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Research Article

Approximating Solution of Fabrizio-Caputo Volterra's Model for Population Growth in a Closed System by Homotopy Analysis Method

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Volterra's model for population growth in a closed system consists in an integral term to indicate accumulated toxicity besides the usual terms of the logistic equation. Scudo in 1971 suggested the Volterra model for a population $u(t)$ of identical individuals to show crowding and sensitivity to "total metabolism": $du/dt = au(t) - bu^2(t) - cu(t)\int_0^t u(s)ds$. In this paper our target is studying the existence and uniqueness as well as approximating the following Caputo-Fabrizio Volterra's model for population growth in a closed system: ${}^{\mathrm{CF}}\mathscr{D}^{\alpha}u(t) = au(t) - bu^2(t) - cu(t)\int_0^t u(s)ds$, $\alpha \in [0,1]$, subject to the initial condition $u(0) = 0$. The mechanism for approximating the solution is Homotopy Analysis Method which is a semianalytical technique to solve nonlinear ordinary and partial differential equations. Furthermore, we use the same method to analyze a similar closed system by considering classical Caputo's fractional derivative. Comparison between the results for these two factional derivatives is also included.

1. Introduction

Malthus was the first economist to propose a systematic theory of population [1] where he gathered experimental data to support his thesis. He proposed the principle that human populations grow exponentially. Consequently, a nonlinear growth equation was introduced into population dynamics by Verhulst [2] to solve the unbounded growth in human population proposed by Malthus. Verhulst introduced the nonlinear term into the rate equation and reached what afterwards became nominated as the logistic equation:

$$\frac{du(t)}{dt} = ku(t)[1 - u(t)]. \tag{1}$$

Volterra's model for population growth in a closed system includes an integral term to indicate accumulated toxicity in addition to the usual terms of the logistic equation. Scudo, in 1971, indicates that Volterra proposed his following model for a population $u(t)$ of identical individuals which exhibits crowding and sensitivity to "total metabolism":

$$\frac{du}{dt} = au(t) - bu^2(t) - cu(t)\int_0^t u(s)\,ds. \tag{2}$$

If the integral term on the right is removed the famous logistic equation with birth rate a and crowding coefficient b appears. The last term contains the integral that indicates the "total metabolism" or total amount of toxins produced since time zero.

The individual death rate is corresponding to this integral, so the population death rate by virtue of toxicity must include a factor u. The existence of the toxic term as a result of the system being closed always causes the population level to fall to zero in the long run, as will be seen shortly. The relative size of the sensitivity to toxins, denoted by c, determines the manner in which the population thrives before its decay.

The tool for describing the behaviour of the equations such as logistic equation must be nonlocal differential equations in time. With this purpose, in the last decades, the Fractional Calculus (FC) allows investigating the nonlocal response of multiple phenomena [3–9]. Fractional derivatives are memory operators which usually represent dissipative effects or damage.

Some fundamental definitions in the context of FC are Erdelyi-Kober, Riesz, Riemann-Liouville, Hadamard, Graunwald-Letnikov, Weyl, Jumarie, or Caputo (see, e.g., [10–15] and references therein). The Riemann-Liouville definition brings about physically unacceptable initial conditions (fractional order initial conditions) [12]. In the Caputo concept, the initial conditions are expressed in terms of integer-order derivatives, so it has physical meaning [13]. These definitions have the disadvantage that their kernel has singularity; this kernel includes memory effects and therefore both definitions cannot precisely describe the full effect of the memory [16]. Due to this problem, Caputo and Fabrizio in [17] introduced a new definition of fractional derivative without singular kernel, the Caputo-Fabrizio (CF) fractional derivative; this derivative possesses very interesting properties which are reviewed in detail in [18]. Caputo and Fabrizio demonstrated that the new derivative involves further properties in comparison with the old version. They indicated, for instance, that it can interpret substance heterogeneities and configurations with different scales, which apparently cannot be investigated with the prominent local theories and also the old versions of fractional derivative. Another utilization is scrutinizing the macroscopic behaviours of some materials, identified with nonlocal communications between atoms. Other applications of the CF fractional derivative can be achieved, for example, in [19–21].

The main aims of this paper are studying the existence and uniqueness as well as approximating the Caputo-Fabrizio fractional population growth:

$$ {}^{\mathrm{CF}}\mathscr{D}^{\alpha}u(t) = au(t) - bu^{2}(t) - cu(t)\int_{0}^{t}u(s)\,ds \qquad \alpha \in [0,1], \tag{3}$$

subject to the initial condition

$$u(0) = 0. \tag{4}$$

Moreover, we shall compare the results with those obtained for the same problem by considering the classical Caputo fractional derivative in (3).

The auxiliary method used here for approximating the solution is a semianalytical technique known as Homotopy Analysis Method to solve nonlinear ordinary and partial differential equations.

Next, we introduce some basic definitions and notations that shall be used in next sections.

