

# SPACES OF LOCALLY HOMOGENEOUS AFFINE SURFACES

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**ABSTRACT.** We examine the topology of various spaces of locally homogeneous affine surfaces which arise from the classification result of Opozda [11] as orbits of the action of  $GL(2, \mathbb{R})$  (Type  $\mathcal{A}$ ) and the  $ax + b$  group (Type  $\mathcal{B}$ ). We determine the topology of the spaces of Type  $\mathcal{A}$  models in relation to the rank of the Ricci tensor. We determine the topology of the spaces of Type  $\mathcal{B}$  models which either are flat or where the Ricci tensor is alternating.

## 1. INTRODUCTION

**1.1. Notational conventions.** An *affine surface* is a pair  $\mathcal{M} = (M, \nabla)$  where  $M$  is a smooth surface and where  $\nabla$  is a torsion free connection on the tangent bundle of  $M$ . Let  $x = (x^1, x^2)$  be a system of local coordinates on  $M$ . Adopt the *Einstein convention* and sum over repeated indices to express  $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}^k \partial_{x^k}$ . The *Christoffel symbols*  $\Gamma = \{\Gamma_{ij}^k\}$  determine the connection in the coordinate chart. Let  $\rho$  be the associated Ricci tensor. The Ricci tensor carries the geometry in dimension 2; an affine surface is flat if and only if  $\rho = 0$ . Since the Ricci tensor of an affine manifold is not necessarily symmetric, let  $\rho_s(X, Y) = \frac{1}{2}\{\rho(X, Y) + \rho(Y, X)\}$  and  $\rho_a(X, Y) = \frac{1}{2}\{\rho(X, Y) - \rho(Y, X)\}$  be the *symmetric* and *alternating* Ricci tensors.

**1.2. Locally homogeneous affine surface geometries.** Work of Opozda [11] shows that any locally homogeneous affine surface  $\mathcal{M}$  is modeled on one of the following geometries.

- **Type  $\mathcal{A}$ .**  $\mathcal{M} = (\mathbb{R}^2, \nabla)$  with constant Christoffel symbols  $\Gamma_{ij}^k = \Gamma_{ji}^k$ . This geometry is homogeneous; the Type  $\mathcal{A}$  connections are the left invariant connections on the Lie group  $\mathbb{R}^2$ .
- **Type  $\mathcal{B}$ .**  $\mathcal{M} = (\mathbb{R}^+ \times \mathbb{R}, \nabla)$  with Christoffel symbols  $\Gamma_{ij}^k = (x^1)^{-1} A_{ij}^k$  where  $A_{ij}^k = A_{ji}^k$  is constant. This geometry is homogeneous; the Type  $\mathcal{B}$  connections are the left invariant connections on the  $ax + b$  group.
- **Type  $\mathcal{C}$ .**  $\mathcal{M} = (M, \nabla)$  where  $\nabla$  is the Levi-Civita connection of the round sphere  $S^2$ .

This result has been applied by many authors. Kowalski and Sekizawa [10] used it to examine Riemannian extensions of affine surfaces, Vanzurova [13] used it to study the metrizability of locally homogeneous affine surfaces, and Důšek [5] used it to study homogeneous geodesics. It plays a central role in the study of locally homogeneous connections with torsion of Arias-Marco and Kowalski [1] (see also [2] for a unified treatment independently of the torsion tensor). Although we will work with the local theory, the compact setting has been examined in [8, 12].

The Ricci tensor  $\rho$  of an affine surface determines the full curvature tensor. In Section 2, we examine the spaces where the Ricci tensor has fixed rank in the

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Type  $\mathcal{A}$  setting. In Section 3, we consider the spaces where either the Ricci tensor vanishes identically or where the Ricci tensor is alternating and non-trivial in the Type  $\mathcal{B}$  setting.

**1.3. Type  $\mathcal{A}$  geometries.** Let  $\mathcal{M}(a, b, c, d, e, f) := (\mathbb{R}^2, \nabla)$  where the Christoffel symbols of  $\nabla$  are constant and given by

$$\begin{aligned} \Gamma_{11}^1 &= a, & \Gamma_{11}^2 &= b, & \Gamma_{12}^1 &= \Gamma_{21}^1 = c, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = d, & \Gamma_{22}^1 &= e, & \Gamma_{22}^2 &= f. \end{aligned} \quad (1.a)$$

This identifies the set of Type  $\mathcal{A}$  geometries with  $\mathbb{R}^6$ . The linear transformations  $T(x^1, x^2) = (a_1^1 x^1 + a_2^1 x^2, a_1^2 x^1 + a_2^2 x^2)$  where  $(a_i^j) \in \text{GL}(2, \mathbb{R})$  act on the set of Type  $\mathcal{A}$  geometries. We say that two Type  $\mathcal{A}$  surface models are *linearly equivalent* if there exists  $T \in \text{GL}(2, \mathbb{R})$  intertwining the two structures. One has that two Type  $\mathcal{A}$  surfaces with non-degenerate Ricci tensor are affine equivalent if and only if they are linearly equivalent (see [3]). On the contrary, there exist Type  $\mathcal{A}$  surfaces with degenerate Ricci tensor which are not linearly equivalent but which nevertheless are affine equivalent. We refer to the discussion in [6] for further details.

We consider the induced action of  $GL(2, \mathbb{R})$  on  $\mathbb{R}^6$  and identify the linear orbit of a Type  $\mathcal{A}$  model  $\mathcal{M}$  with  $\mathcal{S}(\mathcal{M}) = GL(2, \mathbb{R})/\mathcal{I}(\mathcal{M})$  where  $\mathcal{I}(\mathcal{M})$  is the isotropy group  $\mathcal{I}(\mathcal{M}) = \{T \in GL(2, \mathbb{R}); T^* \mathcal{M} = \mathcal{M}\}$ .

