= \frac{\partial}{\partial l}\left\{\Omega \exp\left[-n_0 l\sigma_{el}^{(1)}(\mathbf{0})\right]\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\left(\exp\left[n_0 l\sigma_{el}^{(1)}(\boldsymbol{x}_t)\right] - 1\right)\right\}\\ &= -n_0\sigma_{el}^{(1)}(\mathbf{0})\hat{\Sigma}_2^{(n)}(\boldsymbol{q}, l) + n_0\Omega\exp\left[-n_0 l\sigma_{el}^{(1)}(\mathbf{0})\right]\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\sigma_{el}^{(1)}(\boldsymbol{x}_t)\\ &\quad + n_0\Omega\exp\left[-n_0 l\sigma_{el}^{(1)}(\mathbf{0})\right]\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\sigma_{el}^{(1)}(\boldsymbol{x}_t)\left(\exp\left[n_0 l\sigma_{el}^{(1)}(\boldsymbol{x}_t)\right] - 1\right),\end{aligned} \tag{2.118}$$

where we subtracted and added a 1 in order to express the result in terms of already defined quantities. The first contribution produces a term proportional to $\Sigma_2^{(n)}(\boldsymbol{q}_t, l)$, whereas the second contribution can be expressed as

$$\exp\left[-n_0 l\sigma_{el}^{(1)}(\mathbf{0})\right]\int d^2\boldsymbol{x}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t} n_0\sigma_{el}^{(1)}(\boldsymbol{x}_t) = n_0\exp\left[-n_0 l\sigma_{el}^{(1)}(\mathbf{0})\right]\left|F_{el}^{(1)}(\boldsymbol{q})\right|^2. \tag{2.119}$$

The third contribution can be expressed, using the definition of $\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)$ and (2.91) and applying the convolution theorem, as

$$n_0\Omega \exp\Big[-n_0 l\sigma_{el}^{(1)}(\boldsymbol{0})\Big]\cdot\int d^2\boldsymbol{x}_t\, e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t}\sigma_{el}^{(1)}(\boldsymbol{x}_t)\Big\{\exp\Big[n_0 l\sigma_{el}^{(1)}(\boldsymbol{x}_t)\Big]-1\Big\}$$
$$= n_0\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q}-\boldsymbol{k},l). \tag{2.120}$$

Consequently, joining the three terms, one finds

$$\frac{\partial\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)}{\partial l} = n_0\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\Big(\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q}-\boldsymbol{k};l)-\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)\Big)$$
$$+\,n_0\Omega\exp\Big[-n_0 l\sigma_t^{(1)}\Big]\left|F_{el}^{(1)}(\boldsymbol{q})\right|^2.$$

The above equation tells us that the number of states with transverse momentum $\boldsymbol{q}$ at $l+\delta l$ is (a) fed with the number of states which were with transverse momentum $\boldsymbol{q}-\boldsymbol{k}$ at l and collided with amplitude $F_{el}^{(1)}(\boldsymbol{k})$ with the layer δl gaining $\boldsymbol{k}$, achieving total amount $\boldsymbol{q}$ and (b) decreased by the number of states which already were with momentum $\boldsymbol{q}$ at l and experimented a collision of amplitude $F_{el}^{(1)}(\boldsymbol{k})$ for any $\boldsymbol{k}$ in the layer δl and (c) the number of states which at $l+\delta l$ did not experiment any collision yet with the medium, so they acquire momentum $\boldsymbol{q}_t$ into their first collision at l. As expected, at large distances l the probability of finding this kind of events is negligible, since the third term is exponentially suppressed with l. For large deeps in the medium $l \gg 1/n_0\sigma_t^{(1)}$ then, one arrives to a Master equation

$$\frac{\partial\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)}{\partial l} = n_0\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}\Big\{\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q}-\boldsymbol{k},l)-\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)\Big\}. \tag{2.121}$$

By performing one of the trivial integrals in the momentum $\boldsymbol{k}$ one finds the total cross section of $n = 1$ center and correspondingly, the actual form of the Moliere's equation

$$\frac{\partial\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)}{\partial l} = -n_0\sigma_t^{(1)}\Sigma_2^{(n)}(\boldsymbol{q},l)+n_0\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\left|F_{el}^{(1)}(\boldsymbol{k})\right|^2\hat{\Sigma}_2^{(n)}(\boldsymbol{q}-\boldsymbol{k},l). \tag{2.122}$$

which was derived following homogeneity arguments to the form of any transport equation. Solution to this equation is given by (2.90) without the -1 term, which is responsible of the discarded boundary term. This term is important, however, for mediums of any size to achieve a convergent form for $\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)$. In the high density regime we notice that since

$$\nabla_t^2\sigma_{el}^{(1)}(\boldsymbol{x}_t)\Big|_{\boldsymbol{x}_t=\boldsymbol{0}} < 0, \tag{2.123}$$

then (2.90) can be asymptotically solved. The exponential term $n_0\sigma_{el}^{(1)}(\boldsymbol{x}_t)$ quickly decays in a short $\boldsymbol{x}$ range with respect to the momentum term $\boldsymbol{q}_t \cdot \boldsymbol{x}_t$, so we only need to know the short $\boldsymbol{x}_t$ behavior of $\sigma_{el}^{(1)}(\boldsymbol{x}_t)$ given by,

$$n_0 l \sigma_{el}^{(1)}(\boldsymbol{x}_t) = n_0 l \sigma_{el}^{(1)}(\boldsymbol{0}) - \frac{1}{2}\hat{q} l \boldsymbol{x}_t^2 + O(\boldsymbol{x}_t^4). \tag{2.124}$$

In this regime of high saturation of centers one easily finds using last expression and equation (2.90), a Fokker-Planck equation whose solution is a Gaussian distribution

$$\begin{aligned}\Sigma_2^{(n)}(\boldsymbol{q}, l) &= \int d^2\boldsymbol{x}\, e^{-i\boldsymbol{x}\cdot\boldsymbol{q}} \left\{ \exp\left[-\frac{1}{2}\hat{q} l \boldsymbol{x}^2\right] - \exp\left[-n_0 l \sigma_{el}^{(1)}(\boldsymbol{0})\right]\right\} \\ &= (2\pi)^2 \left\{ \frac{1}{2\pi\hat{q} l} \exp\left[-\frac{\boldsymbol{q}^2}{2\hat{q} l}\right] - \delta^{(2}(\boldsymbol{q}_t) \exp\left[-n_0 l \sigma_{el}^{(1)}(\boldsymbol{0})\right]\right\}. \end{aligned} \tag{2.125}$$

In this asymptotic region the -1 term is not needed anymore except to take into account border effects. However, since we are assuming $n_0 l \gg 1$ within this approximation we neglect the second term and write from here onwards

$$\Sigma_G^{(n)}(\boldsymbol{q}_t, l) = \frac{2\pi}{\hat{q} l} \exp\left[-\frac{\boldsymbol{q}_t^2}{2\hat{q} l}\right]. \tag{2.126}$$

The obtained distribution is now normalized meaning that the border term does not contribute. Under this approximation the averaged squared momentum after traversing a distance l verifies $\langle \boldsymbol{q}^2 \rangle = 2\hat{q} l$ but leads to a high suppression of the long $\boldsymbol{q}$ tail of the original distribution (2.90).

2.6 The p_t^2 average value

Let us compute the average value of the transverse momentum after a single collision. If the coupling $g = Ze^2$ is low enough to guarantee that the amplitude can be given at leading order we write

$$\sigma_{tot}^{(1)} = \int \frac{d^2\boldsymbol{p}_t}{(2\pi)^2} \frac{(4\pi g)^2}{(\boldsymbol{p}_t^2 + \mu_d^2)^2} = \frac{4\pi g^2}{\mu_d^2}, \tag{2.127}$$

so the average value is just

$$\langle \boldsymbol{p}_t^2 \rangle^{(1)} = \frac{1}{\sigma_{tot}^{(1)}} \int \frac{d^2\boldsymbol{q}_t}{(2\pi)^2}\, \boldsymbol{q}_t^2 \left|F_{el}^{(1)}(\boldsymbol{q}_t)\right|^2 = 4\pi\mu_d^2 \int \frac{d^2\boldsymbol{p}_t}{(2\pi)^2} \frac{\boldsymbol{p}_t^2}{(\boldsymbol{p}_t^2 + \mu_d^2)^2}. \tag{2.128}$$

In this optical approximation the integral is logarithmically divergent, since omitting the energy conservation delta in $M^{(1)}_{s_f s_i}(p_f, p_i)$ leads to an infinite range of perpendicular momentum. Using the energy conservation delta we actually find

$$\boldsymbol{q}^2 = (\boldsymbol{p}_1 - \boldsymbol{p}_0)^2 = \beta^2 E^2 + \beta^2 E^2 - 2\beta^2 E^2 \cos\theta = 4E^2 \sin^2\theta/2, \tag{2.129}$$

so we have

$$\begin{aligned}\langle \boldsymbol{p}_t^2 \rangle^{(1)} &\equiv \frac{1}{\sigma^{(1)}_{tot}} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} \, (\boldsymbol{p}_f - \boldsymbol{p}_i)^2 \left| M^{(1)}_{s_f s_i}(p_f, p_i) \right|^2 \\ &= \mu_d^2 \int d\theta \frac{\sin^3\theta/2 \cos\theta/2}{\left(\sin^2\theta/2 + \left(\frac{\mu_d}{2E}\right)^2\right)^2} = \mu_d^2 \left(2\log\left(\frac{2E}{\mu_d}\right) - 1 \right).\end{aligned} \tag{2.130}$$

In order to compute the momentum change after passing a length l through a medium we will consider that $R \gg \mu_d^{-1}$ so the coherent contribution reduces to a forward delta and it can be neglected. We have then

$$\langle \boldsymbol{p}_t^2 \rangle^{(n)} = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \left(\boldsymbol{q}^2 \hat{\Sigma}^{(n)}_2(\boldsymbol{q}, l) + \boldsymbol{q}^2 \hat{\Pi}^{(n)}_2(\boldsymbol{q}, l) \right) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \, \boldsymbol{q}^2 \hat{\Sigma}^{(n)}_2(\boldsymbol{q}, l). \tag{2.131}$$

We will look for a momentum additivity rule by defining a diffusion equation for $\langle \boldsymbol{q}^2(l) \rangle$. One finds, deriving with respect to depth l,

$$\begin{aligned}\frac{\partial}{\partial l} \langle \boldsymbol{p}_t^2 \rangle^{(n)} &= \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \, \boldsymbol{q}^2 \left(\frac{\partial}{\partial l} \hat{\Sigma}^{(n)}_2(\boldsymbol{q}^2, l) \right) = n_0 \exp\left(-n_0 l \sigma^{(1)}_t\right) \langle \boldsymbol{p}_t^2 \rangle^{(1)} \\ &- n_0 \sigma^{(1)}_t \langle \boldsymbol{p}_t^2 \rangle^{(n)} + n_0 \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \, \boldsymbol{q}^2 \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \, \left| F^{(1)}_{el}(\boldsymbol{k}^2) \right|^2 \hat{\Sigma}^{(n)}_2(\boldsymbol{q} - \boldsymbol{k}, l).\end{aligned} \tag{2.132}$$

By defining $\boldsymbol{q}' = \boldsymbol{q} - \boldsymbol{k}$ the above relation simplifies to

$$\begin{aligned}\frac{\partial}{\partial l} \langle \boldsymbol{p}_t^2 \rangle^{(n)} &= n_0 \sigma^{(1)}_t \left(1 - \exp\left[-n_0 l \sigma^{(1)}_t\right] \right) \langle \boldsymbol{p}_t^2 \rangle^{(1)} + n_0 \exp\left[-n_0 l \sigma^{(1)}_t\right] \langle \boldsymbol{p}_t^2 \rangle^{(1)} \\ &= n_0 \sigma^{(1)}_t \langle \boldsymbol{p}_t^2 \rangle^{(1)} \equiv 2\hat{q},\end{aligned} \tag{2.133}$$

which implies that the average squared momentum change is additive in the traveled length

$$\left\langle \boldsymbol{p}_t^2(l) \right\rangle = n_0 \sigma^{(1)}_t \langle \boldsymbol{p}_t^2 \rangle^{(1)} l = 2\hat{q} l. \tag{2.134}$$

Here, we can measure the length or depth into the medium l in units of the quantity $\lambda = 1/n_0 \sigma^{(1)}_t$. The number of times $\eta = l/\lambda$ we walk into the medium the average transverse momentum is just η times the transverse momentum change in one collision (2.130). Correspondingly the quantity $\eta = l/\lambda$ is interpretable as the average number of collisions and λ can be understood as the mean free path, that is, the average distance between two consecutive collisions. Notice that the attenuation exponents

$$P(l) = \exp\Big(- n_0 \sigma_t^{(1)} l \Big), \tag{2.135}$$

which appear in the normalization of the incoherent contribution (2.90) acquire now the sense of a probability distribution. Namely, the probability of penetrating into the medium a length l without undergoing any collision, so $1 - P(l)$ is just the normalization of the incoherent contribution, namely the probability of penetrating l and having, at least, one collision. The medium parameter

$$\hat{q} = \frac{1}{2} n_0 \sigma_t^{(1)} \langle \boldsymbol{p}_t^2 \rangle^{(1)}, \tag{2.136}$$

is known as the transport coefficient. For a screened Coulomb interaction and at lowest order in the coupling we find

$$\hat{q} \simeq 4\pi g^2 n_0 \left(\log\left(\frac{2E}{\mu_d} \right) - \frac{1}{2} \right), \tag{2.137}$$

which means that, in principle, it has to be fixed in terms of the initial energy. If inserted in the Gaussian approximation (2.126) the transport parameter $\hat{q}$ severely modifies the momentum distribution to compensate the exponentially suppressed $\boldsymbol{p}_t$ tail of the Gaussian distribution while matching the average squared momentum transfer.

2.7 Beyond eikonal scattering

We have given a picture of multiple scattering departing from a pure eikonal limit, in which the longitudinal dimension of the medium is never resolved since longitudinal momentum change verifies $q_3 = 0$. The optical phase (1.42) appearing in amplitude (2.12) can be approximated at leading order in q_3, since the slow q_3 dependence of a single interaction $A_0^{(1)}(\boldsymbol{q}) \simeq A_0^{(1)}(\boldsymbol{q}_t, 0)$ can be neglected compared with the rapid phase oscillations $q_3 y_3$ at large distances. This leads to a combination of step functions

$$\begin{aligned}\int_{-\infty}^{y_3} ds\, A_0^{(n)}(\boldsymbol{y}_t, s) &\simeq \sum_{i=1}^{n} \int_{-\infty}^{y_3} ds \int \frac{dq_3}{(2\pi)} e^{-iq_3(s-r_3^i)} \int \frac{d^2\boldsymbol{q}_t}{(2\pi)^2} e^{-iq\cdot(y_t - r_t^i)} \hat{A}_0^{(1)}(\boldsymbol{q}_t, 0) \\ &= \sum_{i=1}^{n} \Theta(y_3 - r_3^i) \chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t^i), \end{aligned} \tag{2.138}$$

where $\Theta(y)$ is the Heaviside step function. The above result says that the wave (1.42) at y_3 is affected by the set of centers at the left of y_3, and changes abruptly at the passage of each center. In this way it preserves the internal structure of the longitudinal organization of the medium although it looses the internal structure of the longitudinal dimensions of each center. We will now consider a set of $n_1 \equiv n(z_1)$ centers laying in a sheet of vanishing thickness $\delta z \to 0$ at coordinate z. The elastic amplitude of this sheet reads from (2.12)

$$M^{(n_1)}_{s_1 s_0}(p_1, p_0) = 2\pi\delta(q^0)\delta^{s_1}_{s_0}\beta \int d^3\boldsymbol{y}\, e^{-iq\cdot y} \frac{\partial}{\partial y_3} \exp\left(-i\frac{g}{\beta}\int_{-\infty}^{y_3} ds\, A_0^{(m)}(\boldsymbol{y})\right)$$
$$= 2\pi\delta(q^0)\delta^{s_1}_{s_0}\beta e^{-iq_3 z} \int d^2\boldsymbol{x}_t\, e^{-i\boldsymbol{q}_t\cdot \boldsymbol{x}_t} \left(\exp\left[-i\frac{g}{\beta}\sum_{i=1}^{m}\chi(\boldsymbol{y}_t - \boldsymbol{r}^i_t)\right] - 1\right), \qquad (2.139)$$

where the integration has been carried by parts. This amplitude reproduces the diffracted part of the wave, so it includes at least one collision. We can write for the total wave the total amplitude $S = M + 1$

$$S^{(n_1)}_{s_1 s_0}(p_1, p_0) = 2\pi\delta(q^0)\delta^{s_1}_{s_0}\beta e^{-iq_3 z} \int d^2\boldsymbol{y}_t\, e^{-i\boldsymbol{q}_t\cdot \boldsymbol{y}_t} \exp\left[-i\frac{g}{\beta}\sum_{i=1}^{m}\chi(\boldsymbol{y}_t - \boldsymbol{r}^i_t)\right], \qquad (2.140)$$

which leaves opened the possibility of non interacting at all in z. We can consider the medium as a set of n layers at $z_1, z_2, \ldots, z_n$ with $n(z_i)$ scattering centers. The total number of centers in the medium is then

$$\sum_{i=1}^{n} n(z_i) \equiv N. \qquad (2.141)$$

The amplitude of emerging with momentum p_n and spin s_n after traversing the n sheets from z_1 to z_n is given at high energies by the convolution

$$S^{(N)}_{s_n s_0}(p_n, p_0) \equiv \left(\prod_{i=1}^{n-1}\int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=0}^{n} S^{n(z_i)}_{s_{i+1} s_i}(p_{i+1}, p_i)\right), \qquad (2.142)$$

sum in repeated indices assumed. From this we simply get

$$M^{(N)}_{s_n s_0}(p_n, p_0) = S^{(N)}_{s_n s_0}(p_n, p_0) - S^{(0)}_{s_n s_0}(p_n, p_0). \qquad (2.143)$$

A path integral representation of the above amplitudes can be written. By integrating in internal momenta and summing over intermediating spins we obtain

$$S^{(N)}_{s_n s_0}(p_n, p_0) = 2\pi\delta(p^0_n - p^0_0)\delta^{s_n}_{s_0}\beta\left(\prod_{i=i}^{n-1}\int \frac{d^2\boldsymbol{p}^t_i}{(2\pi)^2}\right)$$
$$\left(\prod_{i=1}^{n}\int d^2\boldsymbol{x}^t_i \exp\left(-i\boldsymbol{q}_i\cdot \boldsymbol{x}_i - i\frac{g}{\beta}\sum_{j=1}^{n(z_i)}\chi(\boldsymbol{x}^t_i - \boldsymbol{r}^t_j)\right)\right), \qquad (2.144)$$

where $\boldsymbol{q}_i = \boldsymbol{p}_i - \boldsymbol{p}_{i-1}$ is a complete 3-momentum change and the energy conservation deltas have been used to fix the longitudinal components as $p_z \simeq p_0^0 - \boldsymbol{p}_t^2/(2p_0^0)$. We rewrite the terms according to

$$-i\sum_{i=1}^{n}\boldsymbol{q}_i\cdot\boldsymbol{x}_i = -i\boldsymbol{p}_n\cdot\boldsymbol{x}_n + i\sum_{i=1}^{n-1}\boldsymbol{p}_i\cdot\delta\boldsymbol{x}_i + i\boldsymbol{p}_0\cdot\boldsymbol{x}_1, \tag{2.145}$$

with $\delta\boldsymbol{x}_i \equiv \boldsymbol{x}_{i+1} - \boldsymbol{x}_i$. If we also define $\delta z_i = z_{i+1} - z_i$ we then obtain

$$S^{(N)}_{s_n s_0}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\delta^{s_n}_{s_0}\beta\left(\prod_{i=1}^{n}\int d^2\boldsymbol{x}_i^t\right)e^{-i\boldsymbol{p}_n\cdot\boldsymbol{x}_n + i\boldsymbol{p}_0\cdot\boldsymbol{x}_1}$$
$$\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\exp\left(+i\boldsymbol{p}_i^t\cdot\delta\boldsymbol{x}_i^t - i\frac{(\boldsymbol{p}_i^t)^2}{2p_0^0}\delta z_i - i\frac{g}{\beta}\sum_{j=1}^{n(z_i)}\chi(\boldsymbol{x}_i^t - \boldsymbol{r}_j^t)\right)\right). \tag{2.146}$$

Upon performing the momentum integrals and taking the $\delta z \to 0$ limit we find

$$\lim_{\delta z\to 0} S^{(N)}_{s_n s_0}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\delta^{s_n}_{s_0}\beta\int d^2\boldsymbol{x}_n^t\int d^2\boldsymbol{x}_1^t e^{-i\boldsymbol{p}_n\cdot\boldsymbol{x}_n + i\boldsymbol{p}_0\cdot\boldsymbol{x}_1}$$
$$\int_{\boldsymbol{x}_t(z_1)}^{\boldsymbol{x}_t(z_n)}\mathcal{D}^2\boldsymbol{x}_t(z)\,\exp\left(i\int_{z_1}^{z_n}dz\left(\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z) - \frac{g}{\beta}\sum_{k=1}^{n(z)}\chi_0^{(1)}(\boldsymbol{x}_t(z) - \boldsymbol{r}_t^i(z))\right)\right), \tag{2.147}$$

Notice that the evaluation of this amplitude for the most probable path is given by the Euler-Lagrange equation

$$\frac{d\boldsymbol{p}_t}{dz} = g\boldsymbol{\nabla}\left(\int_{-\infty}^{+\infty}dt\,A_0^{n(z)}(\boldsymbol{x} + \beta t)\right), \tag{2.148}$$

which is a classical trajectory approximation assuming a straight propagation. The average of the square of (2.143) over medium configurations can be written as before as the sum of an incoherent and a coherent contribution

$$\left\langle M^{(N)*}_{s_n s_0}(p_n, p_0)M^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle = \Sigma_2^{(N)}(p_n, p_0) + \Pi_s^{(n)}(p_n, p_0), \tag{2.149}$$

where the incoherent contribution is given by the quantity

$$\Sigma_2^{(N)}(p_n, p_0) = \left\langle\left(S^{(N)}_{s_n s_0}(p_n, p_0)\right)^* S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle - \left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle^*\left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle, \tag{2.150}$$

and the coherent contribution is given by the averaged amplitude squared

$$\Pi_2^{(N)}(p_n, p_0) = \left|\left\langle S^{(N)}_{s_n s_0}(p_n, p_0) - S^{(0)}_{s_n s_0}(p_n, p_0)\right\rangle\right|^2. \tag{2.151}$$

The cancellation of the longitudinal phases in the macroscopic limit, since the transverse momentum in the conjugated amplitude equals the transverse momentum in the amplitude, leads to an incoherent contribution of the form

$$\Sigma_2^{(N)}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\beta_p\delta_{s_0}^{s_n}\pi R^2 \prod_{i=1}^{n-1}\left(\int \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t}\right)$$
$$\times\left\{\prod_{i=1}^{n}\exp\left(\delta z_i n_0(z_i)\left(\sigma_{el}^{(1)}(\boldsymbol{x}_i^t) - \sigma_{el}^{(1)}(\boldsymbol{0})\right)\right) - \prod_{i=1}^{n}\exp\left(-\delta z_i n_0(z_i)\sigma_{el}^{(1)}(\boldsymbol{0})\right)\right\}, \tag{2.152}$$

where we divided by time $2\pi\delta(0) \equiv T$ and incoming flux β. Correspondingly the integration in internal momenta is trivial and the internal structure of the scattering distribution is lost. By taking the $\delta z_i \to 0$ limit we obtain

$$\Sigma_2^{(N)}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\beta\delta_{s_0}^{s_n}\pi R^2 \exp\left(-\sigma_{el}^{(1)}(0)\int_{z_1}^{z_n} dz\, n_0(z)\right)$$
$$\times\int d^2\boldsymbol{x}_t e^{-i(\boldsymbol{p}_n^t-\boldsymbol{p}_0^t)\cdot\boldsymbol{x}_t}\left(\exp\left(\sigma_{el}^{(1)}(\boldsymbol{x}_t)\int_{z_1}^{z_n} dz\, n_0(z)\right) - 1\right). \tag{2.153}$$

The evaluation of the coherent contribution follows the same steps. The average of the amplitude produces

$$\left\langle S_{s_n s_0}^{(N)}(p_n, p_0)\right\rangle = 2\pi\delta(p_n^0 - p_0^0)\beta\delta_{s_0}^{s_n}\prod_{i=1}^{n-1}\left(\frac{i\delta z_i}{2\pi p_0^0}\right)\prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_t^i\right) \tag{2.154}$$
$$\times\exp\left(-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t + i\sum_{i=1}^{n-1}\frac{p_0^0}{2}\left(\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\right)^2\delta z_i + \sum_{i=1}^{n}\delta z_i n_0(z_i)\pi_{el}^{(1)}(\boldsymbol{x}_i^t) + i\boldsymbol{p}_0^t\boldsymbol{x}_1^t\right),$$

where in analogy with the function $\sigma_{el}^{(1)}(\boldsymbol{x})$ for the incoherent contribution we defined the Fourier transform of the single elastic amplitude convoluted with the window function of the medium as

$$\pi_{el}^{(1)}(\boldsymbol{x}) \equiv \int \frac{d^2\boldsymbol{q}_t}{(2\pi)^2}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}F_{el}^{(1)}(\boldsymbol{q})W_\Omega(\boldsymbol{q}, R). \tag{2.155}$$

By taking the $\delta z \to 0$ limit the above average transforms into a path integral in the transverse plane where the time variable is the z position as

$$\left\langle S_{s_n s_0}^{(N)}(p_n, p_0)\right\rangle = 2\pi\delta(p_n^0 - p_0^0)\beta_p\delta_{s_0}^{s_n}\int d^2\boldsymbol{x}_n^t e^{-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t}\int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\boldsymbol{x}_1^t}$$
$$\int \mathcal{D}^2\boldsymbol{x}_t(z)\exp\left(i\int_{z_1}^{z_n} dz\left(\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z) - in_0(z)\pi_{el}^{(1)}\left(\boldsymbol{x}_t(z)\right)\right)\right). \tag{2.156}$$

Then the beyond eikonal evaluation of the coherent contribution is given by

$$\Pi_2^{(N)}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\beta\delta_{s_0}^{s_n}\Bigg| \int d^2\boldsymbol{x}_n^t e^{-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t} \int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\boldsymbol{x}_1^t} \tag{2.157}$$
$$\times \int \mathcal{D}^2\boldsymbol{x}_t(z) \exp\left(i\int_{z_1}^{z_n} dz\, \frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z)\right)\left(\exp\left(\int_{z_1}^{z_n} dz\, n_0(z)\pi_{el}^{(1)}\big(\boldsymbol{x}_t(z)\big)\right) - 1\right)\Bigg|^2,$$

where we divided by time and incoming flux. The above beyond eikonal results will become useful when evaluating emission processes occurring in a multiple scattering scenario. As we will see in the next chapter, when an energy gap is considered between the state of the traveling particle and its conjugate, corresponding to an energy carried by the emitted particle, a non vanishing longitudinal phase will lead to coherence effects in the intensity spectrum.

3

High energy emission

In the previous chapters we worked out the basic concepts and tools towards an evaluation of the amplitude and the intensity of a high energy fermion transiting from (p_α, s_α) to (p_β, s_β) due to the effect of the multiple scattering sources in a medium. We will consider now a single photon bremsstrahlung occurring while this multiple scattering develops. For this purpose we take advantage of the fact that, for sufficiently low photon energies, the elastic amplitude has to factorize from the emission amplitude [5]. In consequence, as we will show, our previous results dealing with the pure elastic problem have to be recovered and be still valid for the evaluation of the radiation of soft quanta off high energy fermions.

The emission intensity in a multiple scattering scenario has been predicted by Ter-Mikaelian [1] and Landau and Pomeranchuk [2] to be suppressed with respect to a naive picture consisting in an incoherent sum of single Bethe-Heitler [6] intensities. Longitudinal phases in the scattering amplitudes regulate the amount of matter which can be considered a single and independent emitter. These phases control the coherence in the sum of all the involved Feynman diagrams and grow with the distance, with the photon energy and with the accumulated angle of the electron-photon pair. When the coherence length extends beyond one scattering length, comprising several collisions on average, the total intensity is not anymore the sum of the single Bethe-Heitler intensities at each collision and, thus, the intensity is substantially reduced. This effect, known as the Landau-Pomeranchuk-Migdal (LPM) suppression, leads to a stopping/energy-loss by bremsstrahlung in condensed media significatively smaller than the one expected for a set of unrelated collisions. The implications of this suppression spanned several fields of high energy physics [3, 38–40] and are still open nowadays. Although the first indirect measurements of this suppression were early given through cosmic ray showers [41] soon after its prediction, it was not until recently that various experiments, first at SLAC [42–44] and then at CERN [45, 46], measured in detail the phenomenon. Recent and renewed interest has also suscited the LPM suppression of gluons in the presence of the QCD media produced in high energy collisions at RHIC and LHC. The formation of new states of the matter, namely the quark-gluon plasma (QGP), are studied, among other probes, through the energy loss pattern of partons while traveling by this colored QCD medium[1].

Owing to the amount of work behind this subject it becomes necessary to introduce a brief historical remark. The first evaluation of this effect was given by Landau [3] with a classical calculation for a medium of semi-infinite length. In his work, the external field interaction is replaced with the Fokker-Planck approximation, leading to a total squared momentum transfer satisfying a Gaussian distribution. The computation of Landau produces a differential energy intensity which vanishes as $\sqrt{\omega}$ in the soft limit $\omega \to 0$ while approaches the incoherent sum of Bethe-Heitler intensities for larger ω. Landau's evaluation was later extended by Migdal [4] to the quantum case by means of a Boltzmann transport equation for an averaged target. As it was, maybe, better and later explained in the rederivation by Bell [47], Migdal's result shown that, except for spinorial corrections in the hard part of the photon spectrum, the LPM suppression is still a classical effect, in correspondence with the classical behavior of the infrared divergence [5, 6].

Following these two seminal works, a finite target evaluation was soon introduced first by Ternovskii [48] by using Migdal's transport approach, and later by Shul'ga and Fomin [49] through the classical limit. In both works radiation is computed in the softest regime $\omega \to 0$ and, unlike to the infinite suppression predicted by Landau and Migdal, the resulting intensity is found to saturate into a Bethe-Heitler power-like law in which the medium coherently acts as a single radiation entity. Both results, however, are only given for this coherent limit and in the saturation regime, in which the electron momentum transfer at the end of the process greatly exceeds its mass in magnitude. As we will later see, this coherent regime, which was not considered neither by Landau nor by Migdal, can be treated as part of a more general picture, corresponding to the Weinberg's soft photon theorem [5].

Very diverse and more recent calculations have appeared since then. We cite the work of Blankenbecler and Drell [50–53], who established a formalism in which the finite target case can be generally computed in the Fokker-Planck approximation. They adopted an approach in which a double integral in longitudinal position accounts for the photon emission point, both in the amplitude and its conjugate, and then the relevant quantity modulating the interference behavior is

[1] A more detailed study of the LPM effect and its phenomenology in QCD will be given in the next chapters.

just the accumulated longitudinal momentum change between these two points. Their result is given for arbitrary lengths, in such a way that, asymptotically for very low ω and in the large length limit, they recover Landau's and Migdal's predictions except for constant factors. Simultaneously to their work, and with the scope of developing a bremsstrahlung formalism for QCD matter, Zakharov [20, 54, 55] also treated the semi-infinite case in the Fokker-Planck approximation by reformulating Migdal's transport approach through path integrals in the transverse plane, leading to the same result. In his approach, however, the finite target implementation [54] is not easy and requires a different treatment. In the same line the BDMPS group [22] following a transport approach very similar to the one by Migdal and, omitting the collinear divergence introduced by the neglection of the electron mass (leading to a fail in reproducing the Bethe-Heitler limit) found a result which was shown [56] to be equivalent to the one by Zakharov in the soft limit. We have to cite also the works of Baier and Katkov [57, 58], who put some focus into accounting the Coulomb corrections to the Fokker-Planck approximation, and also in computing the finite size case, among other interesting situations. Their finite size result is comparable to the one found by Ternovskii, but frequently the various cases of study lack a general formulation and require different treatments. Similar predictions were later given by Wiedemann and Gyulassy [19], extending Zakharov path integral result to account for the angular dependence and the finite size case in a very similar way to the finite size evaluation by Blankenbecler and Drell. For further details on the history behind these works extensive and interesting reviews [59, 60] have been written.

A comprehensive non-perturbative QFT description of the emission of quanta for the general case of a finite/structured target, which takes into account not only the LPM effect, but dielectric and transition radiation effects as a particular cases of the same phenomena, admitting an evaluation for general interactions beyond the Fokker-Planck approximation and which takes care of the angular distributions is still missing. Reasons for this situation are diverse and probably related to the complexity and particularities of the vast part of the calculations [59] and the usual approach of considering *ab initio* the intensity for an averaged target of infinite transverse size, instead of an adequate quantum mechanical definition at the level of the amplitude. As a result the diagrammatic structure is hidden, the transverse coherence effects are never considered and the soft photon limit and length constraints are often misunderstood.

Taking these considerations into account, in Section 3.1 we find the emission amplitude and we briefly explain the LPM effect by taking the simpler classical limit. The main purpose of this section is to relate the LPM effect and the Weinberg's soft photon theorem as part of the same coherence phenomena. With these ideas in mind we define then in Section 3.2 the quantum amplitude [61] in terms of the scattering amplitudes found in the previous chapter, with the aim of taking the square of this amplitude in Section 3.3 in order to evaluate the photon intensity. We show that the intensity can be split into a coherent and an incoherent contribution, related to transverse interference effects, just as we did with the pure elastic case. The coherent contribution will be found to be a pure quantum mechanical contribution which does not admit a statistical interpretation, whereas the incoherent contribution results into a mixed contribution which, in the infinite transverse size limit for the medium, reproduces the macroscopic/classical limit and thus admits an statistical interpretation. The resulting expression in the discrete limit, which is the central result of the present work, admits a numerical evaluation by Monte Carlo methods for general interactions, arbitrary medium lengths, angular distributions and medium effects in the photon dispersion relation. In Section 3.4 we take the continuous limit and we find that our result is equivalent to a path integral formulation which can be solved in the Fokker-Planck approximation. This result will be used as a check for the numerical evaluation of our discretized approach. Migdal's/Zakharov and BDMPS results are recovered as two particular cases.

3.1 Amplitude and the classical LPM effect

We consider the amplitude of emission of a photon due to the effect of the multiple scattering sources in a medium. Let the photon be considered free after the emission and let the 4-momentum be denoted by $k \equiv (\omega, \boldsymbol{k})$. Medium effects in the energy momentum relation or refractive index can be introduced by defining an effective photon mass m_γ. Then the photon's velocity reads

$$\beta_k = \sqrt{1 - m_\gamma^2/\omega^2}, \tag{3.1}$$

and the momentum can be written as $k = \omega(1, \beta_k \hat{\boldsymbol{k}})$, where $\hat{\boldsymbol{k}}$ is the unitary vector in the direction of $\boldsymbol{k}$. A photon with this momentum k and polarization λ is given by

$$A_\mu^\lambda(x) = \mathcal{N}(k)\epsilon_\mu^\lambda e^{ik\cdot x}, \tag{3.2}$$

where $\mathcal{N}(k) = \sqrt{4\pi/2\omega}$ is the normalization. The new electron state under the effect of this emitted photon $\psi_\gamma^{(n)}(x)$ can be always written as the superposition

$$\psi_\gamma^{(n)}(x) = \psi_i^{(n)} + e\int d^4y S_F^{(n)}(x-y)\gamma^\mu A_\mu^\lambda(y)\psi_\gamma^{(n)}(y), \tag{3.3}$$

where $\psi_i^{(n)}(y)$ is a solution to the electron state in absence of interaction with the emitted photon $A_\mu^\lambda(y)$, but subject to the interaction with the photons from the external field (1.13) of the medium. It verifies then

$$\left(i\gamma^\mu\partial_\mu - gA_0^{(n)}(x) - m\right)\psi_i^{(n)}(x) = 0, \tag{3.4}$$

where the external field $A_0^{(n)}(x)$ is given by (1.8) and $S_F^{(n)}(x)$ is the adequate propagator, given in particular by (1.30) at high energies. From the equation (3.3) we easily find the required relation

$$\left(i\gamma^\mu\partial_\mu - gA_0^{(n)}(x) - m\right)S_F^{(n)}(x-y) = \delta^4(x-y). \tag{3.5}$$

The wave (3.3) can be always expanded in the original basis of scattered states with the (n) sources. Correspondingly, the amplitude of finding the wave $\psi_\gamma^{(n)}(x)$ in the final scattered state $\psi_f^{(n)}(x)$ is given, using (3.3), by

$$\mathcal{S}_{em}^{(n)} = \int d^3\boldsymbol{x}\,\psi_f^{(n),\dagger}(x)\psi_i^{(n)}(x) + e\int d^3\boldsymbol{x}\,\psi_f^{(n),\dagger}(x)S_F^{(n)}(x-y)A_\mu^\lambda(y)\psi_\gamma^{(n)}(y). \tag{3.6}$$

The first term is just the pure elastic scattering contribution. Indeed by expanding the scattered states as in (2.3) and taking into account the unitarity of the S-matrix we just find

$$\int d^3\boldsymbol{x}\,\psi_f^{(n),\dagger}(x)\psi_i^{(n)}(x) = S_{s_f s_i}^{(n)}(p_f, p_i). \tag{3.7}$$

We are instead interested in the second contribution, which constitutes the radiative correction to the elastic scattering. By using (1.30), we find the conjugate relation

$$\bar{\psi}_f^{(n)}(y) \equiv i\int d^3\boldsymbol{x}\,\psi_f^{(n),\dagger}(x)S_F^{(n)}(x-y), \tag{3.8}$$

and since $\gamma_0^2 = 1$ then

$$\mathcal{S}_{em}^{(n)} = S_{s_f s_i}^{(n)}(p_f, p_i) - ie\int d^4y\bar{\psi}_f^{(n)}(y)\gamma^\mu A_\mu^\lambda(y)\psi_\gamma^{(n)}(y). \tag{3.9}$$

We then define the amplitude of going from (p_i, s_i) to (p_f, s_f) *while* emitting a photon, neglecting the pure elastic contribution, as the contribution

$$\mathcal{M}_{em}^{(n)} \equiv -ie\int d^4y\,\bar{\psi}_f^{(n)}(y)\gamma^\mu A_\mu^\lambda(y)\psi_i^{(n)}(y) + O(e^2), \tag{3.10}$$

where we expanded in the last step $\psi_\gamma^{(n)}(x) = \psi_i^{(n)}(x) + O(e)$, thus the above relation is only valid at leading order in $e = \sqrt{\alpha}$ for the interaction between the emitted photon and the electron. Evaluating (3.10) for a single source $(n = 1)$

leads to the well-known Bethe-Heitler bremsstrahlung amplitude [6]. We are interested, instead, in the phenomenology of the multiple scattering case, which leads to interference effects in the squared amplitude (3.10), as predicted by Ter-Mikaelian [1] and Landau and Pomeranchuk [2, 3]. In order to understand these interferences we make use of the classical behavior of the infrared divergence [5], i.e. since (3.10) has a pole in $\omega \to 0$ the number of photons diverges and a classical evaluation holds [62]. In that case we can replace the electron current by the classical path

$$J_k(x) \equiv \bar{\psi}_f^{(n)}(x)\gamma_k\psi_i^{(n)}(x) \to J_k(x) = v_k(t)\,\delta^3(\boldsymbol{x} - \boldsymbol{x}(t)), \tag{3.11}$$

where $\boldsymbol{v}(t) \equiv \dot{\boldsymbol{x}}(t)$ is the electron velocity. Then (3.10) acquires the classical form

$$\mathcal{M}_{em}^{(n)} = -ie\mathcal{N}(k)\int_{-\infty}^{+\infty} dt\left(\frac{\boldsymbol{k}}{\omega} \times \boldsymbol{v}(t)\right)\exp\left(i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}(t)\right), \tag{3.12}$$

where the polarization vector is assumed to be normalized and orthogonal to the photon momentum $\boldsymbol{\epsilon}^\lambda(k)\cdot\boldsymbol{k} = 0$. Equation (3.12) can be understood as a sum over all the instants t at which the photon can be emitted. To see this fact we discretize the electron trajectory [47] as $\boldsymbol{v}_j$ for $j = 1, \ldots, n_c + 1$ and piecewise path $\boldsymbol{x}_j = \boldsymbol{x}_{j-1} + \boldsymbol{v}_{j-1}(t_j - t_{j-1})$, where n_c is the number of collisions. From (3.12) we get

$$\mathcal{M}_{em}^{(n)} = e\mathcal{N}(k)\frac{1}{\omega}\sum_{j=1}^{n_c}\boldsymbol{\delta}_j\, e^{i\varphi_j}, \tag{3.13}$$

which is a sum of n_c single Bethe-Heitler amplitudes [6] of the form

$$\boldsymbol{\delta}_j \equiv \boldsymbol{k}\times\left(\frac{\boldsymbol{v}_{j+1}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_{j+1}} - \frac{\boldsymbol{v}_j}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_j}\right), \tag{3.14}$$

interfering with a phase $i\varphi_j \equiv i\omega t_j - i\boldsymbol{k}\cdot\boldsymbol{x}_j$. The evaluation of the square of (3.13) leads to a total emission intensity between the photon's solid angle Ω_k and $\Omega_k + d\Omega_k$ given by

$$\omega\frac{dI}{d\omega d\Omega_k} = \frac{e^2}{(2\pi)^2}\left(\sum_{i=1}^{n_c}\boldsymbol{\delta}_i^2 + 2\,\mathrm{Re}\sum_{i=1}^{n_c}\sum_{j=1}^{i-1}\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j e^{i\varphi_{ij}}\right), \tag{3.15}$$

where we have split the sum in a diagonal and non diagonal contribution. It is clear that the phase between two emission points controls the sum in (3.15). By now we assume for simplicity that $\beta_k = 1$, then the phase difference can be written as

$$i\varphi_j^i \equiv ik_\mu(x_j^\mu - x_i^\mu) = i\int_{z_i}^{z_j} dz\frac{k_\mu p^\mu(z)}{p_0} = i\omega(1-\beta_p)\int_{z_i}^{z_j} dz + i\omega\int_{z_i}^{z_j} dz\frac{\delta\boldsymbol{p}^2(z)}{2\beta_p p_0^2}, \tag{3.16}$$

where $p \equiv (p_0, \boldsymbol{p})$ is the electron 4-momentum, its velocity is then given by

$$\beta_p = \sqrt{1 - m_e^2/p_0^2}, \tag{3.17}$$

and $\delta \boldsymbol{p}^2(z)$ the accumulated momentum change at z as measured with respect to the photon direction $\hat{\boldsymbol{k}}$. This phase grows with ω, with the emission angle and of course with accumulated distance in general. When the phase between each two consecutive collisions satisfies $\varphi_j^{j+1} \gg 1$ then the high oscillatory behavior cancels the non-diagonal term in (3.15) and

$$\omega \frac{dI_{sup}}{d\omega d\Omega_k} = \frac{e^2}{(2\pi)^2} \left(\sum_{i=1}^{n_c} \boldsymbol{\delta}_i^2 \right), \tag{3.18}$$

which is a totally incoherent sum of n_c single Bethe-Heitler [6] intensities. In this regime all the available deflections of the electron emit as independent sources of bremsstrahlung. This occurs at frequencies larger than ω_s which by using the condition $\langle \varphi_j^{j+1} \rangle \simeq 1$ and Eqs. (3.16) and (2.134) produces

$$\omega_s \simeq \frac{1}{\lambda} \frac{p_0^2}{m_e^2 + \hat{q}l}, \tag{3.19}$$

where λ is the average distance between two consecutive collisions or the mean free path, l the medium length and $l \gg \lambda$ was assumed. In this limit the radiation behaves as if the n_c scattering centers were separated an infinite distance and the total bremsstrahlung cross section is just the sum of the individual cross sections. On the contrary, if the phase is minimal so it satisfies $\varphi_0^{n_c} \ll 1$ we find from (3.13)

$$\omega \frac{dI_{inf}}{d\omega d\Omega_k} = \frac{e^2}{(2\pi)^2} \left(\sum_{i=1}^{n_c} \boldsymbol{\delta}_i \right)^2, \tag{3.20}$$

which is a Bethe-Heitler intensity with initial velocity $\boldsymbol{v}_1$ and final velocity $\boldsymbol{v}_{n_c+1}$, thus corresponding to the coherent deflection with all the medium. Indeed for negligible phases one finds

$$\sum_{i=1}^{n_c} \boldsymbol{\delta}_i = \boldsymbol{k} \times \left(\frac{\boldsymbol{v}_{n_c+1}}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}_{n_c+1}} - \frac{\boldsymbol{v}_1}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}_1} \right), \tag{3.21}$$

and the internal structure of scattering becomes irrelevant for the process. This occurs for frequencies lower than ω_c, where this frequency is given by $\langle \varphi_0^{n_c} \rangle \simeq 1$ thus by using (2.134) and (3.16) we find

$$\omega_c \simeq \frac{1}{l} \frac{p_0^2}{m_e^2 + \hat{q}l}. \tag{3.22}$$

In this regime the only independent source of bremsstrahlung is the medium as a whole, and the emitted photons are the first and the last ones, which is nothing but the soft photon theorem [5][2]. The LPM effect refers, then, to the suppression from the incoherent plateau (3.18) of maximal intensity to the coherent plateau (3.20) of minimal intensity, and both regimes are given by a Bethe-Heitler like law. Observe that by using (3.19) and (3.22) the ratio of the characteristic frequencies is just the average number of collisions, $\omega_s/\omega_c = n_c$. A schematic representation of this behavior is depicted in Figure 3.1. It results interesting to

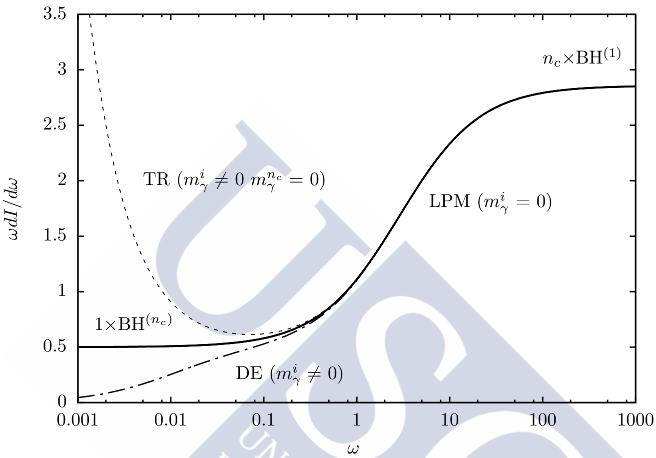


Figure 3.1: Schematic representation of the regimes of the radiation and the LPM effect (solid line), the lower plateau is given by (3.20) and the upper plateau is given by (3.18). Also shown are the dielectric effect (dot dashed line) and the transition radiation effect (dotted line).

relate this phase interference with the uncertainty relation between longitudinal distance and longitudinal momentum change. Phases and denominators at (3.13) agree with the infrared pole of the off-shell electron propagator,

$$(p+k)^2 - m^2 = p^2 + k^2 + 2p \cdot k - m^2 = 2p_\mu k^\mu, \tag{3.23}$$

By rewriting the above relation in terms of the elastic momentum change of an off-shell electron $q = (0, \boldsymbol{q})$ with $q = p_f + k - p_i$ we find

$$(p+q) - m^2 = p^2 + q^2 + 2p \cdot q - m^2 \approx 2p \cdot q = -2\boldsymbol{p} \cdot \boldsymbol{q}, \tag{3.24}$$

where we used the fact that at high energies $\boldsymbol{p} \gg \boldsymbol{q}$. Following this, phase (3.16) agrees, at high energies, with the accumulation of momentum change in

[2]In some LPM literature this is known as the Ternovskii-Shul'ga-Fomin [48, 63] term, however their result restricts to the saturation regime $n_c \gg 1$ under the Fokker-Planck approximation for the distribution of $\boldsymbol{v}_i$.

the dominant direction of $\boldsymbol{p}$, which can be taken along the z direction and we can write

$$i\varphi_j^i = -i\int_{t_i}^{t_j} dt\hat{\boldsymbol{p}}\cdot\boldsymbol{q}(t) = -i\int_{t_i}^{t_j} dt\, q_z(t). \tag{3.25}$$

We assume for simplicity q_z constant in the following interpretation. Then when $1/q_z$ becomes larger than the medium length l the medium is seen as a infinitely thin sheet, being impossible for the traveling electron and photon pair to resolve its interior structure. On the other hand, when $1/q_z$ is of the order or smaller than l, the internal medium structure in the distance $1/q_z$ can not be resolved and the amplitude is the incoherent sum of its coherent lq_z parts. This sum saturates when $1/q_z$ is of the order of the average distance between scatterings.

On the other hand medium effects in the photon dispersion relation severely change this picture in the soft limit. In that case the emitted photon field can be considered to satisfy a Helmholtz equation with a source term

$$\left(\nabla^2 - \frac{\partial^2}{\partial^2 x_0}\right)\boldsymbol{E}(x) = 4\pi\frac{\partial \boldsymbol{J}(x)}{\partial x_0}, \tag{3.26}$$

where the medium current is given by $\boldsymbol{J} = n_0 Ze\dot{\boldsymbol{r}}$. Then by using $m_e\ddot{\boldsymbol{r}} = e\boldsymbol{E}$ we find

$$\omega^2 - \boldsymbol{k}^2 = \frac{4\pi n_0 Ze^2}{m_e} \equiv m_\gamma^2. \tag{3.27}$$

Since $k = \omega(1, \beta_k\hat{\boldsymbol{k}})$ and $p = p_0(1, \beta_p\hat{\boldsymbol{p}})$ we can write for denominators at (3.14) and for the phase (3.16)

$$\frac{k_\mu p^\mu}{p_0} = \omega(1-\beta_k\beta_p\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) = \omega(1-\beta_k) + \omega\beta_k(1-\beta_p\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}). \tag{3.28}$$

For the energies considered here we will assume always $\omega \gg m_\gamma$. Then, with the required accuracy, we notice that β_k can be made unity in the second term of the right hand side but $(1-\beta_k) \simeq m_\gamma^2/2\omega^2$ so

$$\left.\frac{k_\mu p^\mu}{p_0}\right|_{m_\gamma} \simeq \frac{m_\gamma^2}{2\omega} + \omega(1-\beta_p\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}) = \frac{m_\gamma^2}{2\omega} + \left.\frac{k_\mu p^\mu}{p_0}\right|_{m_\gamma=0}. \tag{3.29}$$

This phase difference further suppresses the coherent plateau at frequencies lower than ω_{de}, when the extra term $m_\gamma^2/2\omega$ becomes of the order of the massless term, i.e. $\omega_{de} \simeq m_\gamma^2 l\omega_c$, since for frequencies lower than ω_{de} (3.14) vanishes. This effect is known as the dielectric suppression. However, if this photon mass becomes local in z due to an structured or a finite target, then the correction $m_\gamma^2/2\omega$ changes for each (3.14). In particular, for finite and homogeneous mediums, one can consider that the first and all the interior photons have $m_\gamma \neq 0$ whereas the last photon propagates in the vacuum and thus $m_\gamma = 0$. By using (3.20) this leads to a big resonance difference in the coherent plateau for frequencies lower than ω_{de} and the intensity is dramatically enhanced. This effect is called transition radiation. Both the dielectric suppression and the transition radiation effect are qualitatively shown in Figure 3.1.