Definition 1 (see [22]). For at least n-times continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo fractional derivative of order $\sigma > 0$ is defined as

$$ {}^{c}D^{\sigma}h(t) = \frac{1}{\Gamma(n-\sigma)}\int_{0}^{t}(t-w)^{n-\sigma-1}g^{(n)}(w)\,dw, \tag{5}$$

where $n = [\sigma] + 1$.

By changing the kernel $(t-s)^{-a}$ by the function $\exp(-(\alpha/(1-\alpha))(t-s))$ and $1/\Gamma(1-\alpha)$ by $M(\alpha)/(1-\alpha)$, one obtains the new Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$, which has been recently introduced by Caputo and Fabrizio in [17]. More precisely, they introduced the following concept.

Definition 2 (see [17]). For $0 < \alpha < 1$, $h \in H^{1}(0,b)$, $b > 0$. The Caputo-Fabrizio fractional derivative is defined by

$$ {}^{\mathrm{CF}}\mathscr{D}^{\alpha}h(t) = \frac{M(\alpha)}{1-\alpha}\int_{0}^{t}e^{-(\alpha/(1-\alpha))(t-s)}h'(s)\,ds, \tag{6}$$

where $H^{1}(0,b)$ is the usual Sobolev space, the set of function in $L^{2}(0,b)$ such that the derivative, in the sense of distributions, is in $L^{2}(0,b)$. $M(\alpha)$ is a normalization constant ($M(1) = M(0) = 1$). For simplicity, we shall consider $M(\alpha) = 1$.

Lemma 3 (see [17]). *Let $0 < \alpha < 1$, and let us consider the following fractional differential equation:*

$$ {}^{\mathrm{CF}}\mathscr{D}^{\alpha}f(t) = u(t), \quad t \geq 0. \tag{7}$$

One can get

$$ f(t) = (1-\alpha)[u(t) - u(0)] + \alpha\int_{0}^{t}u(s)\,ds + f(0), \qquad t \geq 0. \tag{8}$$

2. Existence and Uniqueness

In this section, we aim to show the existence and uniqueness of solution of the fractional Volterra population growth model (3). By Lemma 3 and using initial conditions we obtain the following result.

Theorem 4. *If $(1-\alpha)L + \alpha TL < 1$, for $t \in J = [0,T]$ and some $L > 0$, then (3) has a unique solution which is zero solution.*

Proof. First, we have

$$ \begin{aligned} u(t) = {} & (1-\alpha)\left[au(t) - bu^{2}(t) - cu(t)\int_{0}^{t}u(s)\,ds\right] \\ & + \alpha\left[\int_{0}^{t}\left(au(s) - bu^{2}(s) - cu(s)\int_{0}^{s}u(x)\,dx\right)ds\right]. \end{aligned} \tag{9}$$

Let

$$ \Theta(t,u) = au(t) - bu^{2}(t) - cu(t)\int_{0}^{t}u(s)\,ds. \tag{10}$$

We have to find a positive real number L such that

$$ \left\|\Theta(t,u) - \Theta(t,v)\right\| \leq L\left\|u - v\right\|. \tag{11}$$

We obtain

$$\begin{aligned}\|\Theta(t,u)-\Theta(t,v)\| &\leq \| \; a(u(t)-v(t))+b\left(v^2(t)\right.\\&\left.-u^2(t)\right)+c\left(v(t)\int_0^t v(s)\,ds-u(t)\int_0^t u(s)\,ds\right)\|\\&\leq a\|u-v\|+b\|u+v\|\|u-v\|+c\left\|v(t)\int_0^t v(s)\,ds\right.\\&\left.\pm v(t)\int_0^t u(s)\,ds-u(t)\int_0^t u(s)\,ds\right\|\leq\|u-v\|\\&+b\|u+v\|\|u-v\|+c\|v\|\int_0^t\|u-v\|\,ds+c\|u-v\|\\&\cdot\int_0^t\|u\|\,ds.\end{aligned} \tag{12}$$

Due to the assumption that u and v are bounded, and $t \in [0,T]$, there exists a positive constant $\eta > 0$ such that $\|v\| \leq \eta$ and $\|u\| \leq \eta$. Thus, we get

$$\|\Theta(t,u)-\Theta(t,v)\| \leq (1+2\eta b+2c\eta T)\|u-v\|. \tag{13}$$

This shows the Lipschitz condition for Θ by taking $L = 1 + 2\eta b + 2c\eta T$. Regarding (9) we can get

$$\begin{aligned}u_0(t)&=t,\\u_n(t)&=(1-\alpha)\Theta(t,u_{n-1})+\alpha\int_0^t\Theta(t,u_{n-1})\,ds.\end{aligned} \tag{14}$$

Now, we set $F_n(t) = u_n(t) - u_{n-1}(t)$. So we have

$$\begin{aligned}F_n(t)&=(1-\alpha)\left(\Theta(t,u_{n-1})-\Theta(t,u_{n-2})\right)\\&+\alpha\int_0^t\left(\Theta(s,u_{n-1})-\Theta(s,u_{n-2})\right)ds.\end{aligned} \tag{15}$$