It was shown in [6] that any flat Type  $\mathcal{A}$  model is linearly equivalent to one of the following:

$$\begin{aligned} \mathcal{M}_0^0 &:= \mathcal{M}(0, 0, 0, 0, 0, 0), & \mathcal{M}_1^0 &:= \mathcal{M}(1, 0, 0, 1, 0, 0), \\ \mathcal{M}_2^0 &:= \mathcal{M}(-1, 0, 0, 0, 0, 1), & \mathcal{M}_3^0 &:= \mathcal{M}(0, 0, 0, 0, 0, 1), \\ \mathcal{M}_4^0 &:= \mathcal{M}(0, 0, 0, 0, 1, 0), & \mathcal{M}_5^0 &:= \mathcal{M}(1, 0, 0, 1, -1, 0). \end{aligned} \quad (1.b)$$

The structure  $\mathcal{M}_0^0$  is a singular cone point. The next result shows that the remaining orbits  $\mathcal{S}(\mathcal{M}_i^0) := GL(2, \mathbb{R}) \cdot \mathcal{M}_i^0$  for  $1 \leq i \leq 5$  glue together to define a smooth 4-dimensional submanifold of  $\mathbb{R}^6$ . Let  $\mathbb{1}$  be the trivial line bundle over the circle  $S^1$ , let  $\mathbb{L}$  be the Möbius line bundle over  $S^1$ , and let  $\mathcal{A}^0 \subset \mathbb{R}^6 \setminus \{0\}$  be the set of all flat Type  $\mathcal{A}$  geometries other than the cone point  $\mathcal{M}_0^0$ .

**Theorem 1.1.**  $\mathcal{A}^0$  is a smooth submanifold of  $\mathbb{R}^6 \setminus \{0\}$  diffeomorphic to the total space of  $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$  minus the zero section.

The Ricci tensor of any Type  $\mathcal{A}$  model is symmetric. Let  $\mathcal{A}_\pm^1 \subset \mathbb{R}^6$  be the set of all Type  $\mathcal{A}$  geometries where the Ricci tensor has rank 1 and is positive semi-definite (+) or negative semi-definite (-). Any element in  $\mathcal{A}_\pm^1$  is linearly equivalent to one of the following, where  $c \in \mathbb{R}$  and  $c_1 \in \mathbb{R} \setminus \{0, -1\}$  (see [3, 6]):

$$\begin{aligned} \mathcal{M}_1^1 &:= \mathcal{M}(-1, 0, 1, 0, 0, 2), \\ \mathcal{M}_2^1(c_1) &:= \mathcal{M}(-1, 0, c_1, 0, 0, 1 + 2c_1), \\ \mathcal{M}_3^1(c_1) &:= \mathcal{M}(0, 0, c_1, 0, 0, 1 + 2c_1), \\ \mathcal{M}_4^1(c) &:= \mathcal{M}(0, 0, 1, 0, c, 2), \\ \mathcal{M}_5^1(c) &:= \mathcal{M}(1, 0, 0, 0, 1 + c^2, 2c). \end{aligned} \quad (1.c)$$

We will see in Lemma 2.3 that the orbit structure of the action of  $GL(2, \mathbb{R})$  on  $\mathcal{A}_\pm^1$  is quite complicated. It is therefore, perhaps, a bit surprising that the set of all orbits  $\cup_{i,c} \mathcal{M}_i^1(c) \cdot GL(2, \mathbb{R})$  is smooth as shown in the following result.

**Theorem 1.2.**  $\mathcal{A}_\pm^1$  is a smooth submanifold of  $\mathbb{R}^6$  diffeomorphic to  $S^1 \times S^1 \times \mathbb{R}^3$ .

The remaining geometries where the Ricci tensor has rank 2 form an open subset  $\mathbb{R}^6 \setminus \{\{0\} \cup \mathcal{A}^0 \cup \mathcal{A}_+^1 \cup \mathcal{A}_-^1\}$ .

These results should be contrasted with the results in [9] where it is shown that any Type  $\mathcal{A}$  affine surface is linearly equivalent to a surface determined by at most two non-zero parameters.

1.4. **Type  $\mathcal{B}$  geometries.** Let  $\mathcal{N}(a, b, c, d, e, f) := (\mathbb{R}^+ \times \mathbb{R}, \nabla)$  where the Christoffel symbols of  $\nabla$  are given by

$$\begin{aligned} \Gamma_{11}^1 &= \frac{a}{x^1}, & \Gamma_{11}^2 &= \frac{b}{x^1}, & \Gamma_{12}^1 &= \Gamma_{21}^1 = \frac{c}{x^1}, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{d}{x^1}, & \Gamma_{22}^1 &= \frac{e}{x^1}, & \Gamma_{22}^2 &= \frac{f}{x^1}. \end{aligned} \quad (1.d)$$

This identifies the space of Type  $\mathcal{B}$  geometries with  $\mathbb{R}^6$ .

The natural structure group here is not the full general linear group, but rather the  $ax + b$  group. We let  $T_{a,b}(x^1, x^2) := (x^1, ax^2 + bx^1)$  define an action of the  $ax + b$  group on  $\mathbb{R}^+ \times \mathbb{R}$ ; this acts on the Type  $\mathcal{B}$  geometries by reparametrization and defines the natural notion of linear equivalence in this setting. Thus, two Type  $\mathcal{B}$  models  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are said to be *linearly equivalent* if and only if there exists an affine transformation of the form  $\Psi(x^1, x^2) = (x^1, a_1^2 x^1 + a_2^2 x^2)$  for  $a_2^2 \neq 0$  intertwining the two structures. It follows from the work in [3, 4] that two Type  $\mathcal{B}$  surfaces which are neither flat nor of Type  $\mathcal{A}$  are affine isomorphic if and only if they are linearly isomorphic. This is a non-trivial observation as there are non-linear affine transformations from one model to another if the dimension of the space of affine Killing vector fields is 4-dimensional or if the geometry is flat and thus the dimension of the space of affine Killing vector fields is 6-dimensional.