3.2 Amplitude in the quantum approach

Since the number of photons diverges at both regimes, Eqs. (3.20) and (3.18), it is clear that, except for hard photon corrections, the previous classical behavior has to be recovered in a quantum evaluation. The electron states entering (3.10) are given by the superposition (2.3)

$$\psi_i^{(n)}(x) = \psi_i^{(0)}(x) + \sum_{s=1,2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} u_s(p) \sqrt{\frac{m_e}{p^0}} e^{-ip\cdot x} M_{ss_0}^{(n)}(p, p_0; z, z_0), \qquad (3.30)$$

for the incoming wave and

$$\psi_f^{(n)}(x) = \psi_f^{(0)}(x) + \sum_{s=1,2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} u_s(p) \sqrt{\frac{m_e}{p^0}} e^{-ip\cdot x} M_{ss_n}^{(n)}(p, p_n; z, z_n), \qquad (3.31)$$

for the outcoming wave, where the elastic amplitudes $M_{s_a s_b}^{(n)}(p_a, p_b; z_a, z_b)$ are beyond eikonal, and thus ordered, evaluations of the scattering amplitudes (2.143) to keep track of the accumulated longitudinal momentum change, as suggested by (3.25). The existence of the unscattered states $\psi_i^{(0)}(x)$ and $\psi_f^{(0)}(x)$ in all the process produces in (3.10) a term of the form

$$\begin{aligned} \mathcal{M}_{em}^{(0)} &\equiv -ie \int d^4y\, \bar{\psi}_f^{(0)}(y)\gamma^\mu A_\mu^\lambda(y)\psi_i^{(0)}(y) \\ &= -ie\mathcal{N}(k)(2\pi)^4\delta^4(p_n + k - p_0) \sqrt{\frac{m}{p_n^0}} \bar{u}_{s_n}(p_n)\epsilon_\mu^\lambda\gamma^\mu u_{s_0}(p_0) \sqrt{\frac{m}{p_0^0}} = 0, \end{aligned} \qquad (3.32)$$

which corresponds to the vacuum emission diagram and vanishes due to energy momentum conservation. Then it has to be explicitely removed, as we would do in a perturbative evaluation, in order to avoid the divergences inserted by its four Dirac deltas. We define, then, the amplitude of emission of a photon *while* interacting at least once with the medium. By inserting (3.30) and (3.31) in (3.10) and subtracting (3.32) we get

$$\mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} = -ie\mathcal{N}(k) \sum_{s,s'} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \left\{ \sqrt{\frac{m}{p_0^0 - \omega}} \bar{u}_s(p) \epsilon^\lambda_\mu \gamma^\mu u_{s'}(p+k) \sqrt{\frac{m}{p_0^0}} \right\}$$
$$\times \int_{-\infty}^{+\infty} dz\, S^{(n)}_{s_n s}(p_n, p; z_n, z) \exp\Big(- iq(p, p+k)z \Big) S^{(n)}_{s' s_0}(p+k, p_0; z, z_0) - \mathcal{M}^{(0)}_{em}. \tag{3.33}$$

The quantity appearing in the phase at z is the longitudinal momentum change at the emission vertex, which reads

$$\begin{aligned} q(p, p+k) &\equiv p^z_{p_0^0-\omega}(\boldsymbol{p}_t) + k^z_\omega(\boldsymbol{k}_t) - p^z_{p_0^0}(\boldsymbol{p}_t + \boldsymbol{k}_t), \\ q(p-k, p) &\equiv p^z_{p_0^0-\omega}(\boldsymbol{p}_t - \boldsymbol{k}_t) + k^z_\omega(\boldsymbol{k}_t) - p^z_{p_0^0}(\boldsymbol{p}_t), \end{aligned} \tag{3.34}$$

where we explicitely subscripted the energy of $\boldsymbol{p}$ and $\boldsymbol{k}$ in the z components of the momenta. This produces in the high energy limit

$$q_z(p, p+k) \simeq -\frac{\omega}{2p_0^0(p_0^0 - \omega)} \left(m_e^2 + \left(\boldsymbol{p}_t - \frac{p_0^0 - \omega}{\omega} \boldsymbol{k}_t \right)^2 \right) = -\frac{k_\mu p^\mu}{p_0^0}, \tag{3.35}$$

if we carry the integration in $\boldsymbol{p}$ at (3.33) in the leg after the emission, of modulus $\beta p^0 = \beta(p_0^0 - \omega)$, or

$$q_z(p-k, p) \simeq -\frac{\omega}{2p_0^0(p_0^0 - \omega)} \left(m_e^2 + \left(\boldsymbol{p}_t - \frac{p_0^0}{\omega} \boldsymbol{k}_t \right)^2 \right) = -\frac{k_\mu p^\mu}{p_0^0 - \omega}, \tag{3.36}$$

had we chosen $\boldsymbol{p}$ in the leg just before the emission, of modulus $\beta p^0 = \beta p_0^0$. Relations (3.35) and (3.36) are just the pole of the off-shell fermionic propagator, after and before the emission point, respectively. As seen from (3.34), in the high energy limit these two cases are related by a shift in the transverse plane $\boldsymbol{p}_t \to \boldsymbol{p}_t \pm \boldsymbol{k}_t$, which equals to cross the emission vertex in one or another direction. We observe at equation (3.33) that the electron can penetrate till any depth z into the medium with amplitude $S^{(n)}_{s' s_0}(p+k, p_0; z, z_0)$ given by (2.142) and energy p_0^0. This amplitude leaves open the possibility that the electron can interact or not with any center between z_0 to z. At that point it emits a photon, loosing a momentum k and thus energy ω, and changing spin from s' to s, the amplitude of this vertex factorizing as

$$f^\lambda_{ss'}(p, p+k) \equiv \sqrt{\frac{m_e}{p_0^0 - \omega}} \bar{u}_s(p) \epsilon^\lambda_\mu \gamma^\mu u_{s'}(p+k) \sqrt{\frac{m_e}{p_0^0}}, \tag{3.37}$$

and continues propagating elastically till z_n with energy $p_0^0-\omega$, interacting or not, with amplitude $S^{(n)}_{s_n s}(p_n, p; z_n, z)$. Finally, we sum over emission points dz, which automatically inserts the adequate poles of the fermion propagators $(k_\mu p^\mu)^{-1}$ at the phase. This is clearly seen by integrating by parts, in which case one gets from (3.33) the alternative expression

$$\begin{aligned}\mathcal{M}^{(n)}_{em}-\mathcal{M}^{(0)}_{em} &= -e\mathcal{N}(k)\sum_{s,s'}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\int_{-\infty}^{+\infty} dz\,\exp\Big(-iq(p,p+k)z\Big) \\ &\times\frac{d}{dz}\left\{S^{(n)}_{s_n s}(p_n,p;z_n,z)\frac{f^\lambda_{ss'}(p,p+k)}{q(p,p+k)}S^{(n)}_{s's_0}(p+k,p_0;z,z_1)\right\}-(n=0).\end{aligned} \tag{3.38}$$

As an illustrative example we compute the simplest case, a set of n centers distributed at equal z_1 coordinate. A perturbative expansion of the beyond eikonal elastic amplitudes as

$$\begin{aligned}S^{(n)}_{s_a s_b}(p_a,p_b;z_a,z_b) &= \delta^{s_a}_{s_b}(2\pi)^3\delta^3(\boldsymbol{p}_a-\boldsymbol{p}_b)+2\pi\delta(p_a^0-p_b^0) \\ &\times\sqrt{\frac{m_e}{p_a^0}}\bar{u}_{s_a}(p_a)\gamma_0 u_{s_b}(p_b)\sqrt{\frac{m_e}{p_b^0}}\frac{-4\pi iZe^2}{\boldsymbol{q}^2+\mu_d^2}\sum_{i=1}^n e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i},\end{aligned} \tag{3.39}$$

produces for (3.38) with $\boldsymbol{q}=\boldsymbol{p}_1+\boldsymbol{k}-\boldsymbol{p}_0$, at leading order in $g=Ze^2$

$$\begin{aligned}\mathcal{M}^{(n)}_{em}-\mathcal{M}^{(0)}_{em} &= e\mathcal{N}(k)2\pi\delta(p_1^0+\omega-p_0^0)\frac{4\pi iZe^2}{\boldsymbol{q}^2+\mu_d^2}\sum_{i=1}^n e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i} \\ &\times\left\{\frac{f^\lambda_{s_1 s'}(p_1,p_1+k)}{q(p_1,p_1+k)}e^{-iq(p_1,p_1+k)z_1}\sqrt{\frac{m_e}{p_0^0}}\bar{u}_{s'}(p_1+k)\gamma_0 u_{s_0}(p_0)\sqrt{\frac{m_e}{p_0^0}}\right. \\ &\left.-\sqrt{\frac{m_e}{p_0^0-\omega}}\bar{u}_{s_1}(p_1)\gamma_0 u_s(p_0-k)\sqrt{\frac{m_e}{p_0^0-\omega}}\frac{f^\lambda_{ss_0}(p_0-k,p_0)}{q(p_0-k,p_0)}e^{-iq(p_0-k,p_0)z_1}\right\}.\end{aligned} \tag{3.40}$$

And with the help of the completeness relation (2.8) and (3.35), (3.36) and (3.37)

$$\begin{aligned}\frac{p_0^0-\omega}{k_\mu p_0^\mu}\sum_s\sqrt{\frac{m_e}{p_0^0-\omega}}u_s(p_0-k)\bar{u}_s(p_0-k)\sqrt{\frac{m_e}{p_0^0-\omega}} &= \frac{\not{p}_0-\not{k}+m_e}{2k_\mu p_0^\mu}, \\ \frac{p_0^0}{k_\mu p_1^\mu}\sum_{s'}\sqrt{\frac{m_e}{p_0^0}}u_{s'}(p_1+k)\bar{u}_{s'}(p_1+k)\sqrt{\frac{m_e}{p_0^0}} &= \frac{\not{p}_1+\not{k}+m_e}{2k_\mu p_1^\mu},\end{aligned} \tag{3.41}$$

we recover the more familiar form of the Bethe-Heitler amplitude. For mediums of non negligible thickness the above amplitude stops being applicable, since each elastic amplitude at z_i carries its own local phase. If we discretize the medium in n sheets and thus p_i from $i=1,\ldots,n$ the beyond eikonal elastic amplitudes can be read from (2.142) or (2.144). By inserting these amplitudes in (3.33) one gets three terms, so that we write

$$\mathcal{M}_{em}^{(n)} = e\mathcal{N}(k)\left(\prod_{i=1}^{n-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\mathcal{M}_f + \mathcal{M}_{int} + \mathcal{M}_i\right). \tag{3.42}$$

The first of them represents photons emitted in the last leg $(+\infty, z_n)$. Indeed using (3.35) we get

$$\mathcal{M}_f \equiv \frac{f^\lambda_{s_n s}(p_n, p_n + k)}{k_\mu p_n^\mu / p_0^0}\exp\left(+i\frac{k_\mu p_n^\mu}{p_0^0}z_n\right)S^{n(z_n)}_{s s_{n-1}}(p_n + k, p_{n-1})\left(\prod_{i=1}^{n-1}S^{n(z_i)}_{s_{i+1}s_i}(q_i)\right), \tag{3.43}$$

with $q_i = p_i - p_{i-1}$ and $S^{n(z_i)}_{s_{i+1}s_i}(q_i) \equiv S^{n(z_i)}_{s_{i+1}s_i}(p_i, p_i)$. The inner term represents photons emitted at any of the (z_1, z_n) internal legs,

$$\begin{aligned}\mathcal{M}_{int} \equiv &-\sum_{k=1}^{n-1}\left(\prod_{i=k+1}^{n}S^{n(z_i)}_{s_i s_{i-1}}(q_i)\right)\frac{f^\lambda_{s_k s'_k}(p_k, p_k + k)}{k_\mu p_k^\mu / p_0^0} \\ &\times\left(\exp\left(+i\frac{k_\mu p_k^\mu}{p_0^0}z_{k+1}\right) - \exp\left(+i\frac{k_\mu p_k^\mu}{p_0^0}z_k\right)\right)S^{n(z_k)}_{s'_k s_{k-1}}(p_k + k, p_{k-1})\left(\prod_{i=1}^{k-1}S^{n(z_i)}_{s_i s_{i-1}}(q_i)\right).\end{aligned} \tag{3.44}$$

Finally the term consisting in photons emitted in the first leg $(z_1, -\infty)$, given by

$$\mathcal{M}_i = -\left(\prod_{i=2}^{n}S^{n(z_i)}_{s_i s_{i-1}}(q_i)\right)S^{n(z_1)}_{s_1 s}(p_1, p_0 - k)\frac{f^\lambda_{s s_0}(p_0 - k, p_0)}{k_\mu p_0^\mu/(p_0^0 - \omega)}\exp\left(+i\frac{k_\mu p_0^\mu}{p_0^0 - \omega}z_1\right). \tag{3.45}$$

Terms (3.43) and (3.45) are the only ones surviving when $\omega \to 0$, since the

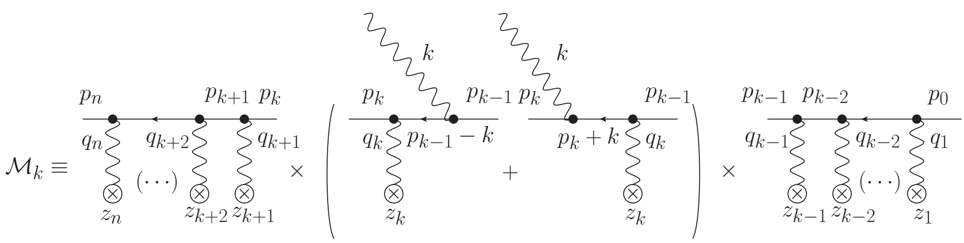


Figure 3.2: Diagrammatic representation of the terms in the sum (3.49).

phase in (3.44) vanish, leading to the soft photon theorem. They correspond to the classical equation (3.21) leading to the coherent plateau in the squared amplitude. A joint expression of these three terms and its vacuum subtraction can be given by reorganizing the sum, which in turns corresponds to the integration by parts at (3.38). This produces

$$\mathcal{M}_{em}^{(n)} = e\mathcal{N}(k)\left(\prod_{i=1}^{n-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\times\sum_{k=1}^{n}\mathcal{M}_k, \tag{3.46}$$

where

$$\begin{aligned}\mathcal{M}_k = &\left(\prod_{i=k+1}^{n} S^{\,n(z_i)}_{s_i s_{i-1}}(q_i)\right)\times\left\{\frac{f^{\lambda}_{s_k s}(p_k, p_k+k)}{k_\mu p_k^\mu/p_0^0}\exp\left(+i\frac{k_\mu p_k^\mu}{p_0^0}z_k\right)S^{\,n(z_k)}_{s s_{k-1}}(p_k+k, p_{k-1})\right.\\ &\left.- S^{\,n(z_k)}_{s_k s}(p_k, p_{k-1}-k)\frac{f^{\lambda}_{s s_{k-1}}(p_{k-1}-k, p_{k-1})}{k_\mu p_{k-1}^\mu/(p_0^0-\omega)}\exp\left(+i\frac{k_\mu p_{k-1}^\mu}{p_0^0-\omega}z_k\right)\right\}\times\left(\prod_{i=1}^{k-1} S^{\,n(z_i)}_{s_i s_{i-1}}(q_i)\right).\end{aligned} \tag{3.47}$$

The Feynman diagram structure of this joint expression is simple and can be seen in Figure 3.2. We notice that with this notation the vacuum subtracted term has been also discretized as

$$\begin{aligned}\mathcal{M}_k^{(0)} = &\left(\prod_{i=k+1}^{n} S^{\,(0)}_{s_i s_{i-1}}(q_i)\right)\times\left\{\frac{f^{\lambda}_{s_k s}(p_k, p_k+k)}{k_\mu p_k^\mu/p_0^0}\exp\left(+i\frac{k_\mu p_k^\mu}{p_0^0}z_k\right)S^{\,(0)}_{s s_{k-1}}(p_k+k, p_{k-1})\right.\\ &\left.- S^{\,(0)}_{s_k s}(p_k, p_{k-1}-k)\frac{f^{\lambda}_{s s_{k-1}}(p_{k-1}-k, p_{k-1})}{k_\mu p_{k-1}^\mu/(p_0^0-\omega)}\exp\left(+i\frac{k_\mu p_{k-1}^\mu}{p_0^0-\omega}z_k\right)\right\}\times\left(\prod_{i=1}^{k-1} S^{\,(0)}_{s_i s_{i-1}}(q_i)\right).\end{aligned} \tag{3.48}$$

Then we write

$$\mathcal{M}_{em}^{(n)} - \mathcal{M}_{em}^{(0)} = e\mathcal{N}(k)\left(\prod_{i=1}^{n-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\times\sum_{k=1}^{n}\left(\mathcal{M}_k - \mathcal{M}_k^{(0)}\right). \tag{3.49}$$

The classical correspondence of this quantum amplitude has been given in (3.13) and is going to be recovered in the averaged square of (3.49) for a medium of macroscopic transverse size.

3.3 Intensity

Relation (3.49) can be squared and then averaged over medium configurations and initial spins, and summed over final states and photon polarizations in order to evaluate the intensity of photons in a particular direction. For that purpose we choose, as with the elastic scattering, a cylinder of transverse area πR^2 and length l. The unpolarized intensity of photons in the energy interval ω and $\omega + d\omega$ and in the solid angle Ω_k and $\Omega_k + d\Omega_k$, per unit of medium transverse area and time, is given by

$$dI \equiv \frac{\omega^2 d\omega d\Omega_k}{(2\pi)^3} \frac{1}{\beta_p T \pi R^2} \frac{1}{2} \sum_{s_n s_0} \sum_{\lambda} \int \frac{d^3\boldsymbol{p}_n}{(2\pi)^3} \left\langle \left| \mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} \right|^2 \right\rangle, \tag{3.50}$$

where we divided by β_p, the incoming electron flux, time T, accounting for time translation invariance, and medium transverse area $\Omega = \pi R^2$. In the process of averaging over medium configurations we notice that

$$\left\langle \left| \mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} \right|^2 \right\rangle = \left\langle \mathcal{M}^{(n)}_{em} \left(\mathcal{M}^{(n)}_{em} \right)^* \right\rangle - \left\langle \mathcal{M}^{(n)}_{em} \right\rangle \left\langle \left(\mathcal{M}^{(n)}_{em} \right)^* \right\rangle + \left| \left\langle \mathcal{M}^{(n)}_{em} \right\rangle - \mathcal{M}^{(0)}_{em} \right|^2 . \tag{3.51}$$

Here we added and subtracted the term $\left\langle \mathcal{M}^{(n)}_{em} \right\rangle \left\langle \left(\mathcal{M}^{(n)}_{em} \right)^* \right\rangle$ as before. Then we define the incoherent contribution to the emission as

$$\Sigma^{(n)}_{em} \equiv \left\langle \mathcal{M}^{(n)}_{em} \left(\mathcal{M}^{(n)}_{em} \right)^* \right\rangle - \left\langle \mathcal{M}^{(n)}_{em} \right\rangle \left\langle \left(\mathcal{M}^{(n)}_{em} \right)^* \right\rangle, \tag{3.52}$$

which as we will shown, in the limit $R \gg \mu_d^{-1}$ can be interpreted in probabilistic terms and, except for quantum corrections in the hard part of the spectrum, leads to the classical behavior of the infrared divergence and the LPM effect. We also find a coherent contribution

$$\Pi^{(n)}_{em} \equiv \left| \left\langle \mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} \right\rangle \right|^2 , \tag{3.53}$$

which is just the averaged emission amplitude squared, and encodes the quantum diffractive behavior of the LPM effect. As we will later show, this contribution in the $R \gg \mu_d^{-1}$ limit can be omitted since it represents the negligible high energy transition radiation of the electron.

Transverse-coherent contribution

The transverse coherent contribution to the intensity consists in the averaged emission amplitude squared. It is given by

$$dI_{coh} \equiv \frac{\omega^2 d\omega d\Omega_k}{(2\pi)^3} \frac{1}{\beta_p T \pi R^2} \frac{1}{2} \sum_{s_n s_0} \sum_{\lambda} \int \frac{d^3\boldsymbol{p}_n}{(2\pi)^3} \Pi^{(n)}_{em}. \tag{3.54}$$

For microscopic mediums it contains the quantum diffractive behavior of the medium boundaries in the transverse plane, thus it can be interpreted as the contribution related to medium transverse coherence. For macroscopic mediums this diffractive behavior, as we will show, constraints the electron propagation to the forward direction, thus this contribution reduces in that limit to the electron transition radiation related to its energy gap and the longitudinal boundaries. Using (3.49) and (3.53) we get

$$\Pi^{(n)}_{em}(k) = e^2\mathcal{N}^2(k)\left|\left(\prod_{i=1}^{n-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\sum_{k=1}^{n}\langle\mathcal{M}_k\rangle - \mathcal{M}^{(0)}_k\right)\right|^2, \qquad (3.55)$$

where the required averages on the right hand side of the above equation affect only to the amplitudes containing centers $\mathcal{M}_k$ and are of the form

$$\begin{aligned}\langle\mathcal{M}_k\rangle = &\left(\prod_{i=k+1}^{n}\left\langle S^{\,n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1})\right\rangle\right)\left(\prod_{i=1}^{k-1}\left\langle S^{\,n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1})\right\rangle\right) \qquad (3.56)\\ &\times\left(\frac{f^{\lambda}_{s_k s}(p_k,p_k+k)}{k_\mu p^\mu_{k-1}/p^0_0}\exp\left(+i\frac{k_\mu p^\mu_k}{p^0_0}z_k\right)\left\langle S^{\,n(z_k)}_{s s_{k-1}}(p_k+k,p_{k-1})\right\rangle\right.\\ &\left.-\left\langle S^{\,n(z_k)}_{s_k s}(p_k,p_{k-1}-k)\right\rangle\frac{f^{\lambda}_{s s_{k-1}}(p_{k-1}-k,p_{k-1})}{k_\mu p^\mu_{k-1}/(p^0_0-\omega)}\exp\left(+i\frac{k_\mu p^\mu_{k-1}}{p^0_0-\omega}z_k\right)\right).\end{aligned}$$

At this point we note that the elastic amplitudes in (3.56) for the layers between n and $k+1$ verify $p^0_i = p^0_0-\omega$, the advanced and retarded elastic amplitudes at the layer k verifies $p^0_k = p^0_0$ and $p^0_k = p^0_0-\omega$, respectively, and the elastic amplitudes between k and 1 verify $p^0_i = p^0_0$. We briefly review the results of the averaging process, *c.f.* Section 2.4. A single coherent average at the layer i of $n(z_i)$ scattering sources over a cylinder of transverse area πR^2 and length δz produces

$$\begin{aligned}\left\langle S^{\,n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1})\right\rangle = &(2\pi)^3\delta(\boldsymbol{q}_i)\delta^{s_i}_{s_{i-1}} + 2\pi\beta_p\delta(q^0_i)\delta^{s_i}_{s_{i-1}}\exp\left(-iq^z_i z_i\right)\\ &\times\int d^2\boldsymbol{x}_t e^{-iq^t_i\cdot\boldsymbol{x}_t}\left(\left\langle\exp\left[-i\frac{g}{\beta_p}\sum_{k=1}^{n(z_i)}\chi^k_0(\boldsymbol{x}_t)\right]\right\rangle-1\right), \qquad (3.57)\end{aligned}$$

where we used the relation $S^{\,n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1}) = M^{n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1})+(2\pi)^3\delta^3(\boldsymbol{p}_i-\boldsymbol{p}_{i-1})\delta^{s_i}_{s_{i-1}}$. In the large but finite $R\gg\mu_d$ limit this average can be well approximated as

$$\int d^2\boldsymbol{x}e^{-iq\cdot\boldsymbol{x}}\left\langle\exp\left[-i\frac{g}{\beta}\sum_{k=1}^{n(z_i)}\chi^k_0(\boldsymbol{x})\right]\right\rangle \simeq \int d^2\boldsymbol{x}e^{-iq\cdot\boldsymbol{x}}\exp\left(n_0(z_i)\delta z\pi^{(1)}_{el}(\boldsymbol{x})\right). \quad (3.58)$$

The function $\pi^{(1)}_{el}(\boldsymbol{x})$ is the Fourier transform of the single elastic amplitude for a collision with a single scattering source coherently distributed at the amplitude level over a cylinder of radius R, and given by

$$\pi^{(1)}_{el}(\boldsymbol{x}) = \int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{+iq\cdot\boldsymbol{x}}F^{(1)}_{el}(\boldsymbol{q})W_{cyl}(\boldsymbol{q},R),\ \ W_{cyl}(\boldsymbol{q},R) = \frac{2\pi R}{|\boldsymbol{q}|}J_1(|\boldsymbol{q}|R), \qquad (3.59)$$

where $W_{cyl}(\boldsymbol{q}, R)$ is the window function of the cylinder and $J_1(x)$ the Bessel function of the first kind. The function $\pi_{el}^{(1)}(\boldsymbol{x})$ has a typical width R, in contrast to dimensions of a single scattering source $r_d = 1/\mu_d$. It results convenient to separate the momentum distribution from the spin content and the longitudinal phases. Then we write

$$\left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1})\right\rangle = \delta^{s_i}_{s_{i-1}} \left(\phi^{(0)}_{coh}(\delta p_i, \delta z) + \phi^{(n)}_{coh}(\delta p_i, \delta z)\right) \exp\left(-i\delta p^z_i z_i\right), \tag{3.60}$$

where we defined the no collision (0) and the collision (n) coherently averaged amplitudes for the matter in δz as

$$\begin{aligned}
&\phi^{(0)}_{coh}(\delta p_i, \delta z) \equiv (2\pi)^3 \delta(\delta \boldsymbol{p}_i), \\
&\phi^{(n)}_{coh}(\delta p_i, \delta z) \equiv 2\pi\beta\delta(\delta p^0_i) \int d^2\boldsymbol{x} e^{-i\delta \boldsymbol{p}^t_i \cdot \boldsymbol{x}} \left(\exp\left(n_0(z_i)\delta z \pi^{(1)}_{el}(\boldsymbol{x})\right) - 1\right).
\end{aligned} \tag{3.61}$$

These distributions act over the electron wave function and can only be interpreted in probabilistic terms in the squared amplitude. They preserve spin, lead to a momentum distribution of typical width $1/R$ and add a local longitudinal phase of the form $\delta p^z_i z_i$ at each step, responsible of modulating the quantum LPM effect. The first section of coherent scatterings produces, by reiterative use of the relation (3.60),

$$\begin{aligned}
\prod_{i=1}^{k-1} \left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1})\right\rangle = \delta^{s_{k-1}}_{s_0} &\left(\prod_{i=1}^{k-1}\left(\phi^{(n)}_{coh}(\delta p_i, \delta z) + \phi^{(0)}_{coh}(\delta p_i, \delta z)\right)\right) \\
&\times \exp\left(-i z_{k-1} p^z_{p^0_0}(\boldsymbol{p}^t_{k-1}) + i\left(\sum_{i=1}^{k-2} \delta z p^z_{p^0_0}(\boldsymbol{p}^t_i)\right) + i z_1 p^z_{p^0_0}(\boldsymbol{p}^t_0)\right),
\end{aligned} \tag{3.62}$$

where $\delta z = z_{i+1} - z_i$ and we rearranged the phase by summing by parts. Subindices in the longitudinal momenta denote the energy, read from the energy conservation deltas either at $\phi^{(n)}_{coh}(\delta p_i)$ or $\phi^{(0)}_{coh}(\delta p_i)$. They fix the longitudinal momentum in terms of the transverse momentum. Similarly the advanced average at the emission layer k produces, using (3.60),

$$\begin{aligned}
\left\langle S^{n(z_k)}_{s s_{k-1}}(p_k + k, p_{k-1})\right\rangle = \delta^{s}_{s_{k-1}} &\left(\phi^{(n)}_{coh}(\delta p_k + k, \delta z) + \phi^{(0)}_{coh}(\delta p_k + k, \delta z)\right) \\
&\times \exp\left(- i z_k p_{p^0_0}(\boldsymbol{p}^t_k + \boldsymbol{k}_t) + i z_k p_{p^0_0}(\boldsymbol{p}^t_{k-1})\right),
\end{aligned} \tag{3.63}$$

whereas the average just after the emission, where the electron energy is now $p_0^0-\omega$, produces

$$\left\langle S^{n(z_k)}_{s_k s}(p_k,p_{k-1}-k)\right\rangle = \delta_s^{s_k}\left(\phi^{(n)}_{coh}(\delta p_k+k,\delta z)+\phi^{(0)}_{coh}(\delta p_k+k,\delta z)\right)$$
$$\times\exp\left(-iz_k p_{p_0^0-\omega}(\boldsymbol{p}_k^t)+iz_k p^z_{p_0^0-\omega}(\boldsymbol{p}^t_{k-1}-\boldsymbol{k}^t)\right). \quad (3.64)$$

For the last set of coherent scatterings with energy $p_0^0-\omega$ we find, using reiteratively (3.60),

$$\prod_{i=k+1}^{n}\left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i,p_{i-1})\right\rangle = \delta^{s_n}_{s_k}\left(\prod_{i=k+1}^{n}\left(\phi^{(n)}_{coh}(\delta p_i,\delta z)+\phi^{(0)}_{coh}(\delta p_i,\delta z)\right)\right)$$
$$\times\exp\left(-iz_n p^z_{p_0^0-\omega}(\boldsymbol{p}_n^t)+i\left(\sum_{i=k+1}^{n-1}\delta z p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t)\right)+iz_{k+1}p^z_{p_0^0-\omega}(\boldsymbol{p}_k^t)\right). \quad (3.65)$$

Since $\omega\ll p_0^0$ we assume that β_p can be taken unaltered in all the process and the photon effect in the elastic amplitude neglected. The leading contribution of the photon to the elastic propagation is enclosed instead in the energy gap of the longitudinal phases and in the loss of $\boldsymbol{k}_t$ at the emission layer z_k. The insertion of (3.62),(3.63), (3.64) and (3.65) in (3.56) produces then a soft photon factorization of the form

$$\langle\mathcal{M}_k\rangle = \left(\prod_{i=1}^{n}\left(\phi^{(n)}_{coh}(\delta p_i,\delta z_i)+\phi^{(0)}_{coh}(\delta p_i,\delta z_i)\right)\right)J_k, \quad (3.66)$$

where we shifted the electron momentum variable from the emission point k onwards, as $\boldsymbol{p}_i^t+\boldsymbol{k}_t\to\boldsymbol{p}_i^t$. The emission current is given by

$$J_k = e^{i\varphi_k}\left\{f_k^{(+)}-f_k^{(-)}\right\}, \quad (3.67)$$

where the phase of each element is given by

$$i\varphi_k \equiv -ip^z_{p_0^0-\omega}(\boldsymbol{p}_n^t-\boldsymbol{k}_t)z_n + i\sum_{i=k}^{n-1}\delta z_i p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t-\boldsymbol{k}_t) - ik_z z_k$$
$$+\,i\sum_{i=1}^{k-1}\delta z_i p^z_{p_0^0}(\boldsymbol{p}_i^t)+ip^z_{p_0^0}(\boldsymbol{p}_0^t)z_1, \quad (3.68)$$

and we defined the following shorthands for the emission vertex together with the corresponding propagator

$$f_k^{(+)}\equiv\frac{f_{s_n s_0}(p_k,p_k+k)}{k_\mu p_k^\mu/p_0^0}, \qquad f_k^{(-)}\equiv\frac{f_{s_n s_0}(p_{k-1}-k,p_{k-1})}{k_\mu p_{k-1}^\mu/(p_0^0-\omega)}. \quad (3.69)$$

The photon longitudinal momentum can be written in the high energy limit as $k_z \simeq \omega - \boldsymbol{k}_t^2/2\omega$. Then the phase can be rearranged as

$$\begin{aligned} i\varphi_k = &+ i\left(\omega\frac{m_e^2}{2p_0^0(p_0^0-\omega)} + i\frac{\boldsymbol{k}_t^2}{2\omega}\right)z_k \\ &+ i\frac{(\boldsymbol{p}_n^t-\boldsymbol{k}_t)^2}{2(p_0^0-\omega)}z_n - i\sum_{i=k}^{n-1}\delta z_i\frac{\left(\boldsymbol{p}_i^t-\boldsymbol{k}_t\right)^2}{2(p_0^0-\omega)} - i\sum_{i=1}^{k-1}\delta z_i\frac{(\boldsymbol{p}_i^t)^2}{2p_0^0} - i\frac{(\boldsymbol{p}_0^t)^2}{2p_0^0}z_1. \end{aligned} \tag{3.70}$$

The boundary terms z_n and z_1 can be omitted if desired since they will cancel in the squared amplitude. Observe that if a term of the form $J_kJ_l^*$ is evaluated for the same electron trajectory in the amplitude and its conjugate we recover the classical phase,

$$J_kJ_l^* \propto \exp\left(+i\sum_{i=l}^{k-1}\delta z_i\left(m_e^2+\left(\boldsymbol{p}_i^t-\frac{p_0^0}{\omega}\boldsymbol{k}_t\right)^2\right)\right) \simeq \exp\left(+i\int_{z_l}^{z_k}dz\,k_\mu x^\mu(z)\right), \tag{3.71}$$

where we used (3.36) and assumed $k > l$. It is straightforward to prove that the term containing no interaction with any of the layers can be written in a similar manner to this one. With the replacement $\phi_{coh}^{(n)} + \phi_{coh}^{(0)} \to \phi_{coh}^{(0)}$ we find

$$\mathcal{M}_k^{(0)} = \left(\prod_{i=1}^{n}\left(\phi_{coh}^{(0)}(\delta p_i,\delta z_i)\right)\right)J_k, \tag{3.72}$$

and we just find the overall vacuum subtraction of the contribution $\langle\mathcal{M}_k\rangle$. Since the sum of $\langle\mathcal{M}_k\rangle - \mathcal{M}_k^{(0)}$ ought to be squared further simplifications can be done. We notice that, *c.f.* Appendix A,

$$\frac{\omega^2}{2}\sum_\lambda\sum_{s_ns_0}\left(f_k^{(+)}-f_k^{(-)}\right)\left(f_l^{(+)}-f_l^{(-)}\right)^* = h_n(y)\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l + h_s(y)\delta_k^s\delta_l^s, \tag{3.73}$$

where the spin flip and non flip currents are given, respectively, by

$$\boldsymbol{\delta}_k^n \equiv \frac{\boldsymbol{k}\times\boldsymbol{p}_k}{k_\mu p_k^\mu} - \frac{\boldsymbol{k}\times\boldsymbol{p}_{k-1}}{k_\mu p_{k-1}^\mu}, \qquad \delta_k^s \equiv \frac{\omega p_k^0}{k_\mu p_k^\mu} - \frac{\omega p_{k-1}^0}{k_\mu p_{k-1}^\mu}. \tag{3.74}$$

Here $y = \omega/p_0^0$ is the fraction of energy carried by the photon, $p_k^0 = p_0^0 - \omega$ and $p_{k-1}^0 = p_0^0$ and the functions $h_n(y)$ and $h_s(y)$ are given by

$$h^n(y) = \frac{1}{2}(1+(1-y)^2), \qquad h^s(y) = \frac{1}{2}y^2. \tag{3.75}$$

This indicates that we can split the emission current into two independent contributions, $\boldsymbol{J}_k^n$ and J_k^s which can be separately squared and then multiplied with $h^n(y)$ and $h^s(y)$. These currents are given by

$$\boldsymbol{J}_k^n \equiv= \frac{1}{\omega^2}\boldsymbol{\delta}_k^n e^{i\varphi_k}, \quad J_k^s \equiv \frac{1}{\omega^2}\delta_k^s e^{i\varphi_k}, \tag{3.76}$$

where the phase φ_k is given by (3.70). Then we have found a transverse coherent average of the emission intensity given by

$$\left(\Pi_{em}^{(n)}\right)_{sn} = \frac{e^2\mathcal{N}^2}{\omega^2}\left|\left(\prod_{i=1}^{n-1}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=1}^{n}\left(\phi_{coh}^{(n)}(\delta p_i,\delta z_i)+\phi_{coh}^{(0)}(\delta p_i,\delta z_i)\right)\right.\right.$$
$$\left.\left.-\prod_{i=1}^{n}\left(\phi_{coh}^{(0)}(\delta p_i,\delta z_i)\right)\right)\left(\sum_{k=1}^{n}\boldsymbol{\delta}_k^n e^{i\varphi_k}\right)\right|^2, \tag{3.77}$$

for the spin no-flip contribution and

$$\left(\Pi_{em}^{(n)}\right)_{sf} = \frac{e^2\mathcal{N}^2}{\omega^2}\left|\left(\prod_{i=1}^{n-1}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=1}^{n}\left(\phi_{coh}^{(n)}(\delta p_i,\delta z_i)+\phi_{coh}^{(0)}(\delta p_i,\delta z_i)\right)\right.\right.$$
$$\left.\left.-\prod_{i=1}^{n}\left(\phi_{coh}^{(0)}(\delta p_i,\delta z_i)\right)\right)\left(\sum_{k=1}^{n}\delta_k^s e^{i\varphi_k}\right)\right|^2, \tag{3.78}$$

for the spin flip contribution. The coherent average contribution to the photon intensity is then given by the sum of these two terms once integrated in the final electron momentum

$$\omega\frac{dI_{coh}}{d\omega d\Omega_k} = \left(\frac{\omega}{2\pi}\right)^3\frac{1}{\pi R^2 T}\int\frac{d^3\boldsymbol{p}_n}{(2\pi)^3}\left(h^n(y)\left(\Pi_{em}^{(n)}\right)_{sn}+h^s(y)\left(\Pi_{em}^{(n)}\right)_{sf}\right). \tag{3.79}$$

A particular case of the above result consists in taking the macroscopic $R\to\infty$ limit. Then $W_{cyl}(\boldsymbol{q},R) = (2\pi)^2\delta^2(\boldsymbol{q})$, $\pi_{el}^{(1)}(\boldsymbol{x}) = F_{el}^{(1)}(\boldsymbol{0})$ and thus

$$\phi_{coh}^{(n)}(\delta p_i,\delta z_i) = (2\pi)^3\delta^3(\delta\boldsymbol{p}_i)\left(\exp\left(n_0(z_i)\delta z_i F_{el}^{(1)}(\boldsymbol{0})\right)-1\right), \tag{3.80}$$

which leads to a pure forward propagation of the electron at the level of the amplitude. Then the only difference of states in the functions $\boldsymbol{\delta}_i^n$ and δ_i^s corresponds to the energy gap of the electron momentum and thus

$$\sum_{k=1}^{n}\delta_k^s e^{i\varphi_k} = \left\{\frac{\beta_f\sin\theta}{1-\beta_f\cos\theta}-\frac{\beta_i\sin\theta}{1-\beta_i\cos\theta}\right\}\sum_{k=1}^{n}e^{i\varphi_k}, \tag{3.81}$$

where θ is the angle between the photon and the initial electron direction $\hat{\boldsymbol{p}}_0$ and β_f and β_i are the final and initial electron velocities. Similarly the spin flip contribution produces

$$\sum_{k=1}^{n} \delta_k^s = e^{i\varphi_k} \left\{ \frac{\beta_f}{1-\beta_f \cos\theta} - \frac{\beta_i}{1-\beta_i \cos\theta} \right\} \sum_{k=1}^{n} e^{i\varphi_k}, \tag{3.82}$$

where the phases are also constrained to the forward direction and given in this case by

$$i\varphi_k = i\omega \frac{m_e^2}{2p_0^0(p_0^0-\omega)} z_k + i\frac{\boldsymbol{k}_t^2}{2\omega} z_k \equiv i\frac{k_\mu p_0^\mu}{p_0^0} z_k, \tag{3.83}$$

as expected. At high energies the negligible energy gap in the velocities β_i and β_f of the current can be neglected, as we already did in the elastic weights. Then $\boldsymbol{\delta}_k^n = 0$ and $\delta_k^s = 0$, which means that at high energies in the macroscopic $R \to \infty$ limit, the coherent average contribution cancels, $\left(\Pi_{em}^{(n)}\right)_{sn} = 0$ and $\left(\Pi_{em}^{(n)}\right)_{sf} = 0$, as expected.

Transverse-incoherent contribution

The transverse-incoherent contribution to the emission intensity corresponds to the averaged squared amplitude. Using (3.49) and (3.52) we get

$$\Sigma_{em}^{(n)} = e^2 \mathcal{N}^2(k) \left(\prod_{i=1}^{n-1} \int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3} \frac{d^3\boldsymbol{u}_i}{(2\pi)^3} \right) \times \sum_{j,k=1}^{n} \left(\langle \mathcal{M}_k \mathcal{M}_j^* \rangle - \langle \mathcal{M}_k \rangle \langle \mathcal{M}_j^* \rangle \right). \tag{3.84}$$

The averages on the right hand side of the above equation are of the form

$$\langle \mathcal{M}_k \mathcal{M}_j^* \rangle = A_{j+1}^n \left((f_j^+)^* A_j^+ - A_j^- (f_j^-)^* \right) A_{k+1}^{j-1} \left(f_k^+ A_k^+ - A_k^- f_k^- \right) A_1^{k-1}, \tag{3.85}$$

$$\langle \mathcal{M}_k \rangle \langle \mathcal{M}_j^* \rangle = B_{j+1}^n \left((f_j^+)^* B_j^+ - B_j^- (f_j^-)^* \right) B_{k+1}^{j-1} \left(f_k^+ B_k^+ - B_k^- f_k^- \right) B_1^{k-1}, \tag{3.86}$$

where f_j^+ and f_j^- are the emission vertex and propagator shorthands given at (3.69) but with the local phases, thus given by

$$f_k^{(+)} \equiv \frac{f_{s_k s}^{\lambda}(p_k, p_k + k)}{k_\mu p_k^\mu / p_0^0} \exp\left(+i \frac{k_\mu p_k^\mu}{p_0^0} \right),$$

$$f_k^{(-)} \equiv \frac{f_{s s_{k-1}}^{\lambda}(p_{k-1} - k, p_{k-1})}{k_\mu p_{k-1}^\mu / (p_0^0 - \omega)} \exp\left(+i \frac{k_\mu p_{k-1}^\mu}{p_0^0 - \omega} \right), \tag{3.87}$$

and the terms A_k^j and B_k^j are shorthands for the incoherent averages of the squared amplitudes corresponding to the layers of scatterers from z_k to z_j. Let δp_i =

$p_i - p_{i-1}$ and $\delta u_i = u_i - u_{i-1}$. If the medium verifies that the transverse dimensions greatly exceed the dimensions of a scatterer, that is, $R \gg \mu_d^{-1}$, we get from (3.49), *c.f.* Section 2.4, elements of the form

$$\left\langle S^{n(z_i)}_{s_i s_{i-1}}(\delta p_i) S^{n(z_i),*}_{r_i r_{i-1}}(\delta u_i)\right\rangle = \delta^{s_{i-1}}_{s_i}\delta^{r_{i-1}}_{r_i}(2\pi)^2\delta^2(\delta \boldsymbol{p}^t_i - \delta \boldsymbol{u}^t_i) \times 2\pi\beta_u\delta(\delta u^0_i)\exp\left(-i\left(\delta p^z_i - \delta u^z_i\right)z_i\right)\phi_{inc}(\delta p_i). \tag{3.88}$$

In (3.88) if we assume that the energy gap due to the photon is negligible in the elastic part $\beta_u \simeq \beta_p \simeq \beta$, then $\phi(q)$ acquires a probabilistic interpretation in all the range and can be split into two contributions

$$\phi_{inc}(q) = e^{-n_0(z_i)\delta z \sigma^{(1)}_t}(2\pi)^3\delta^3(\boldsymbol{q}) + 2\pi\delta(q_0)\beta_p\hat{\Sigma}_2(\boldsymbol{q},\delta z) \equiv \phi^{(0)}_{inc}(q) + \phi^{(n)}_{inc}(q), \tag{3.89}$$

where $n_0(z_i)$ is the density of scattering centers at the layer at z_i of thickness δz, the single elastic cross section $\sigma^{(1)}_t$ is given by (2.54) at arbitrary coupling or by (2.52) at small coupling and the collisional distribution after an incoherent scattering with the layer $\hat{\Sigma}_2(\boldsymbol{q},\delta z)$ is given by (2.90) without the $\Omega = \pi R^2$ factor. We have found, then, the probability of no colliding with the matter in δz, given by $\exp\left(-n_0(z_i)\delta z \sigma^{(1)}_t\right)$, times the corresponding forward distribution $\delta^3(\boldsymbol{q}_i)$, or the incoherent contribution $\hat{\Sigma}_2(\boldsymbol{q},\delta z)$ in case of scattering with the matter in δz. In this transport spin is preserved and the transverse momentum change in the conjugated amplitude $\delta \boldsymbol{u}^t_i$ equals the one in the amplitude $\delta \boldsymbol{p}^t_i$, as a result of the macroscopic limit $R \gg \mu_d^{-1}$. By using the energy conservation deltas the longitudinal phases are of the form

$$\delta p^z_i - \delta u^z_i \equiv \left(p^z(\boldsymbol{p}^t_i) - p^z(\boldsymbol{p}^t_{i-1})\right) - \left(u^z(\boldsymbol{u}^t_i) - u^z(\boldsymbol{u}^t_{i-1})\right), \tag{3.90}$$

Similarly, the coherent averages of the kind B are found to be

$$\left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1})\right\rangle \left\langle S^{n(z_i),*}_{r_i r_{i-1}}(u_i, u_{i-1})\right\rangle = \delta^{s_{i-1}}_{s_i}\delta^{r_{i-1}}_{r_i}(2\pi)^2\delta^2(\delta \boldsymbol{p}^t_i - \delta \boldsymbol{u}^t_i) \times 2\pi\beta\delta(\delta u^0_i)\exp\left(-i\left(\delta p^z_i - \delta u^z_i\right)z_i\right)\phi^{(0)}_{inc}(\delta p_i). \tag{3.91}$$

With these tools we are in position of evaluating all the required terms. The first path of elastic scatterings at (3.85) corresponds to the passage from the beginning of the medium z_1 to previous layer where the photon is emitted in one of the amplitudes, i.e. z_{k-1}

$$A^{k-1}_1 \equiv \left(\prod_{i=1}^{k-1}\left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1}) S^{n(z_i),*}_{r_i r_{i-1}}(u_i, u_{i-1})\right\rangle\right). \tag{3.92}$$

Using (3.88) we find $p_i^0 = u_i^0 \equiv p_0^0$ for $i = 1, \ldots, k-1$ and, since $\boldsymbol{u}_0^t = \boldsymbol{p}_0^t$ then from (3.88) we get $\boldsymbol{u}_i^t = \boldsymbol{p}_i^t$ for $i = 1, \ldots, k-1$. The longitudinal phase with these constraints then vanishes

$$\delta p_i^z - \delta u_i^z = \left(p^z_{p_0^0}(\boldsymbol{p}_i^t) - p^z_{p_0^0}(\boldsymbol{p}_{i-1}^t)\right) - \left(p^z_{p_0^0}(\boldsymbol{u}_i^t) - p^z_{p_0^0}(\boldsymbol{u}_{i-1}^t)\right) = 0. \tag{3.93}$$

At (3.88) we use $(2\pi)^2\delta^2(\boldsymbol{u}_i^t - \boldsymbol{p}_i^t)2\pi\beta_u\delta(u_i^0 - p_0^0) = (2\pi)^3\delta^3(\boldsymbol{u}_i - \boldsymbol{p}_i)$ so for the set of scatterings from z_1 to z_{k-1} we find a simple convolution of incoherent elastic scatterings without longitudinal phases,

$$A_1^{k-1} = \delta_{s_0}^{s_{k-1}}\delta_{s_0}^{r_{k-1}}\left(\prod_{i=1}^{k-1}\phi_{inc}(\delta p_i) \times (2\pi)^3\delta^3(\boldsymbol{u}_i - \boldsymbol{p}_i)\right). \tag{3.94}$$

The averages at z_k are given by two different terms. The term corresponding to a retarded emission after the collision is given by

$$A_k^+ = \left\langle S_{ss_{k-1}}^{n(z_k)}(p_k + k, p_{k-1})\ S_{r_k r_{k-1}}^{n(z_k),*}(u_k, u_{k-1})\right\rangle, \tag{3.95}$$

where both amplitudes are on-shell with $p_{k-1}^0 = u_{k-1}^0 = p_0^0$. Since we inherit $\boldsymbol{u}_{k-1} = \boldsymbol{p}_{k-1}$ from (3.94) we find, using (3.88), that $\boldsymbol{p}_k^t + \boldsymbol{k}^t = \boldsymbol{u}_k^t$. The longitudinal phase in (3.88) then vanishes again since

$$\delta p_k^z - \delta u_k^z = \left(p^z_{p_0^0}(\boldsymbol{p}_k^t + \boldsymbol{k}^t) - p^z_{p_0^0}(\boldsymbol{p}_{k-1}^t)\right) - \left(p^z_{p_0^0}(\boldsymbol{u}_k^t) - p^z_{p_0^0}(\boldsymbol{u}_{k-1}^t)\right) = 0. \tag{3.96}$$

Then using $(2\pi)^2\delta^2(\boldsymbol{p}_k^t + \boldsymbol{k}^t - \boldsymbol{u}_k^t)(2\pi)\beta\delta(p_k^0 + \omega - u_k^0) = (2\pi)^3\delta^3(\boldsymbol{p}_k + \boldsymbol{k} - \boldsymbol{u}_k)$ we find

$$A_k^+ = \delta_{s_{k-1}}^{s}\delta_{r_{k-1}}^{r_k}\phi_{inc}(\delta p_k + k) \times (2\pi)^3\delta^3(\boldsymbol{p}_k + \boldsymbol{k} - \boldsymbol{u}_k). \tag{3.97}$$

The term corresponding to an advanced emission to the collision at z_k is instead given by

$$A_k^- = \left\langle S_{s_k s}^{n(z_k)}(p_k, p_{k-1} - k)\ S_{r_k r_{k-1}}^{n(z_k),*}(u_k, u_{k-1})\right\rangle, \tag{3.98}$$

where one of the amplitudes is on-shell with $p_k^0 = p_0^0 - \omega$ whereas the other with $u_k^0 = p_0^0$. As before from (3.94) we use $\boldsymbol{u}_{k-1} = \boldsymbol{p}_{k-1}$ and thus from (3.88) we get $\boldsymbol{p}_k^t + \boldsymbol{k}^t = \boldsymbol{u}_k^t$. In this case the longitudinal phase can be rearranged as

$$\begin{aligned}\delta p_k^z - \delta u_k^z &= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_k^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{k-1}^t - \boldsymbol{k}_t)\right) - \left(p^z_{p_0^0}(\boldsymbol{u}_k^t) - p^z_{p_0^0}(\boldsymbol{u}_{k-1}^t)\right)\\ &= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_k^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{k-1}^t - \boldsymbol{k}_t)\right) - \left(p^z_{p_0^0}(\boldsymbol{p}_k^t + \boldsymbol{k}^t) - p^z_{p_0^0}(\boldsymbol{p}_{k-1}^t)\right).\end{aligned} \tag{3.99}$$

By adding and subtracting $\pm p^z_\omega(\boldsymbol{k}_t)$ and using (3.35) and (3.36) this can be written as

$$
\begin{aligned}
\delta p_k^z - \delta u_k^z &= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_k^t) + p^z_\omega(\boldsymbol{k}_t) - p^z_{p_0^0}(\boldsymbol{p}_k^t + \boldsymbol{k}^t)\right) \\
&\quad - \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_{k-1}^t - \boldsymbol{k}_t) + p^z_\omega(\boldsymbol{k}_t) - p^z_{p_0^0}(\boldsymbol{p}_{k-1}^t)\right) = -\frac{k_\mu p_k^\mu}{p_0^0} + \frac{k_\mu p_{k-1}^\mu}{p_0^0 - \omega},
\end{aligned}
\tag{3.100}
$$

so that the advanced emission term at z_k reduces to an incoherent elastic scattering carrying an extra longitudinal phase,

$$
A_k^- = \delta_s^{s_k}\delta_{r_{k-1}}^{r_k}\phi_{inc}(\delta p_k + k)\times(2\pi)^3\delta^3(\boldsymbol{p}_k + \boldsymbol{k} - \boldsymbol{u}_k)\exp\left(+i\frac{k_\mu p_k^\mu}{p_0^0}z_k - i\frac{k_\mu p_{k-1}^\mu}{p_0^0 - \omega}z_k\right).
\tag{3.101}
$$

The intermediate step of the scatterings corresponds to the passage of the electron from the layer immediately after the emission z_{k+1} in one of the amplitudes, to the layer before the emission z_{j-1} the other amplitude. It is given by

$$
A_{k+1}^{j-1} = \left(\prod_{i=k+1}^{j-1}\left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1}) S^{n(z_i),*}_{r_i r_{i-1}}(u_i, u_{i-1})\right\rangle\right).
\tag{3.102}
$$

In this section of scatterings one of the amplitudes is on-shell with $p_i^0 = p_0^0 - \omega$ but the other with $u_i^0 = p_0^0$. We inherit the boundary condition $\boldsymbol{p}_k + \boldsymbol{k} = \boldsymbol{u}_k$ either from (3.97) or (3.101) and then we find from (3.88) $\boldsymbol{p}_i^t + \boldsymbol{k}^t = \boldsymbol{u}_i^t$ for $i = k+1, \ldots, j-1$. The longitudinal phase produces

$$
\begin{aligned}
\delta p_i^z - \delta u_i^z &= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{i-1}^t)\right) - \left(p^z_{p_0^0}(\boldsymbol{u}_i^t) - p^z_{p_0^0}(\boldsymbol{u}_{i-1}^t)\right) \\
&= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{i-1}^t)\right) - \left(p^z_{p_0^0}(\boldsymbol{p}_i^t + \boldsymbol{k}^t) - p^z_{p_0^0}(\boldsymbol{p}_{i-1}^t + \boldsymbol{k}^t)\right).
\end{aligned}
\tag{3.103}
$$

As before we can add and subtract $p^z_\omega(\boldsymbol{k}^t)$ and reorganize the terms as in (3.34) in order to obtain, using (3.35),

$$
\begin{aligned}
\delta p_i^z - \delta u_i^z &= \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t) + p^z_\omega(\boldsymbol{k}^t) - p^z_{p_0^0}(\boldsymbol{p}_i^t + \boldsymbol{k}^t)\right) \\
&\quad - \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_{i-1}^t) + p^z_\omega(\boldsymbol{k}^t) - p^z_{p_0^0}(\boldsymbol{p}_{i-1}^t + \boldsymbol{k}^t)\right) = -\frac{k_\mu p_i^\mu}{p_0^0} + \frac{k_\mu p_{i-1}^\mu}{p_0^0}.
\end{aligned}
\tag{3.104}
$$

Correspondingly the phase can be rearranged in the well known classical form

$$
\begin{aligned}
-i\sum_{i=k+1}^{j-1}(\delta p_i^z - \delta u_i^z)z_i &= -i\sum_{i=k+1}^{j-1}\frac{k_\mu p_{i-1}^\mu}{p_0^0}z_i + i\sum_{i=k+1}^{j-1}\frac{k_\mu p_i^\mu}{p_0^0}z_i \\
&= +i\frac{k_\mu p_{j-1}^\mu}{p_0^0}z_{j-1} - i\sum_{i=k+1}^{j-2}\frac{k_\mu p_i^\mu}{p_0^0}(z_{i+1} - z_i) - i\frac{k_\mu p_k^\mu}{p_0^0}z_{k+1}.
\end{aligned}
\tag{3.105}
$$

This passage from z_{k+1} to z_{j-1} produces then the largest contribution to the longitudinal phase accumulation,

$$A_{k+1}^{j-1} = \delta_{s_k}^{s_{j-1}} \delta_{r_k}^{r_{j-1}} \left(\prod_{i=k+1}^{j-1} (2\pi)^3 \delta^3(\boldsymbol{p}_i + \boldsymbol{k} - \boldsymbol{u}_i) \times \phi_{inc}(\delta p_i) \right) \times \exp\left(+i \frac{k_\mu p_{j-1}^\mu}{p_0^0} z_{j-1} - i \sum_{i=k+1}^{j-2} \frac{k_\mu p_i^\mu}{p_0^0} (z_{i+1} - z_i) - i \frac{k_\mu p_k^\mu}{p_0^0} z_{k+1} \right). \tag{3.106}$$

The term corresponding to a retarded emission after the collision at z_j is given by the incoherent average

$$A_j^+ = \left\langle S_{s_j s_{j-1}}^{n(z_j)}(p_j, p_{j-1})\ S_{r r_{j-1}}^{n(z_j),*}(u_j + k, u_{j-1}) \right\rangle. \tag{3.107}$$

Here one amplitude is on-shell with $p_{j-1}^0 = p_0^0 - \omega$ and the other is instead with the initial energy $u_{j-1}^0 = p_0^0$. Since from (3.106) we can set $\boldsymbol{p}_{j-1} + \boldsymbol{k} = \boldsymbol{u}_{j-1}$ then using (3.88) we find $\boldsymbol{p}_j^t = \boldsymbol{u}_j^t$ and thus

$$\begin{aligned}\delta p_j^z - \delta u_j^z &= \left(p_{p_0^0-\omega}^z(\boldsymbol{p}_j^t) - p_{p_0^0-\omega}^z(\boldsymbol{p}_{j-1}^t) \right) - \left(p_{p_0^0}^z(\boldsymbol{u}_j^t + \boldsymbol{k}^t) - p_{p_0^0}^z(\boldsymbol{u}_{j-1}^t) \right) \\ &= \left(p_{p_0^0-\omega}^z(\boldsymbol{p}_j^t) - p_{p_0^0-\omega}^z(\boldsymbol{p}_{j-1}^t) \right) - \left(p_{p_0^0}^z(\boldsymbol{p}_j^t + \boldsymbol{k}^t) - p_{p_0^0}^z(\boldsymbol{p}_{j-1}^t + \boldsymbol{k}^t) \right).\end{aligned} \tag{3.108}$$

By adding and subtracting $p_\omega^z(\boldsymbol{k}^t)$ and using (3.35) we find again a longitudinal phase

$$\begin{aligned}(\delta p_j^z - \delta u_j^z) &= \left(p_{p_0^0-\omega}^z(\boldsymbol{p}_j^t) + p_\omega^z(\boldsymbol{k}^t) - p_{p_0^0}^z(\boldsymbol{p}_j^t + \boldsymbol{k}^t) \right) \\ &- \left(p_{p_0^0-\omega}^z(\boldsymbol{p}_{j-1}^t) + p_\omega^z(\boldsymbol{k}^t) - p_{p_0^0}(\boldsymbol{p}_{j-1}^t + \boldsymbol{k}^t) \right) = -\frac{k_\mu p_j^\mu}{p_0^0} + \frac{k_\mu p_{j-1}^\mu}{p_0^0},\end{aligned} \tag{3.109}$$

so this term produces

$$A_j^+ = \delta_{s_{j-1}}^{s_j} \delta_{r_{j-1}}^{r} (2\pi)^3 \delta^3(\boldsymbol{p}_j - \boldsymbol{u}_j) \times \phi(\delta p_j) \exp\left(+i \frac{k_\mu p_j^\mu}{p_0^0} z_j - i \frac{k_\mu p_{j-1}^\mu}{p_0^0} z_j \right). \tag{3.110}$$

Similarly the term corresponding to an advanced emission previous to the collision at z_j is given by

$$A_j^- = \left\langle S_{s_j s_{j-1}}^{n(z_j)}(p_j, p_{j-1}) S_{s_j' s'}^{n(z_j),*}(u_j, u_{j-1} - k) \right\rangle \tag{3.111}$$

where now both amplitudes are already on-shell with the final energy $p_j^0 = u_j^0 = p_0^0 - \omega$. As before from (3.106) we set $\boldsymbol{p}_{j-1} + \boldsymbol{k} = \boldsymbol{u}_{j-1}$ and then $\boldsymbol{p}_j^t = \boldsymbol{u}_j^t$ so the phase vanishes

$$\delta p_j^z - \delta u_j^z = \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_j^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{j-1}^t)\right) - \left(p^z_{p_0^0-\omega}(\boldsymbol{u}_j^t) - p^z_{p_0^0-\omega}(\boldsymbol{u}_{j-1}^t - \boldsymbol{k})\right) = 0, \tag{3.112}$$

and we find for this term

$$A_j^- = \delta_{s_{j-1}}^{s_j}\delta_r^{r_j}(2\pi)^3\delta^3(\boldsymbol{p}_j - \boldsymbol{u}_j) \times \phi(\delta p_j) \tag{3.113}$$

Finally, the last step of scatterings from z_{j+1} to z_n is given by the incoherent averages

$$A_{j+1}^n = \left(\prod_{i=j+1}^{n} \left\langle S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1}) S^{n(z_i),*}_{r_i r_{i-1}}(u_i, u_{i-1})\right\rangle\right). \tag{3.114}$$

In this case using either (3.110) or (3.113) the relation $\boldsymbol{p}_i + \boldsymbol{k} = \boldsymbol{u}_i + \boldsymbol{k}$ holds for $i = j, \dots, n$ and the longitudinal phase vanishes in all the range since

$$\delta p_i^z - \delta u_i^z = \left(p^z_{p_0^0-\omega}(\boldsymbol{p}_i^t) - p^z_{p_0^0-\omega}(\boldsymbol{p}_{i-1}^t)\right) - \left(p^z_{p_0^0-\omega}(\boldsymbol{u}_i^t) - p^z_{p_0^0-\omega}(\boldsymbol{u}_{i-1}^t)\right) = 0, \tag{3.115}$$

so we get a simple convolution of incoherent elastic averages without longitudinal phases, as it was the case of the electron path from z_1 to z_{k-1},

$$A_{j+1}^n = \delta_{s_j}^{s_n}\delta_{r_j}^{s_n} \times \left(\prod_{i=j+1}^{n} (2\pi)^3\delta^3(\boldsymbol{u}_i - \boldsymbol{p}_i) \times \phi(\delta p_i)\right). \tag{3.116}$$

Upon inserting (3.94) (3.97) (3.101) (3.106) (3.110) (3.113) and (3.116) in (3.85) we find

$$\begin{aligned}
\left\langle \mathcal{M}_k^a \mathcal{M}_j^{a,*}\right\rangle = &\left(\frac{f^{\lambda,*}_{s_n s_0}(p_j, p_j + k)}{k_\mu p_j^\mu / p_0^0} - \frac{f^{\lambda,*}_{s_n s_0}(p_{j-1} - k, p_{j-1})}{k_\mu p_{j-1}^\mu/(p_0^0 - \omega)}\right) \\
&\times \exp\left(-i\sum_{i=k}^{j-1}\frac{k_\mu p_i^\mu}{p_0^0}(z_{i+1} - z_i)\right)\left(\frac{f^{\lambda}_{s_n s_0}(p_k, p_k + k)}{k_\mu p_k^\mu / p_0^0} - \frac{f^{\lambda}_{s_n s_0}(p_{k-1} - k, p_{k-1})}{k_\mu p_{k-1}^\mu/(p_0^0 - \omega)}\right) \\
&\times \left(\prod_{i=j}^{n}\phi_{inc}(\delta p_i) \times (2\pi)^3\delta^3(\boldsymbol{p}_i - \boldsymbol{u}_i)\right)\left(\prod_{i=k+1}^{j-1}\phi_{inc}(\delta p_i) \times (2\pi)^3\delta^3(\boldsymbol{p}_i + \boldsymbol{k} - \boldsymbol{u}_i)\right) \\
&\times \left(\phi_{inc}(\delta p_k + k) \times (2\pi)^3\delta^3(\boldsymbol{p}_k + \boldsymbol{k} - \boldsymbol{u}_k)\right)\left(\prod_{i=1}^{k-1}\phi_{inc}(\delta p_i) \times (2\pi)^3\delta^3(\boldsymbol{p}_i - \boldsymbol{u}_i)\right)
\end{aligned} \tag{3.117}$$