Taking the norm of the latter equation gives

$$\begin{aligned}\|F_n(t)\|&\leq(1-\alpha)\|\Theta(t,u_{n-1})-\Theta(t,u_{n-2})\|\\&+\alpha\int_0^t\|\Theta(s,u_{n-1})-\Theta(s,u_{n-2})\|\,ds\\&\leq(1-\alpha)L\|u_{n-1}-u_{n-2}\|\\&+\alpha L\int_0^t\|u_{n-1}-u_{n-2}\|\,ds\\&\leq(1-\alpha)L\|F_{n-1}\|+\alpha L\int_0^t\|F_{n-1}\|\,ds.\end{aligned} \tag{16}$$

Then, by using the recursive principle it yields

$$\|F_n(t)\| \leq \left[((1-\alpha)L)^n+(\alpha Lt)^n\right]u_0(t), \tag{17}$$

which proves that the solution exists and is continuous. We show that $\overline{u}(t)$ defined as

$$\overline{u}(t)=\lim_{n\to\infty}u_n(t), \tag{18}$$

which is a solution for (3). Let $Z_n(t) = \overline{u}(t) - u_n(t)$. We know Z_n should tend to zero as $n \to \infty$. We can see

$$\begin{aligned}\|\overline{u}(t)-u_{n+1}(t)\|&\leq(1-\alpha)\|\Theta(t,\overline{u})-\Theta(t,u_n)\|\\&+\alpha\int_0^t\|\Theta(s,\overline{u})-\Theta(s,u_n)\|\,ds\\&\leq(1-\alpha)L\|\overline{u}-u_n\|\\&+\alpha L\int_0^t\|\overline{u}-u_n\|\,ds\\&\leq(1-\alpha)L\|\overline{u}-u_n\|\\&+\alpha L\int_0^t\|\overline{u}-u_n\|\,ds\\&\leq(1-\alpha)L\|Z_n\|+\alpha L\int_0^t\|Z_n\|\,ds.\end{aligned} \tag{19}$$

So, the right hand side gives

$$\lim_{n\to\infty}u_n=\overline{u}. \tag{20}$$

We can take $\overline{u}$ as a solution of (3) that is continuous. Furthermore, applying the Lipschitz condition for Θ, we have

$$\begin{aligned}&\overline{u}(t)-(1-\alpha)(\Theta(t,\overline{u}))-\alpha\int_0^t\Theta(s,\overline{u})\,ds\\&=Z_n+(1-\alpha)\left(\Theta(t,\overline{u}_n)\right)+\alpha\int_0^t\Theta(s,\overline{u}_n)\,ds\\&-(1-\alpha)(\Theta(t,\overline{u}))-\alpha\int_0^t\Theta(s,\overline{u})\,ds.\end{aligned} \tag{21}$$

So

$$\begin{aligned}&\left\|\overline{u}(t)-(1-\alpha)(\Theta(t,\overline{u}))-\alpha\int_0^t\Theta(s,\overline{u})\,ds\right\|\\&\leq\|Z_n\|+(1-\alpha)\|\Theta(t,\overline{u}_n)-\Theta(t,\overline{u})\|\\&+\alpha\int_0^t\|\Theta(s,\overline{u}_n)-\Theta(s,\overline{u})\|\,ds\\&\leq\|Z_n\|+(1-\alpha)L\|\overline{u}_n-\overline{u}\|+\alpha tL\|\overline{u}_n-\overline{u}\|.\end{aligned} \tag{22}$$

Passing to the limit when $n \to \infty$ and regarding the initial condition, we obtain

$$\overline{u}(t)=(1-\alpha)(\Theta(t,\overline{u}))+\alpha\int_0^t\Theta(s,\overline{u})\,ds. \tag{23}$$

For uniqueness we consider u and v to be two different solutions of (3); then, the Lipschitz condition for Θ gives the following result:

$$\|u-v\|\leq(1-\alpha)L\|u-v\|+\alpha TL\|u-v\|, \tag{24}$$

rearranged to be

$$\|u-v\|(1-(1-\alpha)L-\alpha TL)\leq 0. \tag{25}$$

Then, $\|u-v\| = 0$ if

$$(1-\alpha) L + \alpha T L < 1. \tag{26}$$

On the other hand, the constant function equal to zero is solution of the equation. Thus, the proof is completed. □

3. Homotopy Analysis Method

The Homotopy Analysis Method operates the theory of the homotopy from topology to generate a convergent series solution for nonlinear systems.

The HAM was first constructed in 1992 by Liao in his Ph.D. Thesis [23] and later altered [24] in 1999 by proposing a nonzero auxiliary parameter, as the convergence-control parameter, h, to construct a homotopy on a differential system in general form [25]. The convergence-control parameter is a nonphysical variable that provides a simple way to certify convergence of a solution series. It is a series expansion method which is not directly dependent on small or large physical parameters. It also provides extreme flexibility to choose the basis functions of the desired solution and the corresponding auxiliary linear operator of the homotopy.