It was shown in [7] that a flat Type  $\mathcal{B}$  model is linearly equivalent to one of the following models:

$$\begin{aligned} \mathcal{N}_0^0 &:= \mathcal{N}(0, 0, 0, 0, 0, 0), & \mathcal{N}_1^0(\pm) &:= \mathcal{N}(1, 0, 0, 0, \pm 1, 0), \\ \mathcal{N}_2^0(c_1) &:= \mathcal{N}(c_1 - 1, 0, 0, c_1, 0, 0), \quad c_1 \neq 0, & \mathcal{N}_3^0 &:= \mathcal{N}(-2, 1, 0, -1, 0, 0), \\ \mathcal{N}_4^0 &:= \mathcal{N}(0, 1, 0, 0, 0, 0), & \mathcal{N}_5^0 &:= \mathcal{N}(-1, 0, 0, 0, 0, 0), \\ \mathcal{N}_6^0(c_2) &:= \mathcal{N}(c_2, 0, 0, 0, 0, 0), \quad c_2 \neq 0, -1. \end{aligned}$$

Let  $\mathcal{B}^0 \subset \mathbb{R}^6$  be the space of flat Type  $\mathcal{B}$  geometries other than the cone point  $\mathcal{N}_0^0$  determined by the origin in  $\mathbb{R}^6$ . Unlike the Type  $\mathcal{A}$  setting described in Theorem 1.1,  $\mathcal{B}^0$  is not a smooth manifold but consists of the union of 3 smooth submanifolds of  $\mathbb{R}^6$  which intersect transversally along the union of 3 smooth curves in  $\mathbb{R}^6$ . Define

$$\begin{aligned} \mathcal{U}_1(r, s) &:= \mathcal{N}(1 + rs^2, -s(1 + rs^2), rs, -rs^2, r, -rs), & \mathcal{B}_1 &:= \text{Range}\{\mathcal{U}_1\}, \\ \mathcal{U}_2(u, v) &:= \mathcal{N}(u, v, 0, 0, 0, 0), & \mathcal{B}_2 &:= \text{Range}\{\mathcal{U}_2\}, \\ \mathcal{U}_3(u, v) &:= \mathcal{N}(u, v, 0, 1 + u, 0, 0), & \mathcal{B}_3 &:= \text{Range}\{\mathcal{U}_3\}. \end{aligned} \quad (1.e)$$

**Theorem 1.3.**  $\mathcal{B}^0 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ .  $\mathcal{B}_2$  and  $\mathcal{B}_3$  are closed smooth surfaces in  $\mathbb{R}^6$  which are diffeomorphic to  $\mathbb{R}^2$  and which intersect transversally along the curve  $\mathcal{N}(-1, v, 0, 0, 0, 0)$  for  $v \in \mathbb{R}$ .  $\mathcal{B}_1$  can be completed to a smooth closed surface  $\tilde{\mathcal{B}}_1$  which intersects  $\mathcal{B}_2$  transversally along the curve  $\mathcal{N}(1, v, 0, 0, 0, 0)$  and which intersects  $\mathcal{B}_3$  transversally along the curve  $\mathcal{N}(0, v, 0, 1, 0, 0)$  for  $v \in \mathbb{R}$ .

In the Type  $\mathcal{B}$  setting, it is possible for the symmetric Ricci tensor  $\rho_s$  to vanish without the geometry being flat; this is not possible in the Type  $\mathcal{A}$  setting. The alternating Ricci tensor,  $\rho_a$ , carries the geometry in this context.

Let  $\mathcal{B}_a$  be the set of all Type  $\mathcal{B}$  structures where  $\rho_s = 0$  but  $\rho_a \neq 0$ . Set

$$\begin{aligned} \mathcal{V}_1(r, s, t) &:= \mathcal{N}(s, t, r, 0, 0, r), \\ \mathcal{V}_2(u, v, w) &:= \mathcal{N}(1 - 2uw + vw^2, w(1 - uw + vw^2), u - vw, -vw^2, v, u + vw) \end{aligned} \quad (1.f)$$

and let  $\mathcal{D}_1 := \text{Range}\{\mathcal{V}_1\}$  and  $\mathcal{D}_2 := \text{Range}\{\mathcal{V}_2\}$ .

**Theorem 1.4.**  $\mathcal{B}_a = \mathcal{D}_1 \cup \mathcal{D}_2$ .  $\mathcal{V}_i$  defines smoothly embedded 3-dimensional submanifolds of  $\mathbb{R}^6$  for  $r \neq 0$  and  $u \neq 0$  which intersect transversally along a smooth 2-dimensional submanifold.

## 2. THE SPACE OF TYPE $\mathcal{A}$ MODELS

Let  $\mathcal{M}(a, b, c, d, e, f) := (\mathbb{R}^2, \nabla)$  be given by Equation (1.a) where the parameters  $(a, b, c, d, e, f)$  are real constants. The associated Ricci tensor is symmetric.

**2.1. The space of flat Type  $\mathcal{A}$  models.** Since the Ricci tensor determines the curvature in dimension two, flat surfaces are determined by a vanishing Ricci tensor. We provide the proof of the first result of the paper as follows.

*The proof of Theorem 1.1.* Let  $\theta \in [0, 2\pi]$  be the usual periodic parameter where we identify 0 with  $2\pi$  to define the circle  $S^1 = (\cos \theta, \sin \theta)$ . Let  $(x^1, x^2, x^3)$  be a point of  $\mathbb{R}^3$ . The bundle  $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$  is then defined by identifying  $(\theta, x^1, x^2, x^3)$  with  $(\theta + \pi, -x^1, x^2, x^3)$ ; this puts the necessary half twist in the first  $x$ -coordinate. We require that  $(x^1, x^2, x^3)$  belongs to  $\mathbb{R}^3 - \{0\}$  to remove the 0-section.

The parametrization of Equation (1.a) is not a very convenient one for studying the Ricci tensor. We make a linear change of coordinates on  $\mathbb{R}^6$  and let  $\mathcal{M}_1(p, q, t, s, v, w)$  be defined by

$$\begin{aligned} \Gamma_{11}^1 &= 2q, & \Gamma_{11}^2 &= p + t, & \Gamma_{12}^1 &= \Gamma_{21}^1 = w, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = q + s, & \Gamma_{22}^1 &= v, & \Gamma_{22}^2 &= p - t. \end{aligned}$$

We substitute these values in Equation (2.d) to obtain

$$\rho = \begin{pmatrix} p^2 + q^2 - s^2 - t^2 - pw - tw & -(p+t)v + (q+s)w \\ -(p+t)v + (q+s)w & qv - sv + (p-t-w)w \end{pmatrix}.$$

We set  $\rho = 0$ . If  $v^2 + w^2 \neq 0$ , we obtain

$$\begin{aligned} p &= (v^2 + w^2)^{-1} \{2svw + t(w^2 - v^2) + w^3\}, \text{ and} \\ q &= (v^2 + w^2)^{-1} \{s(v^2 - w^2) + vw(2t + w)\}. \end{aligned} \quad (2.a)$$