The evaluation of term $\langle \mathcal{M}_k^a\rangle\langle \mathcal{M}_j^{a,*}\rangle$ at (3.86) follows the same steps as (3.85). Since the kinematical conditions in (3.88) equally hold for for the averages (3.91), we only have to replace $\phi_{inc}(\delta p_i)$ with the no collision distribution $\phi^{(0)}_{inc}(\delta p_i)$ in the above expression. We get

$$
\begin{aligned}
\langle \mathcal{M}^a_k \rangle \left\langle \mathcal{M}^{a,*}_j \right\rangle &= \left(\frac{f^{\lambda,*}_{s_n s_0}(p_j, p_j + k)}{k_\mu p^\mu_j / p^0_0} - \frac{f^{\lambda,*}_{s_n s_0}(p_{j-1} - k, p_{j-1})}{k_\mu p^\mu_{j-1}/(p^0_0 - \omega)} \right) \qquad (3.118) \\
&\times \exp\left(-i \sum_{i=k}^{j-1} \frac{k_\mu p^\mu_i}{p^0_0}(z_{i+1} - z_i) \right) \left(\frac{f^{\lambda}_{s_n s_0}(p_k, p_k + k)}{k_\mu p^\mu_k / p^0_0} - \frac{f^{\lambda}_{s_n s_0}(p_{k-1} - k, p_{k-1})}{k_\mu p^\mu_{k-1}/(p^0_0 - \omega)} \right) \\
&\times \left(\prod_{i=j}^{n} \phi^{(0)}_{inc}(\delta p_i) \times (2\pi)^3 \delta^3(\boldsymbol{p}_i - \boldsymbol{u}_i) \right) \left(\prod_{i=k+1}^{j-1} \phi^{(0)}_{inc}(\delta p_i) \times (2\pi)^3 \delta^3(\boldsymbol{p}_i + \boldsymbol{k} - \boldsymbol{u}_i) \right) \\
&\times \left(\phi^{(0)}_{inc}(\delta p_k + k) \times (2\pi)^3 \delta^3(\boldsymbol{p}_k + \boldsymbol{k} - \boldsymbol{u}_k) \right) \left(\prod_{i=1}^{k-1} \phi^{(0)}_{inc}(\delta p_i) \times (2\pi)^3 \delta^3(\boldsymbol{p}_i - \boldsymbol{u}_i) \right).
\end{aligned}
$$

With this result we can integrate in the auxiliary momentum $\boldsymbol{u}_i$. Since $\boldsymbol{u}_n \equiv \boldsymbol{p}_n$ there is an extra 3-Dirac delta accounting for time translation invariance and transverse homogeneity. In addition if we assume that $\omega \ll p^0_0$ then we can neglect the momentum carried by the photon in the elastic distributions and a current is found which factorizes as

$$
\begin{aligned}
\Sigma^{(n)}_{em} &= e^2 \mathcal{N}^2(k)(2\pi)^2 \delta^{(2}(\mathbf{0}) 2\pi \beta_p \delta(0) \qquad (3.119) \\
&\times \left(\prod_{i=1}^{n-1} \int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3} \right) \left\{ \left(\prod_{i=1}^{n} \phi^{(n)}_{inc}(\delta p_i) + \phi^{(0)}_{inc}(\delta p_i) \right) - \left(\prod_{i=1}^{n} \phi^{(0)}_{inc}(\delta p_i) \right) \right\} \left| \sum_{k=1}^{n} J_k \right|^2,
\end{aligned}
$$

where we defined the emission current as

$$
J_k \equiv \left(\frac{f^{\lambda}_{s_n s_0}(p_k - k, p_k)}{k_\mu p^\mu_k / (p^0_0 - \omega)} - \frac{f^{\lambda}_{s_n s_0}(p_{k-1} - k, p_{k-1})}{k_\mu p^\mu_{k-1}/(p^0_0 - \omega)} \right) \exp\left(-i \sum_{i=1}^{k-1} \frac{k_\mu p^\mu_i}{(p^0_0 - \omega)} \delta z \right). \qquad (3.120)
$$

By identifying $2\pi\delta(0) \equiv T$ and $(2\pi)^2\delta(\mathbf{0}) \equiv \pi R^2$ we can finally integrate in the electron final momentum $\boldsymbol{p}_n$ in order to obtain

$$
\begin{aligned}
\omega \frac{dI_{inc}}{d\omega d\Omega} &= \frac{e^2}{(2\pi)^2} \left(\prod_{i=1}^{n} \int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3} \right) \left\{ \left(\prod_{i=1}^{n} \phi^{(n)}_{inc}(\delta p_i) + \phi^{(0)}_{inc}(\delta p_i) \right) - \left(\prod_{i=1}^{n} \phi^{(0)}_{inc}(\delta p_i) \right) \right\} \\
&\times \frac{\omega^2}{2} \sum_{\lambda,s} \left| \sum_{k=1}^{n} J_k \right|^2. \qquad (3.121)
\end{aligned}
$$

The resulting expression is just the evaluation over the elastic distribution at each layer, given either by $\phi^{(n)}_{inc}(\delta p_i, \delta z)$ in case of collision or by $\phi^{(0)}_{inc}(\delta p_i, \delta z)$ in case of no collision, of the current $\sum J_k$, which except for spin corrections in the hard part of the spectrum agrees with the classical current. Finally, we remove the squared no collision emission diagram, which corresponds to no colliding at any of the layers. Further simplifications can be done for the unpolarized and spin averaged intensity as we did in the coherent contribution. In that case one can define two effective currents, *c.f.* Appendix A, one corresponding to spin preserving vertices and agreeing with the classical current, the other corresponding to spin flipping vertices, correcting the classical contribution for hard photons $y = \omega/p_0^0 \sim 1$,

$$\omega\frac{dI_{inc}}{d\omega d\Omega} = \frac{e^2}{(2\pi)^2}\left(\prod_{i=1}^{n}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left\{\left(\prod_{i=1}^{n}\phi_{inc}^{(n)}(\delta p_i) + \phi_{inc}^{(0)}(\delta p_i)\right) - \left(\prod_{i=1}^{n}\phi_{inc}^{(0)}(\delta p_i)\right)\right\}$$
$$\times\left(h^n(y)\left|\sum_{k=1}^{n}\boldsymbol{\delta}_k^n\exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p_i^\mu}{p_0^0-\omega}\delta z\right)\right|^2 + h^s(y)\left|\sum_{k=1}^{n}\delta_k^s\exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p_i^\mu}{p_0^0-\omega}\delta z\right)\right|^2\right), \tag{3.122}$$

where the spin non flip currents agree with the classical result (3.14)

$$\boldsymbol{\delta}_k \equiv \boldsymbol{k}\times\left(\frac{\boldsymbol{p}_k}{k_\mu p_k^\mu} - \frac{\boldsymbol{p}_{k-1}}{k_\mu p_{k-1}^\mu}\right), \tag{3.123}$$

and the spin flip currents are given by

$$\delta_k \equiv \frac{p_0^0\omega}{\omega p_0^0 - \boldsymbol{k}\cdot\boldsymbol{p}_k} - \frac{p_0^0\omega}{\omega p_0^0 - \boldsymbol{k}\cdot\boldsymbol{p}_{k-1}}. \tag{3.124}$$

The weighting functions of the two contributions are given by $h^n(y) = (1 + (1 - y)^2)/2$ and $h^s(y) = y^2/2$. The phases the single Bethe-Heitler amplitudes (3.123) and (3.124), agreeing in this incoherent average with the classical phases, are responsible of causing the interferences in their squared sum, leading to the LPM, the dielectric and the transition radiation effects. The interference pattern is essentially the same as the one discussed in the classical correspondence at Section 3.2. The composition of the elastic propagations at each layer produces a squared momentum transfer additive in the traveled length. If δl verifies $\delta l \leq \lambda$ where $\lambda = 1/n_0\sigma_t^{(1)}$ is the average mean free path, using (3.89) and (2.90) we arrive at (2.134)

$$\frac{\partial}{\partial l}\langle\delta\boldsymbol{p}^2(l)\rangle = n_0\sigma_t^{(1)}\langle\delta\boldsymbol{p}^2(\delta l)\rangle \equiv 2\hat{q}, \tag{3.125}$$

where we define the transport coefficient $\hat{q}$. This allows to relate the momentum transfer in a length l with the momentum transfer in a single collision δl, which for the Debye screened interaction produces (2.130)

$$\langle\delta\boldsymbol{p}^2(\delta l)\rangle = \left(2\log\left(\frac{2p_0^0}{\mu_d}\right) - 1\right)\mu_d^2 \equiv \eta\mu_d^2, \tag{3.126}$$

and the long tail introduced a substantial correction η to the naive expected value μ_d^2 after a single collision and a maximum momentum transfer of $2p_0^0$ is allowed due to the energy conservation delta in (3.89). However, since electron momenta appears convoluted in (3.122) together with the Bethe-Heitler functions $\boldsymbol{\delta}_k$ at (3.123) or δ_k at (3.124), the momentum transfer can be approximated as of the order $|\delta \boldsymbol{p}| \simeq 2.5 m_e$ and by using $\mu_d = \alpha m_e Z^{1/3}$ we obtain instead

$$\eta = 2\left(\log\left(\frac{2.5m_e}{\mu_d}\right) - \frac{1}{2}\right) = 2\left(\log\left(\frac{2.5}{\alpha Z^{1/3}}\right) - \frac{1}{2}\right). \tag{3.127}$$

This can be numerically checked and matches Bethe's [6, 64] $\eta = 2\log(183/Z^{1/3})$ prediction within less than a 3% of deviation in the range $Z = (1, 100)$. The choice of η defines $\hat{q}$ and thus the Fokker-Planck approximation (2.126) for the scattering distribution $\hat{\Sigma}_2(\delta \boldsymbol{p}, z)$ in (3.89). However, since this fix is only valid for $\delta l \leq \lambda$ this approximation is valid only in the incoherent plateau, where the single scattering regime holds. A local with ω definition of η and $\hat{q}$ has to be employed in order to match the Debye screened intensity in all the spectrum. In this way, a single Fokker-Planck approximation can not be used unless the medium length is assumed infinite, in which case the lower plateau can be neglected and the fail of the Fokker-Planck approximation becomes irrelevant.

Intensity (3.122) is suitable for a numerical evaluation for finite size targets under a general interaction. A Monte Carlo code has been developed in which the electron is assumed to describe a piece-wise zig-zag path [47] where the step size is taken as $\delta z = 0.1\lambda$. This leads to paths going from $\sim$ 1500 steps for the shortest mediums to $\sim$ 250000 steps for the largest. The integration in the electron momenta transforms into an average over the paths under the elastic weight (3.89). In a typical run around 10^4 paths had to be computed in order to obtain reasonable precision, spanning 50 photon frequencies and 100 photon angles. Several medium materials and lengths were chosen in order to compare with SLAC [44] and CERN [46] data.

We can follow an alternative approach which helps to qualitatively understand the behavior of the found intensity if we observe that (3.122) is just a sum of single Bethe-Heitler amplitudes $\boldsymbol{\delta}_k$ carrying a phase. These amplitudes are summed, squared and then weighted by the incoherent averages (3.89) of the electron elastic intensity once the photon energy gap effect is neglected in its velocity. This agrees with the classical intensity (3.15) and thus produces crossed terms $\sim \boldsymbol{\delta}_k \boldsymbol{\delta}_j e^{i\varphi_k^j}$ interfering with a phase φ_k^j. We define the distance or coherence length in which $\delta l = z_j - z_k$ in which φ_k^j becomes larger than unity. Using (3.16) and (3.125) we get

$$\varphi_k^j = \frac{1}{p_0^0 - \omega}\int_{z_k}^{z_j} dz\, k_\mu p^\mu(z) \simeq \frac{\omega}{2p_0^0(p_0^0 - \omega)}\left(m_e^2 \delta l + \hat{q}(\delta l)^2\right) = 1, \tag{3.128}$$

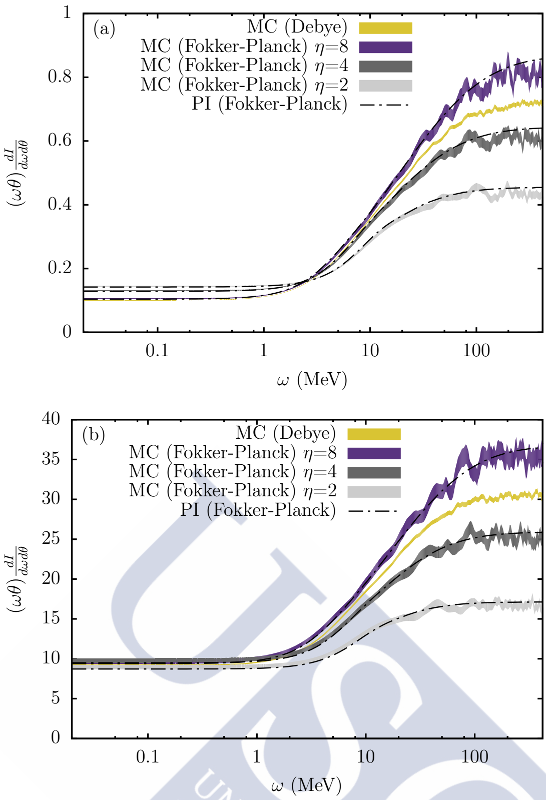


Figure 3.3: Differential intensity of photons in the angle $\theta = 0.01/\gamma_e$ (a) and $\theta = 0.5/\gamma_e$ (b) radiated from electrons of $p_0^0 = 8$ GeV, $\gamma_e = p_0^0/m$, after traversing an Au sheet of $l = 0.0023$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown in solid yellow line for the Debye interaction and for the Fokker-Planck approximation with $\eta = 8$ (purple), $\eta = 4$ (dark grey) and $\eta = 2$ (light grey). Also shown with dot-dashed lines the respective continuous limits of (3.122), leading to the path integral in the Fokker-Planck approximation (3.178).

which by inversion produces a coherence length modulated by the photon frequency

$$\delta l(\omega) \equiv \frac{m_e^2}{2\hat{q}}\left(\sqrt{1+\frac{8\hat{q}p_0^0(p_0^0-\omega)}{m_e^4\omega}}-1\right). \tag{3.129}$$

This defines two characteristic frequencies of the LPM interference: the frequency ω_c at which the coherence length becomes of the order of the medium length $\delta l(\omega_c) = l$ thus $\omega_c \simeq p_0^0(p_0^0-\omega)/(m_e^2 l + \hat{q}l^2)$ and the frequency ω_s at which the coherence length becomes of the order of a mean free path $\delta l(\omega_s) = \lambda$ thus $\omega_s \simeq p_0^0(p_0^0-\omega)/(m_e^2\lambda + \hat{q}\lambda^2)$. Since for $\delta l \geq l$ there are no sources of scattering we further impose to (3.129) $\delta l(\omega) = l$ for $\omega \geq \omega_c$. In a coherence length the phase can be neglected so that the internal structure of scattering becomes irrelevant, *c.f.* Section 2.1, and the centers in $\delta l(\omega)$ act like a single scattering source with a charge equivalent to the total matter in $\delta l(\omega)$. Since there are $l/\delta l(\omega)$ of these coherence lengths for a given ω we simply write

$$\omega \frac{dI_{inc}}{d\omega}(l) = \frac{l}{\delta l(\omega)} e^2 \frac{d\Omega_k}{(2\pi)^2} \int \frac{d^3\delta \boldsymbol{p}}{(2\pi)^3} \left(h^n(y)|\boldsymbol{\delta}_1^n|^2 + h^s(y)|\delta_1^s|^2 \right) \phi_{inc}^{(n)}(\delta \boldsymbol{p}, \delta l(\omega)). \tag{3.130}$$

By inserting (3.89) and integrating in the photon solid angle Ω_k one gets

$$\omega \frac{dI_{inc}}{d\omega}(l) = \frac{l}{\delta l(\omega)} \frac{e^2}{\pi^2} \int_0^\pi d\theta \sin(\theta) F(\theta) \hat{\Sigma}_2(\delta \boldsymbol{p}, \delta l(\omega)), \tag{3.131}$$

where $|\delta \boldsymbol{p}| = 2p_0^0 \beta \sin(\theta)$ is the electron momentum change and the $F(\theta)$ function is given by

$$F(\theta) = \Big[\frac{1 - \beta^2 \cos\theta}{2\beta \sin(\theta/2) \sqrt{1 - \beta^2 \cos^2(\theta/2)}} \times \log \Big[\frac{\sqrt{1 - \beta^2 \cos^2(\theta/2)} + \beta \sin(\theta/2)}{\sqrt{1 - \beta^2 \cos^2(\theta/2)} - \beta \sin(\theta/2)} \Big] - 1 \Big]. \tag{3.132}$$

This last integral can be numerically evaluated both for the Debye screened interaction for $\Sigma_2(\delta \boldsymbol{p}, \delta z)$ and for its Fokker-Planck approximation (2.126), and the resulting values for $\omega \gg \omega_s$ and $\omega \ll \omega_c$, i.e. for the incoherent and coherent plateaus, respectively, are exact. In the Fokker-Planck approximation one can further write an useful interpolating function for these two asymptotic values,

$$\omega \frac{dI_{inc}}{d\omega}(l) = \frac{l}{\delta l(\omega)} \frac{2e^2}{\pi} \frac{1 + n_m(\omega)}{3A + n_m(\omega)} \log \Big(1 + A n_m(\omega) \Big), \tag{3.133}$$

where $n_m(\omega) \simeq 2\hat{q}\delta l(\omega)/m_e^2$ is a measure of the number of transverse masses acquired in a coherence length and $A = e^{-(1+\gamma)}$ where γ is Euler's constant. Then the qualitative behavior of the LPM effect is as follows. For frequencies lower than ω_c the coherence length extends beyond the medium length and then the radiation intensity consists in the difference of the first and last photon diagrams squared. In this coherent plateau the vanishing phase causes trivial convolutions of the elastic distributions at each layer and it can be shown that the total distribution is then given by $\Sigma_2^{(n)}(\delta \boldsymbol{p}, l)$, as stated in (3.130). For frequencies larger than ω_c using (3.129) the number of independent emitters $l/\delta l(\omega)$ grows with $\sqrt{\omega}$ whereas the charge of each emitter logarithmically decreases with $\log(1/\sqrt{\omega})$. This enhancement from the coherence plateau stops around ω_s, when the coherence length acquires the minimum thickness of matter $\lambda = 1/n_0\sigma_t^{(1)}$ to produce radiation, since in absence of collision both (3.123) and (3.124) vanish. In this incoherence plateau the radiation consists in the incoherent superposition of the $n_c = l/\lambda$ single Bethe-Heitler intensities, where n_c is the average number of collisions. This maximal decoupling of the intensity can be diagrammatically understood as if the phase difference introduced by the internal fermion lines is so big that the electron emerging of an emission after a collision is real and incoherent with the next emission diagram.

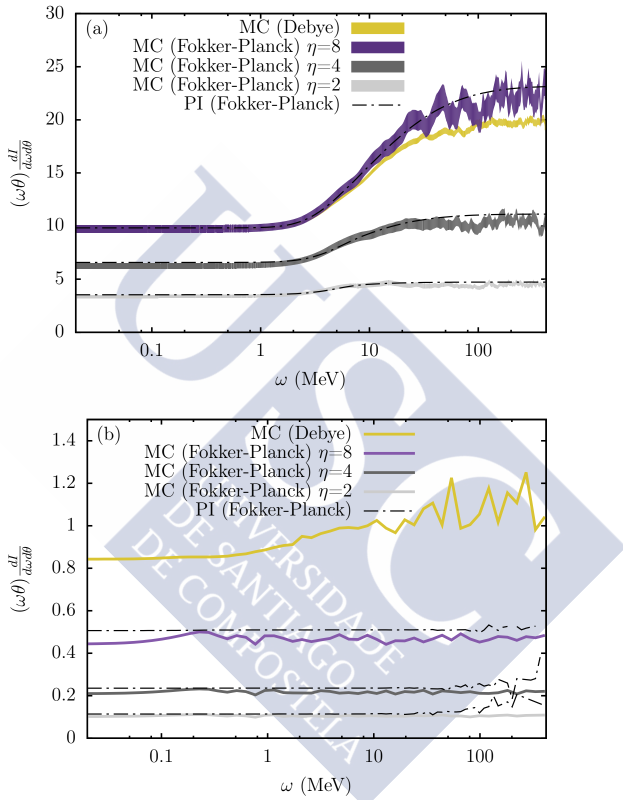


Figure 3.4: Differential intensity of photons in the angle $\theta = 2/\gamma_e$ (a) and $\theta = 10/\gamma_e$ (b) radiated from electrons of $p_0^0 = 8$ GeV, $\gamma_e = p_0^0/m$, after traversing a Gold sheet of $l = 0.0023$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown in solid yellow line for the Debye interaction and for the Fokker-Planck approximation with $\eta = 8$ (purple), $\eta = 4$ (dark grey) and $\eta = 2$ (light grey). Also shown with dot-dashed lines the respective continuous limits of (3.122), leading to the path integral in the Fokker-Planck approximation (3.178).

The radiation intensity coming from electrons of $p_0^0 = 8$ GeV for several photon angles is depicted in Fig. 3.3 and Fig. 3.4 both for the Debye screened interaction and the Fokker-Planck evaluations of (3.122). A target of Gold of $l = 0.0023$ cm is chosen which corresponds to an average of $n_c = 862$ collisions. For this element we obtain a screening mass estimate of $\mu_d = 16$ KeV and a transport parameter of $\hat{q} = (\eta/2)\times 1.89$ KeV^3 with $\eta \simeq 8$ in order to match the angle-integrated incoherent plateau and comparable to our estimate from (3.127) $\eta \simeq 7.76$. As it can be clearly seen the Fokker-Planck approximation which matches the incoherent (single) emission integrated spectrum, mismatches the unintegrated spectrum. At the lower angles the Fokker-Planck approximation overestimates the intensity by a $\sim 20\%$ while at the larger angles the Fokker-Planck approximation underestimates the intensity. For the larger angle $\theta = 10\gamma_e^{-1}$ in particular we find that only half of the real emission is taken into account. In Fig. 3.5, Fig. 3.6 and Fig. 3.7 we show the angle-integrated

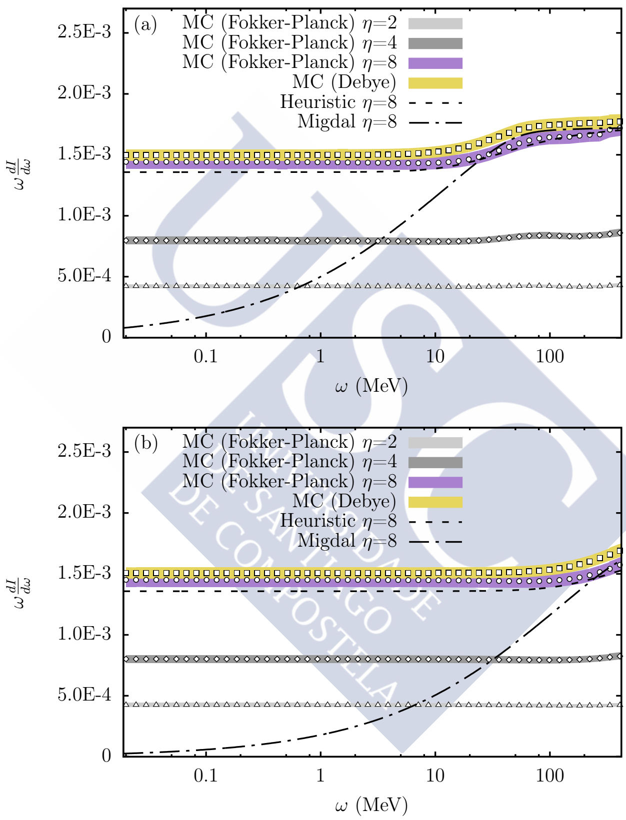


Figure 3.5: Differential intensity of photons radiated from electrons of $p_0^0 = 8$ GeV (a) and $p_0^0 = 25$ GeV (b) after traversing a Gold sheet of $l = 0.00038$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 8$ (purple and circles), $\eta = 4$ (dark grey and diamonds) and $\eta = 2$ (light grey and triangles). Also shown Migdal prediction (3.122) (dot-dashed line) with $\eta = 8$ and our heuristic formula for finite size targets in the Fokker-Planck approximation (dashed line).

spectrum for Gold targets of $l = 0.00038$ cm, $l = 0.0023$ cm and $l = 0.02$ cm, which correspond to an average number of $n_c = 142$, $n_c = 862$ and $n_c = 7502$ collisions, respectively. Two electron energies are shown, $p_0^0 = 8$ Gev and $p_0^0 =$

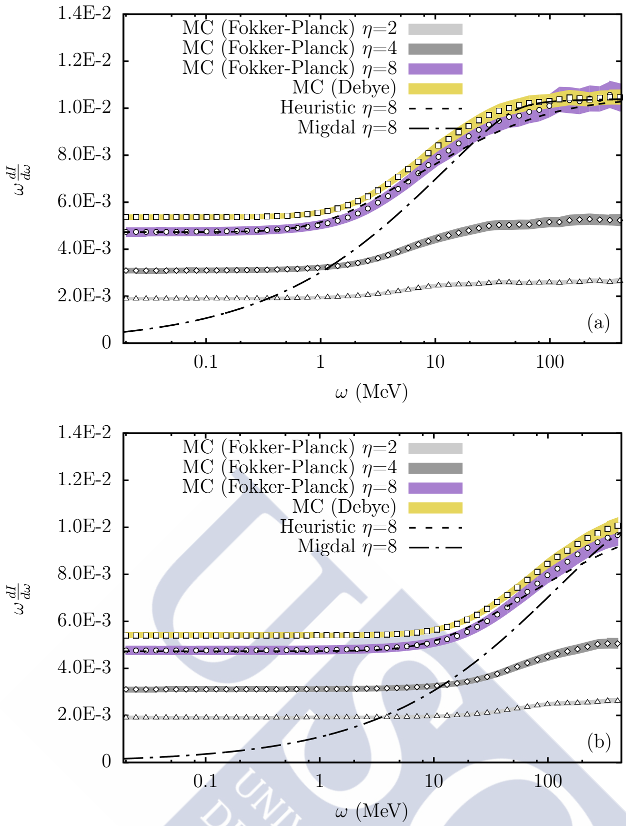


Figure 3.6: Differential intensity of photons radiated from electrons of $p_0^0 = 8$ GeV (a) and $p_0^0 = 25$ GeV (b) after traversing a Gold sheet of $l = 0.0023$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 8$ (purple and circles), $\eta = 4$ (dark grey and diamonds) and $\eta = 2$ (light grey and triangles). Also shown Migdal prediction (3.122) with $\eta = 8$ (dot-dashed line) and our heuristic formula for finite size targets in the Fokker-Planck approximation (dashed line).

25 GeV, and we present the Debye screened interaction and the Fokker-Planck evaluations of (3.122). For $l = 0.00038$ cm the predicted characteristic frequencies are $\omega_c = 8$ MeV and $\omega_s = 1.1$ GeV for electrons of $p_0 = 8$GeV, and $\omega_c = 80$ MeV and $\omega_s = 11$ GeV for electrons of $p_0 = 25$ GeV. Since a small number of collisions is occurring and the medium finiteness is taken into account, the difference between the coherent plateau and the incoherent plateau is small. For $l = 0.0023$ cm the characteristic frequencies (3.19) and (3.22) are given by $\omega_c = 0.48$ MeV and $\omega_s = 418$ MeV for electrons of $p_0 = 8$ GeV, and $\omega_c = 4.7$ MeV and $\omega_s = 4$ GeV for electrons of $p_0 = 25$ GeV. For $l = 0.02$ cm we obtain $\omega_c = 8$ KeV and $\omega_s = 60$ MeV for electrons of $p_0 = 8$ GeV, and $\omega_c = 80$ keV and $\omega_s = 588$ MeV for electrons of $p_0 = 25$ GeV. We also show our heuristic formula

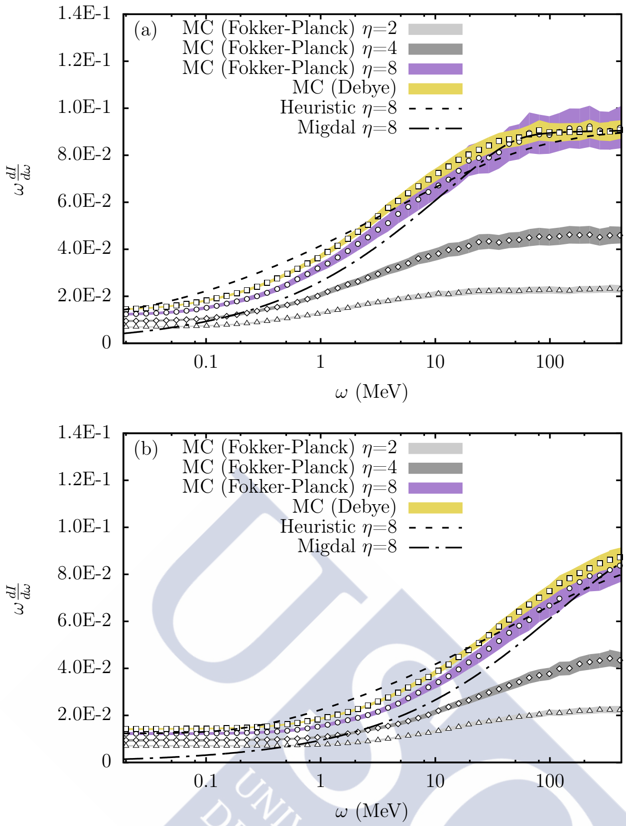


Figure 3.7: Differential intensity of photons radiated from electrons of $p_0^0 = 8$ GeV (a) and $p_0^0 = 25$ GeV (b) after traversing a Gold sheet of $l = 0.0023$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 8$ (purple and circles), $\eta = 4$ (dark grey and diamonds) and $\eta = 2$ (light grey and triangles). Also shown Migdal prediction (3.122) with $\eta = 8$ (dot-dashed line) and our heuristic formula for finite size targets in the Fokker-Planck approximation (dashed line).

(3.130) in the Fokker-Planck approximation, producing a reasonable agreement with (3.122), in particular in the coherence and incoherence plateaus, where it becomes exact. In Fig. 3.8 we show the angle-integrated spectrum for a Carbon target of $l = 0.41$ cm which produces an average number of $n_c = 9521$ collisions, for electron energies of $p_0^0 = 8$ GeV and $p_0^0 = 25$ GeV. Estimates of $\mu_d = 6.8$ KeV, $\hat{q} = (\eta/2) \times 2.10 \cdot 10^{-2}$ KeV^3 and $\eta = 11$ are obtained. The evaluation of (3.122) in the Debye screened interaction is shown together with its Fokker-Planck approximation. The characteristic frequencies are given by $\omega_c = 1$ KeV and $\omega_s = 11$ MeV for electrons of $p_0^0 = 8$ GeV, and $\omega_c = 11$ KeV and $\omega_s = 108$ MeV for electrons of $p_0^0 = 25$ GeV. The introduction of medium effects in the photon dispersion relation leads to the dielectric and the transition radiation effects, *c.f.*

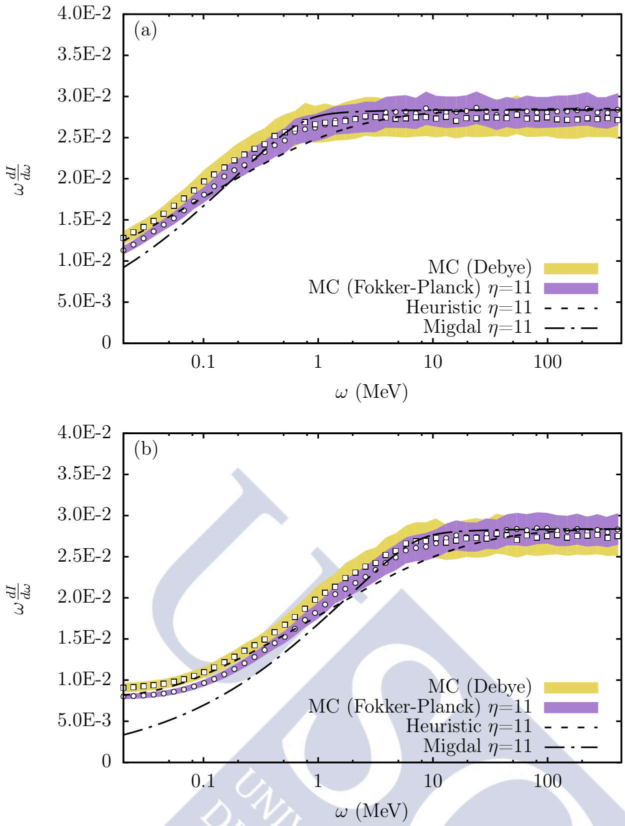


Figure 3.8: Differential intensity of photons radiated from electrons of $p_0^0 = 8$ GeV (a) and $p_0^0 = 25$ GeV (b) after traversing a Carbon sheet of $l = 0.41$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 11$ (purple and circles). Also shown Migdal prediction (3.122) with $\eta = 11$ (dot-dashed line) and our heuristic formula for finite size targets in the Fokker-Planck approximation (dashed line).

Section 3.2. The photon plasma frequency ω_p can be thought as an effective photon mass m_γ which introduces an extra term $m_\gamma^2/2\omega$ in the resonances $k_\mu p^\mu(z)$ at the phases and propagators. Then, a strong suppression occurs for frequencies lower than ω_{de}, which is defined as the frequency at which the extra term becomes of the same order than the phase evaluated at $m_\gamma = 0$, i.e. $\omega_{de}^2 = m_\gamma^2 l \omega_c$. In the particular case that the medium is finite and the photon mass cannot be assumed global for all the emission diagrams, the last photon verifies $m_\gamma = 0$ and thus the intensity is instead dramatically enhanced for frequencies $\omega \leq \omega_{de}$. In Fig.3.9 we show this dielectric and transition radiation effects of (3.122) for a Gold target of $l = 0.0023$ cm, which introduces an effective photon mass of $m_\gamma = 0.08$ KeV for all photons except for the one in the last leg $m_\gamma = 0$. The predicted characteristic frequencies are $\omega_{de} = 0.6$ MeV for $p_0^0 = 8$ GeV and $\omega_{de} = 1.9$ MeV

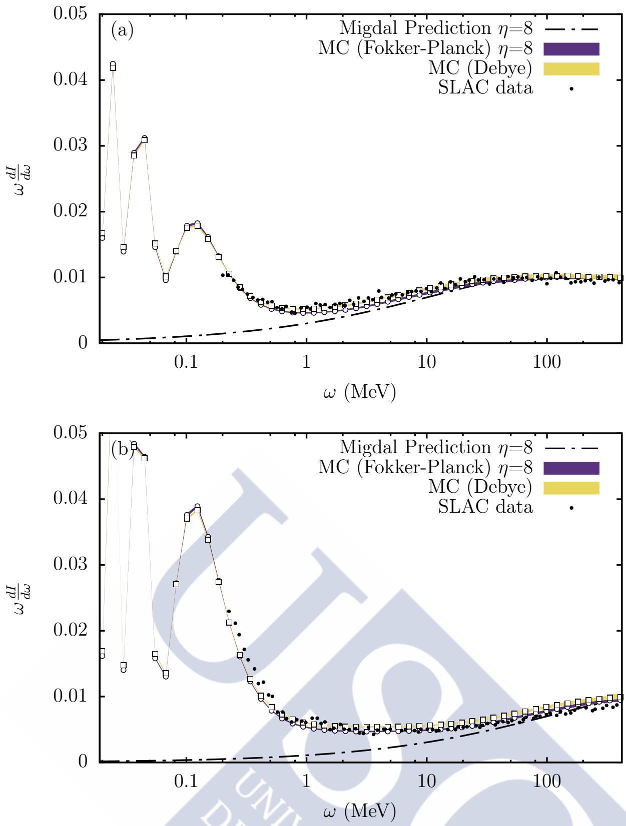


Figure 3.9: Differential intensity of photons radiated from electrons of $p_0^0 = 8$ GeV (a) and $p_0^0 = 25$ GeV (b) after traversing a Gold sheet of $l = 0.0023$ cm, as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 8$ (purple and circles). Also shown Migdal prediction (3.122) with $\eta = 8$ (dot-dashed line) and SLAC data.

for $p_0^0 = 25$ GeV. As it can be seen, for frequencies lower than ω_{de} a dramatic enhancement of the coherent plateau occurs, since the last leg diagram stops to be compensated due to the dielectric suppression of the first leg diagram and the phase interference between them grows as $m_\gamma^2/2\omega$. Also shown is the SLAC experimental data [44] for the same target, being in very good agreement with our evaluation. In Fig. 3.10 we show our results for an Iridium target of $l = 0.0128$ cm and electrons of $p_0^0 = 149$ GeV and a Copper target of $l = 0.063$ and electrons of $p_0^0 = 207$ GeV. Also shown is the CERN data [46] for the same scenario, which covers only a small part of the suppression zone. Slight differences are found in the LPM between theoretical predictions and the experimental data.

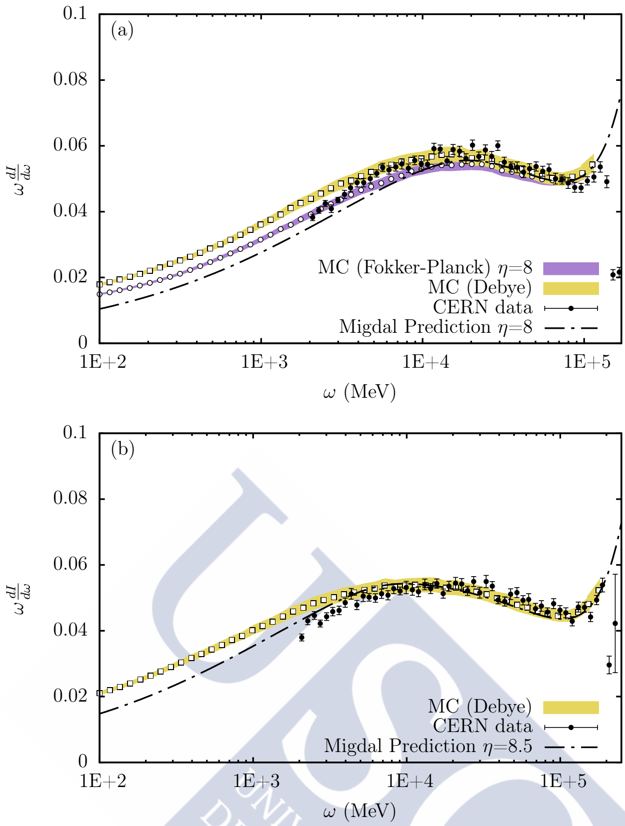


Figure 3.10: Differential intensity of photons radiated from electrons of $p_0^0 = 149$ GeV traversing a sheet of Iridium of $l = 0.0128$ cm (a) and electrons of $p_0^0 = 207$ GeV traversing a sheet of Copper of $l = 0.063$ cm (b), as a function of the photon energy. Monte Carlo evaluation of (3.122) is shown for the Debye interaction (yellow and squares) and for the Fokker-Planck approximation with $\eta = 8$ (purple and circles). Also shown Migdal prediction (3.122) with $\eta = 8$ (dot-dashed line) and SLAC data.

3.4 Continuous limit: a path integral

Several formalisms have been developed whose approach consists in a path integral redefinition of the Boltzmann method used by Migdal. These works rely on the integrability of the resulting path integrals in the Fokker-Planck/Gaussian approximation for the interaction. While this provides a powerful tool to evaluate the intensity we have shown that for finite size targets and for the angular distributions of the final particles the differences between the Fokker-Planck approximation and a direct evaluation of (3.122) under a Debye screened interaction cannot be reconciled into a single definition of the medium transport properties, given by $\hat{q}$. In this section we will show that our result (3.122) in the $\delta z \to 0$ continuous limit leads to a path integral formulation which agrees with the finite size predictions of [19] except for the vacuum regularization, which is strictly required to reproduce the right opacity/perturbative expansion but otherwise irrelevant in the large n_c limit and thus for the Fokker-Planck approximation. It also recuperates the result of Migdal [4], Zakharov [55] and the BDMPS group [22] in the $l \to \infty$ limit and under the right kinematical relaxations in the photon angular integration.

Transverse incoherent average

We will restrict our evaluation to the spin non flip contribution. The spin flip case follows the same steps. Intensity (3.122) is split into two contributions, the first with the incoherent elastic weights including collisions $\phi^{(n)}_{inc}(\delta p_i) + \phi^{(0)}_{inc}(\delta p_i)$ and the second with the elastic weights omitting collisions $\phi^{(0)}_{inc}(\delta p_i)$. We write

$$\omega\frac{dI}{d\omega d\Omega} = \omega\frac{dI^{(n)}}{d\omega d\Omega} - \omega\frac{dI^{(0)}}{d\omega d\Omega}. \tag{3.134}$$

In order to take the continuous limit we notice that a single Bethe-Heitler amplitude transforms into the infinitesimal change

$$\boldsymbol{\delta}_k = \boldsymbol{k} \times \left(\frac{\boldsymbol{p}_k}{k_\mu p^\mu_k} - \frac{\boldsymbol{p}_{k-1}}{k_\mu p^\mu_{k-1}}\right) \to \frac{\partial}{\partial z}\left(\frac{\boldsymbol{k}\times \boldsymbol{p}(z_k)}{k_\mu p^\mu(z_k)}\right). \tag{3.135}$$

This suggests rearranging the sum (integrating by parts) in order to find for $\delta z \ll 1$

$$\begin{aligned}\sum_{k=1}^{n} \boldsymbol{\delta}^n_k \exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p^\mu_i}{p^0_0}\delta z\right) &= \frac{\boldsymbol{k}\times\boldsymbol{p}_n}{k_\mu p^\mu_n}\exp\left(-i\sum_{i=1}^{n-1}\frac{k_\mu p^\mu_i}{p^0_0}\right)\\ &+i\frac{\delta z}{p^0_0}\sum_{k=1}^{n-1}\boldsymbol{k}\times\boldsymbol{p}_k \exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p^\mu_i}{p^0_0}\delta z\right) - \frac{\boldsymbol{k}\times\boldsymbol{p}_0}{k_\mu p^\mu_0}.\end{aligned} \tag{3.136}$$

The first term corresponds to the photon emitted in the final leg, and in the continuous limit corresponds to the integration between $[l,\infty)$. The middle term are just the internal photons and correspond to the integration in $(0,l)$. The third term is the photon emitted in the first leg and corresponds to the integration in $(-\infty,0]$. The boundary photons can be taken into account by extending the integration of the interior term to $(-\infty,+\infty)$ and neglecting the $+\infty$ and the $-\infty$ contribution. The square of the interior sum can be written as

$$\begin{aligned}&\left|\sum_{k=1}^{n}\boldsymbol{\delta}^n_j \exp\left(-i\sum_{i=1}^{k-1}\frac{k_\mu p^\mu_i}{p^0_0}\delta z\right)\right|^2 \\ &= \frac{1}{(p^0_0)^2}2\,\mathrm{Re}\sum_{j=1}^{n-1}\delta z\sum_{k=1}^{j-1}\delta z\,\boldsymbol{k}\times\boldsymbol{p}_j \exp\left(-i\sum_{i=k}^{j-1}\frac{k_\mu p^\mu_i}{p^0_0}\delta z\right)\boldsymbol{k}\times\boldsymbol{p}_k.\end{aligned} \tag{3.137}$$

The presence of the phase from z_k to z_{j-1} suggests a separation of (3.122) into three zones. We also separately consider the contribution arising with the averages $\phi_{inc}^{(n)}(\delta p_i)+\phi_{inc}^{(0)}(\delta p_i)$ from the contribution arising with the averages $\phi_{inc}^{(0)}(\delta p_i)$. For this purpose we write

$$\omega\frac{dI_{inc}^{(n)}}{d\omega d\Omega}=\frac{e^2}{(2\pi)^2}h^n(y)\frac{1}{(p_0^0)^2}2\,\mathrm{Re}\sum_{j=1}^{n}\delta z\sum_{k=1}^{j-1}\delta z\int\frac{d^3\boldsymbol{p}_n}{(2\pi)^3}$$
$$\int\frac{d^3\boldsymbol{p}_j}{(2\pi)^3}\int\frac{d^3\boldsymbol{p}_k}{(2\pi)^3}\,P_{inc}^{(n)}(\boldsymbol{p}_n,\boldsymbol{p}_j)\;\boldsymbol{k}\times\boldsymbol{p}_j\;P_{\gamma}^{(n)}(\boldsymbol{p}_j,\boldsymbol{p}_k)\;\boldsymbol{k}\times\boldsymbol{p}_k\;P_{inc}^{(n)}(\boldsymbol{p}_k,\boldsymbol{p}_0), \tag{3.138}$$

and

$$\omega\frac{dI_{inc}^{(0)}}{d\omega d\Omega}=\frac{e^2}{(2\pi)^2}h^n(y)\frac{1}{(p_0^0)^2}2\,\mathrm{Re}\sum_{j=1}^{n}\delta z\sum_{k=1}^{j-1}\delta z\int\frac{d^3\boldsymbol{p}_n}{(2\pi)^3}$$
$$\int\frac{d^3\boldsymbol{p}_j}{(2\pi)^3}\int\frac{d^3\boldsymbol{p}_k}{(2\pi)^3}\,P_{inc}^{(0)}(\boldsymbol{p}_n,\boldsymbol{p}_j)\;\boldsymbol{k}\times\boldsymbol{p}_j\;P_{\gamma}^{(0)}(\boldsymbol{p}_j,\boldsymbol{p}_k)\;\boldsymbol{k}\times\boldsymbol{p}_k\;P_{inc}^{(n)}(\boldsymbol{p}_k,\boldsymbol{p}_0). \tag{3.139}$$

The functions $P_{inc}^{(n)}(\boldsymbol{p}_a,\boldsymbol{p}_b)$ are the electron probability for going from $\boldsymbol{p}_b$ to $\boldsymbol{p}_a$ including the no collision events. At high energies then the convolution

$$P_{inc}^{(n)}(\boldsymbol{p}_a,\boldsymbol{p}_b)\equiv\left(\prod_{i=b+1}^{a-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=b+1}^{a}\left(\phi_{inc}^{(n)}(\delta p_i)+\phi_{inc}^{(0)}(\delta p_i)\right)\right), \tag{3.140}$$

and the functions $P_{inc}^{(0)}(\boldsymbol{p}_a,\boldsymbol{p}_b)$ are the probability of not colliding thus producing a distribution $\boldsymbol{p}_a=\boldsymbol{p}_b$, given by

$$P_{inc}^{(0)}(\boldsymbol{p}_a,\boldsymbol{p}_b)\equiv\left(\prod_{i=b+1}^{a-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=b+1}^{a}\left(\phi_{inc}^{(0)}(\delta p_i)\right)\right). \tag{3.141}$$

The functions $P_{\gamma}^{(n)}(\boldsymbol{p}_a,\boldsymbol{p}_b)$ and $P_{\gamma}^{(0)}(\boldsymbol{p}_a,\boldsymbol{p}_b)$ convolute the former electron probabilities for going from $\boldsymbol{p}_b$ to $\boldsymbol{p}_a$ with the internal phase (3.34) introduced by the photon emission at z_k and z_j. These terms are instead given by

$$P_{\gamma}^{(n)}(\boldsymbol{p}_j,\boldsymbol{p}_k)\equiv\left(\prod_{i=k+1}^{j-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=k+1}^{j}\left(\phi_{inc}^{(n)}(\delta p_i)+\phi_{inc}^{(0)}(\delta p_i)\right)\right)\exp\left(-i\sum_{i=k+1}^{j-1}\frac{k_\mu p_i^\mu}{p_0^0}\delta z\right), \tag{3.142}$$

and

$$P_{\gamma}^{(0)}(\boldsymbol{p}_j,\boldsymbol{p}_k)\equiv\left(\prod_{i=k+1}^{j-1}\int\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=k+1}^{j}\left(\phi_{inc}^{(0)}(\delta p_i)\right)\right)\exp\left(-i\sum_{i=k+1}^{j-1}\frac{k_\mu p_i^\mu}{p_0^0}\delta z\right). \tag{3.143}$$

Here we join $\phi^{(0)}_{inc}(\delta p_i) + \phi^{(n)}_{inc}(\delta p_i)$ into a single expression. Using (3.89) we find

$$\phi^{(0)}_{inc}(\delta p_i) + \phi^{(n)}_{inc}(\delta p_i) = 2\pi\beta_p \delta(\delta p_i^0) \times \int d^2\boldsymbol{x}_i^t e^{-i\delta \boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t} \exp\left(-n_0(z_i)\delta z\left(\sigma^{(1)}_{el}(\mathbf{0}) - \sigma^{(1)}_{el}(\boldsymbol{x}_i^t)\right)\right). \tag{3.144}$$

The momentum integrations at $P^{(n)}_{inc}(\boldsymbol{p}_n, \boldsymbol{p}_j)$ and $P^{(n)}_{inc}(\boldsymbol{p}_k, \boldsymbol{p}_0)$ are trivial, as expected, since in the absence of longitudinal phases the convolution of infinitesimal eikonal transports has to respect additivity. One finds for the first set of scatterings using (3.144) and (3.140)

$$P^{(n)}_{inc}(\boldsymbol{p}_n, \boldsymbol{p}_j) = 2\pi\beta_p(p_n^0 - p_j^0) \times \int d^2\boldsymbol{x}_n^t e^{-i(\boldsymbol{p}_n^t-\boldsymbol{p}_j^t)\cdot\boldsymbol{x}_n^t} \exp\left(-\left(\sigma^{(1)}_{el}(\mathbf{0}) - \sigma^{(1)}_{el}(\boldsymbol{x}_n^t)\right)\sum_{i=j}^{n}\delta z\, n_0(z_i)\right), \tag{3.145}$$

and for the last set of scatterings

$$P^{(n)}_{inc}(\boldsymbol{p}_k, \boldsymbol{p}_0) = 2\pi\beta_p(p_k^0 - p_0^0) \times \int d^2\boldsymbol{x}_k^t e^{-i(\boldsymbol{p}_k^t-\boldsymbol{p}_0^t)\cdot\boldsymbol{x}_k^t} \exp\left(-\left(\sigma^{(1)}_{el}(\mathbf{0}) - \sigma^{(1)}_{el}(\boldsymbol{x}_k^t)\right)\sum_{i=1}^{k}\delta z\, n_0(z_i)\right). \tag{3.146}$$

The momentum integration in the function $P^{(n)}_{\gamma}(\boldsymbol{p}_j, \boldsymbol{p}_k)$ has to be performed, however, taking into account the internal longitudinal phase. By using (3.34) we find

$$\sum_{i=k+1}^{j-1}\frac{k_\mu p_i^\mu}{p_0^0}\delta z = \frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j - z_{k+1}) + \frac{\omega}{2p_0^0(p_0^0-\omega)}\sum_{i=k+1}^{j-1}\delta z\left(\boldsymbol{p}_i^t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t\right)^2. \tag{3.147}$$

This suggest reorganizing the phase in the elastic weights in the same way

$$-i\boldsymbol{p}_j^t\cdot\boldsymbol{x}_j^t + i\sum_{i=k+1}^{j-1}\left(\boldsymbol{p}_i^t\cdot\delta\boldsymbol{x}_i^t\right) + i\boldsymbol{p}_k^t\cdot\boldsymbol{x}_{k+1}^t - \sum_{i=k+1}^{j-1}\delta z n_0(z_k)\left(\sigma^{(1)}_{el}(\mathbf{0}) - \sigma^{(1)}_{el}(\boldsymbol{x}_i^t)\right), \tag{3.148}$$

where $\delta \boldsymbol{x}_i^t = \boldsymbol{x}_{i+1}^t - \boldsymbol{x}_i^t$. This organization produces for $P_\gamma(\boldsymbol{p}_j, \boldsymbol{p}_k)$ then

$$P_\gamma^{(n)}(\boldsymbol{p}_j, \boldsymbol{p}_k) = 2\pi\beta_e \delta(p_j^0 - p_k^0) \exp\left(-i\frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j - z_{k+1})\right)\left(\prod_{i=k+1}^{j} \int d^2\boldsymbol{x}_i^t\right)$$
$$\left(\prod_{i=k+1}^{j-1} \int \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right) \exp\left(-i\boldsymbol{p}_j^t \cdot \boldsymbol{x}_j^t - i\sum_{i=k+1}^{j-1} \delta z \left(\frac{\omega}{2p_0^0(p_0^0-\omega)}\left(\boldsymbol{p}_i^t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t\right)^2\right.\right.$$
$$\left.\left. -\boldsymbol{p}_i^t \cdot \frac{\delta \boldsymbol{x}_i}{\delta z}\right) + \sum_{i=k+1}^{j-1} \delta z\, n_0(z_k)\Big(- \sigma_{el}^{(1)}(\boldsymbol{0}) + \sigma_{el}^{(1)}(\boldsymbol{x}_k^t)\Big) + i\boldsymbol{p}_k^t \cdot \boldsymbol{x}_{k+1}\right). \tag{3.149}$$

By integrating in the momentum variables we obtain

$$\left(\prod_{i=k+1}^{j-1} \int \frac{d^2\boldsymbol{p}_i}{(2\pi)^2}\right) \exp\left(-i\sum_{i=k+1}^{j-1} \delta z \left(\frac{\omega}{2p_0^0(p_0^0-\omega)}\left(\boldsymbol{p}_i^t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t\right)^2 - \boldsymbol{p}_i^t \cdot \frac{\delta \boldsymbol{x}_i}{\delta z}\right)\right)$$
$$= \left(\prod_{i=k+1}^{j-1} \frac{p_0^0(p_0^0-\omega)}{2\pi i\omega\delta z}\right) \exp\left(i\sum_{i=k+1}^{j-1} \delta z\Big(\frac{p_0^0(p_0^0-\omega)}{2\omega}\left(\frac{\delta \boldsymbol{x}_i}{\delta z}\right)^2 + \frac{p_0^0-\omega}{\omega}\boldsymbol{k} \cdot \frac{\delta \boldsymbol{x}_i}{\delta z}\Big)\right). \tag{3.150}$$

The derivative term $\delta \boldsymbol{x}_k/\delta z$ can be directly integrated since it does not depend on the path. By taking the $\delta z \to 0$ limit we find

$$P_\gamma^{(n)}(\boldsymbol{p}_j, \boldsymbol{p}_k) = 2\pi\beta_e \delta(p_j^0 - p_k^0) \exp\left(-i\frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j - z_k)\right) \int d^2\boldsymbol{x}_j^t \int d^2\boldsymbol{x}_k^t$$
$$\exp\left(-i\boldsymbol{p}_j^t \cdot \boldsymbol{x}_j^t + i\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t \cdot \boldsymbol{x}_j^t\right) \hat{P}_\gamma(\boldsymbol{x}_j^t, \boldsymbol{x}_j^t) \exp\left(+i\boldsymbol{p}_k^t \cdot \boldsymbol{x}_k^t - i\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t \cdot \boldsymbol{x}_k^t\right), \tag{3.151}$$

where we defined the function $\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t, \boldsymbol{x}_k^t)$, the Fourier transform of $P_\gamma^{(n)}(\boldsymbol{p}_j, \boldsymbol{p}_k)$, as the path integral

$$\hat{P}_\gamma(\boldsymbol{x}_j^t, \boldsymbol{x}_k^t) \equiv \int_{\boldsymbol{x}_i^t}^{\boldsymbol{x}_j^t} \mathcal{D}\boldsymbol{x}_t^2(z) \exp\left(i\int_{z_k}^{z_j} dz \left(\frac{p_0^0(p_0^0-\omega)}{2\omega}\dot{\boldsymbol{x}}_t{}^2(z)\right.\right.$$
$$\left.\left. + in_0(z)\Big(\sigma_{el}^{(1)}(\boldsymbol{0}) - \sigma_{el}^{(1)}(\boldsymbol{x}_t(z))\Big)\right)\right). \tag{3.152}$$

Further simplifications can be done still at (3.138). We notice that since at high energies $\boldsymbol{k} \simeq (\boldsymbol{k}_t, \omega - \boldsymbol{k}_t^2/2\omega)$ and $\boldsymbol{p} \simeq (\boldsymbol{p}_t, p_0^0 - \omega - \boldsymbol{p}_t^2/2(p_0^0 - \omega))$ the polarizations can be written as

$$(\boldsymbol{k} \times \boldsymbol{p}_j)(\boldsymbol{k} \times \boldsymbol{p}_i) \simeq \omega^2\Big(\boldsymbol{p}_j^t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}_t\Big)\Big(\boldsymbol{p}_i^t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}_t\Big). \tag{3.153}$$

Since these terms appear also in the exponentials of $P_\gamma(\boldsymbol{p}_j, \boldsymbol{p}_k)$ they can be introduced as in (3.138) with derivatives,

$$
\begin{aligned}
(\boldsymbol{k}\times\boldsymbol{p}_j)P_\gamma(\boldsymbol{p}_j,\boldsymbol{p}_k)(\boldsymbol{k}\times\boldsymbol{p}_i) &= 2\pi\beta_e\delta(p_j^0-p_k^0)\exp\left(-i\frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j-z_k)\right) \\
\times\omega^2\int d^2\boldsymbol{x}_j^t\int d^2\boldsymbol{x}_k^t &\exp\left(-i\boldsymbol{p}_j^t\cdot\boldsymbol{x}_j^t+i\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_j^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma(\boldsymbol{x}_j^t,\boldsymbol{x}_j^t)\right) \\
&\times\exp\left(+i\boldsymbol{p}_k^t\cdot\boldsymbol{x}_k^t-i\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right).
\end{aligned}
\tag{3.154}
$$

The remaining integrations in intermediate momenta $\boldsymbol{p}_j$ and $\boldsymbol{p}_k$ and final momentum $\boldsymbol{p}_n$ integrations are now trivial. If we define the Fourier transform of $P_{inc}^{(n)}(\boldsymbol{p}_k,\boldsymbol{p}_0)$ as

$$
\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)=\exp\left(-\left(\sigma_{el}^{(1)}(\boldsymbol{0})-\sigma_{el}^{(1)}(\boldsymbol{x}_k^t)\right)\int_0^{z_k}dz\,n_0(z)\right),
$$

we finally obtain

$$
\begin{aligned}
\omega\frac{dI^{(n)}}{d\omega d\Omega_k} &= \left(\frac{\omega e}{2\pi}\right)^2\frac{h^n(y)}{(p_0^0)^2}2\,\mathrm{Re}\int_{-\infty}^{+\infty}dz_j\int_{-\infty}^{z_j}dz_k\exp\left(-i\frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j-z_k)\right) \\
&\int d^2\boldsymbol{x}_k^t\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\left(\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t-\boldsymbol{p}_0^t\right)\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0}.
\end{aligned}
\tag{3.155}
$$