Now, by using HAM to approximate the solution of the model (3), in view of properties of Caputo-Fabrizio fractional integrals, we assume that $u(t)$ can be expressed by the functions

$$\left\{ t^{\alpha m \pm n} e^{(\alpha p \pm q)t} : m, n, p, q = 0, 1, 2, \ldots \right\}. \tag{27}$$

It is essential to know that we have a great freedom to choose auxiliary parameters in HAM. From base functions denoted by (27) and the initial condition given in (4), it is convenient to choose

$$u_0(t) = t, \tag{28}$$

as the initial approximation of $u(t)$, as well as

$$\mathscr{L}[\varphi] = {}^{\mathrm{CF}}\mathscr{D}^{\alpha}[\varphi], \tag{29}$$

as the auxiliary linear operator.

Let h denote a nonzero auxiliary parameter. We prepare the HAM deformation equation as

$$(1-q)\mathscr{L}[\Psi(t,q) - u_0(t)] = hq\mathscr{N}[\Psi(t,q)], \tag{30}$$

subject to the initial condition

$$\Psi(0,q) = 0, \tag{31}$$

where $q \in [0,1]$ is an embedding parameter and $\mathscr{N}[\Psi(t,q)]$ is a nonlinear operator, given by

$$\begin{aligned} \mathscr{N}[\Psi(t,q)] = {}& {}^{\mathrm{CF}}\mathscr{D}^{\alpha}[\Psi(t,q)] - a\Psi(t,q) \\ & + b\Psi^2(t,q) + c\Psi(t,q)\int_0^t \Psi(s,q)\,ds. \end{aligned} \tag{32}$$

Evidently, when $q = 0$, the solution of (30) is

$$\Psi(t,0) = u_0(t). \tag{33}$$

When $q = 1$, (30) is the same as the original equation (3), provided

$$\Psi(t,1) = u(t). \tag{34}$$

Thus, $\Psi(t,q)$ alters from the initial approximation $u_0(t)$ to the exact solution $u(t)$. Expanding Taylor's series with respect to q, we have

$$\Psi(t,q) = \Psi(t,0) + \sum_{k=1}^{\infty} u_k(t) q^k, \tag{35}$$

where

$$u_k(t) = \frac{1}{k!}\frac{\partial^k \Psi(t,q)}{\partial q^k}\bigg|_{q=0}. \tag{36}$$

For briefness, one can define the vector

$$\vec{u}_k = \{u_0, u_1, \ldots, u_k\}. \tag{37}$$

Differentiating the HAM deformation equation (30) k times with respect to q, then setting $q = 0$, and finally dividing them by $k!$, we attain the kth-order deformation equation

$$\mathscr{L}[u_k(t) - \chi_k u_{k-1}(t)] = hH(t)\mathscr{R}_k[\vec{u}_{k-1}], \tag{38}$$

subject to the initial condition

$$u_k(0) = 0, \tag{39}$$

where

$$\begin{aligned} \mathscr{R}_k[\vec{u}_{k-1}] = {}& {}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1} - au_{k-1} + b\sum_{i=0}^{k-1} u_i u_{k-1-i} \\ & + c\sum_{i=0}^{k-1} u_i \int_0^t u_{k-1-i}\,ds, \end{aligned} \tag{40}$$

$$\chi_k = \begin{cases} 0 & k = 1, \\ 1 & k > 1, \end{cases}$$

and $H(t) = 1$.

Next, our target is to reach a recursive equation to be implemented, for example, in MATLAB, in order to provide an approximation of the solution.

From (38) we have

$$\begin{aligned} {}^{\mathrm{CF}}\mathscr{D}^{\alpha}[u_k(t) - \chi_k u_{k-1}(t)] = h\Bigg[& {}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1} \\ & + b\left(\sum_{i=0}^{k-1} u_i u_{k-1-i}\right) + c\left(\sum_{i=0}^{k-1} u_i \int_0^t u_{k-1-i}\,ds\right) \\ & - au_{k-1}\Bigg], \end{aligned} \tag{41}$$

which implies that

$$\begin{aligned}&{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_k(t)-\chi_k\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}(t)\\&\quad=h\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}+hb\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\qquad+hc\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}ds\right)-hau_{k-1};\end{aligned}\tag{42}$$

then,

$$\begin{aligned}{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_k(t)&=(h+\chi_k)\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}(t)\\&\quad+hb\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+hc\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}ds\right)-hau_{k-1}.\end{aligned}\tag{43}$$

Now, by Lemma 3 we have

$$\begin{aligned}u_k(t)&=(1-\alpha)\Bigg[(h+\chi_k)\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}(t)\\&\quad+hb\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)+hc\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}dx\right)\\&\quad-hau_{k-1}\Bigg]+\alpha\int_0^t\Bigg[(h+\chi_k)\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}(s)\\&\quad+hb\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)+hc\left(\sum_{i=0}^{k-1}u_i\int_0^s u_{k-1-i}dx\right)\\&\quad-hau_{k-1}\Bigg]ds.\end{aligned}\tag{44}$$