If  $v^2 + w^2 = 0$ , we obtain a single equation

$$p^2 + q^2 - s^2 - t^2 = 0. \quad (2.b)$$

We introduce polar coordinates  $v = r \cos(\theta)$  and  $w = r \sin(\theta)$  to remove the singularity at  $(v, w) = (0, 0)$  in Equation (2.a). We may then combine Equation (2.a) and Equation (2.b) into a single expression:

$$\begin{aligned} p &= p(\theta, r, s, t) := r \sin^3(\theta) + s \sin(2\theta) - t \cos(2\theta), \\ q &= q(\theta, r, s, t) := r \cos(\theta) \sin^2(\theta) + s \cos(2\theta) + t \sin(2\theta). \end{aligned} \quad (2.c)$$

We assume  $(r, s, t) \neq (0, 0, 0)$  to avoid the trivial structure  $\mathcal{M}_0^0$  as the parametrization of Equation (2.c) is singular there. We have  $\theta \in [0, 2\pi]$  and  $(r, s, t) \in \mathbb{R}^3 - \{0\}$ ; since we are permitting  $r$  to be negative in polar coordinates, we must identify  $(\theta, r)$  with  $(\theta + \pi, -r)$  and obtain thereby the bundle  $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$  minus the zero section over  $[0, \pi]$ .  $\square$

**Remark 2.1.** The isotropy subgroups of the structures  $\mathcal{M}_i^0$  vary with  $i$  and the dimension of the orbit space varies correspondingly. We list below the associated

isotropy subgroups.

$$\begin{aligned}\mathcal{I}(\mathcal{M}_0^0) &= \text{GL}(2, \mathbb{R}), \\ \mathcal{I}(\mathcal{M}_1^0) &= \{T : T(x^1, x^2) = (x^1, ax^2) \text{ for } a \neq 0\}, \\ \mathcal{I}(\mathcal{M}_2^0) &= \{id, T\}, \text{ where } T(x^1, x^2) = (-x^2, -x^1), \\ \mathcal{I}(\mathcal{M}_3^0) &= \{T : T(x^1, x^2) = (ax^1, x^2) \text{ for } a \neq 0\}, \\ \mathcal{I}(\mathcal{M}_4^0) &= \{T : T(x^1, x^2) = (a^2x^1 + bx^2, ax^2) \text{ for } a \neq 0, b \in \mathbb{R}\}, \\ \mathcal{I}(\mathcal{M}_5^0) &= \{T : T(x^1, x^2) = (x^1, \pm x^2)\}.\end{aligned}$$

**2.2. The space of Type  $\mathcal{A}$  models with rank-one Ricci tensor.** If the Ricci tensor has rank 1, we can make a linear change of coordinates to ensure  $\rho$  is a multiple of  $dx^2 \otimes dx^2$ . We first establish Theorem 1.2. We then examine the isotropy groups of the models in Equation (1.c) to determine the orbits of the Type  $\mathcal{A}$  models which are not Type  $\mathcal{B}$ .

**Lemma 2.2.** *Let  $\mathcal{M}$  be a Type  $\mathcal{A}$  model which is not flat. Then  $\rho$  is a multiple of  $dx^2 \otimes dx^2$  if and only if  $b = 0$  and  $d = 0$ .*

*Proof.* A direct computation shows

$$\rho = \begin{pmatrix} (a-d)d + b(f-c) & cd - be \\ cd - be & c(f-c) + (a-d)e \end{pmatrix}. \quad (2.d)$$

Consequently, if  $b = 0$  and if  $d = 0$ , then  $\rho$  is a multiple of  $dx^2 \otimes dx^2$ . Conversely, assume  $\rho$  is a multiple of  $dx^2 \otimes dx^2$  or, equivalently,  $-bc + ad - d^2 + bf = 0$  and  $cd - be = 0$ . We wish to show  $b = d = 0$ .

**Case 1.** Suppose that  $d \neq 0$ . The equations are homogeneous so we may assume  $d = 1$  and hence  $c = be$ . Substituting these values yields  $\rho_{11} = -1 + a - b^2e + bf = 0$ . Thus  $a = 1 + b^2e + bf$ . This yields  $\rho = 0$  so this case is impossible as we assumed  $\mathcal{M}$  was not flat.

**Case 2.** Suppose that  $b \neq 0$ . Again, we may assume  $b = 1$  so  $e = cd$ . We compute  $\rho_{11} = f - c + ad - d^2$ . Setting this to zero again yields  $\rho = 0$  which is impossible.  $\square$

*Proof of Theorem 1.2.* Let  $\mathcal{A}_{\pm,0}^1$  be the space of all Type  $\mathcal{A}$  models where the Ricci tensor is a non-zero multiple of  $dx^2 \otimes dx^2$  where the  $\pm$  refers to whether  $\rho_{22}$  is positive or negative. By Lemma 2.2, we set  $b = d = 0$  and obtain  $\rho_{22} = -c^2 + ae + cf$ . We make a change of variables setting

$$a = q + v, \quad b = 0, \quad c = u + p, \quad d = 0, \quad e = q - v, \quad f = 2p.$$

We then have  $\rho_{22} = (p^2 + q^2 - u^2 - v^2)dx^2 \otimes dx^2$  so we may identify

$$\begin{aligned}\mathcal{A}_{+,0}^1 &= \{\Gamma(p, q, u, v) : p^2 + q^2 > u^2 + v^2\}, \\ \mathcal{A}_{-,0}^1 &= \{\Gamma(p, q, u, v) : p^2 + q^2 < u^2 + v^2\}.\end{aligned}$$

We examine  $\mathcal{A}_{-,0}^1$  as the analysis of  $\mathcal{A}_{+,0}^1$  is the same after interchanging the roles of  $(p, q)$  and  $(u, v)$ . Let  $\mathcal{D}^2 := \{(U, V) \in \mathbb{R}^2 : U^2 + V^2 < 1\}$  be the open disk in  $\mathbb{R}^2$ . Let  $-\mathcal{M}$  be the Type  $\mathcal{A}$  model  $\mathcal{M}(-a, -b, -c, -d, -e, -f)$ . We construct a diffeomorphism  $\Phi$  from  $S^1 \times \mathbb{R}^+ \times \mathcal{D}^2$  to  $\mathcal{A}_{-,0}^1$  by setting  $u = r \cos \theta$ ,  $v = r \sin \theta$ ,  $p = rU$ ,  $q = rV$ . For  $r > 0$ ,  $\theta \in S^1$ , and  $U^2 + V^2 < 1$  we have

$$\mathcal{M} = \mathcal{M}(r(\sin(\theta) + V), 0, r(\cos(\theta) + U), 0, r(V - \sin(\theta)), 2rU).$$

It is clear that  $-\mathcal{M}(\theta, r, U, V) = \mathcal{M}(\theta + \pi, r, -U, -V)$ .