The vacuum overall subtraction at (3.122) has to be still evaluated in this continuous limit. However, it is easy to see that we only have to make $\sigma_{el}^{(1)}(\boldsymbol{x})=0$ in the above expression, i.e. replace the functions $P_{inc}^{(n)}(\boldsymbol{x})$ by $P_{inc}^{(0)}(\boldsymbol{x})$ and $P_\gamma^{(n)}(\boldsymbol{x})$ by $P_\gamma^{(0)}(\boldsymbol{x})$ so we arrive at

$$
\begin{aligned}
\omega\frac{dI^{(0)}}{d\omega d\Omega_k} &= \left(\frac{\omega e}{2\pi}\right)^2\frac{h^n(y)}{(p_0^0)^2}2\,\mathrm{Re}\int_{-\infty}^{+\infty}dz_j\int_{-\infty}^{z_j}dz_k\exp\left(-i\frac{\omega m_e^2}{2p_0^0(p_0^0-\omega)}(z_j-z_k)\right) \\
&\int d^2\boldsymbol{x}_k^t\hat{P}^{(0)}(\boldsymbol{x}_k^t)\exp\left(-i\left(\frac{p_0^0-\omega}{\omega}\boldsymbol{k}^t-\boldsymbol{p}_0^t\right)\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(0)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0}.
\end{aligned}
\tag{3.156}
$$

This term provides the right perturbative/opacity expansion of the intensity and becomes necessary when a small number of collisions is expected. It also becomes necessary for an evaluation beyond the Fokker-Planck/Gaussian limit of the path integrals, if it were possible, since it guarantees the convergency at large impact parameters. The Fokker-Planck approximation, however, does not require such a regularizator, since in the large number of collisions the elastic incoherent averages $\Sigma_2^{(n)}(\boldsymbol{q},\delta z)$ rapidly converge at large $\boldsymbol{x}$. We present now the Fokker-Planck approximation of this continuous limit. Since the amplitude has been split in three zones in the z integration, the intensity leads to nine zones. Six of them, however, are related by conjugation with the interchange of z_j and z_k. So we find,

$$\int_l^\infty\int_l^\infty+\int_{-\infty}^0\int_{-\infty}^0+\left(\int_l^\infty\int_{-\infty}^0+\int_{-\infty}^0\int_l^\infty\right)+\int_0^l\int_0^l \\ +\left(\int_0^l\int_{-\infty}^0+\int_{-\infty}^0\int_0^l\right)+\left(\int_0^l\int_l^\infty+\int_l^\infty\int_0^l\right). \tag{3.157}$$

From here onwards we take the soft photon limit thus $p_0^0-\omega\simeq p_0^0$ and $h^n(y)\simeq 1$ for simplicity. We also set the z axis in the initial electron direction so $\boldsymbol{p}_0^t=\boldsymbol{0}$. Using (3.155) the first term (a), representing the square of the photon emitted at the last leg, is given by

$$\omega\frac{dI_a^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p_0^0}\right)^2\int_l^\infty dz_j\int_l^\infty dz_k\exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j-z_k)\right) \\ \times\int d^2\boldsymbol{x}_k^t\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0}. \tag{3.158}$$

We will assume constant density $n_0(z)\equiv n_0$. In the Fokker-Planck approximation for $\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)$ and $\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)$ we truncate at leading order the interactions in $\boldsymbol{x}$

$$\exp\Big(-n_0\delta z\sigma_{el}^{(1)}(\boldsymbol{0})+n_0\delta z\sigma_{el}^{(1)}(\boldsymbol{x})\Big)\approx\exp\left(-\frac{1}{2}\hat{q}\boldsymbol{x}^2\right), \tag{3.159}$$

and the fall off of the interaction at large $\boldsymbol{x}$ is now guaranteed due to the screening neglection. The resulting momentum distribution is Gaussian. Similarly in the propagation with phase $\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j,\boldsymbol{x}_k)$ we find a Gaussian path integral

$$\exp\left(i\int_{z_k}^{z_j}dz\left(\frac{(p_0^0)^2}{2\omega}\dot{\boldsymbol{x}}_t{}^2(z)+in_0\left(\sigma_{el}^{(1)}(\boldsymbol{0})-\sigma_{el}^{(1)}(\boldsymbol{x}_t(z))\right)\right)\right) \\ \approx\exp\left(i\int_0^l dz\left(\frac{1}{2}m_{ef}\dot{\boldsymbol{x}}^2(z)-\frac{1}{2}m_{ef}\Omega^2\boldsymbol{x}^2\right)\right),$$

whose effective mass can be read in the kinetic term $m_{ef}=(p_0^0)^2/\omega$ and then the harmonic oscillator frequency Ω has to be defined as

$$m_{ef}\Omega^2=-i\hat{q}\to\Omega=\frac{1-i}{\sqrt{2}}\sqrt{\frac{\hat{q}\omega}{(p_0^0)^2}}. \tag{3.160}$$

With these definitions we simply outline the result of integrating (3.158),

$$\omega\frac{dI_a^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\int_0^\infty dz\,z\exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2\eta_a^{(1)}-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_a^{(2)}\right)\left(\boldsymbol{k}_t^2\eta_a^{(3)}+\eta_a^{(4)}\right), \tag{3.161}$$

where the functions $\eta_a^{(i)}$ are defined as

$$\eta_a^{(1)}\equiv z,\ \eta_a^{(2)}\equiv\frac{z}{1+iz\Omega^2 l},\ \eta_a^{(3)}\equiv\frac{1}{(1+iz\Omega^2 l)^3},\ \eta_a^{(4)}\equiv\frac{2i\omega\Omega}{(1+iz\Omega^2 l)^2}. \tag{3.162}$$

Similarly to this term we find the contribution (b) corresponding to the photon emerging from the first leg squared. It is given by

$$\omega\frac{dI_b^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p_0^0}\right)^2\int_{-\infty}^0 dz_j\int_{-\infty}^0 dz_k\exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j-z_k)\right)$$
$$\times\int d^2\boldsymbol{x}_k^t\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0}. \tag{3.163}$$

It produces

$$\omega\frac{dI_b^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\int_0^\infty dz\,z\exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2\eta_b^{(1)}-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_b^{(2)}\right)\left(\boldsymbol{k}_t^2\eta_b^{(3)}+\eta_b^{(4)}\right), \tag{3.164}$$

where as a consistency check the functions $\eta_b^{(i)}$ have to be given by the $\hat{q}=0$ evaluation of the previous functions $\eta_a^{(i)}$. Indeed

$$\eta_b^{(1)}=z,\ \eta_b^{(2)}=z,\ \eta_b^{(3)}=1,\ \eta_b^{(4)}=0. \tag{3.165}$$

Correspondingly we can integrate in the longitudinal space to obtain

$$\omega\frac{dI_b^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\boldsymbol{k}_t^2\int_0^\infty dz\,z\exp\left(-\left(\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2+\frac{\boldsymbol{k}_t^2}{2\omega}\right)z\right)=\frac{\left(\frac{e}{2\pi}\right)^2\boldsymbol{k}_t^2}{\left(\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2+\frac{\boldsymbol{k}^2}{2\omega}\right)^2}.$$

Observe that this is just $(\boldsymbol{k}\times\boldsymbol{p}_0)^2/(k_\mu p_0^\mu)^2$, as expected. The next term (c) corresponds to the interference between the photon emitted in the last leg and the photon emitted in the first leg. It is given by

$$\omega\frac{dI_c^{(n)}}{d\omega d\Omega_k}=\left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p_0^0}\right)^2\int_l^\infty dz_j\int_{-\infty}^0 dz_k\exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j-z_k)\right)$$
$$\times\int d^2\boldsymbol{x}_k^t\hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0}+c.c., \tag{3.166}$$

and produces

$$\omega \frac{dI_c^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2 2\,\mathrm{Re} \int_0^\infty dz_j \int_0^\infty dz_k \exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2 \eta_c^{(1)} - \frac{\boldsymbol{k}_t^2}{2\omega}\eta_c^{(2)}\right) \times \left(\boldsymbol{k}_t^2 \eta_c^{(3)} + \eta_c^{(4)}\right), \tag{3.167}$$

where the functions $\eta_c^{(i)}$ are given by

$$\eta_c^{(1)} = il + z_j + z_k,\ \eta_c^{(2)} = \frac{i\sin(\Omega l) + \Omega\cos(\Omega l)(z_j + z_k) + iz_j z_k \Omega^2 \sin(\Omega l)}{\Omega\cos(\Omega l) + iz_j\Omega^2\sin(\Omega l)}$$
$$\eta_c^{(3)} = \frac{\Omega^2}{\left(\Omega\cos(\Omega l) + iz_j\Omega^2\sin(\Omega l)\right)^2},\ \eta_c^{(4)} = 0. \tag{3.168}$$

The term (d) contains all the photons emitted from the internal legs and their respective interferences. It is given by

$$\omega \frac{dI_d^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2 \left(\frac{\omega}{p_0^0}\right)^2 \int_0^l dz_j \int_0^l dz_k \exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j - z_k)\right)$$
$$\times \int d^2\boldsymbol{x}_k^t \hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t) \exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t \cdot \boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial \boldsymbol{x}_j^t} \cdot \frac{\partial}{\partial \boldsymbol{x}_k^t} \hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t, \boldsymbol{x}_k^t)\right)\Bigg|_{\boldsymbol{x}_j=0} + c.c.\ . \tag{3.169}$$

The integration produces, using $\delta z \equiv z_j - z_k$,

$$\omega \frac{dI_d^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2 2\,\mathrm{Re} \int_0^l dz_j \int_0^{z_j} dz_k \exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2 \eta_d^{(1)} - \frac{\boldsymbol{k}_t^2}{2\omega}\eta_d^{(2)}\right) \times \left(\boldsymbol{k}_t^2 \eta_d^{(3)} + \eta_d^{(4)}\right), \tag{3.170}$$

where the functions $\eta_d^{(i)}$ are given by $\eta_d^{(1)} = i(z_j - z_k)$ and

$$\eta_d^{(2)} = \frac{i\sin\left(\Omega(z_j - z_k)\right)}{\Omega\cos\left(\Omega(z_j - z_k)\right) - z_k\Omega^2\sin\left(\Omega(z_j - z_k)\right)},$$
$$\eta_d^{(3)} = \frac{\cos\left(\Omega(z_j - z_k)\right)}{\left(\cos\left(\Omega(z_j - z_k)\right) - z_k\Omega\sin\left(\Omega(z_j - z_k)\right)\right)^3},$$
$$\eta_d^{(4)} = \frac{2i\omega\Omega^2 z_k}{\left(\cos\left(\Omega(z_j - z_k)\right) - z_k\Omega\sin\left(\Omega(z_j - z_k)\right)\right)^2}. \tag{3.171}$$

The term (e) containing the interference between the internal photons and the photon emitted in the first leg is given by

$$\omega\frac{dI_e^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p_0^0}\right)^2\int_0^l dz_j\int_{-\infty}^0 dz_k \exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j-z_k)\right) \tag{3.172}$$
$$\times\int d^2\boldsymbol{x}_k^t \hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\bigg|_{\boldsymbol{x}_j=0} + c.c. \quad .$$

This term produces

$$\omega\frac{dI_e^{(n)}}{d\omega d\Omega_k} = -\left(\frac{e}{2\pi}\right)^2 2\,\mathrm{Re}\int_0^l dz_j\int_0^\infty dz_k\exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2\eta_e^{(1)}-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_e^{(2)}\right)$$
$$\times\left(\boldsymbol{k}_t^2\eta_e^{(3)}+\eta_e^{(4)}\right), \tag{3.173}$$

where the functions $\eta_e^{(i)}$ are given by

$$\eta_e^{(1)} = i(z_j-z_k),\ \eta_e^{(2)} = -iz_k + i\frac{\sin(\omega z_j)}{\Omega\cos(\Omega z_j)},\ \eta_e^{(3)} = \frac{1}{\cos^2(\Omega z_j)},\ \eta_e^{(4)} = 0. \tag{3.174}$$

And finally the term (f) containing the interferences between the photon emitted in the last leg with the internal photons. It is given by

$$\omega\frac{dI_f^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p_0^0}\right)^2\int_0^l dz_j\int_l^\infty dz_k \exp\left(-i\frac{\omega m_e^2}{2(p_0^0)^2}(z_j-z_k)\right) \tag{3.175}$$
$$\times\int d^2\boldsymbol{x}_k^t \hat{P}_{inc}^{(n)}(\boldsymbol{x}_k^t)\exp\left(-i\frac{p_0^0}{\omega}\boldsymbol{k}^t\cdot\boldsymbol{x}_k^t\right)\left(\frac{\partial}{\partial\boldsymbol{x}_j^t}\cdot\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_\gamma^{(n)}(\boldsymbol{x}_j^t,\boldsymbol{x}_k^t)\right)\bigg|_{\boldsymbol{x}_j=0} + c.c.,$$

and produces using $t = (l - z_k)$

$$\omega\frac{dI_f^{(n)}}{d\omega d\Omega_k} = \left(\frac{e}{2\pi}\right)^2 2\,\mathrm{Im}\int_0^l dz_j\int_0^\infty dz_k\exp\left(-\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2\eta_f^{(1)}-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_f^{(2)}\right)$$
$$\times\left(\boldsymbol{k}_t^2\eta_f^{(3)}+\eta_f^{(4)}\right), \tag{3.176}$$

where the functions $\eta_f^{(i)}$ are given by $\eta_f^{(1)} = i\delta + z_k$ where $\delta = l - z_j$ and

$$\eta_f^{(2)} = \frac{i\sin(\Omega\delta)+z_k\Omega\cos(\Omega\delta)}{\Omega\cos(\Omega\delta)-z_j\Omega^2\sin(\Omega\delta)+iz_k\Omega^2\left(\sin(\Omega\delta)+z_j\Omega\cos(\Omega\delta)\right)}$$
$$\eta_f^{(3)} = \frac{\cos(\Omega\delta)+iz_k\sin(\Omega\delta)}{\left(\cos(\Omega\delta)-z_j\Omega\sin(\Omega\delta)+iz_k\Omega\left(\sin(\Omega\delta)+z_j\Omega\cos(\Omega\delta)\right)\right)^3}$$
$$\eta_f^{(3)} = \frac{2i\omega\Omega^2 z_j}{\left(\cos(\Omega\delta)-z_j\Omega\sin(\Omega\delta)+iz_k\Omega\left(\sin(\Omega\delta)+z_j\Omega\cos(\Omega\delta)\right)\right)^2}. \tag{3.177}$$

The sum of the above six contributions constitute the path integral version of the intensity in the Fokker-Planck approximation for mediums of arbitrary length

$$\omega \frac{dI_{inc}^{(n)}}{d\omega d\Omega_k} - \omega \frac{dI_{inc}^{(0)}}{d\omega d\Omega_k} = \sum_{i=a}^{f} \omega \frac{dI_i^{(n)}}{d\omega d\Omega_k}. \tag{3.178}$$

In Fig. 3.3 and Fig. 3.4 we show the evaluation of these six path integrals (3.178) for several photon angles and a Gold target of $l = 0.0023$ cm and an electron energy of $p_0^0 = 8$ GeV. We find a complete agreement with the direct numerical evaluation of the discretized approach (3.122), as expected since we have chosen for that purpose $\delta z \ll \lambda = 1/n_0\sigma_t^{(1)}$.

If the semi-infinite medium length is to be considered the separation of the space in three zones stops having sense. We can define in that case the quantity

$$\omega \frac{dI_{inc}^{(n)}}{d\omega d\Omega_k} \equiv \int_0^l \int_0^l + \int_{-\infty}^0 \int_{-\infty}^0 = \omega \frac{dI_d^{(n)}}{d\omega d\Omega_k} + \omega \frac{dI_b^{(n)}}{d\omega d\Omega_k}. \tag{3.179}$$

The integration in $\boldsymbol{k}_t$ without kinematical restrictions can be easily be done using $\omega^2 d\Omega_k = d^2\boldsymbol{k}_t$, producing

$$\frac{1}{\omega^2} \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} \exp\left(-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_d^{(2)}\right)\left(\boldsymbol{k}_t^2\eta_d^{(3)} + \eta_d^{(4)}\right) = -\frac{1}{\pi} \frac{\Omega^2}{\sin^2\left(\Omega(z_j - z_k)\right)}, \tag{3.180}$$

and

$$\frac{1}{\omega^2} \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} \exp\left(-\frac{\boldsymbol{k}_t^2}{2\omega}\eta_b^{(2)}\right)\left(\boldsymbol{k}_t^2\eta_b^{(3)} + \eta_b^{(4)}\right) = \frac{1}{\pi} \frac{1}{(z_j - z_k)^2}. \tag{3.181}$$

Correspondingly for the particular case $l \to \infty$ we find

$$\omega \frac{dI_{inc}^{(n)}}{d\omega} = \frac{2e^2}{\pi} \mathrm{Re} \int_0^\infty dz_j \int_0^{z_j} dz_k \exp\left(-i\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2 (z_j - z_k)\right)$$
$$\times \left(\frac{1}{(z_j - z_k)^2} - \frac{\Omega^2}{\sin^2\left(\Omega(z_j - z_k)\right)}\right), \tag{3.182}$$

which is Migdal's result [4, 47] for the $\phi(s)$ function. By performing one of the trivial integrals, in which a factor proportional to l arises, and by rotating the contour of integration to avoid oscillations, we find a suitable expression for the intensity for a medium of length $l \to \infty$, of the form

$$\omega \frac{dI_{inc}^{(n)}}{d\omega} = l \frac{2e^2}{\pi} \frac{|\Omega|}{\sqrt{2}} \int_0^\infty dz \exp\left(-\frac{z}{\sqrt{2}s}\right)$$
$$\times \left(\sin\left(\frac{z}{\sqrt{2}s}\right) + \cos\left(\frac{z}{\sqrt{2}s}\right)\right)\left(\frac{1}{z^2} - \frac{1}{\sinh^2(z)}\right), \tag{3.183}$$

where the parameter s is given by

$$s \equiv \frac{2}{\omega}\left(\frac{p_0^0}{m_e}\right)^2 |\Omega| = \frac{2p_0^0}{m_e^2}\sqrt{\frac{\hat{q}}{\omega}}. \tag{3.184}$$

An useful approximant to the above integral within less than a 1% of deviation in all the range is given by

$$\omega\frac{dI_{inc}^{(n)}}{d\omega} \simeq l\frac{2e^2}{3\pi}\sqrt{\frac{\hat{q}\omega}{(p_0^0)^2}}\, s\frac{1-1.52s^4+5.8s^5}{1+2.44s^5+2.73s^6}. \tag{3.185}$$

The other relevant prescription for the infinite length approximation is alternatively given by the quantity

$$\left|\int_{-\infty}^{l}\right|^2 - \left|\int_{-\infty}^{0}\right|^2 = \int_0^l\int_0^l + 2\,\mathrm{Re}\int_{-\infty}^{0}\int_0^l \equiv \omega\frac{dI_d^{(n)}}{d\omega d\Omega_k} + \omega\frac{dI_e^{(n)}}{d\omega d\Omega_k}, \tag{3.186}$$

As with the prescription leading to Migdal solution, since the final leg photon disappears when $l \to \infty$ and the initial photon keeps impaired, it has been removed in order to avoid a $\log(\omega)$ divergence in the angular integration. The angle integration of the second term (e) on the right hand side produces

$$\omega\frac{dI_e^{(n)}}{d\omega} = \frac{2e^2}{\pi}\,\mathrm{Re}\int_0^l dz_j\int_0^\infty dz_k \exp\left(-i\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2(z_j-z_k)\right) \times \frac{\Omega^2}{\left(\sin\left(\Omega z_j\right)-\Omega z_k\cos\left(\Omega z_j\right)\right)^2}. \tag{3.187}$$

Then by taking $l \to \infty$ we find

$$\omega\frac{dI^{(n)}}{d\omega} = \frac{2e^2}{\pi}\,\mathrm{Re}\int_0^\infty dz_j\int_0^\infty dz_k \exp\left(-i\frac{\omega}{2}\left(\frac{m_e}{p_0^0}\right)^2(z_j-z_k)\right) \times\left(\frac{\Omega^2}{\left(\sin\left(\Omega z_j\right)-\Omega z_k\cos\left(\Omega z_j\right)\right)^2} - \frac{\Omega^2}{\sin^2\left(\Omega(z_j-z_k)\right)}\right). \tag{3.188}$$

For the particular albeit unrealistic case in which the fermion mass can be neglected the integral in z_k can be performed and we find

$$\omega\frac{dI_d^{(n)}}{d\omega} = \frac{2e^2}{\pi}\Omega\,\mathrm{Re}\int_0^l dz\,\frac{\cos(\Omega z)}{\sin(\Omega z)}, \tag{3.189}$$

for the first term and

$$\omega \frac{dI_e^{(n)}}{d\omega} = -\frac{2e^2}{\pi}\Omega\,\mathrm{Re}\int_0^l dz \frac{1}{\cos(\Omega z)\sin(\Omega z)}. \tag{3.190}$$

Correspondingly

$$\begin{aligned}\omega\frac{dI_d^{(n)}}{d\omega} + \omega\frac{dI_e^{(n)}}{d\omega} &= \frac{2e^2}{\pi}\Omega\,\mathrm{Re}\int_0^l dz\left(\frac{\cos(\Omega z)}{\sin(\Omega z)} - \frac{1}{\cos(\Omega z)\sin(\Omega z)}\right)\\ &= \frac{2e^2}{\pi}\,\mathrm{Re}\left[\log\left(\cos\left(\Omega l\right)\right)\right],\end{aligned} \tag{3.191}$$

which is the BDMPS result [22]. Unfortunately, the neglection of the projectile mass leads to a continuous enhancement of the intensity in the regime of maximal interference, i.e. $\omega \gg 1$, instead of the expected Bethe-Heitler incoherent plateau (3.18). Then the BDMPS result does not reproduce neither Weinberg's soft photon theorem nor the Bethe-Heitler cross section.

In Figures 3.5, 3.6, 3.7, 3.8, 3.9 and 3.10 the approximant (3.185) to the Migdal prediction (3.183) is shown together our evaluations in the Debye screened interaction and the Fokker-Planck approximation for (3.122). For small size targets, compromising less than $n_c = 10^4$ collisions on average the Migdal prediction is not adequate, since Weinberg's soft photon theorem is not considered.

Transverse coherent average

A reorganization of the coherent average currents in (3.77) and (3.78) can be done. We will restrict to the spin non flip contribution, since the spin flipping contribution follows the same steps. We also consider the limit $\omega \to 0$ in all the quantities except in the phases. Then we notice that

$$\sum_{k=1}^{n} \delta_k^n e^{i\varphi_k} = \frac{\boldsymbol{k}\times\boldsymbol{p}_n}{k_\mu p_n^\mu} e^{i\varphi_n} + \sum_{k=1}^{n-1}\frac{\boldsymbol{k}\times\boldsymbol{p}_k}{k_\mu p_k^\mu}\left(e^{i\varphi_k} - e^{i\varphi_{k+1}}\right) - \frac{\boldsymbol{k}\times\boldsymbol{p}_0}{k_\mu p_0^\mu} e^{i\varphi_1}. \tag{3.192}$$

In the continuous limit the above relation is just an integration by parts, where the two boundary terms correspond to an extension of the integration to $(-\infty, z_1]$ and $[z_n, \infty)$ of the interior term. This interior term is given by

$$\sum_{k=1}^{n-1}\frac{\boldsymbol{k}\times\boldsymbol{p}_k}{k_\mu p_k^\mu}\left(e^{i\varphi_k} - e^{i\varphi_{k+1}}\right) = \sum_{k=1}^{n-1}\frac{\boldsymbol{k}\times\boldsymbol{p}_k}{k_\mu p_k^\mu} e^{i\varphi_k}\left(1 - e^{i\varphi_{k+1} - i\varphi_k}\right). \tag{3.193}$$

Following (3.70) and (3.36) the phase difference is found to be

$$i\varphi_{k+1} - i\varphi_k = \frac{i\omega(z_{k+1} - z_k)}{2p_0^0(p_0^0 - \omega)}\left(m_e^2 + \left(\boldsymbol{p}_k^t - \frac{p_0^0}{\omega}\boldsymbol{k}_t\right)^2\right) = i\frac{k_\mu p_k^\mu}{p_0^0 - \omega}\delta z. \tag{3.194}$$

By taking the $\delta z \to 0$ limit then we find

$$\sum_{k=1}^{n-1} \frac{\boldsymbol{k} \times \boldsymbol{p}_k}{k_\mu p_k^\mu} e^{i\varphi} \left(1 - \exp\left(i \frac{k_\mu p_k^\mu}{p_0^0 - \omega} \delta z\right)\right) \simeq -\frac{i}{p_0^0} \sum_{k=1}^{n-1} \delta z\, \boldsymbol{k} \times \boldsymbol{p}_k e^{i\varphi_k}. \tag{3.195}$$

The quantity to evaluate at (3.77) is just the convolution of the above current with the elastic weights. For the elements containing interaction this quantity can be always reorganized as

$$\begin{aligned}
&\left(\prod_{i=1}^{n-1} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=1}^{n} \left(\phi_{coh}^{(n)}(\delta p_i, \delta z_i) + \phi_{coh}^{(0)}(\delta p_i, \delta z_i)\right)\right)\left(\sum_{k=1}^{n} \boldsymbol{\delta}_k^n e^{i\varphi_k}\right) \\
&= -\frac{i}{p_0^0} \exp\left(i \frac{\omega m_e^2 z_k}{2p_0^0(p_0^0 - \omega)} + i \frac{\boldsymbol{k}_t^2 z_k}{2\omega}\right) \int \frac{d^3\boldsymbol{p}_k}{(2\pi)^3} P_{coh}^{(n)}(\boldsymbol{p}_n, \boldsymbol{p}_k) \boldsymbol{k} \times \boldsymbol{p}_k P_{coh}^{(n)}(\boldsymbol{p}_k, \boldsymbol{p}_0),
\end{aligned} \tag{3.196}$$

where the functions $P_{coh}^{(n)}(\boldsymbol{p}_a, \boldsymbol{p}_b)$ act at the level of the amplitude and thus cannot be interpreted in probabilistic terms. If we define

$$p_n^z = -\frac{(\boldsymbol{p}_n^t - \boldsymbol{k}_t)^2}{2(p_0^0 - \omega)}, \quad p_k^z = -\frac{(\boldsymbol{p}_k^t - \boldsymbol{k})^2}{2(p_0^0 - \omega)}, \quad p_0^z = -\frac{(\boldsymbol{p}_0^t)^2}{2p_0^0}, \tag{3.197}$$

and complete the boundary terms to total three momenta as $\boldsymbol{p}_t \cdot \boldsymbol{x}_t + p_z z \equiv \boldsymbol{p} \cdot \boldsymbol{x}$ the propagators are given by Fourier transforms as follows

$$\begin{aligned}
P_{coh}^{(n)}(\boldsymbol{p}_k, \boldsymbol{p}_0) \equiv& 2\pi\beta_p \delta(p_k^0 - p_0^0) \left(\prod_{i=1}^{k-1} \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\left(\prod_{i=1}^{k} \frac{d^2\boldsymbol{x}_i^t}{(2\pi)^2}\right) e^{-i\boldsymbol{p}_k \cdot \boldsymbol{x}_k + i\boldsymbol{p}_0 \cdot \boldsymbol{x}_1} \\
&\times \exp\left(-i \sum_{i=1}^{k-1} \delta z \left(\frac{(\boldsymbol{p}_i^t)^2}{2p_0^0} - \boldsymbol{p}_i^t \cdot \frac{\delta \boldsymbol{x}_i^t}{\delta z}\right) + \sum_{i=1}^{k} \delta z n_0(z_i) \pi_{el}^{(1)}(\boldsymbol{x}_i^t)\right),
\end{aligned} \tag{3.198}$$

and

$$\begin{aligned}
P_{coh}^{(n)}(\boldsymbol{p}_n, \boldsymbol{p}_k) \equiv& 2\pi\beta_p \delta(p_n^0 - p_k^0) \left(\prod_{i=k+1}^{n-1} \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\left(\prod_{i=k+1}^{n} \frac{d^2\boldsymbol{x}_i^t}{(2\pi)^2}\right) e^{-i\boldsymbol{p}_n \cdot \boldsymbol{x}_n + i\boldsymbol{p}_k \cdot \boldsymbol{x}_{k+1}} \\
&\times \exp\left(-i \sum_{i=k+1}^{n-1} \delta z \left(\frac{(\boldsymbol{p}_i^t - \boldsymbol{k}_t)^2}{2(p_0^0 - \omega)} - \boldsymbol{p}_i^t \cdot \frac{\delta \boldsymbol{x}_i^t}{\delta z}\right) + \sum_{i=k+1}^{n} \delta z n_0(z_i) \pi_{el}^{(1)}(\boldsymbol{x}_i^t)\right),
\end{aligned} \tag{3.199}$$

where $\delta\boldsymbol{x}_i^t \equiv \boldsymbol{x}_{i+1}^t - \boldsymbol{x}_i^t$. The integration in internal momenta of the first passage produces

$$\left(\prod_{i=1}^{k-1} \frac{ip_0^0}{2\pi\delta z}\right)\left(\prod_{i=2}^{k-1} \frac{d^2\boldsymbol{x}_i^t}{(2\pi)^2}\right)\exp\left(+i\sum_{i=1}^{k-1}\delta z_i \frac{p_0^0}{2}\left(\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\right)^2 + \sum_{i=1}^{k}\delta z_i n_0(z_i)\pi_{el}^{(1)}(\boldsymbol{x}_i^t)\right)$$
$$\to \int \mathcal{D}^2\boldsymbol{x}_t(z)\exp\left(i\int_{z_1}^{z_k} dz\left(\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z) - in_0(z)\pi_{el}^{(1)}(\boldsymbol{x}_t(z))\right)\right) \quad (3.200)$$

whereas the second passage produces

$$\left(\prod_{i=k+1}^{n-1} \frac{i(p_0^0-\omega)}{2\pi\delta z}\right)\left(\prod_{i=k+2}^{n-1} \frac{d^3\boldsymbol{x}_i^t}{(2\pi)^2}\right)\exp\left(+i\sum_{i=k+1}^{n-1}\delta z_i\left(\frac{p_0^0-\omega}{2}\left(\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\right)^2 + \boldsymbol{k}_t\cdot\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\right)\right.$$
$$\left. + \sum_{i=k+1}^{n}\delta z_i n_0(z_i)\pi_{el}^{(1)}(\boldsymbol{x}_i^t)\right) \to \exp\left(+i\boldsymbol{k}_t\cdot(\boldsymbol{x}_n^t - \boldsymbol{x}_{k+1}^t)\right)$$
$$\times \int \mathcal{D}^2\boldsymbol{x}_t(z)\exp\left(i\int_{z_{k+1}}^{z_n} dz\left(\frac{p_0^0-\omega}{2}\dot{\boldsymbol{x}}_t^2(z) - in_0(z)\pi_{el}^{(1)}(\boldsymbol{x}_t(z))\right)\right). \quad (3.201)$$

The remaining integration in $\boldsymbol{p}_k$ can be done with some care. We notice that

$$\int \frac{d^2\boldsymbol{p}_k^t}{(2\pi)^2}\left(\boldsymbol{p}_t - \frac{p_0^0}{\omega}\boldsymbol{k}_t\right)\exp\left(-i\boldsymbol{k}_t\cdot\boldsymbol{x}_{k+1}^t + i\boldsymbol{p}_k^t\cdot(\boldsymbol{x}_{k+1}^t - \boldsymbol{x}_k^t) - i\delta z\frac{(\boldsymbol{p}_k^t)^2}{2p_0^0}\right)$$
$$= \left(i\frac{\partial}{\partial\boldsymbol{x}_k^t} - i\frac{p_0^0}{\omega}\left(\frac{\partial}{\partial\boldsymbol{x}_{k+1}^t} + \frac{\partial}{\partial\boldsymbol{x}_k^t}\right)\right)e^{-i\boldsymbol{k}_t\cdot\boldsymbol{x}_{k+1}^t}\frac{ip_0^0}{2\pi\delta z}\exp\left(+i\frac{(\boldsymbol{x}_{k+1}^t - \boldsymbol{x}_k^t)^2}{2p_0^0\delta z}\right). \quad (3.202)$$

In the limit $\delta z \to 0$ we find the representation of the Dirac delta function and then

$$\int \frac{d^3\boldsymbol{p}_k}{(2\pi)^3}P_{coh}^{(n)}(\boldsymbol{p}_n,\boldsymbol{p}_k)\boldsymbol{k}\times\boldsymbol{p}_k P_{coh}^{(n)}(\boldsymbol{p}_k,\boldsymbol{p}_0) = i\omega 2\pi\beta_p\delta(p_n^0 - p_0^0)\int d^2\boldsymbol{x}_n^t e^{i(\boldsymbol{k}_t - \boldsymbol{p}_n^t)\cdot\boldsymbol{x}_n^t}$$
$$\times \int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\cdot\boldsymbol{x}_1^t}\int d^2\boldsymbol{x}_k^t e^{-i\boldsymbol{k}_t\cdot\boldsymbol{x}_k^t}\left\{\hat{P}_{coh}^{(n)}(\boldsymbol{x}_n^t,\boldsymbol{x}_k^t)\left(\left(\frac{p_0^0}{\omega} - 1\right)\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_{coh}^{(n)}(\boldsymbol{x}_k^t,\boldsymbol{x}_1^t)\right)\right.$$
$$\left. + \left(\frac{p_0^0}{\omega}\;\frac{\partial}{\partial\boldsymbol{x}_k^t}\hat{P}_{coh}^{(n)}(\boldsymbol{x}_n^t,\boldsymbol{x}_k^t)\right)\hat{P}_{coh}^{(n)}(\boldsymbol{x}_k^t,\boldsymbol{x}_1^t)\right\}, \quad (3.203)$$

or alternatively

$$i\omega 2\pi\beta_p\delta(p_n^0 - p_0^0)\int d^2\boldsymbol{x}_n^t e^{-i(\boldsymbol{p}_n^t - \boldsymbol{k}_t)\cdot\boldsymbol{x}_n^t}\int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\cdot\boldsymbol{x}_1^t} \quad (3.204)$$
$$\times \int d^2\boldsymbol{x}_k^t e^{-i\boldsymbol{k}_t\cdot\boldsymbol{x}_k^t}\hat{P}_{coh}^{(n)}(\boldsymbol{x}_n^t,\boldsymbol{x}_k^t)\left\{-\frac{\boldsymbol{k}_t p_0^0}{\omega} - i\frac{\partial}{\partial\boldsymbol{x}_k^t}\right\}\hat{P}_{coh}^{(n)}(\boldsymbol{x}_k^t,\boldsymbol{x}_1^t),$$

where the functions $\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_k, \boldsymbol{x}^t_1)$ an $\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_n, \boldsymbol{x}^t_k)$ are the path integrals

$$\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_n, \boldsymbol{x}^t_k) \equiv \int \mathcal{D}^2\boldsymbol{x}_t(z) \exp\left(i\int_{z_k}^{z_n} dz\left(\frac{p^0_0-\omega}{2}\dot{\boldsymbol{x}}^2_t(z) - in_0(z)\pi^{(1)}_{el}(\boldsymbol{x}_t(z))\right)\right), \tag{3.205}$$

and

$$\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_k, \boldsymbol{x}^t_1) \equiv \int \mathcal{D}^2\boldsymbol{x}_t(z) \exp\left(i\int_{z_1}^{z_k} dz\left(\frac{p^0_0}{2}\dot{\boldsymbol{x}}^2_t(z) - in_0(z)\pi^{(1)}_{el}(\boldsymbol{x}_t(z))\right)\right). \tag{3.206}$$

It is easy to note that the term involving no collisions corresponds to making $n = 0$ in the above expressions. Then using (3.77), (3.79), (3.196) and (3.203), we find the non flip transverse coherent contribution in the continuous limit

$$\begin{aligned}
\omega\frac{dI^{coh}}{d\omega d\Omega_k} &= \left(\frac{e}{2\pi}\right)^2\left(\frac{\omega}{p^0_0}\right)^2\frac{1}{\pi R^2}\int\frac{d^2\boldsymbol{p}^t_n}{(2\pi)^2} \\
&\times\left|\int dz_k \exp\left(i\frac{\omega m_e^2 z_k}{2p^0_0(p^0_0-\omega)} + i\frac{\boldsymbol{k}_t^2 z_k}{2\omega}\right)\int d^2\boldsymbol{x}^t_n e^{-i(\boldsymbol{p}^t_n-\boldsymbol{k}_t)\cdot\boldsymbol{x}^t_n}\int d^2\boldsymbol{x}^t_1 e^{+i\boldsymbol{p}^t_0\cdot\boldsymbol{x}^t_1}\right. \\
&\times\left.\int d^2\boldsymbol{x}^t_k e^{-i\boldsymbol{k}_t\cdot\boldsymbol{x}^t_k}\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_n, \boldsymbol{x}^t_k)\left\{-\frac{\boldsymbol{k}_t p^0_0}{\omega} - i\frac{\partial}{\partial\boldsymbol{x}^t_k}\right\}\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_k, \boldsymbol{x}^t_1) - (n=0)\right|^2.
\end{aligned} \tag{3.207}$$

Since the Fourier transform of $\pi^{(1)}_{el}(\boldsymbol{x}_t)$ is an even function of $\boldsymbol{q}_t$, the function has a saddle point in $\boldsymbol{x}_t = 0$. By assuming a constant density $n_0(z) = n_0$ a series truncation at high number of collisions holds and

$$-in_0\pi^{(1)}_{el}(\boldsymbol{x}_t) \simeq -\frac{1}{2}\hat{q}\boldsymbol{x}^2_t, \tag{3.208}$$

where by using (3.59) we find

$$\hat{q} = -\frac{\partial^2}{\partial^2\boldsymbol{x}_t}\pi^{(1)}_{el}(\boldsymbol{x}_t)\bigg|_{\boldsymbol{x}_t=0} = 4\pi gR\int_0^\infty dq\frac{q^2}{q^2+\mu_d^2}J_1(qR) = 4\pi g\mu_d RK_1(\mu_d R), \tag{3.209}$$

where $K_1(x)$ is the modified Bessel function. We observe that the transverse coherent average leads to an equivalent charge given by the average interaction potential. In this Fokker-Planck approximation the path integrals have a Gaussian form and thus they are solvable,

$$\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_n, \boldsymbol{x}^t_k) \equiv \int \mathcal{D}^2\boldsymbol{x}_t(z) \exp\left(i\int_{z_k}^{z_n} dz\left(\frac{p^0_0-\omega}{2}\dot{\boldsymbol{x}}^2_t(z) - \frac{1}{2}(p^0_0-\omega)\Omega_f^2\boldsymbol{x}^2_t(z)\right)\right), \tag{3.210}$$

and

$$\hat{P}^{(n)}_{coh}(\boldsymbol{x}^t_k, \boldsymbol{x}^t_1) \equiv \int \mathcal{D}^2\boldsymbol{x}_t(z) \exp\left(i\int_{z_1}^{z_k} dz\left(\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z) - \frac{1}{2}p_0^0\Omega_i^2\boldsymbol{x}_t^2(z)\right)\right), \quad (3.211)$$

with real oscillator frequencies given by

$$\Omega_f = \sqrt{\frac{4\pi g\mu_d RK_1(\mu_d R)}{p_0^0 - \omega}}, \quad \Omega_i = \sqrt{\frac{4\pi g\mu_d RK_1(\mu_d R)}{p_0^0}}. \quad (3.212)$$

In the limit $R \to \infty$ both $\hat{q}$ and the oscillator frequencies vanish and the coherent contribution vanishes when the z integration in all the space $(-\infty, \infty)$ is performed.

Part II

QCD

4

High energy multiple scattering in QCD

In the following we consider a multiple scattering process for a high energy, asymptotically free parton, traveling through colored condensed media. The purpose of this chapter is to summarize a QCD equivalent of the formalism already developed in the preceding chapters for QED. The main change with respect to a QED scenario is, of course, the introduction of the color structure. We only briefly review the SU(3) algebra relations required for our purpose. In Section 4.1 we shortly look for the form of the color field representing the medium, which will be considered static and classical. As expected the resulting interaction will have matrix structure in the color space of the target partons as the main difference with respect to the scalar field of QED. In Section 4.2 we derive the high energy integration of the scattering amplitude and its main properties, in the same approach we took for the QED case. The amplitude is shown to inherit the matrix structure of the target parton vertices and incorporate the matrix structure of the traveling parton vertex. We show that even the squared single elastic amplitude, after averaged over color target space, does not allow a non-perturbative functional form in the high energy limit. In Section 4.3 we obtain the probability of finding the quark in some state after a multiple scattering process with a color and space averaged medium consisting in n partons, as a function of the single $n = 1$ case. We show that an splitting into an incoherent and a coherent contribution, containing the probabilistic and the interference behavior, respectively, is required in the microscopic limit, as we have already found in the QED case. The incoherent contribution at low n leads to the independent superposition of the n single squared scattering amplitudes, is positive defined and then admits a statistical interpretation, whereas the coherent contribution leads in the same low density regime to the $\sim n^2$ transverse interferences between the different partons. We show that both contributions admit a functional form for arbitrary n, in such a way that the low collision limit is recovered. These forms are suitable for a numerical evaluation under the realistic interaction, and particularly simple in their Fokker-Planck approximations. The existence of the non-negligible coherent contribution leads to an enhancement of the differential elastic cross sections at low momentum changes for mediums of finite transverse size. This enhancement depends on the geometry of the medium and contains the diffractive behavior of its boundaries. The enhancement grows for large n and tightens up for large medium transverse sizes R, in such a way that for mediums of infinite transverse area with respect to the parton direction it constraints to a pure forward contribution, which thus does not contribute to the averaged squared momentum change. In this limit, then, the incoherent behavior of the scattering is found and the macroscopical limit recovered. Both the incoherent and the coherent terms are shown to satisfy a transport equation for the probability and for the amplitude, respectively. Finally in Section 4.4 we extend the same results to a beyond eikonal evaluation of the amplitudes and their squares, with the scope of using them later in the study of some inelastic processes like gluon bremsstrahlung.

4.1 Fields of color

Let us consider a single quark in the target medium making a transition from the state $\psi_i(x)$, characterized by momentum p_i, spin s_i and color a_i, to the state $\psi_j(x)$, with p_j, s_j and a_j. The transition is mediated by a gluon, a field with structure of unitary, hermitian and traceless matrix t_α [65] of the SU(N$_c$) algebra

$$[t_\alpha, t_\beta] = i f^\gamma_{\alpha\beta} t_\gamma, \tag{4.1}$$

where the number of colors $N_c = 3$. Let also Latin letters run in the quark color dimension $a = 1, ..., N_c$ and Greek letters in the gluon color dimension $\alpha = 1, ..., N_c^2 - 1$. The structure constants can be expressed, using the Jacobi identity, as the matrix elements of an adjoint representation of SU(N$_c$),

$$(T_\alpha)^\gamma_\beta = -i f^\gamma_{\alpha\beta}, \tag{4.2}$$

satisfying the same algebra. The gluon carries an unit of color c_j and one of anticolor $\bar{c}_i$, an unit of spin $s_i - s_j$ and a fraction of momentum $p_i - p_j$ off the quark. The matrix elements of the current of this process are given by

$$(J^\mu_\alpha(x))^{a_j}_{a_i} = g_s \bar{\psi}_j(x)\gamma^\mu t_\alpha \psi_i(x) = g_s (t_\alpha)^{a_j}_{a_i} \sqrt{\frac{m}{p^0_j}} \bar{u}_{s_j}(p_j)\gamma^\mu u_{s_i}(p_i) \sqrt{\frac{m}{p^0_i}} e^{+i(p_j - p_i)x}, \tag{4.3}$$

where the quantity $g_s(t_\alpha)^{a_j}_{a_i} \equiv \sqrt{\alpha_s}(t_\alpha)^{a_j}_{a_i}$ is to be interpreted as a colored charge and α_s is the strong coupling constant. In particular for a static current we would find $\left(J^0_\alpha(x)\right)^{a_k}_{a_i} = g_s(t_\alpha)^{a_j}_{a_i}$ and $\boldsymbol{J}_\alpha(x) = \boldsymbol{0}$. The charge or strength of the process depends then on the particular color transition choice of the target quark and the color of the intermediating gluon. From this colored current, as in ordinary electrodynamics, emanates a colored field or gluon, which is just the propagation of the source current to any other point

$$A^\mu_\alpha(x) = \int d^4y D^{\mu\nu}_{\alpha\beta}(x-y)J^\nu_\beta(y). \tag{4.4}$$

sum over repeated indices assumed. From here onwards the gluon will be considered massive, with effective mass μ_d, in order to account for screening effects at large distances in the QCD medium. The propagator in the Feynman gauge for this gluon is given by

$$\hat{D}^{\mu\nu}_{\alpha\beta}(q) = \delta_{\alpha\beta}\, g^{\mu\nu}\frac{4\pi}{-q^2+\mu_d^2}. \tag{4.5}$$

From equations (4.3), (4.4) and (4.5) then we easily obtain

$$A^\mu_\alpha(x) = \frac{4\pi g_s}{-(p_j-p_i)^2+\mu_d^2}\sqrt{\frac{m}{p_j^0}}\bar{u}_{s_j}(p_j)t_\alpha\gamma^\mu u_{s_i}(p_i)\sqrt{\frac{m}{p_i^0}}e^{+i(p_j-p_i)x}. \tag{4.6}$$

This gluon is ought to be absorbed by another traveling quark while performing the transition from the state $\psi_k(x)$ to $\psi_l(x)$. The amplitude of this process is given by

$$M = -ig_s\int d^4x\, \bar{\psi}_l(x)t_\alpha\gamma_\mu\psi_k(x)A^\mu_\alpha(x), \tag{4.7}$$

Then omitting the momentum conservation delta and defining $q = p_j - p_i = p_l - p_k$ we find

$$M \propto \sqrt{\frac{m}{p_l^0}}\bar{u}_{s_l}(p_l)t_\alpha\gamma_\mu u_{s_k}(p_k)\sqrt{\frac{m}{p_k^0}}\frac{4\pi i g_s^2}{q^2-\mu_d^2}\sqrt{\frac{m}{p_j^0}}\bar{u}_{s_j}(p_j)t_\alpha\gamma^\mu u_{s_l}(p_l)\sqrt{\frac{m}{p_l^0}}. \tag{4.8}$$

Compared to its QED analogous, this amplitude incorporates the matrix structure of both quarks. Their elements can be written in general as

$$\left(M\right)^{a_j b_j}_{a_i b_i} \propto g_s^2(t_\alpha)^{a_j}_{a_i}(t_\alpha)^{a_l}_{a_k} = \frac{1}{2}\delta^{a_k}_{a_j}\delta^{a_l}_{a_i} - \frac{1}{6}\delta^{a_i}_{a_j}\delta^{a_k}_{a_l}. \tag{4.9}$$

The particular color transition of the target and traveling parton is normally not observed and we should average over initial configurations and sum over final states in the square of M. We introduce the notation $d = N_c$ for the color dimension of quarks and and $d_A = N_c^2 - 1$ for the color dimension of gluons. The average squared charge C_R for a qq scattering at leading order in g_s^2 is then

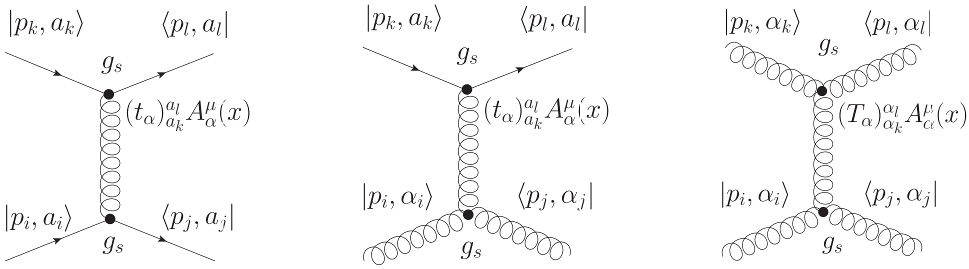


Figure 4.1: Left: The qq interaction. Center: The qg interaction. Right: The gg interaction.

$$C_R^{qq} \equiv \frac{1}{d}\sum_{a_j a_i}\frac{1}{d}\sum_{a_l a_k}(t_\alpha)^{a_j}_{a_i}(t^\dagger_\beta)^{a_i}_{a_j}(t_\alpha)^{a_l}_{a_k}(t^\dagger_\beta)^{a_k}_{a_l} = \frac{1}{d^2}\,\mathrm{Tr}^2(t_\alpha t_\beta) = \frac{N_c^2-1}{4N_c^2} = \frac{2}{9}. \qquad (4.10)$$

Here we used the hermiticity of the t_α matrices and the normalization $t_\alpha = \lambda_\alpha/2$, where λ_α are the Gell-Mann matrices [65], so $\mathrm{Tr}(t_\alpha t_\beta) = T_f\delta_{\alpha\beta}$ where $T_f = 1/2$ is the first invariant or Casimir. Similarly, for a qg the gluon emitted by a target gluon is given by the replacement $(t_\alpha)^{a_j}_{a_i} \to (T_\alpha)^{\alpha_j}_{\alpha_i} = -if^{\alpha_j}_{\alpha\alpha_i}$. We find an averaged charge of

$$C_R^{qg} \equiv \frac{1}{d}\sum_{\alpha_j\alpha_i}\frac{1}{d_A}\sum_{a_l a_k}(t_\alpha)^{a_l}_{a_k}(t^\dagger_\beta)^{a_k}_{a_l}(T_\alpha)^{\alpha_j}_{\alpha_i}(T^\dagger_\beta)^{\alpha_i}_{\alpha_j} = \frac{1}{dd_A}\,\mathrm{Tr}(t_\alpha t_\beta)\,\mathrm{Tr}(T_\alpha T_\beta) = T_f = \frac{1}{2}, \qquad (4.11)$$

where we used the adjoint representation normalization $\mathrm{Tr}(T_\alpha T_\beta) = C_A\delta_{\alpha\beta}$, with $C_A = N_c$. And finally the case of a gluon scattered by a target gluon produces using the same procedure

$$C_R^{gg} \equiv \frac{1}{d_A}\sum_{\alpha_j\alpha_i}\frac{1}{d_A}\sum_{\alpha_k\alpha_l}(T_\alpha)^{\alpha_j}_{\alpha_i}(T^\dagger_\beta)^{\alpha_i}_{\alpha_j}(T_\alpha)^{\alpha_l}_{\alpha_k}(T^\dagger_\beta)^{\alpha_k}_{\alpha_l} = \frac{1}{d_A^2}\,\mathrm{Tr}^2(T_\alpha T_\beta) = \frac{N_c^2}{N_c^2-1} = \frac{9}{8}. \qquad (4.12)$$

In Fig. 4.1 these three different diagrams of scattering scattering are shown. The above results can be reproduced also in the classical and static field limit. For target quarks heavier than the average momentum change scale μ_d, or for moving quarks with a typical energy $\langle p_i^0\rangle$ much greater than μ_d but much smaller than the energy of the traveling parton, quarks in the target can be considered almost at rest, $p_i^0 \simeq m_q$ or $p_i^0 \simeq \langle p_i^0\rangle$ respectively. The recoil can be considered negligible $p_j \simeq p_i$ and we find

$$\sqrt{\frac{m_q}{p_j^0}}\bar{u}_{s_j}(p_j)\gamma_0 u_{s_i}(p_i)\sqrt{\frac{m_q}{p_i^0}} \approx 1, \qquad \sqrt{\frac{m_q}{p_j^0}}\bar{u}_{s_j}(p_j)\gamma_k u_{s_i}(p_i)\sqrt{\frac{m_q}{p_i^0}} \approx \frac{(p_i)_k}{m_q} \ll 1, \qquad (4.13)$$

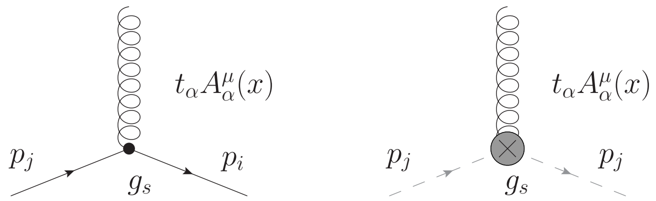


Figure 4.2: Left: The field created by a quantum current $\langle i|j\rangle$. Right: The field as a classic and static current where $p_i \approx p_j$.

and then the gluon field simplifies to

$$A^\mu_\alpha(x) = \delta^\mu_0 \frac{4\pi g_s t_\alpha}{-(p_j - p_i)^2 + \mu_d^2} e^{+i(p_j - p_i)x}. \tag{4.14}$$

In this approximation the slow varying momentum dependence of the spinor terms is assumed. After coherently forming the superposition in momentum and integrating in time we find

$$\langle A^\mu_\alpha(x)\rangle_{q,t} \equiv \delta^\mu_0 g_s t_\alpha \int \frac{d^4q}{(2\pi)^4} \int dt \frac{4\pi}{-q^2 + \mu_d^2} e^{iq\cdot x} = \delta^\mu_0 g_s t_\alpha \frac{1}{|\boldsymbol{x}|} e^{-\mu_d|\boldsymbol{x}|}, \tag{4.15}$$

where the last integral has been done with the help of Cauchy's theorem. So the classical field of a particle of charge $g_s(t_\alpha)^{a_j}_{a_i}$ at $\boldsymbol{x} = 0$ is recovered, i.e. $J^0_\alpha(x) = g_s t_\alpha \delta^{(3}(\boldsymbol{x})$ and $\boldsymbol{J}_\alpha = \boldsymbol{0}$. Under these simplifications our colored medium, then, will be characterized from here onwards as the classical gluon field of a set of n static quarks, of the form

$$A^0_\alpha(x) = \sum_{k=1}^{n} \frac{g_s t_\alpha}{|\boldsymbol{x} - \boldsymbol{r}_k|} e^{-\mu_d|\boldsymbol{x} - \boldsymbol{r}_k|}. \tag{4.16}$$

For a mixed scenario consisting of gluons and quarks the coupling of the gluon content can be simply replaced with the adjoint representation $(t_\alpha) \to (T_\alpha)$ in this classical approximation. In our following derivations, for simplicity, we will restrict to a medium exclusively composed of quarks.