Hence, we can conclude

$$\begin{aligned}u_k(t)&=(1-\alpha)h\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+c\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}dx\right)-au_{k-1}\Bigg]\\&\quad+\alpha h\int_0^t\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+c\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}dx\right)-au_{k-1}\Bigg]ds+(h\\&\quad+\chi_k)\left[(1-\alpha)\,{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}+\alpha\int_0^t{}^{\mathrm{CF}}\mathscr{D}^{\alpha}u_{k-1}(s)\,ds\right].\end{aligned}\tag{45}$$

By definition of the Caputo-Fabrizio fractional derivative, we get

$$\begin{aligned}u_k(t)&=(1-\alpha)h\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+c\left(\sum_{i=0}^{k-1}u_i\int_0^t u_{k-1-i}dx\right)-au_{k-1}\Bigg]\\&\quad+\alpha h\int_0^t\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+c\left(\sum_{i=0}^{k-1}u_i\int_0^s u_{k-1-i}dx\right)-au_{k-1}\Bigg]ds+(h\\&\quad+\chi_k)\Bigg[e^{-(\alpha/(1-\alpha))t}\int_0^t e^{(\alpha/(1-\alpha))s}u_{k-1}'(s)\,ds+\frac{\alpha}{1-\alpha}\\&\quad\cdot\int_0^t e^{-(\alpha/(1-\alpha))s}\int_0^s e^{(\alpha/(1-\alpha))w}u_{k-1}'(w)\,dw\,ds\Bigg].\end{aligned}\tag{46}$$

Now, using integration by parts as well as the initial condition we obtain

$$\begin{aligned}u_k(t)&=(1-\alpha)h\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)+c\Bigg(\sum_{i=0}^{k-1}u_i\\&\quad\cdot\int_0^t u_{k-1-i}dx\Bigg)-au_{k-1}\Bigg]\\&\quad+\alpha h\int_0^t\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)\\&\quad+c\left(\sum_{i=0}^{k-1}u_i\int_0^s u_{k-1-i}dx\right)-au_{k-1}\Bigg]ds+(h\\&\quad+\chi_k)\Bigg[e^{-(\alpha/(1-\alpha))t}\Bigg(e^{(\alpha/(1-\alpha))t}u_{k-1}(t)-\frac{\alpha}{1-\alpha}\\&\quad\cdot\int_0^t e^{(\alpha/(1-\alpha))s}u_{k-1}(s)\,ds\Bigg)+\frac{\alpha}{1-\alpha}\\&\quad\cdot\int_0^t e^{-(\alpha/(1-\alpha))s}\Bigg(e^{(\alpha/(1-\alpha))s}u_{k-1}(s)\\&\quad-\frac{\alpha}{1-\alpha}\int_0^w e^{(\alpha/(1-\alpha))w}u_{k-1}(w)\,dw\Bigg)ds\Bigg].\end{aligned}\tag{47}$$

So,

$$\begin{aligned}u_k(t)&=(1-\alpha)h\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)+c\Bigg(\sum_{i=0}^{k-1}u_i\\&\quad\cdot\int_0^t u_{k-1-i}dx\Bigg)-au_{k-1}\Bigg]\\&\quad+\alpha h\int_0^t\Bigg[b\left(\sum_{i=0}^{k-1}u_iu_{k-1-i}\right)+c\Bigg(\sum_{i=0}^{k-1}u_{\mathrm{i}}\end{aligned}$$

$$\cdot \int_0^s u_{k-1-i} dx \Big) - a u_{k-1} \Big] ds + (h + \chi_k) \cdot \Big[\Big(u_{k-1}(t) - \frac{\alpha}{1-\alpha} \cdot e^{-(\alpha/(1-\alpha))t} \int_0^t e^{(\alpha/(1-\alpha))s} u_{k-1}(s)\, ds \Big) + \frac{\alpha}{1-\alpha} \cdot \int_0^t \Big(u_{k-1}(s) - \frac{\alpha}{1-\alpha} \cdot e^{-(\alpha/(1-\alpha))s} \int_0^s e^{(\alpha/(1-\alpha))w} u_{k-1}(w)\, dw \Big) ds \Big]. \tag{48}$$

Then, we realize

$$u_k(t) = (1-\alpha) hb \left(\sum_{i=0}^{k-1} u_i u_{k-1-i} \right) + (1-\alpha) \cdot hc \left(\sum_{i=0}^{k-1} u_i \int_0^t u_{k-1-i} dx \right) - [(1-\alpha) ha] u_{k-1} + \alpha hb \int_0^t \left(\sum_{i=0}^{k-1} u_i u_{k-1-i} \right) ds + \alpha hc \int_0^t \left(\sum_{i=0}^{k-1} u_i \int_0^s u_{k-1-i} dx \right) ds - \alpha ha \int_0^t u_{k-1} ds + (h+\chi_k) u_{k-1}(t) - \frac{(h+\chi_k)\alpha}{1-\alpha} \cdot e^{-(\alpha/(1-\alpha))t} \int_0^t e^{(\alpha/(1-\alpha))s} u_{k-1}(s)\, ds + \frac{(h+\chi_k)\alpha}{1-\alpha} \cdot \int_0^t u_{k-1}(s)\, ds - (h+\chi_k) \left(\frac{\alpha}{1-\alpha} \right)^2 \cdot \int_0^t e^{-(\alpha/(1-\alpha))s} \int_0^s e^{(\alpha/(1-\alpha))w} u_{k-1}(w)\, dw\, ds. \tag{49}$$