Let  $\tilde{\mathcal{M}}$  be an arbitrary Type  $\mathcal{A}$  model with  $\text{Rank}\{\rho_{\tilde{\mathcal{M}}}\} = 1$  and  $\rho_{\tilde{\mathcal{M}}}$  negative semi-definite. We may express

$$\rho_{\tilde{\mathcal{M}}} = \lambda(\cos(\phi)dx^2 - \sin(\phi)dx^1) \otimes (\cos(\phi)dx^2 - \sin(\phi)dx^1)$$

for  $\lambda < 0$ . Here  $\phi$  is only defined modulo  $\pi$  instead of the usual  $2\pi$ . Let

$$T_\phi(x^1, x^2) = (\cos(\phi)x^1 + \sin(\phi)x^2, -\sin(\phi)x^1 + \cos(\phi)x^2).$$

be the associated rotation so that  $T_\phi^*(dx^2) = -\sin(\phi)dx^1 + \cos(\phi)dx^2$  and thus  $(T_\phi)_*\tilde{\mathcal{M}}$  belongs to  $\mathcal{A}_{-,0}^1$ . We then have

$$\mathcal{A}_-^1 = \{\mathbb{R}/(2\pi\mathbb{Z}) \times \mathcal{A}_{-,0}^1\}/(\phi, \mathcal{M}) \sim (\phi + \pi, -\mathcal{M})$$

where the gluing reflects the fact that when  $\phi = \pi$  we have replaced  $(x^1, x^2)$  by  $(-x^1, -x^2)$  and thus changed the sign of the Christoffel symbols. Using our previous parametrization of  $\mathcal{A}_{-,0}^1$ , this yields

$$\mathcal{A}_-^1 = (\mathbb{R}^2/(2\pi\mathbb{Z})^2) \times \mathbb{R}^+ \times \mathcal{D}^2/\{(\phi, \theta, r, U, V) \sim (\phi + \pi, \theta + \pi, r, -U, -V)\}.$$

After setting  $\tilde{\theta} = \theta + \phi$ , we can rewrite this equivalence relation in the form

$$(\phi, \tilde{\theta}, r, U, V) \sim (\phi + \pi, \tilde{\theta}, r, -U, -V).$$

The variable  $\tilde{\theta}$  now no longer plays a role in the gluing. After replacing  $\mathbb{R}^+$  by  $\mathbb{R}$  and  $\mathcal{D}^2$  by  $\mathbb{R}^2$ , we see  $\mathcal{A}_-^1$  is diffeomorphic to  $S^1 \times S^1 \times \mathbb{R}^3$  modulo the relation

$$(\phi, \tilde{\theta}, x_1, x_2, x_3) \sim (\phi + \pi, \tilde{\theta}, x_1, -x_2, -x_3).$$

These gluing relations define the total space of the bundle  $\mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}$  over  $(S^1, \phi)$ . Since  $\mathbb{L} \oplus \mathbb{L}$  is diffeomorphic to the trivial 2-plane bundle  $\mathbb{1} \oplus \mathbb{1}$ , we obtain finally that  $\mathcal{A}_-^1$  is diffeomorphic to  $S^1 \times S^1 \times \mathbb{R}^3$ .  $\square$

We adopt the notation of Equation (1.c) to describe the orbits of the models  $\mathcal{M}_i^1(\cdot)$  in the following lemma.

**Lemma 2.3.**

- (1)  $\mathcal{I}(\mathcal{M}_1^1) = \{\text{id}\}$ .
- (2)  $\mathcal{I}(\mathcal{M}_2^1(c_1)) = \{\text{id}\}$  if  $c_1 \neq -\frac{1}{2}$ .
- (3)  $\mathcal{I}(\mathcal{M}_2^1(-\frac{1}{2})) = \{\text{id}, T\}$ , where  $T(x^1, x^2) = (x^1 + x^2, -x^2)$ .
- (4)  $\mathcal{I}(\mathcal{M}_3^1(c_1)) = \{T : T(x^1, x^2) = (v^{-1}x^1, x^2) \text{ for } v \in \mathbb{R} \setminus \{0\}\}$ .
- (5)  $\mathcal{I}(\mathcal{M}_4^1(c)) = \{T : T(x^1, x^2) = (x^1 - wx^2, x^2) \text{ for } w \in \mathbb{R}\}$ , if  $c \neq 0$ .
- (6)  $\mathcal{I}(\mathcal{M}_4^1(0)) = \{T : T(x^1, x^2) = (v^{-1}(x^1 - wx^2), x^2) \text{ for } w \in \mathbb{R}, v \in \mathbb{R} \setminus \{0\}\}$ .
- (7)  $\mathcal{I}(\mathcal{M}_5^1(c)) = \{\text{id}\}$ , if  $c \neq 0$ .
- (8)  $\mathcal{I}(\mathcal{M}_5^1(0)) = \{\text{id}, T\}$  where  $T(x^1, x^2) = (x^1, -x^2)$ .