4.2 Scattering amplitude

From (4.16) a medium consisting of quarks and/or gluons will be characterized as an external field with one non vanishing time independent component, namely $A^0_\alpha(x)$. The quark state at high energies under this interaction, assuming that the asymptotic state has momentum p_i placed along the x_3 direction, *c.f* Section 1.2, is given by (1.29)

$$\psi^{(n)}(x) = \left(1 + \frac{i}{2p_i^0}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{1}{2p_i^0}\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)\varphi_s^{(n)}(x), \tag{4.17}$$

where $\varphi_s^{(n)}(x)$ is a stationary and high energy, Schrodinger-like, solution to the equation (1.43) with asymptotically free initial condition. We write then

$$\frac{\partial\varphi_s^{(n)}(x)}{\partial x_3} = \left(ip_i^3 - \frac{i}{\beta_p}g_s t_\alpha A_\alpha^0(\boldsymbol{x})\right)\varphi_s^{(n)}(x), \qquad \lim_{x_3\to-\infty}\varphi_s^{(n)}(x) = \varphi_s^{(0)}(x), \tag{4.18}$$

where β_p is the quark velocity and $\varphi_s^{(0)}(x)$ a free solution of momentum p_i. The solution to this matrix equation (4.18) with that initial condition is given by the ordered exponential

$$\varphi_s^{(n)}(x) = \mathcal{P}\exp\left(-i\frac{g_s}{\beta_p}\int_{-\infty}^{x_3} dx_3'\, t_\alpha A_\alpha^0(\boldsymbol{x})\right)\varphi_s^{(0)}(x). \tag{4.19}$$

Noticing that the term $\boldsymbol{\alpha}\cdot\boldsymbol{p}$ is canceled when the derivative of the free part is taken, we find the quark equivalent of (1.42)

$$\psi^{(n)}(x) = \left\{\left(1 + i\frac{\gamma_k\gamma_0}{2p_i^0}\partial_k\right)\mathcal{P}\exp\left(-i\frac{g_s}{\beta_p}\int_{-\infty}^{x_3} dx_3'\, t_\alpha A_\alpha^0(\boldsymbol{x})\right)\right\}\psi^{(0)}(x). \tag{4.20}$$

We will explicitely write the states with the color vector. For the asymptotic initial quark in particular

$$\psi^{(0)}(x) = e^{-ip_i\cdot x}\sqrt{\frac{m_q}{p_i^0}}u_{s_i}^{a_i}(p_i). \tag{4.21}$$

Let the eikonal phase in the high energy integration of the wave be denoted as

$$W^{(n)}(\boldsymbol{x}, p_i) \equiv \mathcal{P}\exp\left(-i\frac{g_s}{\beta_p}\int_{-\infty}^{x_3} dx_3'\, t_\alpha A_\alpha^0(\boldsymbol{x})\right). \tag{4.22}$$

From here onwards path ordering is going to be implicitly assumed. The scattering amplitude of finding the quark in a state with momentum p_f due to the external field actuation can be found [66] by using the Lippmann-Schwinger relation

$$\psi^{(n)}(x) = \psi^{(0)}(x) + \int d^4y S_F(x-y)\gamma_0 g_s t_\alpha A_\alpha^0(\boldsymbol{y})\psi^{(n)}(y) = \psi^{(0)}(x) + \psi_{diff}^{(n)}(x), \tag{4.23}$$

where the Feynman-Stueckelberg quark propagator is given by

$$S_F^q(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{\not{p} + m_q}{p^2 - m_q^2} = \frac{1}{i} \sum_{s,a} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \times \left(\sqrt{\frac{m_q}{p^0}} u_s^a(p) \otimes \bar{u}_a^s(p) \sqrt{\frac{m_q}{p^0}} \right) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{4.24}$$

Here we restricted to energy positive solutions and we used the completeness relation in spin and in color

$$\sum_{s,a} u_s^a(p) \otimes \bar{u}_a^s(p) = \frac{\not{p} + m_q}{2m_q}. \tag{4.25}$$

Then, the diffracted part in the Lippmann-Schwinger equation produces a superposition of states of momentum p_f, spin s_f and color a_f of the form

$$\psi_{diff}^{(n)}(x) = \sum_{s_f,a_f} \int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3} e^{-i\boldsymbol{p}_f\cdot\boldsymbol{x}} \sqrt{\frac{m_q}{p_f^0}} u_{s_f}^{a_f}(p_f) \left(M_{s_f s_i}^{(n)}(p_f, p_i) \right)_{a_i}^{a_f}, \tag{4.26}$$

so that we write for the amplitude of finding the quark in this final state f as

$$\left(M_{s_f s_i}^{(n)}(p_f, p_i) \right)_{a_i}^{a_f} \equiv \left\langle a_f \right| \int d^4y e^{+i(p_f - p_i)y} \times \sqrt{\frac{m_q}{p_f^0}} \bar{u}_{s_f}(p_f)\gamma_0 \left(-ig_s t_\alpha A_\alpha^0(\boldsymbol{y}) \right) \left\{ \left(1 + i\frac{\gamma_k\gamma_0}{2p_i^0}\partial_k \right) W^{(n)}(\boldsymbol{y}, p_i) \right\} u_{s_i}(p_i) \sqrt{\frac{m_q}{p_i^0}} \left| a_i \right\rangle. \tag{4.27}$$

The integration of the above equation in the high energy limit follows exactly the same steps as those presented at Section 2.1. In the high energy limit, provided that the Glauber condition (1.23) holds, we can neglect the $1/2p_i^0$ operator correction. By defining $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ and neglecting by now the beyond eikonal corrections in q_z one easily finds

$$M_{s_f s_i}^{(n)}(p_f, p_i) = 2\pi\delta(p_f^0 - p_i^0)\beta_p \sqrt{\frac{m_q}{p_f^0}} \bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i) \sqrt{\frac{m_q}{p_i^0}} \times \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t} \left(\exp\left[-i\frac{g_s}{\beta_p} \int_{-\infty}^{+\infty} dy_3\, t_\alpha A_\alpha^0(\boldsymbol{y}) \right] - 1 \right). \tag{4.28}$$

The spinorial product can be reduced to a spin conservation delta in the high energy limit

$$\sqrt{\frac{m_q}{p_f^0}} \bar{u}_{s_f}(p_f)\gamma_0 u_{s_i}(p_i) \sqrt{\frac{m_q}{p_i^0}} \simeq \delta_{s_i}^{s_f} \tag{4.29}$$

For convenience we define the integral part of (4.28) as

$$F_{el}^{(n)}(\boldsymbol{q}_t) \equiv \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{y}_t}\left(\exp\left[-i\frac{g_s}{\beta_p}\int_{-\infty}^{+\infty} dy_3\, t_\alpha A_\alpha^0(\boldsymbol{y})\right]-1\right), \tag{4.30}$$

since spin and energy are preserved and $F_{el}^{(n)}(\boldsymbol{q})$ contains the relevant momentum distribution after the scattering. We introduce now the notation shorthands

$$i\frac{g_s}{\beta_p}\int_{-\infty}^{+\infty} dy_3 t_\alpha A_\alpha^0(\boldsymbol{y}) = i\frac{g_s^2}{\beta_p}t_\alpha t_\alpha\sum_{k=1}^{n}\chi_0^{(1)}(\boldsymbol{y}_t-\boldsymbol{r}_t^k) = i\frac{g_s^2}{\beta_p}t_\alpha t_\alpha\sum_{k=1}^{n}\chi_0^k(\boldsymbol{y}_t). \tag{4.31}$$

The color transitions of the target and traveling quarks are expected to be evaluated in $F_{el}^{(n)}(\boldsymbol{q})$ and usually color-averaged in its square. For a color averaged single target quark ($n = 1$) performing the color transition $b_i \to b_f$ we obtain the following useful optical theorem

$$\begin{aligned}\frac{1}{N_c}\int\frac{d^2\boldsymbol{q}_t}{(2\pi)^2}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)F_{el}^{(1)}(\boldsymbol{q}_t)\right) &= \frac{1}{N_c}\int d^2\boldsymbol{x}_t\left(\exp\left(+i\frac{g_s^2}{\beta_p}t_\beta^\dagger t_\beta^\dagger\chi_0^1(\boldsymbol{x}_t)\right)-1\right)_{b_f}^{b_i}\\ \times\left(\exp\left(-i\frac{g_s^2}{\beta_p}t_\alpha t_\alpha\chi_0^1(\boldsymbol{x}_t)\right)-1\right)_{b_i}^{b_f} &= -\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{0})+F_{el}^{(1)}(\boldsymbol{0})\right),\end{aligned} \tag{4.32}$$

where the trace refers from here onwards to the colors in the target space only. A symmetrical relation can be written for the color average in the projectile or for both averages together, in general. We notice that the color average of $|F_{el}^{(1)}(\boldsymbol{q})|^2$ does not admit, however, a reexponentiation. In order to see this fact we expand the amplitude using (4.30) and (4.16). We get

$$F_{el}^{(1)}(\boldsymbol{q}) = -i\frac{g_s^2}{\beta_p}t_{\alpha_1}t_{\alpha_1}\hat{A}_0^{(1)}(\boldsymbol{q}) - \frac{g_s^4}{2\beta_p^2}t_{\alpha_2}t_{\alpha_2}t_{\alpha_1}t_{\alpha_1}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{A}_0^{(1)}(\boldsymbol{k})\hat{A}_0^{(1)}(\boldsymbol{q}-\boldsymbol{k}) + \ldots, \tag{4.33}$$

where the Fourier transform of the single field satisfies $\hat{A}_0^{(1)}(\boldsymbol{q}) = 4\pi/(\boldsymbol{q}^2+\mu_d^2)$ for the Debye screened interaction (4.16). The total color average of the squared amplitude produces

$$\begin{aligned}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)_{a_f b_f}^{a_i b_i}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right)_{a_i b_i}^{a_f b_f} &= \frac{g_s^{4}}{\beta_p}^{2} C_R^{(2)}\left|\hat{A}_0^{(1)}(\boldsymbol{q})\right|^2 + \frac{g_s^6}{2\beta_p^3}C_R^{(3)}\hat{A}_0^{(1)}(\boldsymbol{q})\\ \times\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{A}_0^{(1)}(\boldsymbol{k})\hat{A}_0^{(1)}(\boldsymbol{q}-\boldsymbol{k}) &+ \frac{g_s^8}{4\beta_p^4}C_R^{(4)}\left(\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{A}_0^{(1)}(\boldsymbol{k})\hat{A}_0^{(1)}(\boldsymbol{q}-\boldsymbol{k})\right)^2 + \ldots.\end{aligned} \tag{4.34}$$

The required color charges are given by operations like (4.10). Indeed for the simplest diagram we find the aforementioned result (4.10)

$$C_R^{(2)} = \frac{1}{N_c^2}\,\mathrm{Tr}(t_{\alpha_1}t_{\beta_1}) = \frac{N_c^2-1}{4N_c^2} = \frac{2}{9}. \tag{4.35}$$

The interference term $C_R^{(3)}$ has to cancel by symmetry under conjugation. We obtain, of course

$$C_R^{(3)} = \frac{1}{N_c^2}\left(\mathrm{Tr}^2(t_{\alpha_1}t_{\beta_2}t_{\beta_1}) - \mathrm{Tr}^2(t_{\alpha_2}t_{\alpha_1}t_{\beta_1})\right) = 0, \tag{4.36}$$

as expected. The next term requires a little more of work. It is given by

$$\begin{aligned} C_R^{(4)} &= \frac{1}{N_c^2}\left(\frac{1}{2}\delta^{a_f}_{b_1}\delta^{b_f}_{a_1} - \frac{1}{6}\delta^{a_f}_{a_1}\delta^{b_f}_{b_1}\right)\left(\frac{1}{2}\delta^{a_1}_{b_i}\delta^{b_1}_{a_i} - \frac{1}{6}\delta^{a_1}_{a_i}\right)(t_{\beta_2})^{b_1}_{b_i}(t_{\beta_2})^{a_i}_{a_1}(t_{\beta_1})^{a_1}_{a_f}(t_{\beta_1})^{b_1}_{b_f} \\ &= \frac{1}{N_c^2}\left(\frac{10}{36}\,\mathrm{Tr}^2(t_{\beta_1}t_{\beta_2}) - \frac{2}{12}\,\mathrm{Tr}(t_{\beta_1}t_{\beta_2}t_{\beta_1}t_{\beta_2})\right) = \frac{2}{27}. \end{aligned} \tag{4.37}$$

It can be shown that the next order squared-diagram would produce $C_R^{(6)} = 22/729$. So that the average color charge of two kicks with the same center $2/27$ is not anymore the square of the averaged color charge of two single kicks $(2/9)^2$, and so on. Since a reexponentiation does not hold, in the high energy formalism we will restrict to leading order evaluations in the coupling of the single amplitudes.

On the other hand, the scattering amplitude for the gluon is found by tracing back the same approach with the gluon propagator in the Feynman gauge. By using the completeness and orthogonality relations

$$\sum_\lambda \epsilon^\lambda_\mu(k)\epsilon^{\lambda *}_\nu(k) = g_{\mu\nu}, \qquad \epsilon^{\lambda_f}_\mu(k_f)\epsilon^\mu_{\lambda_i}(k_i) = -\delta^{\lambda_f}_{\lambda_i}, \tag{4.38}$$

one arrives at a very similar form to (4.28)

$$\begin{aligned} M^{(n)}_{\lambda_f\lambda_i}(k_f,k_i) &= 2\pi\delta(\delta k^0)\beta_k\epsilon^{\lambda_f *}_\mu(k_f)\epsilon^\mu_{\lambda_i}(k_i) \\ &\times\int d^2\boldsymbol{y}_t e^{-i\delta\boldsymbol{k}_t\cdot\boldsymbol{y}_t}\left(\exp\left[-i\frac{g_s}{\beta_p}\int_{-\infty}^{+\infty} dy_3\, T_\alpha A^0_\alpha(\boldsymbol{y})\right] - 1\right). \end{aligned} \tag{4.39}$$

Similarly to the quark case, in the high energy limit we find $k_f \simeq k_i$ and thus we obtain a polarization conservation delta. The gluon dynamics in this sense is similar to the quark except for the coupling definitions, and contained in the function $F^{(n)}_{el}(\boldsymbol{q})$.

4.3 Multiple scattering effects

Amplitude (4.28) can be squared and averaged over multiple scatterers configurations. As in the QED case we find an incoherent and a coherent contribution, but with some differences related to the color behavior. It results illustrative to obtain this transverse interference behavior by means of an expansion in the number of collisions prior to a direct evaluation of the full expression. An inspection of (4.30) suggests defining

$$\Gamma_k \equiv \exp\left(-i\frac{g_s^2}{\beta_p} t_\alpha t_\alpha \chi_0^k(\boldsymbol{y}_t)\right) - 1. \tag{4.40}$$

Let us denote by b_f^k and b_i^k the final and initial color, respectively, of the target quark k, then omitting the indices in the left hand side

$$F_{el}^{(n)}(\boldsymbol{q}) = \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\prod_{k=1}^{n} \left((\Gamma_k)_{b_i^k}^{b_f^k} + \delta_{b_i^k}^{b_f^k} \right) - \prod_{k=1}^{n} \delta_{b_i^k}^{b_f^k} \right). \tag{4.41}$$

It is now easy to expand the amplitude in the number of collisions. We notice

$$\begin{aligned} F_{el}^{(n)}(\boldsymbol{q}) &= \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\sum_{k=1}^{n} (\Gamma_k)_{b_i^k}^{b_f^k} \prod_{j\neq k} \delta_{b_i^j}^{b_f^j} + \sum_{k=1}^{n} \sum_{j=k+1}^{n} (\Gamma_k)_{b_i^k}^{b_f^k} (\Gamma_j)_{b_i^j}^{b_f^j} \prod_{l\neq j\neq k} \delta_{b_i^l}^{b_f^l} + \dots \right) \\ &= \sum_{k=1}^{n} I_k + \sum_{k=1}^{n} \sum_{j=k+1}^{n} I_{kj} + \cdots . \end{aligned} \tag{4.42}$$

The single collision contribution can be rewritten as

$$\begin{aligned} \sum_{k=1}^{n} I_k &= \sum_{k=1}^{n} \left(\prod_{j\neq k} \delta_{b_i^j}^{b_f^j} \right) \int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{x}_t} \left(\exp\left(-i\frac{g_s^2}{\beta_p} t_\alpha t_\alpha \chi_0^k(\boldsymbol{y}_t)\right) - 1 \right)_{b_i^k}^{b_f^k} \\ &= \sum_{k=1}^{n} \left(\prod_{j\neq k} \delta_{b_i^j}^{b_f^j} \right) e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^k} \left(F_{el}^{(1)}(\boldsymbol{q}_t) \right)_{b_i^k}^{b_f^k} , \end{aligned} \tag{4.43}$$

which is the expected result, consisting in the transition of the target quark k causing the collision and the no transition of the rest of quarks. Due to the high energy limit any term of higher order can be always written as a convolution

$$I_{kj} = \left(\prod_{l\neq j\neq k} \delta_{b_i^k}^{b_f^l} \right) \int \frac{d^2\boldsymbol{k}_t}{(2\pi)^2} e^{-i\boldsymbol{k}_t\cdot\boldsymbol{r}_t^k - i(\boldsymbol{q}_t-\boldsymbol{k}_t)\cdot\boldsymbol{r}_t^j} \left(F_{el}^{(1)}(\boldsymbol{k}_t) \right)_{b_i^k}^{b_f^k} \left(F_{el}^{(1)}(\boldsymbol{q}_t - \boldsymbol{k}_t) \right)_{b_i^j}^{b_f^j} , \tag{4.44}$$

which is easily interpretable also in terms of single processes. We can consider now the squared amplitude, which with this notation is given by

$$F_{el}^{(n)\dagger}(\boldsymbol{q})F_{el}^{(n)}(\boldsymbol{q}) = \sum_{k=1}^{n}\sum_{j=1}^{n} I_j^\dagger I_k + \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{l=j+1}^{n}\left(I_{jl}^\dagger I_k + I_k^\dagger I_{jl}\right) + \cdots . \tag{4.45}$$

The first contribution is given by the square of the single collision amplitudes, which is given by terms as

$$I_j^\dagger I_k = \left(\prod_{l\neq k}\delta_{b_i^l}^{b_f^l}\right)\left(\prod_{m\neq j}\delta_{b_f^m}^{b_i^m}\right)\left(e^{+i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^j}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)_{b_f^j}^{b_i^j}\right)\times\left(e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t^k}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right)_{b_i^k}^{b_f^k}\right). \tag{4.46}$$

The sum of this terms can be always arranged into a diagonal and a non diagonal contribution following

$$\sum_{k=1}^{n}\sum_{j=1}^{n} I_j^\dagger I_k = \sum_{k=1}^{n} I_k^\dagger I_k + \sum_{k=1}^{n}\sum_{j\neq k}^{n} I_j^\dagger I_k, \tag{4.47}$$

which produces for the diagonal contribution

$$\sum_{k=1}^{n} I_k^\dagger I_k = n\left(\prod_{j\neq k}^{n}\delta_{b_i^j}^{b_f^j}\delta_{b_f^j}^{b_i^j}\right)\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)_{b_f^k}^{b_i^k}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right)_{b_i^k}^{b_f^k} = nN_c^{n-1}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)F_{el}^{(1)}(\boldsymbol{q}_t)\right), \tag{4.48}$$

where trace refers to the target color space. By dividing by the dimension of the target color space and by perturbatively expanding the amplitudes up to first order in the coupling (4.33) we get

$$\frac{1}{N_c^n}\sum_{k=1}^{n} I_k^\dagger I_k \simeq \frac{n}{N_c}\frac{g_s^4}{\beta_p^2}t_\alpha t_\beta\,\mathrm{Tr}(t_\alpha t_\beta)\left(\hat{A}_0^{(1)}(\boldsymbol{q})\right)^2 = n\frac{g_s^4}{\beta_p^2}\frac{N_c^2-1}{4N_c^2}\left(A_0^{(1)}(\boldsymbol{q})\right)^2, \tag{4.49}$$

which is n, the number of independent collisions, times the squared amplitude of a single collision. Notice that the color average in the target produces an average color conservation in the projectile $\delta_{a_i}^{a_f}$. For the non-diagonal contribution we find

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j\neq k}^{n} I_j^\dagger I_k &= \sum_{k=1}^{n}\sum_{j\neq k}^{n}\left(\prod_{l\neq k}^{n}\delta_{b_i^l}^{b_f^l}\right)\left(\prod_{m\neq j}^{n}\delta_{b_f^m}^{b_i^m}\right)e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{r}_t^k-\boldsymbol{r}_t^j)}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)_{b_f^j}^{b_i^j}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right)_{b_i^k}^{b_f^k}\\ &= N_c^{n-2}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right)\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{r}_t^k-\boldsymbol{r}_t^j)}\right).\end{aligned} \tag{4.50}$$

Unlike to the QED case, by using (4.33) we get that the first order in the coupling vanishes due to the traceless t_α matrices, so at leading order in the coupling

$$\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}_t)\right)\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{q}_t)\right) = \left(-\frac{1}{2}\frac{g_s^4}{\beta_p^2}t_\alpha t_\beta\,\mathrm{Tr}(t_\alpha t_\beta)\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{A}_0^{(1)}(\boldsymbol{k})\hat{A}_0^{(1)}(\boldsymbol{q}-\boldsymbol{k})\right)^2. \tag{4.51}$$

By averaging over target spatial configurations, for simplicity over a solid cylinder of section $\Omega = \pi R^2$ we have

$$\left\langle\sum_{k=1}^{n}\sum_{j\neq k}^{n} e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{r}_t^k-\boldsymbol{r}_t^j)}\right\rangle_\Omega = n(n-1)\left(\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t e^{-i\boldsymbol{q}_t\cdot\boldsymbol{r}_t}\right)^2 = n(n-1)\left(\frac{w(\boldsymbol{q},R)}{\pi R^2}\right)^2. \tag{4.52}$$

where $w(\boldsymbol{q},R)$ is the window function given at (2.77). Then we get

$$\frac{1}{N_c^n}\sum_{k=1}^{n}\sum_{j\neq k}^{n}\left\langle I_j^\dagger I_k\right\rangle_\Omega \simeq n^2\left(\frac{1}{2}\frac{g_s^4}{\beta_p^2}\frac{N_c^2-1}{4N_c^2}\frac{w(\boldsymbol{q},R)}{\pi R^2}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\hat{A}_0^{(1)}(\boldsymbol{k})\hat{A}_0^{(1)}(\boldsymbol{q}-\boldsymbol{k})\right)^2, \tag{4.53}$$

where n was assumed large so that $n(n-1)\simeq n^2$. Observe that the diagonal contribution produced a term of order n and α_s^2, the number of independent collisions times the leading order of a color averaged single collision, whereas the non diagonal contribution produces a term of order n^2 and α_s^4, which measures the n^2 different interferences between single collisions with different scattering centers. By performing the last integral in momentums the final result can be written explicitely as

$$\begin{aligned}\left\langle\left|F_{el}^{(n)}(\boldsymbol{q})\right|^2\right\rangle \cong\, & n\frac{N_c^2-1}{4N_c^2}\left(\frac{\alpha_s}{\beta_p}\frac{4\pi}{\boldsymbol{q}^2+\mu_d^2}\right)^2 \\ & + \left(n\frac{N_c^2-1}{4N_c^2}\frac{w(\boldsymbol{q},R)}{\pi R^2}\frac{1}{2}\left(\frac{\alpha_s}{\beta_p}\right)^2\frac{16\pi\,\mathrm{arcsinh}\left(\frac{q}{2\mu_d}\right)}{q\sqrt{q^2+4\mu_d^2}}\right)^2.\end{aligned} \tag{4.54}$$

For convenience we give the window functions for a solid cylinder and for a cylinder with normalized Gaussian decaying density, of a typical width R^2,

$$\frac{w_c(q,R)}{\pi R^2} \equiv 2\frac{J_1(qR)}{qR}, \qquad \frac{w_g(q,R)}{\pi R^2} \equiv \exp\left(-\frac{q^2R^2}{2}\right). \tag{4.55}$$

Notice that in the limit $R\to\infty$ we recover always the forward propagation $\delta^2(\boldsymbol{q})$ in the non-diagonal contribution, as expected due to transverse homogeneity. In

Figure 4.3 the above result is shown for a typical QCD medium. As it can be seen, although the non-diagonal contribution is a higher order correction in the coupling to the diagonal contribution, for $n = 10$ and higher number of constituents its contribution in the soft scattering zone $q \sim 1/R$ is the dominant one. For mediums of large length the above discussion is not enough, since the scattering typically involves more than one collision. For the next order evaluations we refer to the procedure at Section 2.4. Since the expansion in the number of collisions is an alternated series, a functional form is required. We define then the shorthand notations

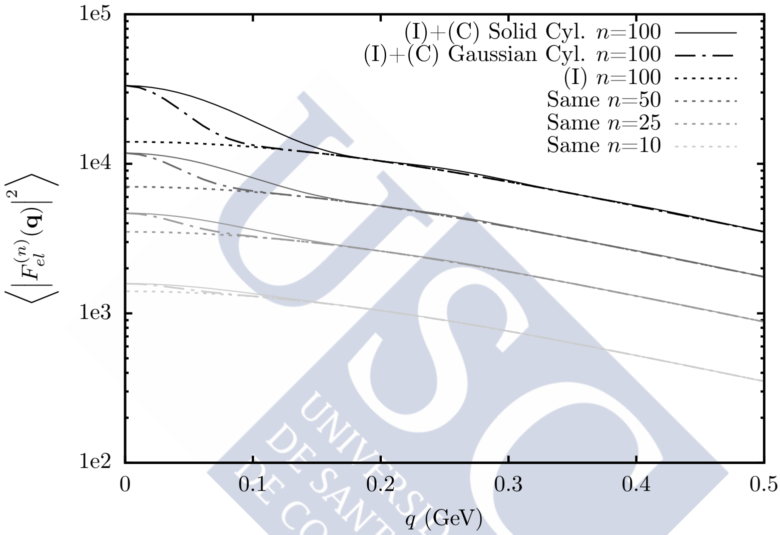


Figure 4.3: Averaged squared elastic amplitude at leading order in the coupling and in the number of collisions as a function of the momentum change, of a quark emerging from a medium with n = 100 quarks (black lines), with Debye screening mass of μ_d = 0.5 GeV and a medium radius of R = 10 r_d. Incoherent (I) contribution (diagonal term) is shown in dotted lines, and incoherent (I) and coherent (C) contributions (diagonal and non diagonal term) are shown for a cylinder with Gaussian decaying density (dot-dashed lines) and a solid cylinder (solid lines). Same results for n = 50 (dark grey), n = 25 (medium grey) and n = 10 (lightest grey).

$$W_0^{(n)}(\boldsymbol{y}) \equiv \exp\left(-i\sum_{k=1}^{n}\frac{g_s^2}{\beta}t_\alpha t_\beta \chi_0^k(\boldsymbol{y})\right) = \prod_{k=1}^{n}\left(\exp\left(-i\frac{g_s^2}{\beta_p}t_\alpha t_\alpha \chi_0^k(\boldsymbol{x})\right)\right) = \prod_{k=1}^{n} W_0^k(\boldsymbol{y}). \tag{4.56}$$

Then the average over target colors and positions can be formally written as

$$\frac{1}{N_c^n}\operatorname{Tr}\left\langle M_{s_f s_i}^{(n)\dagger}(p_f, p_i) M_{s_f s_i}^{(n)}(p_f, p_i)\right\rangle = 2\pi\delta(q^0)\beta_p \delta_{s_i}^{s_f}\frac{1}{N_c^n}\operatorname{Tr}\left\langle F_{el}^{(n)\dagger}(\boldsymbol{q}) F_{el}^{(n)}(\boldsymbol{q})\right\rangle, \tag{4.57}$$

where the trace refers to the independent color average/traces of the different n centers. We divided by an overall infinite $T = 2\pi\delta(0)$ factor, related to time translation invariance, and by the quark incoming flux β_p. We have to evaluate

$$\frac{1}{N_c^n}\,\mathrm{Tr}\left(\left\langle F_{el}^{(n)\dagger}(\boldsymbol{q})F_{el}^{(n)}(\boldsymbol{q})\right\rangle\right) = \int d^2\boldsymbol{x}\int d^2\boldsymbol{y}e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} \times \frac{1}{N_c^n}\,\mathrm{Tr}\left\langle\left(W_0^{(n)\dagger}(\boldsymbol{y})-1\right)\left(W_0^{(n)}(\boldsymbol{x})-1\right)\right\rangle. \tag{4.58}$$

The average on the right hand side, by using our notation, factorizes as

$$\frac{1}{N_c^n}\,\mathrm{Tr}\left\langle\left(W_0^{(n)\dagger}(\boldsymbol{y})-1\right)\left(W_0^{(n)}(\boldsymbol{x})-1\right)\right\rangle = \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})W_0^{k}(\boldsymbol{x})\right\rangle\right) - \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right) - \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k}(\boldsymbol{x})\right\rangle\right) + 1. \tag{4.59}$$

As learned from the QED case and the statistical relation $\langle x^2\rangle = \langle x\rangle^2 + \sigma^2$, our previous discussion regarding to a splitting into an incoherent and a coherent contributions suggests adding and subtracting now the term

$$\pm\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{(1)\dagger}(\boldsymbol{y})\right\rangle\right)\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{(1)}(\boldsymbol{x})\right\rangle\right). \tag{4.60}$$

Then we can pair the following terms as

$$\begin{aligned} 1 &- \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right) - \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k}(\boldsymbol{x})\right\rangle\right) \\ &+ \prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right)\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{(1)}(\boldsymbol{x})\right\rangle\right) \\ = &\left(\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right) - 1\right)\left(\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{(1)}(\boldsymbol{x})\right\rangle\right) - 1\right), \end{aligned} \tag{4.61}$$

in such a way that we found the usual splitting into a coherent and an incoherent contribution

$$\frac{1}{N_c}\,\mathrm{Tr}\left(\left\langle F_{el}^{(n)\dagger}(\boldsymbol{q})F_{el}^{(n)}(\boldsymbol{q})\right\rangle\right) = \hat{\Pi}_2^{(n)}(\boldsymbol{q}) + \hat{\Sigma}_2^{(n)}(\boldsymbol{q}), \tag{4.62}$$

where the coherent contribution is given by the color/space averaged amplitude squared

$$\hat{\Pi}_2^{(n)}(\boldsymbol{q}) \equiv \left(\int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^k(\boldsymbol{x})\right\rangle\right)-1\right)\right)^\dagger \\ \times\left(\int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^k(\boldsymbol{x})\right\rangle\right)-1\right)\right), \tag{4.63}$$

and the incoherent contribution, which admits an statistical and classical interpretation as a deviation σ^2 from the quantum coherent contribution, is given by

$$\hat{\Sigma}_2^{(n)}(\boldsymbol{q}) \equiv \int d^2\boldsymbol{x}\int d^2\boldsymbol{y} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left(\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})W_0^k(\boldsymbol{x})\right\rangle\right)\right. \\ \left. -\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right)\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^k(\boldsymbol{x})\right\rangle\right)\right). \tag{4.64}$$

The operation $\langle\star\rangle$ indicates an average over medium space configuration in a cylinder of transverse area $\Omega = \pi R^2$ and length l. The factorization in single averages provides a great simplification. For the incoherent contribution the first term in (4.64) produces

$$\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})W_0^k(\boldsymbol{x})\right\rangle\right) \equiv \left(\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t \frac{1}{N_c}\operatorname{Tr} W_0^{k\dagger}(\boldsymbol{y})W_0^k(\boldsymbol{x})\right)^n. \tag{4.65}$$

Since $\chi_0^k(\boldsymbol{x})$ as a function of $\boldsymbol{x}-\boldsymbol{r}_k$ vanishes for distances larger than μ_d we can take the exponential approximation even for low n, *c.f.* Section 2.4, provided that the condition $\mu_d^2 \ll R^2$ is satisfied. By adding and subtracting 1 and assuming a constant density n_0 then

$$\lim_{R\to\infty}\left(1+\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t\left[\frac{1}{N_c}\operatorname{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})W_0^1(\boldsymbol{x})\right)-1\right]\right)^n \\ \simeq \lim_{R\to\infty}\exp\left\{n_0 l\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\operatorname{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})W_0^1(\boldsymbol{x})\right)-1\right)\right\}. \tag{4.66}$$

By taking the $R\to\infty$ limit and rearranging the terms in the exponential we obtain

$$\lim_{R\to\infty}\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\operatorname{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})W_0^1(\boldsymbol{x})\right)-1\right)= \tag{4.67}$$

$$\frac{1}{N_c}\operatorname{Tr}\left(F_{el}^{(1)\dagger}(0)\right)+\frac{1}{N_c}\operatorname{Tr}\left(F_{el}^{(1)}(0)\right)+\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{1}{N_c}\operatorname{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right).$$

which by using the optical theorem (4.32) simplifies to

$$\int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\left(e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}-1\right)\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right), \tag{4.68}$$

Similarly for the second term in the incoherent contribution (4.64) we find

$$\begin{aligned}
&\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y})\right\rangle\right)\prod_{k=1}^{n}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^{k}(\boldsymbol{x})\right\rangle\right) &(4.69)\\
&\equiv\left(\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})\right)\right)^n\left(\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1}(\boldsymbol{x})\right)\right)^n\\
&\simeq \exp\left\{n_0 l\left(\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})\right)-1\right)+\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1}(\boldsymbol{x})\right)-1\right)\right)\right\}.
\end{aligned}$$

The term in the exponential, after identifying the single color-space averaged elastic amplitudes and by using (4.32), can be rewritten as

$$\begin{aligned}
&\lim_{R\to\infty}\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1\dagger}(\boldsymbol{y})\right)-1\right)+\lim_{R\to\infty}\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\,\mathrm{Tr}\left(W_0^{1}(\boldsymbol{x})\right)-1\right)\\
&=\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{0})\right)+\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{0})\right)=-\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right).
\end{aligned} \tag{4.70}$$

Finally the incoherent contribution (4.64) can be written as

$$\begin{aligned}
\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l)&=\pi R^2\exp\left[-n_0 l\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right)\right] &(4.71)\\
&\times\int d^2\boldsymbol{x}e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\left\{\exp\left[n_0 l\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right)\right]-1\right\},
\end{aligned}$$

which as expected is a solution to the Moliere transport equation [36, 37] for QCD matter with boundary condition $\Sigma_2^{(1)}(\boldsymbol{q},0)=0$, i.e. in absence of matter the diffracted wave vanishes. It is easy to show that this transport equation is given by

$$\begin{aligned}
\frac{\partial}{\partial l}\Sigma_2^{(n)}(\boldsymbol{q},l)&=n_0\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{k})F_{el}^{(1)}(\boldsymbol{k})\right)\left(\Sigma_2^{(n)}(\boldsymbol{q}-\boldsymbol{k},l)-\Sigma_2^{(n)}(\boldsymbol{q},l)\right)\\
&+\frac{n_0}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{k})F_{el}^{(1)}(\boldsymbol{k})\right)\pi R^2\exp\left[-n_0 l\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})F_{el}^{(1)}(\boldsymbol{q})\right)\right].
\end{aligned} \tag{4.72}$$

The interpretation is straightforward: the number of states with momentum transfer $\boldsymbol{q}$ at a depth l grows with the number of states with $\boldsymbol{q}-\boldsymbol{k}$ at $l-\delta l$, experimenting a single collision acquiring $\boldsymbol{k}$, and decreases with the number of states with already $\boldsymbol{q}$ at l experimenting a single collision loosing arbitrary momentum. The extra boundary term, correcting the Moliere result, accounts for the probability of penetrating till l without interacting and experimenting a single collision of momentum transfer $\boldsymbol{q}$. Notice also that an expansion in the number of collisions leads to the diagonal contribution, as expected from an incoherent average. Indeed

$$\begin{aligned}\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l) &= \pi R^2 \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(n_0 l \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{+i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{N_c} \mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{k}) F_{el}^{(1)}(\boldsymbol{k})\right) + \cdots \right) \\ &= \frac{\pi R^2 n_0 l}{N_c} \mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}) F_{el}^{(1)}(\boldsymbol{q})\right) + \cdots = \frac{n}{N_c} \mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}) F_{el}^{(1)}(\boldsymbol{q})\right) + \cdots, \end{aligned} \tag{4.73}$$

which is just the first diagonal term (4.48) found in the initial discussion. The Fourier transforms of the single differential elastic cross sections can be given in closed form order to order in the coupling α_s. Using (4.33) we find

$$\begin{aligned}\sigma_{el}^{(1)}(\boldsymbol{x}) &= \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{N_c} \mathrm{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q}) F_{el}^{(1)}(\boldsymbol{q})\right) \\ &= \frac{4\pi g_s^4}{N_c \beta_p^2} t_\beta t_\alpha \,\mathrm{Tr}(t_\alpha t_\beta) \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\hat{A}_0^{(1)}(\boldsymbol{q})\right)^2 = \frac{4\pi g_s^4}{\beta_p^2 \mu_d^2} \frac{N_c^2 - 1}{4N_c^2} \mu_d |\boldsymbol{x}| K_1(\mu_d |\boldsymbol{x}|). \end{aligned} \tag{4.74}$$

Observe that these functions still have the remaining and trivial matrix structure in the color of the projectile. After averaging over the traveling quark colors $\mathrm{Tr}\,\sigma_{el}^{(1)}(\boldsymbol{0})/N_c \equiv \sigma_t^{(1)}$ we obtain the (scalar) total single elastic cross section. Finally we write for the incoherent term

$$\hat{\Sigma}_2^{(n)}(\boldsymbol{q},l) = \pi R^2 \exp\left[-n_0 l \sigma_{el}^{(1)}(\boldsymbol{0})\right] \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\exp\left[n_0 l \sigma_{el}^{(1)}(\boldsymbol{x})\right] - 1 \right), \tag{4.75}$$

which preserves color in the traveling quark so the incoherent scattering in QCD, for a color averaged medium behaves, except for the coupling strength, exactly as in the QED scenario. For a medium length verifying $\delta l \lesssim \lambda$, where the mean free path is given by $\lambda = 1/n_0 \sigma_{el}^{(1)}(\boldsymbol{0}) \equiv 1/n_0 \sigma_t^{(1)}$, a perturbative truncation of the distribution is enough. The average momentum transfer is then given by

$$\left\langle \boldsymbol{q}^2(\delta l) \right\rangle = \frac{\int d^2\boldsymbol{q}\, \boldsymbol{q}^2 \Sigma_2^{(n)}(\boldsymbol{q},\delta l)}{\int d^2\boldsymbol{q}\, \Sigma_2^{(n)}(\boldsymbol{q},\delta l)} \cong \frac{1}{\sigma_{el}^{(1)}(\boldsymbol{0})} \int d^2\boldsymbol{q} \boldsymbol{q}^2 \frac{1}{N_c} \mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{q}) F_{el}^{(1)}(\boldsymbol{q})\right), \tag{4.76}$$

which is just the normalized incoherent sum of the averages of the single collision distributions. The above expression diverges at large $\boldsymbol{q}$, as expected. However, by using the energy conservation delta at (4.57), instead of only (4.59), we find

$$\left\langle \boldsymbol{q}^2(\delta l) \right\rangle = \mu_d^2 \left(2 \log\left(\frac{2p_0^0}{\mu_d}\right) - 1 \right). \tag{4.77}$$

For mediums of larger size we can use the transport/Moliere equation (4.72). We then find an equivalent equation for the squared momentum transfer ,

$$\frac{\partial}{\partial l}\left\langle \boldsymbol{q}^2(l)\right\rangle = n_0\sigma_{el}^{(1)}(\mathbf{0})\left\langle \boldsymbol{q}^2(\delta l)\right\rangle \equiv 2\hat{q}, \tag{4.78}$$

so that the medium transport parameter $\hat{q}$ reads

$$\hat{q} = \frac{n_0}{2}\left(\frac{4\pi g_s^4}{\beta_p^2\mu_d^2}\frac{N_c^2-1}{4N_c^2}+\cdots\right)\times\mu_d^2\left(2\log\left(\frac{2p_0^0}{\mu_d}\right)-1\right). \tag{4.79}$$

The coherent contribution (4.63), on the other hand, consists in the averaged amplitude (4.28) squared, encoding the diffractive behavior of the medium. For its evaluation we observe

$$\begin{aligned}\prod_{k=1}^{n}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^k(\boldsymbol{x})\right\rangle\right) &= \left(1+\frac{1}{\pi R^2}\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\operatorname{Tr}\left(W_0^1(\boldsymbol{x}_t)\right)-1\right)\right)^n\\ &\simeq \exp\left[n_0 l\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\operatorname{Tr}\left(W_0^1(\boldsymbol{x})-1\right)\right)\right].\end{aligned} \tag{4.80}$$

By using (4.30) and (2.75) the term in the exponential can be written as a function $\pi_{el}^{(1)}(\boldsymbol{x})$

$$\int_\Omega d^2\boldsymbol{r}_t\left(\frac{1}{N_c}\operatorname{Tr}\left(W_0^{(1)}(\boldsymbol{x})-1\right)\right) = \int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}w(\boldsymbol{q},R)\frac{1}{N_c}\operatorname{Tr}F_{el}^{(1)}(\boldsymbol{q}) \equiv \pi_{el}^{(1)}(\boldsymbol{x}), \tag{4.81}$$

where the window function $w(\boldsymbol{q},R)$ can be given in general for any geometry. In particular for the cylinder or the Gaussian decaying cylinder, as we noted in (2.77). Since at all orders the color traces produce color conservation in the projectile, the coherent contribution is given by a perfect square times the unit matrix

$$\hat{\Pi}_2^{(1)}(\boldsymbol{q}) = \left|\int d^2\boldsymbol{x}e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\left(\exp\left(n_0 l\pi_{el}^{(1)}(\boldsymbol{x})\right)-1\right)\right|^2. \tag{4.82}$$

Observe that an expansion in the number of collisions produces

$$\int d^2\boldsymbol{x}_t e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\left(\exp\left(n_0 l\pi_{el}^{(1)}(\boldsymbol{x})\right)-1\right) = n_0 l w(\boldsymbol{q},R)\frac{1}{N_c}\operatorname{Tr}F_{el}^{(1)}(\boldsymbol{q})+\cdots, \tag{4.83}$$

and then at first order we obtain

$$\hat{\Pi}_2^{(1)}(\boldsymbol{q}) = n^2\left(\frac{w(\boldsymbol{q},R)}{\pi R^2}\right)^2\frac{1}{N_c}\operatorname{Tr}\left(F_{el}^{(1)\dagger}(\boldsymbol{q})\right)\frac{1}{N_c}\operatorname{Tr}\left(F_{el}^{(1)}(\boldsymbol{q})\right)+\cdots, \tag{4.84}$$

which is the non-diagonal contribution (4.50) we have found in the single scattering regime. The typical width of the function $\pi_{el}^{(1)}(\boldsymbol{x})$ is R, instead of the width $r_d = 1/\mu_d$, the dimensions of a single scattering center, of the function $\sigma_{el}^{1}(\boldsymbol{x})$. Consequently, the typical momentum change in a coherent scattering is always smaller than μ_d provided that $R > r_d = 1/\mu_d$, as we already encountered in the expansion in the number of collisions. For $R \gg r_d$, a leading order in α_s evaluation of this function is possible

$$\pi_{el}^{(1)}(\boldsymbol{x}) = -\frac{\alpha_s^2}{2\beta_p^2}\frac{N_c^2-1}{4N_c^2}1\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{i\boldsymbol{q}\cdot\boldsymbol{x}}w(\boldsymbol{q},R)\frac{16\pi\,\mathrm{arcsinh}\left(\frac{q}{2\mu_d}\right)}{q\sqrt{q^2+\mu_d^2}}+\cdots. \tag{4.85}$$

In particular we find for a cylinder with Gaussian decaying density, in the saddle point approximation,

$$\pi_{el}^{(1)}(\boldsymbol{x}) = -\frac{\alpha_s^2}{2\beta_p^2}\frac{N_c^2-1}{4N_c^2}\frac{\pi}{\mu_d^2}\exp\left(-\frac{\boldsymbol{x}^2}{2R^2}\right)+\cdots, \tag{4.86}$$

which has less than a $\sim 2\%$ of deviation of the actual value for $R/r_d > 10$ and for $x/R < 4$. We also notice that, as expected, when $R \to \infty$, $\pi_{el}^{(1)}(\boldsymbol{x})$ is independent on $\boldsymbol{x}$ and we recover the pure forward contribution. An evolution equation at the level of the amplitude can be written

$$\begin{aligned}\frac{\partial}{\partial l}\Pi_1^{(n)}(\boldsymbol{q}) = n_0\int\frac{d^2\boldsymbol{k}_t}{(2\pi)^2}w(\boldsymbol{k},R)\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{k})\right)\Pi_1^{(n)}(\boldsymbol{q}-\boldsymbol{k}) \\ +\,w(\boldsymbol{q},R)\frac{1}{N_c}\,\mathrm{Tr}\left(F_{el}^{(1)}(\boldsymbol{q})\right).\end{aligned} \tag{4.87}$$

which can not be interpreted in probabilistic terms since $\Pi_2^{(n)}(\boldsymbol{q}) = \Pi_1^{(n)\dagger}(\boldsymbol{q})\Pi_1^{(n)}(\boldsymbol{q})$.

4.4 Beyond eikonal scattering

We will now assume that the q_z component of the momentum change cannot be omitted, either in order to take into account the $1/p_0$ corrections of the preceding pure eikonal results or in order to consider the interference between amplitudes with different on-shell states, as is the case of inelastic processes. In that case the phase accumulated by the dropped term $q_z z$ can be large at large distances, and as we have seen in the QED case, lead to interference phenomena such as the LPM effect. The amplitude for a set of n quarks placed sharing equal z coordinate can be written as

$$M_{s_f s_i}^{(n)}(p_f,p_i) = 2\pi\delta(\delta p_i^0)\delta_{s_i}^{s_f}\beta_p\int d^3\boldsymbol{y}e^{-i\boldsymbol{q}\cdot\boldsymbol{y}}\frac{\partial}{\partial y_3}\exp\left(-i\frac{g_s}{\beta_p}\int_{-\infty}^{y_3}dy_3' t_\alpha A_\alpha^0(\boldsymbol{y})\right), \tag{4.88}$$

where the eikonal phase reads, using (4.16),

$$\int_{-\infty}^{y_3} dy'_3 \, t_\alpha A^0_\alpha(\boldsymbol{y}) = g_s t_\alpha t_\alpha \sum_{i=k}^{n} \int_{-\infty}^{y_3} dy'_3 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{4\pi}{\boldsymbol{q}^2 + \mu_d^2} e^{-i\boldsymbol{q}\cdot(\boldsymbol{y}-\boldsymbol{r}_k)}. \tag{4.89}$$

We will assume that $\boldsymbol{q}^2 \simeq \boldsymbol{q}_t^2$ so that the longitudinal structure of each center is neglected, but we keep q_z in the phase since y_3 can be arbitrarily large and we want to keep the longitudinal structure of the medium. This approximation produces

$$\begin{aligned}\int_{-\infty}^{y_3} dy'_3 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{4\pi}{\boldsymbol{q}^2 + \mu_d^2} e^{-i\boldsymbol{q}\cdot(\boldsymbol{y}-\boldsymbol{r}_k)} &\simeq \int_{-\infty}^{y_3} dy'_3 \delta(y'_3 - r_3^k) \\ \times \int \frac{d^2\boldsymbol{q}_t}{(2\pi)^2} \frac{4\pi}{\boldsymbol{q}_t^2 + \mu_d^2} e^{-i\boldsymbol{q}_t\cdot(\boldsymbol{y}_t-\boldsymbol{r}_t^k)} &= \Theta(y_3 - r_3^k)\chi_0^{(1)}(\boldsymbol{y}_t - \boldsymbol{r}_t^k) \equiv \Theta^k(y_3)\chi_0^k(\boldsymbol{y}_t).\end{aligned} \tag{4.90}$$

Since we placed the quarks at the same z coordinate we can easily integrate by parts the amplitude, which takes the simpler form

$$M^{(n)}_{s_f s_i}(p_f, p_i) = \left(S^{(n)}_{s_f s_i}(p_f, p_i) - S^{(0)}_{s_f s_i}(p_f, p_i)\right), \tag{4.91}$$

where the eikonal amplitude, including the no collision amplitude, is given as usual by

$$S^{(n)}_{s_f s_i}(p_f, p_i) \equiv 2\pi\beta_p \delta(\delta p_i^0)\delta^{s_f}_{s_i} \int d^2\boldsymbol{y}_t e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \exp\left(-i\frac{g_s^2}{\beta_p} t_\alpha t_\alpha \sum_{k=1}^{n} \chi_0^k(\boldsymbol{y})\right), \tag{4.92}$$

where the total three momentum change $\boldsymbol{q}\cdot\boldsymbol{x}$ appears now in the phase. We can consider the medium as a set of n layers of quarks at $z_1, z_2 \ldots z_n$, of $n(z_i)$ quarks respectively. The total number of quarks in the medium is defined as

$$\sum_{i=1}^{n} n(z_i) \equiv N. \tag{4.93}$$

At high energies the amplitude of finding the quark with momentum p_n after the passage through these layers is given by the convolution of single layer amplitudes, then

$$S^{(N)}_{s_n s_0}(p_n, p_0) = \sum_s \left(\prod_{i=1}^{n-1} \int \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=1}^{n} S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1})\right), \tag{4.94}$$

where sum over intermediate spins and ordering is assumed. So that we find

$$M^{(N)}_{s_n s_0}(p_n, p_0) = S^{(N)}_{s_n s_0}(p_n, p_0) - S^{(0)}_{s_n s_0}(p_n, p_0). \tag{4.95}$$

At this point we notice that a path integral expression for the amplitude exists by taking the separation between layers $\delta z_k \to 0$. The resulting expression is, however, uninteresting out of a formal context, since the interaction carries the matrix structure in the traveling and target color spaces. We instead take the intensity and average over target colors, which leads to color preservation in the projectile. By splitting as before in a incoherent and a coherent contribution we obtain

$$\left\langle M^{(N)\dagger}_{s_n s_0}(p_n, p_0) M^{(N)}_{s_n s_0}(p_n, p_0) \right\rangle = \Sigma_2^{(N)}(p_n, p_0) + \Pi_s^{(n)}(p_n, p_0), \tag{4.96}$$

where the coherent contribution is given by the averaged amplitudes squared

$$\Pi_2^{(N)}(p_n, p_0) = \left\langle S^{(N)\dagger}_{s_n s_0}(p_n, p_0) - S^{(0)\dagger}_{s_n s_0}(p_n, p_0) \right\rangle \left\langle S^{(N)}_{s_n s_0}(p_n, p_0) - S^{(0)}_{s_n s_0}(p_n, p_0) \right\rangle, \tag{4.97}$$

and the incoherent contribution instead by

$$\Sigma_2^{(N)}(p_n, p_0) = \left\langle S^{(N)\dagger}_{s_n s_0}(p_n, p_0) S^{(N)}_{s_n s_0}(p_n, p_0) \right\rangle - \left\langle S^{(N)\dagger}_{s_n s_0}(p_n, p_0) \right\rangle \left\langle S^{(N)}_{s_n s_0}(p_n, p_0) \right\rangle. \tag{4.98}$$

The first term of the incoherent contribution can be written, after dividing by the incoming quark flux and the infinite factor $T = 2\pi\delta(0)$ accounting for time translation invariance, as the product

$$\begin{aligned} &\left\langle S^{(N)\dagger}_{s_n s_0}(p_n, p_0) S^{(N)}_{s_n s_0}(p_n, p_0) \right\rangle = 2\pi\beta_p \delta(p_n^0 - p_0^0)\delta^{s_n}_{s_0} \\ &\times \prod_{i=1}^{n-1} \left(\int \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2} \int \frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2} \exp\left(-i \frac{(\boldsymbol{p}_i^t)^2 - (\boldsymbol{u}_i^t)^2}{2p_0^0} \delta z_i \right) \right) \\ &\times \prod_{i=1}^{n} \left(\int d^2\boldsymbol{x}_i^t \int d^2\boldsymbol{y}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t + i\delta\boldsymbol{u}_i^t\cdot\boldsymbol{y}_i^t} \prod_{k=1}^{n(z_i)} \left(\frac{1}{N_c} \mathrm{Tr} \left\langle W_0^{k\dagger}(\boldsymbol{y}_t) W_0^k(\boldsymbol{x}_t) \right\rangle \right) \right), \end{aligned} \tag{4.99}$$

where $\delta z_i \equiv z_{i+1} - z_i$ and $\delta\boldsymbol{p}_i = \boldsymbol{p}_i - \boldsymbol{p}_{i-1}$ and the internal momentum in the conjugated amplitude has been denoted by $\boldsymbol{u}_i$. The bottom line is just a product of averages at each layer of length δz_i and density $n_0(z_i)$ like the one computed in the previous section, thus we write

$$\begin{aligned} &\int d^2\boldsymbol{x}_i^t \int d^2\boldsymbol{y}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t + i\delta\boldsymbol{u}_i^t\cdot\boldsymbol{y}_i^t} \prod_{k=1}^{n(z_i)} \left(\frac{1}{N_c} \mathrm{Tr} \left\langle W_0^{k\dagger}(\boldsymbol{y}_t) W_0^k(\boldsymbol{x}_t) \right\rangle \right) \\ &\quad = (2\pi)^2 \delta^2(\delta\boldsymbol{p}_i^t - \delta\boldsymbol{u}_i^t) \int d^2\boldsymbol{x}_t^i e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t} \exp\left(\delta z_i n_0(z_i) \left(\sigma_{el}^{(1)}(\boldsymbol{x}_i^t) - \sigma_{el}^{(1)}(\boldsymbol{0}) \right) \right). \end{aligned} \tag{4.100}$$

Similarly for the second term in the incoherent contribution we find

$$\left\langle S^{(N)\dagger}_{s_n s_0}(p_n, p_0)\right\rangle \left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle = 2\pi\beta_p \delta(p_n^0 - p_0^0)\delta^{s_n}_{s_0} \tag{4.101}$$
$$\times \prod_{i=1}^{n-1}\left(\int \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\int \frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\exp\left(-i\frac{(\boldsymbol{p}_i^t)^2-(\boldsymbol{u}_i^t)^2}{2p_0^0}\delta z_i\right)\right)$$
$$\times \prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_i^t\int d^2\boldsymbol{y}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t+i\delta\boldsymbol{u}_i^t\cdot\boldsymbol{y}_i^t}\prod_{k=1}^{n(z_i)}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y}_t)\right\rangle \operatorname{Tr}\left\langle W_0^{k}(\boldsymbol{x}_t)\right\rangle\right)\right),$$

where as before by using the results of the preceding section we obtain for each of the single layer averages

$$\int d^2\boldsymbol{x}_i^t\int d^2\boldsymbol{y}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t+i\delta\boldsymbol{u}_i^t\cdot\boldsymbol{y}_i^t}\prod_{k=1}^{n(z_i)}\left(\frac{1}{N_c}\operatorname{Tr}\left\langle W_0^{k\dagger}(\boldsymbol{y}_t)\right\rangle\left\langle W_0^{k}(\boldsymbol{x}_t)\right\rangle\right) \tag{4.102}$$
$$= (2\pi)^2\delta^2(\delta\boldsymbol{p}_i^t-\delta\boldsymbol{u}_i^t)\int d^2\boldsymbol{x}_t^i e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t}\exp\Big(-\delta z_i n_0(z_i)\sigma_{el}^{(1)}(\mathbf{0})\Big).$$

We can now proceed to integrate in the conjugate momentums $\boldsymbol{u}_i^t$ observing that since $\boldsymbol{u}_0^t \equiv \boldsymbol{p}_0^t$ and, by using the transverse momentum deltas, the longitudinal phase vanishes

$$\prod_{i=1}^{n-1}\left(\int \frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\exp\left(-i\frac{(\boldsymbol{p}_i^t)^2-(\boldsymbol{u}_i^t)^2}{2p_0^0}\delta z_i\right)\right)\prod_{i=1}^{n}\left((2\pi)^2\delta^2(\delta\boldsymbol{p}_i^t-\delta\boldsymbol{u}_i^t)\right) = (2\pi)^2\delta^2(\mathbf{0}), \tag{4.103}$$

as expected by symmetry arguments for a medium of infinite transverse size. Then the incoherent contribution is given simply by

$$\Sigma_2^{(N)}(p_n,p_0) = 2\pi\delta(p_n^0-p_0^0)\beta_p\delta^{s_n}_{s_0}\pi R^2\prod_{i=1}^{n-1}\left(\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t}\right)$$
$$\times\left\{\prod_{i=1}^{n}\exp\left(\delta z_i n_0(z_i)\left(\sigma_{el}^{(1)}(\boldsymbol{x}_i^t)-\sigma_{el}^{(1)}(\mathbf{0})\right)\right)-\prod_{i=1}^{n}\exp\Big(-\delta z_i n_0(z_i)\sigma_{el}^{(1)}(\mathbf{0})\Big)\right\}. \tag{4.104}$$

We notice that the integration in the internal momentum variables is trivial and the internal scattering structure is lost. The joint action of the squared averaged amplitudes at each layer simply convolute without phase and the pure eikonal limit remains valid. By taking the $\delta z_i \to 0$ limit we obtain

$$\Sigma_2^{(N)}(p_n,p_0) = 2\pi\delta(p_n^0-p_0^0)\beta_p\delta^{s_n}_{s_0}\pi R^2\exp\left(-\sigma_{el}^{(1)}(\mathbf{0})\int_{z_1}^{z_n}dz\, n_0(z)\right)$$
$$\times\int d^2\boldsymbol{x}_t e^{-i(\boldsymbol{p}_n^t-\boldsymbol{p}_0^t)\cdot\boldsymbol{x}_t}\left(\exp\left(\sigma_{el}^{(1)}(\boldsymbol{x}_t)\int_{z_1}^{z_n}dz\, n_0(z)\right)-1\right), \tag{4.105}$$

which except for the varying local density is our previous result. For the coherent contribution the scenario is however different if we do not take the $R \to \infty$ limit. We have

$$\left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle = 2\pi\delta(p_n^0 - p_0^0)\beta_p \delta^{s_n}_{s_0} \\ \times \prod_{i=1}^{n-1}\left(\int \frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_i^t e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^t}\prod_{k=1}^{n_0(z_i)}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^k(\boldsymbol{x}_k^t)\right\rangle\right)\right) \\ \times \exp\left(+i\frac{(\boldsymbol{p}_n^t)^2}{2p_0^0}z_n - i\sum_{i=1}^{n-1}\delta z_i\frac{(\boldsymbol{p}_i^t)^2}{2p_0^0} - i\frac{(\boldsymbol{p}_0^t)^2}{2p_0^0}z_1\right). \tag{4.106}$$

The total kinematical phase must be rearranged as follows

$$-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t + i\frac{(\boldsymbol{p}_n^t)^2}{2p_0^0}z_n + i\sum_{i=1}^{n-1}\boldsymbol{p}_i^t\cdot\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\delta z_i - i\sum_{i=1}^{n}\frac{(\boldsymbol{p}_i^t)^2}{2p_0^0}\delta z_i - i\frac{(\boldsymbol{p}_0^t)^2}{2p_0^0}z_1 + i\boldsymbol{p}_0^t\boldsymbol{x}_1^t, \tag{4.107}$$

and the medium contribution is given by the results of the previous section

$$\prod_{k=1}^{n_0(z_i)}\left(\frac{1}{N_c}\,\mathrm{Tr}\left\langle W_0^k(\boldsymbol{x}_k^t)\right\rangle\right) = \exp\left(\delta z_i n_0(z_i)\pi_{el}^{(1)}(\boldsymbol{x}_i^t)\right). \tag{4.108}$$

Then we can perform now the integral in internal momenta. If we omit the two boundary terms in z_n and z_0, which otherwise will cancel when taking the square of the amplitude, we find

$$\left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle = 2\pi\delta(p_n^0 - p_0^0)\beta_p \delta^{s_n}_{s_0}\prod_{i=1}^{n-1}\left(\frac{i\delta z_i}{2\pi p_0^0}\right)\prod_{i=1}^{n}\left(\int d^2\boldsymbol{x}_t^i\right) \tag{4.109}$$
$$\times \exp\left(-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t + i\sum_{i=1}^{n-1}\frac{p_0^0}{2}\left(\frac{\delta\boldsymbol{x}_i^t}{\delta z_i}\right)^2\delta z_i + \sum_{i=1}^{n}\delta z_i n_0(z_i)\pi_{el}^{(1)}(\boldsymbol{x}_i^t) + i\boldsymbol{p}_0^t\boldsymbol{x}_1^t\right),$$

which by taking the $\delta z \to 0$ limit transforms into a path integral in the transverse plane with time variable the z position as

$$\left\langle S^{(N)}_{s_n s_0}(p_n, p_0)\right\rangle = 2\pi\delta(p_n^0 - p_0^0)\beta_p \delta^{s_n}_{s_0}\int d^2\boldsymbol{x}_n^t e^{-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t}\int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\boldsymbol{x}_1^t} \\ \int \mathcal{D}^2\boldsymbol{x}_t(z)\exp\left(i\int_{z_1}^{z_n}dz\left(\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z) - in_0(z)\pi_{el}^{(1)}\big(\boldsymbol{x}_t(z)\big)\right)\right). \tag{4.110}$$

Similarly the second term in the coherent contribution is simply found by doing N=0 in the above relation, thus by dividing by the incoming quark flux and a factor $T = 2\pi\delta(0)$ accounting time translation invariance, we finally write

$$\Pi_2^{(N)}(p_n, p_0) = 2\pi\delta(p_n^0 - p_0^0)\beta_p\delta_{s_0}^{s_n}\left|\int d^2\boldsymbol{x}_n^t e^{-i\boldsymbol{p}_n^t\cdot\boldsymbol{x}_n^t}\int d^2\boldsymbol{x}_1^t e^{+i\boldsymbol{p}_0^t\boldsymbol{x}_1^t}\right. \tag{4.111}$$

$$\left.\times\int\mathcal{D}^2\boldsymbol{x}_t(z)\exp\left(i\int_{z_1}^{z_n}dz\,\frac{p_0^0}{2}\dot{\boldsymbol{x}}_t^2(z)\right)\left(\exp\left(\int_{z_1}^{z_n}dz\,n_0(z)\pi_{el}^{(1)}\left(\boldsymbol{x}_t(z)\right)\right)-1\right)\right|^2,$$

which either in the $p_0^0 \to \infty$ or the $R \to \infty$ limits transforms into our previous eikonal result.

5

High energy emission in QCD

The LPM effect in a QCD multiple scattering scenario has become an useful tool of indirectly observing the properties of the hadronic matter at extremely high temperatures. High energy collisions of heavy nuclei at the RHIC [67–70] and the LHC [71–73] hadron colliders are expected to reproduce the required extreme conditions for the QGP formation. Among other probes, the energy loss of hard jets constituents [74–78] produced in these collisions can be used to trace back the relevant characteristics of the formed QCD medium. While traveling through the QGP, a high energetic quark or gluon may undergo a collisional process with the medium constituents. The leading contribution to the energy loss suffered in this multiple collision scenario is of radiative nature [77]. As we have already seen, the radiation of quanta leading to this energy loss is going to be severely modified by the existence of multiple sources of scattering [1, 2, 4], compared to an emission scenario consisting in an incoherent sum of single collision intensities [6, 79]. Hence a systematic study of the intensity of gluon bremsstrahlung in media and the related parton energy loss shall provide indicative signs of the QGP formation and its characteristics.