Finally, we have the following equation:

$$u_k(t) = (1-\alpha) hb \left(\sum_{i=0}^{k-1} u_i u_{k-1-i} \right) + [(h+\chi_k) - (1-\alpha) ha] u_{k-1} + (1-\alpha) \cdot hc \left(\sum_{i=0}^{k-1} u_i \int_0^t u_{k-1-i} dx \right) + \alpha hc \int_0^t \left(\sum_{i=0}^{k-1} u_i \int_0^s u_{k-1-i} dx \right) ds + \alpha hb \int_0^t \left(\sum_{i=0}^{k-1} u_i u_{k-1-i} \right) ds + \left(\frac{(h+\chi_k)\alpha}{1-\alpha} - \alpha ha \right) \int_0^t u_{k-1} ds - \frac{(h+\chi_k)\alpha}{1-\alpha} \cdot e^{-(\alpha/(1-\alpha))t} \int_0^t e^{(\alpha/(1-\alpha))s} u_{k-1}(s)\, ds - (h+\chi_k) \cdot \left(\frac{\alpha}{1-\alpha} \right)^2 \cdot \int_0^t e^{-(\alpha/(1-\alpha))s} \int_0^s e^{(\alpha/(1-\alpha))w} u_{k-1}(w)\, dw\, ds. \tag{50}$$

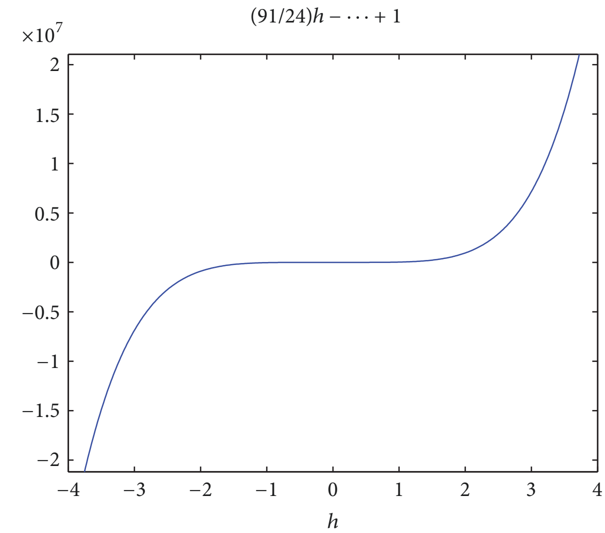

FIGURE 1: The h-curves obtained from the 6-order HAM approximate solution for Fabrizio-Caputo logistic equation.

As suggested by Liao [25], the appropriate region for h is a horizontal line segment. We can investigate the influence of h on the convergence of u, by plotting the curve of it versus h, as shown in Figure 1. It seems that the best choice can be in the region $[-1, 1]$. We can also see the results of the 8th order analytical approximations for $u(t)$ given by HMA for Fabrizio-Caputo differential equation in Figure 2.

Xu [26] has also studied the Caputo form of this equation for nonzero initial conditions. The result for this equation with zero initial conditions is as follows:

$$u_k(t) = (h+\chi_k) u_{k-1} - (h+\chi_k) \sum_{i=0}^{n-1} u_{k-1}^{(i)}(0) \frac{t^i}{i!} + hJ^{\alpha} \left[b \sum_{i=0}^{k-1} u_i u_{k-1-i} - a u_{k-1} + c \sum_{i=0}^{k-1} u_i \int_0^t u_{k-1-i} ds \right]. \tag{51}$$

As shown in Figure 3, it seems that the best choice for h can be in the region $[-1, 1]$. The 8th order analytical approximations for $u(t)$ given by HMA for the differential equation by considering the classical Caputo fractional derivative is shown

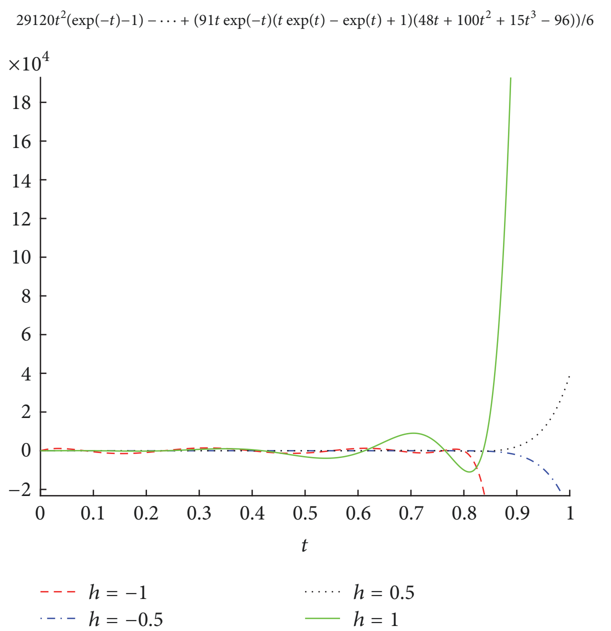

FIGURE 2: The 8th order analytical approximations for $u(t)$ given by HMA for Fabrizio-Caputo logistic differential equation.