*Proof.* Suppose  $T \in \mathcal{I}(\mathcal{M}_i^1(\cdot))$ . The Ricci tensor of  $\mathcal{M}_i^1(\cdot)$  is a non-zero multiple of  $dx^2 \otimes dx^2$ . Since  $T$  must preserve the Ricci tensor,  $T(dx^2) = \pm dx^2$ . This implies  $(y^1, y^2) = T(x^1, x^2) = (v^{-1}(x^1 - wx^2), \varepsilon x^2)$  for  $\varepsilon = \pm 1$ . Then

$$\begin{aligned} dy^1 &= v^{-1}(dx^1 - wdx^2), & dy^2 &= \varepsilon dx^2, & \partial_{y^1} &= v\partial_{x^1}, & \partial_{y^2} &= \varepsilon(w\partial_{x^1} + \partial_{x^2}), \\ {}^y\Gamma_{11}^1 &:= v({}^x\Gamma_{11}^1 - w {}^x\Gamma_{11}^2), \\ {}^y\Gamma_{11}^2 &:= v^2\varepsilon {}^x\Gamma_{11}^2, \\ {}^y\Gamma_{12}^1 &:= \varepsilon({}^x\Gamma_{12}^1 + w({}^x\Gamma_{11}^1 - {}^x\Gamma_{12}^2 - w {}^x\Gamma_{11}^2)), \\ {}^y\Gamma_{12}^2 &:= v({}^x\Gamma_{12}^2 + w {}^x\Gamma_{11}^2), \\ {}^y\Gamma_{22}^1 &:= \frac{1}{v}({}^x\Gamma_{22}^1 + w(2 {}^x\Gamma_{12}^1 - {}^x\Gamma_{22}^2)) + w^2({}^x\Gamma_{11}^1 - 2 {}^x\Gamma_{12}^2) - w^3 {}^x\Gamma_{11}^2, \\ {}^y\Gamma_{22}^2 &:= \varepsilon({}^x\Gamma_{22}^2 + 2w {}^x\Gamma_{12}^2 + w^2 {}^x\Gamma_{11}^2). \end{aligned}$$

**Case 1.**  $\mathcal{M}_1^1 = \mathcal{M}(-1, 0, 1, 0, 0, 2)$  and  $T^*\mathcal{M}_1^1 = \mathcal{M}(-v, 0, \varepsilon(1 - w), 0, -\frac{w^2}{v}, 2\varepsilon)$ . Examining  $\Gamma_{11}^1$  and  $\Gamma_{22}^2$  yields  $\varepsilon = 1$  and  $v = 1$ . Examining  $\Gamma_{22}^1$  yields  $w = 0$ .

**Case 2.** We have  $c \notin \{0, -1\}$ ,  $\mathcal{M}_2^1(c) = \mathcal{M}(-1, 0, c, 0, 0, 1 + 2c)$ , and

$$T^*\mathcal{M}_2^1(c) = \mathcal{M}(-v, 0, \varepsilon(c - w), 0, -\frac{1}{v}(w + w^2), (1 + 2c)\varepsilon).$$

Examining  $\Gamma_{11}^1$  yields  $v = 1$ . Suppose  $c \neq -\frac{1}{2}$ . Examining  $\Gamma_{22}^2$  yields  $\varepsilon = 1$ . Since  $\varepsilon = 1$ , examining  $\Gamma_{12}^1$  yields  $w = 0$ . Suppose  $c = -\frac{1}{2}$ . Examining  $\Gamma_{12}^1$  and  $\Gamma_{22}^1$  yields  $(\varepsilon, w) = (1, 0)$  or  $(\varepsilon, w) = (-1, -1)$ .

**Case 3.** We have  $c \notin \{0, -1\}$ ,  $\mathcal{M}_3^1(c) = \mathcal{M}(0, 0, c, 0, 0, 1 + 2c)$ , and

$$T^*\mathcal{M}_3^1(c) = \mathcal{M}(0, 0, c\varepsilon, 0, -\frac{w}{v}, (1 + 2c)\varepsilon).$$

Examining  $\Gamma_{12}^1$  yields  $\varepsilon = 1$ . Examining  $\Gamma_{22}^1$  yields  $w = 0$ . There is then no condition on  $v$ .

**Case 4.**  $\mathcal{M}_4^1(c) = \mathcal{M}(0, 0, 1, 0, c, 2)$  and  $T^*\mathcal{M}_4^1(c) = \mathcal{M}(0, 0, \varepsilon, 0, \frac{c}{v}, 2\varepsilon)$ . Examining  $\Gamma_{22}^2$  yields  $\varepsilon = 1$ . There is no condition on  $w$ . If  $c \neq 0$ , examining  $\Gamma_{22}^1$  yields  $v = 1$ ; if  $c = 0$ , there is no condition on  $v$ .

**Case 5.**  $\mathcal{M}_5^1(c) = \mathcal{M}(1, 0, 0, 0, 1 + c^2, 2c)$  and

$$T^*\mathcal{M}_5^1(c) = \mathcal{M}(v, 0, w\varepsilon, 0, \frac{1}{v}(1 + (c - w)^2), 2c\varepsilon).$$

Examining  $\Gamma_{11}^1$  shows  $v = 1$ . Examining  $\Gamma_{12}^1$  shows  $w = 0$ . If  $c \neq 0$ , examining  $\Gamma_{22}^2$  shows  $\varepsilon = 1$ . If  $c = 0$ , we obtain  $\varepsilon = \pm 1$ .  $\square$

The general linear group  $\text{GL}(2, \mathbb{R})$  acts on the space  $\mathbb{R}^6$  of all Type  $\mathcal{A}$  geometries via change of coordinates. Let  $\text{GL}_+(2, \mathbb{R})$  be the subgroup of matrices with positive determinant. If  $\mathcal{M}$  is a Type  $\mathcal{A}$  model with  $\text{Rank}\{\rho\}(\mathcal{M}) = 2$ , then the associated space of affine Killing vector fields is 2-dimensional and  $\mathcal{M}$  does not also admit a Type  $\mathcal{B}$  structure [3]. But there are Type  $\mathcal{A}$  models with  $\text{Rank}\{\rho\} = 1$  which also admit Type  $\mathcal{B}$  structures. Let  $\mathcal{O}_\pm^1 \subset \mathcal{A}_\pm^1$  be the set of Type  $\mathcal{A}$  models with  $\text{Rank}\{\rho\} = 1$  and which do not admit Type  $\mathcal{B}$  structures.