Early studies of the intensity of gluon bremsstrahlung in a multiple scattering context exist [39, 75, 76]. The main difficulties introduced with respect to the QED case we have reviewed in Chapter 3 are related to the non abelian nature of QCD, namely the color structure and the gluon ability to reinteract with the medium. As we will show, these circumstances lead to a LPM interference in the soft limit dominated by the gluon rescattering, in analogy with the dielectric and transition radiation effects occurring in QED and shown at Figure 3.2. A first attempt for a QCD evaluation was given at [39] directly using the Migdal prediction [4] but without including gluon rescattering, and a more detailed evaluation was then given at [40, 80] without gluon rescattering effect in the longitudinal phase. Semi-infinite medium calculations in the Fokker-Planck approximation for the angle integrated intensity, either following a transport approach [21] or a path integral formalism [20, 55] were soon presented including the full gluon rescattering effect. These works can be shown to be the QCD equivalent to the Boltzmann transport approach [47] used by Migdal [4] with the adequate approximations [56]. Further extensions including the angular dependence, finite, cold or expanding target scenarios were later developed [54, 81–87] although usually lacking a general formulation, until a finite size evaluation of the angle dependence of the spectrum was given [19, 88–91] generalizing the previous results. The intensity of gluon bremsstrahlung within this framework [91–116] was then widely used to make predictions of the quenching behavior of the hadron spectra due to a multiple collision process with confined or deconfined QCD matter. Several other distinct frameworks were also developed at the same time or soon after the previous works, among them a reaction operator formalism for finite media [117–120] and for QGP [121, 122], a high twist formalism for finite nuclei [123, 124] and a finite field theory framework [125–127]. Applications of these results were also presented to make finite plasma, cold nuclear matter and heavy flavor suppression predictions at RHIC and LHC, see for instance [128–130]. An excellent comparison of all these frameworks was given at [131] and very good and detailed reviews on the subject have been written [132–137].

The aim of this chapter is to develop a formalism [138] in which the intensity of gluons in a multiple scattering scenario can be evaluated for a general interaction with the constituents of a finite or structured medium, and in which the angular spectrum is also considered. We will basically follow the steps we took at Chapter 3 for the QED case, to construct a non-perturbative description of the emission amplitude. At the level of the amplitude the diagrammatic structure becomes clear and the soft photon theorem [5, 139, 140] for finite media can be understood, together with the LPM effect, as part of the same coherence phenomena. The arising gluon intensity, which is the central result of this chapter, will lead, under the adequate color averaged interaction of the gluon and its Fokker-Planck approximation, to the well known results at [19–21]. In order to make a direct connection with the previous results, the results obtained for the multiple soft scattering case are easily adapted to implement an approximation for the emission scenario after a first hard collision.

In Section 5.1 we define amplitude of emission of a gluon under the effect of the medium as a set of static and classical Debye screened sources, as we explained at Section 4.1. The diagrammatic structure of the bremsstrahlung amplitude for multiple collisions is explained and the Bertsch-Gunion limit [79] recovered for the single collision scenario. In Section 5.2 we compute the intensity of gluons for a color and space averaged target with finite length l but for simplicity with macroscopical transverse size $R \to \infty$. In this approximation the transverse coherent contributions to the multiple scattering distributions are neglected. A color averaged effective interaction for the gluon is used which translates into Debye screened single interactions agreeing with previous results [19, 20, 141]. The resulting intensity is shown to satisfy a QED like form, where the LPM interfering phase, however, is governed by the gluon rescattering, and then the role of the traveling fermion is played by the gluon. Emission in a multiple soft scattering scenario will be shown to be dominated by a longitudinal coherent contribution, corresponding to the first and last leg gluons of the process, and lightly enhanced for small gluon energies as a result of the LPM effect, and then suppressed in the soft regime due to the mass suppression effect. An energy-loss formalism for that scenario will be developed using energy conservation constraints. Finally, the radiation due to a multiple soft collision scenario after a hard collision will be approximated as a particular subset of terms. The intensity in that case, except for the suppression in the soft limit related to a non vanishing gluon mass, will be shown to reproduce the well known results [19, 21, 91] if the Fokker-Planck approximation is used.

5.1 Amplitude

Let us consider an ideal scenario for an on-shell quark coming from the infinity and going to the infinity, while undergoing a multiple scattering process with a QCD medium. The medium will be composed of N static and classical sources characterized by (4.16), extending from z_1 to z_n in the initial quark direction. In QCD the amplitude of emission (3.10) of a single gluon, *c.f.* Sections 3.1 and 4.2, is given by

$$\mathcal{M}_{em}^{(N)} \equiv -ig_s \int d^4x \psi_f^{(N)}(x) A_{\mu,\alpha}^{(N)}(x) t_\alpha \gamma^\mu \psi_i^{(N)}(x), \tag{5.1}$$

where $\psi_f^{(N)}(x)$, $A_{\mu,\alpha}^{(N)}(x)$ and $\psi_i^{(N)}(x)$ are the states of the outcoming quark, the emitted gluon and the incoming quark, respectively. The initial and final quark asymptotic states are denoted by $\psi_i^{(0)}(x)$ and $\psi_f^{(0)}(x)$, with momentum, spin and color (p_0, s_0, a_0) and (p_n, s_n, a_n), respectively

$$\psi_i^{(0)}(x) \equiv \sqrt{\frac{m_q}{p_0^0}} u_{s_0}^{a_0}(p_0) e^{-ip_0\cdot x}, \qquad \psi_f^{(0)}(x) \equiv \sqrt{\frac{m_q}{p_n^0}} u_{s_n}^{a_n}(p_n) e^{-ip_n\cdot x}, \tag{5.2}$$

where we used spinor conventions given at Appendix A and m_q is the quark mass. The gluon final asymptotic state is written $A_{\mu,\alpha_n}^{(0)}(x)$, with momentum, polarization and color $(k_n, \lambda_n, \alpha_n)$

$$A_{\mu,\alpha_n}^{(0)}(x) = \mathcal{N}_k \epsilon_{\mu,\alpha_n}^{\lambda_n}(k_n) e^{-ik_n\cdot x}, \tag{5.3}$$

where the normalization, given in Gaussian units, reads $\mathcal{N}_k = \sqrt{4\pi/2\omega}$, ω is the gluon energy and the polarization vector reads $\epsilon^{\lambda_n}(k_n)$. The high energy states at any coordinate x under the effect of the N sources of the medium can be written as a superposition of diffracted states of the form

$$\psi_i^{(N)}(x) = \psi_i^{(0)}(x) + \int \frac{d^3\boldsymbol{p}_k}{(2\pi)^3} \sqrt{\frac{m_q}{p_k^0}} u_{s_k}^{a_k}(p_k) e^{-ip_k\cdot x} \left(M_q^{(N)}(p_k, p_0; z_k, z_1)\right)_{a_0 s_0}^{a_k s_k}, \quad (5.4)$$

sum over repeated indices assumed. The matrix elements of $M_q^{(N)}(p_k, p_0; z_k, z_1)$ are the beyond eikonal elastic amplitude (4.91) of finding the quark in the state (p_k, s_k, a_k) after traversing the colored matter in the interval (z_1, z_k). These amplitudes carry local and ordered longitudinal phases, responsible of regulating the coherence in the squared amplitude leading to the LPM effect. Similarly for the final quark we write

$$\psi_f^{(N)}(x) = \psi_f^{(0)}(x) + \int \frac{d^3\boldsymbol{p}_k}{(2\pi)^3} \sqrt{\frac{m_q}{p_k^0}} u_{s_k}^{a_k}(p_k) e^{-ip_k\cdot x} \left(M_q^{(N)}(p_k, p_n; z_k, z_n)\right)_{a_n s_n}^{a_k s_k}. \quad (5.5)$$

On contrary to QED case (3.10), in QCD the gluon is allowed to reinteract with the medium once it is emitted, then we write also for its state

$$A_{\mu,\alpha}^{(N)}(x) = A_\mu^{(0)}(x)\delta_\alpha^{\alpha_n} + \int \frac{d^3\boldsymbol{k}_k}{(2\pi)^3} \mathcal{N}_k \epsilon_\mu^{\lambda_k}(k) e^{-ik\cdot x} \left(M_g^{(N)}(k_k, k_n; z_k, z_n)\right)_{a_f \lambda_f}^{a_k \lambda_k}, \quad (5.6)$$

where the matrix elements of $M_g^{(N)}(k_k, k_n; z_k, z_n)$ are the beyond eikonal elastic amplitude (4.91) of finding the gluon in the state $(k_n, \lambda_n, \alpha_n)$ after traversing the colored matter in the interval (z_k, z_n). The emitted gluon will be considered massive, with a plasma frequency given by $\omega_p = m_g$ to account for medium effects in the dispersion relation. Then the gluon 4-momentum is denoted by $k = \omega(1, \beta_k \boldsymbol{k})$, where the gluon velocity is given by

$$\beta_k = \sqrt{1 - m_g^2/\omega^2}. \quad (5.7)$$

The longitudinal polarization related to $m_g \neq 0$ will be, however, neglected. The effective gluon mass can be considered of the order of μ_d, the effective screening mass of the external field of the medium [132, 142–145]. As with the QED analogous (3.10), the insertion of (5.4), (5.5) and (5.6) in (5.1) produces a term containing the three unscattered states, representing the emission amplitude in the vacuum

$$\begin{aligned} \mathcal{M}_{em}^{(0)} &= -ig_s \mathcal{N}_k \int d^4x \psi_f^{(0)}(x) A_{\mu,\alpha}^{(0)}(x) t_\alpha \gamma^\mu \psi_i^{(0)}(x) \quad (5.8) \\ &= -ig_s \mathcal{N}_k (2\pi)^4 \delta^4(p_n + k_n - p_i) \left(t_{\alpha_n}\right)_{a_0}^{a_n} \sqrt{\frac{m_q}{p_n^0}} \bar{u}_{s_n}(p_n) \gamma^\mu \epsilon_\mu^{\lambda_n}(k) u_{s_0}(p_0) \sqrt{\frac{m_q}{p_0^0}}, \end{aligned}$$

which vanishes due to energy momentum conservation. This diagram has to be subtracted from the full amplitude in order to avoid spurious divergences in the evaluation of the non-perturbative scattering amplitudes. Then we define the amplitude of emission *while* interacting with the medium. By using $S = 1 + M$ and inserting (5.4), (5.5) and (5.6) in (5.1) and subtracting (5.8) we obtain

$$\begin{aligned}\mathcal{M}_{em}^{(N)}-\mathcal{M}_{em}^{(0)} = -ig_s\mathcal{N}_k \int \frac{d^3\boldsymbol{p}_k}{(2\pi)^3}\int \frac{d^3\boldsymbol{k}_k}{(2\pi)^3}\int_{-\infty}^{+\infty} dz \exp\Big(-iq(p,p+k)z_k\Big)\\ \times f^{\lambda_k}_{s'_k s_k}(p_k, p_k+k_k)\Big(S_g^{(N)}(k_n,k;z_n,z_k)\Big)^{\alpha_n}_{\alpha_k}\\ \times \Big(S_q^{(N)}(p_n,p;z_n,z_k)\Big)^{a_n s_n}_{a'_k s'_k}\Big(t_{\alpha_k}\Big)^{a'_k}_{a_k}\Big(S_q^{(N)}(p_k+k_k,p_0;z_k,z_1)\Big)^{a_k s_k}_{a_0 s_0} - \mathcal{M}_{em}^{(0)},\end{aligned} \tag{5.9}$$

where the quark and gluon elastic amplitudes S_q and S_g are the respective beyond eikonal and ordered evaluations outlined at (4.94). The shorthand notation $f^{\lambda_k}_{s'_k s_k}(p_k, p_k+k_k)$ refers to the emission vertex, which factorizes and reads

$$f^{\lambda_k}_{s'_k s_k}(p_k, p_k+k_k) \equiv \sqrt{\frac{m_q}{p_0^0-\omega}}\bar{u}_{s'_k}(p_k)\epsilon^{\lambda_k}_\mu(k_k)\gamma^\mu u_{s_k}(p_k+k_k)\sqrt{\frac{m_q}{p_0^0}}. \tag{5.10}$$

If $p^z_{p_0^0}(\boldsymbol{p}^t)$ denotes the longitudinal momentum of a parton state with energy p_0^0 and transverse momentum $\boldsymbol{p}^t$, the local longitudinal phase arising at the emission point is given by

$$\begin{aligned} q(p_k, p_k+k_k) &\equiv p^z_{p_0^0-\omega}(\boldsymbol{p}^t_k) + k^z_\omega(\boldsymbol{k}^t_k) - p^z_{p_0^0}(\boldsymbol{p}^t_k+\boldsymbol{k}^t_k),\\ q(p_k-k_k, p_k) &\equiv p^z_{p_0^0-\omega}(\boldsymbol{p}^t_k-\boldsymbol{k}^t_k) + k^z_\omega(\boldsymbol{k}^t_k) - p^z_{p_0^0}(\boldsymbol{p}^t_k),\end{aligned} \tag{5.11}$$

depending on which of the intermediate quark momenta $\boldsymbol{p}_k$, either at (5.4) or (5.5), is chosen to perform the remaining integral in (5.9). The relation of (5.11) with the pole of an off-shell quark propagator for the leg after or before the emission can be seen by taking the high energy limit, $\beta_k \cong 1$ and $\beta_p \cong 1$, respectively, where β_p is the velocity of the quark in the QCD medium. Indeed we find

$$q_z(p, p+k) \simeq -\frac{\omega}{2p_0^0(p_0^0-\omega)}\left(m_q^2 + \left(\boldsymbol{p}_t - \frac{p_0^0-\omega}{\omega}\boldsymbol{k}_t\right)^2\right) - \frac{m_g^2}{2\omega} \simeq -\frac{k_\mu p^\mu}{p_0^0}, \tag{5.12}$$

if we carry the integration with $\boldsymbol{p}$, the momentum of the leg after the emission, of modulus $\beta_p p^0 = \beta_p(p_0^0-\omega)$, or

$$q_z(p-k, p) \simeq -\frac{\omega}{2p_0^0(p_0^0-\omega)}\left(m_q^2 + \left(\boldsymbol{p}_t - \frac{p_0^0}{\omega}\boldsymbol{k}_t\right)^2\right) - \frac{m_g^2}{2\omega} \simeq -\frac{k_\mu p^\mu}{p_0^0-\omega}, \tag{5.13}$$

if $\boldsymbol{p}$ corresponds to the quark momentum at the leg before the emission, of modulus $\beta_p p^0 = \beta_p p_0^0$. It is interesting to compare (5.12) and (5.13) with the QED analogous (3.35) and (3.36), and observe that in the soft $\omega \ll p_0^0$ limit the resonances (5.12) and (5.13) are dominated by the gluon momentum change and mass. The interpretation of the emission amplitude (5.9) is straightforward from right to left. The quark can be found at a depth z_k with energy p_0^0 and a given amplitude of being with deflected state (p_k, s_k, a_k). At that point it emits a gluon of color α_k, loosing momentum k_k, changing spin from s_k to s'_k and color from a_k to a'_k. The amplitude of this emission vertex (5.10) factorizes from the elastic propagation as

$$\left\{ \sqrt{\frac{m_q}{p_0^0 - \omega}}\, \bar{u}_{s'_k}(p_k) \epsilon_\mu^{\lambda_k}(k_k) \gamma^\mu u_{s_k} k(p_k + k_k) \sqrt{\frac{m_q}{p_0^0}} \right\} \left(t_{\alpha_k}\right)_{a_k}^{a'_k}. \tag{5.14}$$

From there onwards the quark and the gluon continue propagating till the end of the medium z_n with a given amplitude of ending at the states (p_n, s_n, a_n) and $(k_n, \lambda_n, \alpha_n)$, respectively. Finally we sum (integrate) over the different points in which the gluon can be emitted, which adequately inserts the corresponding off-shell propagators (5.12) and (5.13). As an illustrative example we consider the single case, i.e. just $N = n(z_1) = n_1$ centers distributed at equal coordinate $z_1 = 0$. Then the elastic amplitudes (4.94) reduce to the single layer case (4.92) and we obtain from (5.9)

$$\begin{aligned} \mathcal{M}_{em}^{(n_1)} = -g_s \mathcal{N} \Bigg\{ \int \frac{d^3\boldsymbol{k}_0}{(2\pi)^3} \frac{p_0^0 - \omega}{k_\mu^0 p_0^\mu} \left(S_g^{(n_1)}(k_1, k_0)\right)_{\alpha_0}^{\alpha_1} \left(S_q^{n_1}(p_1, p_0 - k_0)\right)_{a_k s_k}^{a_1 s_1} \left(t_{\alpha_0}\right)_{a_0}^{a_k} \\ \times f_{s_k s_0}^{\lambda_0}(p_0 - k_0, p_0) - \frac{p_0^0}{k_\mu^1 p_1^\mu} f_{s_1 s_k}^{\lambda_1}(p_1, p_1 + k_1) \left(t_{\alpha_1}\right)_{a_k}^{a_1} \left(S_q^{(n_1)}(p_1 + k_1, p_0)\right)_{a_0 s_0}^{a_k s_k} \Bigg\}. \end{aligned} \tag{5.15}$$

We observe that in order to obtain $\mathcal{M}_{em}^{(0)}$ we simply have to evaluate the above expression in $n_1 = 0$. The leading order of (4.92) in g_s^2 produces for the quark

$$\begin{aligned} \left(S_q^{(n_1)}(p_1, p_0)\right)_{a_0 s_0}^{a_1 s_1} &= (2\pi)^3 \delta^3(\boldsymbol{q}_q) \delta_{s_0}^{s_1} \delta_{a_0}^{a_1} + 2\pi \delta(q_q^0) \\ &\times \sqrt{\frac{m_q}{p_1^0}}\, \bar{u}_{s_1}(p_1) \gamma_0 u_{s_0}(p_0) \sqrt{\frac{m_q}{p_0^0}} \left(-i g_s^2 \left(t_\alpha\right)_{a_0}^{a_1} t_\alpha A_0^{(1)}\left(\boldsymbol{q}_q^t\right) \sum_{i=1}^{n_1} e^{-i \boldsymbol{q} \cdot \boldsymbol{r}_i} + \cdots \right), \end{aligned} \tag{5.16}$$

where the momentum change is $\boldsymbol{q}_q = \boldsymbol{p}_1 - \boldsymbol{p}_0$ and $A_0^{(1)}(\boldsymbol{q}) = 4\pi/(\boldsymbol{q}^2 + \mu_d^2)$ is the Fourier transform of the external field (4.16). Similarly for the gluon we find, by changing to the adjoint representation, *c.f.* Section 4.2,

$$\left(S_g^{(n_1)}(k_1,k_0)\right)^{\alpha_1\lambda_1}_{\alpha_0\lambda_0} = (2\pi)^3\delta^3(\boldsymbol{q}_g)\delta^{\lambda_1}_{\lambda_0}\delta^{\alpha_1}_{\alpha_0} + 2\pi\delta(q_g^0) \times \epsilon^{\lambda_1}_{\mu}(k_1)\epsilon^{\mu}_{\lambda_0}(k_0)\left(-ig_s^2\left(T_\alpha\right)^{\alpha_1}_{\alpha_0} t_\alpha A_0^{(1)}\left(\boldsymbol{q}_g^t\right)\sum_{i=1}^{n_1} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i} + \cdots\right), \tag{5.17}$$

where the momentum change is denoted $\boldsymbol{q}_g = \boldsymbol{k}_1 - \boldsymbol{k}_0$. The subtraction of $\mathcal{M}^{(0)}_{em}$ removes the trivial amplitude, i.e. no collision at any point. In the pure high energy limit the spinor and polarization content of the elastic amplitudes reduce to conservation deltas. In that case we find after inserting (5.16) and (5.17) in (5.15) and using $\left(T_\alpha\right)^{\alpha_1}_{\alpha_0} t_{\alpha_0} = [t_{\alpha_1}, t_\alpha]$

$$\mathcal{M}^{(n_1)}_{em} - \mathcal{M}^{(0)}_{em} = -ig_s^3\mathcal{N}_k A_0^{(1)}(\boldsymbol{q}_g)\sum_{i=1}^{n_1} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i}\left(\frac{p_0^0}{k^1_\mu p_1^\mu} f^{\lambda_1}_{s_1 s_0}(p_1, p_1+k_1)\left(t_{\alpha_1}t_\alpha\right)^{a_1}_{a_0} - \frac{p_0^0-\omega}{k^1_\mu p_0^\mu} f^{\lambda_1}_{s_1 s_0}(p_0-k_1,p_0)\left(t_\alpha t_{\alpha_1}\right)^{a_1}_{a_0} - \frac{p_0^0-\omega}{k^0_\mu p_0^\mu} f^{\lambda_0}_{s_1 s_0}(p_0-k_0,p_0)\left[t_{\alpha_1},t_\alpha\right]^{a_1}_{a_0}\right)t_\alpha, \tag{5.18}$$

with $\boldsymbol{q} \equiv \boldsymbol{p}_1 + \boldsymbol{k}_1 - \boldsymbol{p}_0$ and then $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_0$. Equation (5.18) will become more familiar assuming that $\omega \ll p_0^0$. In that case we can omit the transverse components $\boldsymbol{p}_1^t$ and $\boldsymbol{p}_0^t$ in (5.12) and (5.13). By expanding the denominators (5.11) and the vertex in transverse and longitudinal components we obtain, in the $\omega \to 0$ limit,

$$\mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} = ig_s^3\mathcal{N}_k\left(\frac{2\boldsymbol{\epsilon}^t_\lambda\cdot\boldsymbol{k}_1^t}{m_g^2+(\boldsymbol{k}_1^t)^2} - \frac{2\boldsymbol{\epsilon}^t_\lambda\cdot\boldsymbol{k}_0^t}{m_g^2+(\boldsymbol{k}_0^t)^2}\right)A_0^{(1)}(\boldsymbol{q}_t)\left[t_\alpha, t_{\alpha_1}\right]^{a_1}_{a_0} t_\alpha\sum_{i=1}^{n_1} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i}, \tag{5.19}$$

whose square is the massive version of the Bertsch-Gunion result [79]. A diagrammatic representation of (5.18) is shown in Figure 5.1. For mediums of larger thickness the above formula does not hold. In that case we can discretize the medium into a set of n layers from z_1 to z_n verifying

$$N = \sum_{i=1}^{n} n(z_i) \equiv \sum_{i=1}^{n} n_i. \tag{5.20}$$

Then the elastic amplitudes for the quark and the gluon are given as convolutions in the form of (4.94). The integration in z of (5.9) produces three different zones so that we write

$$\mathcal{M}^{(N)}_{em} = g_s\mathcal{N}_k\left(\prod_{i=1}^{n-1}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3}\right)\left(\prod_{i=k}^{n-1}\frac{d^3\boldsymbol{k}_i}{(2\pi)^3}\right)\left(\mathcal{M}_f + \mathcal{M}_{int} + \mathcal{M}_i\right). \tag{5.21}$$

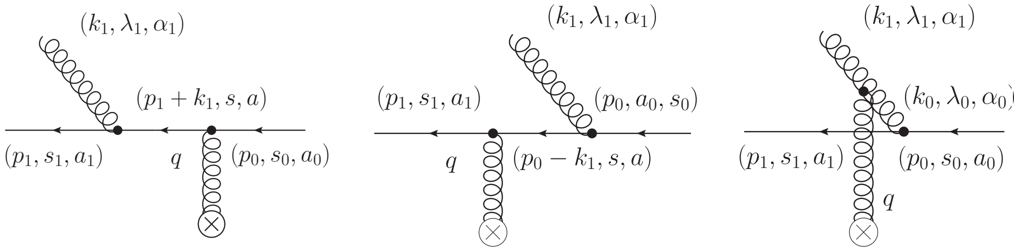


Figure 5.1: The Feynman diagrams appearing in the leading order expansion in the coupling constant g_s^2 of $\mathcal{M}_{em}^{(n)} - \mathcal{M}_{em}^{(0)}$ in the single collision regime (5.18).

The first term corresponds to the integration of (5.9) in the $(z_n, +\infty)$ region and contains the gluon emitted in the last leg. It reads

$$\mathcal{M}_f \equiv \left(t_{\alpha_n}\right)^{a_n}_{a'_n} \frac{f^{\lambda_n}_{s_n s'_n}(p_n, p_n + k_n)}{k^n_\mu p^\mu_n / p^0_0} \exp\left(+i\frac{k^n_\mu p^\mu_n}{p^0_0} z_n\right)\left(S^{(n_n)}_{s'_n s_{n-1}}(\delta p_n + k_n)\right)^{a'_n}_{a_{n-1}} \times\left(\prod_{i=1}^{n-1}\left(S^{(n_i)}_{s_i s_{i-1}}(\delta p_i)\right)^{a_i}_{a_{i-1}}\right), \tag{5.22}$$

where $\delta p_i \equiv p_i - p_{i-1}$ is the momentum change of the quark at the layer at z_i and the abbreviations $S^{n(z_i)}_{s_i s_{i-1}}(\delta p_i) \equiv S^{n(z_i)}_{s_i s_{i-1}}(p_i, p_{i-1})$ have been used. The integration of (5.9) in the inner region (z_0, z_n) contains all the gluons emitted from internal legs. Indeed we find

$$\mathcal{M}_{int} \equiv -\sum_{k=1}^{n-1}\left(\prod_{i=k+1}^{n}\left(S^{(n_i)}_{s_i s_{i-1}}(\delta p_i)\right)^{a_i}_{a_{i-1}}\left(S^{(n_i)}_{\lambda_i \lambda_{i-1}}(\delta k_i)\right)^{\alpha_i}_{\alpha_{i-1}}\right)\left(t_{\alpha_k}\right)^{a_k}_{a'_k}\frac{f^{\lambda_k}_{s_k s'_k}(p_k, p_k + k_k)}{k^k_\mu p^\mu_k / p^0_0}\times \left(\exp\left(i\frac{k^k_\mu p^\mu_k}{p^0_0} z_{k+1}\right) - \exp\left(i\frac{k^k_\mu p^\mu_k}{p^0_0} z_k\right)\right)\left(S^{(n_k)}_{s'_k s_{k-1}}(\delta p_k + k_k)\right)^{a'_k}_{a_{k-1}}\left(\prod_{i=1}^{k-1}\left(S^{(n_i)}_{s_i s_{i-1}}(\delta p_i)\right)^{a_i}_{a_{i-1}}\right). \tag{5.23}$$

Finally the term containing the gluon emitted from the first leg is given by the integration of (5.9) in the region $(-\infty, z_0)$. It produces

$$\mathcal{M}_i \equiv -\left(\prod_{i=2}^{n}\left(S^{(n_i)}_{s_i s_{i-1}}(\delta p_i)\right)^{a_i}_{a_{i-1}}\left(S^{(n_i)}_{\lambda_i \lambda_{i-1}}(\delta k_i)\right)^{\alpha_i}_{\alpha_{i-1}}\right) \tag{5.24}$$
$$\times\left(S^{(n_1)}_{s_1 s'_0}(\delta p_1 + k_1)\right)^{a_1}_{a'_0}\left(S^{(n_1)}_{\lambda_1 \lambda_0}(\delta k_1)\right)^{\alpha_1}_{\alpha_0}\left(t_{\alpha_0}\right)^{a'_0}_{a_0}\frac{f^{\lambda_0}_{s'_0 s_0}(p_0 - k_0, p_0)}{k^0_\mu p^\mu_0/(p^0_0 - \omega)}\exp\left(+i\frac{k^0_\mu p^\mu_0}{p^0_0 - \omega} z_1\right).$$

Similarly to the QED case, in the soft gluon approximation $\omega \ll p^0_0$ the first term (5.22) and the last term (5.24) are the only ones surviving when $\omega \to 0$,

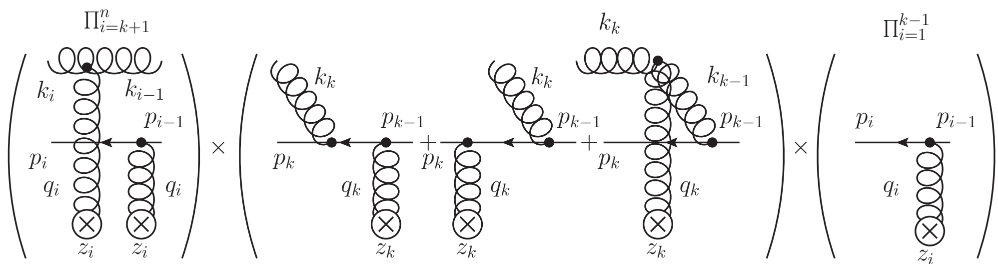


Figure 5.2: The Feynman diagrams [138] appearing in an expansion in the coupling constant g_s^2 of $\mathcal{M}_{em}^{(n)} - \mathcal{M}_{em}^{(0)}$ in the multiple collision regime (5.29).

since the phases, which are of the form (5.12) or (5.13), vanish, and then the internal sum (5.23) cancels [5, 62, 139, 140]. Notice, however, that previously to this cancellation the phase has an enhancement $1/\omega$ due to the m_g and $\boldsymbol{k}_k^t$ terms, which means that incoherence may occur in the sum of the amplitudes in the soft regime, previous to the phase cancellation, in analogy with the dielectric and transition radiation effects of QED. In the regime of vanishing phases the coherent plateau is found, and the medium emits as a single entity of equivalent charge the amount of colored matter in (z_1, z_n). The three terms (5.22), (5.23) and (5.24) can be reunited into a single expression. For that purpose we simplify now the notation by extracting the energy, spin and polarization conservation deltas from the elastic amplitudes as follows

$$\begin{aligned}
\left(S^{(n_i)}_{s_i s_{i-1}}(\delta p_i)\right)^{a_i}_{a_{i-1}} &= \delta^{s_i}_{s_{i-1}} 2\pi\beta_p \delta(\delta p_i^0) \left(S^{\,i}_{q}(\delta \boldsymbol{p}_i)\right)^{a_i}_{a_{i-1}} \\
\left(S^{(n_i)}_{\lambda_i \lambda_{i-1}}(\delta k_i)\right)^{\alpha_i}_{\alpha_{i-1}} &= \delta^{\lambda_i}_{\lambda_{i-1}} 2\pi\beta_k \delta(\delta k_i^0) \left(S^{\,i}_{g}(\delta \boldsymbol{k}_i)\right)^{\alpha_i}_{\alpha_{i-1}} .
\end{aligned} \tag{5.25}$$

We define also the f_k^+ and f_k^-, φ_k^+ and φ_k^- shorthands for the advanced and retarded propagator together with the spinorial vertices, and the local phases, respectively, at the emission point z_k

$$f_k^+ \equiv \frac{f^{\lambda_n}_{s_n s_0}(p_k, p_k + k_k)}{k^k_\mu p^\mu_k / p^0_0}, \; f_k^- \equiv \frac{f^{\lambda_n}_{s_n s_0}(p_k - k_k, p_k)}{k^k_\mu p^\mu_k / (p^0_0 - \omega)}, \varphi_k^+ = \frac{k^k_\mu p^\mu_k}{p^0_0} z_k, \varphi_k^- = \frac{k^k_\mu p^\mu_k}{p^0_0 - \omega} z_k. \tag{5.26}$$

With this notation the sum of all the gluons intervening in the amplitude can be written as

$$\mathcal{M}^{(N)}_{em} = g_s N_k 2\pi\beta_p \delta(p_n^0 + \omega - p_0^0) \left(\prod_{i=1}^{n-1} \int \frac{d^2 \boldsymbol{p}_i^t}{(2\pi)^2} \right) \left(\prod_{i=k}^{n-1} \int \frac{d^2 \boldsymbol{k}_i^t}{(2\pi)^2} \right) \times \sum_{k=1}^{n} \mathcal{M}_k, \tag{5.27}$$

where a single Bethe-Heitler element of gluon bremsstrahlung is given by

$$\mathcal{M}_k = \prod_{i=k+1}^{n}\left(\left(S_q^{\,i}(\delta\boldsymbol{p}_i)\right)^{a_i}_{a_{i-1}}\left(S_g^{\,i}(\delta\boldsymbol{k}_i)\right)^{\alpha_i}_{\alpha_{i-1}}\right)\left(f_k^{+}e^{i\varphi_k^{+}}\left(t_{\alpha_k}\right)^{a_k}_{a'_k}\left(S_q^{\,k}(\delta\boldsymbol{p}_k+\boldsymbol{k}_k)\right)^{a'_k}_{a_{k-1}}\right.$$
$$\left. - f_{k-1}^{-}e^{i\varphi_k^{-}}\left(S_g^{\,k}(\delta\boldsymbol{k}_k)\right)^{\alpha_k}_{\alpha_{k-1}}\left(S_q^{\,k}(\delta\boldsymbol{p}_k+\boldsymbol{k}_{k-1})\right)^{a_k}_{a'_{k-1}}\left(t_{\alpha_{k-1}}\right)^{a'_{k-1}}_{a_{k-1}}\right)\prod_{i=1}^{k-1}\left(S_q^{\,i}(\delta\boldsymbol{k}_k)\right)^{a_i}_{a_{i-1}}. \tag{5.28}$$

We observe that the vacuum term is given by the evaluation of the former expression in $(n_i) = 0$ for all i. Thus we finally write

$$\mathcal{M}_{em}^{(N)} - \mathcal{M}_{em}^{(0)} = g_s\mathcal{N}_k 2\pi\beta_p\delta(p_n^0+\omega-p_0^0)$$
$$\times\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\right)\left(\prod_{i=k}^{n-1}\int\frac{d^2\boldsymbol{k}_i^t}{(2\pi)^2}\right)\times\sum_{k=1}^{n}\left(\mathcal{M}_k-\mathcal{M}_k^{(0)}\right). \tag{5.29}$$

Amplitude (5.29) is just a sum of single Bethe-Heitler/Bertsch-Gunion elements in a QCD scenario. A diagrammatic representation is depicted in Figure 5.2. The internal phases in the scattering amplitudes are responsible of modulating the interference behavior leading to the LPM effect in the square of (5.29).

5.2 Intensity and energy loss

The amplitude (5.29) must be squared and averaged over medium configurations, and summed and averaged over final and initial states respectively. The medium will be chosen as a solid cylinder with transverse area πR^2 and length $l = z_n - z_1$. The transverse dimensions will be assumed much larger than the dimensions of a single scatterer, i.e. $R \gg \mu_d^{-1} = r_d$, where r_d is the Debye radius of the plasma, in such a way that the transverse boundary effects can be neglected. The unpolarized differential intensity of gluons in the energy interval ω and $\omega + d\omega$, in the solid angle Ω_k and $\Omega_k + d\Omega_k$, per unit of quark incoming flux, time and transverse area is given by

$$dI \equiv \frac{\omega^2 d\omega d\Omega_k}{(2\pi)^3}\frac{1}{\beta_p T\pi R^2}\frac{1}{2}\sum_{s_n s_0}\sum_{\lambda_n}\frac{1}{N_c}\sum_{a_n a_0}\sum_{\alpha_n}$$
$$\times\int\frac{d^3\boldsymbol{p}_n}{(2\pi)^3}\left\langle\left(\mathcal{M}_{em}^{(N)}-\mathcal{M}_{em}^{(0)}\right)^*\left(\mathcal{M}_{em}^{(N)}-\mathcal{M}_{em}^{(0)}\right)\right\rangle, \tag{5.30}$$

where the operation $\langle\star\rangle$ denotes an average over target configurations, both in color and spatial dimensions. The form of (5.29) as $M - 1$ suggests, as with the elastic case, an splitting into a incoherent and a coherent contribution. We then

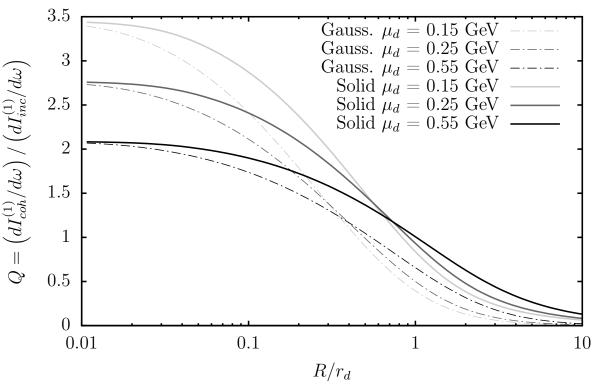


Figure 5.3: Quotient between the transverse coherent and transverse incoherent intensities for the single scattering regime out of the suppression zone $\omega \gg m_g$ as a function of the transverse size of the QCD medium R in units of the Debye radius of the plasma $r_d = \mu_d^{-1}$, for a medium of $n_0 T_f$ = 8 fm^{-3} corresponding to a transport parameter $\hat{q}$ = 0.98 GeV^2/fm, and a gluon mass of m_g = 0.15 GeV, for a Gaussian decaying density (dot-dashed lines) and an uniform density in a solid cylinder (solid lines) for different Debye screenings, as marked.

proceed as before by adding and subtracting the term $\left\langle \mathcal{M}^{(N)}_{em} \right\rangle^* \left\langle \mathcal{M}^{(N)}_{em} \right\rangle$ in which case we obtain

$$\frac{1}{2} \sum_{s_n s_0} \sum_{\lambda_n} \frac{1}{N_c} \sum_{a_n a_0} \sum_{\alpha_n} \left\langle \left(\mathcal{M}^{(N)}_{em} - \mathcal{M}^{(0)}_{em} \right)^* \left(\mathcal{M}^{(N)}_{em} - \mathcal{M}^{(0)}_{em} \right) \right\rangle = \Sigma^{(N)}_{em} + \Pi^{(N)}_{em}, \tag{5.31}$$

where the coherent part, which contains the diffractive and quantum contribution of the medium boundaries to the emission intensity, is given by the averaged emission amplitude squared,

$$\Pi^{(N)}_{em} \equiv \frac{1}{2} \sum_{s_n s_0} \sum_{\lambda_n} \frac{1}{N_c} \sum_{a_n a_0} \sum_{\alpha_n} \left\langle \mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} \right\rangle^* \left\langle \mathcal{M}^{(n)}_{em} - \mathcal{M}^{(0)}_{em} \right\rangle, \tag{5.32}$$

whereas the incoherent part, which contains a contribution which in the macroscopic limit $R \gg \mu_d^{-1}$ can be interpreted in probabilistic terms, is given by

$$\Sigma^{(N)}_{em} \equiv \frac{1}{2} \sum_{s_n s_0} \sum_{\lambda_n} \frac{1}{N_c} \sum_{a_n a_0} \sum_{\alpha_n} \left\langle \left(\mathcal{M}^{(N)}_{em} \right)^* \mathcal{M}^{(N)}_{em} \right\rangle - \left\langle \mathcal{M}^{(N)}_{em} \right\rangle^* \left\langle \mathcal{M}^{(N)}_{em} \right\rangle. \tag{5.33}$$

For simplicity we take then $R \to \infty$ and neglect the boundary contribution of the coherent part (5.32). In that limit the coherent part produces a set of pure forward scatterings so that each $\mathcal{M}_k$ vanishes and so it does the sum (5.29). This effect can be seen in Fig. 5.3, where the ratio between the transverse-coherent and the transverse-incoherent intensities is shown, for the single scattering regime, as a function of the transverse size of the medium. Then the relevant contribution

(5.33) can be rewritten, using (5.29), as

$$
\begin{aligned}
\Sigma_{em}^{(N)} &= g_s^2 N_k^2 \beta_p T 2\pi\beta_p \delta(p_n^0+\omega-p_0^0)\frac{1}{2}\sum_{s_n s_0}\sum_{\lambda_n}\frac{1}{N_c}\sum_{a_n a_0}\sum_{\alpha_n}\left(\prod_{i=k-1}^{n-1}\int\frac{d^2\boldsymbol{k}_i^t}{(2\pi)^2}\right)\\
&\times\left(\prod_{i=j-1}^{n-1}\int\frac{d^2\boldsymbol{v}_i^t}{(2\pi)^2}\right)\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\int\frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\sum_{j,k=1}^{n}\left(\left\langle\mathcal{M}_j^*\mathcal{M}_k\right\rangle-\left\langle\mathcal{M}_j^*\right\rangle\left\langle\mathcal{M}_k\right\rangle\right).
\end{aligned}
\tag{5.34}
$$

The quark and gluon momenta in the amplitude have been denoted by $\boldsymbol{p}_i$ and $\boldsymbol{k}_i$, respectively, and by $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ in the conjugated amplitude. Similarly the quark and gluon color index will be denoted by a_i and α_i in the amplitude and by b_i and β_i in the conjugated amplitude. With this notation we notice that $b_0 = a_0$ and $b_n = a_n$, $\boldsymbol{p}_0 = \boldsymbol{u}_0$ and $\boldsymbol{p}_n = \boldsymbol{u}_n$ for the observed initial and final states of the quark, respectively, and $\alpha_n = \beta_n$ and $\boldsymbol{k}_n = \boldsymbol{v}_n$ for the observed final state of the gluon. The averaged squared terms on the right hand side of (5.34) can be written as

$$
\begin{aligned}
\left\langle\mathcal{M}_j^*\mathcal{M}_k\right\rangle &= A_{j+1}^n\left(f_j^{+*}e^{-i\varphi_j^+}\left(t_{\beta_j}\right)_{b_j}^{b'_j}A_j^+ - A_j^+ f_{j-1}^{-*}e^{-i\varphi_j^-}\left(t_{\beta_{j-1}}\right)_{b'_{j-1}}^{b_{j-1}}\right)A_{j+1}^{k-1}\\
&\times\left(f_k^+ e^{i\varphi_k^+}\left(t_{\alpha_k}\right)_{a'_k}^{a_k}A_k^+ - A_k^- f_{k-1}^- e^{i\varphi_k^-}\left(t_{\alpha_{k-1}}\right)_{a_{k-1}}^{a'_{k-1}}\right)A_1^{k-1},
\end{aligned}
\tag{5.35}
$$

where the A_k^j are abbreviations denoting the averaged squared elastic amplitudes for the layers from z_k to z_j. The squared average terms in the right hand side of (5.34) are written similarly as

$$
\begin{aligned}
\left\langle\mathcal{M}_j^*\right\rangle\left\langle\mathcal{M}_k\right\rangle &= B_{j+1}^n\left(f_j^{+*}e^{-i\varphi_j^+}\left(t_{\beta_j}\right)_{b_j}^{b'_j}B_j^+ - B_j^+ f_{j-1}^{-*}e^{-i\varphi_j^-}\left(t_{\beta_{j-1}}\right)_{b'_{j-1}}^{b_{j-1}}\right)B_{j+1}^{k-1}\\
&\times\left(f_k^+ e^{i\varphi_k^+}\left(t_{\alpha_k}\right)_{a'_k}^{a_k}B_k^+ - B_k^- f_{k-1}^- e^{i\varphi_k^-}\left(t_{\alpha_{k-1}}\right)_{a_{k-1}}^{a'_{k-1}}\right)B_1^{k-1},
\end{aligned}
\tag{5.36}
$$

where B_k^j denote the averaged amplitudes squared for the layers from z_k to z_j. Both at (5.35) and (5.36) we have used the fact that the averages factorize layer to layer due to the z-ordered form of the beyond eikonal elastic amplitudes (4.94). The first set of scatterings corresponds to the passage of the quark state (q) and its conjugate state ($\bar{q}$) from the layer at z_1 to the on at z_k,

$$
A_1^{k-1} = \prod_{i=1}^{k-1}\left(\frac{1}{N_c^{n_i}}\operatorname{Tr}\left\langle\left(S_q^{i\dagger}(\delta\boldsymbol{u}_i)\right)_{b_i}^{b_{i-1}}\left(S_q^{i}(\delta\boldsymbol{p}_i)\right)_{a_{i-1}}^{a_i}\right\rangle\right) = \prod_{i=1}^{k-1}A_i. \tag{5.37}
$$

We refer to the Section 4.3 for the details of the averaging process and leave implicit the color indices and the ordering. For a single layer average we find

$$A_i = (2\pi)^2 \delta^2(\delta \boldsymbol{p}_i^t - \delta \boldsymbol{u}_i^t) \exp\Big(-i\big(\delta p_i^z - \delta u_i^z\big) z_i \Big) \times \phi_{q\bar{q}}^{(n_i)}(\delta \boldsymbol{p}_i^t), \tag{5.38}$$

where δp_i and δu_i are the momentum change of the q and $\bar{q}$ states and the elastic distribution $\phi_{q\bar{q}}^{(n_i)}(\delta \boldsymbol{p}_i^t)$ satisfies

$$\phi_{q\bar{q}}^{(n_i)}(\delta \boldsymbol{p}_i^t) = \int d^2\boldsymbol{x}_i^{q\bar{q}} \exp\Big(-i\delta \boldsymbol{p}_i^t \cdot \boldsymbol{x}_i^{q\bar{q}} + n_0(z_i)\delta z_i \big(\sigma_{q\bar{q}}^{(1)}(\boldsymbol{x}_i^{q\bar{q}}) - \sigma_{q\bar{q}}^{(1)}(\boldsymbol{0}) \big) \Big). \tag{5.39}$$

Here $n_0(z_i)$ and δz_i are the density and thickness of the target quark layer, respectively, and $\sigma_{q\bar{q}}^{(1)}(\boldsymbol{0})$ the Fourier transform of the single elastic amplitude of a quark given at (4.74). For the interaction (4.16) we find at leading order

$$\sigma_{q\bar{q}}^{(1)}(\boldsymbol{x}) \equiv \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{N_c} \mathrm{Tr}\left(F_q^{(1)\dagger}(\boldsymbol{q}) F_q^{(1)}(\boldsymbol{q}) \right) = \frac{4\pi g_s^4}{\beta_p^2 \mu_d^2} \frac{N_c^2 - 1}{4N_c^2} \mu_d |\boldsymbol{x}| K_1(\mu_d |\boldsymbol{x}|), \tag{5.40}$$

where $K_1(x)$ is the modified Bessel function. We observe that (5.40) preserves the color of the quark states q and $\bar{q}$ hence any multiple interaction like (5.39) and any convolution of (5.39) does too. The macroscopic limit $R \to \infty$ and the condition $\boldsymbol{u}_0^t = \boldsymbol{p}_0^t$ leads to an observed quark trajectory satisfying $\boldsymbol{u}_i^t = \boldsymbol{p}_i^t$ for $i = 1, \ldots, k-1$. This fact together with energy conservation $u_i^0 = p_i^0 = p_0^0$ leads to a vanishing longitudinal phase in all the interval

$$\delta p_i^z - \delta u_i^z = \Big(p_{p_0^0}^z(\boldsymbol{p}_i^t) - p_{p_0^0}^z(\boldsymbol{p}_{i-1}^t) \Big) - \Big(p_{p_0^0}^z(\boldsymbol{u}_i^t) - p_{p_0^0}^z(\boldsymbol{u}_{i-1}^t) \Big) = 0. \tag{5.41}$$

Correspondingly for the entire passage from z_1 to z_{k-1} we find, using reiteratively (5.38),

$$A_1^{k-1} = \prod_{i=1}^{k-1} A_i = \prod_{i=1}^{k-1} \Big((2\pi)^2 \delta^2(\boldsymbol{p}_{i-1}^t - \boldsymbol{u}_{i-1}^t) \times \phi_{q\bar{q}}^{(n_i)}(\delta \boldsymbol{p}_i^t) \Big). \tag{5.42}$$

The next average corresponds to the layer at z_k where the gluon is emitted in the amplitude and then two possibilities arise. The first one corresponds to an advanced collision previous to the emission, and it is given by

$$A_k^+ \equiv \frac{1}{N_c^{n_k}} \mathrm{Tr} \left\langle \Big(S_q^{k\dagger}(\delta \boldsymbol{u}_k) \Big)_{b_k}^{b_{k-1}} \Big(S_q^k(\delta \boldsymbol{p}_k + \boldsymbol{k}_k) \Big)_{a_{k-1}}^{a'_k} \right\rangle. \tag{5.43}$$

Both amplitudes are yet on-shell with the initial quark $p^0_{k-1} = u^0_{k-1} = p^0_0$ and from (5.42) we inherit $\boldsymbol{u}^t_{k-1} = \boldsymbol{p}^t_{k-1}$, so we find, using (5.38), that $\boldsymbol{p}^t_k + \boldsymbol{k}^t_k = \boldsymbol{u}^t_k$ so that the longitudinal phase at (5.38) vanishes again

$$\delta p^z_k - \delta u^z_k = \Big(p^z_{p^0_0}(\boldsymbol{p}^t_k + \boldsymbol{k}^t_k) - p^z_{p^0_0}(\boldsymbol{p}^t_{k-1})\Big) - \Big(p^z_{p^0_0}(\boldsymbol{u}^t_k) - p^z_{p^0_0}(\boldsymbol{u}^t_{k-1})\Big) = 0. \tag{5.44}$$

Then the averaged squared amplitude of the advanced collision produces

$$A^+_k = (2\pi)^2\delta^2(\boldsymbol{p}^t_k + \boldsymbol{k}^t_k - \boldsymbol{u}^t_k) \times \phi^{(n_i)}_{q\bar{q}}(\delta\boldsymbol{p}^t_k + \boldsymbol{k}^t_k). \tag{5.45}$$

The other possibility at the emission layer at z_k is that the collision occurs just after the emission. Since the gluon can reinteract with the target quarks for this contribution we get the average of three elastic amplitudes, of the form

$$A^-_k \equiv \frac{1}{N_c^{n_k}}\,\mathrm{Tr}\left\langle\left(S^{k\dagger}_q(\delta\boldsymbol{u}_k)\right)^{b_{k-1}}_{b_k}\left(S^k_q(\delta\boldsymbol{p}_k + \boldsymbol{k}_{k-1})\right)^{a_k}_{a'_{k-1}}\left(S^k_g(\delta\boldsymbol{k}_k)\right)^{\alpha_k}_{\alpha_{k-1}}\right\rangle. \tag{5.46}$$

From (5.42) we obtain $\boldsymbol{u}^t_{k-1} = \boldsymbol{p}^t_{k-1}$ but now one amplitude is on-shell with the initial quark and the other with the final quark, $p^0_{k-1} + \omega = u_{k-1} = p^0_0$. From the averaging process it can be shown that $\boldsymbol{u}^t_k = \boldsymbol{p}^t_k + \boldsymbol{k}^t_k$ and then the longitudinal phase is given by

$$\begin{aligned}\delta p^z_k + \delta k^z_k - \delta u^z_k &= \Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_k) + p_\omega(\boldsymbol{k}^t_k) - p^z_{p^0_0}(\boldsymbol{p}^t_k + \boldsymbol{k}^t_k)\Big) \\ &-\Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_{k-1} - \boldsymbol{k}^t_{k-1}) + p^z_\omega(\boldsymbol{k}^t_{k-1}) - p^z_{p^0_0}(\boldsymbol{p}^t_{k-1})\Big) = -\frac{k^k_\mu p^\mu_k}{p^0_0} + \frac{k^{k-1}_\mu p^\mu_{k-1}}{p^0_0 - \omega},\end{aligned} \tag{5.47}$$

where we have used (5.12) and (5.13). The average procedure follows the same steps, leading to

$$A^-_k = (2\pi)^2\delta^2(\boldsymbol{p}^t_k + \boldsymbol{k}^t_k - \boldsymbol{u}^t_k)\exp\left(i\frac{k^k_\mu p^\mu_k}{p^0_0}z_k - i\frac{k^{k-1}_\mu p^\mu_{k-1}}{p^0_0 - \omega}z_k\right)\phi^{(n_k)}_{q\bar{q}g}(\delta\boldsymbol{p}^t_k + \boldsymbol{k}^t_{k-1}, \delta\boldsymbol{k}^t_k), \tag{5.48}$$

where the elastic distribution for the three states $\phi^{(n_k)}_{q\bar{q}g}(\delta\boldsymbol{p}^t_k, \delta\boldsymbol{k}^t_k)$ can be shown to satisfy

$$\phi^{(n_k)}_{q\bar{q}g}(\delta\boldsymbol{p}^t_k, \delta\boldsymbol{k}^t_k) \equiv \int d^2\boldsymbol{x}^{q\bar{q}}_k \int d^2\boldsymbol{x}^{g\bar{q}}_k \exp\Big(-i\delta\boldsymbol{p}^t_k\cdot\boldsymbol{x}^{q\bar{q}}_k - i\delta\boldsymbol{k}^t_k\cdot\boldsymbol{x}^{g\bar{q}}_k\Big) \tag{5.49}$$

$$\times\exp\bigg(n_0(z_i)\delta z_i\Big(\varphi^{(1)}_g(\boldsymbol{0}) + \Big(\sigma_{q\bar{q}}\big(\boldsymbol{x}^{q\bar{q}}_k\big) - \sigma_{q\bar{q}}(\boldsymbol{0})\Big) + \sigma_{g\bar{q}}\big(\boldsymbol{x}^{g\bar{q}}_k\big) + \sigma_{gq}\big(\boldsymbol{x}^{g\bar{q}}_k - \boldsymbol{x}^{q\bar{q}}_k\big)\Big)\bigg).$$

Here the Fourier transform of the single squared amplitude of a quark $\sigma^{(1)}_{q\bar{q}}(\boldsymbol{x})$ is given by (5.40) whereas the Fourier transform of the crossed amplitudes are given by

$$\sigma^{(1)}_{gq}(\boldsymbol{x}) \equiv \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{N_c} \operatorname{Tr}\left(F^{(1)}_g(\boldsymbol{q}) F^{(1)}_q(\boldsymbol{q})\right)$$
$$= \frac{4\pi g_s^4}{\beta_p \beta_k \mu_d^2} \frac{1}{2N_c} T_\alpha t_\alpha \mu_d |\boldsymbol{x}| K_1(\mu_d|\boldsymbol{x}|) + \cdots, \tag{5.50}$$

which represents a collision affecting the (q) and (g) states with a single quark, leaving $(\bar{q})$ unaltered, and similarly a single collision affecting the (g) and $(\bar{q})$ states leaving the state (q) unaltered is given by

$$\sigma^{(1)}_{g\bar{q}}(\boldsymbol{x}) \equiv \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{N_c} \operatorname{Tr}\left(F^{(1)}_g(\boldsymbol{q}) F^{(1)\dagger}_q(\boldsymbol{q})\right)$$
$$= -\frac{4\pi g_s^4}{\beta_p \beta_k \mu_d^2} \frac{1}{2N_c} T_\alpha t_\alpha \mu_d |\boldsymbol{x}| K_1(\mu_d|\boldsymbol{x}|) + \cdots. \tag{5.51}$$

We observe that while (5.40) preserves the color of the traveling (q) and $(\bar{q})$ states, both (5.50) and (5.51) allow arbitrary and unobserved color rotations of the intermediate states. The unpaired single elastic cross section appearing in (5.49) has been denoted by

$$\varphi^{(1)}_g(\boldsymbol{0}) \equiv \frac{1}{N_c} \operatorname{Tr}\left(F^{(1)}_g(\boldsymbol{0})\right) = -\frac{g_s^4}{2\beta_k^2} T_f \frac{4\pi}{\mu_d^2} + \cdots, \tag{5.52}$$

which satisfies $\varphi^{(1)}_g(\boldsymbol{0}) = -\sigma^{(1)}_{g\bar{g}}(\boldsymbol{0})/2$ at leading order, where $\sigma^{(1)}_{g\bar{g}}(\boldsymbol{0})$ is the single elastic cross section of a gluon. The next passage is given by the intermediate scatterings of (q), $(\bar{q})$ and (g) states through the layers from z_{k+1} to z_{j-1}. It is given by

$$A^{j-1}_{k+1} \equiv \prod_{i=k+1}^{j-1} \left(\frac{1}{N_c^{n_i}} \operatorname{Tr} \left\langle \left(S^{i\dagger}_q(\delta\boldsymbol{u}_i)\right)^{b_{i-1}}_{b_i} \left(S^{i}_q(\delta\boldsymbol{p}_i)\right)^{a_i}_{a_{i-1}} \left(S^{i}_g(\delta\boldsymbol{k}_i)\right)^{\alpha_i}_{\alpha_{i-1}} \right\rangle \right) = \prod_{i=k+1}^{j-1} A_i. \tag{5.53}$$

Either from (5.45) or (5.48) we inherit the condition $\boldsymbol{p}^t_k + \boldsymbol{k}^t_k = \boldsymbol{u}^t_k$ and the averaging produces then $\boldsymbol{p}^t_i + \boldsymbol{k}^t_i = \boldsymbol{u}^t_i$ for $i = k+1, \ldots, j-1$. In this passage the quark state (q) is on-shell with the final quark but $(\bar{q})$ is on-shell yet with the initial quark, thus we find a longitudinal phase of the form

$$\delta p^z_i + \delta k^z_i - \delta u^z_i = \left(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_i) + p_\omega(\boldsymbol{k}^t_i) - p^z_{p^0_0}(\boldsymbol{p}^t_i + \boldsymbol{k}^t_i)\right) \tag{5.54}$$
$$- \left(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_{i-1}) + p^z_\omega(\boldsymbol{k}^t_{i-1}) - p^z_{p^0_0}(\boldsymbol{p}^t_{i-1} + \boldsymbol{k}^t_{i-1})\right) = -\frac{k^i_\mu p^\mu_i}{p^0_0} + \frac{k^{i-1}_\mu p^\mu_{i-1}}{p^0_0}.$$