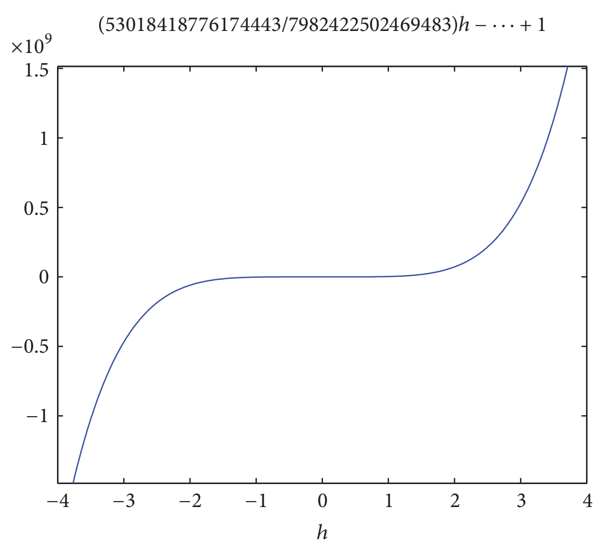

FIGURE 3: The h-curves obtained from the 6th-order HAM approximate solution for Caputo logistic equation.

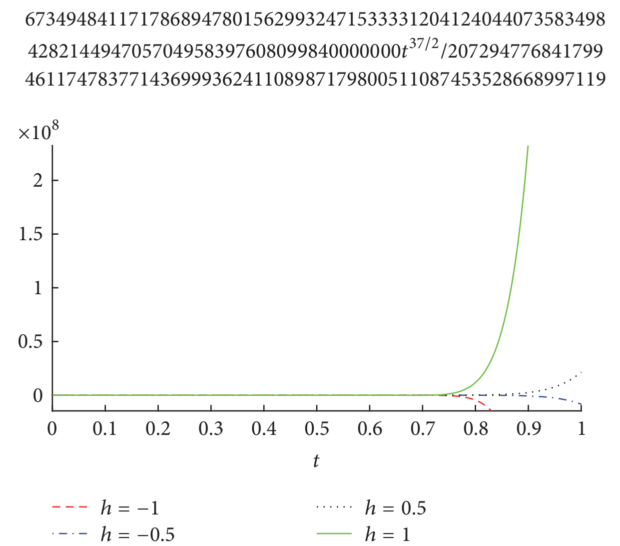

FIGURE 4: The 8th order analytical approximations for $u(t)$ given by HMA by considering the classical Caputo fractional derivative.

in Figure 4. This approximation is for $a = b = c = 10$ and $\alpha = 0.5$. We also get here $u_0(t) = t$.

As a result, we can see that the solutions are similar with more oscillation in the case of Caputo-Fabrizio type of the population growth.

4. MATLAB Codes

Here, we prepare codes written by MATLAB related to plotting the figures you can see as the results of approximation by HAM.

MATLAB code for Figure 1:

```
clc; clear all; close all;
syms t u x s h z
q = 0.5;
a = 10;
b = 10;
c = 10;
for j = 2 : 6
if j == 2
z(j) = 0
else z(j) = 1
end
end
for j = 2 : 6
u(1) = t;
g = 0
for i = 1 : (j − 1)
f = u(i) ∗ u(j − i)
g = g + f;
end
l = 0
for i = 1 : (j − 1)
m = u(i) ∗ int(u(j − i), 0, t)
l = l + m;
end
u(j) = ((h + z(j) − ((1 − q) ∗ h ∗ a)) ∗ u(j − 1))
+ (((1 − q) ∗ h ∗ a) ∗ l) + ((q ∗ h ∗ c) ∗ int(l, 0, t))
+ (((((h + z(j)) ∗ q)/(1 − q)) − (q ∗ h ∗ a)) ∗
int(u(j − 1), 0, t)) + ((q ∗ h ∗ b) ∗ int(g, 0, t)) −
((((h + z(j)) ∗ q)/(1 − q)) ∗ exp(−(q/(1 − q)) ∗
t) ∗ int(exp((q/(1 − q)) ∗ s) ∗ u(j − 1), s, 0, t)) +
```

```
(((1-q)*h*b)*g)-((h+z(j))*((q/(1-q))^2)*
int(exp(-(q/(1-q))*s)*(int(exp((q/(1-q))*
x)*u(j-1),x,0,s)),s,0,t))
end
usum = t
for
j = 2 : 6
usum = usum + u(j)
end
syms p o
p(t) = usum
o(h) = p(1)
ezplot(o(h),[-4,4])
```