**Theorem 2.4.**

- (1)  $\mathcal{O}_-^1$  is empty; every element of  $\mathcal{A}_-^1$  also admits Type  $\mathcal{B}$  structure.
- (2)  $\text{GL}_+(2, \mathbb{R})$  acts without fixed points on  $\mathcal{O}_+^1$ . The action admits a section  $s : \mathbb{R} \rightarrow \mathcal{O}_+^1$  so  $\mathcal{O}_+^1 = \text{GL}_+(2, \mathbb{R}) \times \mathbb{R}$  is a principal fiber bundle over  $\mathbb{R}$ .

*Proof.* Results of [3] show that the models  $\mathcal{M}_i^1(\cdot)$  for  $1 \leq i \leq 4$  also admit Type  $\mathcal{B}$  structures while the models  $\mathcal{M}_5^1(c)$  do not. The Ricci tensor associated to  $\mathcal{M}_i^1(\cdot)$  is given by:

$$\begin{aligned} \rho^{\mathcal{M}_1^1} &= dx^2 \otimes dx^2, & \rho^{\mathcal{M}_2^1} &= c_1(1 + c_1)dx^2 \otimes dx^2, \\ \rho^{\mathcal{M}_3^1} &= c_1(1 + c_1)dx^2 \otimes dx^2, & \rho^{\mathcal{M}_4^1} &= dx^2 \otimes dx^2, \\ \rho^{\mathcal{M}_5^1} &= (1 + c^2)dx^2 \otimes dx^2. \end{aligned}$$

If  $\rho \leq 0$ , then it follows that  $i = 2$  or  $i = 3$  and  $c \in (-1, 0)$ . Thus any element of  $\mathcal{A}_-^1$  admits a Type  $\mathcal{B}$  structure which proves Assertion (1).

Let  $\mathfrak{M}_5^1 = \cup_c \mathcal{M}_5^1(c)$ ; this is a smooth curve in  $\mathbb{R}^6$ . Type  $\mathcal{A}$  models which are linearly equivalent to  $\mathcal{M}_1^1$ ,  $\mathcal{M}_2^1(c_1)$  for  $c_1 + c_1^2 > 0$ ,  $\mathcal{M}_3^1(c_1)$  for  $c_1 + c_1^2 > 0$ , or  $\mathcal{M}_4^1(c)$  all admit Type  $\mathcal{B}$  structures and have  $\rho \geq 0$ . Thus we may identify the structures  $\mathcal{O}_{1,+}$  which do not admit Type  $\mathcal{B}$  structures with  $\text{GL}(2, \mathbb{R}) \cdot \mathfrak{M}_5^1$ . Let  $T(x^1, x^2) := (x^1, -x^2)$ . We have  $T\mathcal{M}_5^1(c) = \mathcal{M}_5^1(-c)$ . Since  $\det(T) = -1$ , we conclude therefore that  $\mathcal{O}_+^1 = \text{GL}_+(2, \mathbb{R}) \cdot \mathfrak{M}_5^1$ . By Lemma 2.3, the action of  $\text{GL}_+(2, \mathbb{R})$  on  $\mathfrak{M}_5^1$  is fixed point free. Assertion (2) follows.  $\square$

### 3. THE SPACE OF TYPE $\mathcal{B}$ CONNECTIONS

Let  $\mathcal{N}(a, b, c, d, e, f) := (\mathbb{R}^+ \times \mathbb{R}, \nabla)$  where the Christoffel symbols of  $\nabla$  are given by (1.d). The Ricci tensor needs not be symmetric in this setting:

$$\rho = (x^1)^{-2} \begin{pmatrix} (a-d+1)d + b(f-c) & cd - be + f \\ c(d-1) - be & -c^2 + fc + (a-d-1)e \end{pmatrix} \quad (3.a)$$

#### 3.1. The space of flat Type $\mathcal{B}$ models.

*The proof of Theorem 1.3.* Let  $\mathcal{N} = \mathcal{N}(a, b, c, d, e, f)$ . We clear denominators in Equation (3.a) and set  $\tilde{\rho}_{ij} = (x^1)^2 \rho_{ij}$ . Adopt the notation of Equation (1.e). A direct computation shows **that** the structures  $\mathcal{U}_i(\cdot)$  are flat. We distinguish cases to establish the converse. We use Equation (3.a) and set  $\tilde{\rho} = 0$ . Since  $\tilde{\rho}_{12} - \tilde{\rho}_{21} = c + f$ ,  $f = -c$ .

**Case 1.** Assume  $e \neq 0$ . Set  $c = rs$ ,  $e = r$ , and  $f = -rs$  for  $r \neq 0$ . Then

$$\tilde{\rho}_{22} = -r(1 - a + d + 2rs^2) \text{ and } \tilde{\rho}_{21} = -r(b + s - ds).$$

We solve these equations to obtain  $a = 1 + d + 2rs^2$  and  $b = (-1 + d)s$ . We have  $\tilde{\rho}_{11} = 2(d + rs^2)$ . Thus  $d = -rs^2$  which gives the parametrization  $\mathcal{U}_1$ .

**Case 2.** Suppose  $e = 0$ . Set  $a = u$ ,  $b = v$ , and  $f = -c$  to obtain

$$\tilde{\rho} = \begin{pmatrix} d(1 + u - d) - 2cv & c(d-1) \\ c(d-1) & -2c^2 \end{pmatrix}.$$

This yields  $c = 0$  and  $d(1 + u - d) = 0$ . If we set  $d = 0$ , we obtain the parametrization  $\mathcal{U}_2$ ; if we set  $d = 1 + u$ , we obtain the parametrization  $\mathcal{U}_3$ . This establishes the first assertion.

The parametrization  $\mathcal{U}_2$  and  $\mathcal{U}_3$  intersect when  $u = -1$ ; the intersection is transversal along the curve  $\mathcal{N}(-1, v, 0, 0, 0, 0)$ . We wish to extend the parametrization  $\mathcal{U}_1$  to study the limiting behavior as  $e \rightarrow 0$ . We distinguish cases.