Making an analogy with the QED classical phase (3.16) obtained in the $R \to \infty$ limit, this suggests a reorganization of the total contribution of the interval as follows

$$-i\sum_{i=k+1}^{j-1}\left(\delta p_i^z+\delta k_i^z-\delta u_i^z\right)z_i = +i\frac{k_\mu^{j-1}p_{j-1}^\mu}{p_0^0}z_{j-1} - i\sum_{i=k+1}^{j-2}\frac{k_\mu^i p_i^\mu}{p_0^0}\delta z_i - i\frac{k_\mu^{k+1}p_{k+1}^\mu}{p_0^0}z_{k+1}, \tag{5.55}$$

where we denoted $\delta z_i \equiv z_{i+1} - z_i$. The average of the set of interactions of this passage can be rewritten then as

$$\begin{aligned} A_{k+1}^{j-1} = &\left(\prod_{i=k+1}^{j-1}(2\pi)^2\delta^2(\boldsymbol{p}_i^t+\boldsymbol{k}_i^t-\boldsymbol{u}_i^t)\times\phi_{q\bar{q}g}^{(n_i)}(\delta\boldsymbol{p}_i^t,\delta\boldsymbol{k}_i^t)\right) \\ &\times\exp\left(+i\frac{k_\mu^{j-1}p_{j-1}^\mu}{p_0^0}z_{j-1} - i\sum_{i=k+1}^{j-2}\frac{k_\mu^i p_i^\mu}{p_0^0}\delta z_i - i\frac{k_\mu^{k+1}p_{k+1}^\mu}{p_0^0}z_{k+1}\right). \end{aligned} \tag{5.56}$$

At z_j a gluon is emitted in the conjugated amplitude. Two possibilities can be contemplated. The first one corresponds to a collision before the emission, which affects then only to the (q), $(\bar{q})$ and (g) states and is given by the average

$$A_j^+ \equiv \frac{1}{N_c^{n_j}}\,\mathrm{Tr}\left\langle\left(S_q^{j\dagger}(\delta\boldsymbol{u}_j+\boldsymbol{v}_j)\right)_{b'_j}^{b_{j-1}}\left(S_q^{j}(\delta\boldsymbol{p}_j)\right)_{a_{j-1}}^{a_j}\left(S_g^{j}(\delta\boldsymbol{k}_j)\right)_{\alpha_{j-1}}^{\alpha_j}\right\rangle. \tag{5.57}$$

From (5.56) we inherit the condition $\boldsymbol{u}_{j-1}^t = \boldsymbol{p}_{j-1}^t + \boldsymbol{k}_{j-1}^t$ so from the elastic average we obtain $\boldsymbol{u}_j^t + \boldsymbol{v}_j^t = \boldsymbol{p}_j^t + \boldsymbol{k}_j^t$. Since $p_{j-1}^0 + \omega = u_{j-1}^0 = p_0^0$ the longitudinal phase reads for this case

$$\begin{aligned} \delta p_j^z + \delta k_j^z - \delta u_j^z = &\left(p_{p_0^0-\omega}^z(\boldsymbol{p}_j^t) + p_\omega(\boldsymbol{k}_j^t) - p_{p_0^0}^z(\boldsymbol{p}_j^t+\boldsymbol{k}_j^t)\right) \\ &-\left(p_{p_0^0-\omega}^z(\boldsymbol{p}_{j-1}^t) + p_\omega^z(\boldsymbol{k}_{j-1}^t) - p_{p_0^0}^z(\boldsymbol{p}_{j-1}^t+\boldsymbol{k}_{j-1}^t)\right) = -\frac{k_\mu^j p_j^\mu}{p_0^0} + \frac{k_\mu^{j-1}p_{j-1}^\mu}{p_0^0}. \end{aligned} \tag{5.58}$$

Then the average of square advanced interaction at z_j produces a term of the form

$$A_j^+ = (2\pi)^2\delta^2(\boldsymbol{p}_j^t+\boldsymbol{k}_j^t-\boldsymbol{u}_j^t-\boldsymbol{v}_j^t)\phi_{q\bar{q}g}(\delta\boldsymbol{p}_j,\delta\boldsymbol{k}_j)\exp\left(+i\frac{k_\mu^j p_j^\mu}{p_0^0}z_j - i\frac{k_\mu^{j-1}p_{j-1}^\mu}{p_0^0}z_j\right). \tag{5.59}$$

The alternative case at z_j corresponds to a collision just after the emission, which thus affects the four states and is given instead by the average of four amplitudes

$$A_j^- \equiv \frac{1}{N_c}\,\mathrm{Tr}\left\langle\left(S_q^{j\dagger}(\delta\boldsymbol{u}_j+\boldsymbol{v}_{j-1})\right)_{b_j}^{b'_{j-1}}\left(S_g^{j\dagger}(\delta\boldsymbol{v}_j)\right)_{\beta_j}^{\beta_{j-1}}\left(S_q^{j}(\delta\boldsymbol{p}_j)\right)_{a_{j-1}}^{a_j}\left(S_g^{j}(\delta\boldsymbol{k}_j)\right)_{\alpha_{j-1}}^{\alpha_j}\right\rangle. \tag{5.60}$$

From (5.56) we find the condition $\boldsymbol{u}^t_{j-1} = \boldsymbol{p}^t_{j-1} + \boldsymbol{k}^t_{j-1}$ and then $\boldsymbol{u}^t_j + \boldsymbol{v}^t_j = \boldsymbol{p}^t_j + \boldsymbol{k}^t_j$. The phase arising in the collision can be reorganized using (5.12) and (5.13) as

$$\begin{aligned}\delta p^z_j + \delta k^z_j - \delta u^z_j - \delta v^z_j &= \Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_j) + p^z_\omega(\boldsymbol{k}^t_j) - p^z_{p^0_0}(\boldsymbol{p}^t_j + \boldsymbol{k}^t_j)\Big) \\ &- \Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_{j-1}) + p^z_\omega(\boldsymbol{k}^t_{j-1}) - p^z_{p^0_0}(\boldsymbol{p}^t_{j-1} + \boldsymbol{k}^t_{j-1})\Big) - \Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_j + \boldsymbol{k}^t_j - \boldsymbol{v}^t_j) \\ &+ p^z_\omega(\boldsymbol{v}^t_j) - p^z_{p^0_0}(\boldsymbol{p}^t_j + \boldsymbol{k}^t_j)\Big) + \Big(p^z_{p^0_0-\omega}(\boldsymbol{p}^t_{j-1} + \boldsymbol{k}^t_{j-1} - \boldsymbol{v}^t_{j-1}) + p^z_\omega(\boldsymbol{v}^t_{j-1}) \\ &- p^z_{p^0_0}(\boldsymbol{p}^t_{j-1} + \boldsymbol{k}^t_{j-1})\Big) = -\frac{k^j_\mu p^\mu_j}{p^0_0} + \frac{k^{j-1}_\mu p^\mu_{j-1}}{p^0_0} + \frac{v^j_\mu u^\mu_j}{p^0_0} - \frac{v^{j-1}_\mu u^\mu_{j-1}}{p^0_0}.\end{aligned} \tag{5.61}$$

Notice that the last two terms can also be defined, if desired in terms of the momentum of the q state, as

$$\frac{v^j_\mu u^\mu_j}{p^0_0} - \frac{v^{j-1}_\mu u^\mu_{j-1}}{p^0_0} = \frac{v^j_\mu (p+k)^\mu_j}{p^0_0 - \omega} - \frac{v^{j-1}_\mu (p+k)^\mu_{j-1}}{p^0_0 - \omega}, \tag{5.62}$$

provided $(p + k)$ refers to a quark 4-momentum of modulus p^0_0 with transverse component $\boldsymbol{p}^t + \boldsymbol{k}^t$. The elastic average at this step can be shown to satisfy then

$$\begin{aligned}A^-_j &= (2\pi)^2\delta^2(\boldsymbol{p}^t_j + \boldsymbol{k}^t_j - \boldsymbol{u}^t_j - \boldsymbol{v}^t_j) \times \phi^{(n_j)}_{q\bar{q}g\bar{g}}(\delta\boldsymbol{p}^t_j, \delta\boldsymbol{k}^t_j, \delta\boldsymbol{v}^t_j) \\ &\times \exp\left(+i\frac{k^j_\mu p^\mu_j}{p^0_0} z_j - i\frac{k^{j-1}_\mu p^\mu_{j-1}}{p^0_0} z_j - i\frac{v^j_\mu u^\mu_j}{p^0_0} z_j + i\frac{v^{j-1}_\mu u^\mu_{j-1}}{p^0_0} z_j\right),\end{aligned} \tag{5.63}$$

where the elastic distribution of the momentum of the three states produces

$$\begin{aligned}\phi^{(n_j)}_{q\bar{q}g\bar{g}}(\delta\boldsymbol{p}^t_j, \delta\boldsymbol{k}^t_j, \delta\boldsymbol{v}^t_j) &\equiv \int d^2\boldsymbol{x}^{q\bar{q}}_j \int d^2\boldsymbol{x}^{g\bar{q}}_j \int d^2\boldsymbol{x}^{\bar{g}\bar{q}}_j \exp\Big[-i\delta\boldsymbol{p}^t_j\cdot\boldsymbol{x}^{q\bar{q}}_j \\ &- i\delta\boldsymbol{k}^t_j\cdot\boldsymbol{x}^{g\bar{q}}_j + i\delta\boldsymbol{v}^t_j\cdot\boldsymbol{x}^{\bar{g}\bar{q}}_j + n_0(z_j)\delta z_j\Big(\sigma^{(1)}_{g\bar{q}}\big(\boldsymbol{x}^{g\bar{q}}_j\big) + \Big(\sigma^{(1)}_{q\bar{q}}\big(\boldsymbol{x}^{q\bar{q}}_j\big) - \sigma^{(1)}_{q\bar{q}}(\boldsymbol{0})\Big) \\ &+ \Big(\sigma^{(1)}_{g\bar{g}}\big(\boldsymbol{x}^{g\bar{g}}_j\big) - \sigma^{(1)}_{g\bar{g}}(\boldsymbol{0})\Big) + \sigma^{(1)}_{gq}\big(\boldsymbol{x}^{q\bar{q}}_j - \boldsymbol{x}^{g\bar{q}}_j\big) + \sigma^{(1)}_{q\bar{g}}\big(\boldsymbol{x}^{q\bar{q}}_j - \boldsymbol{x}^{\bar{g}\bar{q}}_j\big) + \sigma^{(1)}_{\bar{q}\bar{g}}\big(\boldsymbol{x}^{\bar{g}\bar{q}}_j\big)\Big)\Big].\end{aligned} \tag{5.64}$$

Here the Fourier transform of the quark single squared amplitude $\sigma^{(1)}_{q\bar{q}}(\boldsymbol{x})$ is given by (5.40), the cross products $\sigma^{(1)}_{gq}(\boldsymbol{x})$ and $\sigma^{(1)}_{g\bar{q}}(\boldsymbol{x})$ are given at (5.50) and (5.51), respectively, and the new cross squared amplitudes appearing in this average are given at leading order for (4.16) by

$$\sigma^{(1)}_{g\bar{g}}(\boldsymbol{x}) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{N_c} \mathrm{Tr}\Big(F^{(1)\dagger}_g(\boldsymbol{q}) F^{(1)}_g(\boldsymbol{q})\Big) = \frac{4\pi g^4_s}{\beta^2_k \mu^2_d} T_f \mu_d |\boldsymbol{x}| K_1(\mu_d|\boldsymbol{x}|), \tag{5.65}$$

which represents the collision with the gluon state (g) and the conjugate state ($\bar{g}$) leaving the rest of states unaltered, and the other two verify $\sigma^{(1)}_{q\bar{g}}(\boldsymbol{x}) = \sigma^{(1)}_{g\bar{q}}(\boldsymbol{x})$ acting however on the q and $\bar{g}$ states, and $\sigma^{(1)}_{\bar{q}\bar{g}}(\boldsymbol{x}) = \sigma^{(1)}_{gq}(\boldsymbol{x})$, but acting on the ($\bar{q}$) and ($\bar{g}$) states. Finally the passage of the quark state (q) and its conjugate ($\bar{q}$), and of the gluon state (g) and its conjugate state ($\bar{g}$), from the layer z_{j+1} to the end z_n is given by the average

$$A^n_{j+1} \equiv \prod_{j+1}^{n} \left(\frac{1}{N_c^{n_i}} \operatorname{Tr} \left\langle \left(S_q^{i\dagger}(\delta \boldsymbol{u}_i) \right)^{b_{i-1}}_{b_i} \left(S_g^{i\dagger}(\delta \boldsymbol{v}_i) \right)^{\beta_{i-1}}_{\beta_i} \left(S_q^{i}(\delta \boldsymbol{p}_i) \right)^{a_i}_{a_{i-1}} \left(S_g^{i}(\delta \boldsymbol{k}_i) \right)^{\alpha_i}_{\alpha_{i-1}} \right\rangle \right). \tag{5.66}$$

As before, the total longitudinal phase in the interval suggests a rearrangement of the form

$$-i \sum_{i=j+1}^{n} \left(\delta p_i^z - \delta u_i^1 + \delta k_i^z - \delta v_i^z \right) z_i = +i \frac{k^n_\mu p^\mu_n}{p^0_0} z_n - i \sum_{i=j+1}^{n-1} \frac{k^i_\mu p^\mu_i}{p^0_0} \delta z_i - i \frac{k^j_\mu p^\mu_j}{p^0_0} z_{j+1}$$
$$- i \frac{v^n_\mu u^\mu_n}{p^0_0} z_n + i \sum_{i=j+1}^{n-1} \frac{v^i_\mu u^\mu_i}{p^0_0} \delta z_i + i \frac{v^j_\mu u^\mu_j}{p^0_0} z_{j+1}. \tag{5.67}$$

The averages of the squared amplitude are of the same kind as the one explained for the previous scattering (5.63), and we obtain

$$A^n_{j+1} = \prod_{i=j+1}^{n} \left((2\pi)^2 \delta^2 (\boldsymbol{p}^t_i + \boldsymbol{k}^t_i - \boldsymbol{u}^t_i - \boldsymbol{v}^t_i) \times \phi^{(n_j)}_{q\bar{q}g\bar{g}}(\delta \boldsymbol{p}^t_j, \delta \boldsymbol{k}^t_j, \delta \boldsymbol{v}^t_j) \right) \tag{5.68}$$
$$\times \exp \left(-i \sum_{i=j+1}^{n-1} \frac{k^i_\mu p^\mu_i}{p^0_0} \delta z_i - i \frac{k^j_\mu p^\mu_j}{p^0_0} z_{j+1} + i \sum_{i=j+1}^{n-1} \frac{v^i_\mu u^\mu_i}{p^0_0} \delta z_i + i \frac{v^j_\mu u^\mu_j}{p^0_0} z_{j+1} \right).$$

Now, from (5.35) the average over initial spins and the sum over final spins and polarizations provides further simplifications. We notice, *c.f.* Appendix A,

$$\frac{1}{2} \sum_{s_n s_0} \sum_{\lambda_n} \left(\left(f^+_j \right)^* t_{\beta_j} \phi^{(n_j)}_{q\bar{q}g} \left(\delta \boldsymbol{p}_j, \delta \boldsymbol{k}_j \right) - \left(f^-_{j-1} \right)^* \phi^{(n_j)}_{q\bar{q}g\bar{g}} \left(\delta \boldsymbol{p}^t_j, \delta \boldsymbol{k}^t_j, \delta \boldsymbol{v}^t_j \right) t_{\beta_{j-1}} \right) \tag{5.69}$$
$$\times \left(f^+_k t_{\alpha_k} \phi^{(0)}_{q\bar{q}} \left(\delta \boldsymbol{p}^t_k \right) - f^-_{k-1} \phi^{(0)}_{q\bar{q}g} \left(\delta \boldsymbol{p}^t_k, \delta \boldsymbol{k}^t_k \right) t_{\alpha_{k-1}} \right) = \frac{1}{\omega^2} \left(h^n(y) \boldsymbol{\delta}^n_j \cdot \boldsymbol{\delta}^n_k + h^s(y) \delta^s_j \delta^s_k \right),$$

where $y = \omega / p^0_0$ is the fraction of energy carried by the gluon and the functions $h^n(y) = (1 + (1 - y)^2)/2$ and $h^s(y) = y^2/2$ are the kinematical weights of the spin no flip and spin flip contributions, which are given by the currents

$$\boldsymbol{\delta}^n_k \equiv \frac{\boldsymbol{k}_k \times \boldsymbol{p}_k}{k^k_\mu p^\mu_k} t_{\beta_k} \phi^{(n_k)}_{q\bar{q}g} \left(\delta \boldsymbol{p}_k, \delta \boldsymbol{k}_k \right) - \frac{\boldsymbol{k}_{k-1} \times \boldsymbol{p}_{k-1}}{k^{k-1}_\mu p^\mu_{k-1}} \phi^{(n_k)}_{q\bar{q}g\bar{g}} \left(\delta \boldsymbol{p}^t_k, \delta \boldsymbol{k}^t_k, \delta \boldsymbol{v}^t_k \right) t_{\beta_{k-1}}, \tag{5.70}$$

and

$$\delta_k^s \equiv \frac{\omega p_0^0}{k_\mu^k p_k^\mu} t_{\beta_k} \phi_{q\bar{q}g}^{(n_k)}\Big(\delta\boldsymbol{p}_k, \delta\boldsymbol{k}_k\Big) - \frac{\omega p_0^0}{k_\mu^{k-1} p_{k-1}^\mu} \phi_{q\bar{q}g\bar{g}}^{(n_k)}\Big(\delta\boldsymbol{p}_k^t, \delta\boldsymbol{k}_k^t, \delta\boldsymbol{v}_k^t\Big) t_{\beta_{k-1}}. \tag{5.71}$$

In the following we will assume that $\omega \ll p_0^0$. In this soft limit the spin flip contribution can be neglected and the kinematical weight of the non flip contribution is given simply by $h^n(y) \simeq 1$. Hence, after inserting equations (5.42), (5.45), (5.48), (5.56), (5.59), (5.63) and (5.68) in (5.35), integrating in the momentum trajectory of the $(\bar{q})$ states, summing over final spin and polarization and average over initial spin we find

$$\begin{aligned}
&\frac{1}{2}\sum_{\lambda,s}\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\Big\langle\mathcal{M}_j^\dagger\mathcal{M}_k\Big\rangle \\
&= (2\pi)^2\delta^2(\mathbf{0})\exp\left(-i\sum_{i=j}^{n-1}\delta z_i\left(\frac{k_\mu^i p_i^\mu}{p_0^0-\omega}-\frac{v_\mu^i p_i^\mu}{p_0^0-\omega}\right)-i\sum_{i=k}^{j-1}\delta z_i\frac{k_\mu^i p_i^\mu}{p_0^0-\omega}\right) \\
&\left(\prod_{i=j+1}^{n}\phi_{q\bar{q}g\bar{g}}^{(n_i)}\Big(\delta\boldsymbol{p}_i,\delta\boldsymbol{k}_i,\delta\boldsymbol{v}_i\Big)\right)\boldsymbol{\delta}_j^n\left(\prod_{i=k+1}^{j-1}\phi_{q\bar{q}g}^{(n_i)}\Big(\delta\boldsymbol{p}_i^t,\delta\boldsymbol{k}_i^t\Big)\right)\boldsymbol{\delta}_k^n\left(\prod_{i=1}^{k-1}\phi_{q\bar{q}}^{(n_i)}\Big(\delta\boldsymbol{p}_i^t\Big)\right).
\end{aligned} \tag{5.72}$$

The extra delta $(2\pi)^2\delta^2(\mathbf{0}) = \pi R^2$, which accounts the space translation invariance in the transverse plane, appears due to the conditions $\boldsymbol{p}_n^t = \boldsymbol{u}_n^t$ and $\boldsymbol{k}_n^t = \boldsymbol{v}_n^t$. The integration in quark momenta has been shifted by $\boldsymbol{k}^t$. The averaged elastic amplitudes squared B_j^k appearing at (5.36) follow the same steps as the previous calculation. Since the same kinematical conditions hold, it can be shown that the elastic distributions (5.39), (5.49) and (5.64) have to be replaced with the squared averaged amplitudes. An elastic average squared of the state (q) and $(\bar{q})$ is given by

$$\phi_{q\bar{q}}^{(0)}\Big(\delta\boldsymbol{p}_i^t\Big) \equiv \exp\Big(-n_0(z_i)\delta z_i\sigma_{q\bar{q}}^{(1)}(\mathbf{0})\Big) \times (2\pi)^2\delta^2(\delta\boldsymbol{p}_i^t). \tag{5.73}$$

Similarly, an elastic average squared of the states (q), (g) and $\bar{q}$ is given instead by

$$\begin{aligned}
\phi_{q\bar{q}g}^{(0)}(\delta\boldsymbol{p}_i^t,\delta\boldsymbol{k}_i^t) \equiv &\int d^2\boldsymbol{x}_k^{q\bar{q}}\int d^2\boldsymbol{x}_k^{g\bar{q}}\exp\Big(-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^{q\bar{q}}-i\delta\boldsymbol{k}_i^t\cdot\boldsymbol{x}_i^{g\bar{q}}\Big) \\
&\times\exp\Big(n_0(z_i)\delta z_i\Big(\varphi_g^{(1)}(\mathbf{0})-\sigma_{q\bar{q}}^{(1)}(\mathbf{0})+\sigma_{gq}^{(1)}\Big(\boldsymbol{x}_i^{g\bar{q}}-\boldsymbol{x}_k^{q\bar{q}}\Big)\Big)\Big),
\end{aligned} \tag{5.74}$$

which produces $\delta\boldsymbol{p}_i^t = \delta\boldsymbol{k}_i^t$. And finally an elastic average squared of the states (q), (g), $(\bar{q})$ and $(\bar{g})$ produces

$$\phi^{(0)}_{q\bar{q}g\bar{g}}(\delta\boldsymbol{p}_i^t, \delta\boldsymbol{k}_i^t, \delta\boldsymbol{v}_i^t) \equiv \int d^2\boldsymbol{x}_i^{q\bar{q}} \int d^2\boldsymbol{x}_i^{g\bar{q}} \int d^2\boldsymbol{x}_i^{\bar{g}\bar{q}} e^{-i\delta\boldsymbol{p}_i^t\cdot\boldsymbol{x}_i^{q\bar{q}} - i\delta\boldsymbol{k}_i^t\cdot\boldsymbol{x}_i^{g\bar{q}} + i\delta\boldsymbol{v}_i^t\cdot\boldsymbol{x}_i^{\bar{g}\bar{q}}}$$

$$\times \exp\left(n_0(z_i)\delta z_i\Big(- \sigma^{(1)}_{q\bar{q}}(\mathbf{0}) - \sigma^{(1)}_{g\bar{g}}(\mathbf{0}) + \sigma^{(1)}_{gq}\Big(\boldsymbol{x}_i^{q\bar{q}} - \boldsymbol{x}_i^{g\bar{q}}\Big) + \sigma^{(1)}_{\bar{g}\bar{q}}\Big(\boldsymbol{x}_i^{\bar{g}\bar{q}}\Big)\Big)\right), \tag{5.75}$$

which produces $\delta\boldsymbol{p}_i^t = \delta\boldsymbol{k}_i^t$. Using these elastic distributions, from (5.36) we get

$$\frac{1}{2}\sum_{\lambda,s}\left(\prod_{i=1}^{n-1}\int \frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\left\langle \mathcal{M}_j^\dagger\right\rangle\left\langle \mathcal{M}_k\right\rangle \tag{5.76}$$

$$= (2\pi)^2\delta^2(\mathbf{0})\exp\left(-i\sum_{i=j}^{n-1}\delta z_i\left(\frac{k^i_\mu p^\mu_i}{p_0^0-\omega} - \frac{v^i_\mu p^\mu_i}{p_0^0-\omega}\right) - i\sum_{i=k}^{j-1}\delta z_i\frac{k^i_\mu p^\mu_i}{p_0^0-\omega}\right)$$

$$\left(\prod_{i=j+1}^{n}\phi^{(0)}_{q\bar{q}g\bar{g}}\Big(\delta\boldsymbol{p}_i, \delta\boldsymbol{k}_i, \delta\boldsymbol{v}_i\Big)\right)\boldsymbol{\delta}_j^n\left(\prod_{i=k+1}^{j-1}\phi^{(0)}_{q\bar{q}g}\Big(\delta\boldsymbol{p}_i^t, \delta\boldsymbol{k}_i^t\Big)\right)\boldsymbol{\delta}_k^n\left(\prod_{i=1}^{k-1}\phi^{(0)}_{q\bar{q}}\Big(\delta\boldsymbol{p}_i^t\Big)\right).$$

The structure of the intensities (5.72) and (5.76) has substantial differences with the QED scenario, (3.117) and (3.118), related to the ability of the gluon to reinteract with the medium constituents. First, due to the form of (5.70) it is not possible yet to factorize the elastic distributions independently of z_k and z_j. Second, the accumulated phase between z_k and z_j, regulating the interference and hence the LPM effect, is now dominated for soft gluons by the accumulated squared momentum change of the gluon and its mass. Indeed, for $\omega \ll p_0^0$ either looking at (5.13) or from $k_\mu p^\mu = \omega p_0^0(1-\beta_k\beta_p\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}})$ the quark can be considered frozen with respect to the gluon rescattering in the initial direction $p^\mu(z) = p^\mu(0)$. Then if we set $\beta_p = 1$ the phase becomes

$$\sum_{i=k}^{j-1}\frac{k^i_\mu p^\mu_i}{p_0^0-\omega} \cong \sum_{i=k}^{j-1}\frac{k^i_\mu p^\mu_0}{p_0^0} = \frac{m_g^2}{(1+\beta_k)\omega}(z_j - z_k) + \sum_{i=k}^{j-1}\frac{\delta\boldsymbol{k}_i^2}{2\beta_k\omega}\delta z_i, \tag{5.77}$$

where $\delta\boldsymbol{k}_i^2$ is the accumulated gluon squared momentum change with respect to the initial quark direction. Third, the unobserved intermediate states of the gluon leads to an extra phase in the interval from z_j to z_n of the form

$$\sum_{i=j}^{n-1}\delta z_i\left(\frac{k^i_\mu p^\mu_i}{p_0^0-\omega} - \frac{v^i_\mu p^\mu_i}{p_0^0-\omega}\right) \neq 0, \tag{5.78}$$

since $v_i \neq k_i$. And third, color rotation is allowed in the states from z_k onwards, since the elastic distributions (5.49) and (5.64) carry a non trivial matrix structure.

We will assume, however, that a color averaged effective interaction for the gluon exists which leads to the same momentum transport as the elastic distributions (5.49) and (5.64) and thus the same phase (5.77). For that purpose we take as an ansatz the color average of (5.72) and (5.76) for the single layer $n = 1$ case. Since an expansion in δz holds, using the relation $n_0(z_1)\delta z_1(2\pi)^2\delta^2(\mathbf{0}) \equiv n_1$ we find from (5.72) a crossed term of the form

$$(2\pi)^2\delta^2(\mathbf{0})\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\phi^{(n_1)}_{q\bar{q}g}\left(\delta\boldsymbol{p},\delta\boldsymbol{k}\right)t_{\alpha_0}\right) \simeq n_1\int d^2\boldsymbol{x}_1^{q\bar{q}}e^{-i\delta\boldsymbol{p}\cdot\boldsymbol{x}_1^{q\bar{q}}}\int d^2\boldsymbol{x}_1^{g\bar{q}}e^{-i\delta\boldsymbol{k}\cdot\boldsymbol{x}_1^{g\bar{q}}}$$
$$\times\left\{\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\left(\varphi_g^{(1)}(\mathbf{0})-\sigma_{q\bar{q}}^{(1)}(\mathbf{0})\right)t_{\alpha_0}\right)+\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{q\bar{q}}^{(1)}\left(\boldsymbol{x}_1^{q\bar{q}}\right)t_{\alpha_0}\right)\right. \tag{5.79}$$
$$\left.+\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{g\bar{q}}^{(1)}\left(\boldsymbol{x}_1^{g\bar{q}}\right)t_{\alpha_0}\right)+\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{gq}^{(1)}\left(\boldsymbol{x}_1^{g\bar{q}}-\boldsymbol{x}_1^{q\bar{q}}\right)t_{\alpha_0}\right)\right\}.$$

The first contribution represents the passage without momentum changes. We are interested instead in the last three terms. Using (5.40) the first of these contributions can be rewritten as

$$\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{q\bar{q}}^{(1)}\left(\boldsymbol{x}_1^{q\bar{q}}\right)t_{\alpha_0}\right)=\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}t_\alpha t_\beta t_{\alpha_1}\right)\frac{\delta_{\alpha\beta}}{2N_c}\frac{4\pi g_s^4}{\beta_p^2\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{q\bar{q}}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{q\bar{q}}\right|\right)$$
$$=-\frac{1}{N_c^2-1}\left(g_s^2\frac{N_c^2-1}{2N_c}\right)\left(\frac{N_c^2-1}{4N_c^2}\frac{4\pi g_s^4}{\beta_p^2\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{q\bar{q}}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{q\bar{q}}\right|\right)\right), \tag{5.80}$$

that is, $1/8$ times the independent color average of the emission vertex $C_f = (N_c^2-1)/2N_c$ times the independent color average of the squared scattering amplitude of a quark with a single quark. The second contribution produces, using (5.51),

$$\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{g\bar{q}}^{(1)}\left(\boldsymbol{x}_1^{g\bar{q}}\right)t_{\alpha_0}\right)=\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_\alpha t_{\alpha_1}t_{\alpha_0}\right)\left(T_\alpha\right)^{\alpha_1}_{\alpha_0}\frac{\delta_{\alpha\beta}}{2N_c}\frac{4\pi g_s^4}{\beta_p\beta_k\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{g\bar{q}}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{g\bar{q}}\right|\right)$$
$$=\frac{N_c^2}{N_c^2-1}\left(g_s^2\frac{N_c^2-1}{2N_c}\right)\left(\frac{N_c^2-1}{4N_c^2}\frac{4\pi g_s^4}{\beta_p\beta_k\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{g\bar{q}}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{g\bar{q}}\right|\right)\right), \tag{5.81}$$

which is $9/8$ times the independent color average of the emission vertex times the color average of the squared scattering amplitude of a quark with a single quark. For the third contribution we get, using (5.50),

$$\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}\sigma_{gq}^{(1)}\left(\boldsymbol{x}_1^{gq}\right)t_{\alpha_0}\right)=-\frac{g_s^2}{N_c}\,\mathrm{Tr}\left(t_{\alpha_1}t_\alpha t_{\alpha_0}\right)\left(T_\alpha\right)^{\alpha_1}_{\alpha_0}\frac{\delta_{\alpha\beta}}{2N_c}\frac{4\pi g_s^4}{\beta_p\beta_k\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{gq}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{gq}\right|\right)$$
$$=\frac{N_c^2}{N_c^2-1}\left(g_s^2\frac{N_c^2-1}{2N_c}\right)\left(\frac{N_c^2-1}{4N_c^2}\frac{4\pi g_s^4}{\beta_p\beta_k\mu_d^2}\mu_d\left|\boldsymbol{x}_1^{gq}\right|K_1\left(\mu_d\left|\boldsymbol{x}_1^{gq}\right|\right)\right), \tag{5.82}$$

which is again the same result as before. Then if we assume the high energy limit both for the gluon and the quark $\beta_k \simeq \beta_p \simeq 1$ we can write an effective elastic distribution for (5.49) given by the replacement

$$\begin{aligned}
&\left(\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{q\bar{q}}_i\right) - \sigma^{(1)}_{q\bar{q}}(\mathbf{0})\right) + \sigma^{(1)}_{g\bar{q}}\left(\boldsymbol{x}^{g\bar{q}}_i\right) + \sigma^{(1)}_{gq}\left(\boldsymbol{x}^{g\bar{q}}_i - \boldsymbol{x}^{q\bar{q}}_i\right) \\
&\to -\frac{1}{8}\left(\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{q\bar{q}}_i\right) - \sigma^{(1)}_{q\bar{q}}(\mathbf{0})\right) + \frac{9}{8}\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{g\bar{q}}_i\right) + \frac{9}{8}\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{g\bar{q}}_i - \boldsymbol{x}^{q\bar{q}}_i\right),
\end{aligned} \tag{5.83}$$

where the (trivial) color structure of $\sigma^{(1)}_{q\bar{q}}(\boldsymbol{x})$ at (4.74) has been dropped. Further simplifications produces the soft limit of the phase. In that case the integrals in the quark momenta can be performed producing $\boldsymbol{x}^{q\bar{q}}_i = 0$ in all the distributions so that $\phi_{q\bar{q}}(\boldsymbol{x}^{q\bar{q}}_i) \to 1$. Indeed for the first set of scatterings we obtain, using (5.39),

$$\int \frac{d^2\delta\boldsymbol{p}_i}{(2\pi)^2}\phi^{(n_i)}_{q\bar{q}}(\delta\boldsymbol{p}^t_i) = 1. \tag{5.84}$$

For the intermediate step of scatterings we find, using (5.49),

$$\begin{aligned}
&\int \frac{d^2\delta\boldsymbol{p}^t_i}{(2\pi)^2}\phi^{(n_i)}_{q\bar{q}g}(\delta\boldsymbol{p}^t_i, \delta\boldsymbol{k}^t_i) \\
&= \int d^2\boldsymbol{x}^{g\bar{q}}_i \exp\left(-i\delta\boldsymbol{k}^t_i\cdot\boldsymbol{x}^{g\bar{q}}_i\right)\exp\left(\varphi^{(1)}_g(\mathbf{0}) + \frac{9}{8}\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{g\bar{q}}_i\right) + \frac{9}{8}\sigma^{(1)}_{qq}\left(\boldsymbol{x}^{g\bar{q}}_i\right)\right).
\end{aligned} \tag{5.85}$$

By summing the two last contributions the above term can be rewritten as the squared scattering amplitude of a gluon with a single target quark

$$\frac{9}{8}\sigma^{(1)}_{qq}\left(\boldsymbol{x}^{g\bar{q}}_i\right) + \frac{9}{8}\sigma^{(1)}_{q\bar{q}}\left(\boldsymbol{x}^{g\bar{q}}_i\right) = \left(\frac{9}{8} + \frac{9}{8}\right)\frac{2}{9}\frac{4\pi g_s^4}{\beta^2\mu_d^2}\mu_d\left|\boldsymbol{x}^{g\bar{q}}_i\right| K_1\left(\mu_d\left|\boldsymbol{x}^{g\bar{q}}_i\right|\right) = \sigma^{(1)}_{g\bar{g}}(\boldsymbol{x}^{g\bar{q}}_i), \tag{5.86}$$

where as before the color structure of $\sigma^{(1)}_{g\bar{g}}(\boldsymbol{x}^{g\bar{q}}_i)$ at (5.65) has been dropped [19, 141, 146]. For the last set of scatterings the integration in the quark momenta produces, using (5.64),

$$\begin{aligned}
\int \frac{d^2\delta\boldsymbol{p}^t_i}{(2\pi)^2}\phi^{(n_i)}_{q\bar{q}g\bar{g}}(\delta\boldsymbol{p}^t_i, \delta\boldsymbol{k}^t_i, \delta\boldsymbol{v}^t_i) &= (2\pi)^2\delta^2(\delta\boldsymbol{v}^t_i - \delta\boldsymbol{k}^t_i) \\
&\times \int d^2\boldsymbol{x}^{g\bar{q}}_i e^{-i\delta\boldsymbol{k}^t_i\cdot\boldsymbol{x}^{g\bar{q}}_i}\exp\left(\sigma^{(1)}_{g\bar{g}}\left(\boldsymbol{x}^{g\bar{g}}_i\right) - \sigma^{(1)}_{g\bar{g}}(\mathbf{0})\right),
\end{aligned} \tag{5.87}$$

which makes the phase vanish in the interval z_j to z_n, since $\boldsymbol{v}^t_i = \boldsymbol{k}^t_i$ for $i = j, \ldots, n$. Under these approximations we assume that the gluon effective interaction is given by the solution of a Moliere/transport equation (4.72) where the single collision distribution is given by the Fourier transform of $\sigma^{(1)}_{g\bar{g}}(\boldsymbol{x}_i^{g\bar{q}})$. From (5.72) we obtain then

$$\frac{1}{2N_c}\sum_{\lambda,s,a_0}\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\int\frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\left(\prod_{i=j-1}^{n-1}\int\frac{d^2\boldsymbol{v}_i^t}{(2\pi)^2}\right)\left\langle\mathcal{M}_j^\dagger\mathcal{M}_k\right\rangle \simeq (2\pi)^2\delta^2(\mathbf{0})\frac{C_f}{\omega^2}$$
$$\exp\left(-i\sum_{i=k}^{j-1}\delta z_i\frac{k^i_\mu p_0^\mu}{p_0^0-\omega}\right)\left(\prod_{i=j+1}^{n}\phi^{(n_i)}_{g\bar{g}}\left(\delta\boldsymbol{k}_i\right)\right)\boldsymbol{\delta}_j^n\left(\prod_{i=k+1}^{j}\phi^{(n_i)}_{g\bar{g}}\left(\delta\boldsymbol{k}_i^t\right)\right)\boldsymbol{\delta}_k^n\left(\prod_{i=1}^{k}\phi^{(n_i)}_{g\bar{g}}\left(\delta\boldsymbol{k}_i^t\right)\right),\tag{5.88}$$

where we extracted $C_f = (N_c^2-1)/2N_c$, the color average of the emission vertex, and for convenience we multiplied by one with a set of normalized gluon distributions for the range $i = 1,\ldots,k-1$. After summing in j and k from (5.88) we obtain

$$\frac{1}{2N_c}\sum_{\lambda,s,a_0}\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\int\frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\left(\prod_{i=j-1}^{n-1}\int\frac{d^2\boldsymbol{v}_i^t}{(2\pi)^2}\right)\sum_{j,k}^{n}\left\langle\mathcal{M}_j^\dagger\mathcal{M}_k\right\rangle$$
$$= (2\pi)^2\delta^2(\mathbf{0})\times\frac{C_f}{\omega^2}\left|\sum_{k=1}^{n}\exp\left(-i\sum_{i=k}^{n-1}\delta z_i\frac{k^i_\mu p_0^\mu}{p_0^0-\omega}\right)\boldsymbol{\delta}_k^n\right|^2\left(\prod_{i=1}^{n}\phi^{(n_i)}_{g\bar{g}}\left(\delta\boldsymbol{k}_i^t\right)\right).\tag{5.89}$$

Under the new color averaged effective elastic distributions the averages appearing in the non flip emission currents (5.70) can be factorized so that

$$\boldsymbol{\delta}_k^n \equiv \frac{\boldsymbol{k}_k\times\boldsymbol{p}_0}{k^k_\mu p_0^\mu}-\frac{\boldsymbol{k}_{k-1}\times\boldsymbol{p}_0}{k^{k-1}_\mu p_0^\mu}.\tag{5.90}$$

Similarly for the elastic averaged amplitudes squared (5.76) we obtain, after summing in j and k,

$$\frac{1}{2N_c}\sum_{\lambda,s,a_0}\left(\prod_{i=1}^{n-1}\int\frac{d^2\boldsymbol{p}_i^t}{(2\pi)^2}\int\frac{d^2\boldsymbol{u}_i^t}{(2\pi)^2}\right)\left(\prod_{i=j-1}^{n-1}\int\frac{d^2\boldsymbol{v}_i^t}{(2\pi)^2}\right)\sum_{j,k}^{n}\left\langle\mathcal{M}_j^\dagger\right\rangle\left\langle\mathcal{M}_k\right\rangle$$
$$= (2\pi)^2\delta^2(\mathbf{0})\times\frac{C_f}{\omega^2}\left|\sum_{k=1}^{n}\exp\left(-i\sum_{i=k}^{n-1}\delta z_i\frac{k^i_\mu p_0^\mu}{p_0^0-\omega}\right)\boldsymbol{\delta}_k^n\right|^2\left(\prod_{i=1}^{n}\phi^{(0)}_{g\bar{g}}\left(\delta\boldsymbol{k}_i^t\right)\right).\tag{5.91}$$

Correspondingly, after identifying $(2\pi)^2\delta^2(\mathbf{0}) \equiv \pi R^2$ and $2\pi\beta_p\delta(0) \equiv \beta_p T$, inserting (5.89) and (5.91) in (5.34), and using (5.30), we finally find an intensity of the form

$$\omega\frac{dI}{d\omega d\Omega_k} = \frac{g_s^2 C_f}{(2\pi)^2}\left(\prod_{i=1}^{n-1}\frac{d^3\boldsymbol{k}_i}{(2\pi)^3}\right)\left\{\left(\prod_{i=1}^{n}\phi^{(n_i)}_{g\bar{g}}(\delta\boldsymbol{k}_i)\right)-\left(\prod_{i=1}^{n}\phi^{(0)}_{g\bar{g}}(\delta\boldsymbol{k}_i)\right)\right\}$$
$$\times\left|\sum_{k=1}^{n}\exp\left(-i\sum_{i=k}^{n-1}\delta z_i\frac{k^i_\mu p_0^\mu}{p_0^0-\omega}\right)\boldsymbol{\delta}_k^n\right|^2.\tag{5.92}$$

Equation (5.92) is the central result of this section. The resulting intensity is the evaluation, over the elastic transports at each of the single layers, of a sum of single Bethe-Heitler currents for the gluon (5.90) carrying each one a phase (5.77) responsible of modulating the LPM effect in a QCD scenario. The elastic distributions appearing in (5.92) have been completed to the original three dimensional elastic distributions and are given by

$$\phi_{g\bar{g}}^{(n_i)}(\delta\boldsymbol{k}_i) \equiv \exp\left(-n_0(z_i)\delta z_i \sigma_{g\bar{g}}^{(1)}(\mathbf{0})\right) \times (2\pi)^3 \delta^3(\delta\boldsymbol{k}_i) + 2\pi\beta_k \delta(\delta k^0)\Sigma_2^{(n_i)}(\delta\boldsymbol{k}_i^t, \delta z_i). \tag{5.93}$$

Equation (5.93) can be interpreted as the probability of no colliding with the layer of density $n_0(z_i)$ and thickness δz_i, given by $\exp\big(-n_0(z_i)\delta z_i \sigma_{g\bar{g}}^{(1)}(\mathbf{0})\big)$, times the forward distribution $(2\pi)^3\delta^3(\delta\boldsymbol{k}_i)$, or the collisional distribution in case of collision, which is given by

$$\Sigma_{g\bar{g}}^{(n_i)}(\boldsymbol{q}, \delta z) \equiv \exp\left(-n_0(z)\delta z \sigma_{g\bar{g}}^{(1)}(\mathbf{0})\right) \int d^2\boldsymbol{x} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \left(\exp\left(n_0(z)\delta z \sigma_{g\bar{g}}^{(1)}(\boldsymbol{x})\right) - 1\right). \tag{5.94}$$

The Fourier transform of the squared scattering amplitude of a gluon with a single quark of the QCD medium $\sigma_{g\bar{g}}^{(1)}(\boldsymbol{x})$ is given by (5.65) but without the color structure. Its color averaged charge T_f equals the sum of the charges of the gq and $g\bar{q}$ interactions (5.51) and (5.50), which govern the gluon interaction between z_k and z_j. In (5.92) the overall no collision case is removed at the end. Its contribution is given by the effective elastic distributions substituting the averaged elastic amplitudes squared (5.36)

$$\phi_{g\bar{g}}^{(0)}(\delta\boldsymbol{k}_i) \equiv \exp\left(-n_0(z_i)\delta z_i \sigma_{g\bar{g}}^{(1)}(\mathbf{0})\right) \times (2\pi)^3 \delta^3(\delta\boldsymbol{k}_i) \tag{5.95}$$

Following (5.93) or (5.95) then the gluon mean free path in the medium $\lambda_{g\bar{g}}$, if we assume an uniform density $n_0 \equiv n_0(z_i)$, is given by

$$\lambda_{g\bar{g}} \equiv \frac{1}{n_0 \sigma_{g\bar{g}}^{(1)}(\mathbf{0})} \tag{5.96}$$

For a medium of vanishing length $l \ll \lambda_{g\bar{g}}$ the cancellation of the elastic distribution (5.93) with the no collision case (5.95) at (5.92) guarantees that the intensity vanishes. For mediums of very large length $l \gg \lambda_{g\bar{g}}$ the subtraction of the overall no collision case (5.95) at (5.92) can be neglected. The motion of the gluon under the effective distributions (5.93) satisfies an additivity rule for the squared momentum change. Indeed, for a step such that $\delta l \lesssim \lambda_{g\bar{g}}$ then the total elastic distribution is given by the incoherent superposition of the single elastic distributions of the centers at δl. The squared momentum change in δl equals, then, the momentum change in a single collision

$$\left\langle \delta\boldsymbol{k}^2(\delta l)\right\rangle \equiv \frac{\lambda_{g\bar{g}}}{\delta l}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\delta\boldsymbol{k}^2\phi^{(n)}_{g\bar{g}}(\delta\boldsymbol{k}) \simeq 2\mu_d^2\left(\log\left(\frac{2\beta_k\omega}{\mu_d}\right)-\frac{1}{2}\right) = \eta(\omega)\mu_d^2. \quad (5.97)$$

where the function $\eta(\omega)$ accounts for the long tail of the Debye interaction. For larger distances l we can use the transport equation [36, 37] satisfied by (5.94) given at (4.72). We then find a gluon analogous of the quark transport (4.78), a transport equation for the gluon squared momentum change

$$\frac{\partial}{\partial l}\left\langle \delta\boldsymbol{k}^2(l)\right\rangle = n_0\sigma^{(1)}_{g\bar{g}}\left\langle \delta\boldsymbol{k}^2(\delta l)\right\rangle \equiv 2\hat{q}. \quad (5.98)$$

where we defined the transport parameter of the medium $\hat{q}$. The presence of the long tail correction $\eta(\omega)$ in $\left\langle \delta\boldsymbol{k}^2(\delta l)\right\rangle$ at (5.97) makes $\hat{q}$ slowly dependent on the gluon energy. An accurate form of $\eta(\omega)$ under gluon bremsstrahlung would require the effect of the convolution of the elastic distributions (5.93) with the $\boldsymbol{\delta}_k$ currents (5.90). Using our previous QED estimates, in a single collision the functions $\boldsymbol{\delta}_k$ substantially reduce the gluon maximal momentum change (5.97) from the kinematical limit $2\beta_k\omega$ to 2-3 times the gluon mass m_g [6]. For a multiple collision scenario the LPM effect modulating the intensity (5.92) has to be taken into account in order to obtain an accurate ω dependence of the medium transport parameter $\hat{q}$ as measured through gluon bremsstrahlung [20].

Equation (5.92) is suitable for a numerical evaluation under a general interaction. A Monte Carlo code has been built where the integration in the gluon internal momenta at (5.92) is performed as a sum over discretized gluon paths. The gluon paths are generated as zig-zag trajectories satisfying the effective elastic distributions (5.93), with a step size δz. The quark is allowed to suffer medium interactions, but can be equally considered frozen for $\omega \ll p_0^0$. The phases and the gluon trajectory are calculated with the exact kinematical constraints, so that energy is always conserved in the collisions. This property becomes relevant for gluon energies ω^2 comparable to the final averaged squared momentum change $2\hat{q}l$. In a typical run the step size is chosen as $0.01\lambda_{g\bar{g}}$ in such a way that the largest medium $l \sim 5$ produces $\gtrsim 10^4$ steps. An array of ~ 100 gluon frequencies and ~ 800 final angles were computed for 10^4 discretized paths in order to guarantee that the uncertainty in all the cases falls below the 10%.

A second approach to qualitatively understand the behavior of intensity (5.92) would consist in an analogy with the QED case. The classical behavior discussed at Section 3.1 suggests interpreting intensity (5.92) as a sum of n single Bethe-Heitler amplitudes $\boldsymbol{\delta}_k$ (5.90) of a gluon being emitted at the point z_k, carrying a relative phase (5.77) producing interferences in the squared emission amplitude. Each gluon appears twice, namely before or after the k-th collision. For mediums of vanishing length gluon momentum homogeneity produces a cancellation

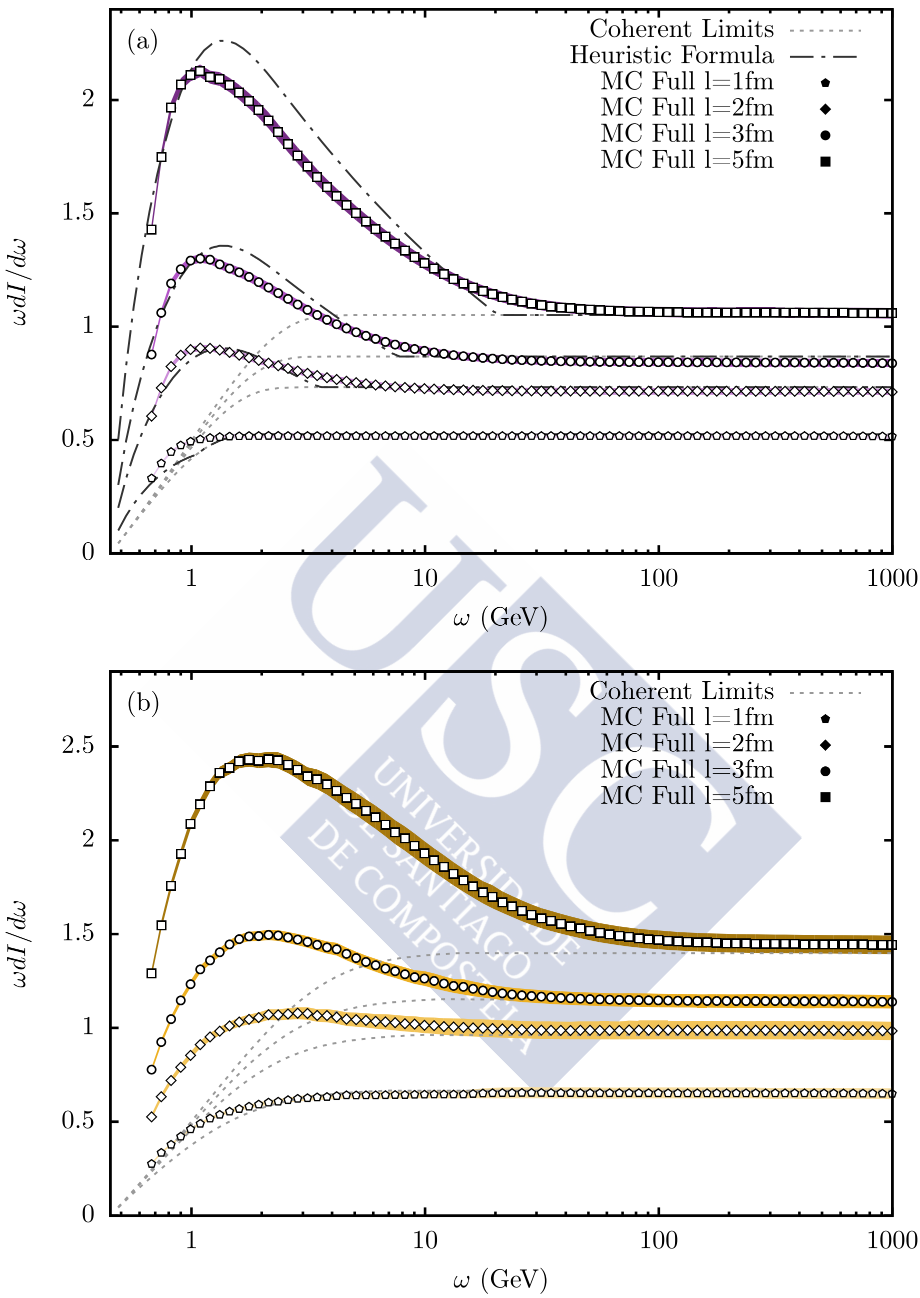


Figure 5.4: Intensity of gluons as a function of the gluon energy [138] emitted after traversing a medium of density $n_0 T_f = 8$ fm^{-3} corresponding to a transport parameter $\hat{q} = 0.98$ GeV^2/fm, for a gluon mass of $m_g = 0.45$ GeV and a Debye screening mass of $\mu_d = 0.45$ GeV. Coupling is set to $\alpha_s = 0.5$. Numerical evaluation of (5.92) is shown for Fokker-Planck approximation (a) and the Debye interaction (b) for several medium lengths, $l = 1$ fm (pentagons), $l = 2$ fm (diamonds), $l = 3$ fm (circles) and $l = 5$ fm (squares). Also shown is the coherent plateau $i_0(l)$ for both interactions (dashed lines) and the heuristic formula (5.103) for the Fokker-Planck approximation (dot-dashed line).

of the sum at (5.92) and intensity vanishes. For mediums of small length, satisfying $l \lesssim \lambda_{g\bar{g}}$, the internal sum cancels and we are left with the first and last gluons. By expanding (5.92) in opacity for a single layer the massive Bertsch-Gunion formula is recovered. For mediums of larger length, compromising several collisions on average, the internal sum in (5.92) becomes relevant and is modulated by the phase. In that case we can pair the sum of the single elements $\boldsymbol{\delta}_k$ in groups separated by a maximum distance $\delta l = z_j - z_m$ such that their relative phase, under the elastic distributions (5.93), acquires a value of the order of unity. This condition reads, using (5.98),

$$\varphi_m^j \equiv \int_{z_m}^{z_j} dz \frac{k_\mu(z) p^\mu(0)}{p_0^0 - \omega} \simeq \frac{m_g^2 \delta l}{(1+\beta_k)\omega} + \frac{\hat{q}(\delta l)^2}{2\beta_k \omega} = 1. \tag{5.99}$$

In the distance δl the partial internal sum between z_m and z_j at (5.92) of the single amplitudes $\boldsymbol{\delta}_k^n$ cancels, since their relative phase is negligible using the condition (5.99), and the first and last gluons of the group are the only ones surviving,

$$\sum_{k=m}^{j} \boldsymbol{\delta}_k e^{i\varphi_k} = e^{i\varphi_m} \sum_{k=m}^{j} \boldsymbol{\delta}_k e^{i\varphi_m^k} \simeq e^{i\varphi_m} \sum_{k=m}^{j} \boldsymbol{\delta}_k = e^{i\varphi_m} \left(\frac{\boldsymbol{k}_j \times \boldsymbol{p}_0}{k_\mu^j p_0^\mu} - \frac{\boldsymbol{k}_{k-1} \times \boldsymbol{p}_0}{k_\mu^{k-1} p_0^\mu} \right), \tag{5.100}$$

where we used (5.90) and (5.99). Then the elements in δl coherently act between themselves but they incoherently interfere with any other group. This defines a coherence length, modulated by ω, which by using (5.99) is given by

$$\delta l(\omega) \equiv \frac{m_g^2}{2\hat{q}} \left(\sqrt{1 + \frac{8\hat{q}\omega}{m_g^4}} - 1 \right), \tag{5.101}$$

This interference is characterized by the frequency ω_c at which the coherence length acquires the maximum value l, given then by $\omega_c \simeq (\hat{q}l + m_g^2)l$, and the characteristic frequency ω_s at which the coherence length acquires the minimum value $\lambda_{g\bar{g}}$ required to produce bremsstrahlung, given then by $\omega_s = m_g^4/\hat{q}$, where $\mu_d \simeq m_g$ was assumed. Since for $\delta l(\omega) > l$ there are not centers producing bremsstrahlung we further impose to (5.101) the condition $\delta l(\omega) = l$ for $\omega > \omega_c$. Following (5.100) in a coherence length the gluon is not able to resolve the scattering structure and the matter in $\delta l(\omega)$ acts like a single and independent scatterer with equivalent charge $n_0 \delta l(\omega)$ for the distribution (5.93). Since in a medium of length l there are $l/\delta l(\omega)$ of these units, we write the approximated formula for the intensity after traversing a distance l

$$\omega \frac{dI(l)}{d\omega} = \frac{l}{\delta l(\omega)} \frac{g_s^2 C_f}{(2\pi)^2} \int d\Omega_k \int \frac{d^3 \boldsymbol{k}_0}{(2\pi)^3} \boldsymbol{\delta}_1^2 \phi_{g\bar{g}}^{(n)}(\delta \boldsymbol{k}_1, \delta l(\omega)). \tag{5.102}$$

By inserting (5.93) in the above equation and integrating in the final solid angle we get

$$\omega \frac{dI(l)}{d\omega} = \frac{\alpha_s C_f}{\pi^2} \frac{l}{\delta l(\omega)} \int_0^{\pi} d\theta \sin(\theta) F(\theta) \Sigma_2(\delta \boldsymbol{k}_1, \delta l(\omega)), \tag{5.103}$$

where the momentum change in the coherence length is given by $|\delta \boldsymbol{k}| = 2\beta_k \omega \sin\theta$ and the function arising in the angular integration is given by

$$\begin{aligned} F(\theta) = \Big[& \frac{1 - \beta_k^2 \cos\theta}{2\beta_k \sin(\theta/2) \sqrt{1 - \beta_k^2 \cos^2(\theta/2)}} \\ & \times \log\Big[\frac{\sqrt{1 - \beta_k^2 \cos^2(\theta/2)} + \beta_k \sin(\theta/2)}{\sqrt{1 - \beta_k^2 \cos^2(\theta/2)} - \beta_k \sin(\theta/2)} \Big] - 1 \Big]. \end{aligned} \tag{5.104}$$

Equation (5.103) becomes exact in the frequency interval $\omega \ll \omega_s$ in which the gluon is able to resolve each of the internal scatterings, which is the totally incoherent superposition of η, the average number of collisions, single Bertsch-Gunion intensities, and in the interval $\omega \gg \omega_c$ in which the gluon is not able to resolve any of the internal scatterings, which is nothing but Weinberg's soft photon theorem [5] assuming always that $p_0^0 \gg \omega$ in all cases. For $\omega \gg \omega_c$ the phase (5.77) vanishes and the medium acts like a single scatterer with an equivalent charge the amount of matter contained in l, following a Bethe-Heitler law $1/\omega$. The radiation in this interval grows with the quotient between the gluon squared momentum change in l and its squared mass. This medium dependent term is a coherent plateau and dominates the radiation and the quark energy loss. The above qualitative behavior can be made quantitative for the Fokker-Planck approximation for (5.65). In that case (5.103) for $\omega \gg \omega_c$ has the asymptotic expressions

$$\lim_{l \to 0} \omega \frac{dI(l)}{d\omega} = \frac{2}{3\pi} \alpha_s C_f \frac{2\hat{q}\delta l}{m_g^2}, \qquad \lim_{l \to \infty} \omega \frac{dI(l)}{d\omega} = \frac{2}{\pi} \alpha_s C_f \left(\log\left(\frac{2\hat{q}\delta l}{m_g^2} \right) - (\gamma + 1) \right). \tag{5.105}$$

An expression interpolating between these two limits can be found

$$i_0(l) \equiv \omega \frac{dI}{d\omega} = \frac{2}{\pi} \alpha_s C_f \frac{1 + \eta}{3A + \eta} \log(1 + A\eta), \tag{5.106}$$

where $A = \exp(-1 - \gamma)$, γ is Euler's constant and $\eta = 2\hat{q}l/m_g^2$ is the average number of collisions in l. For frequencies below ω_c the gluon starts to resolve the internal structure of the scattering and radiation decouples into $l/\delta l(\omega)$ elements of equivalent charge the amount of matter in $\delta l(\omega)$. Then in this regime intensity, using (5.101), grows as $l/\delta l(\omega) \simeq \sqrt{\hat{q}/\omega}$ times a Bethe-Heitler law $1/\omega$ with a slow logarithmic charge decrease $\log(\delta l\omega) \sim \log(\omega/\hat{q})$. This enhancement stops at ω_s, where $\delta l(\omega)$ acquires the minimum average value $\lambda_{g\bar{g}}$ to produce a collision and thus radiation. This would consists in the totally incoherent sum of the $l/\lambda_{g\bar{g}}$ single emitters, but suppression due to a vanishing velocity β_k quickly cancels this enhancement.