MATLAB code for Figure 2:

```
clc; clear all; close all;

syms t u x s h z
q = 0.5;
a = 10;
b = 10;
c = 10;
for j = 2 : 8
if j == 2
z(j) = 0
else z(j) = 1
end
end
for j = 2 : 8
u(1) = t;
g = 0
for i = 1 : (j-1)
f = u(i)*u(j-i)
g = g + f;
end
l = 0
for i = 1 : (j-1)
m = u(i)*int(u(j-i),0,t)
l = l + m;
end
u(j) = ((h+z(j)-((1-q)*h*a))*u(j-1))
+ (((1-q)*h*c)*l) + ((q*h*c)*int(l,0,t))
+ (((((h+z(j))*q)/(1-q)) - (q*h*a))*
int(u(j-1),0,t)) + ((q*h*b)*int(g,0,t)) -
((((h+z(j))*q)/(1-q))*exp(-(q/(1-q))*
t)*int(exp((q/(1-q))*s)*u(j-1),s,0,t)) +
(((1-q)*h*b)*g)-((h+z(j))*((q/(1-q))^2)*
int(exp(-(q/(1-q))*s)*(int(exp((q/(1-q))*
x)*u(j-1),x,0,s)),s,0,t))
end
usum = t
for j = 2 : 8
usum = usum + u(j)
end
syms p
p(h) = usum;
figure;
hold on;
p1 = ezplot(p(-1),[0,1])
p2 = ezplot(p(-0.5),[0,1])
p3 = ezplot(p(0.5),[0,1])
p4 = ezplot(p(1),[0,1])
set (p1, "color", "r", "linestyle", "--")
set (p2, "color", "b", "linestyle", "-.")
set (p3, "color", "k", "linestyle", ":")
set (p4, "color", "g", "linestyle", "-")
legend ("h = -1", "h = -0.5", "h = 0.5", "h =
1")
```

MATLAB code for Figure 3:

```
clc; clear all; close all;

syms t u x s h z n
q = 0.5; a = 10; n = 1; b = 10; c = 10;
u(1) = t;
for j = 2 : 4
if j == 2
z(j) = 0
else z(j) = 1
end
end
for j = 2 : 4
g = 0
for i = 1 : (j-1)
f = u(i)*u(j-i)
g = g + f
end
l = 0
for i = 1 : (j-1)
m = u(i)*int(u(j-i),0,t)
l = l + m
end
w = 0
for i = 1 : n
r(t) = diff(u(j-1),i-1)
y(i) = (r(0)*(t(i-1)))/(factorial(i-1))
w = w + y(i)
end
u(j) = ((h+z(j))*u(j-1)) - ((h+z(j))*
w) + (((h*b)/gamma(q))*int((power((t-
s),(q-1)))*g,s,0,t)) - (((h*a)/gamma(q))*
int((power((t-s),(q-1)))*u(j-1),s,0,t)) +
(((h*c)/gamma(q))*int((power((t-s),(q-
1)))*l,s,0,t))
```

```
end
v = t
for j = 2 : 4
v = v + u(j)
end
syms p o
p(t) = v
o(h) = p(1)
ezplot(o(h), [−4, 4])
```

MATLAB code for Figure 4:

clc; clear all; close all;

```
syms t u x s h z n
q = 0.5; a = 10; n = 1; b = 10; c = 10;
u(1) = t;
for j = 2 : 8
if j == 2
z(j) = 0
else z(j) = 1
end
end
for j = 2 : 8
g = 0
for i = 1 : (j − 1)
f = u(i) ∗ u(j − i)
g = g + f
end
l = 0
for i = 1 : (j − 1)
m = u(i) ∗ int(u(j − i), 0, t)
l = l + m
end
w = 0
for i = 1 : n
r(t) = diff(u(j − 1), i − 1)
y(i) = (r(0) ∗ (t(i − 1)))/(factorial(i − 1))
w = w + y(i)
end
u(j) = ((h + z(j)) ∗ u(j − 1)) − ((h + z(j)) ∗ w) + (((h ∗ b)/gamma(q)) ∗ int((power((t − s), (q − 1))) ∗ g, s, 0, t)) − (((h ∗ a)/gamma(q)) ∗ int((power((t − s), (q − 1))) ∗ u(j − 1), s, 0, t)) + (((h ∗ c)/gamma(q)) ∗ int((power((t − s), (q − 1))) ∗ l, s, 0, t))
end
v = t
for j = 2 : 8
v = v + u(j)
end
syms p
p(h) = v;
figure;
hold on;
p1 = ezplot(p(−1), [0, 1])
p2 = ezplot(p(−0.5), [0, 1])
p3 = ezplot(p(0.5), [0, 1])
p4 = ezplot(p(1), [0, 1])
set (p1, "color", "r", "linestyle", "−−")
set (p2, "color", "b", "linestyle", "−.")
set (p3, "color", "k", "linestyle", ":")
set (p4, "color", "g", "linestyle", "−")
legend ("h = −1", "h = −0.5", "h = 0.5", "h = 1")
```

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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