**Case A.** Suppose  $\lim_{n \rightarrow \infty} \mathcal{U}_1(r_n, s_n) \in \text{Range}\{\mathcal{U}_2\}$ . We have

$$\begin{aligned} \lim_{n \rightarrow \infty} 1 + r_n s_n^2 &= u, & \lim_{n \rightarrow \infty} -s_n(1 + r_n s_n^2) &= v, & \lim_{n \rightarrow \infty} -r_n s_n &= 0, \\ \lim_{n \rightarrow \infty} -r_n s_n^2 &= 0, & \lim_{n \rightarrow \infty} r_n &= 0, & \lim_{n \rightarrow \infty} -r_n s_n &= 0. \end{aligned}$$

These equations imply  $u = 1$ ,  $\lim_{n \rightarrow \infty} r_n = 0$ ,  $\lim_{n \rightarrow \infty} s_n = -v$ . Thus we may simply set  $r = 0$  to obtain a transversal intersection along the curve  $\mathcal{N}(1, v, 0, 0, 0, 0)$ .

**Case B.** Suppose  $\lim_{n \rightarrow \infty} \mathcal{U}_1(r_n, s_n) \in \text{Range}\{\mathcal{U}_3\}$ . We have

$$\begin{aligned} \lim_{n \rightarrow \infty} 1 + r_n s_n^2 &= u, & \lim_{n \rightarrow \infty} -s_n(1 + r_n s_n^2) &= v, & \lim_{n \rightarrow \infty} -r_n s_n &= 0, \\ \lim_{n \rightarrow \infty} -r_n s_n^2 &= 1 + u, & \lim_{n \rightarrow \infty} r_n &= 0, & \lim_{n \rightarrow \infty} -r_n s_n &= 0. \end{aligned}$$

These equations imply  $u = 0$ ,  $\lim_{n \rightarrow \infty} r_n = 0$ , and  $\lim_{n \rightarrow \infty} r_n s_n^2 = -1$ . We change variables setting  $r = -t^2$  and  $s = \frac{1}{t} + w$  to express

$$\mathcal{U}_1(-t^2, \frac{1}{t} + w) = \mathcal{N} \left( \begin{array}{ccc} -tw(2 + tw), & w(2 + 3tw + t^2 w^2), & -t(1 + tw), \\ (1 + tw)^2, & -t^2, & t(1 + tw) \end{array} \right).$$

We may now safely set  $t = 0$  to obtain the intersection with  $\text{Range}\{\mathcal{U}_3\}$  along the curve  $\mathcal{N}(0, 2w, 0, 1, 0, 0)$ .  $\square$

**3.2. Type  $\mathcal{B}$  models with alternating Ricci tensor.** It was shown in [3] that any Type  $\mathcal{B}$  model with alternating Ricci tensor is linearly equivalent to one of the following models:

$$\begin{aligned} \mathcal{N}_1(c) &:= \mathcal{N}(0, c, 1, 0, 0, 1), & \text{for } c \in \mathbb{R}, \\ \mathcal{N}_2(c, \pm) &:= \mathcal{N}(1 \mp c^2, c, 0, \mp c^2, \pm 1, \pm 2c), & \text{for } c > 0. \end{aligned}$$

The proof of Theorem 1.4. Adopt the notation of Equation (1.f). It is clear that  $\mathcal{V}_1$  defines a smooth 3-dimensional submanifold of  $\mathbb{R}^6$ . To see similarly that  $\mathcal{V}_2$  is smooth, we note that we can recover  $u = \frac{1}{2}(c + f)$  and  $v = e$ . If  $v \neq 0$ , then  $w = \frac{1}{v}(f - u)$  while if  $v = 0$ ,  $w = \frac{1}{2u}(1 - a)$ . Thus  $\mathcal{V}_2$  is 1-1; it is not difficult to verify that the Jacobian determinant is non-zero. This shows that  $\mathcal{V}_2$  also defines a smooth 3-dimensional submanifold of  $\mathbb{R}^6$ . We set  $v = 0$  and  $u = r$  to see that  $\mathcal{V}_1$  and  $\mathcal{V}_2$  intersect along the surface  $v = 0$ ,  $u = r$ ,  $s = 1 - 2uw$  and  $t = w(1 - uw)$ . A direct computation shows that the associated Ricci tensors are non-trivial and alternating:

$$\tilde{\rho}_{\mathcal{V}_1} = r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \tilde{\rho}_{\mathcal{V}_2} = u \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let  $\mathcal{N}$  be a Type  $\mathcal{B}$  model with  $\rho_s = 0$  and  $\tilde{\rho}_{a,12} = \frac{c+f}{2} \neq 0$ . We distinguish two cases.

**Case 1.** Suppose  $e = 0$ . Set  $c = 2r - f$  for  $r \neq 0$ . Setting the  $\rho_s = 0$  yields

$$\begin{aligned} \rho_{s,11} : 0 &= d(1 + a - d) + 2b(f - r), & \rho_{s,12} : 0 &= (1 - d)f + r(2d - 1), \\ \rho_{s,22} : 0 &= -2(f^2 - 3fr + 2r^2). \end{aligned}$$

We solve the equation  $-2(f^2 - 3fr + 2r^2) = 0$  to obtain  $f = r$  or  $f = 2r$ . Setting  $f = 2r$  yields  $\rho_{s,12} : 0 = r$  which is false. Thus  $f = r$ . We obtain  $\rho_{s,12} = 2dr$  so  $d = 0$ . Set  $a = s$  and  $b = t$  to obtain the parametrization  $\mathcal{V}_1$ .

**Case 2.** Set  $c = 2u - f$  and  $e = v$  for  $u \neq 0$  and  $v \neq 0$ . We obtain

$$\begin{aligned} \rho_{s,11} : 0 &= d(1 + a - d) + 2b(f - u), \\ \rho_{s,12} : 0 &= (1 - d)f - u + 2du - bv, \\ \rho_{s,22} : 0 &= -2f^2 + 6fu - 4u^2 - (1 - a + d)v. \end{aligned}$$

Setting  $\rho_{s,12} = 0$  and  $\rho_{s,22} = 0$  yields  $a = \frac{1}{v}(2f^2 - 6fu + 4u^2 + v + dv)$  and  $b = \frac{1}{v}(f - df - u + 2du)$ . We obtain  $\rho_{s,11} = \frac{1}{v}(2(f^2 - 2fu + u^2 + dv))$ . This implies that  $d = -\frac{(f-u)^2}{v}$ . Setting  $f = vw + u$  yields the parametrization  $\mathcal{V}_2$ . This parametrization can be extended safely to  $v = 0$ ; we require  $u \neq 0$  to ensure  $\rho_a \neq 0$ .  $\square$

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