In Figure 5.4 we show the angular integrated intensity of gluons of m_g = 0.45 GeV emitted from a high energy quark after performing a multiple collision with a medium with screening μ_d = 0.45 GeV and density $n_0 T_f$ = 8 fm^{-3}, which corresponds to a transport parameter $\hat{q}$ = 0.98 GeV^2/fm. From here onwards we choose α_s = 0.5. The numerical evaluation of (5.92) is shown for the Debye interaction (4.16) and its Fokker-Planck approximation. Also shown are the coherent plateaus $i_0(l)$ for both cases and the heuristic formula (5.103) in the Fokker-Planck approximation. As we can see the coherent plateaus of the numerical evaluations match the analytical expression (5.103) for both interactions. The heuristic formula provides a reasonable approximation to the LPM effect in the rest of the range. The results are shown for medium lengths of l = 1,2,3 and 5 fm, corresponding to $\eta \sim$ 10,20,30 and 50 collisions and characteristic frequencies $\omega_c \simeq$ 1, 4, 9 and 25 GeV, respectively. We see that although the shortest medium compromises 10 collisions the LPM effect is negligible since ω_c falls in the mass suppression zone. The enhancement from the coherent plateau starts to be noticeable for larger lengths.

In Figures 5.5 and 5.6 we show the results for two medium lengths l =1 and 5 fm, two medium densities $n_0 T_f$ = 1 and 8 fm^{-3}, and two gluon masses m_g = 0.15 and 0.45 GeV, for the Debye interaction and its Fokker-Planck approximation. We see that for the same parameters the Debye interaction produces more radiation than its Fokker-Planck approximation. This disagreement can be cast into a redefinition of $\hat{q}$ at the cost of making it dependent on the length l and the screening mass μ_d of the medium. We also notice that the enhancement zone is wider in the Debye interaction than in the Fokker-Planck approximation, starting for larger ω_c. This difference can be understood as the effect of the long-tail of the Debye interaction on $\hat{q}$ and thus on ω_c.

In Figure 5.7 we show the asymptotic emission intensity $i_0(l)$ for large ω, or coherent plateau, as a function of the medium length for two gluon masses m_g = 0.15 and 0.55 GeV and several medium screening masses μ_d, both for the Debye interaction and its Fokker-Planck approximation keeping the transport parameter $\hat{q}$ constant. As we can see the ratio of the two asymptotic intensities in the Debye interaction and its Fokker-Planck approximation is not constant so that one cannot recast the difference into a redefinition of $\hat{q}$ independently of the medium properties.

Finally, a third approach for evaluating (5.92) would consist in taking the

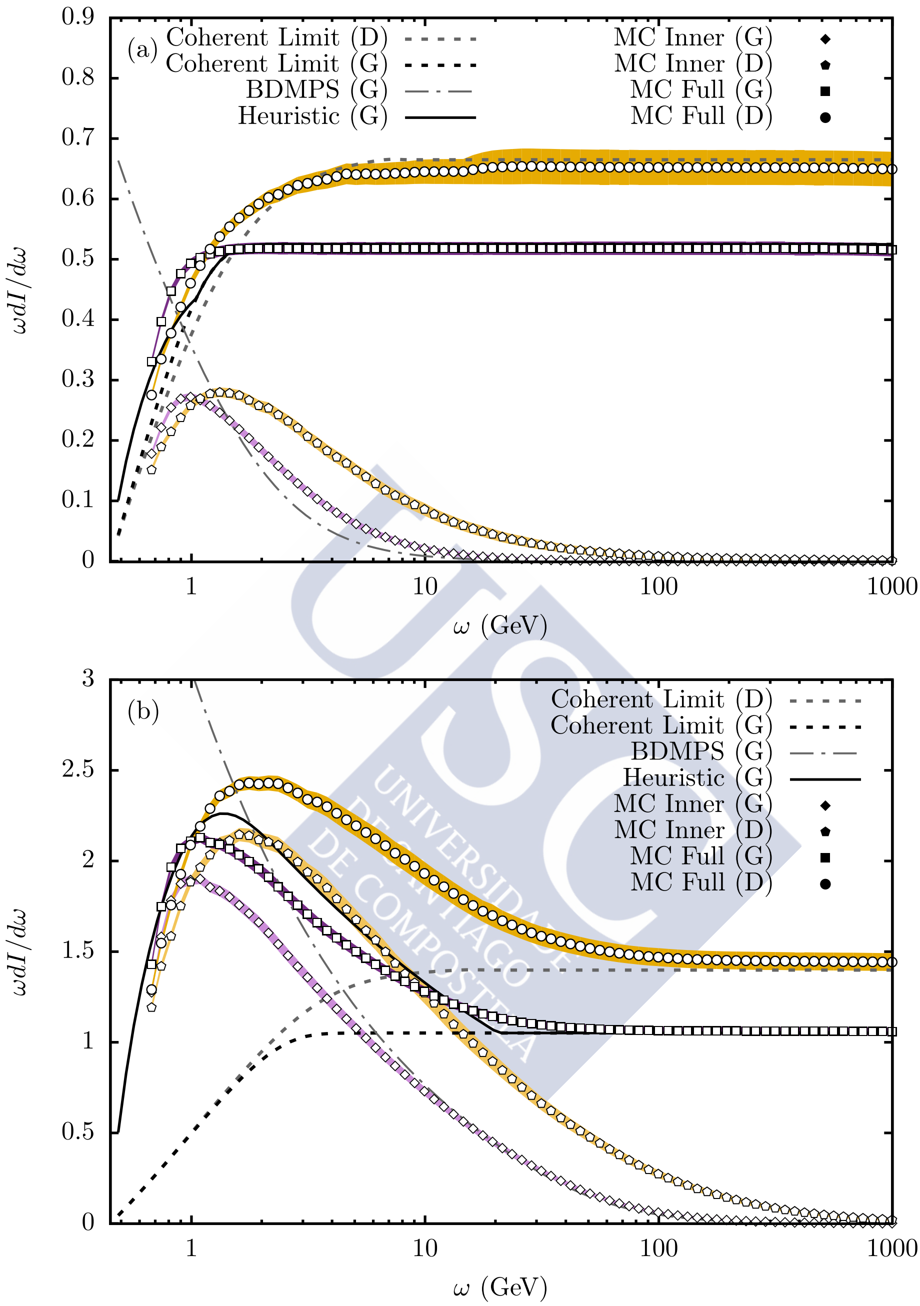


Figure 5.5: Intensity of gluons of m_g = 0.45 GeV as a function of the gluon energy [138] for a medium of $n_0 T_f = 8$ fm^{-3} corresponding to a transport parameter $\hat{q} = 0.98$ GeV^2/fm, and medium screening mass μ_d = 0.45 GeV. Results are shown for the Monte Carlo evaluation of (5.92) both for the Debye interaction (circles) and the Fokker-Planck approximation (squares). Also shown is our heuristic formula (3.130) (solid black line) and the coherent plateau of the Debye interaction (grey dashed line) and of the Fokker-Planck approximation (black dashed line). BDMPS result is also shown (dot-dashed line) and the evaluation of (5.92) only with the inner terms in the Debye (pentagons) and the Fokker-Planck (diamonds) interactions.

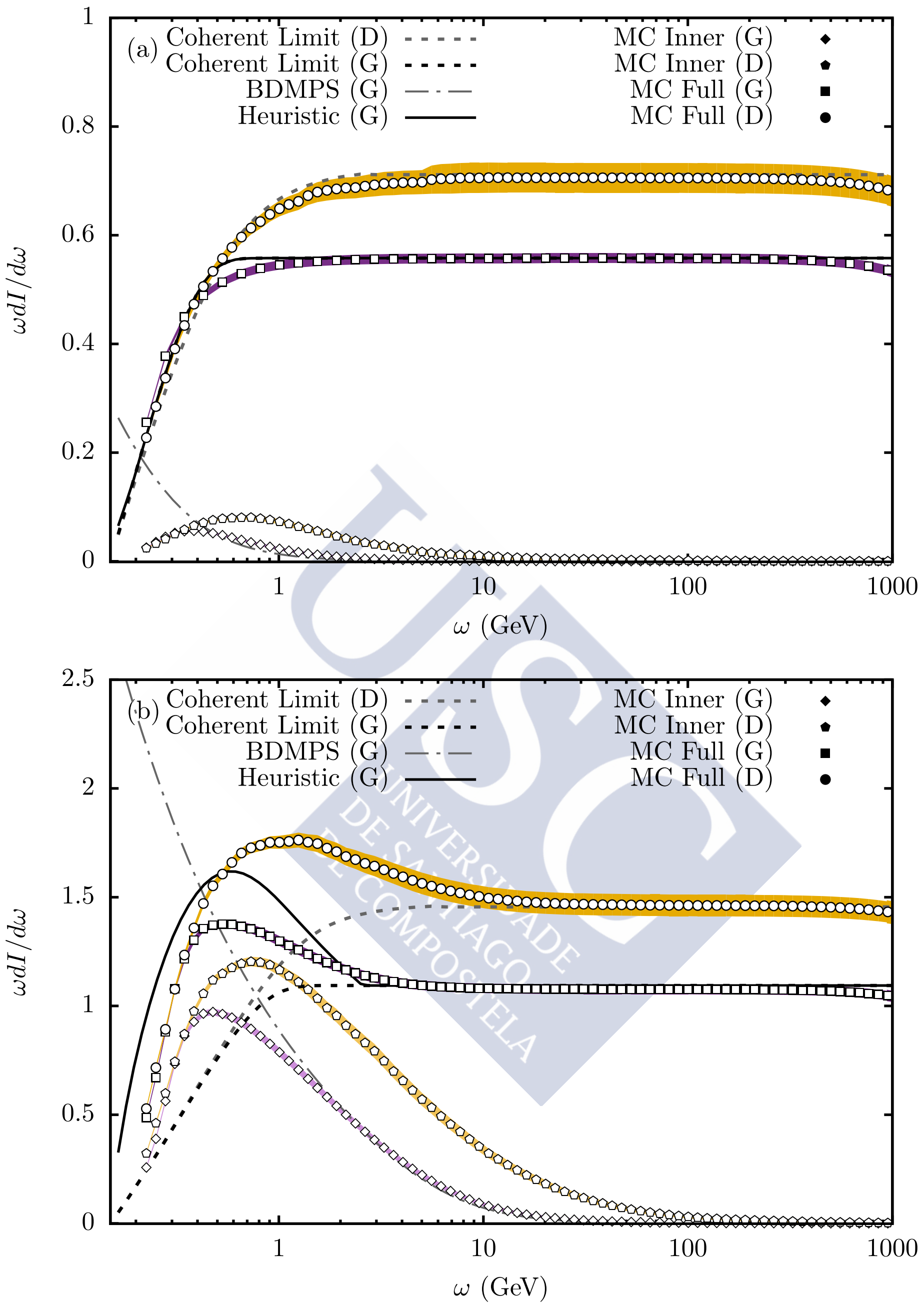


Figure 5.6: Intensity of gluons of $m_g = 0.15$ GeV as a function of the gluon energy [138] for a medium of $n_0 T_f = 1$ fm^{-3} corresponding to a transport parameter $\hat{q} = 0.12$ GeV^2/fm, and a medium screening mass of $\mu_d = 0.15$ GeV. Results are shown for the Monte Carlo evaluation of (5.92) both for the Debye interaction (circles) and the Fokker-Planck approximation (squares). Also shown is our heuristic formula (3.130) (solid black line) and the coherent plateau of the Debye interaction (grey dashed line) and of the Fokker-Planck approximation (black dashed line). BDMPS result is also shown (dot-dashed line) and the evaluation of (5.92) only with the inner terms in the Debye (pentagons) and the Fokker-Planck (diamonds) interactions.

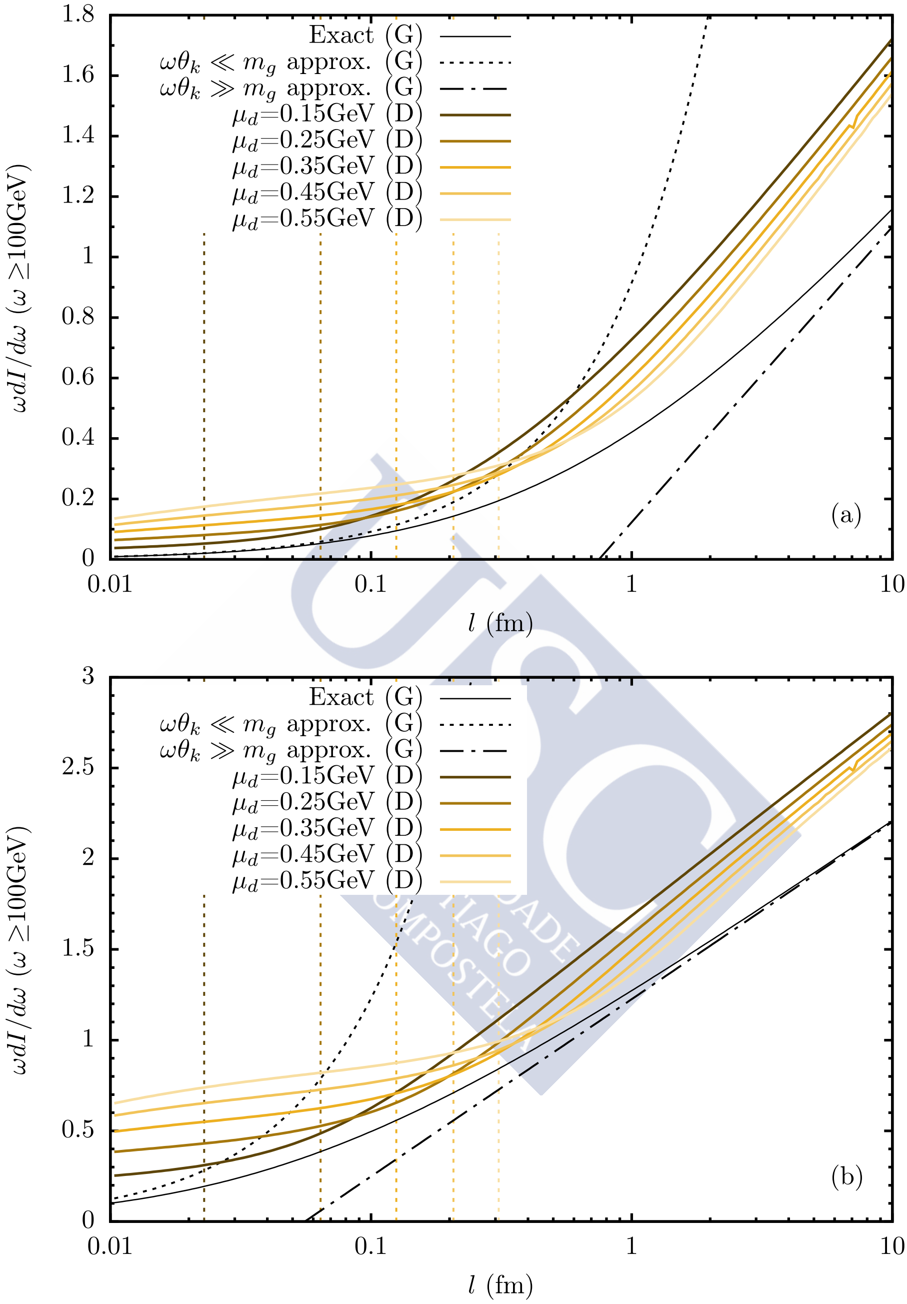


Figure 5.7: Asymptotic emission intensity $i_0(l)$ at as a function of the medium length [138] for a medium of density $n_0 T_f = 8$ fm^{-3} corresponding to a transport parameter of $\hat{q} = 0.98$ GeV^2/fm (α_s = 0.5). Results are shown (a) for a gluon mass of $m_g = 0.55$ GeV and (b) $m_g = 0.15$ GeV. Debye interaction is shown in solid lines for different screening masses, as marked, and the Fokker-Planck approximation is shown in solid black line. Also shown are the small and large medium approximation of the Fokker-Planck approximation. Vertical dashed lines mark the transition between large and small media for each screening mass $\eta = 1$.

$\delta z \to 0$. In that case a Boltzmann transport equation can be found [4, 22, 47] or an equivalent path integral in transverse coordinates [19, 20, 88]. In both cases the Fokker-Planck approximation produces a differential equation or a quadratic path integral, respectively, which are solvable. While this method leads to reasonably simple results, we have already shown that for finite size targets the differences between the Fokker-Planck approximation and the Debye interaction cannot be recast into a simple redefinition of the transport parameter $\hat{q}$ independently of the medium properties and the gluon energy. In QED the equivalence between the $\delta z \to 0$ limit of the intensity (3.122) and the path integral formalism has been proven at Section 3.4. We notice that QCD intensity (5.92) is equivalent to the QED intensity (3.122) if the electron role is played by a gluon moving in the opposite direction. Indeed by comparing the soft gluon QCD phase (5.77) with the QED phase (3.16) we have to replace $\omega/(p_0^0)^2 \to 1/\omega$ so that

$$\frac{\omega}{2p_0^0(p_0^0-\omega)}m_e^2 \to \frac{m_g^2}{2\omega}, \quad \Omega = \frac{1-i}{\sqrt{2}}\sqrt{\frac{\hat{q}\omega}{(p_0^0)^2}} \to \Omega_g = \frac{1-i}{\sqrt{2}}\sqrt{\frac{\hat{q}}{\omega}}. \tag{5.107}$$

Similarly the coupling of the gluon to the external field and the coupling of the squared emission vertex has to be replaced, respectively, as

$$(Ze^2)^2 \to g_s^4 T_f, \quad e^2 \to g_s^2 C_f. \tag{5.108}$$

Notice that the final passage of the gluon from z_j to l is affected by collisions, while the electron final momentum has been integrated out and is the initial passage from z_1 to z_k the one affected by collisions. Correspondingly, the integral in space has to be reversed so that

$$z = \infty \to z = -\infty, \quad z = 0 \to z = l \quad z = l \to z = 0. \tag{5.109}$$

With these replacements the QCD equivalents of (3.178) are straightforward to obtain. We give only here the required expressions for the semi-infinite length approximations of Migdal/Zakharov [4, 20] and the BDMPS group [22] for the angle integrated intensity. Using the replacement (5.109) the Migdal/Zakharov approximation (3.179) transforms to

$$\omega\frac{dI_{inc}^{(n)}}{d\omega d\Omega_k} \equiv \int_0^l\int_0^l + \int_l^\infty\int_l^\infty = \omega\frac{dI_d^{(n)}}{d\omega d\Omega_k} + \omega\frac{dI_b^{(n)}}{d\omega d\Omega_k}. \tag{5.110}$$

The final result of integrating in angle and longitudinal positions (5.110) can be directly written using (3.182) and prescriptions (5.107) and (5.108). We obtain in the $l \to \infty$ limit

$$\begin{aligned}\omega\frac{dI_d^{(n)}}{d\omega} + \omega\frac{dI_b^{(n)}}{d\omega} = l\frac{2}{\pi}g_s^2 C_f\frac{|\Omega|_g}{\sqrt{2}}\int_0^\infty dz\exp\left(-\frac{z}{\sqrt{2}s}\right) \\ \times\left(\sin\left(\frac{z}{\sqrt{2}s}\right)+\cos\left(\frac{z}{\sqrt{2}s}\right)\right)\left(\frac{1}{z^2}-\frac{1}{\sinh^2(z)}\right),\end{aligned} \tag{5.111}$$

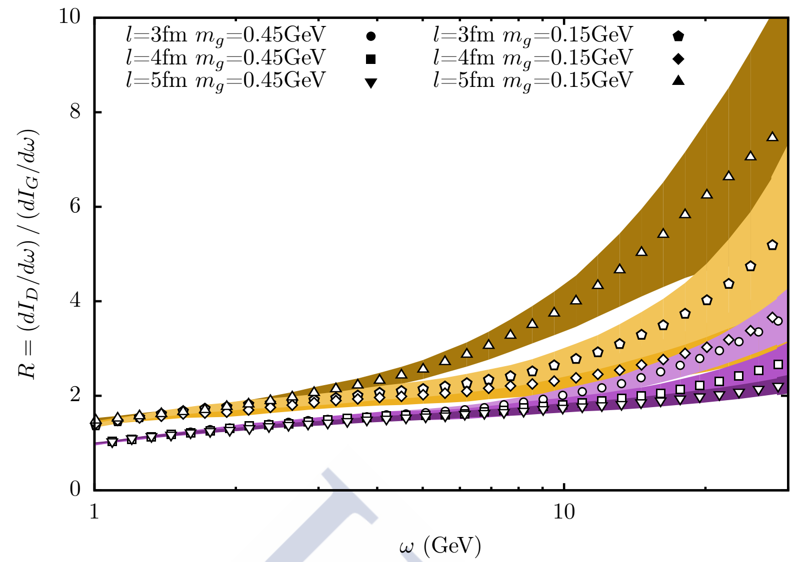


Figure 5.8: Ratio between the inner terms corresponding to the BDMPS prescription of (3.122) for the Debye interaction and its Fokker-Planck approximation for several lengths l and gluon masses m_g, as marked. Results are shown for a medium of $n_o T_f = 8$ fm^{-3} corresponding to a transport parameter $\hat{q} = 0.98$ GeV^2/fm. Debye screening is set to $\mu_d = m_g$ and coupling $\alpha_s = 0.5$.

where the parameter s is given for QCD by $s = 2\omega\Omega_g/m_g^2$, with Ω_g given at (5.107). Similarly using the prescription (5.109) the BDMPS approximation (3.186) transforms to

$$\omega\frac{dI^{(n)}_{inc}}{d\omega d\Omega_k} \equiv \left|\int_0^\infty\right|^2 - \left|\int_l^\infty\right|^2 = \int_0^l\int_0^l + 2\,\mathrm{Re}\int_{-\infty}^0\int_0^l \equiv \omega\frac{dI^{(n)}_d}{d\omega d\Omega_k} + \omega\frac{dI^{(n)}_e}{d\omega d\Omega_k}. \tag{5.112}$$

The final result of integrating in angle and longitudinal positions can be directly written using (3.191) and prescriptions (5.107) and (5.108), obtaining in the $m_g = 0$ case the BDMPS result

$$\omega\frac{dI^{(n)}_d}{d\omega} + \omega\frac{dI^{(n)}_e}{d\omega} = \frac{2g_s^2 C_f}{\pi}\,\mathrm{Re}\left[\log\left(\cos\left(\Omega_g L\right)\right)\right] \tag{5.113}$$

with Ω_g as before given at (5.107). In Figures 5.5 and 5.6 the BDMPS result (5.113) is shown together with our evaluation of the intensity (5.92) including only the inner gluons indicated by the BDMPS definition (5.112). Neither the BDMPS subset for (5.92) nor the BDMPS result itself (5.113) consider the emission in the coherent limit since the gluon emitted in the original quark direction and the last gluon are not included. Correspondingly, this prescription has to be considered only as an approximation for the radiation scenario in the transverse direction after a hard collision. Our results match the BDMPS result at high energies for the Fokker-Planck approximation, only with differences for the small l or big m_g limits, as expected. We observe that at low ω the kinematical restriction in the integration in $\boldsymbol{k}_t$ and the suppression effect due to the gluon mass make the intensity vanish. The Debye interaction still produces more radiation compared to its Fokker-Planck approximation, having a much slower fall off as $\sim 1/\omega$ for $\omega \gg \omega_c$ in contrast to the $1/\omega^2$ behavior of the BDMPS result. This can be seen in Figure 5.8 where we show the ratio between the intensity for the Debye interaction and its Fokker-Planck approximation for a medium of $n_0 T_f = 8$ fm^3 corresponding to a transport parameter of $\hat{q} = 0.98$ GeV^2/fm and two gluon masses and Debye screenings $m_g = \mu_d$ 0.15 and 0.45 GeV. The ratio is not constant and strongly depends on the mass and energy of the gluon, going from ~ 1 (~ 1.5) at low energies for $m_g = 0.45$ GeV ($m_g = 0.15$ GeV) to ~ 2.2 (~ 8) at larger energies. A single change in the $\hat{q}$ parameter can not fit the Debye results in all the range. A change of $\hat{q} \to 3.5\hat{q}$ fits the result for the case $n_0 T_f = 8$ fm^{-3}, a gluon/Debye mass of $\mu_d = 0.15$ GeV and large lengths above $l = 3$ fm, with an error of 20%. For a gluon/Debye mass of $\mu_d = 0.45$ GeV the scale factor is 2.8 instead, with an error of 40%. This enhancement of the realistic spectrum by a factor 3-4 times larger than the well known Fokker-Planck approximations [19–21, 91] for the same medium characteristics may suggest that single gluon total cross sections [147] much larger than the leading order expectations $\sigma^{(1)}_{g\bar{g}}(\mathbf{0})$ ~ 1.5 mb (~ 4 mb) with $\alpha_s = 0.3$ (0.5), supporting the idea of a strong coupled nature of the formed QGP, may not be required to obtain the medium transport parameters $\hat{q}$ favored by the data [67, 148].

Energy loss

For a quark coming from the infinity and going to the infinity, in a multiple soft collision scenario, energy loss is dominated by the large ω behavior of the intensity $i_0(l)$. This term comes from the emission amplitude of the first and last gluons emitted in the medium and produces an energy loss proportional to the initial quark energy, p_0^0. Multiple gluon emission can be taken into account by assuming independent emissions [149]. Let us define the probability density $f(l, \Delta)$ of having a total energy loss Δ at a distance l. The increment in the probability $f(l, \Delta)$ after traveling a distance δl is fed with the number of states having lost $\Delta - \omega$ at l and emptied with the number of states already with Δ at l, which leads to the transport equation

$$\frac{\partial f(l, \Delta)}{\partial l} = \int_0^{p_0^0} d\omega \frac{dI(l)}{d\omega dl}(f(l, \Delta - \omega) - f(l, \Delta)). \tag{5.114}$$

Equation (5.114) can be solved by defining the Laplace transform of $f(l, \Delta)$, we call $\varphi(l, p)$, which produces the transformed equation

$$\int_0^\infty f(l,\Delta)e^{-p\Delta} = \varphi(l,p) \to \frac{\partial\varphi(l,p)}{\partial l} = \varphi(l,p)\int_0^{p_0^0} d\omega \frac{dI(l)}{d\omega dl}(e^{-p\omega}-1), \quad (5.115)$$

which leads to the inverse Laplace transform

$$f(l,\Delta) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{+i\infty+\sigma} dp\, e^{p\Delta} \exp\Big(-\int_0^{p_0^0} \frac{dI(l)}{d\omega}\big(1-e^{-p\omega}\big)\Big). \quad (5.116)$$

This equation can be simplified if we assume that the spectrum can be well approximated as constant $i_0(l)$, in which case one finds the following expression suitable for a numerical evaluation

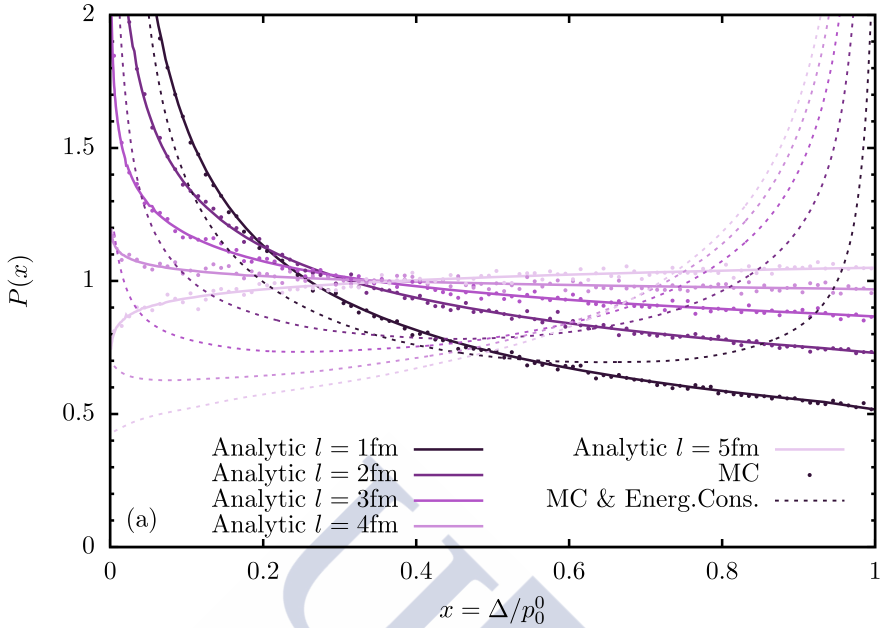
$$f(x,l) = \frac{1}{\pi}\int_0^\infty ds\ \exp\big(-i_0(l)C(s)\big)\ \cos\big(xs - i_0(l)S(s)\big), \quad (5.117)$$

where $x = \Delta/p_0^0$ is the total fraction of energy loss, the sum of the single energy losses $y = \omega/p_0^0$, and the functions

$$C(s) = \int_0^s du\frac{1-\cos(u)}{u}, \qquad S(s) = \int_0^s du\frac{\sin(u)}{u},$$

are related to the sine and cosine integrals. For finite quark energies, p_0^0, this probability can not be correct, since the hypothesis of independent gluon emission is not valid for energy losses $\Delta \sim p_0^0$ and (5.114) does not hold. On the other hand, for small energy losses the approximation should be valid. Therefore, one can take the probability distribution (5.117) and normalize it up to $x = 1$. This approximation, however, underestimates the energy loss, specially large energy loses are strongly underestimated. In order to circumvent the problem a different approach would be to simulate a Monte Carlo code which exactly takes into account energy conservation at each step. A Poisson process can be generated in this way which takes as input the single gluon intensity given at (5.92).

In Figure 5.9 we show the probability of losing a fraction of energy $x = \Delta/p_0^0$ assuming a constant spectrum given by the coherent plateau $i_0(l)$ for several medium lengths under the Debye interaction and its Fokker-Planck approximation. The evaluation of (5.117) assuming single independent emissions is shown together with the Monte Carlo evaluation of the Poisson process assuming energy conservation. In order to check the Monte Carlo evaluation the single independent emission has also been implemented in the simulation, exactly matching the evaluation of (5.117). For small $i_0(l)$ the multiple emission spectrum approaches the single emission spectrum, recovering the Bethe-Heitler power law $1/x^\alpha$ with



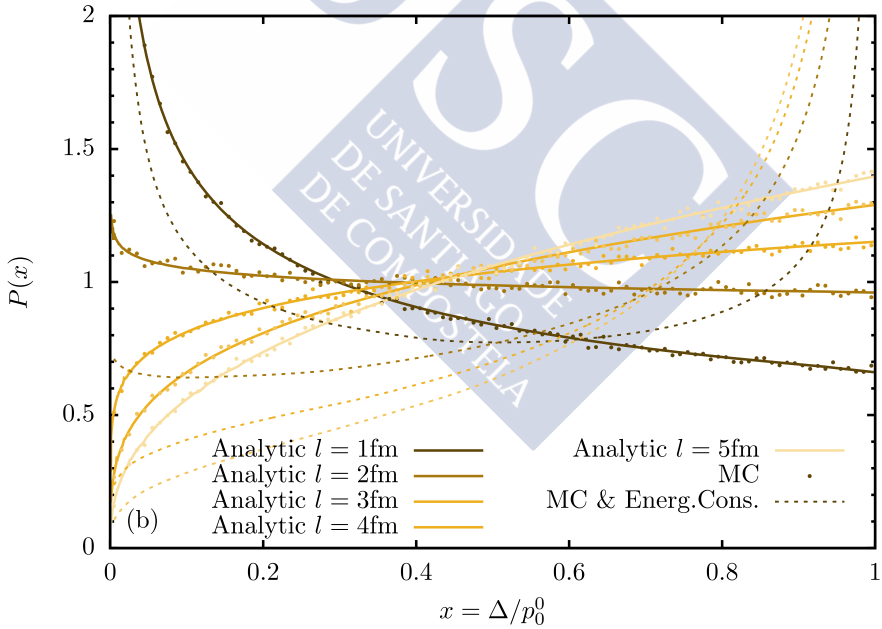


Figure 5.9: Probability of a fraction of energy loss $x \equiv \Delta/p_0^0$ for several medium lengths assuming the coherent plateau spectrum $i_0(l)$. Results are shown for a medium of density $n_0 T_f = 8$ fm^{-3}, corresponding to a transport parameter of $\hat{q} = 0.98$ GeV^2/fm, a gluon mass of $m_g = 0.45$ GeV and a screening of $\mu_d = 0.45$ GeV. Coupling is set to α_s =0.5. The Fokker-Planck approximation (a) and the Debye interaction (b) are shown for several medium lengths, as marked. Evaluation of (5.117) is shown in solid line, while the Monte Carlo evaluation of the Poisson process is shown assuming independent emissions (dots) or assuming dependent emissions/energy conservation (dashed lines).

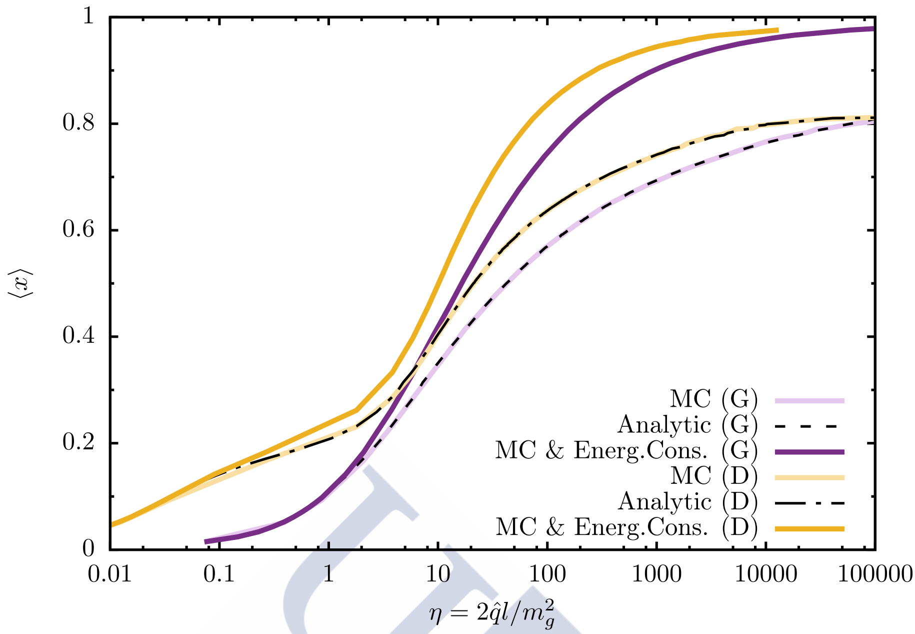


Figure 5.10: Average fraction of energy loss as a function of the average number of collisions assuming a multiple emission scenario for the coherent plateau intensity $i_0(l)$. Results are shown for a medium of $n_0 T_f = 8$ fm^{-3} corresponding to $\hat{q} = 0.98$ GeV^2/fm, $\alpha_s = 0.5$. Gluon mass and Debye screening are set to m_g=μ_d= 0.45 GeV^2/fm. Direct evaluation of the independent emission case is show for the Debye interaction (dot-dashed line) and its Fokker-Planck approximation (dashed line). Monte Carlo evaluation of the independent emission case is shown for the Debye interaction (light yellow) and for the Fokker-Planck approximation (light purple). Monte Carlo evaluation for the dependent emission scenario with energy conservation constraint is also shown for the Debye case (yellow) and the Fokker-Planck case (purple).

$\alpha = 1$. For larger $i_0(l)$ the multiple emission possibilities gradually enhance the probability of having larger energy losses, thus the average power α decreases. We observe that energy conservation constraints further enhance the probability of having larger energy losses.

In Figure 5.10 we show the average fraction of energy loss assuming the coherent plateau for the spectrum $i_0(l)$, both in the Debye interaction and its Fokker-Planck approximation, as a function of the average number of collisions. We observe that for the same parameters the average fraction of energy loss is larger for the Debye interaction than for the Fokker-Planck approximation, as expected from the results at (5.7), due to the long tail of the Debye potential. We also notice that for more than ~ 10 collisions the energy loss is substantial $\sim$ 60%.

In Figure 5.11 we show the average fraction of remaining energy as a function of the initial quark energy, assuming the full intensity (5.92). We observe that the LPM effect translates into a slight deviation from the constant energy loss plateau

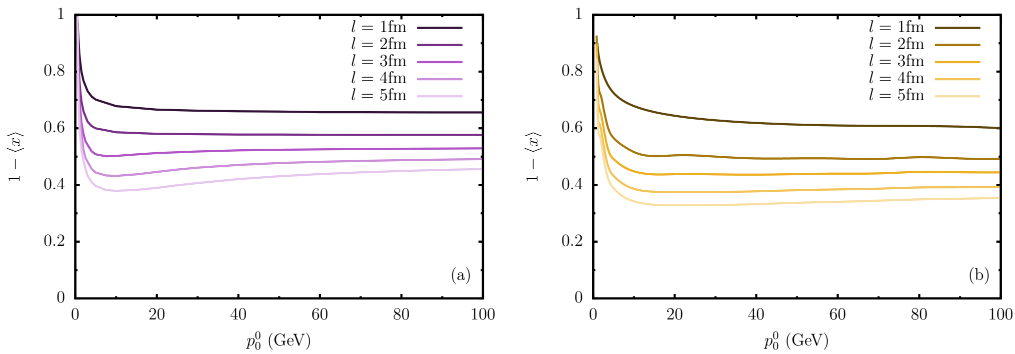


Figure 5.11: Average fraction of remaining energy $1 - \langle x \rangle$ as a function of the quark initial energy p_0^0 assuming the full spectrum given by (5.92), for a medium of density $n_0 T_f = 8$ fm^{-3} corresponding to a transport parameter of $\hat{q} = 0.98$ GeV^2/fm, a gluon mass of $m_g = 0.45$ GeV and a Debye screening of $\mu_d = 0.45$ GeV. Coupling is set to $\alpha_s = 0.5$. Both the Fokker-Planck approximation (a) and in the Debye interaction (b) are shown for several medium lengths, as marked.

caused by the coherent contribution $i_0(l)$ for low energy quarks. Quarks of energy near ω_c experiment larger energy losses than high energetic quarks. For energies approaching the gluon mass the average fraction of remaining energy approaches to 1 as expected due to bremsstrahlung suppression.

Conclusions

- A high energy approximation of the state of a fermion under a classical and static external field has been written. The state becomes ordered in the asymptotic initial direction, causing that sources placed at coordinate $x_3 < z_3$ affect the state of the particle at coordinate z_3. Due to this fact, beyond eikonal amplitudes preserve the internal scattering structure and longitudinal phases, then, cause interferences in the square of the amplitudes.

- The averaged squared scattering amplitude can be always split into a transverse coherent and a transverse incoherent contribution. The first encodes the diffractive and quantum effect of the medium transverse boundaries (non diagonal terms) and the second leads to the probabilistic interpretation of the multiple scattering (diagonal terms). An evolution, and a transport equation, can be written for both contributions. Moliere's result is recovered for $R \to \infty$. In this limit the averaged squared momentum change becomes additive in the traveled length for any interaction.

- The resulting elastic distributions for a Debye screened interaction with the medium lead to Rutherford tails of the form $1/\boldsymbol{q}^4$ which cannot be accounted for with a Fokker-Planck/Gaussian approximation. The averaged squared momentum change can be accounted at expenses of making the medium transport parameter $\hat{q}$ dependent, albeit slowly, on the particle energy and the screening.

- A formalism for the emission intensity in a multiple scattering scenario has been implemented and evaluated under a realistic interaction, with the angular dependence, and taking into account structured/finite targets and its effect on the photon dispersion relations. Results are in good agreement with the experimental data of SLAC and CERN, and show that Weinberg's soft photon theorem saturates the LPM suppression.

- Within the Fokker-Planck approximation our result recovers Migdal Zakharov result in the $l \to \infty$ limit and Wiedemann and Gyulassy results for finite l. The Fokker-Planck approximation, however, overestimates the angle-unintegrated spectrum at lower photon angles but underestimates it at large angles. For the integrated spectrum, the Fokker-Planck approximation requires a length l and frequency ω dependent definition, unless the very large limit of collisions is taken $\eta \geq 10^4$. This becomes critical for the energy loss estimations for particles of energies below p_0^{lpm}, i.e. total suppression.

- A QCD evaluation analogous to the QED scenario is found for the multiple scattering in a medium. The traceless color matrices lead to a higher order in α_s coherent contribution.

- A formalism for the intensity in a multiple scattering scenario has been implemented within the QCD formalism which admits an evaluation for a general interaction. The transverse coherent corrections may become relevant for mediums of R$\leq$ 4-5 fm. For $R \to \infty$ and with the adequate approximations for the color averaged gluon interactions, within the Fokker-Planck approximations, Zakharov result is recovered when $l \to \infty$, the BDMPS subset when $m_g \to 0$ and Salgado and Wiedemann results for the general case.

- Exact kinematical integration has been taken, causing corrections in the gluon in the soft regime. Soft gluon suppression is found due to mass gluon effects. The Fokker-Planck approximation underestimates the Debye interaction intensity and the difference can not be cast into a single definition of the medium transport properties through $\hat{q}$. In the BDMPS subset, Fokker-Planck approximation underestimates the radiation by a factor $\sim$ 3-4 which may suggest that larger gluon elastic cross sections, leading to the hypothesis of strongly coupled QGP, may not be required to match the data.

List of Figures

A

Spinors and polarizations

The spinor conventions used through this work are now going to be explicitely written, since in various cases of our interest some derivations going beyond the usual polarization and spin sum tricks are required [150]. A free solution to the Dirac equation is given by

$$\psi(x) = \mathcal{N}(p)u_s(p)e^{-ip\cdot x}, \tag{A.1}$$

where $\mathcal{N}(p)$ is some normalization and $u_s(p)$ a free spinor. In order to obtain the exact form of this spinor we write for the Dirac matrices

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \tag{A.2}$$

where $\boldsymbol{\sigma}$ are Pauli matrices. This equation can be easily solved for a free electron at rest. By calling $u_s(m, \mathbf{0})$ its solution we have

$$(p_\mu\gamma^\mu - m)u_s(m, \boldsymbol{p}) = (m\gamma^0 - m)u_s(m, \mathbf{0}) = 0. \tag{A.3}$$

We trivially obtain two orthogonal solutions, labeled with s=1,2 and given by

$$u_s(m, \mathbf{0}) = \begin{bmatrix} \varphi_s \\ 0 \end{bmatrix}, \; \varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{A.4}$$

In order to construct a free solution for an arbitrary p we observe

$$(p_\mu\gamma^\mu - m)(p_\nu\gamma^\nu + m)u_s(m, \mathbf{0}) = (p^2 - m^2)u_s(m, \mathbf{0}) = 0, \tag{A.5}$$

so, up to an overall constant, we have found the solution for $\boldsymbol{p} \neq 0$ in terms of the rest solution, indeed

$$u_s(m, \boldsymbol{p}) = (p_\mu\gamma^\mu + m)u_s(m, \mathbf{0}) = \begin{bmatrix} (p_0 + m)\varphi_s \\ (\boldsymbol{\sigma} \cdot \boldsymbol{p})\varphi_s \end{bmatrix}. \tag{A.6}$$

We will be using the normalization convention

$$\delta_{s_1 s_2} \equiv \bar{u}_{s_1}(p) u_{s_2}(p). \tag{A.7}$$

For our previous solutions we obtain

$$\begin{aligned} \bar{u}_{s_1}(m,p) u_{s_2}(m,p) &= \varphi^\dagger_{s_1}((p_0+m)^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2)\varphi_{s_2} \\ &= ((p_0+m)^2 - \boldsymbol{p}^2)\delta_{s_1 s_2} = m(m+2p_0)\delta_{s_1 s_2}, \end{aligned} \tag{A.8}$$

where we used the relations $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \boldsymbol{p}^2 + i(\boldsymbol{p}\times\boldsymbol{p})\cdot\boldsymbol{\sigma} = \boldsymbol{p}^2$ and $\varphi^\dagger_{s_1}\varphi_{s_2} = \delta_{s_1 s_2}$. This requires us to define the spinors in A.1 according to

$$u_s(p) = \sqrt{\frac{p_0+m}{2m}} \begin{bmatrix} \varphi_s \\ \frac{\sigma\cdot p}{p_0+m}\varphi_s \end{bmatrix}. \tag{A.9}$$

In order to obtain $\mathcal{N}(p)$ we require current conservation, that is, a free spinor has to remain normalized. This condition reads

$$\int d^4x\, \bar{\psi}_2(x)\gamma^0\psi_1(x) = (2\pi)^4\delta^4(p_2-p_1)\mathcal{N}(p_1)\mathcal{N}(p_2)\bar{u}_{s_2}(p_2)\gamma_0 u_{s_1}(p_1). \tag{A.10}$$

From this condition, taking the $p_0 \gg m$ limit in the arising normalization, one finds

$$\mathcal{N}(p) = \sqrt{\frac{m}{p_0}}, \qquad \mathcal{N}(p)u_s(p) \simeq \sqrt{\frac{1}{2}} \begin{bmatrix} \varphi_s \\ \frac{\sigma\cdot p}{p_0+m}\varphi_s \end{bmatrix}. \tag{A.11}$$

Once we explicitely know (A.1) we are in position to compute some required unpolarized cross sections. For a photon whose polarization vector is $\epsilon^\lambda_\mu(k)$ and $\boldsymbol{p}$ and $\boldsymbol{u}$ of modulus $\beta(p^0_0 - \omega)$, where β is the velocity, we evaluate the quantity

$$\begin{aligned} h_k(p,u) \equiv \frac{1}{2}\sum_{s_f s_i}\sum_{\lambda} &\sqrt{\frac{m}{p^0_0-\omega}}\bar{u}_{s_f}(p)\epsilon^\lambda_\mu(k)\gamma^\mu u_{s_i}(p+k)\sqrt{\frac{m}{p^0_0}} \\ &\times\left(\sqrt{\frac{m}{p^0_0-\omega}}\bar{u}_{s_f}(u)\epsilon^\lambda_\mu(k)\gamma^\mu u_{s_i}(u+k)\sqrt{\frac{m}{p^0_0}}\right)^*. \end{aligned} \tag{A.12}$$

where we sum over final spin and polarization and average over initial spins. A diagrammatic representation of this vertex is shown in Figure A.1. First we note that

$$\sum_{\lambda=1,2}\epsilon^\lambda_\mu(k)\epsilon^\lambda_\nu(k) = \delta_{ij} - \frac{k_i k_j}{\omega^2}, \qquad i,j = 1,2,3, \tag{A.13}$$

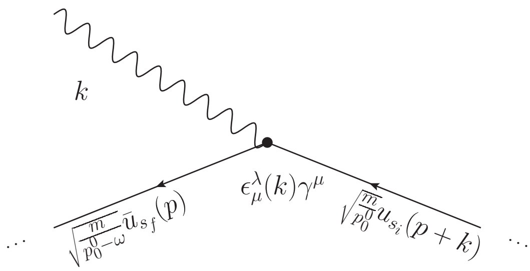


Figure A.1: Diagrammatic representation of the emission vertex.

so, if we place the z axis in the photon direction we take advantage of a further simplification

$$\sum_{\lambda=1,2} \epsilon_\mu^\lambda(k)\epsilon_\nu^\lambda(k) = \delta_{ij} - \delta_{i3}\delta_{j3}. \tag{A.14}$$

In this reference frame we only sum over transverse directions $i = 1, 2$ and then, sum over repeated spin indices assumed

$$\begin{aligned} h_k(p,u) = \frac{1}{2}\sum_{i=1,2} &\sqrt{\frac{m}{p_0^0-\omega}}\bar{u}_{s_f}(p)\gamma^i u_{s_i}(p+k)\sqrt{\frac{m}{p_0^0}} \\ &\times \sqrt{\frac{m}{p_0^0}}\bar{u}_{s_i}(u+k)\gamma^i u_{s_f}(u)\sqrt{\frac{m}{p_0^0-\omega}}. \end{aligned} \tag{A.15}$$

These spin sums can be done in the usual way or by directly using the explicit forms of the spinors, in order to obtain, using (A.11),

$$\sqrt{\frac{m}{p_0-\omega}}\bar{u}_{s_f}(p)\gamma^i u_{s_i}(p+k)\sqrt{\frac{m}{p_0}} = \frac{1}{2}\varphi_{s_f}^\dagger\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\boldsymbol{k})}{p_0+m} + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\sigma_i}{p_0-\omega+m}\right)\varphi_{s_i}, \tag{A.16}$$

and

$$\sqrt{\frac{m}{p_0}}\bar{u}_{s_i}(u+k)\gamma^i u_{s_f}(u)\sqrt{\frac{m}{p_0-\omega}} = \frac{1}{2}\varphi_{s_i}^\dagger\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot\boldsymbol{u}}{p_0-\omega+m} + \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{u}+\boldsymbol{k})\sigma_i}{p_0+m}\right)\varphi_{s_f}. \tag{A.17}$$

We then find a trace,

$$\begin{aligned} h_k(p,u) &= \frac{1}{2}\sum_{s_f,s_i}\frac{1}{4}\varphi_{s_f}^\dagger\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\boldsymbol{k})}{p_0+m} + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\sigma_i}{p_0-\omega+m}\right)\varphi_{s_i} \\ &\qquad \times \varphi_{s_i}^\dagger\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot\boldsymbol{u}}{p_0-\omega+m} + \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{u}+\boldsymbol{k})\sigma_i}{p_0+m}\right)\varphi_{s_f} \\ &= \frac{1}{8}\mathrm{Tr}\left\{\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot(\boldsymbol{p}+\boldsymbol{k})}{p_0+m} + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\sigma_i}{p_0-\omega+m}\right)\left(\frac{\sigma_i\boldsymbol{\sigma}\cdot\boldsymbol{u}}{p_0-\omega+m} + \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{u}+\boldsymbol{k})\sigma_i}{p_0+m}\right)\right\}. \end{aligned} \tag{A.18}$$

In order to compute the required four traces above we use the following relations satisfied by the Pauli matrices

$$\mathrm{Tr}\,(\sigma_i\sigma_j\sigma_k\sigma_l) = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{A.19}$$

We easily find, then for the first term

$$\frac{1}{(p_0+m)(p_0-\omega+m)}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\mathrm{Tr}(\sigma_i\sigma_j\sigma_i\sigma_k)u_j(p+k)_k = \frac{-4u_3(p+k)_3}{(p_0+m)(p_0-\omega+m)}, \tag{A.20}$$

for the second term,

$$\frac{1}{(p_0+m)(p_0-\omega+m)}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\mathrm{Tr}(\sigma_j\sigma_i\sigma_k\sigma_i)(u+k)_j p_k = \frac{-4p_3(u+k)_3}{(p_0+m)(p_0-\omega+m)}, \tag{A.21}$$

for the third term,

$$\frac{1}{(p_0+m)^2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\mathrm{Tr}(\sigma_j\sigma_i\sigma_i\sigma_k)(u+k)_j(p+k)_k = \frac{4}{(p_0+m)^2}(\boldsymbol{u}+\boldsymbol{k})\cdot(\boldsymbol{p}+\boldsymbol{k}), \tag{A.22}$$

and for the last term,

$$\frac{1}{(p_0-\omega+m)^2}\sum_{i=1}^{2}\sum_{j,k=1}^{3}\mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_i)u_j p_k = \frac{4}{(p_0-\omega+m)^2}\boldsymbol{u}\cdot\boldsymbol{p}. \tag{A.23}$$

So, by joining the four terms and expanding in transverse and longitudinal directions we simplify to

$$\begin{aligned} h_k(p,u) =& \frac{1}{2}\left(\frac{1}{p_0+m}(p+k)_3 - \frac{1}{p_0-\omega+m}p_3\right)\left(\frac{1}{p_0+m}(u+k)_3 - \frac{1}{p_0-\omega+m}u_3\right) \\ &+ \frac{1}{2(p_0+m)^2}(\boldsymbol{p}+\boldsymbol{k})_t\cdot(\boldsymbol{u}+\boldsymbol{k})_t + \frac{1}{2(p_0-\omega+m)^2}\boldsymbol{p}_t\cdot\boldsymbol{u}_t. \end{aligned} \tag{A.24}$$

Since the reference frame in which $\boldsymbol{k}$ points in the z direction has been chosen, then $\boldsymbol{k}_t = 0$, and $\boldsymbol{p}_t$ and $\boldsymbol{u}_t$ refers to the transverse components with respect to $\boldsymbol{k}$. In a general frame these components are given by

$$\boldsymbol{p}_t = \boldsymbol{p} - \left(\boldsymbol{p}\cdot\frac{\boldsymbol{k}}{\omega}\right)\frac{\boldsymbol{k}}{\omega}, \quad \boldsymbol{u}_t = \boldsymbol{u} - \left(\boldsymbol{u}\cdot\frac{\boldsymbol{k}}{\omega}\right)\frac{\boldsymbol{k}}{\omega}, \tag{A.25}$$

so that

$$(\boldsymbol{p}+\boldsymbol{k})_t\cdot(\boldsymbol{u}+\boldsymbol{k})_t=\boldsymbol{p}_t\cdot\boldsymbol{u}_t=\left(\boldsymbol{p}-\left(\boldsymbol{p}\cdot\frac{\boldsymbol{k}}{\omega}\right)\frac{\boldsymbol{k}}{\omega}\right)\cdot\left(\boldsymbol{u}-\left(\boldsymbol{u}\cdot\frac{\boldsymbol{k}}{\omega}\right)\frac{\boldsymbol{k}}{\omega}\right)$$
$$=\boldsymbol{p}\cdot\boldsymbol{u}-\frac{1}{\omega^2}(\boldsymbol{p}\cdot\boldsymbol{k})(\boldsymbol{u}\cdot\boldsymbol{k})=\left(\boldsymbol{p}\times\frac{\boldsymbol{k}}{\omega}\right)\cdot\left(\boldsymbol{u}\times\frac{\boldsymbol{k}}{\omega}\right). \quad (A.26)$$

The transverse contribution in $h_k(p,u)$ produces, then, the soft and classical result related then to spin no flip amplitudes

$$\sum_\lambda \epsilon^\lambda_\mu(k)\epsilon^\lambda_\nu(k)p^\mu u^\nu=(\boldsymbol{p}\times\hat{\boldsymbol{k}})\cdot(\boldsymbol{u}\times\hat{\boldsymbol{k}}), \quad (A.27)$$

times a kinematical hard photon correction we call $h^n(y)$, where $y=\omega/p_0^0$ is the fraction of energy carried by the photon,

$$\frac{1}{2}\frac{1}{(p_0+m)^2}(\boldsymbol{p}+\boldsymbol{k})_t\cdot(\boldsymbol{u}+\boldsymbol{k})_t+\frac{1}{2}\frac{1}{(p_0-\omega+m)^2}\boldsymbol{p}_t\cdot\boldsymbol{u}_t \quad (A.28)$$
$$\simeq\frac{1}{2}\frac{p_0^2+(p_0-\omega)^2}{p_0^2(p_0-\omega)^2}\left(\boldsymbol{p}\times\frac{\boldsymbol{k}}{\omega}\right)\cdot\left(\boldsymbol{q}\times\frac{\boldsymbol{k}}{\omega}\right)\equiv h_n(y)\left(\hat{\boldsymbol{p}}\times\frac{\boldsymbol{k}}{\omega}\right)\cdot\left(\hat{\boldsymbol{u}}\times\frac{\boldsymbol{k}}{\omega}\right),$$

and where we re-absorbed the modulus of $|\boldsymbol{p}|=\beta(p_0-\omega)$ and $|\boldsymbol{u}|=\beta(p_0-\omega)$ in $h_n(y)$, and in the high energy limit $p_0\gg m$ and β=1. Similarly for the photon longitudinal contribution, we observe, similarly taking β=1,

$$\frac{1}{p_0+m}(p+k)_z-\frac{1}{p_0+m-\omega}p_z\simeq\left(1-\frac{m}{p_0}\right)-\left(1-\frac{m}{p_0-\omega}\right)=\frac{m\omega}{p_0(p_0-\omega)} \quad (A.29)$$

so that we find a contribution which corrects the classical contribution only in the hard part of the spectrum

$$\frac{1}{2}\left(\frac{1}{p_0^0+m}(p+k)_z-\frac{1}{p_0^0-\omega+m}p_z\right)\left(\frac{1}{p_0^0+m}(u+k)_z-\frac{1}{p_0^0-\omega+m}u_z\right)$$
$$\simeq\frac{1}{2}\frac{m^2\omega^2}{p_0^2(p_0-\omega)^2}. \quad (A.30)$$

Finally, joining the two contributions we find

$$h_k(p,u)=h_n(y)\left(\hat{\boldsymbol{p}}\times\frac{\boldsymbol{k}}{\omega}\right)\cdot\left(\hat{\boldsymbol{u}}\times\frac{\boldsymbol{k}}{\omega}\right)+h_s(y), \quad (A.31)$$

where the kinematical weights are given by

$$h_n(y)=\frac{1}{2}\frac{(p_0^0)^2+(p_0^0-\omega)^2}{(p_0^0)^2}=1-y+\frac{1}{2}y^2,\ h_s(y)=\frac{1}{2}\left(\frac{m\omega}{p_0(p_0-\omega)}\right)^2\simeq\frac{1}{2}y^2, \quad (A.32)$$

where the hard limit has been taken $p_0 - \omega \simeq m$ in the $h_s(y)$ function. The above form of the hard corrections to the unpolarized and spin averaged current is the well known result [4, 6].

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